

# Notes on Initial Algebras and Terminal Coalgebras

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## **Chapter 1**

# **Preliminaries**

## 1.1 Limits and colimits

Through this section let  $\mathcal{A}, \mathcal{D}$  be categories with  $\mathcal{D}$  small. We recall what limits and colimits are, some significant examples (particularly products and coproducts) and some of their basic properties. We also introduce some notation. For detailed proofs see [Bor08, Chapter 2].

**Definition 1.1.1.** Given a functor  $F: \mathcal{D} \rightarrow \mathcal{A}$  a **cone** on  $F$  is an object  $C \in \mathcal{A}$  and a family of arrows  $(\pi_D: C \rightarrow FD)_{D \in \mathcal{D}}$  such that for every arrow  $d: D_1 \rightarrow D_2$  of  $\mathcal{D}$  we have

$$Fd \circ \pi_{D_1} = \pi_{D_2}. \quad (1.1.1)$$

The arrows  $\pi_D$  are called **projections** of the cone, the object  $C$  the **vertex**.

**Definition 1.1.2.** Given a functor  $F: \mathcal{D} \rightarrow \mathcal{A}$  a **limit** of  $F$  is a cone  $(L, (\pi_D)_{D \in \mathcal{D}})$  such that for every other cone  $(C, (\rho_D)_{D \in \mathcal{D}})$  there is a unique arrow  $m: C \rightarrow L$  such that

$$\rho_D = \pi_D \circ m \quad \text{for every } D \in \mathcal{D}.$$

**Proposition 1.1.3.** When a functor  $F$  has a limit that limit is unique (up to isomorphism).

The following proposition is a way of proving equality of two arrows into a limit. It is most useful when working in abstract categories where arrows are not (generally) functions.

**Proposition 1.1.4.** Let  $(L, (\pi_D)_{D \in \mathcal{D}})$  be the limit of  $F$  and consider two arrows  $f, g: C \rightarrow L$ . If for every  $D \in \mathcal{D}$ ,  $\pi_D \circ f = \pi_D \circ g$  then  $f = g$ .

The definitions of cone and of limit are dualized to yield those of cocone and colimit.

**Definition 1.1.5.** Given a functor  $F: \mathcal{D} \rightarrow \mathcal{A}$  a **cocone** on  $F$  is an object  $C \in \mathcal{A}$  and a family of arrows  $(\sigma_D: FD \rightarrow C)_{D \in \mathcal{D}}$  such that for every arrow  $d: D_1 \rightarrow D_2$  of  $\mathcal{D}$  we have

$$\sigma_{D_2} \circ Fd = \sigma_{D_1}. \quad (1.1.2)$$

The arrows  $\sigma_D$  are called **coprojections** of the cocone.

**Definition 1.1.6.** Given a functor  $F: \mathcal{D} \rightarrow \mathcal{A}$  a **colimit** of  $F$  is a cocone  $(L, (\sigma_D)_{D \in \mathcal{D}})$  such that for every other cocone  $(C, (\tau_D)_{D \in \mathcal{D}})$  there is a unique arrow  $m: L \rightarrow C$  such that

$$\tau_D = m \circ \sigma_D \quad \text{for every } D \in \mathcal{D}.$$

Propositions 1.1.3 and 1.1.4 are dualized as follows.

**Proposition 1.1.7.** When a functor  $F$  has a colimit that colimit is unique (up to isomorphism).

**Proposition 1.1.8.** Let  $(L, (\sigma_D)_{D \in \mathcal{D}})$  be the colimit of  $F$  and consider two arrows  $f, g: L \rightarrow C$ . If for every  $D \in \mathcal{D}$ ,  $f \circ \sigma_D = g \circ \sigma_D$  then  $f = g$ .

cone

projection, vertex

limit

cocone

coprojection

colimit

## Products and coproducts

We now turn to two particular classes of (co)limits: products and coproducts. Recall that a discrete category is a category that has no non-identity arrow.

**Definition 1.1.9.** Given a functor  $F: \mathcal{D} \rightarrow \mathcal{A}$  where  $\mathcal{D}$  is some discrete category a limit of  $F$  is called a **product** while a colimit a **coproduct**.

product, coproduct

**Notation 1.1.10.** Notice that to give a functor from a discrete category to  $\mathcal{A}$  is equivalent to picking an element of  $\mathcal{A}$  for every element of  $\mathcal{D}$ . We thus speak of the “product of  $A$  and  $B$ ” for  $A, B \in \mathcal{A}$  without explicit reference to any functor. Moreover we denote the product of  $A$  and  $B$  in  $\mathcal{A}$ , when it exists, by  $A \times B$ . Similarly we speak of the coproduct of two elements of  $\mathcal{A}$  and denote it by  $A + B$  when it exists.

**Proposition 1.1.11.** In a category, when the interested (co)products exists, we have that

1.  $A \times B$  is isomorphic to  $B \times A$ ;
2.  $A + B$  is isomorphic to  $B + A$ ;
3.  $(A \times B) \times C$  is isomorphic to  $A \times (B \times C)$ ;
4.  $(A + B) + C$  is isomorphic to  $A + (B + C)$ .

In light of this proposition we write  $A \times B \times C$  and  $A + B + C$  with no parenthesis and similarly for any finite number of factors or addenda.

**Notation 1.1.12.** When we take the (co)product of an infinite number of objects of  $\mathcal{A}$  we use the notation  $\prod_{i \in I} A_i$  (for products) and  $\coprod_{i \in I} A_i$  (for coproducts). Notice that the order of the factors/addenda does not matter.

**Notation 1.1.13.** Let  $(L = \prod_{i \in I} A_i, (\pi_i)_{i \in I})$  be a product in  $\mathcal{A}$ . Then by Proposition 1.1.4 there is a one-to-one correspondence between arrows  $f: C \rightarrow L$  and cones  $(C, (\rho_i: C \rightarrow A_i)_{i \in I})$ . Moreover any collection of arrows  $(\rho_i: C \rightarrow A_i)_{i \in I}$  satisfies condition (1.1.1) so it is a cone. This means that to give an arrow  $f: C \rightarrow L$  is equivalent to giving a family of arrows  $(\rho_i: C \rightarrow A_i)_{i \in I}$  so we write

$$f = \langle \rho_1, \rho_2, \rho_3, \dots \rangle; \quad (1.1.3)$$

given a convenient ordering on  $I$ .

By a dual argument we obtain that arrows  $f: L \rightarrow C$  out of a coproduct  $L = \coprod_{i \in I} A_i$  are in one-to-one correspondence with families of arrows  $(\tau_i: A_i \rightarrow C)$  and write

$$f = [\tau_1, \tau_2, \tau_3, \dots] \quad (1.1.4)$$

for a convenient ordering on  $I$ .

**Notation 1.1.14.** Let  $A, B, C, D$  be objects of  $\mathcal{A}$  such that  $A \times B, C \times D$  exist and let  $f: A \rightarrow C, g: B \rightarrow D$  be arrows. With reference to the following diagram we denote  $\langle f \circ \pi_1, g \circ \pi_2 \rangle$  by  $f \times g$ . Dually if  $A + B, C + D$  exist we denote  $[\sigma_1 \circ f, \sigma_2 \circ g]$  by  $f + g$ .

$f \times g, f + g$

$$\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\uparrow \pi_1 & & \uparrow \\
A \times B & \xrightarrow{f \times g} & C \times D \\
\downarrow \pi_2 & & \downarrow \\
B & \xrightarrow{g} & D
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow & & \downarrow \sigma_1 \\
A + B & \xrightarrow{f+g} & C + D \\
\uparrow & & \uparrow \sigma_2 \\
B & \xrightarrow{g} & D
\end{array}$$

The next proposition shows how the notation introduced above for products interacts with the composition. It is most useful for performing calculations; we will use it without explicit reference particularly in the proof of the Primitive Recursion Theorem (see 2.3.3).

**Proposition 1.1.15.** Consider the following diagram.

$$\begin{array}{ccccc}
& & & A & \xrightarrow{g_1} & D \\
& & f_1 \nearrow & \uparrow & & \uparrow \\
D & \xrightarrow{h} & C & \xrightarrow{\langle f_1, f_2 \rangle} & A \times B & \xrightarrow{g_1 \times g_2} & D \times E \\
& & f_2 \searrow & \downarrow & & \downarrow \\
& & & B & \xrightarrow{g_2} & E
\end{array}$$

We have that

1.  $(g_1 \times g_2) \circ \langle f_1, f_2 \rangle = \langle g_1 \circ f_1, g_2 \circ f_2 \rangle$ ;
2.  $\langle f_1, f_2 \rangle \circ h = \langle f_1 \circ h, f_2 \circ h \rangle$ .

### Initial and terminal objects

**Definition 1.1.16.** An **initial object** for a category  $\mathcal{A}$  is an object  $0 \in \mathcal{A}$  such that for every other object  $A \in \mathcal{A}$  there is exactly one arrow from  $0$  to  $A$ . Dually a **terminal object** for  $\mathcal{A}$  is an object  $1 \in \mathcal{A}$  such that for every other object  $A \in \mathcal{A}$  there is exactly one arrow from  $A$  to  $1$ .

initial object

terminal object

**Remark 1.1.17.** Initial objects, when they exist, are colimits of the functor from the empty category into  $\mathcal{A}$ ; as such they are all isomorphic and we speak of *the* initial object. Dually terminal objects are limits of the functor from the empty category and are unique as well.

### (Co)limit-preserving functors

**Definition 1.1.18.** A functor  $G: \mathcal{A} \rightarrow \mathcal{B}$  **preserve limits** if, for every small category  $\mathcal{D}$  and functor  $F: \mathcal{D} \rightarrow \mathcal{A}$ , when the limit of  $F$  exists the limit of  $G \circ F$  is obtained by applying  $G$  to the limit cone. That is: if  $(L, (\pi_D)_{D \in \mathcal{D}})$  is the limit of  $F$  then  $(GL, (G\pi_D)_{D \in \mathcal{D}})$  is the limit of  $G \circ F$ . As expected  $G$  is said to preserve colimits when, for every small category  $\mathcal{D}$  and functor  $F: \mathcal{D} \rightarrow \mathcal{A}$ , when the colimit of  $F$  exists then the colimit of  $G \circ F$  is the image through  $G$  of the colimit cocone.

limit preservation

colimit preservation

By restricting the definition above to a specific small category  $\mathcal{D}$  we obtain the notion of a functor that preserves  $\mathcal{D}$ -(co)limits. We will be interested in functors that preserve  $\omega$ -(co)limits where  $\omega$  is the set of natural numbers with the usual ordering regarded as a category.

## 1.2 Other notations

**Notation 1.2.1.** When we are working with a category  $\mathcal{A}$  that has enough structure<sup>1</sup> we will often define endofunctors  $F: \mathcal{A} \rightarrow \mathcal{A}$  by their action on objects only such as

$$FX = X \times X + 1.$$

This means that the functor  $F$  sends objects  $A$  to  $A \times A + 1$  (where 1 is the terminal object of  $\mathcal{A}$ ) and arrows  $f$  to  $f \times f + \text{id}_1$ .

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<sup>1</sup>i.e. that has all the (co)limits needed for our definition to make sense.

## **Chapter 2**

# **Algebras for endofunctors**



## 2.1 $F$ -algebras

Though this section let  $\mathcal{A}$  be an arbitrary category that we call the **base category** and  $F: \mathcal{A} \rightarrow \mathcal{A}$  an endofunctor.

**Definition 2.1.1.** An **algebra** for  $F$  (or an  **$F$ -algebra**) is a pair  $(A, \alpha)$  where  $A$  is an object of  $\mathcal{A}$  (that we call the **carrier** or **base object**) and  $\alpha$  is an arrow of type  $FA \rightarrow A$  (that we call the **structure** of the algebra).

$F$ -algebra

As one should expect  $F$ -algebras form a category of their own.

**Definition 2.1.2.** Given  $F$ -algebras  $(A, \alpha)$  and  $(B, \beta)$  a **morphism** between them is an arrow  $f: A \rightarrow B$  of  $\mathcal{A}$  such that

$$\begin{array}{ccc} FA & \xrightarrow{\alpha} & A \\ \downarrow Ff & & \downarrow f \\ FB & \xrightarrow{\beta} & B \end{array}$$

commutes.

**Remark 2.1.3.** Given morphisms of algebras  $f: (A, \alpha) \rightarrow (B, \beta)$  and  $g: (B, \beta) \rightarrow (C, \gamma)$  their composition is exactly  $g \circ f$ ; one can easily check that  $g \circ f$  makes the diagram above commute using that  $f$  and  $g$  are morphisms of  $F$ -algebras and that  $F$  is a functor. The identity on an algebra  $(A, \alpha)$  is exactly  $\text{id}_A$ .

We denote the category of  $F$ -algebras and morphisms of  $F$ -algebras by  $\text{Alg}F$ .

$\text{Alg}F$

**Example 2.1.4.** In  $\text{Set}$  let  $1$  be the terminal object  $\{*\}$  and consider the functor

$$FX = X + 1.$$

To give an arrow  $\alpha: A+1 \rightarrow A$  is equivalent to giving two functions  $\alpha_0: 1 \rightarrow A$ ,  $\alpha_1: A \rightarrow A$ ; that is  $\alpha = [\alpha_1, \alpha_0]$ . So an algebra for  $F$  is set  $A$  with a constant  $\alpha_0$  and a unary operation  $\alpha_1$ . A morphism of  $F$ -algebras is a function  $f$  such that

$$\begin{array}{ccc} A + 1 & \xrightarrow{[\alpha_1, \alpha_0]} & A \\ \downarrow f + \text{id}_1 & & \downarrow f \\ B + 1 & \xrightarrow{[\beta_1, \beta_0]} & B \end{array}$$

commutes. This translates, using Proposition 1.1.8, into the following two conditions

$$f \circ \alpha_1 = \beta_1 \circ f, \quad f \circ \alpha_0 = \beta_0.$$

Which together give that  $f$  preserves the constant and commutes with the unary operation of the algebras.

**Example 2.1.5.** Similarly algebras for the endofunctor  $FX = X \times X + 1$  are sets equipped with a constant and a binary operation. Morphisms of such algebras preserve the constant and commute with the operation. Indeed if  $(A, a, *_A)$  and  $(B, b, *_B)$  are two  $F$ -algebras a morphism  $f: A \rightarrow B$  is a function such that

1.  $f(a) = b$ ;
2.  $f(x *_A y) = f(x) *_B f(y)$  for all  $x, y \in A$ .

This is reminiscent of structures such as groups and monoids.

**Example 2.1.6.** Let  $\Sigma_1$  be a set; then algebras for  $FX = \Sigma_1 \times X$  are sets with a unary operation for every element  $\sigma \in \Sigma_1$ . Indeed let  $A$  be an  $F$ -algebra and let  $\alpha: \Sigma_1 \times A \rightarrow A$  be its structure; for a fixed  $\sigma \in \Sigma_1$  we have that  $\alpha(\sigma, -)$  is a unary operation on  $A$ . Morphisms of  $F$ -algebras in this case commute with all such unary operations.

Examples 2.1.4, 2.1.5, 2.1.6 can be generalized with the introduction of polynomial functors over  $\mathbf{Set}$ . This captures the idea of a  $\Sigma$ -algebra from universal algebra and provides a generalization to other base categories with sufficient structure.

**Definition 2.1.7.** A **signature**  $\Sigma$  is a collection of sets  $(\Sigma_n)_{n < \omega}$  where  $\Sigma_n$  is called the set of  $n$ -ary symbols. The symbols in  $\Sigma_0$  are also called constant symbols (or symbols for constants).

signature

**Definition 2.1.8.** Given a signature  $\Sigma$  a  **$\Sigma$ -algebra** is a set  $A$  together with interpretations for every symbol of  $\Sigma$ . That is: to every  $\sigma \in \Sigma_0$  we associate a fixed element of  $A$  and to every  $\sigma \in \Sigma_n$  an  $n$ -ary function on  $A$ . We denote the interpretation of  $\sigma$  in  $A$  by  $\sigma^A$ .

 $\Sigma$ -algebra

Given two  $\Sigma$ -algebras  $A$  and  $B$  a **morphism** of  $\Sigma$ -algebras is a function  $f: A \rightarrow B$  such that

$$f\sigma^A = \sigma^B \quad (2.1.1)$$

for every  $\sigma \in \Sigma_0$  and

$$f(\sigma^A(x_1, \dots, x_n)) = \sigma^B(f(x_1), \dots, f(x_n)) \quad (2.1.2)$$

for every  $\sigma \in \Sigma_n$  and  $x_1, \dots, x_n \in A$ .

We will now observe how  $\Sigma$ -algebras can be realized as algebras for specific endofunctors. To every signature we associate a **polynomial functor**  $H_\Sigma: \mathbf{Set} \rightarrow \mathbf{Set}$  that operates as follows on objects  $X$  and arrows  $f$ .

polynomial functor

$$\begin{aligned} H_\Sigma X &= \coprod_{n < \omega} \Sigma_n \times X^n \\ H_\Sigma f &= \coprod_{n < \omega} \text{id}_{\Sigma_n} \times \underbrace{f \times \dots \times f}_{n \text{ times}} \end{aligned}$$

Then if  $A$  is a  $\Sigma$ -algebra we can define functions  $\alpha_n: \Sigma_n \times A^n \rightarrow A$  naturally as

$$\alpha_n(\sigma, x_1, \dots, x_n) := \sigma^A(x_1, \dots, x_n) \quad (2.1.3)$$

and then  $\alpha := [\alpha_0, \alpha_1, \dots]$  makes  $A$  into a  $H_\Sigma$ -algebra. Conversely if we have a  $H_\Sigma$ -algebra  $(A, \alpha)$  we can define an interpretation of every symbol by reading (2.1.3) the other way around.

For morphisms consider the following square that we know commutes if  $f$  is a morphism of  $H_\Sigma$ -algebras.

$$\begin{array}{ccc} \coprod_{n < \omega} \Sigma_n \times A^n & \xrightarrow{\alpha} & A \\ \downarrow H_\Sigma f & & \downarrow f \\ \coprod_{n < \omega} \Sigma_n \times B^n & \xrightarrow{\beta} & B \end{array}$$

Since the diagonal of the square is an arrow out of a coproduct we know from Proposition 1.1.8 that  $f \circ \alpha$  and  $\beta \circ H_\Sigma f$  are equal when composed with the coprojections of the cocone and this can happen if and only if conditions (2.1.1) and (2.1.2) hold.

We now turn to continuous algebras.

**Definition 2.1.9.** A **complete partial order** (or a **CPO**) is a poset  $(A, \sqsubseteq)$  in which all  $\omega$ -chains

$$a_1 \sqsubseteq a_2 \sqsubseteq a_3 \sqsubseteq \dots$$

have a join i.e. a least upper bound; we denote it by  $\bigvee_{n < \omega} a_n$ . A function  $f: (A, \sqsubseteq) \rightarrow (B, \sqsubseteq)$  is **continuous** if

1. it is monotone i.e.  $a \sqsubseteq b$  in  $A$  implies  $f(a) \sqsubseteq f(b)$  in  $B$ ;
2. given an  $\omega$ -chain  $a_1 \sqsubseteq a_2 \sqsubseteq a_3 \sqsubseteq \dots$  we have  $f(\bigvee_{n < \omega} a_n) = \bigvee_{n < \omega} f(a_n)$ .

Note that by 1 any chain in  $A$  is preserved by  $f$  and since  $B$  is a CPO it must have a join; condition 2 forces that join to be the image of the join of the original chain under  $f$ .

**Remark 2.1.10.** CPOs and continuous functions form a category CPO. In CPO (co)products are formed as in Set and have, for products, the pointwise order and, for coproducts, each component keeps its own ordering and elements in different components are never comparable. Moreover CPO has a terminal object 1 that is the singleton  $\{*\}$  with the only possible partial order on it.

**Example 2.1.11.** Consider the functor  $FX = X + 1$  on CPO. Algebras for  $F$  are CPOs with a constant and a continuous unary operation. Similarly algebras for the functor  $FX = X \times X + 1$  are CPOs with a constant and a continuous binary operation. The continuity of the operation (which we denote here as  $*$ ) in this case gives us that if

$$a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \dots \quad \text{and} \quad b_0 \sqsubseteq b_1 \sqsubseteq b_2 \sqsubseteq \dots$$

are chains then  $(a_0, b_0) \sqsubseteq (a_1, b_1) \sqsubseteq (a_2, b_2) \sqsubseteq \dots$  is a chain in the product and

$$\bigvee_{n < \omega} a_n * \bigvee_{n < \omega} b_n = \bigvee_{n < \omega} a_n * b_n.$$

complete partial  
order

continuous function

**Example 2.1.12.** Now let  $FX = X_\perp$  be the functor that adds to a CPO a bottom element (i.e. an element  $\perp$  such that  $\perp \sqsubseteq x$  for every  $x$  in the poset) and that sends a continuous function  $f: X \rightarrow Y$  to  $f_\perp: X_\perp \rightarrow Y_\perp$  defined as

$$f_\perp(x) = \begin{cases} f(x) & \text{if } x \neq \perp \\ \perp & \text{if } x = \perp \end{cases}.$$

An algebra  $(A, \alpha)$  for  $F$  has a unary continuous function  $\alpha_1: A \rightarrow A$  and a constant  $\alpha_\perp$  such that  $\alpha_\perp \sqsubseteq \alpha(a)$  for every  $a \in A$  by monotony of  $\alpha$ . Comparing this to Example 2.1.11 by changing the functor we now have a condition on our constant.

Finally let  $\text{CPO}_\perp$  be the category of CPOs with a bottom element and continuous functions that preserve the bottom i.e. such that  $f(\perp) = \perp$ . We call such functions **strict continuous functions**.

strict continuous  
function

**Remark 2.1.13.** In  $\text{CPO}_\perp$  products work as in CPO (the bottom element is naturally the one whose components are all  $\perp$ ) but the construction of coproducts must be changed. Indeed coproducts in  $\text{CPO}_\perp$  are formed as in CPO but the bottoms of all the components are unified into a single element. The initial object is still the singleton as it clearly has a bottom.

**Remark 2.1.14.** Notice that the functor  $FX = X + 1$  on  $\text{CPO}_\perp$  is the identity functor so its algebras are CPOs with a bottom and a strict continuous unary operation. If we want a constant on top of the unary operation we can consider the functor  $FX = X_\perp + 1_\perp$ ,  $Ff = f_\perp + \text{id}_{1_\perp}$ .

## 2.2 Initial Algebras

**Definition 2.2.1.** An **initial algebra** for an endofunctor  $F$  is an initial object in  $\text{Alg}F$ . When an initial algebra exists we denote it by  $(\mu F, \iota)$ .

initial algebra

**Remark 2.2.2.** If  $0$  is the initial object of  $\mathcal{A}$  and  $F$  preserves initial objects then  $(0, \text{id}_0)$  is the initial algebra for  $F$ .

We will now look at some non-trivial examples of initial algebras.

**Example 2.2.3.** Consider the functor  $FX = X + 1$  on  $\text{Set}$ . The initial algebra for  $F$  is  $(\mathbb{N}, \iota)$  where  $\iota = [\iota_1, \iota_0]$  with  $\iota_0 = 0$  and  $\iota_1$  the successor function on the naturals. Indeed let  $(B, \beta)$  be another algebra with  $\beta = [\beta_1, \beta_0]$  and consider the function  $f: \mathbb{N} \rightarrow B$  defined as

$$\begin{cases} f(0) & = \beta_0 \\ f(n+1) & = \beta_1(f(n)) \end{cases} \quad (2.2.1)$$

This is a morphism of algebras:

1.  $f(0) = \beta_0$ ;
2.  $(f \circ \iota_1)(n) = f(n+1) = (\beta_1 \circ f)(n)$  for all  $n \in \mathbb{N}$  so  $f \circ i_1 = \beta_1 \circ f$ ;

and clearly unique because any other morphism must satisfy 1 and 2, thus a simple induction gives us that it must be  $f$ .

**Example 2.2.4.** Let  $\mathcal{P}_f: \text{Set} \rightarrow \text{Set}$  be the finite power-set functor. This functor sends a set  $X$  to  $\mathcal{P}_f X = \{Z \subseteq X : Z \text{ is finite}\}$  and a function  $f: X \rightarrow Y$  to the function  $\mathcal{P}_f f$  that sends a finite subset of  $X$  to its (obviously finite)  $f$ -image in  $Y$ . Let  $V_\omega$  be the set of hereditarily finite sets. That is:

finite power-set  
functor

1.  $\emptyset \in V_\omega$ ;
2. if  $X_1, \dots, X_n \in V_\omega$  then  $\{X_1, \dots, X_n\} \in V_\omega$ .

We will see in Example 3.1.15 that  $(V_\omega, \text{id})$  is the initial algebra of  $\mathcal{P}_f$ .

**Example 2.2.5.** Let  $\mathcal{A}$  be a category with countable coproducts. If  $A \in \mathcal{A}$  we write  $\mathbb{N} \bullet A$  for the coproduct of  $A$  with itself countably many times. We shall now consider the functor  $FX = X + A$  for which we prove the initial algebra to be  $\mathbb{N} \bullet A$ .

First we need an algebra structure  $\iota$  on  $\mathbb{N} \bullet A$ . If  $\text{in}_k: A \rightarrow \mathbb{N} \bullet A$  is the  $k$ -th coprojection then we set

$$\iota = [\alpha_1, \text{in}_0]: (\mathbb{N} \bullet A) + A \rightarrow \mathbb{N} \bullet A \quad (2.2.2)$$

where  $\alpha_1$  is obtained from the universal property of the coproduct applied to the cone  $(\text{in}_k)_{1 \leq k < \omega}$ . We thus have that the following triangles commute for every  $k < \omega$ .

$$\begin{array}{ccc} \mathbb{N} \bullet A & \xrightarrow{\alpha_1} & \mathbb{N} \bullet A \\ & \nwarrow \text{in}_k \quad \nearrow \text{in}_{k+1} & \\ & A & \end{array}$$

Now let  $(B, \beta)$  with  $\beta = [\beta_1, \beta_0]$  be an algebra and  $f$  a morphism from  $(\mathbb{N} \bullet A, \iota)$ . We have that the following square commutes.

$$\begin{array}{ccc} \mathbb{N} \bullet A + A & \xrightarrow{[\alpha_1, \text{in}_0]} & \mathbb{N} \bullet A \\ \downarrow f + \text{id} & & \downarrow f \\ B + A & \xrightarrow{[\beta_1, \beta_0]} & B \end{array}$$

So any such  $f$  must be such that

1.  $f \circ \text{in}_0 = \beta_0$ ;
2.  $f \circ \text{in}_{k+1} = f \circ \alpha_1 \circ \text{in}_k = \beta_1 \circ f \circ \text{in}_k$  for all  $k \in \mathbb{N}$ .

This gives us that  $[\beta_0, \beta_1 \circ \beta_0, \beta_1 \circ \beta_1 \circ \beta_0, \dots]$  is the unique morphism from  $(\mathbb{N} \bullet A, \iota)$  to  $(B, \beta)$  which proves our claim.

Finally we describe initial algebras for polynomial functors in two equivalent ways: using closed terms and using trees.

**Definition 2.2.6.** Let  $\Sigma$  be a signature. The set of **closed term** for  $\Sigma$  is defined inductively as follows:

closed term

1. a constant symbol is a closed term;
2. an expression of the form  $\sigma(t_1, \dots, t_n)$  where  $\sigma \in \Sigma_n$  and  $t_1, \dots, t_n$  are closed terms is a closed term.

**Remark 2.2.7.** Let  $\mu H_\Sigma$  be the set of all closed terms; this is naturally a  $\Sigma$ -algebra:

1. if  $\sigma$  is a constant symbol then  $\sigma^{\mu H_\Sigma}$  is  $\sigma$  itself but regarded as a term;
2. if  $\sigma$  is an  $n$ -ary symbol then it defines an  $n$ -ary operation on  $\mu H_\Sigma$  that takes terms  $t_1, \dots, t_n$  to the term  $\sigma(t_1, \dots, t_n)$ .

**Remark 2.2.8.** Let  $A$  be a  $\Sigma$ -algebra and  $t$  a closed term. We can define an evaluation function  $\text{ev}: \mu H_\Sigma \rightarrow A$  as follows:

1. if  $t$  is a constant symbol  $\sigma$  then  $\text{ev}(t) = \sigma^A$ ;
2. if  $t$  is of the form  $\sigma(t_1, \dots, t_n)$  then  $\text{ev}(t) = \sigma^A(\text{ev}(t_1), \dots, \text{ev}(t_n))$ .

Clearly  $\text{ev}$  is a morphism and moreover it is unique so  $\mu H_\Sigma$  is the initial algebra for  $H_\Sigma$ .

**Definition 2.2.9.** Given a signature  $\Sigma$  a  $\Sigma$ -**tree** is an ordered tree where every node of  $k$  children is labelled by a  $k$ -ary symbol of  $\Sigma$ . As an example if  $\sigma_0$  is a constant symbol,  $\sigma_1$  a 2-ary symbol,  $\sigma_2$  a 3-ary symbol and  $\sigma_3$  a 1-ary symbol then the following are valid  $\Sigma$ -trees.



**Remark 2.2.10.** Every  $n$ -ary symbol of  $\Sigma$  defines an  $n$ -ary operation on  $\Sigma$ -trees which takes  $n$  trees to the tree obtained by connecting each root to a new node labelled by  $\sigma$ , which becomes the root of a new tree. We call this operation **tree-tupling** (below an example of tree-tupling induced by a 3-ary symbol  $\sigma$ ). Note that the order of the trees matters.

tree-tupling



**Proposition 2.2.11.** The initial algebra  $\mu H_\Sigma$  is the algebra of finite  $\Sigma$ -trees with tree-tupling.

*Proof.* Let  $T$  be the algebra of finite  $\Sigma$ -trees with tree-tupling; we shall find an isomorphism  $f: \mu H_\Sigma \rightarrow T$  of algebras. Define  $f$  by structural recursion as follows

1. if  $t$  is a constant term  $\sigma$  let  $f(t)$  be the tree formed by a single node labelled by  $\sigma$ ;
2. if  $t$  is a term of the form  $\sigma(t_1, \dots, t_n)$  let  $f(\sigma)$  be the tree obtained by tree-tupling  $f(t_1), \dots, f(t_n)$  with the new root labelled by  $\sigma$ .

This function is a morphism by definition and has an inverse, defined similarly by structural recursion, which is again a morphism.  $\square$

**Example 2.2.12.** By the discussion above the initial algebra for the functor  $FX = X \times X + 1$  on Set is the algebra of all finite binary trees.

**Example 2.2.13.** Consider the functor  $FX = B \times X + 1$  on Set with  $B \in \text{Set}$ . Algebras for  $F$  are sets with a constant and  $|B|$  unary operations. By the discussion above we know that  $\mu F$  is the algebra of finite trees for the signature  $\Sigma = (\Sigma_0 = \{*\}, \Sigma_1 = B)$  so its elements are “linear” trees such as



with  $a, b, c \in B$ . We immediately deduce that  $\mu F$  can also be realized as the set of words over  $B$ .

**Remark 2.2.14.** Notice the importance of constant symbols. Indeed if a polynomial functor on Set has no constant then it preserves the initial object  $0$  so by Remark 2.2.2 the initial algebra is trivial. Equivalently one can observe that, without constant symbols, the set of closed terms is empty.

**Example 2.2.15.** Consider  $FX = X_{\perp}$  on  $\text{CPO}_{\perp}$ . Let  $\mathbb{N}^{\top}$  be the set of the natural numbers with an added topmost element  $\infty$  ordered naturally. Notice that  $(\mathbb{N}^{\top})_{\perp}$  is isomorphic (as an order) to  $\mathbb{N}^{\top}$  and consider the successor function  $s: (\mathbb{N}^{\top})_{\perp} \rightarrow \mathbb{N}^{\top}$  with  $s(\infty) = \infty$ . We claim this is the initial algebra for  $F$ .

Let  $(A, \alpha)$  be another  $F$ -algebra and  $f: (\mathbb{N}^{\top}, s) \rightarrow (A, \alpha)$  a morphism. Because  $f$  is an arrow of  $\text{CPO}_{\perp}$  it must be strict so  $f(0) = \perp$  and because it is a morphism we must have  $f(s(n)) = \alpha(f(n))$ ; this defines  $f$  inductively on  $\mathbb{N}$ . Finally we recall that  $f$  must be continuous (as an arrow of  $\text{CPO}_{\perp}$ ) so

$$f(\infty) = f\left(\bigvee_{n < \omega} n\right) = \bigvee_{n < \omega} (f(n)).$$

This implies that there is really a unique morphism from  $(\mathbb{N}^{\top}, s)$  to  $(A, \alpha)$  so  $(\mathbb{N}^{\top}, s)$  is initial.

We conclude this section with the a classic lemma of Lambek’s which gives necessary conditions for functors to have an initial algebra.

**Definition 2.2.16.** A **fixed point** of an endofunctor  $F$  is an element  $A \in \mathcal{A}$  that is isomorphic to  $FA$ .

**Lemma 2.2.17 (Lambeck's Lemma).** An initial algebra for  $F$  is always a fixed point.

*Proof.* Let  $(\mu F, \iota)$  be the initial algebra of  $F$  so there is a unique morphism

$$f: (\mu F, \iota) \rightarrow (F(\mu F), F\iota).$$

We have that the following diagram commutes

$$\begin{array}{ccc} F(\mu F) & \xrightarrow{\iota} & \mu F \\ \downarrow Ff & & \downarrow f \\ F(F(\mu F)) & \xrightarrow{F\iota} & F(\mu F) \\ \downarrow F\iota & & \downarrow \iota \\ F(\mu F) & \xrightarrow{\iota} & \mu F \end{array}$$

so  $\iota \circ f$  is an endomorphism of algebras on the initial algebra; hence it is  $\text{id}_{\mu F}$ . Now

$$f \circ \iota = F\iota \circ Ff = F(\iota \circ f) = F(\text{id}_{\mu F}) = \text{id}_{F(\mu F)}$$

so  $\iota$  is an isomorphism. This shows that  $\mu F$  is a fixed point of  $F$ .  $\square$

**Remark 2.2.18.** We know from Cantor's Theorem that there is no surjection from a given set  $X$  into its power set  $\mathcal{P}X$  so the functor  $\mathcal{P}$  on  $\text{Set}$  has no fixed point hence no initial algebra.

Lambeck's lemma can be seen as a generalization of the following order-theoretic lemma.

**Definition 2.2.19.** Let  $(P, \sqsubseteq)$  be a poset and  $f: P \rightarrow P$  a monotone function. An element  $x \in P$  is a **pre-fixed point** of  $f$  if  $f(x) \sqsubseteq x$ .

**Lemma 2.2.20.** Given a poset  $(P, \sqsubseteq)$  and a monotone function  $f: P \rightarrow P$  let  $A = \{x \in P: f(x) \sqsubseteq x\}$  be the set of pre-fixed points of  $f$ . If  $\bar{a}$  is the meet of  $A$  then  $\bar{a}$  is the least fixed point of  $f$ .

*Proof.* If  $x \in A$  then  $\bar{a} \sqsubseteq x$  and  $f(\bar{a}) \sqsubseteq f(x) \sqsubseteq x$  follows from monotony of  $f$  and definition of  $A$ . This shows that  $f(\bar{a})$  is a lower bound for  $A$  so  $f(\bar{a}) \sqsubseteq \bar{a}$ . Now by applying  $f$  again we obtain  $f(f(\bar{a})) \sqsubseteq f(\bar{a})$  which gives us  $f(\bar{a}) \in A$  and thus  $\bar{a} \sqsubseteq f(\bar{a})$ . Moreover  $\bar{a}$  must be the least fixed point of  $f$  since all fixed points are pre-fixed points.  $\square$

This lemma then gives us the following well-known theorem for free.

**Theorem 2.2.21 (Knaster-Tarski Theorem).** Let  $(L, \sqsubseteq)$  be a complete lattice. Then every monotone function of  $L$  has a fixed point.



*Proof.* Let  $f$  be a monotone function on  $L$  and  $A$  be the set of pre-fixed points of  $f$ . It must have a meet  $\bar{a}$  because the lattice is complete so by the Lemma we have a fixed point.  $\square$

The generalization of order theoretic results such as Lemma 2.2.20 will be a recurrent theme. Indeed we can see orders as categories in the usual way, order-preserving functions as functors and pre-fixed points as algebras for these functors.

## 2.3 Recursion and Induction

Recursion is a way of defining a function  $f$  on the natural numbers by specifying a value for  $f(0)$  and a way of deriving the value of  $f(n+1)$  from the value of  $f(n)$ . Recursion, however, can also be used to define functions on (among other things) trees; in this case it is usually called *structural* induction (we have an example of this in Remark 2.2.8). Here we show how one can use the concept of an initial algebra to provide a general definition of recursion of which the cited examples as particular cases.

**Definition 2.3.1.** Let  $F$  be an endofunctor over  $\mathcal{A}$  with an initial algebra  $\mu F$ . A morphism  $f: \mu F \rightarrow A$  of  $\mathcal{A}$  is **recursively specified** if there exists an algebra structure  $\alpha: FA \rightarrow A$  on  $A$  that makes  $f$  into an algebra homomorphism (actually the unique algebra homomorphism).

recursively specified morphism

**Example 2.3.2.** Consider the functor  $FX = X \times X + 1$  on  $\mathbf{Set}$  and let  $T$  be its initial algebra that we know from Example 2.2.12 is the algebra of finite binary trees with tree-tupling. Let  $h: T \rightarrow \mathbb{N}$  be the function that assigns 0 to the root-only tree and, given a tree  $t$  obtained by tree-tupling of  $t_1$  and  $t_2$ ,  $h(t) = 1 + \max\{h(t_1), h(t_2)\}$ . We now show that  $h$  is recursively specified according to our definition.

We need an algebra structure on  $\mathbb{N}$  i.e. an arrow  $\alpha: \mathbb{N} \times \mathbb{N} + 1 \rightarrow \mathbb{N}$ . Since  $\alpha$  is an arrow out of a coproduct we have  $\alpha = [\alpha_1, \alpha_0]$  so we can let

$$\begin{aligned}\alpha_0 &= 0; \\ \alpha_1(n, m) &= 1 + \max\{n, m\}.\end{aligned}$$

Indeed if  $\iota_1$  is tree-tupling and  $\iota_0$  is the root-only tree the following square trivially commutes thus  $h$  is recursively specified.

$$\begin{array}{ccc} T \times T + 1 & \xrightarrow{[\iota_1, \iota_0]} & T \\ \downarrow h \times h + \text{id}_1 & & \downarrow h \\ \mathbb{N} \times \mathbb{N} + 1 & \xrightarrow{[\alpha_1, \alpha_0]} & \mathbb{N} \end{array}$$

The following theorem is a generalization of the classic recursion with parameters theorem.

**Theorem 2.3.3 (Primitive Recursion).** Assume the base category  $\mathcal{A}$  to have finite products and let  $F$  be an endofunctor with an initial algebra  $\mu F$ . Then for every  $\alpha: F(A \times$

$\mu F) \rightarrow A$  there is a unique  $h: \mu F \rightarrow A$  such that the following square commutes.

$$\begin{array}{ccc} F(\mu F) & \xrightarrow{\iota} & \mu F \\ \downarrow F\langle h, \text{id}_{\mu F} \rangle & & \downarrow h \\ F(A \times \mu F) & \xrightarrow{\alpha} & A \end{array} \quad (2.3.1)$$

*Proof.* Let  $\pi_1, \pi_2$  be the projections of the product  $A \times \mu F$  and consider the arrow

$$\bar{\alpha}: F(A \times \mu F) \xrightarrow{\langle \text{id}_{F(A \times \mu F)}, F\pi_2 \rangle} F(A \times \mu F) \times F(\mu F) \xrightarrow{\alpha \times \iota} A \times \mu F.$$

This gives an algebra structure on  $A \times \mu F$  so let  $\bar{h}: \mu F \rightarrow A \times \mu F$  be the unique algebra homomorphism given by the initiality of  $\mu F$ .

$$\begin{array}{ccccc} F(\mu F) & \xrightarrow{\iota} & & & \mu F \\ \downarrow F\bar{h} & & (1) & & \downarrow \bar{h} \\ F(A \times \mu F) & \xrightarrow{\langle \text{id}_{F(A \times \mu F)}, F\pi_2 \rangle} & F(A \times \mu F) \times F(\mu F) & \xrightarrow{\alpha \times \iota} & A \times \mu F \\ \downarrow F\pi_2 & \swarrow \pi_2 & & (2) & \downarrow \pi_2 \\ F(\mu F) & \xrightarrow{\iota} & & & \mu F \end{array} \quad (2.3.2)$$

The outer square in (2.3.2) commutes because (1), (2) and (3) do. Indeed (1) commutes because  $\bar{h}$  is an algebra homomorphism, (2) by Notation 1.1.14 and (3) because of Notation 1.1.13. By functoriality of  $F$  we then have that  $\pi_2 \circ \bar{h}$  is an endomorphism on the initial algebra thus  $\pi_2 \circ \bar{h} = \text{id}_{\mu F}$ .

Now set  $h := \pi_1 \circ \bar{h}$  so  $\bar{h} = \langle h, \text{id}_{\mu F} \rangle$ . Extending (1) by  $\pi_1$  we obtain the following diagram that we know commutes.

$$\begin{array}{ccccccc} F(\mu F) & \xrightarrow{\iota} & & & \mu F & & \\ \downarrow F\bar{h} = F\langle h, \text{id}_{\mu F} \rangle & & & & \downarrow \bar{h} & \searrow h & \\ F(A \times \mu F) & \xrightarrow{\langle \text{id}_{F(A \times \mu F)}, F\pi_2 \rangle} & F(A \times \mu F) \times F(\mu F) & \xrightarrow{\alpha \times \iota} & A \times \mu F & \xrightarrow{\pi_1} & A \end{array}$$

But notice that

$$\pi_1 \circ (\alpha \times \iota) \circ \langle \text{id}_{F(A \times \mu F)}, F\pi_2 \rangle = \pi_1 \circ \langle \alpha, \iota \circ F\pi_2 \rangle = \alpha.$$

so we have (2.3.1).

To prove uniqueness consider  $h: \mu F \rightarrow A$  homomorphism of algebras such that (2.3.1) commutes. We claim that  $\bar{h} = \langle h, \text{id}_{\mu F} \rangle$  so that  $h = \pi_1 \circ \bar{h}$ ; thus uniqueness. In order to do it we show that  $\langle h, \text{id}_{\mu F} \rangle$  is an algebra homomorphism and conclude it must be  $\bar{h}$  by

initiality of  $\mu F$ . Indeed we have

$$\begin{aligned}
 \pi_1 \circ (\alpha \times \iota) \circ \langle \text{id}_{F(A \times \mu F)}, F\pi_2 \rangle \circ F\langle h, \text{id}_{\mu F} \rangle &= \alpha \circ \text{id}_{F(A \times \mu F)} \circ F\langle h, \text{id}_{\mu F} \rangle \\
 &= h \circ \iota \\
 &= \pi_1 \circ \langle h, \text{id}_{\mu F} \rangle \circ \iota; \\
 \\
 \pi_2 \circ (\alpha \times \iota) \circ \langle \text{id}_{F(A \times \mu F)}, F\pi_2 \rangle \circ F\langle h, \text{id}_{\mu F} \rangle &= \iota \circ F\pi_2 \circ F\langle h, \text{id}_{\mu F} \rangle \\
 &= \iota \circ F(\pi_2 \circ \langle h, \text{id}_{\mu F} \rangle) \\
 &= \iota \circ F\text{id}_{\mu F} \\
 &= \iota \circ \text{id}_{F(\mu F)} \\
 &= \iota \\
 &= \pi_2 \circ \langle h, \text{id}_{\mu F} \rangle \circ \iota;
 \end{aligned}$$

so we conclude by Proposition 1.1.4.  $\square$

Recall the classic recursion with parameters theorem. We shall prove it using Theorem 2.3.3; for a classical proof see [And19, Theorem 18.1].

**Theorem 2.3.4 (Recursion with parameters).** Let  $A$  be a set,  $a \in A$  and  $f: A \times \mathbb{N} \rightarrow A$  be a function. Then there is a function  $g: \mathbb{N} \rightarrow A$  such that

$$\begin{cases} g(0) = a \\ g(n+1) = f(g(n), n) \end{cases} .$$

*Proof.* Consider the functor  $FX = X + 1$  on Set and form the arrow

$$\alpha: F(A \times \mu F) = A \times \mathbb{N} + 1 \xrightarrow{[f, a]} A$$

where we abuse notation and let  $a: 1 \rightarrow A$  be the function that picks  $a \in A$ . Now by Theorem 2.3.3 there is a function  $g: \mathbb{N} \rightarrow A$  such that (2.3.1) commutes. This gives us the diagram

$$\begin{array}{ccc}
 \mathbb{N} + 1 & \xrightarrow{[\iota_1, \iota_0]} & \mathbb{N} \\
 \downarrow \langle g, \text{id}_{\mathbb{N}} \rangle + \text{id}_1 & & \downarrow g \\
 A \times \mathbb{N} + 1 & \xrightarrow{[f, a]} & A
 \end{array} \tag{2.3.3}$$

that yields the conditions

1.  $g \circ \iota_0 = a \circ \text{id}_1$ ;
2.  $g \circ \iota_1 = f \circ \langle g, \text{id}_{\mathbb{N}} \rangle$ .

Now 1 gives us  $g(0) = a$  and 2 that

$$g(n+1) = (g \circ \iota_1)(n) = (f \circ \langle g, \text{id}_{\mathbb{N}} \rangle)(n) = f(g(n), n) \quad \text{for all } n < \omega.$$

So  $g$  has the required properties.  $\square$

The induction principle, that if a subset of  $\mathbb{N}$  contains 0 and is closed under the successor operation then it is  $\mathbb{N}$ , can also be generalized to the more general setting of  $F$ -algebras. Recall that a subobject in a category is represented by a monomorphism. We call subalgebras subobjects in  $\text{Alg}F$ .

**Theorem 2.3.5 (Induction Principle).** Let  $F$  be an endofunctor and  $\mu F$  its initial algebra. Then every  $m: (B, \beta) \rightarrow (\mu F, \iota)$  monic is an isomorphism.

*Proof.* By initiality there is a unique homomorphism of  $F$ -algebras  $h: (\mu F, \iota) \rightarrow (B, \beta)$ . Now  $m \circ h$  is an endomorphism of the initial object of  $\text{Alg}F$  so  $m \circ h = \text{id}_{\mu F}$ , but notice that we have  $m$  monic and split-epi thus an isomorphism.  $\square$

## **Chapter 3**

# **Initial algebras from finitary iteration**

### 3.1 Adámek's Theorem

We begin this section by recalling Kleene's Fixed Point Theorem and its proof with the goal of generalizing it to the categorical setting.

**Theorem 3.1.1 (Kleene's Fixed Point Theorem).** Let  $A$  be a CPO with bottom  $\perp$ ; then every continuous function  $F: A \rightarrow A$  has a least fixed point  $\mu F = \sup_{n < \omega} F^n(\perp)$ <sup>1</sup>.

*Proof.* Consider the  $\omega$ -chain  $\perp \leq F(\perp) \leq F^2(\perp) \leq \dots$  and let  $\mu F$  be its join. By continuity  $F(\mu F) = \bigvee_{n < \omega} F(F^n(\perp))$  but we know that  $\bigvee_{n < \omega} F^n(\perp) = \bigvee_{n < \omega} F^{n+1}(\perp)$  so  $\mu F = F(\mu F)$ , a fixed point.

Now let  $F(x) \leq x$  be a pre-fixed point. As  $\perp \leq x$  we have  $F(\perp) \leq F(x) \leq x$  and by induction we obtain that  $F^n(\perp) \leq x$  for all  $n < \omega$ . But this shows that  $x$  is an upper bound so  $\mu F \leq x$  by definition and, finally,  $\mu F$  must be the least fixed point because it is less than any pre-fixed point and fixed points are trivially pre-fixed points.  $\square$

Unsurprisingly we shall replace CPOs with bottom by categories with an initial object and endofunctions by endofunctors. The  $\omega$ -chain used in the proof of Theorem 3.1.1 becomes a diagram and the continuity condition become a colimit-preservation one.

**Definition 3.1.2.** The **initial-algebra  $\omega$ -chain** of an endofunctor  $F$  is the diagram

initial-algebra  
 $\omega$ -chain

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} F^3 0 \xrightarrow{F^3!} \dots \quad (3.1.1)$$

where we denote by  $!$  the unique arrow out of the initial object. This diagram can be realized as a functor from  $\omega$  regarded as a category.

**Remark 3.1.3.** Let  $(A, \alpha)$  be an  $F$ -algebra. Then it induces a canonical cocone  $(\alpha_n: F^n 0 \rightarrow A)_{n < \omega}$  on (3.1.1) by

$$\begin{aligned} \alpha_0 &= ! \\ \alpha_{n+1} &= F F^n 0 \xrightarrow{F \alpha_n} F A \xrightarrow{\alpha} A. \end{aligned}$$

To check that the  $\alpha_n$ 's are a cocone we prove  $\alpha_n = \alpha_{n+1} \circ F^n!$  by induction on  $n$ . When  $n = 0$  the condition becomes  $! = \alpha_1 \circ !$  which is true because both sides are arrows out of the initial object. Now suppose the condition holds for  $n - 1$ :

$$\begin{aligned} \alpha_{n+1} \circ F^n! &= \alpha \circ F \alpha_n \circ F^k! && \text{definition of } \alpha_{n+1} \\ &= \alpha \circ F(\alpha_n \circ F^{n-1}!) && \text{functoriality of } F \\ &= \alpha \circ F \alpha_{n-1} && \text{inductive hypothesis} \\ &= \alpha_n && \text{definition of } \alpha_n. \end{aligned}$$

**Remark 3.1.4.** If we apply  $F$  to the initial-algebra  $\omega$ -chain we obtain

$$F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} F^3 0 \xrightarrow{F^3!} \dots \quad (3.1.2)$$

<sup>1</sup>We write  $F^n$  for  $\underbrace{F \dots F}_{n \text{ times}}$  and  $F^0$  is the identity function. We use a similar notation for functors.

This  $\omega$ -chain has the same colimit as the original one. Indeed since the first element of (3.1.1) is the initial object there is an obvious one-to-one correspondence between cocones on (3.1.1) and cocones on the new chain above; and the same factorization morphisms work.

We are ready to state and prove the main theorem of this chapter.

**Theorem 3.1.5 (Adámek).** Let  $\mathcal{A}$  be a category with an initial object 0 and colimits of  $\omega$ -chains and  $F$  an endofunctor that preserves  $\omega$ -colimits. Then  $F$  has an initial algebra  $\mu F = \text{colim}_{n < \omega} F^n 0$ .

*Proof.* Let  $(\mu F, (c_n: F^n 0 \rightarrow \mu F)_{n < \omega})$  be the colimit of the initial-algebra  $\omega$ -chain. Since  $F$  preserves  $\omega$ -colimits we have that  $(F(\mu F), (Fc_n: F^{n+1} 0 \rightarrow F(\mu F))_{n < \omega})$  is the colimit of (3.1.2) but  $(c_{n+1}: F^{n+1} 0 \rightarrow \mu F)_{n < \omega}$  is a cocone on (3.1.2) so there is a unique arrow  $\iota$  from  $F(\mu F)$  to  $\mu F$  such that

$$\iota \circ Fc_n = c_{n+1} \quad \text{for all } n < \omega. \quad (3.1.3)$$

This gives an  $F$ -algebra structure to  $\mu F$  that we now check is initial.

Let  $(A, \alpha)$  be a generic  $F$ -algebra. We have the induced cocone  $(\alpha_n: F^n 0 \rightarrow \mu F)_{n < \omega}$  as in Remark 3.1.3 and this gives a unique arrow  $h: \mu F \rightarrow A$  such that  $h \circ c_n = \alpha_n$  for  $n < \omega$ . We shall now prove that  $h$  is a morphism of  $F$ -algebras and that it is unique.

To prove that  $h \circ \iota = \alpha \circ Fh$  we check that both arrows are factorization morphisms for the cocone  $(\alpha_{n+1}: F^{n+1} 0 \rightarrow A)_{n < \omega}$ . In practice we shall check that

$$h \circ \iota \circ Fc_n = \alpha_{n+1} = \alpha \circ Fh \circ Fc_n$$

for all  $n < \omega$ . For the first equality notice that  $h \circ \iota \circ Fc_n = h \circ c_{n+1} = \alpha_{n+1}$ . For the second equality consider the following diagram

$$\begin{array}{ccc} F^{n+1}0 & \xrightarrow{\alpha_{n+1}} & A \\ Fc_n \downarrow & \searrow F\alpha_n & \uparrow \alpha \\ F(\mu F) & \xrightarrow{Fh} & FA \end{array} .$$

The upper-right triangle commutes by definition of  $\alpha_{n+1}$  (see Remark 3.1.3). The lower-left triangle commute because  $h$  is the factorization arrow and by functoriality of  $F$ . Then, as the whole square commutes,  $\alpha \circ Fh \circ Fc_n = \alpha_{n+1}$ . This proves that  $h: \mu F \rightarrow A$  is a homomorphism of  $F$ -algebras.

To prove uniqueness suppose there is an arrow  $k: \mu F \rightarrow A$  such that it is also an  $F$ -algebra homomorphism i.e.  $k \circ \iota = \alpha \circ Fk$ . We will show that  $k \circ c_n = \alpha_n$  for all  $n < \omega$  and conclude by uniqueness of  $h$ . Working by induction, if  $n = 0$  then  $k \circ c_0 = \alpha_0$

is trivial because both arrows are out of the initial object. Now suppose that  $k \circ c_n = \alpha_n$ :

$$\begin{aligned}
 k \circ c_{n+1} &= k \circ \iota \circ Fc_n && (3.1.3) \\
 &= \alpha \circ Fk \circ Fc_n && k \text{ homomorphism} \\
 &= \alpha \circ F(k \circ c_n) && F \text{ functor} \\
 &= \alpha \circ F\alpha_n && \text{inductive hypothesis} \\
 &= \alpha_{n+1} && \text{definition of } \alpha_{n+1}.
 \end{aligned}$$

This concludes the proof.  $\square$

**Remark 3.1.6.** Looking at the proof of Theorem 3.1.5 we notice that it isn't really necessary for  $\mathcal{A}$  to have colimits of *all*  $\omega$ -chains or for  $F$  to preserve *all* of them as we care for these properties only when applied to the initial-algebra  $\omega$ -chain. However in practice it may be more convenient to check the general properties rather than the particular case.

We will now look at some examples of functors to which one can apply Adámek's Theorem. We urge the reader to notice that the Theorem does not only guarantee the existence of an initial algebra, but also that such algebra is a colimit. By looking at the diagram in question (the initial-algebra  $\omega$ -chain) we can often build an intuition of what the initial algebra is "internally".

**Remark 3.1.7.** We shall prove that in a category  $\mathcal{A}$  with binary coproducts, a terminal object  $1$  and colimits of  $\omega$ -chains, the functor  $FX = X + 1$  preserves colimits of  $\omega$ -chains.

Consider an  $\omega$ -chain

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \xrightarrow{a_3} \dots \quad (3.1.4)$$

and let  $(A, \alpha_i : A_i \rightarrow A)_{i < \omega}$  be its colimit. We want to prove that  $(A + 1, \alpha_i + \text{id}_1)_{i < \omega}$  is the colimit of

$$A_0 + 1 \xrightarrow{a_0 + \text{id}_1} A_1 + 1 \xrightarrow{a_1 + \text{id}_1} A_2 + 1 \xrightarrow{a_2 + \text{id}_1} A_3 + 1 \xrightarrow{a_3 + \text{id}_1} \dots \quad (3.1.5)$$

By functoriality of  $F$  we have immediately that it is a cocone on (3.1.5) so we only need to show it is universal. Let  $(B, \beta_i : A_i + 1 \rightarrow B)_{i < \omega}$  be a cocone on (3.1.5) and note that for all  $i < \omega$ ,  $\beta_i = [\beta_i^1, \beta_i^2]$  with  $\beta_i^1 : A_i \rightarrow B$  and  $\beta_i^2 : 1 \rightarrow B$ . By composition with the coprojections of the  $A_i + 1$ 's we obtain that

1.  $(B, \beta_i^1)_{i < \omega}$  is a cocone on (3.1.4);
2.  $(B, \beta_i^2)_{i < \omega}$  is a cocone on

$$1 \xrightarrow{\text{id}_1} 1 \xrightarrow{\text{id}_1} 1 \xrightarrow{\text{id}_1} 1 \xrightarrow{\text{id}_1} \dots \quad (3.1.6)$$

Now (3.1.4) has colimit  $(A, \alpha_i)_{i < \omega}$  and (3.1.6) has colimit  $(1, \gamma_i = \text{id}_1)_{i < \omega}$  so we obtain unique arrows  $m^1 : A \rightarrow B$  and  $m^2 : 1 \rightarrow B$  such that

$$m^1 \circ \alpha_i = \beta_i^1 \quad \text{and} \quad m^2 \circ \text{id}_1 = \beta_i^2 \quad \text{for all } i < \omega.$$



Finally we set  $m = [m^1, m^2]$  and this is a factorization arrow for  $(B, \beta_i)_{i < \omega}$  as the following calculation shows

$$\begin{aligned} m \circ (\alpha_i + \text{id}_1) &= [m^1, m^2] \circ (\alpha_i + \text{id}_1) \\ &= [m^1 \circ \alpha_i, m^2 \circ \text{id}_1] \\ &= [\beta_i^1, \beta_i^2] \\ &= \beta_i. \end{aligned}$$

We only need to show that such  $m : A + 1 \rightarrow B$  is unique. Let  $n : A + 1 \rightarrow B$  be such that  $n \circ (\alpha_i + \text{id}_1) = \beta_i$  for all  $i < \omega$ . Since  $n$  is an arrow out of a coproduct we have  $n = [n^1, n^2]$  with  $n^1 : A \rightarrow B, n^2 : 1 \rightarrow B$ . Then from the factorization condition we obtain  $[n^1 \circ \alpha_i, n^2 \circ \text{id}_1] = [\beta_i^1, \beta_i^2]$  and, by composing with the coprojections,  $n^1 \circ \alpha_i = \beta_i^1$  and  $n^2 \circ \text{id}_1 = \beta_i^2$  for all  $i < \omega$ . But now we must have  $n^1 = m^1$  and  $n^2 = m^2$  because  $m^1, m^2$  are unique. Finally  $n = m$  and  $F$  preserves colimits of  $\omega$ -chains.

By a similar argument one can show that all polynomial functors preserve colimits of  $\omega$ -chains.

**Example 3.1.8.** In a category with binary coproducts and an initial object  $1$  the initial algebra for  $FX = X + 1$ , if it exists, is called the **natural numbers object** (or **NNO**) of the category. Notice that a NNO, being an initial algebra, is unique so the use of “the” is justified. Explicitly a NNO is a pair  $(N, \iota : N + 1 \rightarrow N)$  such that for every other pair  $(A, \alpha : A + 1 \rightarrow A)$  there is a unique  $h : N \rightarrow A$  such that

natural numbers object

$$\begin{array}{ccc} N + 1 & \xrightarrow{\iota} & N \\ \downarrow h + \text{id}_1 & & \downarrow h \\ A + 1 & \xrightarrow{\alpha} & A \end{array}$$

commutes. By Adámek's Theorem the NNO is the colimit of the  $\omega$ -chain

$$0 \longrightarrow 1 \longrightarrow 1 + 1 \longrightarrow 1 + 1 + 1 \longrightarrow \dots$$

So if  $\mathcal{A} = \text{Set}$  the chain above becomes the sequence of natural numbers, where each  $n \in \mathbb{N}$  is regarded as the set  $\{0, \dots, n-1\}$ , and  $N$  will thus be exactly  $\mathbb{N}$ ; hence the name “natural numbers object”.

**Example 3.1.9.** On  $\text{Pos}$ , the category of partially ordered sets and order-preserving functions, consider the functor  $FX = X_\perp$ . Its initial-algebra  $\omega$ -chain is

$$\emptyset \longrightarrow \perp \longrightarrow \begin{array}{c} \bullet \\ | \\ \perp \end{array} \longrightarrow \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \perp \end{array} \longrightarrow \dots$$

So the  $n$ -th element of the  $\omega$ -chain can be thought of as the set  $n = \{0, \dots, n-1\}$  with the natural order upon it while the connecting morphisms are inclusions. The colimit is  $\mathbb{N}$  with the standard ordering.

**Definition 3.1.10.** A function  $f : M \rightarrow M'$  between metric spaces  $(M, d_M), (M', d_{M'})$  is **non-expanding** if  $d_{M'}(f(x), f(y)) \leq d_M(x, y)$  for all  $x, y \in M$ . A metric space  $(M, d_M)$  is 1-bounded if  $d_M(x, y) \leq 1$  for all  $x, y \in M$ . We denote by **MS** the category of 1-bounded metric spaces and non-expanding maps.

non-expanding  
map

**MS**

**Remark 3.1.11.** Coproducts in **MS** are disjoint unions where each summand retains its metric and points in different summands have distance 1. The product of  $M, M' \in \mathbf{MS}$  is the cartesian product  $M \times M'$  endowed with the maximum metric

$$d_{M \times M'}((x, x'), (y, y')) = \max\{d_M(x, y), d_{M'}(x', y')\}.$$

Moreover **MS** has a terminal object  $1 = \{*\}$  i.e. the singleton with its unique metric.

**Example 3.1.12.** We can now consider  $FX = X + 1$  on **MS**. The objects of the initial-algebra  $\omega$ -chain are again the natural numbers  $n = \{0, \dots, n-1\}$ , but this time regarded as discrete spaces (that is: spaces where all points have distance 1) and the connecting arrows are the inclusions. Thus the colimit, and initial algebra of  $F$ , is  $\mathbb{N}$  endowed with the discrete metric.

Now consider the functor  $FX = \frac{1}{2}X + 1$  where we denote by  $\frac{1}{2}X$  the space obtained by scaling the metric of  $X$  by a factor of  $\frac{1}{2}$ . This functor preserves colimits of  $\omega$ -chains by Remark 3.1.7, mutatis mutandis. Its initial-algebra  $\omega$ -chain in the following sequence of spaces



where 0 always denotes the new point added by the functor. The metric on  $F^n 0$  is

$$d(i, j) = \begin{cases} 0 & \text{if } i = j \\ 2^{-\min\{i, j\}} & \text{otherwise} \end{cases}$$

and the connecting morphisms are (non-expanding) inclusions. We then obtain that  $\mu F$  is again  $\mathbb{N}$ , but this time endowed with the metric  $d$  above.

In the previous examples we have seen how Adámek's Theorem allows us to look at “finite pieces” of the initial algebra. The next one shows that we can use it also to understand how the structure on the initial algebra works.

**Example 3.1.13.** Consider  $FX = X \times X + 1$  on **Set**. This functor preserves colimits of  $\omega$ -chains and we know from Example 2.2.12 that  $\mu F$  is the algebra of all finite binary trees. We will now apply Adámek's Theorem to obtain a “constructive” reason for why this is so.

If we denote the pair  $(x, y) \in X \times X$  by the tree  $\begin{array}{c} \wedge \\ x \quad y \end{array}$  we have that

$$F^0 0 = \emptyset,$$

$$F^1 0 = \emptyset \times \emptyset + 1 = 1 = \{\bullet\},$$

$$F^2 0 = 1 \times 1 + 1 = \left\{ \bullet, \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right\},$$

$$F^3 0 = (1 \times 1 + 1) \times (1 \times 1 + 1) + 1 = \left\{ \bullet, \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}, \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}, \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \right\}, \dots$$

so  $F^n 0$  is the set of binary trees of height less than  $n$ . The connecting arrows  $F^n 0 \rightarrow F^{n+1} 0$  are inclusions and so the colimit of the chain, and thus the initial algebra, is the union  $\bigcup_{n < \omega} F^n 0$  i.e. the set of all finite binary trees.

Recall (again from Example 2.2.12) that the structure  $\iota : \mu F \times \mu F + 1 \rightarrow \mu F$  is given by tree-tupling. This information too can be obtained by Adámek's Theorem. Recall from the proof that, if  $(\mu F, c_n : F^n 0 \rightarrow \mu F)_{i < \omega}$  is the colimit of the initial-algebra  $\omega$ -chain, then  $\iota$  is the unique arrow such that

$$c_n = \iota \circ F c_{n-1} \quad \text{for all } 0 < n < \omega \quad (3.1.7)$$

and, moreover, the  $c_n$ 's are inclusions. Now fix  $0 < k < \omega$  and consider a tree  $\begin{array}{c} \diagup \quad \diagdown \\ t_1 \quad t_2 \end{array}$  in

$F^k 0$  that we also regard as the pair  $(t_1, t_2)$  for  $t_1, t_2 \in F^{k-1} 0$ . We have

$$c_k \left( \begin{array}{c} \diagup \quad \diagdown \\ t_1 \quad t_2 \end{array} \right) = \begin{array}{c} \diagup \quad \diagdown \\ t_1 \quad t_2 \end{array} \in \mu F$$

$$\iota \circ F c_{k-1}((t_1, t_2)) = \iota \left( \underbrace{(t_1, t_2)}_{\in \mu F \times \mu F + 1} \right)$$

so by (3.1.7)  $\iota$  is indeed tree-tupling.

**Example 3.1.14.** For a finite signature  $\Sigma$  the associated polynomial functor  $H_\Sigma$  preserves colimits of  $\omega$ -chains so we can build  $\mu F$ , the set of finite  $\Sigma$ -trees, as the colimit of

$$0 \longrightarrow H_\Sigma 0 \longrightarrow H_\Sigma^2 0 \longrightarrow H_\Sigma^3 0 \longrightarrow \dots$$

Now  $H_\Sigma^n 0$  can be regarded as the set of finite  $\Sigma$ -trees of height less than  $n$  while the connecting arrows of the chain are inclusions; thus  $\mu F = \bigcup_{n < \omega} H_\Sigma^n 0$  i.e.  $\mu F$  is the set of all finite  $\Sigma$ -trees. By an argument similar to that of Example 3.1.13 one can show that the structure  $\iota : H_\Sigma \mu F \rightarrow \mu F$  is really tree-tupling, as we know from Proposition 2.2.11.

**Example 3.1.15.** The finite power-set functor  $\mathcal{P}_f : \text{Set} \rightarrow \text{Set}$  preserves colimits of  $\omega$ -chains so it has an initial algebra  $\mu \mathcal{P}_f$  that is the colimit of the following  $\omega$ -chain

$$\emptyset \longrightarrow \{\emptyset\} \longrightarrow \{\emptyset, \{\emptyset\}\} \longrightarrow \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \longrightarrow \dots$$

where the arrows are inclusions. We thus obtain that  $\mu F = \bigcup_{n < \omega} \mathcal{P}_f^n 0 = V_\omega$ ; moreover notice that  $\mathcal{P}_f V_\omega$  is precisely  $V_\omega$ . Now let  $(V_\omega, c_n : V_n \rightarrow V_\omega)_{i < \omega}$  be the colimit cocone; we know that  $c_n = \iota \circ \mathcal{P}_f c_{n-1}$  for all  $0 < n < \omega$ . Fix  $0 < n < \omega$  and consider  $\{A_1, \dots, A_k\} \in V_n$ ; we have

$$\mathcal{P}_f c_{n-1}(\{A_1, \dots, A_k\}) = \{c_{n-1}(A_1), \dots, c_{n-1}(A_k)\} = \{A_1, \dots, A_k\}$$

because  $c_{n-1} : V_{n-1} \rightarrow V_\omega$  is an inclusion. Then

$$\iota \circ \mathcal{P}_f c_{n-1}(\{A_1, \dots, A_k\}) = c_n(\{A_1, \dots, A_k\}) \Rightarrow \iota(\{A_1, \dots, A_k\}) = \{A_1, \dots, A_k\}$$

so we obtain  $\iota = \text{id}_{V_\omega}$ .

## 3.2 Algebraically Complete Categories

**Definition 3.2.1.** A category  $\mathcal{A}$  is **algebraically complete** if every endofunctor on it has an initial algebra.

algebraically complete category

**Remark 3.2.2.** By the Knaster-Tarski Theorem (see Theorem 2.2.21) every complete lattice is an algebraically complete category.

The following Theorem shows that, even for categories with very little structure (the existence of products is enough), being algebraically complete is a very strong condition. The theorem was first proven by Adámek and Koubek in [V79] (1979); we present a neat proof by Freyd, published in [Fre91] (1991).

**Theorem 3.2.3.** Let  $\mathcal{A}$  be an algebraically complete category with products. Then  $\mathcal{A}$  is a preorder.

*Proof.* Towards a contradiction suppose  $\mathcal{A}$  is not a preorder i.e. there is some hom-set  $\mathcal{A}(A, B)$  that has at least two elements. Consider the endofunctor

$$F : \mathcal{A} \xrightarrow{\mathcal{A}(A, -)} \mathbf{Set} \xrightarrow{\mathbf{Set}(-, 2)} \mathbf{Set}^{\text{op}} \xrightarrow{B^{(-)}} \mathcal{A}$$

where  $B^{(-)} : \mathbf{Set}^{\text{op}} \rightarrow \mathcal{A}$  is the functor that takes a set  $M$  to the object

$$B^M = \underbrace{B \times \dots \times B}_{|M| \text{ times}}.$$

We will write  $FD = B^{S(D)}$  for  $S(D) = 2^{\mathcal{A}(A, D)}$ . Note that  $F$  has no fixed point. Indeed assume  $D \cong FD$  then we have

$$\begin{aligned} \mathcal{A}(A, D) &\cong \mathcal{A}(A, FD) && \text{functors preserve isomorphisms} \\ &\cong \mathcal{A}(A, B^{S(D)}) \\ &\cong \mathcal{A}(A, B)^{S(D)} && \text{hom-functor preserves limits} \end{aligned}$$

but the cardinality of the latter is at least  $2^{|S(D)|} = 2^{2^{|\mathcal{A}(A, D)|}}$  that is impossible by Cantor's Theorem. Now Lambek's Lemma (see Lemma 2.2.17) gives that  $F$  has no initial algebra thus  $\mathcal{A}$  is not algebraically complete; contradiction.  $\square$

## **Chapter 4**

# **Initial algebras from transfinite iteration**

## 4.1 Colimits of chains

Through this chapter we regard ordinals  $j \in \text{Ord}$  as linear orders and thus as categories. A  $j$ -chain in some category  $\mathcal{A}$  is a functor  $C : j \rightarrow \mathcal{A}$ ; we denote the elements of the chain by  $C_i$ , for  $i \in j$ , and its arrows by  $c_{i,i'} : C_i \rightarrow C_{i'}$  for  $i \leq i' < j$ . We say that  $\mathcal{A}$  has colimits of chains if for all  $j \in \text{Ord}$  every  $j$ -chain has a colimit; this includes the existence of an initial object (as the colimit of the empty chain) but does not include colimits of Ord-chains.

$j$ -chain

**Lemma 4.1.1.** Let  $j$  be an ordinal and  $C : j \rightarrow \mathcal{A}$  a  $j$ -chain into a category  $\mathcal{A}$  with colimits of chains. Now let  $j' \subseteq j$  be cofinal in  $j$  i.e. such that for every  $i \in j$  there is some  $i' \in j'$  such that  $i \leq i'$  and denote by  $C'$  the restriction of  $C$  to  $j'$ . Then  $C$  and  $C'$  have the same colimit.

*Proof.* Notice that there is a canonical bijection between cocones on  $C$  and cocones on  $C'$ . Given a cocone  $(A, \alpha_i : C_i \rightarrow A)_{i \in j}$  on  $C$  throwing away all the indexes not in  $j'$  yields  $(A, \alpha_i : C_i \rightarrow A)_{i \in j'}$ : a cocone on  $C'$ . Conversely given  $(A, \alpha_i : C_i \rightarrow A)_{i \in j'}$  we need to define  $\alpha_i : C_i \rightarrow A$  for all  $i \in j \setminus j'$ ; but there's only one way to do so. Indeed by cofinality let  $i' \in j'$  be such that  $i \leq i'$ ; then by the cocone condition we must set  $\alpha_i = \alpha_{i'} \circ c_{i,i'}$ . This is independent of the choice of  $i'$  since if  $i \leq i' < i''$  with  $i', i'' \in j'$  then we have

$$\alpha_i = \alpha_{i'} \circ c_{i,i'} = \alpha_{i''} \circ c_{i',i''} \circ c_{i,i'} = \alpha_{i''} \circ c_{i,i''}.$$

Thus  $(A, \alpha_i : C_i \rightarrow A)_{i \in j'}$  can be extended to the whole chain in only one way.

Let  $(L, \sigma_i : C_i \rightarrow L)_{i \in j}$  be the colimit of  $C$ ; then  $(L, \sigma_i : C_i \rightarrow L)_{i \in j'}$  is the colimit of  $C'$ . Indeed let  $(A, \alpha_i : C_i \rightarrow A)_{i \in j'}$  be a cocone on  $C'$  that we extend to a cocone on  $C$  as explained above. Now, since  $L$  is the colimit, let  $m : L \rightarrow A$  be the unique factorization arrow such that

$$m \circ \sigma_i = \alpha_i \quad \text{for all } i \in j.$$

Particularly for all  $i \in j'$  so  $m$  is a factorization of  $(A, \alpha_i : C_i \rightarrow A)_{i \in j'}$  through  $(L, \sigma_i : C_i \rightarrow L)_{i \in j'}$ . To prove it is unique let  $n : L \rightarrow A$  be such that  $n \circ \sigma_i = \alpha_i$  for all  $i \in j'$ ; we will show that  $n$  actually works for all  $i \in j$  and thus must be  $m$  by uniqueness. Let  $i \in j \setminus j'$  and  $i' \in j'$  such that  $i \leq i'$  by cofinality, then we have

$$n \circ \sigma_i = n \circ \sigma_{i'} \circ c_{i,i'} = \alpha_{i'} \circ c_{i,i'} = \alpha_i.$$

For the converse let  $(L, \sigma_i : C_i \rightarrow L)_{i \in j'}$  be the colimit of  $C'$ . One can prove that its unique extension  $(L, \sigma_i : C_i \rightarrow L)_{i \in j}$  is the colimit of  $C$  by an argument similar to the one above. Finally we have that  $C$  and  $C'$  have the same colimit (modulo restriction/extension) and, moreover, the factorization arrows are the same.  $\square$

Thanks to this lemma we can safely calculate colimits of chains, or apply Proposition 1.1.8, on convenient cofinal subchains. This will allow us to “skip” limit ordinals because given  $j$  limit the subset  $\{i \in j : i \text{ successor}\}$  is cofinal in  $j$ .

## 4.2 Transfinite iteration

The main result of this section is again a generalization to the categorical setting of an order-theoretic result; this time from Zermelo (see [Zer04]). Let  $P$  be a chain-complete poset (particularly it has a bottom element  $\perp$ ) and  $f : P \rightarrow P$  an order-preserving map. We can define an Ord-indexed sequence by transfinite recursion as follows

$$\begin{aligned} f^0(\perp) &= \perp && \text{base case;} \\ f^{j+1}(\perp) &= f(f^j(\perp)) && \text{for all ordinals } j; \\ f^j(\perp) &= \bigvee_{i < j} f^i(\perp) && \text{for limit ordinals } j. \end{aligned}$$

This sequence is clearly a chain.

**Theorem 4.2.1 (Zermelo).** Let  $P$  be a chain-complete poset. Then every order-preserving  $f : P \rightarrow P$  has a lest fixed point  $\mu f$  and  $\mu f = f^j(\perp)$  for some  $j \in \text{Ord}$ .

*Proof.* By Hartog's Lemma (see [Gol96, Theorem 8.18] for a proof) there is an ordinal  $i$  such that there is no injection  $i \rightarrow P$ . Particularly the function  $i \rightarrow P$  defined by  $j \mapsto f^j(\perp)$  is not injective thus there are  $j < k < i$  such that  $f^j(\perp) = f^k(\perp)$  and we have  $f^j(\perp) \leq f^{j+1}(\perp) \leq f^k(\perp)$  because  $(f^j(\perp))_{j \in \text{Ord}}$  is a chain. So  $f(f^j(\perp)) = f^{j+1}(\perp) = f^j(\perp)$  hence  $f^j(\perp)$  is a fixed point.

Now let  $x = f(x)$  be another fixed point. By transfinite induction we can prove  $x \geq f^i(\perp)$  for all  $i \in \text{Ord}$  (similarly to what we did in Theorem 3.1.1) thus when  $i = j$  we obtain  $x \geq f^j(\perp)$ . This proves that  $f^j(\perp)$  is indeed the least fixed point of  $f$ .  $\square$

In order to generalize this theorem and its proof we begin by defining a transfinite version of the initial-algebra  $\omega$ -chain. This will be a functor  $W : \text{Ord} \rightarrow \mathcal{A}$  into a category  $\mathcal{A}$  with colimits of chains. It is defined by transfinite recursion as follows.

$$\begin{aligned} W_0 &= 0 && \text{base case;} \\ W_{j+1} &= FW_j && \text{for all ordinals } j; \\ W_j &= \text{colim}_{i < j} W_i && \text{for all limit ordinals } j. \end{aligned}$$

As for arrows  $w_{0,1}$  is the unique arrow  $0 \rightarrow W_1 = F0$  and  $w_{i+1,j+1} = Fw_{i,j}$  for all  $i, j \in \text{Ord}$ . When  $j$  is limit we have  $W_j = \text{colim}_{i < j} W_i$  so we let  $\{w_{i,j}\}_{i < j}$  be the arrows forming the colimit cocone. We are only missing  $w_{j,j+1}$  for  $j$  limit. Such arrows are obtained using the universal property of  $W_j$ . First notice that we can leave out the initial object 0 and all the limit ordinals without changing the colimit i.e.

$$W_j = \text{colim}_{i < j} W_i = \text{colim}_{i < j \text{ successor}} W_i$$

by Lemma 4.1.1. Now  $(Fw_{i,j} : FW_i = W_{i+1} \rightarrow FW_j = W_{j+1})_{i < j}$  is a cocone on this reduced diagram. Thus we obtain a unique factorization arrow  $W_j \rightarrow W_{j+1}$  that we clare to be our  $w_{j,j+1}$  so the definition of our transfinite chain can continue. Notice also that, being the factorization arrow,  $w_{j,j+1}$  is such that

$$w_{j,j+1} \circ w_{i+1,j} = Fw_{i,j} \quad \text{for all } i < j. \quad (4.2.1)$$

We now generalize Remark 3.1.3 to the transfinite case.

**Remark 4.2.2.** Let  $(A, \alpha)$  be an  $F$ -algebra; then it induces a cocone  $(\alpha_i : W_i \rightarrow A)_{i \in \text{Ord}}$  on  $W$  as follows.

$$\begin{aligned} \alpha_0 &= ! && \text{base case;} \\ \alpha_{i+1} &= \alpha \circ F\alpha_i && \text{for all ordinals } i; \end{aligned}$$

for limit ordinals  $j$  we define  $\alpha_j : W_j \rightarrow A$  to be the unique factorization arrow such that  $\alpha_j \circ w_{i,j} = \alpha_i$  for all  $i < j$ . In order for this construction to work we need to prove that, at stage  $j$ ,  $(\alpha_i : W_i \rightarrow A)_{i < j}$  is a cocone i.e. that

$$\alpha_i \circ w_{k,i} = \alpha_k \quad (4.2.2)$$

for all  $k < i < j$ . Assuming that all  $\alpha_i$ 's for  $i < j$  have been constructed we have that:

1. for  $w_{0,1}$  condition (4.2.2) is trivial;
2. for  $w_{k,i}$  with  $i$  limit it is trivial by construction of  $\alpha_i$ ;
3. for  $w_{i,i+1}$  with  $i$  limit we can prove (4.2.2) holds by showing

$$\alpha_{i+1} \circ w_{i,i+1} \circ w_{k,i} = \alpha_i \circ w_{k,i} = \alpha_k$$

for all  $k < i$  and conclude by uniqueness of  $\alpha_i$ . It is enough to prove the relation for all  $k < i$  successors by Lemma 4.1.1 and indeed we have

$$\begin{aligned} \alpha_{i+1} \circ w_{i,i+1} \circ w_{k,i} &= \alpha \circ F\alpha_i \circ w_{i,i+1} \circ w_{k,i} && \text{definition of } \alpha_{i+1} \\ &= \alpha \circ F\alpha_i \circ Fw_{k-1,i} && (4.2.1) \\ &= \alpha \circ F(\alpha_i \circ w_{k-1,i}) && \text{functoriality of } F \\ &= \alpha \circ F\alpha_{k-1} && \text{definition of } \alpha_i \\ &= \alpha_k && \text{definition of } \alpha_k. \end{aligned}$$

4. for  $w_{i,i+1}$  with  $i$  non-limit we proceed inductively like in Remark 3.1.3; the base case is provided by item 1 if  $i$  is a natural number and by item 3 otherwise.

Finally we conclude that  $(\alpha_i : W_i \rightarrow A)_{i \in \text{Ord}}$  is well defined and a cocone on  $W$  since, as we just proved, for every  $j$  limit  $(\alpha_i : W_i \rightarrow A)_{i < j}$  is a cocone.

**Definition 4.2.3.** We say that the initial-algebra chain **converges in  $\lambda$  steps** if  $w_{\lambda, \lambda+1}$  is an isomorphism and that it converges in **exactly  $\lambda$  steps** when  $\lambda$  is the least such ordinal.

convergence of  
the initial-algebra  
chain

We are now ready to state and prove the transfinite version of Adámek's Theorem.

**Theorem 4.2.4.** Let  $\mathcal{A}$  be a category with colimits of chains and  $F$  an endofunctor. If the initial-algebra chain of  $F$  converges in  $j$  steps then  $W_j$  is the initial algebra  $\mu F$  with structure given by

$$(w_{j,j+1})^{-1} : W_{j+1} = FW_j \rightarrow W_j.$$



*Proof.* We only need to prove that  $W_j$  with the algebra structure given above is initial. To this end let  $(A, \alpha)$  be an  $F$ -algebra and let  $h$  be the arrow  $\alpha_j : W_j \rightarrow A$  of the cocone induced by  $(A, \alpha)$  in the sense of Remark 4.2.2. We shall prove that  $h$  is the unique  $F$ -algebra homomorphism from  $W_j$  to  $A$ .

$$\begin{array}{ccc}
 W_{j+1} = FW_j & \xrightarrow{(w_{j,j+1})^{-1}} & W_j \\
 \downarrow Fh = F\alpha_j & \searrow \alpha_{j+1} & \downarrow h = \alpha_j \\
 FA & \xrightarrow{\alpha} & A
 \end{array}$$

In the above diagram the lower left triangle commute by construction of  $\alpha_{j+1}$  and the upper right one because the  $\alpha_i$ 's form a cocone. Thus the outer square commutes i.e.  $h$  is an homomorphism.

To prove that  $h$  is unique assume that  $k : W_j \rightarrow A$  is another homomorphism. We will show by transfinite induction that  $h \circ w_{i,j} = k \circ w_{i,j}$  for all  $i \leq j$  and then conclude by choosing  $i = j$ . Notice that  $h \circ w_{i,j} = \alpha_j \circ w_{i,j} = \alpha_i$  so we are really interested in proving

$$\alpha_i = k \circ w_{i,j} \quad (4.2.3)$$

for all  $i \leq j$ . If  $i = 0$  then (4.2.3) is obvious because both sides are arrows out of the initial object. If we have  $\alpha_i = k \circ w_{i,j}$  then

$$\begin{aligned}
 \alpha_{i+1} &= \alpha \circ F\alpha_i && \text{definition of } \alpha_i \\
 &= \alpha \circ F(k \circ w_{i,j}) && \text{inductive hypothesis} \\
 &= \alpha \circ Fk \circ w_{i+1,j+1} && \text{fctoriality of } F \\
 &= k \circ (w_{j,j+1})^{-1} \circ w_{i+1,j+1} && k \text{ is an homomorphism} \\
 &= k \circ w_{i+1,j}.
 \end{aligned}$$

Finally if  $i$  is limit we know  $\alpha_i$  is the unique arrow such that  $\alpha_i \circ w_{s,i} = \alpha_s$  for all  $s < i$ ; so if we show that  $k \circ w_{i,j}$  does the same then we have (4.2.3). Indeed for  $s < i$  we have

$$\begin{aligned}
 k \circ w_{i,j} \circ w_{s,i} &= k \circ w_{s,j} \\
 &= h \circ w_{s,j} && \text{inductive hypothesis since } s < i \\
 &= \alpha_j \circ w_{s,j} \\
 &= \alpha_s && \text{the } \alpha_i \text{'s are a cocone.}
 \end{aligned}$$

This concludes the proof.  $\square$

**Corollary 4.2.5.** Let  $\mathcal{A}$  be a category with colimits of chains and  $F$  an endofunctor. If  $F$  preserves colimits of  $\lambda$ -chains for some limit ordinal  $\lambda$  then the initial algebra chain of  $F$  converges in  $\lambda$  steps. Therefore we have  $\mu F = W_\lambda$ .

*Proof.* Since  $W_\lambda = \text{colim}_{i < \lambda} W_i$  and  $F$  preserves colimits of  $\lambda$ -chains we have that  $FW_\lambda = \text{colim}_{i < \lambda} FW_i$  with colimit cocone  $(Fw_{i,\lambda} = w_{i+1,\lambda+1} : W_{i+1} \rightarrow W_{\lambda+1} = FW_\lambda)_{i < \lambda}$ . We thus obtain a unique morphism  $\iota : W_{\lambda+1} = FW_\lambda \rightarrow W_\lambda$  such that  $\iota \circ Fw_{i,\lambda} = w_{i+1,\lambda}$  for

all  $i < \lambda$  (similarly to the proof of Theorem 3.1.5). Combining this universal property with that of  $w_{\lambda, \lambda+1}$  (see 4.2.1) we obtain the following two identities for all  $i < \lambda$ .

$$\begin{aligned} \iota \circ Fw_{i, \lambda} = w_{i+1, \lambda} &\implies \iota \circ w_{\lambda, \lambda+1} \circ w_{i+1, \lambda} = \text{id}_{W_\lambda} \circ w_{i+1, \lambda} \\ w_{\lambda, \lambda+1} \circ w_{i+1, \lambda} = Fw_{i, \lambda} &\implies w_{\lambda, \lambda+1} \circ \iota \circ Fw_{i, \lambda} = \text{id}_{W_{\lambda+1}} \circ Fw_{i, \lambda} \end{aligned}$$

And, by Proposition 1.1.8,  $\iota \circ w_{\lambda, \lambda+1} = \text{id}_{W_\lambda}$  and  $w_{\lambda, \lambda+1} \circ \iota = \text{id}_{W_{\lambda+1}}$  so the initial-algebra chain converges in  $\lambda$  steps. We conclude by Theorem 4.2.4.  $\square$

Clearly for  $\lambda = \omega$  we obtain Adámek's Theorem. We now give an example of a functor that has initial-algebra convergent in more than  $\omega$  steps; showing that Theorem 4.2.4 is more powerful than Adámek's Theorem.

**Example 4.2.6.** Recall that trees are simple graphs such that every two distinct vertexes are connected by a unique path; we generally mark one vertex as the **root** of the tree and draw them with the root on top as we have been doing so far. Trees with no infinite paths are called **well-founded**. When a tree  $t$  is well-founded we define its **height** by

$$h(t) = \sup\{h(t') + 1 : t' \text{ maximal proper subtree of } t\}.$$

well-founded tree

height of a tree

Thus finite trees (that are always well-founded) have height equal to the maximum length of a root-to-leaf path. For other trees the height is an infinite ordinal:  $T_1$  below is constructed by linking to the root a copy of every path of finite length; thus the height of  $T_1$  is  $\omega$ . On the other hand  $T_2$  has  $T_1$  as the only maximal proper subtree so its height is  $\omega + 1$ .



We have seen in Example 3.1.13 that  $FX = X \times X + 1$  on Set satisfies the hypothesis of Adámek's Theorem 3.1.5 thus from Corollary 4.2.5 its initial-algebra chain converges in  $\omega$  steps to the algebra of finite binary trees. Now consider the functor  $GX = X^\omega + 1$  on Set. Its initial-algebra chain is given by

$$W_i = \{\text{all countably branching trees } t \text{ such that } h(t) < i\}$$

and the connecting arrows are inclusions. Indeed  $W_0 = \emptyset$ ,  $W_1 = \{t_0\}$  where  $t_0$  is the root-only tree (the unique tree of height 0),  $W_2 = W_1^\omega + 1$  contains the root-only tree, of height 0, and the unique countably branching tree



of height 1. Generally  $W_{i+1} = (W_i)^\omega + 1$  gives us that the elements of  $W_{i+1}$  are the root-only tree and all trees



where  $s_n \in W_i$  for every  $n < \omega$ . Since the connecting maps  $w_{i,i+1}$  are inclusions at limit steps we take unions so, for example,  $W_\omega$  is the set of all countably branching trees of height less than  $\omega$  i.e. all countably branching trees of finite height. However, as opposed to what happens for  $F$ , the chain does not stop at  $\omega$  because one can take a sequence  $t_0, t_1, t_2, \dots$  of trees with heights cofinal in  $\omega$ ; then the tree



has height  $\omega$  i.e. it is an element of  $W_{\omega+1} \setminus W_\omega$ . Indeed for every countable ordinal  $i < \omega_1$ , by the same argument, one has that  $W_i \subset W_{i+1}$ . The chain then converges at the first uncountable ordinal  $\omega_1$  because, given a sequence of trees  $t_0, t_1, t_2, \dots$  such that  $t_n \in W_{i_n}$  for  $i_n < \omega_1$  then the tree (4.2.4) has height  $\sup\{i_0, i_1, i_2, \dots\}$  that is a countable ordinal, and thus  $t \in W_{\omega_1}$ . So the initial-algebra  $\mu G$  is the set of all well-founded countable branching trees.

### 4.3 Smooth monomorphisms and a converse to Lambeck's Lemma

From Theorem 4.2.4 we know that convergence of the initial-algebra chain implies existence of the initial algebra; but what about the converse? Are all initial algebras obtainable by iteration? The following example shows that this is not the case.

**Example 4.3.1.** A graph  $G$  is a pair  $(V_G, E_G)$  where  $V_G$  is a set of vertexes and  $E_G$  a symmetric binary relation on  $V_G$ . When  $x E_G y$  we say that  $x$  and  $y$  are adjacent. A graph morphism is a function  $f : G \rightarrow G'$  such that  $x E_G y$  implies  $f(x) E_{G'} f(y)$  for every  $x, y \in V_G$ . Graphs and graph morphisms form the category  $\text{Gra}$ .

We say that a graph  $G$  has a loop if there is some vertex  $x$  such that  $x E_G x$  and that it is loop-free otherwise. For a cardinal  $\kappa$  we denote by  $K_\kappa$  the loop-free graph on  $\kappa$  vertexes such that any two distinct vertexes are always adjacent; these graphs are called complete graphs. Given a loop-free graph  $G$  we define its chromatic number  $\chi(G)$  to be the smallest cardinal such that there is a morphism from  $G$  to  $K_{\chi(G)}$ ; notice that  $\chi(K_\kappa) = \kappa$ . Finally we can consider the following endofunctor on  $\text{Gra}$ :

$$FG = \begin{cases} 1 & \text{if } G \text{ has loops} \\ K_{\kappa^+} & \text{if } G \text{ is loop-free} \end{cases}$$

where 1 is the loop on a single vertex and  $\kappa^+$  is the smallest cardinal bigger than  $\kappa = \chi(G) + \aleph_1$ . As for arrows if we have  $f : G \rightarrow G'$  and  $G, G'$  are loop-free then  $Ff$  is the inclusion between complete graphs, otherwise if  $G$  has loops so must  $G'$  and  $Ff$  is  $\text{id}_1$ , finally if  $G$  is loop-free and  $G'$  has a loop then  $Ff$  is the constant function to 1.

If  $G$  is loop-free then there are no arrows from  $FG$  to  $G$  because the existence of such an arrow implies  $\chi(FG) \leq \chi(G)$  but we have  $\chi(FG) > \chi(G)$  by construction of  $F$ . Then every  $F$ -algebra is of the form  $1 \rightarrow G$  with  $G$  a graph that has a loop. This means that  $F$ -algebras are graphs  $G$  with a marked loop and it is clear that  $\text{id}_1 : F1 = 1 \rightarrow 1$  is the initial algebra. However the initial-algebra chain of  $F$  never converges because its elements have increasing cardinalities.

However, in the case of  $\text{Set}$ , we have the following result that, together with Lambek's Lemma, immediately implies that every initial algebra over  $\text{Set}$  is obtained by iteration.

**Theorem 4.3.2.** Let  $F$  be an endofunctor over  $\text{Set}$  such that there is some set  $A$  and an injective function  $\alpha : FA \rightarrow A$ ; then the initial-algebra chain of  $F$  converges.

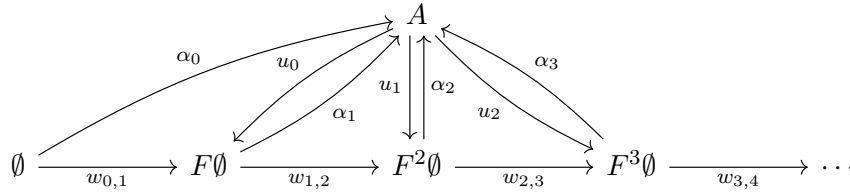
*Proof.* The implication is trivial if  $F\emptyset = \emptyset$  (Remark 2.2.2) so we only need to prove the theorem under the assumption that  $F\emptyset \neq \emptyset$ .

Our  $\alpha : FA \rightarrow A$  is an  $F$ -algebra so it induces a canonical cocone  $(\alpha_i : W_i \rightarrow A)_{i \in \text{Ord}}$  over the initial-algebra chain  $(W_i)_{i \in \text{Ord}}$  as of Remark 4.2.2; we will prove that  $\alpha_\omega$  is injective.

Since  $F\emptyset \neq \emptyset$  then, upon the action of the functor, we obtain a function  $F\emptyset \rightarrow FA$  with non-empty domain from the unique  $\emptyset \rightarrow A$  so  $FA \neq \emptyset$  and, as we have  $\alpha$ , by the same argument we get that  $A \neq \emptyset$ . Now  $\alpha$  is injective so it splits i.e. there exists  $\alpha' : A \rightarrow FA$  such that  $\alpha' \circ \alpha = \text{id}_{FA}$  and, since  $F\emptyset \neq \emptyset$ , let  $u_0$  be any function of type  $A \rightarrow F\emptyset$ . Then we define

$$u_{n+1} = Fu_n \circ \alpha' \quad \text{for } 0 < n < \omega$$

and obtain the following situation.



We have that  $w_{n,n+1} = u_n \circ \alpha_n$  holds for  $n < \omega$  by induction. The base case is obtained by noticing that  $u_0 \circ \alpha_0$  and  $w_{0,1}$  are both arrows of type  $\emptyset \rightarrow F\emptyset$  and  $\emptyset$  is the initial object. The inductive step follows from the following calculation using the definitions of  $\alpha'$ ,  $u_{n+1}$ ,  $\alpha_{n+1}$  and the inductive hypothesis:

$$w_{n+1,n+2} = Fw_{n,n+1} = Fu_n \circ F\alpha_n = (Fu_n \circ \alpha') \circ (\alpha \circ F\alpha_n) = u_{n+1} \circ \alpha_{n+1}.$$

Finally let  $x, y \in W_\omega$  such that  $\alpha_\omega(x) = \alpha_\omega(y)$ . Since  $W_\omega = \text{colim}_{n < \omega} F^n\emptyset$  there is some  $k < \omega$  and  $x', y' \in F^k\emptyset$  such that  $w_{k,\omega}(x') = x$  and  $w_{k,\omega}(y') = y$ . Then, as  $(\alpha_i : W_i \rightarrow A)_{i \in \text{Ord}}$  is a cocone,  $\alpha_k = \alpha_\omega \circ w_{k,\omega}$  so  $\alpha_k(x') = \alpha_k(y')$  and we have

$$w_{k+1,\omega} \circ u_k \circ \alpha_k = w_{k+1,\omega} \circ w_{k,k+1} = w_{k,\omega}$$

that implies

$$x = w_{k,\omega}(x') = (w_{k+1,\omega} \circ u_k \circ \alpha_k)(x') = (w_{k+1,\omega} \circ u_k \circ \alpha_k)(y') = w_{k,\omega}(y') = y$$

and this completes the proof that  $\alpha_\omega$  is injective. Now we can prove that the initial-algebra chain converges.

Since  $\alpha_\omega$  is injective and has non-empty domain there is a function  $\beta : A \rightarrow W_\omega$  such that  $\beta \circ \alpha_\omega = \text{id}_A$ . Because of this  $F\alpha_\omega$  is also injective (functors preserve identities and composition) and so must be  $\alpha_{\omega+1} = \alpha \circ F\alpha_\omega$ . If we define  $v = \beta \circ \alpha_{\omega+1}$  we obtain

$$\alpha_\omega \circ v = \alpha_\omega \circ \beta \circ \alpha_{\omega+1} = \alpha_{\omega+1}$$

from which

$$\alpha_\omega \circ v \circ w_{\omega,\omega+1} = \alpha_{\omega+1} \circ w_{\omega,\omega+1} = \alpha_\omega \implies v \circ w_{\omega,\omega+1} = \text{id}_{W_\omega};$$

$$\alpha_{\omega+1} \circ w_{\omega,\omega+1} \circ v = \alpha_\omega \circ v = \alpha_{\omega+1} \implies w_{\omega,\omega+1} \circ v = \text{id}_{W_{\omega+1}}.$$

So  $w_{\omega,\omega+1}$  is an isomorphism and the initial-algebra chain converges.  $\square$

Now we'd like to generalize Theorem 4.3.2. Notice that  $\alpha$ ,  $\alpha_\omega$  and  $\alpha_{\omega+1}$  being injective (mono) plays a crucial role in the argument above; this suggests that we should look into monomorphisms and, particularly, subobjects.

**Definition 4.3.3.** Fixed an object  $A$  in a category  $\mathcal{A}$  a **subobject** of  $A$  is represented by a monomorphism  $m : B \rightarrowtail A$ . If we have two monomorphisms  $m_B : B \rightarrowtail A, m_C : C \rightarrowtail A$  we write  $m_B \leq m_C$  if there is some  $k : B \rightarrow C$  such that  $m_B = m_C \circ k$  (notice that this  $k$  too is mono). When we have  $m_B \leq m_C \leq m_B$  the  $k$  above is an isomorphism and we say  $m_B$  and  $m_C$  represent the same subobject. Thus we can consider the collection of equivalence classes of subobjects of  $A$  that is partially ordered by  $\leq$ . If  $\mathcal{A}$  is such that every object  $A \in \mathcal{A}$  has only a set of subobjects then  $\mathcal{A}$  is called **well-powered**.

subobject

well-powered category

We can restrict ourselves to a class  $\mathcal{M}$  of monomorphisms and speak of  $\mathcal{M}$ -subobjects i.e. subobjects represented by mono in  $\mathcal{M}$ . When a given object  $A$  has only a set of  $\mathcal{M}$ -subobjects we write  $\text{Sub}_{\mathcal{M}}(A, \leq)$  for the poset of  $\mathcal{M}$ -subobjects of  $A$ ; if  $\mathcal{M}$  is the class of all monomorphisms we also write simply  $\text{Sub}(A, \leq)$ . Notice that we do not require the monomorphisms that witness  $\leq$  to be in  $\mathcal{M}$ .

$\text{Sub}_{\mathcal{M}}(A, \leq)$

From here onwards let  $\mathcal{M}$  denote a class of monomorphisms (of some generic category  $\mathcal{A}$  or otherwise clear from the context) containing all isomorphisms and closed under composition.

$\mathcal{M}$

**Definition 4.3.4.** We say that an object  $A$  has **smooth  $\mathcal{M}$ -subobjects** if  $\text{Sub}_{\mathcal{M}}(A, \leq)$  is a chain-complete poset (particularly it is a set) where joins of chains are given by colimits of the corresponding chain of subobjects. We say that the class  $\mathcal{M}$  itself is **smooth** if every object has smooth  $\mathcal{M}$ -subobjects. When the class of all monomorphisms of a given category is smooth we say that the category has **smooth monomorphisms**.

smooth  $\mathcal{M}$ -subobjects

smooth class of monomorphisms

**Remark 4.3.5.** To be more precise consider an ordinal  $\lambda$  and a chain of  $\mathcal{M}$ -subobjects  $(m_i : A_i \rightarrowtail A)_{i < \lambda}$  i.e. we have arrows  $a_{i,j} : A_i \rightarrow A_j$  (that need not be in  $\mathcal{M}$ ) that witness  $m_i \leq m_j$  for  $i \leq j < \lambda$ . By definition if  $A$  has smooth  $\mathcal{M}$ -subobjects then there is an  $\mathcal{M}$ -subobject  $m : C \rightarrowtail A$  that is the join of our chain in  $\text{Sub}_{\mathcal{M}}(A, \leq)$ . Particularly we

have that  $m_i \leq m$  for all  $i < \lambda$  so there are morphisms  $c_i : A_i \rightarrow C$  such that  $m \circ c_i = m_i$  for all  $i < \lambda$ . These morphisms form a cocone over our chain: given  $a_{i,j}$  for  $i < j$  we have

$$m \circ c_j \circ a_{i,j} = m_j \circ a_{i,j} = m_i = m \circ c_i$$

and so  $c_j \circ a_{i,j} = c_i$  because  $m$  is mono. In order to have smooth  $\mathcal{M}$ -subobjects this cocone must be a colimit in  $\mathcal{A}$ .

This gives us another way of interpreting Definition 4.3.4. If we consider  $\text{Sub}_{\mathcal{M}}(A, \leq)$  as a subcategory of  $\mathcal{A}$  then  $A$  has smooth  $\mathcal{M}$ -subobjects when  $\text{Sub}_{\mathcal{M}}(A, \leq)$  has colimits of chains (i.e. joins) and these colimits do not change once we move from  $\text{Sub}_{\mathcal{M}}(A, \leq)$  to the whole category  $\mathcal{A}$ .

**Remark 4.3.6.** Given an object  $A$  with smooth  $\mathcal{M}$ -subobjects then  $\text{Sub}_{\mathcal{M}}(A, \leq)$  must have a join for the empty chain. Such join has to be the colimit of the empty chain in  $\mathcal{A}$  and thus the ambient category must have an initial object  $0$ . Moreover the unique arrow  $0 \rightarrow A$  is in  $\mathcal{M}$ . We conclude that for a category to have a smooth class of mono  $\mathcal{M}$  it is necessary for it to have an initial object and all arrows out of such initial object must be in  $\mathcal{M}$ .

**Example 4.3.7.** Set has smooth monomorphisms.

**Example 4.3.8.** The category  $\text{Mon}$  of monoids has smooth monomorphisms. Indeed it is well-powered ( $\text{Sub}(M, \leq)$  is always a set) and we know that, given a chain of submonoids of a given monoid  $M$ , the union of the chain (i.e. its colimit in  $\text{Mon}$ ) is again a submonoid of  $M$  and clearly the join of the chain in  $\text{Sub}(M, \leq)$ . Similarly the categories of groups and vector spaces have smooth monomorphisms.

**Example 4.3.9.** On the other hand monomorphisms are not smooth in the category of rings. Indeed its initial object is  $\mathbb{Z}$  and there are non-monomorphisms with domain  $\mathbb{Z}$  (e.g.  $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  for some  $n \in \mathbb{N}$ ) so by Remark 4.3.5 the category of rings does not have smooth monomorphisms.

We now work our way towards a sufficient condition for the convergence of the initial-algebra chain.

**Definition 4.3.10.** Given a category and a class  $\mathcal{M}$  of monomorphisms we say that an endofunctor  $F$  **preserves**  $\mathcal{M}$  if for every  $m \in \mathcal{M}$  we have  $Fm \in \mathcal{M}$ .

$\mathcal{M}$ -preserving  
functor

**Definition 4.3.11.** An  $\mathcal{M}$ -**pre-fixed point** of an endofunctor  $F$  is an object  $A$  together with a monomorphism  $FA \rightarrowtail A$  in  $\mathcal{M}$ .

$\mathcal{M}$ -pre-fixed point

**Theorem 4.3.12.** Let  $F$  be an  $\mathcal{M}$ -preserving endofunctor over a well-powered category with smooth  $\mathcal{M}$ -subobjects and  $A$  an  $\mathcal{M}$ -pre-fixed point of  $F$ . Then the initial-algebra chain of  $F$  converges.

*Proof.* Let's call  $\alpha$  the monomorphism in  $\mathcal{M}$  that makes  $A$  into an  $\mathcal{M}$ -pre-fixed point of  $F$  i.e.  $\alpha : FA \rightarrowtail A$  and consider the function  $f : \text{Sub}_{\mathcal{M}}(A, \leq) \rightarrow \text{Sub}_{\mathcal{M}}(A, \leq)$  defined by

$$f(B \xrightarrow{u} A) = (FB \xrightarrow{Fu} FA \xrightarrow{\alpha} A).$$

Notice that this function is well-defined as  $F$  preserves  $\mathcal{M}$  and  $\mathcal{M}$  is by definition closed under composition; moreover  $f$  is also monotone. Indeed let  $B \leq C$  be two  $\mathcal{M}$ -subobjects of  $A$  and  $w : B \rightarrowtail C$  that witnesses  $B \leq C$  i.e

$$\begin{array}{ccc} B & \xrightarrow{u} & A \\ \downarrow w & \nearrow v & \\ C & & \end{array}$$

commutes. By applying  $F$  and adjoining  $\alpha$  we obtain

$$\begin{array}{ccccc} FB & \xrightarrow{Fu} & FA & \xrightarrow{\alpha} & A \\ \downarrow Fw & \nearrow Fv & & & \\ FC & & & & \end{array}$$

with  $Fu, Fw$  mono because  $F$  preserves  $\mathcal{M}$  and  $Fu = Fv \circ Fw$ ; so  $f(B) \leq f(C)$  in  $\text{Sub}_{\mathcal{M}}(A, \leq)$ .

Now by assumption  $\text{Sub}_{\mathcal{M}}(A, \leq)$  is chain-complete and we can apply Zermelo's Fixed Point Theorem 4.2.1 to obtain that  $f$  has a fixed point  $\mu f = f^i(\perp)$  for some ordinal  $i$ . Moreover the canonical cocone  $(\alpha_j : W_j \rightarrow A)_{j \in \text{Ord}}$  induced as in 4.2.2 by the algebra  $\alpha : FA \rightarrowtail A$  is such that  $\alpha_j = f^j(\perp)$  for all  $j \in \text{Ord}$ ; this is clear for the successor step, while the limit step follows from the fact that joins of chains in  $\text{Sub}_{\mathcal{M}}(A, \leq)$  are colimits of the corresponding chains of subobjects because  $A$  has smooth  $\mathcal{M}$ -subobjects. Then from  $f(f^i(\perp)) = f^i(\perp)$ , the definition of fixed point, we obtain that  $\alpha_i$  and  $\alpha_{i+1}$  represent the same subobject of  $A$ . Finally we know that  $\alpha_i = \alpha_{i+1} \circ w_{i,i+1}$  and that, since  $\alpha_i$  and  $\alpha_{i+1}$  represent the same subobject, there is an arrow  $v : W_{i+1} \rightarrow W_i$  such that  $\alpha_{i+1} = \alpha_i \circ v$  (witnessing  $\alpha_{i+1} \leq \alpha_i$ ). Now the final part of the proof of Theorem 4.3.2 applies mutatis mutandis.  $\square$

None of the hypothesis of Theorem 4.3.12 can be dropped, as the following examples show.

**Example 4.3.13.** Consider  $\text{Ord}^\top$ : the category of all ordinals ordered by inclusion and with an extra object  $\top$  bigger than any other. As endofunctor consider  $F\lambda = \lambda + 1$  for  $\lambda \in \text{Ord}$  and  $F\top = \top$  and take  $\mathcal{M}$  to be the class of all mono that, in this case, is simply the class of all arrows thus preserved by  $F$ . The initial-algebra chain is the chain of all ordinals that clearly does not converge. Theorem 4.3.12 cannot be applied to  $\text{Ord}^\top$  and  $F$  because  $\text{Ord}^\top$  is not well-powered; indeed  $\top$  has a proper class of subobjects (all the ordinals).

**Example 4.3.14.** Consider the endofunctor  $FX = X + 1$  on  $\text{Set}$  and the category  $\text{Alg}F$  of  $F$ -algebras and their morphisms; recall from Example 2.2.3 that  $(\mathbb{N}, s, 0)$ , where  $s$  is the successor function, is the initial algebra for  $F$ . Then  $\text{Alg}F$  does not have smooth monomorphisms as there are  $F$ -algebras, such as  $(\mathbb{Z}/n\mathbb{Z}, s, 0)$ , that do not have  $\mathbb{N}$  as subobject; thus the hypothesis of Theorem 4.3.12 do not hold. Indeed we can find a functor on  $\text{Alg}F$  that has no initial algebra while having a fixed point and preserving monomorphisms.

Denote by  $\mathcal{P}_0 X$  the power set of  $X$  without the empty set  $\emptyset$ . This is a functor on  $\mathbf{Set}$  with the natural assignment on arrows that sends  $f : X \rightarrow Y$  to  $\mathcal{P}_0 f : \mathcal{P}_0 X \rightarrow \mathcal{P}_0 Y$  mapping  $A \in \mathcal{P}_0 X$  to  $f(A)$ . This construction gives a functor on  $\mathbf{Alg}^F$  that sends an algebra  $(A, f, c)$  to  $(\mathcal{P}_0 A, \mathcal{P}_0 f, \{c\})$ . Now  $\mathcal{P}_0$  preserves monomorphisms, the terminal algebra  $\{*\}$  is a fixed point and, up to isomorphism, the only one by cardinality arguments. However  $\{*\}$ , as  $\mathcal{P}_0$ -algebra, is not initial: if  $(A, f, c)$  is such that  $f(c) \neq c$  then there is no  $F$ -algebra morphism from  $\{*\}$  to  $A$  and thus no  $\mathcal{P}_0$ -algebra morphism from  $\{*\}$  (as  $\mathcal{P}_0$ -algebra) to any  $\mathcal{P}_0$ -algebra with carrier  $(A, f, c)$ .

**Example 4.3.15.** Consider the category  $\mathbf{Set} \times \mathbf{Set}$  that is well-powered and has smooth monomorphism. As functor take

$$F(X, Y) = \begin{cases} (\emptyset, 1) & \text{if } X \neq \emptyset \\ (\emptyset, \mathcal{P}Y) & \text{if } X = \emptyset \end{cases}$$

defined on arrows by using  $\mathcal{P}$  when we fall under the second case.  $F$  does not preserve monomorphisms (e.g. if we denote with 1, 2, 3 the sets of cardinality 1, 2, 3 then the mono  $(\emptyset, 2) \rightarrow (3, 3)$  maps to  $(\emptyset, \mathcal{P}2) \rightarrow (\emptyset, 1)$  that is not a mono) but has pre-fixed points such as  $F(1, 1) = (\emptyset, 1) \rightarrow (1, 1)$ . However  $F$  cannot have any fixed point (thus any initial-algebra and so the initial-algebra chain cannot converge) because, if  $X \neq \emptyset$ ,  $F(X, Y) = (\emptyset, 1)$  is never isomorphic to  $(X, Y)$  and, if  $X = \emptyset$ ,  $F(X, Y) = (\emptyset, \mathcal{P}Y)$  and  $(X, Y)$  can't be isomorphic by Cantor's Theorem.

Finally we can put all the work we have done so far into a single theorem, the Initial Algebra Theorem which is, among other things, a converse to Lambeck's Lemma.

**Theorem 4.3.16 (Initial Algebra Theorem).** Let  $\mathcal{A}$  be a well-powered category,  $\mathcal{M}$  a smooth class of monomorphisms and  $F$  a functor preserving  $\mathcal{M}$ . Then the following conditions are equivalent:

1. the initial-algebra chain for  $F$  converges;
2.  $F$  has an initial-algebra;
3.  $F$  has a fixed point;
4.  $F$  has a pre-fixed point.

Moreover, if any of the above equivalent conditions holds,  $\mu F$  is an  $\mathcal{M}$ -subobject of every  $\mathcal{M}$ -pre-fixed point of  $F$ .

*Proof.* We have that 1 implies 2 by Theorem 4.2.4; 2 implies 3 by Lambeck's Lemma and 3 implies 4 since, by definition,  $\mathcal{M}$  contains all isomorphisms. Finally 4 implies 1 is exactly Theorem 4.3.12 and from its proof we obtain that  $\mu F$  is an  $\mathcal{M}$ -subobject of every  $\mathcal{M}$ -pre-fixed point of  $F$ .  $\square$



## **Chapter 5**

# **Coalgebras for endofunctors**

## 5.1 $F$ -coalgebras

Though this section let  $\mathcal{A}$  be an arbitrary base category and  $F : A \rightarrow A$  an endofunctor. We shall study  $F$ -coalgebras, the dual concept to  $F$ -algebras.

**Definition 5.1.1.** A **coalgebra** for  $F$  (or a  **$F$ -coalgebra**) is a pair  $(A, \alpha)$  where  $A$  is an object of  $\mathcal{A}$  and  $\alpha$  is an arrow of type  $A \rightarrow FA$ .

$F$ -coalgebra

The notion of morphism between coalgebras is as expected.

**Definition 5.1.2.** Given  $F$ -coalgebras  $(A, \alpha), (B, \beta)$  a morphism between them is an arrow  $f : A \rightarrow B$  of  $\mathcal{A}$  such that

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & FA \\ \downarrow f & & \downarrow Ff \\ B & \xrightarrow{\beta} & FB \end{array}$$

commutes.

As with algebras, coalgebras on  $F$  also form a category that we indicate with  $\text{Coalg}F$ .

$\text{Coalg}F$

**Example 5.1.3.** Over  $\text{Set}$  coalgebras for  $FX = X + 1$  are called **systems with termination**. Given a coalgebra  $\alpha : A \rightarrow A + 1$  we think of  $A$  as a set of **states** and of  $\alpha$  as the **dynamic** of the system. Indeed to give an arrow  $\alpha : A \rightarrow A + 1$  is, for every  $a \in A$ , to deem it a **terminal state** if  $\alpha(a)$  is in the right summand or to specify a **next state**  $\alpha(a)$  in the left summand. We can represent such systems with directed graphs as follows, with terminal states marked as  $\bullet^T$ .

systems with termination



If  $(A, \alpha), (B, \beta)$  are two systems with termination a morphism between them is, according to Definition 5.1.2, a function  $f$  such that

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A + 1 \\ \downarrow f & & \downarrow f + \text{id}_1 \\ B & \xrightarrow{\beta} & B + 1 \end{array}$$

commutes i.e. a function  $f$  such that

1.  $a \in A$  is terminal if and only if  $f(a) \in B$  is;
2. if  $a$  is not terminal then  $\beta(f(a)) = f(\alpha(a))$  i.e. the next state relationship is preserved by  $f$ .

**Example 5.1.4.** As coalgebras for  $FX = X \times X + 1$  on Set we obtain a system in which a given state  $x$  either is terminal or has two possible next states  $(x_0, x_1)$ . We can think of such systems as having next states based on a binary input.

Similarly if we fix a set  $\Sigma$  of possible inputs and consider the functor  $FX = X^\Sigma + 1$  we obtain systems with termination with the next state based on inputs in  $\Sigma$ .

**Example 5.1.5.** Let  $\Sigma$  be a set of inputs. A **deterministic automata** is composed of a set  $S$  of states, a transition function  $\delta_s : S \rightarrow S$  for every  $s \in \Sigma$  and a set  $A \subset S$  of **accepting states**. In order to obtain deterministic automata as coalgebras for a functor we group the  $\delta_s$ 's into a single function  $\delta : S \rightarrow S^\Sigma$  assigning to a state  $t$  the map  $s \mapsto \delta_s(t)$  and consider the characteristic function  $\chi : S \rightarrow \{0, 1\}$  of  $A$ . So deterministic automata are coalgebras for the functor

$$FX = \{0, 1\} \times X^\Sigma$$

over Set.

**Example 5.1.6.** As above let  $\Sigma$  be a set of inputs. A **non-deterministic automata**, similarly to a deterministic one, is made of a set of states  $S$  with a subset  $A$  of accepting states but the transition functions have codomain  $\mathcal{P}(S)$  i.e.

$$\delta_s : S \rightarrow \mathcal{P}(S) \quad \text{for every } s \in \Sigma.$$

Non-deterministic automata are coalgebras for the functor

$$FX = \{0, 1\} \times (\mathcal{P}(S))^\Sigma$$

over Set.

**Example 5.1.7.** Given a set  $\Sigma$  of inputs and a set  $\Gamma$  of outputs a coalgebra for the functor  $FX = \Gamma \times X^\Sigma$  over Set is called a **Moore automata**. Every state  $x$  of such a coalgebra is labeled by an element  $\gamma \in \Gamma$  (we say that  $x$  emits output  $\gamma$ ) and has next states based on inputs in  $\Sigma$  like in a deterministic automata.

**Example 5.1.8.** Now consider a signature  $\Sigma$  and its associated polynomial functor  $H_\Sigma$  as in Definition 2.1.8. A coalgebra for  $H_\Sigma$  is given by an arrow

$$\alpha : A \rightarrow H_\Sigma A = \coprod_{n < \omega} \Sigma_n \times A^n$$

that, in the case of Set, we think of as made of two functions:

- a function head :  $A \rightarrow \coprod_{n < \omega} \Sigma_n$  that assigns to each state  $a \in A$  a symbol of  $\Sigma$ ;
- a function body :  $A \rightarrow A^{<\omega}$  that assigns to a state  $a \in A$  a sequence  $(y_0, \dots, y_{n-1})$  of next states where  $n$  is the arity of head( $a$ ). We also write  $\text{body}(a, i)$  for  $y_i$ .

Systems with termination (Example 5.1.3) are obtained when the signature has one constant symbol and one unary symbol. If  $\Sigma$  is the (finite) set of inputs of a deterministic automata (Example 5.1.5) the signature with two  $|\Sigma|$ -ary symbols can be used to realize deterministic automatas as a coalgebras for a polynomial functor. Similarly Moore automata can be recovered as coalgebras for appropriate polynomial functors.

deterministic  
automata  
  
accepting state

non-deterministic  
automata

Moore automata

## 5.2 Terminal Coalgebras

We have seen that the initial  $F$ -algebra is quite interesting and we will see now that its dual, the terminal  $F$ -coalgebra, is interesting as well.

**Definition 5.2.1.** A **terminal coalgebra** for an endofunctor  $F$  is a terminal object in  $\text{Coalg} F$ . When it exists we usually denote it by  $(\eta F, \tau)$ .

terminal coalgebra

**Example 5.2.2.**  $FX = X + 1$  has terminal coalgebra  $(\mathbb{N}^\top, \tau)$  with structure  $\tau$  given by the following

$$0 \longleftarrow 1 \longleftarrow 2 \longleftarrow \dots \quad \infty \curvearrowright$$

and 0 as terminal state. Given any other system with termination ( $F$ -coalgebra)  $A$  we have  $b_A : A \rightarrow \mathbb{N}^\top$  defined as follows. For every  $x \in A$  we set

1.  $b_A(x) = 0$  if  $x$  is a terminal state;
2.  $b_A(x) = n$  if there is a finite chain  $(x, x_1, x_2, \dots, x_n)$  such that  $x_{i+1}$  is the next state of  $x_i$  and  $x_n$  is terminal;
3.  $b_A(x) = \infty$  otherwise.

This function clearly preserves terminal and next states so it is a morphism of coalgebras; moreover it is unique because  $\mathbb{N}^\top$  has exactly one terminal state, exactly one sequence of  $n$  next states ending in a terminal state for every  $n$  and exactly one infinite sequence of next states. This proves that  $\mathbb{N}^\top$  is indeed the terminal coalgebra for  $F$ . We call  $b_A(x)$  the **behaviour** of the state  $x$  of  $A$  and it corresponds with the number of iterations needed to reach a terminal state starting from  $x$  (or  $\infty$  if we cannot reach a terminal state starting from  $x$ ).

**Example 5.2.3.** Coalgebras for  $FX = \Sigma \times X$  on  $\text{Set}$  are transition systems with outputs: each state  $x \in A$  has a next state  $\alpha_1(x) \in A$  and emits an output  $\alpha_0(x) \in \Sigma$ ; thus the coalgebra structure is given by  $\alpha = \langle \alpha_0, \alpha_1 \rangle : A \rightarrow \Sigma \times A$ . Notice that every state  $x \in A$  generates an infinite sequence

$$(x_n)_{n < \omega} = (x_0 = \alpha_0(x), x_1 = (\alpha_0 \circ \alpha_1)(x), x_2 = (\alpha_0 \circ \alpha_1 \circ \alpha_1)(x), \dots)$$

of outputs; this is an element of  $\Sigma^\omega$ , the set of infinite sequences (also called **streams**) over  $\Sigma$ . We can make  $\Sigma^\omega$  into an  $F$ -coalgebra with

$\Sigma$ -streams

$$\tau = \langle \text{head}, \text{tail} \rangle : \Sigma^\omega \rightarrow \Sigma \times \Sigma^\omega$$

where  $\text{head}(x_n)_{n < \omega} = x_0$  and  $\text{tail}(x_n)_{n < \omega} = (x_n)_{0 < n < \omega}$ . Thus the function  $b_A$  that sends  $x \in A$  to  $(x_n)_{n < \omega} \in \Sigma^\omega$  makes

$$\begin{array}{ccc} A & \xrightarrow{\langle \alpha_0, \alpha_1 \rangle} & \Sigma \times A \\ \downarrow b_A & & \downarrow \text{id}_\Sigma \times b_A \\ \Sigma^\omega & \xrightarrow{\langle \text{head}, \text{tail} \rangle} & \Sigma \times \Sigma^\omega \end{array} \quad (5.2.1)$$

commute so it is a coalgebra morphism. This  $b_A$  is unique since, if a second morphism  $g$  that makes (5.2.1) commute exists, one can prove that  $b_A(x)$  and  $g(x)$  are equal by induction over the indexes of the streams. Thus  $(\Sigma^\omega, \langle \text{head}, \text{tail} \rangle)$  is the terminal coalgebra.

**Example 5.2.4.** For a little variation consider the functor  $FX = \Sigma \times X + 1$  over  $\text{Set}$  with  $\Sigma \in \text{Set}$ . Coalgebras for  $F$  can be thought of as systems with termination where non-terminal states emit an output in  $\Sigma$ .

Tracing a state gives either a finite word over  $\Sigma$ , if we eventually hit a terminal state, or a stream over  $\Sigma$  if we hit none. Let  $\Sigma^\infty = \Sigma^* + \Sigma^\omega$  where  $\Sigma^*$  is the set of finite words over  $\Sigma$ . Again  $\tau = \langle \text{head}, \text{tail} \rangle : \Sigma^\infty \rightarrow \Sigma \times \Sigma^\infty + 1$  as in Example 5.2.3, with the added caveat of sending the empty word to  $*$  in the right summand, is the terminal coalgebra.

**Example 5.2.5.** As in Example 5.1.5 consider a set of inputs  $\Sigma$  and the functor  $FX = \{0, 1\} \times X^\Sigma$  over  $\text{Set}$  which realizes deterministic automatas as coalgebras. Consider  $\mathcal{P}(\Sigma^*)$  the set of formal languages over  $\Sigma$ . We say  $L \in \mathcal{P}(\Sigma^*)$  is an accepting language if it contains the empty word  $\varepsilon$ . Given a language  $L \in \mathcal{P}(\Sigma^*)$  and an input  $s \in \Sigma$  we define its next state as its Brzozowski derivative

$$s^{-1}L = \{w \in \Sigma^* : sw \in L\};$$

this makes  $\mathcal{P}(\Sigma^*)$  into a coalgebra.

Now if  $A$  is an automata and  $x_0 \in A$  a state we can consider the language  $L(x_0)$  accepted by  $x_0$ ; that is the set of words  $w = w_1 \dots w_n \in \Sigma^*$  such that the sequence

$$x_0 \xrightarrow{\delta_{w_1}} x_1 \xrightarrow{\delta_{w_2}} \dots \xrightarrow{\delta_{w_n}} x_n \quad (5.2.2)$$

ends in an accepting state. We notice that

1.  $L(x)$  is accepting in  $\mathcal{P}(\Sigma^*)$  if and only if  $x$  is accepting in  $A$  (see Example 5.1.5);
2. for every input  $s \in \Sigma$  if  $y = \delta_s(x)$  then  $L(y) = s^{-1}L(x)$ ;

so  $L : A \rightarrow \mathcal{P}(\Sigma^*)$  is a morphism of coalgebras. Moreover  $L$  is unique: let  $f : A \rightarrow \mathcal{P}(\Sigma^*)$  be a morphism of coalgebras; the following points show that  $f(x_0) \subseteq L(x_0)$  and  $L(x_0) \subseteq f(x_0)$  for a fixed  $x_0 \in A$ . Thus  $f(x_0) = L(x_0)$  for every  $x_0 \in A$  and  $f = L$ . We have that  $\mathcal{P}(\Sigma^*)$  is the terminal coalgebra for  $F$ .

- a. Let  $w_1 \dots w_n = w \in f(x_0)$ ; in order to prove that  $w \in L(x_0)$  we need to show that (5.2.2) ends in an accepting state. We have that  $f(x_1) = f(\delta_{w_1}(x_0)) = w_1^{-1}f(x_0)$  because  $f$  is a morphism and, by iterating, we obtain  $f(x_n) = w_n^{-1} \dots w_1^{-1}f(x_0)$ . Finally since  $w \in f(x_0)$  we get  $\varepsilon \in f(x_n)$  i.e.  $f(x_n)$  is accepting in  $\mathcal{P}(\Sigma^*)$  and so must be  $x_n$  in  $A$  because  $f$  is a morphism.
- b. Let  $w_1 \dots w_n = w \in L(x_0)$  thus we know that (5.2.2) ends in an accepting state;  $x_n$  accepting gives  $f(x_n)$  accepting, because  $f$  is a morphism, and thus  $\varepsilon \in f(x_n)$ . Moreover, again because  $f$  is a morphism, we have  $f(x_n) = f(\delta_{w_n}(x_{n-1})) = w_n^{-1}f(x_{n-1})$  and, since  $\varepsilon \in f(x_n)$ , we obtain  $w_n \in f(x_{n-1})$ . Iteration of this argument gives  $w_1 \dots w_n = w \in f(x_0)$ .

**Example 5.2.6.** Recall from Example 5.1.7 that Moore automata are coalgebras for  $FX = \Gamma \times X^\Sigma$  over Set. Consider the set  $\Gamma^{\Sigma^*}$  of functions from  $\Sigma^*$  to  $\Gamma$  with the following coalgebra structure:  $f \in \Gamma^{\Sigma^*}$  emits output  $f(\varepsilon)$  and, for input  $s \in \Sigma$ , has next state  $g \in \Gamma^{\Sigma^*}$  such that  $g(w) = f(sw)$ . The reader can check for himself that this coalgebra is terminal or derive it from the following proposition.

**Proposition 5.2.7.** Let  $\Sigma$  be a signature and let  $T_\Sigma$  be the set of all  $\Sigma$ -trees (finite and infinite ones) equipped with the  $H_\Sigma$ -coalgebra structure that sends a tree



to the element  $(\sigma, (t_0, \dots, t_{n-1}))$  of  $H_\Sigma(T_\Sigma)$ . Notice that this is the inverse of tree-tupling (see Remark 2.2.10) so let it be denoted by  $\tau^{-1}$ . The terminal coalgebra of  $H_\Sigma$  is  $(T_\Sigma, \tau^{-1})$ .

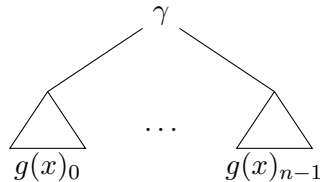
*Proof.* Let  $(A, \alpha)$  be a  $H_\Sigma$ -coalgebra and consider the function  $h : A \rightarrow T_\Sigma$  that sends  $x \in A$  to the tree  $t_x$  constructed recursively as



where  $\alpha(x) = (\sigma, y_0, \dots, y_{n-1})$ . This is clearly a function such that

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & H_\Sigma A \\ \downarrow h & & \downarrow H_\Sigma h \\ T_\Sigma & \xrightarrow{\tau^{-1}} & H_\Sigma T_\Sigma \end{array}$$

commutes i.e. a coalgebra morphism. Now let  $g : A \rightarrow T_\Sigma$  be another morphism and  $x \in A$  such that  $\alpha(x) = (\sigma, y_0, \dots, y_{n-1})$  as above. Then  $g(x)$  is a tree



such that

$$(\gamma, g(x)_0, \dots, g(x)_{n-1}) = (\tau^{-1} \circ g)(x) = (H_\Sigma g \circ \alpha)(x) = (\sigma, g(y_0), \dots, g(y_{n-1}))$$

so  $\gamma = \sigma$  and  $g(x)_i = g(y_i)$  for  $i < n$  i.e.  $g(x)$  and  $t_x$  have the same root and the  $i$ -th subtree of  $g(x)$  is  $g(y_i)$ . We can iterate this process to obtain that  $g(x) = t_x$ , thus  $g$  must be  $h$ , and  $(T_\Sigma, \tau^{-1})$  is the terminal coalgebra.  $\square$

The necessary condition of Lambeck's Lemma 2.2.17 dualizes as well.

**Lemma 5.2.8 (Lambeck's Lemma).** A terminal coalgebra for  $F$ , if it exists, is always a fixed point.

*Proof.* Dual to that of Lemma 2.2.17. □

**Remark 5.2.9.** The power-set functor  $\mathcal{P}$  has no terminal coalgebra because it has no fixed point. Similarly the functor  $FX = \{0, 1\} \times (\mathbb{P}X)^\Sigma$ , which gives non-deterministic automata as coalgebras (see Example 5.1.6), does not have a terminal coalgebra.

Lambeck's Lemma for coalgebras can be interpreted as a generalization of an order-theoretic lemma just as its version for algebras.

**Definition 5.2.10.** Let  $(P, \sqsubseteq)$  be a poset and  $f : P \rightarrow P$  a monotone function. An element  $x \in P$  is a **post-fixed point** if  $x \sqsubseteq f(x)$ .

**Lemma 5.2.11.** Given a poset  $(P, \sqsubseteq)$  and a monotone function  $f : P \rightarrow P$  let  $A = \{x \in P : x \sqsubseteq f(x)\}$  be the set of post-fixed points of  $f$ . If  $\bar{a}$  is the join of  $A$  then  $\bar{a}$  is the biggest fixed-point of  $f$ .

*Proof.* A classical proof can be obtained by dualizing the one of Lemma 2.2.20. Otherwise it follows from Lambeck's Lemma 5.2.8 once one notices that post-fixed points for  $f$  are just  $f$ -coalgebras. □

The Knaster-Tarski Theorem 2.2.21 obviously follows from this version of the lemma as well since the hypothesis guarantees the existence of the join of  $A$ .

**Remark 5.2.12.** For functors over  $\mathbf{Set}$ , the existence of a terminal coalgebra implies the existence of an initial algebra. Indeed if  $F$  is an endofunctor over  $\mathbf{Set}$  with terminal coalgebra  $\nu F$  then by Lambeck's Lemma 5.2.8  $\nu F$  is a fixed point and now Theorem 4.3.2 guarantees the existence of the initial algebra  $\mu F$ . The converse is false since, as it was found in [AK95, Example 7] by Adámek and Koubek, there is a set functor that has a fixed point (and thus an initial algebra) but does not have a terminal coalgebra (we shall see this in Example 6.2.18). Thus the existence of a terminal coalgebra, even for set functors, is a more complicated matter than the existence of an initial algebra and a version of the Initial Algebra Theorem 4.3.16 for coalgebras is not possible, even for  $\mathbf{Set}$ .

## 5.3 Corecursion and Bisimulation

We strive now to dualize what we have seen about recursion and induction in Section 2.3. Recursion is dualized straightforwardly but we don't do the same for induction. Instead of coinduction we will discuss bisimulation, a concept that has many applications, for instance, in modal logic (see [Bv07, Section 3]).

**Definition 5.3.1.** Let  $F$  be an endofunctor over some category  $\mathcal{A}$  with a terminal coalgebra  $\nu F$ . A morphism  $f : A \rightarrow \nu F$  of  $\mathcal{A}$  is **corecursively specified** if there is a coalgebra structure  $\alpha : A \rightarrow FA$  on  $A$  that makes  $f$  into a morphism of coalgebras (actually  $f$  is unique since  $\nu F$  is terminal).

corecursively specified morphism

**Example 5.3.2.** As for Example 5.2.2 the terminal coalgebra for  $FX = X + 1$  over Set is  $(\mathbb{N}^\top, \tau)$ . We shall see that the function  $\text{add} : \mathbb{N}^\top \times \mathbb{N}^\top \rightarrow \mathbb{N}^\top$  defined as addition of natural numbers extended by  $x + \infty = \infty$ ,  $\infty + x = \infty$  and  $\infty + \infty = \infty$  is corecursively specified. In order to do so we need a coalgebra structure on  $\mathbb{N}^\top \times \mathbb{N}^\top$  that induces  $\text{add}$  as the unique coalgebra morphism to  $\mathbb{N}^\top$ . Consider then the coalgebra given by the following structure  $\alpha : \mathbb{N}^\top \times \mathbb{N}^\top \rightarrow \mathbb{N}^\top \times \mathbb{N}^\top + 1$ :

1. the only terminal state is  $(0, 0)$ ;
2. if  $(x, y)$  is not terminal then we set

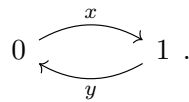
$$\alpha(x, y) = \begin{cases} (0, \tau(y)) & \text{if } x = 0 \text{ (hence } y \neq 0) \\ (\tau(x), y) & \text{if } x \neq 0 \end{cases}.$$

Now let  $b_A : \mathbb{N}^\top \times \mathbb{N}^\top \rightarrow \mathbb{N}^\top$  be the unique morphism to the terminal coalgebra and notice that a state  $(x, y)$ , under iterated applications of  $\alpha$ , either reaches  $(0, 0)$  in  $x + y$  steps or generates an infinite sequence exactly when  $x = \infty$  or  $y = \infty$ . Thus we have that  $b_A = \text{add}$ .

**Example 5.3.3.** Consider the functor  $FX = \Sigma \times X$  over Set with  $\Sigma$  a fixed set of outputs. We know from Example 5.2.3 that the terminal coalgebra for  $F$  is  $\Sigma^\omega$ , the set of  $\Sigma$ -streams, with structure given by  $\tau = \langle \text{head}, \text{tail} \rangle$ .

A single stream can be identified with the function  $f : 1 \rightarrow \Sigma^\omega$  that picks it so we say that a stream is corecursively specified if its  $f : 1 \rightarrow \Sigma^\omega$  is corecursively specified. Coalgebra structures on 1 are functions  $\alpha : 1 \rightarrow \Sigma \times 1$  so the only corecursively specifiable streams are the constant ones.

Other kinds of streams can be corecursively specified as well. For example consider the set  $2 = \{0, 1\}$  with the coalgebra structure given by  $\alpha : 2 \rightarrow \Sigma \times 2$  defined by  $\alpha(0) = (x, 1)$  and  $\alpha(1) = (y, 0)$  for  $x, y \in \Sigma$ . Graphically this coalgebra can be represented as



Since  $\Sigma^\omega$  is the terminal coalgebra there is a unique  $h : 2 \rightarrow \Sigma^\omega$  such that

$$\langle \text{head}, \text{tail} \rangle \circ h = (\text{id}_\Sigma \times h) \circ \alpha.$$

If we write  $h(0) = (a_0, a_1, a_2, \dots)$  and  $h(1) = (b_0, b_1, b_2, \dots)$  from the above we obtain

1.  $\text{head}(h(0)) = (\pi_\Sigma \circ (\text{id}_\Sigma \times h))(\alpha(0))$  hence  $a_0 = x$ ;
2.  $\text{head}(h(1)) = (\pi_\Sigma \circ (\text{id}_\Sigma \times h))(\alpha(1))$  hence  $b_0 = y$ ;
3.  $\text{tail}(h(0)) = (\pi_{\Sigma^\omega} \circ (\text{id}_\Sigma \times h))(\alpha(0)) = h(1)$  hence  $(a_1, a_2, \dots) = (b_0, b_1, \dots)$ ;
4.  $\text{tail}(h(1)) = (\pi_{\Sigma^\omega} \circ (\text{id}_\Sigma \times h))(\alpha(1)) = h(0)$  hence  $(b_1, b_2, \dots) = (a_0, a_1, \dots)$ ;



where  $\pi_\Sigma, \pi_{\Sigma^\omega}$  are the projections of  $\Sigma \times \Sigma^\omega$ . Thus we must have

$$h(0) = (x, y, x, y, \dots) \text{ and } h(1) = (y, x, y, x, \dots).$$

So alternating streams can also be corecursively constructed.

Finally we look at an example of a corecursively specifiable function on streams: the zip function. Given two streams  $x = (x_0, x_1, \dots), y = (y_0, y_1, \dots)$  we define a new stream

$$\text{zip}(x, y) = (x_0, y_0, x_1, y_1, \dots).$$

We give  $\Sigma^\omega \times \Sigma^\omega$  a coalgebra structure  $\alpha = \langle \alpha_{\text{head}}, \alpha_{\text{tail}} \rangle$  by defining

$$\begin{aligned} \alpha_{\text{head}} : \Sigma^\omega \times \Sigma^\omega &\longrightarrow \Sigma \\ (x, y) &\longmapsto \text{head}(x) \end{aligned}$$

$$\begin{aligned} \alpha_{\text{tail}} : \Sigma^\omega \times \Sigma^\omega &\longrightarrow \Sigma^\omega \times \Sigma^\omega \\ (x, y) &\longmapsto (y, \text{tail}(x)) \end{aligned}$$

and notice that the following squares commute

$$\begin{array}{ccc} \Sigma^\omega \times \Sigma^\omega & \xrightarrow{\alpha_{\text{head}}} & \Sigma \\ \downarrow \text{zip} & & \downarrow \text{id} \\ \Sigma^\omega & \xrightarrow{\text{head}} & \Sigma \end{array} \quad \begin{array}{ccc} \Sigma^\omega \times \Sigma^\omega & \xrightarrow{\alpha_{\text{tail}}} & \Sigma^\omega \times \Sigma^\omega \\ \downarrow \text{zip} & & \downarrow \text{zip} \\ \Sigma^\omega & \xrightarrow{\alpha_{\text{tail}}} & \Sigma^\omega \end{array} .$$

Thus we have that zip is corecursively specified.

**Theorem 5.3.4 (Primitive Corecursion).** Assume the category  $\mathcal{A}$  has finitite coproducts and  $F$  is an endofunctor on  $\mathcal{A}$  with a terminal coalgebra. Then for every  $\alpha : X \rightarrow F(X + \nu F)$  there is a unique  $h : X \rightarrow \nu F$  such that the following square commutes

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & F(X + \nu F) \\ \downarrow h & & \downarrow F[h, \text{id}_{\nu F}] \\ \nu F & \xrightarrow{\tau} & F(\nu F) \end{array}$$

*Proof.* Dual to that of Theorem 2.3.3. □

**Example 5.3.5.**

**Definition 5.3.6.** Let  $(A, \alpha)$  and  $(B, \beta)$  be two  $F$ -coalgebras for a functor  $F$  over a category  $\mathcal{A}$  with binary products. A **bisimulation** between them is a relation

$$\langle r_A, r_B \rangle : R \rightharpoonup A \times B$$

bisimulation

together with a coalgebra structure  $\rho$  on  $R$  that makes  $r_A$  and  $r_B$  into morphisms of coalgebras i.e. the following squares

$$\begin{array}{ccc} R & \xrightarrow{\rho} & FR \\ \downarrow r_A & & \downarrow Fr_A \\ A & \xrightarrow{\alpha} & FA \end{array} \quad \begin{array}{ccc} R & \xrightarrow{\rho} & FR \\ \downarrow r_B & & \downarrow Fr_B \\ B & \xrightarrow{\beta} & FB \end{array}$$

commute. When the two coalgebras are the same, say  $(A, \alpha)$ , we speak of a bisimulation on  $(A, \alpha)$ .

**Remark 5.3.7.** In the above definition by saying that  $\langle r_A, r_B \rangle$  is a relation we simply mean that it is a monomorphism i.e. it represents a subobject of  $A \times B$ .

The intuition behind the definition is that two states in two different coalgebras related by a bisimulation have the same behaviour. An example should clarify the matter.

**Example 5.3.8.** Consider a pair of systems with termination (see Example 5.1.3)

$$\alpha : A \rightarrow A + 1 \text{ and } \beta : B \rightarrow B + 1.$$

Then a bisimulation between them is a relation  $R \subseteq A \times B$ , with coalgebra structure  $\rho$  on  $R$ , such that if  $xRy$  then

1.  $x$  is terminal if and only if  $y$  is terminal;
2. if  $x, y$  are not terminal and  $x', y'$  are their respective next states then  $x'Ry'$ .

Indeed by the definition of bisimulation we have that the following squares in **Set** commute.

$$\begin{array}{ccc} R & \xrightarrow{\rho} & R + 1 \\ \downarrow r_A & & \downarrow r_A + \text{id}_1 \\ A & \xrightarrow{\alpha} & A + 1 \end{array} \quad \begin{array}{ccc} R & \xrightarrow{\rho} & R + 1 \\ \downarrow r_B & & \downarrow r_B + \text{id}_1 \\ B & \xrightarrow{\beta} & B + 1 \end{array} \quad (5.3.1)$$

Since we are working in **Set** and  $R \subseteq A \times B$  we can assume that  $r_A, r_B$  are (restrictions of) the projections on  $A$  and  $B$  so

- a. since  $r_A$  is a coalgebra morphism we have that  $(x, y)$  is terminal in  $R$  if and only if  $x$  is terminal in  $A$  and, similarly using  $r_B$ ,  $(x, y)$  is terminal in  $R$  if and only if  $y$  is terminal in  $B$ ; thus we obtain 1 above;
- b. if  $x, y$  are not terminal then, since  $r_A, r_B$  are morphisms, we obtain

$$(r_A \circ \rho)(x, y) = (\alpha \circ r_A)(x, y) = \alpha(x) = x',$$

$$(r_B \circ \rho)(x, y) = (\beta \circ r_B)(x, y) = \beta(y) = y'.$$

Thus  $\rho(x, y) = (x', y')$  and we have 2.

On the other hand if 1 and 2 hold we can make  $R$  into a coalgebra  $\rho : R \rightarrow R + 1$  such that  $\rho(x, y) = (x', y')$  if  $x$  is not terminal and  $\rho(x, y) = *$  in the right summand if  $x$  is terminal. The projections  $r_A, r_B$  then clearly make the squares 5.3.1 commute since they preserve next-states and preserve and reflect terminal ones.

In virtue of conditions 1 and 2 we obtain immediately that the unique morphisms  $b_A : A \rightarrow \mathbb{N}^\top, b_B : B \rightarrow \mathbb{N}^\top$  into the terminal coalgebra are such that, for every bisimulation  $R, xRy$  implies  $b_A(x) = b_B(y)$  which formalizes the idea of bisimulations “linking” states with the same behaviour.

**Definition 5.3.9.** The **diagonal subobject**  $\Delta_A$  of  $A \times A$  is the one represented by  $\langle \text{id}_A, \text{id}_A \rangle : A \rightarrowtail A \times A$ . diagonal subobject

**Remark 5.3.10.** The diagonal subobject  $\Delta_A$  is always a bisimulation on any coalgebra  $(A, \alpha)$ . A bisimulation not contained in  $\Delta_A$  is a **proper bisimulation** while one contained in  $\Delta_A$  is **improper**. Moreover notice that a relation  $\langle r_1, r_2 \rangle : R \rightarrowtail A \times A$  is improper if  $r_1 = r_2$ . proper/improper bisimulation

Finally we can state and prove the coinduction principle.

**Theorem 5.3.11 (Coinduction Principle).** Let  $F$  be an endofunctor with a terminal coalgebra  $\nu F$ . There are no proper bisimulations on  $\nu F$ .

*Proof.* Let  $r = \langle r_1, r_2 \rangle : R \rightarrowtail \nu F \times \nu F$  be a bisimulation i.e.  $R$  has a coalgebra structure such that  $r_1$  and  $r_2$  are coalgebra morphisms. By terminality of  $\nu F$  we must have  $r_1 = r_2$  and thus the bisimulation is improper.  $\square$

The coinduction principle ties into our interpretation of bisimulations linking states that have the same behaviour by saying that no two states in the terminal coalgebra have the same behaviour. Terminal coalgebras we can thus think of as containing all possible behaviours “exactly once”.

## **Chapter 6**

# **Iteration for Coalgebras**

## 6.1 Terminal Coalgebras from Finitary Iteration

The goal of this section is to dualize to coalgebras what has been done in Chapter 3. Towards this goal we assume to be working in a category  $\mathcal{A}$  with a terminal object 1.

**Definition 6.1.1.** The **terminal-coalgebra**  $\omega^{\text{op}}$ -chain is the diagram

terminal-algebra  
 $\omega^{\text{op}}$ -chain

$$1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \xleftarrow{F^2!} F^3 1 \xleftarrow{F^3!} \cdots \quad (6.1.1)$$

where  $!$  is the unique arrow into the terminal object.

**Remark 6.1.2.** Every coalgebra  $\alpha : A \rightarrow FA$  induces a canonical cone  $(\alpha_n : A \rightarrow F^n 1)_{n < \omega}$  over (6.1.1) defined as following

$$\begin{aligned} \alpha_0 &= ! \\ \alpha_{n+1} &= A \xrightarrow{\alpha} FA \xrightarrow{F\alpha_n} F^{n+1} 1. \end{aligned}$$

Proof that this is really a cone is dual to Remark 3.1.3.

Adámek's Theorem 3.1.5 dualizes straightforwardly as well.

**Theorem 6.1.3.** Let  $\mathcal{A}$  be a category with a terminal object and limits of  $\omega^{\text{op}}$ -chains and  $F$  an endofunctor preserving  $\omega^{\text{op}}$ -limits. Then  $F$  has terminal coalgebra  $\nu F = \lim_{n < \omega} F^n 1$ . The structure of  $\nu F$  is the unique  $\tau : \nu F \rightarrow F(\nu F)$  such that

$$Fl_n \circ \tau = l_{n+1}, \quad n < \omega \quad (6.1.2)$$

where  $(l_n : \nu F \rightarrow F^n 1)_{n < \omega}$  is the limit cone of (6.1.1).

*Proof.* Dual to that of Theorem 3.1.5. □

Just as for algebras this result can be used directly to prove the existence of a terminal coalgebra but it also tells us that such coalgebra is a limit. We can often use this to our advantage to better understand the structure of the terminal coalgebra.

**Remark 6.1.4.** Given a signature  $\Sigma$  the corresponding polynomial functor  $H_\Sigma$  over Set preserves limits of  $\omega^{\text{op}}$ -chains. Recall that  $H_\Sigma$  is defined by

$$\begin{aligned} H_\Sigma X &= \coprod_{n < \omega} \Sigma_n \times X^n \\ H_\Sigma f &= \coprod_{n < \omega} \text{id}_{\Sigma_n} \times f^n \quad \text{with } f^n = \underbrace{f \times \cdots \times f}_{n \text{ times}}. \end{aligned}$$

Consider an  $\omega^{\text{op}}$ -chain  $(A_i, \alpha_i)_{i < \omega}$  with limiting cone

$$\begin{array}{c} A \\ \swarrow \alpha_0 \quad \downarrow \alpha_1 \quad \searrow \alpha_2 \\ A_0 \xleftarrow{a_0} A_1 \xleftarrow{a_1} A_2 \xleftarrow{a_2} \cdots \end{array}$$

we will prove that

$$\begin{array}{c}
 H_\Sigma A \\
 \swarrow H_\Sigma \alpha_0 \quad \downarrow H_\Sigma \alpha_1 \quad \searrow H_\Sigma \alpha_2 \\
 H_\Sigma A_0 \xleftarrow{H_\Sigma a_0} H_\Sigma A_1 \xleftarrow{H_\Sigma a_1} H_\Sigma A_2 \xleftarrow{H_\Sigma a_2} \cdots
 \end{array}$$

is the limit cone of  $(H_\Sigma A_i, H_\Sigma a_i)_{i < \omega}$ . In order to do so we need to show that for any cone

$$\begin{array}{c}
 B \\
 \swarrow \beta_0 \quad \downarrow \beta_1 \quad \searrow \beta_2 \\
 H_\Sigma A_0 \xleftarrow{H_\Sigma a_0} H_\Sigma A_1 \xleftarrow{H_\Sigma a_1} H_\Sigma A_2 \xleftarrow{H_\Sigma a_2} \cdots
 \end{array}$$

there is a unique  $f : B \rightarrow H_\Sigma A$  such that  $\beta_i = H_\Sigma \alpha_i \circ f$  for every  $i < \omega$ .

Fix  $x \in B$  and notice that  $H_\Sigma A_0 = \coprod_{n < \omega} \Sigma_n \times A_0^n$  so  $\beta_0(x) \in \Sigma_m \times A_0^m$  for a unique  $m < \omega$ . Moreover since  $(B, \beta_i)_{i < \omega}$  is a cone and  $H_\Sigma a_i = \coprod_{n < \omega} \text{id}_{\Sigma_n} \times a_i^n$  we must have  $\beta_i(x) \in \Sigma_m \times A_i^m$  for every  $i < \omega$  and the same  $m$ . This means that we can write  $B$  as  $\coprod_{n < \omega} B_n$  with

$$B_n = \{x \in B : \beta_0(x) \in \Sigma_n \times A_0^n\}.$$

Fixing  $n < \omega$  we obtain that

$$\begin{array}{c}
 B_n \\
 \swarrow \beta_0 \quad \downarrow \beta_1 \quad \searrow \beta_2 \\
 \Sigma_n \times A_0^n \xleftarrow{\text{id}_{\Sigma_n} \times a_0^n} \Sigma_n \times A_1^n \xleftarrow{\text{id}_{\Sigma_n} \times a_1^n} \Sigma_n \times A_2^n \xleftarrow{\text{id}_{\Sigma_n} \times a_2^n} \cdots
 \end{array}$$

is a cone over  $(\Sigma_n \times A_i^n, \text{id}_{\Sigma_n} \times a_i^n)_{i < \omega}$  that has limit  $(\Sigma_n \times A^n, \text{id}_{\Sigma_n} \times \alpha_i^n)_{i < \omega}$  because the functor  $GX = \Sigma_n \times X^n$  preserves limits of  $\omega^{\text{op}}$ -chains. Thus there is a unique  $f_n : B_n \rightarrow \Sigma_n \times A^n$  such that

$$\beta_i = (\text{id}_{\Sigma_n} \times \alpha_i^n) \circ f_n. \quad (6.1.3)$$

Finally we can define  $f : B \rightarrow H_\Sigma A$  by setting

$$f(x) = f_n(x) \quad \text{when } x \in B_n.$$

This function is such that  $\beta_i = H_\Sigma \alpha_i \circ f$  for every  $i < \omega$  since  $B = \coprod_{n < \omega} B_n$  and we have (6.1.3) for every  $n < \omega$ . Moreover if  $g : B \rightarrow H_\Sigma A$  is such that  $\beta_i = H_\Sigma \alpha_i \circ g$  for every  $i < \omega$  then its restriction  $g|_{B_n}$  satisfies  $\beta_i = (\text{id}_{\Sigma_n} \times \alpha_i^n) \circ g|_{B_n}$  for every  $i < \omega$  but then  $g|_{B_n} = f_n$  by uniqueness of  $f_n$ . Thus  $g$  and  $f$  agree on every  $B_n$  and so we must have  $f = g$ .

**Example 6.1.5.** Consider  $FX = X + 1$  over  $\text{Set}$ . The terminal-coalgebra  $\omega^{\text{op}}$ -chain for  $F$  is

$$1 \xleftarrow{!} 1 + 1 \xleftarrow{F!} 1 + 1 + 1 \xleftarrow{F^2!} 1 + 1 + 1 + 1 \xleftarrow{F^3!} \cdots$$

so the  $n$ -th object of the chain is the set  $n + 1 = \{0, 1, \dots, n\}$  of  $n + 1$  elements and the connecting functions  $F^n! : n + 2 \rightarrow n + 1$  are such that

$$(F^n!)(i) = \begin{cases} i & \text{if } i \leq n \\ n & \text{if } i = n + 1 \end{cases}.$$

Then the limit is the set containing all the elements  $x \in \prod_{n < \omega} (n+1)$  such that  $(F^n!)(x_{n+1}) = x_n$  by how limits are computed in Set (see [Bor08, Chapter 2, Section 8]). Such elements are either the sequence  $(0, 1, 2, 3, \dots)$  or they have the form  $(0, 1, \dots, k, k, \dots)$ . Sequences of the second kind we identify with the number  $k$  and  $(0, 1, 2, 3, \dots)$  we call  $\infty$ ; this way we obtain  $\mathbb{N}^\top$ . By using (6.1.2) we obtain that the coalgebra structure  $\tau : \mathbb{N}^\top \rightarrow \mathbb{N}^\top + 1$  sends  $\infty$  to itself, 0 to the unique point in the right hand side and  $k$  to  $k - 1$ ; thus we have a new explanation of Exampe 5.2.2.

**Example 6.1.6.** Consider the functor  $FX = \Sigma \times X$  over Set, where  $\Sigma$  is a fixed set. We can identify  $F\Sigma = \Sigma \times 1$  with  $\Sigma$  and thus the terminal-coalgebra  $\omega^{\text{op}}$ -chain is

$$1 \xleftarrow{!} \Sigma \xleftarrow{F!} \Sigma \times \Sigma \xleftarrow{F^2!} \Sigma \times \Sigma \times \Sigma \xleftarrow{F^3!} \dots$$

where the connecting functions are the projections onto the left-most factor. The limit is the object  $\Sigma^\omega$  of  $\Sigma$ -streams with limit projections  $l_n : \Sigma^\omega \rightarrow \Sigma^n$  such that  $l_n$  is the projection on the first  $n$ -th entries of a  $\Sigma$ -stream. The coalgebra structure  $\tau : \Sigma^\omega \rightarrow \Sigma \times \Sigma^\omega$  is given by  $\tau = \langle \text{head}, \text{tail} \rangle$  with

- $\text{head}(x_0, x_1, x_2, x_3, \dots) = x_0$ ;
- $\text{tail}(x_0, x_1, x_2, x_3, \dots) = (x_1, x_2, x_3, \dots)$ .

This is a new explanation of Example 5.2.3.

We can use Theorem 6.1.3 to obtain an alternative proof of Proposition 5.2.7. To achieve this goal we begin by giving an explicit description of the terminal-coalgebra  $\omega^{\text{op}}$ -chain of a polynomial functor.

**Remark 6.1.7.** Let  $\Sigma$  be a signature,  $H_\Sigma$  the corresponding polynomial functor (over Set) and  $\Sigma_\perp$  a new signature obtained by adding to  $\Sigma$  a new constant symbol  $\perp$ . We shall see that the  $n$ -th object of the termial-coalgebra  $\omega^{\text{op}}$ -chain is

$$H_\Sigma^n(1) = \{T : T \text{ is a } \Sigma_\perp\text{-tree of height at most } n \text{ and all leaves at height } n \text{ labeled by } \perp\}.$$

In order to prove it we begin by identifying  $1 = \{*\} = H_\Sigma^0(1)$  with the  $\Sigma_\perp$  tree formed by a single node (the root) labeled by  $\perp$ . Now  $H_\Sigma^1(1) = \coprod_k \Sigma_k \times 1^k$  can be identified with the set of all  $\Sigma_\perp$  trees of height at most 1 that have the root labeled by some  $k$ -ary symbol of  $\Sigma$  and the  $k$  leaves (that are at height 1) labeled by  $\perp$ . Similarly  $H_\Sigma^2(1) = \coprod_k \Sigma_k \times (H_\Sigma^1(1))^k$  can be identified with the set of  $\Sigma_\perp$ -trees that have root labeled by a  $k$ -ary symbol of  $\Sigma$  and whose  $k$  subtrees are picked from  $H_\Sigma^1(1)$ ; thus elements of  $H_\Sigma^2(1)$  have height at most 2 and leaves at height 2 (if any) all labeled by  $\perp$ . By iterating we obtain that  $H_\Sigma^n(1)$  is precisely the set of all  $\Sigma_\perp$ -trees of height at most  $n$  with leaves at height  $n$  labeled by  $\perp$ . The connecting functions  $H_\Sigma^{n+1}(1) \rightarrow H_\Sigma^n(1)$  cut out the layer of  $\perp$ -labeled leaves and re-label every leaf at height  $n$  with  $\perp$ .

*Alternative proof of Proposition 5.2.7.* Since  $H_\Sigma$  preserves limits of  $\omega^{\text{op}}$ -chains we have that  $\nu H_\Sigma = \lim_{n < \omega} H_\Sigma^n(1)$ . Let  $v_{n,m} : H_\Sigma^n(1) \rightarrow H_\Sigma^m(1)$  for  $m \leq n < \omega$  be the connecting functions of the terminal coalgebra  $\omega^{\text{op}}$ -chain.

We shall first prove that  $\lim_{n < \omega} H_\Sigma^n(1)$  is the set  $T_\Sigma$  of all  $\Sigma$ -trees with projections  $l_n : T_\Sigma \rightarrow H_\Sigma^n(1)$  that cut a tree at height  $n$  and re-label every leaf at level  $n$  with  $\perp$ . In order to do so we just need to show that for every sequence  $(t_n : t_n \in H_\Sigma^n(1))_{n < \omega}$  such that

$$v_{n,m}(t_n) = t_m, \quad m \leq n < \omega \quad (6.1.4)$$

there is a unique  $t \in T_\Sigma$  such that  $l_n(t) = t_n$  for every  $n < \omega$ . Such  $t$  is constructed inductively as follows:

- the root of  $t$  is the root of  $t_1$  which, by (6.1.4), is also the root of  $t_m$  for all  $1 \leq m$ ;
- the  $n$ -th layer of  $t$  is labeled as the  $n$ -th layer of  $t_{n+1}$  which, by (6.1.4), is the same as the  $n$ -th layer of every  $t_m$  for  $n \leq m$ ;

and it is unique since the condition  $l_n(t) = t_n$  (for all  $n < \omega$ ) forces  $t$  and  $t_n$  to agree on all layers of height less than  $n$ .

The coalgebra structure  $\tau : T_\Sigma \rightarrow H_\Sigma T_\Sigma$  is, by (6.1.2), the unique function such that

$$l_{m+1} = H_\Sigma l_m \circ \tau, \quad m < \omega.$$

Tree-tupling  $\bar{\tau} : H_\Sigma T_\Sigma \rightarrow H_\Sigma$  satisfies  $l_{m+1} \circ \bar{\tau} = H_\Sigma l_m$  because cutting a tree-tupling at the  $(m+1)$ -th layer and re-labeling the leaves (this is  $l_{m+1} \circ \bar{\tau}$ ) is the same as cutting and re-labeling every maximal subtree at their  $m$ -th layer (this is  $H_\Sigma l_m$ ). This gives us that  $\tau$  is the inverse of  $\bar{\tau}$ .  $\square$

## 6.2 Terminal Coalgebras from Infinitary Iteration

We shall now look at dualization of results from Chapter 4. As mentioned in Remark 5.2.12, even for  $\text{Set}$ , the existence of a terminal coalgebra is a more complicated matter than the existence of an initial algebra so we should not expect coalgebraic versions of Theorem 4.3.2 or of Theorem 4.3.16. We will, however, present some conditions that guarantee a terminal coalgebra for functors over  $\text{Set}$ ; these conditions require non-trivial set-theoretic machinery or even the Generalized Continuum Hypothesis (GCH).

We work with functors from  $\text{Ord}^{\text{op}}$  into a given category  $\mathcal{A}$ . Such diagrams should in theory be called *cochains*, being the dual of chains as defined in Section 4.1; however it is customary to simply call them **chains** as the context is generally enough to differentiate between the two kinds. With this in mind a category  $\mathcal{A}$  is said to have limits of chains if for all  $j \in \text{Ord}$  all diagrams  $j^{\text{op}} \rightarrow \mathcal{A}$  have a limit. This includes the limit of the empty chain, so  $\mathcal{A}$  must have a terminal object 1.

**Definition 6.2.1.** Let  $\mathcal{A}$  be a category with limits of chains and  $F$  an endofunctor over  $\mathcal{A}$ . The **terminal-coalgebra chain** is the  $\text{Ord}^{\text{op}}$ -chain in  $\mathcal{A}$  with objects  $V_j$  and arrows  $v_{j,i}$

terminal-coalgebra  
chain



(with  $j \geq i$ ) defined by transfinite recursion as follows.

$$\begin{aligned} V_0 &= 1 && \text{base case;} \\ V_{j+1} &= FV_j && \text{for all ordinals } j; \\ V_j &= \lim_{i < j} V_i && \text{for all limit ordinals } j; \end{aligned}$$

As for arrows we proceed by dualizing what we have done for the initial-algebra chain:  $v_{1,0}$  is the unique arrow  $V_1 \rightarrow 1$  and  $v_{i+1,j+1} = Fv_{i,j}$  for every  $i \geq j$  in Ord. When  $j$  is a limit ordinal we let  $\{v_{j,i}\}_{i < j}$  be the limit cone of  $W_j$  and, finally,  $v_{j+1,j}$  is obtained as the unique factorization arrow such that

$$v_{j,i+1} \circ v_{j+1,j} = Fv_{j,i} \quad \text{for all } i < j.$$

**Remark 6.2.2.** Dual to Remark 4.2.2 every  $F$ -coalgebra  $(A, \alpha)$  induces a canonical cone  $(\alpha_i : A \rightarrow V_i)_{i \in \text{Ord}}$  over the terminal-coalgebra chain given by

$$\begin{aligned} \alpha_0 &= ! && \text{base case;} \\ \alpha_{i+1} &= F\alpha_i \circ \alpha && \text{for all ordinals } i; \end{aligned}$$

and, for limit ordinals  $j$ ,  $\alpha_j : A \rightarrow V_j$  is the unique factorization arrow such that  $v_{j,i} \circ \alpha_j = \alpha_i$  for all  $i < j$  obtained using the universal property of  $V_j$ . Proof that this is well-defined is dual to what we did in Remark 4.2.2.

**Definition 6.2.3.** We say that the terminal-coalgebra chain **converges in  $\lambda$  steps** if  $v_{\lambda+1,\lambda}$  is an isomorphism. It converges in **exactly  $\lambda$  steps** if  $\lambda$  is the least such ordinal.

convergence of the  
terminal-coalgebra  
chain

**Theorem 6.2.4.** Let  $\mathcal{A}$  be a category with limits of chains and  $F$  an endofunctor. If the terminal-coalgebra chain converges in  $\lambda$  steps then  $V_\lambda$  is the terminal coalgebra and its structure is given by

$$(v_{\lambda+1,\lambda})^{-1} : V_\lambda \rightarrow V_{\lambda+1} = FV_\lambda.$$

*Proof.* Dual to that of Theorem 4.2.4 □

**Corollary 6.2.5.** Let  $\mathcal{A}$  be a category with limits of chains and  $F$  an endofunctor. If  $F$  preserves limits of  $\lambda$ -chains for some limit ordinal  $\lambda$  then the terminal-coalgebra chain of  $F$  converges in  $\lambda$  steps. Therefore  $\nu F = V_\lambda$ .

*Proof.* Dual to that of Corollary 4.2.5. □

**Remark 6.2.6.** For  $\lambda = \omega$  Corollary 6.2.5 becomes Theorem 6.1.3 and we have examples of terminal-coalgebra chains that converge in  $\omega$  steps in those generated by polynomial functors because of Remark 6.1.4.

We now have a condition, preservation of limits  $\lambda^{\text{op}}$ -chain for some limit ordinal  $\lambda$ , which implies convergence of the terminal-coalgebra chain and thus existence of a terminal coalgebra. However this condition isn't satisfied by many common functors (e.g. the finite power-set functor  $\mathcal{P}_f$  does not preserve limits of  $\omega^{\text{op}}$ -chains but, as we shall see in Example 6.2.16, it has a terminal coalgebra). However for functors over Set there is another condition which guarantees convergence of the terminal-coalgebra chain; in order to study it we need to recall some set-theoretic machinery.

**Definition 6.2.7.** The **cofinality** of a cardinal  $\lambda$  is the cardinality of the smallest cofinal subset of  $\lambda$ . If  $\lambda$  is infinite and its cofinality is  $\lambda$  itself then we say that  $\lambda$  is a **regular cardinal**.

cofinality

regular cardinal

**Notation 6.2.8.** Until the end of the section we will, for the sake of brevity, deliberately confuse a subset  $Y \subseteq X$  with its inclusion function  $y : Y \rightarrow X$ . We shall also write  $z \cap y$  for the inclusion  $z \cap y : Z \cap Y \rightarrow X$  of the intersection of two subsets  $z : Z \rightarrow X, y : Y \rightarrow X$ ; and similarly for the union of subsets.

**Definition 6.2.9.** Given a regular cardinal  $\lambda$  we say that a functor  $F$  over  $\mathbf{Set}$  is  **$\lambda$ -accessible**<sup>1</sup> if for every set  $X$  and  $x \in FX$  there is a subset  $m : M \rightarrow X, |M| < \lambda$  such that  $x \in \text{im}(Fm)$ .

We can think of a  $\lambda$ -accessible functor as of a functor whose behaviour is determined by its action on sets of cardinality less than  $\lambda$ .

**Example 6.2.10.** Functors that are  $\aleph_0$ -accessible are also called **finitary set functors**. An example of a finitary set functor is  $\mathcal{P}_f$ , the finite power-set functor. Indeed let  $X$  be a set and  $x \in \mathcal{P}_f(X)$ ; this means that  $x$  is a finite subset of  $X$  so let us consider  $m : x \rightarrow X$ . It is clear that  $x \in \text{im}(\mathcal{P}_f m)$  so we have that  $\mathcal{P}_f$  is finitary.

**Example 6.2.11.** The power-set functor  $\mathcal{P}$  is not accessible for any regular cardinal. This is, in hindsight, because of Theorem 6.2.14 and Remark 5.2.9 but can be proved directly. Indeed let  $\lambda$  be a regular cardinal and  $X$  a set of cardinality  $\lambda$ . Then if  $m : M \rightarrow X$  is a subset of  $X$  such that  $X \in \text{im}(\mathcal{P}m)$  then we must have  $M = X$  so  $|M| = \lambda$  and thus  $\mathcal{P}$  is not  $\lambda$ -accessible.

On the other hand, for a regular cardinal  $\lambda$ , the bounded power-set functor

$$\mathcal{P}_\lambda(X) = \{Y \subseteq X : |Y| < \lambda\}$$

with the natural assignment on arrows is  $\lambda$ -accessible. Proving this is similar to what we have done for the finite power-set functor in Example 6.2.10.

**Example 6.2.12.** The functor  $FX = X^n$  is  $\lambda$ -accessible for every cardinal  $n < \lambda$ . Moreover one can prove that the coproduct of accessible functors is itself accessible so a polynomial functor  $H_\Sigma$  is  $\lambda$ -accessible provided that all arities of the symbols of the signature  $\Sigma$  are less than  $\lambda$ .

In order to prove that being  $\lambda$ -accessible implies existence of a terminal coalgebra we need the following lemma on set functors.

**Lemma 6.2.13.** Every functor  $F$  over  $\mathbf{Set}$  preserves non-empty finite intersections.

*Proof.* Refer to [Jiř21, Theorem B.2.2]. □

<sup>1</sup>Questa teoricamente è la definizione di funtore “ $\lambda$ -bound” mentre “ $\lambda$ -accessible” significa che il funtore preserva colimiti  $\lambda$ -filtrati (ed è una condizione che si può definire per categorie oltre a  $\mathbf{Set}$ ). Tuttavia nel caso di funtori su  $\mathbf{Set}$  le due condizioni sono equivalenti e dunque usiamo “ $\lambda$ -accessible” perché è il nome più comune nella letteratura recente. È meglio precisare questa cosa o semplicemente usare “ $\lambda$ -accessible” come se nulla fosse (considerando che comunque non usiamo mai che un  $\lambda$ -accessible preserva colimiti  $\lambda$ -filtrati direttamente, ma usiamo solo la condizione equivalente in Definizione 6.2.9)?

**Theorem 6.2.14.** Given a regular cardinal  $\lambda$  the terminal-coalgebra chain of a  $\lambda$ -accessible functor  $F$  over  $\mathbf{Set}$  converges in  $\lambda + \lambda$  steps.

*Proof.* First notice that if  $F1 = \emptyset$  then  $F$  must be the constant functor with value  $\emptyset$  and the theorem holds. Thus we may assume  $F1 \neq \emptyset$  so there is an arrow of type  $1 \rightarrow F1$ ; but this is a coalgebra and it induces a cone  $(\alpha_i : 1 \rightarrow V_i)_{i \in \mathbf{Ord}}$  by Remark 6.2.2 which guarantees that every element of the terminal-coalgebra chain is non-empty.

The proof proceeds in two steps: in (1) we prove that every  $v_{j,i} : V_j \rightarrow V_i$  for  $j \geq i \geq \lambda$  is a monomorphism and in (2) that  $v_{\lambda+\lambda+1, \lambda+\lambda}$  is an isomorphism, thus the terminal-coalgebra chain converges in  $\lambda + \lambda$  steps.

- (1) It is sufficient to prove that  $v_{\lambda+1, \lambda}$  is mono since, in  $\mathbf{Set}$ , every mono with non-empty domain splits so  $Fv_{\lambda+1, \lambda} = v_{\lambda+2, \lambda+1}$  is mono as well and we can conclude that every  $v_{i, \lambda}$  with  $i \geq \lambda$  is a monomorphism through a transfinite induction. This gives that all  $v_{j,i}$  with  $j \geq i \geq \lambda$  are mono.

To prove that  $v_{\lambda+1, \lambda}$  is a monomorphism we shall see that two distinct elements  $x, y \in V_{\lambda+1}$  remain distinct under application of  $v_{\lambda+1, \lambda}$ . Since  $F$  is  $\lambda$ -accessible there are subsets

$$u_x : U_x \rightarrow V_\lambda, \quad u_y : U_y \rightarrow V_\lambda$$

of  $V_\lambda$  such that  $|U_x|, |U_y| < \lambda$  and  $x \in \text{im}(Fu_x), y \in \text{im}(Fu_y)$ . If  $u : U \rightarrow V_\lambda$  is the union of  $U_x, U_y$  then we have  $|U| < \lambda$  and  $x, y \in \text{im}(Fu)$ . Moreover  $(v_{\lambda, i})_{i < \lambda}$  is a limit cone so, by the universal property, every pair of distinct elements of the vertex  $a, b \in V_\lambda$  (and particularly if  $a, b \in U$ ) remain distinct under  $v_{\lambda, i}$  for at least one  $i < \lambda$ ; this is true for every kind of limit cone in  $\mathbf{Set}$ , but for ones over  $\omega^{\text{op}}$  it is also the case that  $a, b$  remain distinct under every  $v_{\lambda, j}$  for  $i \leq j < \lambda$ . Now let us chose for every pair  $(a, b)$  of distinct elements of  $U$  an ordinal  $i_{(a,b)} < \lambda$  such that  $v_{\lambda, i_{(a,b)}}$  keeps  $a$  and  $b$  distinct; the set of such ordinals has cardinality at most  $|U \times U| < \lambda$  and thus can't be cofinal because  $\lambda$  is a regular cardinal. This means that there is  $i_0 < \lambda$  such that every pair of distinct elements of  $U$  remains distinct under  $v_{\lambda, i_0}$  i.e.  $v_{\lambda, i_0} \circ u$  is a monomorphism.

Now, since  $U \neq \emptyset$ ,  $v_{\lambda, i_0} \circ u$  splits and we have that  $Fv_{\lambda, i_0} \circ Fu$  is a monomorphism as well thus, as  $x, y \in \text{im}(Fu)$ ,  $x$  and  $y$  are kept distinct by  $Fv_{\lambda, i_0}$ . Finally

$$Fv_{\lambda, i_0} = v_{\lambda+1, i_0+1} = v_{\lambda, i_0+1} \circ v_{\lambda+1, \lambda}$$

gives that  $v_{\lambda+1, \lambda}$  must keep  $x$  and  $y$  distinct too.

- (2) To prove that  $v_{\lambda+\lambda+1, \lambda}$  is an isomorphism it is enough to prove that  $F$  preserves the limit  $V_{\lambda+\lambda}$  of the chain  $(V_i)_{i < \lambda+\lambda}$ . This because the chains  $(V_i)_{i < \lambda+\lambda}$  and  $(FV_i)_{i < \lambda+\lambda}$  have the same limit by the dual of Lemma 4.1.1. By the same lemma we can disregard the first  $\lambda$  members of this chain and write

$$V_{\lambda+\lambda} = \lim_{i < \lambda} V_{\lambda+i}$$

and so by (1) the connecting morphisms as well as the ones forming the limiting cone are all mono. Let us write  $v_i$  for  $v_{\lambda+i, \lambda}$  and particularly  $v_\lambda$  for  $v_{\lambda+\lambda, \lambda}$ . Since we are

working in **Set** the limit of such a chain is the intersection, thus  $V_{\lambda+\lambda} = \bigcap_{i < \lambda} V_{\lambda+i}$ . We shall now prove that  $F$  preserves this intersection i.e. that given an  $x \in FV_\lambda$  such that  $x$  lies in the image of  $Fv_i$  for all  $i < \lambda$  then  $x$  must lie in the image of  $Fv_\lambda$  as well.

Fix an  $x \in FV_\lambda$  such that  $x \in \text{im}(Fv_i)$  for every  $i < \lambda$ . Since  $F$  is  $\lambda$ -accessible there is a subset  $u : U \rightarrow V_\lambda$  of  $V_\lambda$  such that  $|U| < \lambda$  and  $x \in \text{im}(Fu)$ . We can assume that  $u \cap v_\lambda$  is non-empty since we can take any element of  $V_{\lambda+\lambda}$  and add it to  $U$  without making its cardinality  $\lambda$  or more. Now for every  $i < \lambda$  we have a mono  $v_{\lambda+\lambda, \lambda+i}$  and

$$\begin{array}{ccc} & V_{\lambda+\lambda} & \\ v_\lambda \swarrow & & \searrow v_{\lambda+\lambda, \lambda+i} \\ V_\lambda & \xleftarrow{v_i} & V_{\lambda+i} \end{array}$$

commutes so  $u \cap v_i$  is non-empty as well; we let  $u_i$  be  $u \cap v_i$ . Now from  $u_i = u \cap v_i$  being non-empty,  $x \in \text{im}(Fu)$ ,  $x \in \text{im}(Fv_i)$  (which we have from choice of  $U$  and hypothesis on  $x$ ) and Lemma 6.2.13 we obtain that  $x \in \text{im}(Fu_i)$  for all  $i < \lambda$ . Finally notice that  $(u_i)_{i < \lambda}$  is a decreasing  $\lambda$ -chain of subsets of  $U$  thus, since  $|U| < \lambda$ , it must converge at some  $i_0 < \lambda$ . For that ordinal  $i_0$  we have  $u_{i_0} = u_\lambda = u \cap v_\lambda \subseteq v_\lambda$  so  $\text{im}(Fu_{i_0}) \subseteq \text{im}(Fv_\lambda)$  and this gives that  $x \in \text{im}(Fv_\lambda)$ .

This concludes the proof.  $\square$

**Remark 6.2.15.** Worrell proved in [Wor05] that there is a functor  $F$  over **Set** whose terminal coalgebra chain converges in  $\omega + \omega$  steps but is not finitary. Thus the condition of Theorem 6.2.14 is not necessary.

**Example 6.2.16.** By Example 6.2.10 and Example 6.2.11 the functors  $\mathcal{P}_\lambda$  all have a terminal coalgebra as well as polynomial functors by Example 6.2.12 (but we already knew that).

As we hinted in Remark 5.2.12 there is a functor with an initial algebra which does not have a terminal coalgebra. With the knowledge that  $\mathcal{P}_f$  has a terminal coalgebra and the following proposition we can construct directly such functor.

**Proposition 6.2.17.** Let  $\mathcal{A}$  be a complete and well-powered category and  $G$  an endofunctor over  $\mathcal{A}$  preserving monomorphisms and having a terminal coalgebra  $\nu G$ . Then every subfunctor  $F$  of  $G$  has a terminal coalgebra  $\nu F$  which is always a subobject of  $\nu G$ .

*Proof.* See [Jir21, Corollary 7.2.10].  $\square$

**Example 6.2.18.** Consider the functor  $\mathcal{P}_{-\aleph_0}(X)$  over **Set** defined by

$$\mathcal{P}_{-\aleph_0}(X) = \{Y \subseteq X : Y \text{ is finite or uncountable}\}$$

on objects and such that if  $f : X \rightarrow Y$  is a function then  $\mathcal{P}_{-\aleph_0}f : \mathcal{P}_{-\aleph_0}(X) \rightarrow \mathcal{P}_{-\aleph_0}(Y)$  is defined by

$$(\mathcal{P}_{-\aleph_0}f)(A) = \begin{cases} f(A) & \text{if } A \text{ is finite or } f|_A \text{ is injective} \\ \emptyset & \text{otherwise} \end{cases}$$

for  $A \in \mathcal{P}_{-\aleph_0}(X)$ .

This functor has  $\mathbb{N}$  as fixed point since  $\mathcal{P}_{-\aleph_0}(\mathbb{N})$  is a countable set. Then by Theorem 4.3.2 has an initial algebra; we shall now see that it does not have a terminal coalgebra.

Note that  $\mathcal{P}_f$  is a subfunctor of  $\mathcal{P}_{-\aleph_0}$  so Proposition 6.2.17 applies: if  $\nu\mathcal{P}_{-\aleph_0}$  exists then it is at least as large as  $\nu\mathcal{P}_f$  which is uncountable. By Lambek's Lemma 5.2.8 we know that  $\nu\mathcal{P}_{-\aleph_0}$  must be a fixed point of  $\mathcal{P}_{-\aleph_0}$  but  $\mathcal{P}_{-\aleph_0}$  has no uncountable fixed points.

Indeed let  $A$  be an uncountable set,  $B$  the set of all countably infinite subsets of  $A$  and  $C = \mathcal{P}_{-\aleph_0}(A)$ ; then

$$|B \cup C| = |\mathcal{P}(A)| > |A|$$

by Cantor's Theorem. The map  $X \mapsto A \setminus X$  is an injection of type  $\mathcal{P}(A) \rightarrow \mathcal{P}(A)$  and, since  $A$  is uncountable, it restricts to an injection  $B \rightarrow C$  which gives  $|B| \leq |C|$ . Now we have

$$|C| = |B| + |C| = |\mathcal{P}(A)|$$

that implies  $|A| < |\mathcal{P}(A)| = |C| = |\mathcal{P}_{-\aleph_0}(A)|$  so  $\mathcal{P}_{-\aleph_0}$  has no uncountable fixed points.

The previous example shows, as already observed, that there is no coalgebraic version of Theorem 4.3.2 for Set. There is, however, a condition concerning the fixed points of a set functor which implies existence of a terminal coalgebra. Unfortunately it requires us to make assumptions independent from ZFC.

**Definition 6.2.19.** A set functor  $F$  has a **fixed point pair** if there is a regular cardinal  $\lambda$  such that  $F$  has a fixed point of cardinality  $\lambda$  and one of cardinality  $\lambda^+$  (the smallest cardinal bigger than  $\lambda$ ).

fixed point pair

**Remark 6.2.20.** Recall that the **Generalized Continuum Hypothesis (GCH)** is the statement that, for every infinite cardinal  $\lambda$ ,  $\lambda^+ = 2^\lambda$ ; where with  $2^\lambda$  we indicate the power-set of  $\lambda$ . The case when  $\lambda = \aleph_0$  is referred to simply as the **Continuum Hypothesis (CH)**. Both are independent from ZFC.

Under GCH if  $\kappa < \lambda$  and  $\lambda$  is a regular cardinal then  $\lambda^\kappa = \lambda$  (see [Jec78, Theorem 5.15 (iii)]). Moreover we recall that we have  $\lambda^\lambda = 2^\lambda$  for every infinite cardinal  $\lambda$  regardless of GCH.

For our goals we need the following technical lemma.

**Lemma 6.2.21.** Let  $Z$  be an infinite pre-fixed point of a functor  $F$  over Set and  $X$  be any set such that  $|X| \leq |Z|$  and  $|X|, |Z|$  have the same cofinality. Then for every  $x \in FX$  there is a subset  $y : Y \rightarrow X$  such that  $|Y| < |X|$  and  $x \in \text{im}(Fy)$ .

*Proof.* Refer to [Jir21, Lemma B.4.8]. □

**Theorem 6.2.22.** Assuming GCH every functor  $F$  with a fixed point pair has a terminal coalgebra.

*Proof.* Let  $\lambda$  be the regular cardinal of Definition 6.2.19. We shall prove that for every element  $x \in V_{\lambda+1}$  there is a subset  $m : M \rightarrow V_\lambda$  such that  $x \in \text{im}(Fm)$  and  $|M| < \lambda$ . This is enough to perform (1) and (2) from the proof of Theorem 6.2.14 (as if  $F$  were  $\lambda$ -accessible) and hence establish the existence of the terminal coalgebra by convergence of the terminal-coalgebra chain. Moreover, again as in proof of Theorem 6.2.14, we can assume  $F1 \neq \emptyset$  thus  $V_i \neq \emptyset$  for every  $i \in \text{Ord}$ . The proof proceeds in three steps.

(1) We show that  $|V_j| \leq \lambda$  for every  $j < \lambda$  with a transfinite induction.

- The base case is trivial.
- For the successor step notice that if  $|V_j| \leq \lambda$  then there is an injection  $m : V_j \hookrightarrow \lambda$  which splits (since  $V_j \neq \emptyset$ ). Thus  $Fm : FV_j \hookrightarrow F\lambda$  also splits (thus is mono) and we have

$$|V_{j+1}| = |FV_j| \leq |F\lambda| = \lambda;$$

where the final equation is guaranteed by  $F$  having a fixed point of cardinality  $\lambda$ .

- For the limit step let  $j < \lambda$  be a limit ordinal and assume  $|V_i| \leq \lambda$  for every  $i < j$ . Let  $\kappa < \lambda$  be the cardinality of  $j$ . Then from  $V_j = \lim_{i < j} V_i$  and how limits are computed in **Set** we obtain that  $V_j$  has cardinality at most that of  $\prod_{i < j} V_i$  which, by induction hypothesis, is at most  $\lambda^\kappa$ . But  $\lambda^\kappa = \lambda$  because we are assuming GCH (Remark 6.2.20).

(2) Now, since  $V_\lambda = \lim_{j < \lambda} V_j$ , we have

$$|V_\lambda| \leq \left| \prod_{j < \lambda} V_j \right| \leq \lambda^\lambda = 2^\lambda = \lambda^+$$

by computation of limits in **Set**, step (1) and GCH. This means that there is a monomorphism  $m : V_\lambda \rightarrow \lambda^+$  which splits, hence  $Fm$  is also a monomorphism and we have  $|FV_\lambda| = |V_{\lambda+1}| \leq |F\lambda^+| = \lambda^+$  because  $F$  has a fixed point of cardinality  $\lambda^+$ .

- (3) Fix  $x \in V_{\lambda+1} = FV_\lambda$ . We apply Lemma 6.2.21 to the fixed point  $Z = \lambda^+$  and  $X = V_\lambda$  to obtain a subset  $m_0 : M_0 \hookrightarrow V_\lambda$  such that  $|M_0| < \lambda^+$  and  $x \in \text{im}(Fm_0)$ . Now let  $x_0 \in FM_0$  be such that  $(Fm_0)(x_0) = x$  and notice that  $|M_0| < \lambda^+$  implies  $|M_0| \leq \lambda$ ; we can now apply Lemma 6.2.21 to the fixed point  $Z = \lambda$  and the set  $M_0$  to obtain a subset  $m_1 : M_1 \hookrightarrow M_0$  with  $x_0 \in \text{im}(Fm_1)$  and  $|M_1| < |M_0| \leq \lambda$ .

Finally  $m = m_0 \circ m_1 : M_1 \hookrightarrow V_\lambda$  is a subset of  $V_\lambda$  such that  $|M_1| < \lambda$  and  $x \in \text{im}(Fm)$  since  $x_0 \in \text{im}(Fm_1)$  and  $x = (Fm_0)(x_0)$ .

This concludes the proof.  $\square$

But even assuming GCH is insufficient to get a sufficient and necessary condition.

**Example 6.2.23.** Consider the set functor defined by

$$FX = X + 1 + \{M \subseteq X : X \text{ uncountable}\}$$

on objects and naturally on arrows. Notice that  $F$  has the same terminal-coalgebra chain as  $G$   $GX = X + 1$  so it has a countable terminal coalgebra. By an argument similar to that of Example 6.2.18 we have that  $F$  does not have uncountable fixed points and hence it has a terminal coalgebra but no fixed point pair.

Moreover trying to force an analogous of Theorem 4.3.2 for terminal coalgebras and fixed point pairs creates conditions independent from ZFC as in the following proposition.

**Proposition 6.2.24.** The Continuum Hypothesis  $\aleph_1 = 2^{\aleph_0}$  holds if and only if every set functor with fixed point pair  $(\aleph_0, \aleph_1)$  has a terminal coalgebra.

*Proof.* The “if” direction is clear from Theorem 6.2.22. For the reverse assume that  $\aleph_1 < 2^{\aleph_0}$  and consider the functor  $\mathcal{P}_{-\aleph_0, \aleph_1}$  defined as

$$\mathcal{P}_{-\aleph_0, \aleph_1}(X) = \{Y \subseteq X : |Y| \neq \aleph_0, \aleph_1\}$$

on objects and naturally on arrows. Now if  $X$  has cardinality  $\aleph_0$  or  $\aleph_1$  then  $\mathcal{P}_{-\aleph_0, \aleph_1}(X) = \mathcal{P}_f(X)$  which gives that both  $\aleph_0$  and  $\aleph_1$  are fixed points; hence  $\mathcal{P}_{-\aleph_0, \aleph_1}$  has fixed point pair  $(\aleph_0, \aleph_1)$ . But, by working as in Example 6.2.18, we can prove that  $\mathcal{P}_{-\aleph_0, \aleph_1}$  has no terminal coalgebra.  $\square$

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