Grothendieck's Galois Theory

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November 8, 2022

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1 Introduction

The Fundamental Theorem of Galois Theory is an old and profound result of field theory. In modern terms it establishes a one-to-one correspondence between the subextensions of a finite Galois extension of a field and the subgroups of the Galois group of that extension. A statement and a "classical" proof of this theorem can be found in [3].

Another similar result, this time in algebraic topology, is the Fundamental Theorem of Covering Theory. This theorems states the existence of a one-to-one correspondence between subgroups of the fundamental group and path-connected covering spaces of a path-connected topological space.

The two statements, at least on an intuitive level, have something in common. Intrigued by this curious fact and moved by a belief that this is no coincidence Grothendieck was able to prove a theorem (here Theorem 3.7) that has both of them as consequences; thus unifying the matter[1]. In this pages we will explore Grothendieck's approach following closely section III of [1], but with some added details (mostly confined in the appendix).

[1]It is noteworthy to say that Grothendieck went even further than that and was able to prove an even more general theorem that, even for the classical case of fields, yields results that weren't previously known. However, we will stick to the base case here; the interested reader can check [1] section IV and V.

The reader is assumed to have a basic knowledge of category theory (categories, functors, natural transformations, adjoint functors, some (co)limits) and of group actions (mostly about transitive actions). We use the notation $H \leq G$ to indicate that H is a subgroup of G and $[-,-]$ for the hom-set $\mathscr{C}(-,-)$.

2 Group actions and adjunctions

Definition 2.1. In a category $\mathscr{C}$ an arrow $f\colon X \to Y$ is a **strict epimorphism** if it is the joint coequalizer of all the pairs of arrows it coequalizes. This means that any arrow $g\colon X \to Z$ such that $g \circ x = g \circ y$ for all $x, y\colon C \to X$ such that $f \circ x = f \circ y$ there exists a unique arrow $h\colon Y \to Z$ such that $h \circ f = g$. Refer to Figure 2.1.

$$\begin{tikzcd}
C \arrow[r, shift left, "y"] \arrow[r, shift right, "x"'] & X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, dashed, "h"] \\
& & Z
\end{tikzcd}$$

Figure 2.1

Remark 2.2. Strict epimorphisms are epimorphisms, as the name implies.

Remark 2.3. If an arrow is both a strict epimorphism and a monomorphism then it is an isomorphism.

Definition 2.4. Let H be a group, A an object of $\mathscr{C}$ and $G = \mathrm{Aut}(A)$ the group of automorphisms of A in $\mathscr{C}$ i.e. the group whose underlying set is the set of isomorphisms of type $A \to A$ of $\mathscr{C}$ and whose operation is composition in $\mathscr{C}$. An **action** of H on A is a group homomorphism $H \to G$.

Notation 2.5. Given an action of a group H on an object A of $\mathscr{C}$ we denote, with a slight abuse of notation, the automorphism of A associated to $h \in H$ with the same letter h.

Definition 2.6. If H acts on A as in Definition 2.4 we define the **quotient** of A by H in $\mathscr{C}$ to be an element A/H of $\mathscr{C}$ equipped with an arrow $q\colon A \to A/H$ such that:

(1) for all $h \in H$ we have $q \circ h = q$,

(2) for any $x\colon A \to X$ such that $x \circ h = x$ for all $h \in H$ there exists a unique arrow $\varphi\colon A/H \to X$ such that $x = \varphi \circ q$.

See also Figure 2.2.

Remark 2.7. Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of "the" quotient of A by H instead of "a" quotient of A by H.

$$
\begin{tikzcd}
h \circlearrowright A \arrow[r, "q"] \arrow[dr, "x"'] & A/H \arrow[d, dashed, "\varphi"] \\
& X
\end{tikzcd}
$$

Figure 2.2

Notation 2.8. Sometimes we use the sentence “the quotient of A by H” to refer to the object A/H, some others to the arrow $q\colon A \to A/H$; the context should be enough to differentiate between the two.

Example 2.9. In the case of the category **Set** of sets and functions if a group H acts on a set A the quotient A/H is the set of orbits of A under the action and the arrow $q : A \to A/H$ is the function that sends each element of A to its orbit.

Remark 2.10. Consider a quotient $q\colon A \to A/H$; by condition (1) above $q \circ h = q = q \circ 1_A$ so q coequalizes all the pairs $(h, 1_A)$, for every $h \in H$. If another arrow $x\colon A \to X$ coequalizes all the pairs that q does then this arrow is such that $x \circ h = x \circ 1_A = x$ for all $h \in H$ and thus, by condition (2), we have a unique factorization $x = \varphi \circ q$. This proves that all quotients are strict epimorphisms.

Notation 2.11. Let G be a group, then with **GSet** we denote the category of G-sets and G-invariant maps.

Remark 2.12. In **GSet** an arrow f is an epimorphisms if and only if it is surjective.

Definition 2.13. A G-set E is said to be **transitive** if for any pair $x, y \in E$ there is a $g \in G$ such that $g \cdot x = y$. Equivalently a G-set is transitive if it has a single orbit i.e. the quotient E/G is a singleton.

Remark 2.14. If E is a transitive G-set and $\varphi : E \to F$ a G-invariant map then it is determined uniquely by its value of any $x \in E$ as for any $y \in E$ we have $y = g \cdot x$ for some $g \in G$ and thus

$$\varphi(y) = \varphi(g \cdot x) = g \cdot \varphi(x).$$

Example 2.15. The underlying set of G (that we also denote with G) is a G-set with the action given by left multiplication; we call this the **canonical action** of G on itself. This is trivially a transitive action and so Remark 2.14 applies: to give a G-invariant map $\varphi : G \to E$ is to give its value on a single element of G; it is customary to give $\varphi(e)$ where e is the neutral element of the group.

Remark 2.16. Let E be a transitive G-set. Fix an $x \in E$ and let φ be the G-invariant map defined by $\varphi(e) = x$. We argue that $\varphi\colon G \to E$ makes E into a quotient of G by the subgroup

$$H = \mathrm{Fix}(x) = \{g \in G\colon g \cdot x = x\}.$$

Indeed by the definition of H and the fact that φ is G-invariant we have

$$(\varphi \circ h)(e) = \varphi(h \cdot e) = h \cdot \varphi(e) = h \cdot x = x = \varphi(e)$$

$$
\begin{tikzcd}
G \arrow[loop left, "h"] \arrow[r, "\varphi"] \arrow[dr, "g"'] & E \cong G/H \arrow[d, dashed, "f"] \\
& F
\end{tikzcd}
$$

Figure 2.3

for all $h \in H$. Moreover let $g\colon G \to F$ satisfy (1) of Definition 2.6; as we discussed above g is entirely determined by the image of e so we obtain (2) defining an arrow $f\colon E \to F$ by $f(x) = g(e)$; this f is unique as arrows out of a transitive G-set are defined by their value on a single element of E. The situation is depicted in Figure 2.3.

We conclude that every transitive G-set is of the form G/H for some $H \leq G$.

Remark 2.17. Let $H \leq G$ be such that the quotient $q : G \to G/H$ exists in **GSet**; then G/H is a transitive G-set. Indeed we know from Remark 2.10 that q is an epimorphism and thus, from Remark 2.12, a surjection. If $x, y \in G/H$ then there are $x', y' \in G$ such that $x = q(x'), y = q(y')$ and, since G is transitive, there is some $g \in G$ such that $g \cdot x' = y'$. Now using that q is G-invariant we obtain

$$g \cdot x = g \cdot q(x') = q(g \cdot x') = q(y') = y$$

and so G/H is transitive.

For the rest of the section fix a category $\mathscr{C}$, an object $A \in \mathscr{C}$ and let $G = \mathrm{Aut}(A)$.

Remark 2.18. Consider a subgroup $H \leq G$ and an object $X \in \mathscr{C}$. H acts on the hom-set $[A, X]$ as follows:

$$
\begin{aligned}
H \times [A, X] &\longrightarrow [A, X] \\
(h, x) &\longmapsto h \cdot x = x \circ h.
\end{aligned}
$$

Remark 2.19. Assume that the action $G \times [A, X] \to [A, X]$ is transitive and let $\mathbf{GSet}^t$ be the category of transitive G-sets (a subcategory of **GSet**). Then we have a functor

$$
\begin{tikzcd}
{[A,-]_G\colon} & \mathscr{C} \arrow[r] & \mathbf{GSet}^t \\
& X \arrow[d, "f"] \arrow[r, mapsto] & {[A,X]_G} \arrow[d, "f_*"] \\
& Y & {[A,Y]_G}
\end{tikzcd}
$$

where we indicate with $[A, X]_G$ the hom-set $[A, X]$ upon which G acts as described in Remark 2.18 and f_* is post-composition with f. It is easy to check that f_* is indeed G-invariant.

Remark 2.20. Consider an object $E \in \mathbf{GSet}^t$, pick an element $x_0 \in E$ and let $H = \mathrm{Fix}(x_0) \leq G$ (the choice of x_0 is irrelevant as E is transitive). Moreover assume that $\mathscr{C}$ has quotients of A by any subgroup of G.

By what we observed in Remark 2.14 we have a bijection between elements of E and arrows of type $G \to E$. Consider then $f \in [A, X]_G$ and its corresponding arrow $\varphi\colon G \to [A, X]_G$; we claim that f factors through A/H if and only if φ factors through $E \cong G/H$ (see Figure 2.4). Now f factors if and only if $f \circ h = f$ for all $h \in H$ and φ factors if and only if $\varphi \circ h = \varphi$ for all $h \in H$. Moreover by using that φ is G-invariant we obtain the relation

$$(\varphi \circ h)(e) = \varphi(h \cdot e) = h \cdot \varphi(e) = h \cdot f = f \circ h$$

which, together with the fact that φ is determined by $\varphi(e)$, makes the two factorization conditions equivalent.

This gives us, for each $X \in \mathscr{C}$ and $E \in \mathbf{GSet}^t$, a bijection

$$[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G). \tag{$*$}$$

Provided that $\mathscr{C}$ has quotients of A by subgroups of G we can define a functor $A \times_G -$ from $\mathbf{GSet}^t$ to $\mathscr{C}$ by setting $A \times_G E = A/H$ for $H = \mathrm{Fix}(x_0)$ and $x_0 \in E$ as above. The behaviour of $A \times_G -$ on arrows is defined as follows. Let $f\colon E \cong G/H \to F \cong G/H'$ be an arrow in $\mathbf{GSet}^t$ with $H = \mathrm{Fix}(x_0)$, $H' = \mathrm{Fix}(f(x_0))$ and $x_0 \in E$. Notice that for all $h \in H$ we have $f(h \cdot x_0) = f(x_0)$ by definition of H and $f(h \cdot x_0) = h \cdot f(x_0)$ by G-invariance of f; thus $H \leq H'$. This means that if $q\colon A \to A/H$ and $q'\colon A \to A/H'$ are the quotients in $\mathscr{C}$ there is a unique arrow $\tilde{f}\colon A/H \to A/H'$ such that $q' = \tilde{f} \circ q$; we set $A \times_G f = \tilde{f}$.

Finally we can rewrite $(*)$ as

$$[A \times_G E, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$$

and, by some routine calculations (see Section 5 for details), wee have an adjunction

$$A \times_G - \dashv [A, -]_G.$$

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r] \arrow[dr, "f"'] & A/H \arrow[d, dashed] \\ & X \end{tikzcd} \qquad \begin{tikzcd} G \arrow[loop left, "h"] \arrow[r] \arrow[dr, "\varphi"'] & E \cong G/H \arrow[d, dashed] \\ & {[A, X]_G} \end{tikzcd}$$

Figure 2.4

Our main problem will be that of finding conditions on $\mathscr{C}$ that make this adjunction into an equivalence of categories.

3 Categorial axiomatization of Galois Theory

Through this section fix a category $\mathscr{C}$, an object $A \in \mathscr{C}$ and denote by G the group $\mathrm{Aut}(A)$.

Definition 3.1. We define the following conditions/axioms.

(1) For every $X \in \mathscr{C}$ there is at least a map of type $A \to X$ and all maps $A \to X$ are strict epimorphisms.

(2) For any subgroup $H \leq \mathrm{Aut}(A)$ the quotient $q\colon A \to A/H$ exists and is preserved by $[A, -]\colon \mathscr{C} \to \mathbf{GSet}$ i.e. if $q : A \to A/H$ is the quotient of A by H in $\mathscr{C}$ then $q_* : [A, A] \to [A, A/H]$ is the quotient of $[A, A]$ by H in $\mathbf{GSet}$.

(3) Every endomorphism of A is an isomorphism i.e. $[A, A] = \mathrm{Aut}(A)$.

Remark 3.2. It is known that if $f \circ g$ is a strict epimorphism then so is f. Thus it follows from Axiom (1) that every arrow $X \to Y$ in $\mathscr{C}$ is a strict epimorphism.

Proposition 3.3. Axiom (1) implies that $[A, -]$ is faithful, reflects monomorphisms and isomorphisms.

Proof. Consider arrows $f, g\colon X \to Y \in \mathscr{C}$ such that $[A, f] = [A, g]$; that is $f_* = g_*$. By Axiom (1) let $h\colon A \to X$ be a third arrow of $\mathscr{C}$ then we have $f_*(h) = g_*(h)$ i.e. $f \circ h = g \circ h$. Again by Axiom (1) h is an epimorphism (really, a strong one) and thus we obtain $f = g$; that is: $[A, -]$ is faithful.

It is well known that every faithful functor reflects monomorphisms. Because of this if $f \in \mathbf{GSet}$ is an isomorphism and $g \in \mathscr{C}$ is such that $[A, g] = f$ then g is a monomorphism too; but, as an arrow of $\mathscr{C}$, g is also a strict epimorphism and thus an isomorphism. □

Remark 3.4. Consider an arrow $x\colon A \to X$ and the epi-mono factorization $x_* = \psi \circ \rho$[2]. With reference to the diagram below $I = [A, A]/H$ with $H = \mathrm{Fix}(x) \leq G$.

$$\begin{tikzcd}
{[A,A]} \arrow[r, "\rho"] \arrow[rr, bend left, "x_*"] & I \arrow[r, "\psi"] & {[A,X]}
\end{tikzcd}$$

Indeed by Axiom (3) $[A, A] = G$ so an arrow from $[A, A]$ is determined by its behaviour on 1_A. Now

$$(\psi \circ \rho \circ h)(1_A) = x_*(1_A \circ h) = x \circ h = x = x_*(1_A) = (\psi \circ \rho)(1_A)$$

that is $\psi \circ \rho \circ h = \psi \circ \rho$ which, by monicness of ψ, implies $\rho \circ h = \rho$ for every $h \in H$. Notice also that ρ is both epi and G-invariant so we can repeat the argument of Remark 2.17 to obtain that I is transitive. With I transitive it is easy to see (as in Remark 2.16) that every $\sigma : [A, A] \to Y$ such that $\sigma \circ h = \sigma$ for every $h \in H$ factors uniquely through I. We conclude that I is really $[A, A]/H$.

Proposition 3.5. Any arrow $x\colon A \to X$ of $\mathscr{C}$ is a quotient of A by $H = \mathrm{Fix}(x) \leq G$ (with respect to the action on $[A, X]$ described in Remark 2.18) i.e. $X = A/H$.

Proof. By choosing $H = \mathrm{Fix}(x)$ we get $x \circ h = x$ for all $h \in H$ and thus there is a unique arrow $\varepsilon\colon A/H \to X$ of $\mathscr{C}$ such that $x = \varepsilon \circ q$, where $q\colon A \to A/H$ is the quotient of A by H (see Figure 3.5).

[2] We recall that **GSet** is a topos and, as such, has epi-mono factorization. The interested reader can find evidence of both facts in [4].

$$
\begin{tikzcd}
A \arrow[r, "q"] \arrow[rr, bend left, "x"] & A/H \arrow[r, "\varepsilon"] & X .
\end{tikzcd}
$$

Figure 3.5

$$
\begin{tikzcd}
{[A,A]} \arrow[r, "q_*"] \arrow[rr, bend left, "x_*"] \arrow[dr, bend right, "\rho"'] & {[A,A/H]} \arrow[r, "\varepsilon_*"] & {[A,X]} \\
& {[A,A]/H} \arrow[u, "\eta"] \arrow[ur, bend right, "\psi"'] &
\end{tikzcd}
$$

Figure 3.6

By applying $[A, -]$ we obtain Figure 3.6 where ρ, ψ are the epi-mono factorization of x_* that we discussed in Remark 3.4 and η is the unique isomorphism between $[A, A/H]$ and $[A, A]/H$ as they are both quotients of $[A, A]$ by H (the first by Axiom (2) and the second by Remark 3.4). This also gives us that $\eta \circ \rho = q_*$. Thus

$$\varepsilon_* \circ \eta \circ \rho = \varepsilon_* \circ q_* = x_* = \psi \circ \rho$$

that by epicness of ρ implies $\varepsilon_* \circ \eta = \psi$. Now since η is an isomorphism we obtain $\varepsilon_* = \psi \circ \eta^{-1}$ and on the right hand side we only have monomorphisms so ε must be one and, by Proposition 3.3, ε in $\mathscr{C}$ is mono as well. Being an arrow of $\mathscr{C}$, by Remark 3.2, ε is also a strict epimorphism and thus an isomorphism. $\square$

Proposition 3.6. The action of $\mathrm{Aut}(A)$ on $[A, X]$ is transitive for all $X \in \mathscr{C}$.

Proof. Consider again Figure 3.6. We proved that ε is an isomorphism and thus ε_* must be one too; moreover η is iso as well so $[A, X] \cong [A, A]/H$. By Axiom (3) we have $[A, A] = \mathrm{Aut}(A)$ so $[A, X] \cong \mathrm{Aut}(A)/H$ and by Remark 2.17 this is a transitive G-set. $\square$

This last proposition gives that $[A, -]$ is actually a functor into $\mathbf{GSet}^t$ and thus allows the discussion of Section 2 to be repeated.

Theorem 3.7. Given a category $\mathscr{C}$ and an object $A \in \mathscr{C}$ such that Axioms (1), (2) and (3) hold there exists an adjunction

$$A \times_G - \dashv [A, -]_G$$

where $G = \mathrm{Aut}(A)$ such that the maps

$$\eta \colon E \cong [A, A]/H \to [A, A/H]$$

$$\varepsilon \colon A/H \to X$$

are isomorphisms. This establishes an equivalence of categories between $\mathscr{C}$ and $\mathbf{GSet}^t$.

Proof. The existence of the adjunction follows from the discussion in Section 2 (which we can repeat thanks to Proposition 3.6), the fact that η and ε are (the components of) the unit and counit of the adjunction is obtained by routine calculations (see Section 5) and the fact that they are isomorphisms follows respectively from Axiom (2) (for η) and Proposition 3.5 (for ε). $\square$

4 Conclusion

Let k be a field and E a finite Galois extension of k. Let $\mathscr{C}$ be the opposite category of the category of subextensions of E (with arrows restricted to the field homomorphisms that fix the base field k) then it is possible to prove that $\mathscr{C}$ satisfies Axioms (1), (2) and (3) with $A = E$. Indeed:

- Axiom (3) is trivially satisfied and $G = \mathrm{Aut}(A, A)$ is the Galois group of E,
- Axiom (2) is satisfied as one can check that, for each $H \leq G$ the fixed field

$$E^H = \{x \in E \colon \forall h \in H\, h(x) = x\}$$

  is the quotient of E by H in $\mathscr{C}$[3],
- Axiom (1) is satisfied because in the category of subextensions of E every object has an arrow to E and every such arrow is a strict monomorphism.

We can then apply Theorem 3.7 to obtain that $\mathscr{C}$ and $\mathbf{GSet}^t$ are equivalent. The correspondence of Galois can then be retrieved by associating to each $F \in \mathscr{C}$ the subgroup $H \leq G$ such that the transitive G-set G/H is precisely the G-set associated to F by the equivalence above.

Theorem 3.7 can also be used to give alternative proofs of other well-known results (such as the correspondence between subgroups of the fundamental group and path-connected covering spaces of a path-connected topological space) that intuitively share some similarities with the case of Galois Theory. Indeed the most beautiful part about this theorem is that, through its very abstract nature, is able to clearly identify, capture and unify these "intuitive similarities" that different pieces of mathematics posses[4].

5 Appendix

5.1 Naturality of $[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$

Fix a category $\mathscr{C}$, an element $A \in \mathscr{C}$ and let $G = \mathrm{Aut}(A)$. Let's indicate with ψ_{EX} the bijection between $[A/H, X]$ and $\mathbf{GSet}^t(E, [A, X]_G)$ described in Remark 2.20; we shall prove that it is natural in both $X \in \mathscr{C}$ and $E \in \mathbf{GSet}^t$.

Naturality in X. Given an arrow $f \colon X \to Y$ of $\mathscr{C}$ we want to prove that the ψ_{EX} are the components of a natural transformation $[A/H, -] \Rightarrow \mathbf{GSet}^t(E, [A, -]_G)$ i.e that the following diagram commutes.

[3] If we prefer to work in the original category of subextensions (not in the dual) this means that E^H is the coquotient of E by H.

[4] Arguably this is the goal of category theory as a whole, making this particular theorem into a quintessential example.

$$\begin{tikzcd}
{[A/H,X]} \arrow[r, "\psi_{EX}"] \arrow[d, "f_*"] & \mathbf{GSet}^t(E,[A,X]_G) \arrow[d, "(f_*)_*"] \\
{[A/H,Y]} \arrow[r, "\psi_{EY}"] & \mathbf{GSet}^t(E,[A,Y]_G)
\end{tikzcd}$$

Recall that here $H = \mathrm{Fix}(x_0)$ for $x_0 \in E$. Pick any $x \in [A/H, X]$ and let $q\colon A \to A/H$ be the quotient arrow (the quotient exists because $\mathscr{C}$ is assumed to have all quotients by subgroups of G); then chasing x through the diagram down the two possible ways yields two arrows in $\mathbf{GSet}^t$ of type $E \to [A,Y]_G$. Since E is transitive arrows out of E are determined uniquely by the image of x_0; keeping this in mind the following computations show that the square commutes.

$$((f_*)_* \circ \psi_{EX})(x)(x_0) = (f_* \circ \psi_{EX}(x))(x_0) = f_*(\psi_{EX}(x)(x_0)) = f_*(x \circ q) = f \circ x \circ q$$

$$(\psi_{EY} \circ f_*)(x)(x_0) = \psi_{EX}(f \circ x)(x_0) = f \circ x \circ q$$

□

Naturality in E. We want to prove that the ψ_{EX} are the components of a natural transformation $[A/-, X] \Rightarrow \mathbf{GSet}^t(-, [A,X]_G)$. Given $f\colon E \cong G/H \to F \cong G/H'$ with $H = \mathrm{Fix}(x_0), H' = \mathrm{Fix}(f(x_0)), x_0 \in E$, using the notation $\tilde{f} = A \times_G f$ (see Remark 2.20), we show the naturality of the following square.

$$\begin{tikzcd}
{[A/H,X]} \arrow[r, "\psi_{EX}"] & \mathbf{GSet}^t(E,[A,X]_G) \\
{[A/H',X]} \arrow[r, "\psi_{FX}"] \arrow[u, "\tilde{f}^*"] & \mathbf{GSet}(F,[A,X]_G) \arrow[u, "f^*"']
\end{tikzcd}$$

As before we chase an $x \in [A/H', X]$ down the two possible paths to get two arrows of type $E \to [A,X]_G$ and prove that they are the same by evaluating them on $x_0 \in E$:

$$(\psi_{EX} \circ \tilde{f}^*)(x)(x_0) = \psi_{EX}(x \circ \tilde{f})(x_0) = x \circ \tilde{f} \circ q = x \circ q',$$

$$(f^* \circ \psi_{FX})(x)(x_0) = (\psi_{FX}(x) \circ f)(x_0) = \psi_{FX}(x)(f(x_0)) = x \circ q'.$$

□

5.2 η and ε are the unit and counit of $A \times_G - \dashv [A, -]_G$

η is the unit. Our adjunction is given by the bijection

$$\psi\colon \mathscr{C}(A \times_G E, X) \cong \mathbf{GSet}^t(E, [A,X]_G)$$

natural in $X \in \mathscr{C}$ and $E \in \mathbf{GSet}^t$. To obtain the unit we set $X = A \times_G E$:

$$\mathscr{C}(A \times_G E, A \times_G E) \cong \mathbf{GSet}^t(E, [A, A \times_G E]_G)$$

and so

$$\mathscr{C}(A/H, A/H) \cong \mathbf{GSet}^t(E, [A, A/H]_G)$$

where $H = \text{Fix}(x) \leq G$ ($x \in E$) by definition of $A \times_G E$. Now (the component at E of) the unit is given by the image of $1_{A/H}$ under this bijection. By the discussion in Section 2 if $q\colon A \to A/H$ is the quotient arrow in $\mathscr{C}$ then $\psi(1_{A/H})$ is the map $E \cong [A, A]_G/H \to [A, A/H]_G$ of $\mathbf{GSet}^t$ that factors the map $G \to E$ that sends e to q. But this last map is q_* and so $\psi(1_{A/H}) = \eta$. $\square$

ε is the counit. Keeping the proof above in mind we set $E = [A, X]$ and obtain

$$\psi\colon \mathscr{C}(A \times_G [A, X], X) \cong \mathbf{GSet}^t([A, X]_G, [A, X]_G)$$

that becomes

$$\mathscr{C}(A/H, X) \cong \mathbf{GSet}^t([A, X]_G, [A, X]_G)$$

with $H = \text{Fix}(x)$ for some $x \in [A, X]$. Notice that since $[A, X] \cong G/H$ by Proposition 3.6 we can consider $1_{[A,X]}$ as a map of type $G/H \to [A, X]$ such that $1_{[A,X]}(e) = x$ (this can be obtained by chasing 1_A around Figure 3.6) there is iso). Now we have that $\psi^{-1}(1_{[A,X]})$ is the arrow $A/H \to X$ of $\mathscr{C}$ that factorizes x i.e. $\psi^{-1}(1_{[A,X]}) = \varepsilon$ as in Figure 3.5. $\square$

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