Notes on Initial Algebras

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Contents

Chapter 1

Preliminaries

1.1 Limits and colimits

Through this section let $\mathscr{A}, \mathscr{D}$ be categories with $\mathscr{D}$ small. We recall what limits and colimits are, some significant examples (particularly products and coproducts) and some of their basic properties. We also introduce some notation. For detailed proofs see [Bor08, Chapter 2].

Definition 1.1.1. Given a functor $F\colon \mathscr{D} \to \mathscr{A}$ a **cone** on F is an object $C \in \mathscr{A}$ and a family of arrows $(\pi_D\colon C \to FD)_{D\in\mathscr{D}}$ such that for every arrow $d\colon D_1 \to D_2$ of $\mathscr{D}$ we have

cone

$$Fd \circ \pi_{D_1} = \pi_{D_2}. \tag{1.1.1}$$

The arrows π_D are called **projections** of the cone, the object C the **vertex**.

projection, vertex

Definition 1.1.2. Given a functor $F\colon \mathscr{D} \to \mathscr{A}$ a **limit** of F is a cone $(L, (\pi_D)_{D\in\mathscr{D}})$ such that for every other cone $(C, (\rho_D)_{D\in\mathscr{D}})$ there is a unique arrow $m\colon C \to L$ such that

limit

$$\rho_D = \pi_D \circ m \quad \text{for every } D \in \mathscr{D}.$$

Proposition 1.1.3. When a functor F has a limit that limit is unique (up to isomorphism).

The following proposition is a way of proving equality of two arrows into a limit. It is most useful when working in abstract categories where arrows are not (generally) functions.

Proposition 1.1.4. Let $(L, (\pi_D)_{D\in\mathscr{D}})$ be the limit of F and consider two arrows $f, g\colon C \to L$. If for every $D \in \mathscr{D}, \pi_D \circ f = \pi_D \circ g$ then $f = g$.

The definitions of cone and of limit are dualized to yield those of cocone and colimit.

Definition 1.1.5. Given a functor $F\colon \mathscr{D} \to \mathscr{A}$ a **cocone** on F is an object $C \in \mathscr{A}$ and a family of arrows $(\sigma_D\colon FD \to C)_{D\in\mathscr{D}}$ such that for every arrow $d\colon D_1 \to D_2$ of $\mathscr{D}$ we have

cocone

$$\sigma_{D_2} \circ Fd = \sigma_{D_1}. \tag{1.1.2}$$

The arrows σ_D are called **coprojections** of the cocone.

coprojection

Definition 1.1.6. Given a functor $F\colon \mathscr{D} \to \mathscr{A}$ a **colimit** of F is a cocone $(L, (\sigma_D)_{D\in\mathscr{D}})$ such that for every other cocone $(C, (\tau_D)_{D\in\mathscr{D}})$ there is a unique arrow $m\colon L \to C$ such that

colimit

$$\tau_D = m \circ \sigma_D \quad \text{for every } D \in \mathscr{D}.$$

Propositions 1.1.3 and 1.1.4 are dualized as follows.

Proposition 1.1.7. When a functor F has a colimit that colimit is unique (up to isomorphism).

Proposition 1.1.8. Let $(L, (\sigma_D)_{D\in\mathscr{D}})$ be the colimit of F and consider two arrows $f, g\colon L \to C$. If for every $D \in \mathscr{D}, f \circ \sigma_D = g \circ \sigma_D$ then $f = g$.

Products and coproducts

We now turn to two particular classes of (co)limits: products and coproducts. Recall that a discrete category is a category that has no non-identity arrow.

Definition 1.1.9. Given a functor $F\colon \mathscr{D} \to \mathscr{A}$ where $\mathscr{D}$ is some discrete category a limit of F is called a **product** while a colimit a **coproduct**.

product, coproduct

Notation 1.1.10. Notice that to give a functor from a discrete category to $\mathscr{A}$ is equivalent to picking an element of $\mathscr{A}$ for every element of $\mathscr{D}$. We thus speak of the "product of A and B" for $A, B \in \mathscr{A}$ without explicit reference to any functor. Moreover we denote the product of A and B in $\mathscr{A}$, when it exists, by $A \times B$. Similarly we speak of the coproduct of two elements of $\mathscr{A}$ and denote it by $A + B$ when it exists.

Proposition 1.1.11. In a category, when the interested (co)products exists, we have that

1. $A \times B$ is isomorphic to $B \times A$;
2. $A + B$ is isomorphic to $B + A$;
3. $(A \times B) \times C$ is isomorphic to $A \times (B \times C)$;
4. $(A + B) + C$ is isomorphic to $A + (B + C)$.

In light of this proposition we write $A \times B \times C$ and $A + B + C$ with no parenthesis and similarly for any finite number of factors or addenda.

Notation 1.1.12. When we take the (co)product of an infinite number of objects of $\mathscr{A}$ we use the notation $\prod_{i \in I} A_i$ (for products) and $\coprod_{i \in I} A_i$ (for coproducts). Notice that the order of the factors/addenda does not matter.

Notation 1.1.13. Let $(L = \prod_{i \in I} A_i, (\pi_i)_{i \in I})$ be a product in $\mathscr{A}$. Then by Proposition 1.1.4 there is a one-to-one correspondance between arrows $f\colon C \to L$ and cones $(C, (\rho_i\colon C \to A_i)_{i \in I})$. Moreover any collection of arrows $(\rho_i\colon C \to A_i)_{i \in I}$ satisfies condition (1.1.1) so it is a cone. This means that to give an arrow $f\colon C \to L$ is equivalent to giving a family of arrows $(\rho_i\colon C \to A_i)_{i \in I}$ so we write

$$f = \langle \rho_1, \rho_2, \rho_3, \dots \rangle; \tag{1.1.3}$$

given a convenient ordering on I.

By a dual argument we obtain that arrows $f\colon L \to C$ out of a coproduct $L = \coprod_{i \in I} A_i$ are in one-to-one correspondence with families of arrows $(\tau_i\colon A_i \to C)$ and write

$$f = [\tau_1, \tau_2, \tau_3, \dots] \tag{1.1.4}$$

for a convenient ordering on I.

Notation 1.1.14. Let A, B, C, D be objects of $\mathscr{A}$ such that $A \times B, C \times D$ exist and let $f\colon A \to C, g\colon B \to D$ be arrows. With reference to the following diagram we denote $\langle f \circ \pi_1, g \circ \pi_2 \rangle$ by $f \times g$. Dually if $A + B, C + D$ exist we denote $[\sigma_1 \circ f, \sigma_2 \circ g]$ by $f + g$.

$f \times g, f + g$

$$
\begin{tikzcd}
A \arrow[r, "f"] & C \\
A \times B \arrow[u, "\pi_1"] \arrow[r, dashed, "f \times g"] \arrow[d, "\pi_2"'] & C \times D \arrow[u] \arrow[d] \\
B \arrow[r, "g"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & C \arrow[d, "\sigma_1"] \\
A + B \arrow[r, dashed, "f + g"] & C + D \\
B \arrow[u] \arrow[r, "g"] & D \arrow[u, "\sigma_2"']
\end{tikzcd}
$$

The next proposition shows how the notation introduced above for products interacts with the composition. It is most useful for performing calculations; we will use it without explicit reference particularly in the proof of the Primitive Recursion Theorem (see 2.3.3).

Proposition 1.1.15. Consider the following diagram.

$$
\begin{tikzcd}
& & A \arrow[r, "g_1"] & D \\
D \arrow[r, "h"] & C \arrow[ur, "f_1"] \arrow[r, "\langle f_1, f_2 \rangle"] \arrow[dr, "f_2"'] & A \times B \arrow[u] \arrow[d] \arrow[r, "g_1 \times g_2"] & D \times E \arrow[u] \arrow[d] \\
& & B \arrow[r, "g_2"] & E
\end{tikzcd}
$$

We have that

1. $(g_1 \times g_2) \circ \langle f_1, f_2 \rangle = \langle g_1 \circ f_1, g_2 \circ f_2 \rangle$;
2. $\langle f_1, f_2 \rangle \circ h = \langle f_1 \circ h, f_2 \circ h \rangle$.

Initial and terminal objects

Definition 1.1.16. An **initial object** for a category $\mathscr{A}$ is an object $0 \in \mathscr{A}$ such that for every other object $A \in \mathscr{A}$ there is exactly one arrow from 0 to $\mathscr{A}$. Dually a **terminal object** for $\mathscr{A}$ is an object $1 \in \mathscr{A}$ such that for every other object $A \in \mathscr{A}$ there is exactly one arrow from A to 1.

initial object

terminal object

Remark 1.1.17. Initial objects, when they exist, are colimits of the functor from the empty category into $\mathscr{A}$; as such they are all isomorphic and we speak of *the* initial object. Dually terminal objects are limits of the functor from the empty category and are unique as well.

(Co)limit-preserving functors

Definition 1.1.18. A functor $G\colon \mathscr{A} \to \mathscr{B}$ **preserve limits** if, for every small category $\mathscr{D}$ and functor $F\colon \mathscr{D} \to \mathscr{A}$, when the limit of F exists the limit of $G \circ F$ is obtained by applying G to the limit cone. That is: if $(L, (\pi_D)_{D \in \mathscr{D}})$ is the limit of F then $(GL, (G\pi_D)_{D \in \mathscr{D}})$ is the limit of $G \circ F$. As expected G is said to preserve colimits when, for every small category $\mathscr{D}$ and functor $F\colon \mathscr{D} \to \mathscr{A}$, when the colimit of F exists then the colimit of $G \circ F$ is the image through G of the colimit cocone.

limit preservation

colimit preservation

By restricting the definition above to a specific small category $\mathscr{D}$ we obtain the notion of a functor that preserves $\mathscr{D}$-(co)limits. We will be interested in functors that preserve ω-(co)limits where ω is the set of natural numbers with the usual ordering regarded as a category.

1.2 Other notations

Notation 1.2.1. When we are working with a category $\mathscr{A}$ that has enough structure [1] we will often define endofunctors $F\colon \mathscr{A} \to \mathscr{A}$ by their action on objects only such as

$$FX = X \times X + 1.$$

This means that the functor F sends objects A to $A \times A + 1$ (where 1 is the terminal object of $\mathscr{A}$) and arrows f to $f \times f + \mathsf{id}_1$.

[1] i.e. that has all the (co)limits needed for our definition to make sense.

Chapter 2

Algebras for endofunctors

2.1 F-algebras

Though this section let $\mathscr{A}$ be an arbitrary category that we call the **base category** and $F\colon \mathscr{A} \to \mathscr{A}$ an endofunctor.

Definition 2.1.1. An **algebra** for F (or an F**-algebra**) is a pair (A, α) where A is an object of $\mathscr{A}$ (that we call the **carrier** or **base object**) and α is an arrow of type $FA \to A$ (that we call the **structure** of the algebra). F-algebra

As one should expect F-algebras form a category of their own.

Definition 2.1.2. Given F-algebras (A, α) and (B, β) a **morphism** between them is an arrow $f : A \to B$ of $\mathscr{A}$ such that

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Ff"] & A \arrow[d, "f"] \\ FB \arrow[r, "\beta"] & B \end{tikzcd}$$

commutes.

Remark 2.1.3. Given morphisms of algebras $f\colon (A, \alpha) \to (B, \beta)$ and $g\colon (B, \beta) \to (C, \gamma)$ their composition is exactly $g \circ f$; one can easily check that $g \circ f$ makes the diagram above commute using that f and g are morphisms of F-algebras and that F is a functor. The identity on an algebra (A, α) is exactly id_A.

We denote the category of F-algebras and morphisms of F-algebras by $\mathsf{Alg}F$. $\mathsf{Alg}F$

Example 2.1.4. In Set let 1 be the terminal object $\{*\}$ and consider the functor

$$FX = X + 1.$$

To give an arrow $\alpha\colon A+1 \to A$ is equivalent to giving two functions $\alpha_0\colon 1 \to A$, $\alpha_1\colon A \to A$; that is $\alpha = [\alpha_1, \alpha_0]$. So an algebra for F is set A with a constant α_0 and a unary operation α_1. A morphism of F-algebras is a function f such that

$$\begin{tikzcd} A+1 \arrow[r, "{[\alpha_1,\alpha_0]}"] \arrow[d, "f+\mathsf{id}_1"] & A \arrow[d, "f"] \\ B+1 \arrow[r, "{[\beta_1,\beta_0]}"] & B \end{tikzcd}$$

commutes. This translates, using Proposition 1.1.8, into the following two conditions

$$f \circ \alpha_1 = \beta_1 \circ f, \quad f \circ \alpha_0 = \beta_0.$$

Which together give that f preserves the constant and commutes with the unary operation of the algebras.

Example 2.1.5. Similarly algebras for the endofunctor $FX = X \times X + 1$ are sets equipped with a constant and a binary operation. Morphisms of such algebras preserve the constant and commute with the operation. Indeed if $(A, a, *_A)$ and $(B, b, *_B)$ are two F-algebras a morphism $f\colon A \to B$ is a function such that

1. $f(a) = b$;
2. $f(x *_A y) = f(x) *_B f(y)$ for all $x, y \in A$.

This is reminiscent of structures such as groups and monoids.

Example 2.1.6. Let Σ_1 be a set; then algebras for $FX = \Sigma_1 \times X$ are sets with a unary operation for every element $\sigma \in \Sigma_1$. Indeed let A be an F-algebra and let $\alpha\colon \Sigma_1 \times A \to A$ be its structure; for a fixed $\sigma \in \Sigma_1$ we have that $\alpha(\sigma, -)$ is a unary operation on A. Morphisms of F-algebras in this case commute with all such unary operations.

Examples 2.1.4, 2.1.5, 2.1.6 can be generalized with the introduction of polynomial functors over Set. This captures the idea of a Σ-algebra from universal algebra and provides a generalization to other base categories with sufficient structure.

Definition 2.1.7. A **signature** Σ is a collection of sets $(\Sigma_n)_{n<\omega}$ where Σ_n is called the set of n-ary symbols. The symbols in Σ_0 are also called constant symbols (or symbols for constants).

signature

Definition 2.1.8. Given a signature Σ a Σ-**algebra** is a set A together with interpretations for every symbol of Σ. That is: to every $\sigma \in \Sigma_0$ we associate a fixed element of A and to every $\sigma \in \Sigma_n$ an n-ary function on A. We denote the interpretation of σ in A by σ^A.

Σ-algebra

Given two Σ-algebras A and B a **morphism** of Σ-algebras is a function $f\colon A \to B$ such that

$$f\sigma^A = \sigma^B \tag{2.1.1}$$

for every $\sigma \in \Sigma_0$ and

$$f(\sigma^A(x_1, \dots, x_n)) = \sigma^B(f(x_1), \dots, f(x_n)) \tag{2.1.2}$$

for every $\sigma \in \Sigma_n$ and $x_1, \dots, x_n \in A$.

We will now observe how Σ-algebras can be realized as algebras for specific endofunctors. To every signature we associate a **polynomial functor** $H_\Sigma\colon \mathsf{Set} \to \mathsf{Set}$ that operates as follows on objects X and arrows f.

polynomial functor

$$H_\Sigma X = \coprod_{n<\omega} \Sigma_n \times X^n$$
$$H_\Sigma f = \coprod_{n<\omega} \mathsf{id}_{\Sigma_n} \times \underbrace{f \times \ldots \times f}_{n \text{ times}}$$

Then if A is a Σ-algebra we can define functions $\alpha_n\colon \Sigma_n \times A^n \to A$ naturally as

$$\alpha_n(\sigma, x_1, \dots, x_n) \coloneqq \sigma^A(x_1, \dots, x_n) \tag{2.1.3}$$

and then $\alpha := [\alpha_0, \alpha_1, \dots]$ makes A into a H_Σ-algebra. Conversely if we have a H_Σ-algebra (A, α) we can define an interpretation of every symbol by reading (2.1.3) the other way around.

For morphisms consider the following square that we know commutes if f is a morphism of H_Σ-algebras.

$$\begin{tikzcd}
\coprod_{n<\omega} \Sigma_n \times A^n \arrow[r, "\alpha"] \arrow[d, "H_\Sigma f"] & A \arrow[d, "f"] \\
\coprod_{n<\omega} \Sigma_n \times B^n \arrow[r, "\beta"] & B
\end{tikzcd}$$

Since the diagonal of the square is an arrow out of a coproduct we know from Proposition 1.1.8 that $f \circ \alpha$ and $\beta \circ H_\Sigma f$ are equal when composed with the coprojections of the cocone and this can happen if and only if conditions (2.1.1) and (2.1.2) hold.

We now turn to continous algebras.

Definition 2.1.9. A **complete partial order** (or a **CPO**) is a poset $(A, \sqsubseteq)$ in which all ω-chains complete partial order

$$a_1 \sqsubseteq a_2 \sqsubseteq a_3 \sqsubseteq \dots$$

have a join i.e. a least upper bound; we denote it by $\bigvee_{n<\omega} a_n$. A function $f\colon (A, \sqsubseteq) \to (B, \sqsubseteq)$ is **continous** if continous function

1. it is monotone i.e. $a \sqsubseteq b$ in A implies $f(a) \sqsubseteq f(b)$ in B;
2. given an ω-chain $a_1 \sqsubseteq a_2 \sqsubseteq a_3 \sqsubseteq \dots$ we have $f(\bigvee_{n<\omega} a_n) = \bigvee_{n<\omega} f(a_n)$.

Note that by 1 any chain in A is preserved by f and since B is a CPO it must have a join; condition 2 forces that join to be the image of the join of the original chain under f.

Remark 2.1.10. CPOs and continous functions form a category CPO. In CPO (co)products are formed as in Set and have, for products, the pointwise order and, for coproducts, each component keeps its own ordering and elements in different components are never comparable. Moreover CPO has a terminal object 1 that is the singleton $\{*\}$ with the only possible partial order on it.

Example 2.1.11. Consider the functor $FX = X + 1$ on CPO. Algebras for F are CPOs with a constant and a continous unary operation. Similarly algebras for the functor $FX = X \times X + 1$ are CPOs with a constant and a continous binary operation. The continuity of the operation (which we denote here as $*$) in this case gives us that if

$$a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \dots \quad \text{and} \quad b_0 \sqsubseteq b_1 \sqsubseteq b_2 \sqsubseteq \dots$$

are chains then $(a_0, b_0) \sqsubseteq (a_1, b_1) \sqsubseteq (a_2, b_2) \sqsubseteq \dots$ is a chain in the product and

$$\bigvee_{n<\omega} a_n * \bigvee_{n<\omega} b_n = \bigvee_{n<\omega} a_n * b_n.$$

Example 2.1.12. Now let $FX = X_\perp$ be the functor that adds to a CPO a bottom element (i.e. an element $\perp$ such that $\perp \sqsubseteq x$ for every x in the poset) and that sends a continous function $f\colon X \to Y$ to $f_\perp\colon X_\perp \to Y_\perp$ defined as

$$f_\perp(x) = \begin{cases} f(x) & \text{if } x \neq \perp \\ \perp & \text{if } x = \perp \end{cases}.$$

An algebra (A, α) for F has a unary continous function $\alpha_1\colon A \to A$ and a constant $\alpha_\perp$ such that $\alpha_\perp \sqsubseteq \alpha(a)$ for every $a \in A$ by monotony of α. Comparing this to Example 2.1.11 by changing the functor we now have a condition on our constant.

Finally let $\mathsf{CPO}_\perp$ be the category of CPOs with a bottom element and continous functions that preserve the bottom i.e. such that $f(\perp) = \perp$. We call such functions **strict continous functions**.

strict continous function

Remark 2.1.13. In $\mathsf{CPO}_\perp$ products work as in CPO (the bottom element is naturally the one whose components are all $\perp$) but the construction of coproducts must be changed. Indeed coproducts in $\mathsf{CPO}_\perp$ are formed as in CPO but the bottoms of all the components are unified into a single element. The initial object is still the singleton as it clearly has a bottom.

Remark 2.1.14. Notice that the functor $FX = X + 1$ on $\mathsf{CPO}_\perp$ is the identity functor so its algebras are CPOs with a bottom and a strict continous unary operation. If we want a constant on top of the unary operation we can consider the functor $FX = X_\perp + 1_\perp$, $Ff = f_\perp + \mathsf{id}_{1_\perp}$.

2.2 Initial Algebras

Definition 2.2.1. An **initial algebra** for an endofunctor F is an initial object in $\mathsf{Alg}F$. When an initial algebra exists we denote it by $(\mu F, \iota)$.

initial algebra

Remark 2.2.2. If 0 is the initial object of $\mathscr{A}$ and F preserves initial objects then $(0, \mathsf{id}_0)$ is the initial algebra for F.

We will now look at some non-trivial examples of initial algebras.

Example 2.2.3. Consider the functor $FX = X + 1$ on Set. The initial algebra for F is $(\mathbb{N}, \iota)$ where $\iota = [\iota_1, \iota_0]$ with $\iota_0 = 0$ and ι_1 the successor function on the naturals. Indeed let (B, β) be another algebra with $\beta = [\beta_1, \beta_0]$ and consider the function $f\colon \mathbb{N} \to B$ defined as

$$\begin{cases} f(0) &= \beta_0 \\ f(n+1) &= \beta_1(f(n)) \end{cases}. \tag{2.2.1}$$

This is a morphism of algebras:

1. $f(0) = \beta_0$;

2. $(f \circ \iota_1)(n) = f(n+1) = (\beta_1 \circ f)(n)$ for all $n \in \mathbb{N}$ so $f \circ i_1 = \beta_1 \circ f$;

and clearly unique because any other morphism must satisfy 1 and 2, thus a simple induction gives us that is must be f.

Example 2.2.4. Let $\mathcal{P}_{\mathsf{f}} \colon \mathsf{Set} \to \mathsf{Set}$ be the finite power-set functor. This functor sends a set X to $\mathcal{P}_{\mathsf{f}} X = \{Z \subseteq X \colon Z \text{ is finite}\}$ and a function $f \colon X \to Y$ to the function $\mathcal{P}_{\mathsf{f}} f$ that sends a finite subset of X to its (obviously finite) f-image in Y. Let V_ω be the set of hereditarily finite sets. That is:

finite power-set functor

1. $\emptyset \in V_\omega$;
2. if $X_1, \ldots, X_n \in V_\omega$ then $\{X_1, \ldots, X_n\} \in V_\omega$.

We will see in Example 3.1.15 that (V_ω, id) is the initial algebra of $\mathcal{P}_{\mathsf{f}}$.

Example 2.2.5. Let $\mathscr{A}$ be a category with countable coproducts. If $A \in \mathscr{A}$ we write $\mathbb{N} \bullet A$ for the coproduct of A with itself countably many times. We shall now consider the functor $FX = X + A$ for which we prove the initial algebra to be $\mathbb{N} \bullet A$.

First we need an algebra structure ι on $\mathbb{N} \bullet A$. If $\mathsf{in}_k \colon A \to \mathbb{N} \bullet A$ is the k-th coprojection then we set

$$\iota = [\alpha_1, \mathsf{in}_0] \colon (\mathbb{N} \bullet A) + A \to \mathbb{N} \bullet A \tag{2.2.2}$$

where α_1 is obtained from the universal property of the coproduct applied to the cone $(\mathsf{in}_k)_{1<k<\omega}$. We thus have that the following triangles commute for every $k < \omega$.

$$\begin{tikzcd}
\mathbb{N} \bullet A \arrow[rr, "\alpha_1"] & & \mathbb{N} \bullet A \\
& A \arrow[ul, "\mathsf{in}_k"] \arrow[ur, "\mathsf{in}_{k+1}"'] &
\end{tikzcd}$$

Now let (B, β) with $\beta = [\beta_1, \beta_0]$ be an algebra and f a morphism from $(\mathbb{N} \bullet A, \iota)$. We have that the following square commutes.

$$\begin{tikzcd}
\mathbb{N} \bullet A + A \arrow[r, "{[\alpha_1, \mathsf{in}_0]}"] \arrow[d, "f + \mathsf{id}"] & \mathbb{N} \bullet A \arrow[d, "f"] \\
B + A \arrow[r, "{[\beta_1, \beta_0]}"] & B
\end{tikzcd}$$

So any such f must be such that

1. $f \circ \mathsf{in}_0 = \beta_0$;
2. $f \circ \mathsf{in}_{k+1} = f \circ \alpha_1 \circ \mathsf{in}_k = \beta_1 \circ f \circ \mathsf{in}_k$ for all $k \in \mathbb{N}$.

This gives us that $[\beta_0, \beta_1 \circ \beta_0, \beta_1 \circ \beta_1 \circ \beta_0, \ldots]$ is the unique morphism from $(\mathbb{N} \bullet A, \iota)$ to (B, β) which proves our claim.

Finally we describe initial algebras for polynomial functors in two equivalent ways: using closed terms and using trees.

Definition 2.2.6. Let Σ be a signature. The set of **closed term** for Σ is defined inductively as follows: closed term

1. a constant symbol is a closed term;
2. an expression of the form $\sigma(t_1, \ldots, t_n)$ where $\sigma \in \Sigma_n$ and $t_1, \ldots, t_n$ are closed terms is a closed term.

Remark 2.2.7. Let μH_Σ be the set of all closed terms; this is naturally a Σ-algebra:

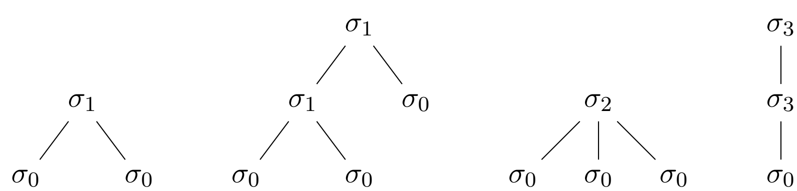
1. if σ is a contant symbol then $\sigma^{\mu H_\Sigma}$ is σ itself but regarded as a term;
2. if σ is an n-ary symbol then it defines an n-ary operation on μH_Σ that takes terms $t_1, \ldots, t_n$ to the term $\sigma(t_1, \ldots, t_n)$.

Remark 2.2.8. Let A be a Σ-algebra and t a closed term. We can define an evaluation function $\mathsf{ev}\colon \mu H_\Sigma \to A$ as follows:

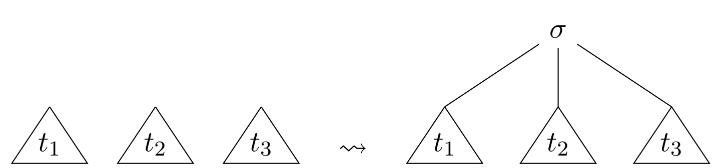
1. if t is a constant symbol σ then $\mathsf{ev}(t) = \sigma^A$;
2. if t is of the form $\sigma(t_1, \ldots, t_n)$ then $\mathsf{ev}(t) = \sigma^A(\mathsf{ev}(t_1), \ldots, \mathsf{ev}(t_n))$.

Clearly ev is a morphism and moreover it is unique so μH_Σ is the initial algebra for H_Σ.

Definition 2.2.9. Given a signature Σ a Σ-**tree** is an ordered tree where every node of k children is labelled by a k-ary symbol of Σ. As an example if σ_0 is a constant symbol, σ_1 a 2-ary symbol, σ_2 a 3-ary symbol and σ_3 a 1-ary symbol then the following are valid Σ-trees.



Remark 2.2.10. Every n-ary symbol of Σ defines an n-ary operation on Σ-trees which takes n trees to the tree obtained by connecting each root to a new node labelled by σ, which becomes the root of a new tree. We call this operation **tree-tupling** (below an example of tree-tupling induced by a 3-ary symbol σ). Note that the order of the trees matters. tree-tupling



Proposition 2.2.11. The initial algebra μH_Σ is the algebra of finite Σ-trees with tree-tupling.

Proof. Let T be the algebra of finite Σ-trees with tree-tupling; we shall find an isomorphism $f\colon \mu H_\Sigma \to T$ of algebras. Define f by structural recursion as follows

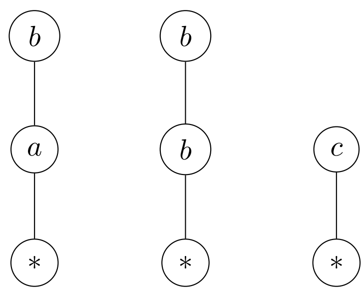
1. if t is a constant term σ let $f(t)$ be the tree formed by a single node labelled by σ;

2. if t is a term of the form $\sigma(t_1, \ldots, t_n)$ let $f(\sigma)$ be the tree obtained by tree-tupling $f(t_1), \ldots, f(t_n)$ with the new root labelled by σ.

This function is a morphism by definition and has an inverse, defined similarly by structural recursion, which is again a morphism. $\square$

Example 2.2.12. By the discussion above the initial algebra for the functor $FX = X \times X + 1$ on Set is the algebra of all finite binary trees.

Example 2.2.13. Consider the functor $FX = B \times X + 1$ on Set with $B \in$ Set. Algebras for F are sets with a constant and $|B|$ unary operations. By the discussion above we know that μF is the algebra of finite trees for the signature $\Sigma = (\Sigma_0 = \{*\}, \Sigma_1 = B)$ so its elements are "linear" trees such as



with $a, b, c \in B$. We immediately deduce that μF can also be realized as the set of words over B.

Remark 2.2.14. Notice the importance of constant symbols. Indeed if a polynomial functor on Set has no constant then it preserves the initial object 0 so by Remark 2.2.2 the initial algebra is trivial. Equivalently one can observe that, without constant symbols, the set of closed terms is empty.

Example 2.2.15. Consider $FX = X_\perp$ on CPO$_\perp$. Let $\mathbb{N}^\top$ be the set of the natural numbers with an added topmost element ∞ ordered naturally. Notice that $(\mathbb{N}^\top)_\perp$ is isomorphic (as an order) to $\mathbb{N}^\top$ and consider the successor function $s\colon (\mathbb{N}^\top)_\perp \to \mathbb{N}^\top$ with $s(\infty) = \infty$. We claim this is the initial algebra for F.

Let (A, α) be another F-algebra and $f\colon (\mathbb{N}^\top, s) \to (A, \alpha)$ a morphism. Because f is an arrow of CPO$_\perp$ it must be strict so $f(0) = \perp$ and because it is a morphism we must have $f(s(n)) = \alpha(f(n))$; this defines f inductively on $\mathbb{N}$. Finally we recall that f must be continous (as an arrow of CPO$_\perp$) so

$$f(\infty) = f\left(\bigvee_{n<\omega} n\right) = \bigvee_{n<\omega} (f(n)).$$

This implies that there is really a unique morphism from $(\mathbb{N}^\top, s)$ to (A, α) so $(\mathbb{N}^\top, s)$ is initial.

We conclude this section with the a classic lemma of Lambeck's which gives necessary conditions for functors to have an initial algebra.

Definition 2.2.16. A **fixed point** of an endofunctor F is an element $A \in \mathscr{A}$ that is isomorphic to FA.

Lemma 2.2.17 (Lambeck's Lemma). An initial algebra for F is always a fixed point.

Proof. Let $(\mu F, \iota)$ be the initial algebra of F so there is a unique morphism

$$f\colon (\mu F, \iota) \to (F(\mu F), F\iota).$$

We have that the following diagram commutes

$$\begin{tikzcd}
F(\mu F) \arrow[r, "\iota"] \arrow[d, "Ff"] & \mu F \arrow[d, "f"] \\
F(F(\mu F)) \arrow[r, "F\iota"] \arrow[d, "F\iota"] & F(\mu F) \arrow[d, "\iota"] \\
F(\mu F) \arrow[r, "\iota"] & \mu F
\end{tikzcd}$$

so $\iota \circ f$ is an endomorphism of algebras on the initial algebra; hence it is $\mathsf{id}_{\mu F}$. Now

$$f \circ \iota = F\iota \circ Ff = F(\iota \circ f) = F(\mathsf{id}_{\mu F}) = \mathsf{id}_{F(\mu F)}$$

so ι is an isomorphism. This shows that μF is a fixed point of F. □

Remark 2.2.18. We know from Cantor's Theorem that there is no surjection from a given set X into its power set $\mathcal{P}X$ so the functor $\mathcal{P}$ on Set has no fixed point hence no initial algebra.

Lambeck's lemma can be seen as a generalization of the following order-theoretic lemma.

Definition 2.2.19. Let $(P, \sqsubseteq)$ be a poset and $f\colon P \to P$ a monotone function. An element $x \in P$ is a **pre-fixed point** of f if $f(x) \sqsubseteq x$.

Lemma 2.2.20. Given a poset $(P, \sqsubseteq)$ and a monotone function $f\colon P \to P$ let $A = \{x \in P\colon f(x) \sqsubseteq x\}$ be the set of pre-fixed points of f. If $\overline{a}$ is the meet of A then $\overline{a}$ is the least fixed point of f.

Proof. If $x \in A$ then $\overline{a} \sqsubseteq x$ and $f(\overline{a}) \sqsubseteq f(x) \sqsubseteq x$ follows from monotony of f and definition of A. This shows that $f(\overline{a})$ is a lower bound for A so $f(\overline{a}) \sqsubseteq \overline{a}$. Now by applying f again we obtain $f(f(\overline{a})) \sqsubseteq f(\overline{a})$ which gives us $f(\overline{a}) \in A$ and thus $\overline{a} \sqsubseteq f(\overline{a})$. Moreover $\overline{a}$ must be the least fixed point of f since all fixed points are pre-fixed points. □

This lemma then gives us the following well-known theorem for free.

Theorem 2.2.21 (Knaster-Tarski Theorem). Let $(L, \sqsubseteq)$ be a complete lattice. Then every monotone function of L has a fixed point.

Proof. Let f be a monotone function on L and A be the set of pre-fixed points of f. It must have a meet $\overline{a}$ because the lattice is complete so by the Lemma we have a fixed point. $\square$

The generalization of order theoretic results such as Lemma 2.2.20 will be a recurrent theme. Indeed we can see orders as categories in the usual way, order-preserving functions as functors and pre-fixed points as algebras for these functors.

2.3 Recursion and Induction

Recursion is a way of defining a function f on the natural numbers by specifying a value for $f(0)$ and a way of deriving the value of $f(n+1)$ from the value of $f(n)$. Recursion, however, can also be used to define functions on (among other things) trees; in this case it is usually called *structural* induction (we have an example of this in Remark 2.2.8). Here we show how one can use the concept of an initial algebra to provide a general definition of recursion of which the cited examples as particular cases.

Definition 2.3.1. Let F be an endofuctor with an initial algebra μF. A morphism $f\colon \mu F \to A$ is **recursively specified** if there exists an algebra structure $\alpha\colon FA \to A$ on A that makes f into an algebra homomorphism (actually the unique algebra homomorphism).

recursively specified morphism

Example 2.3.2. Consider the functor $FX = X \times X + 1$ on Set and let T be its initial algebra that we know from Example 2.2.12 is the algebra of finite binary trees with tree-tupling. Let $h\colon T \to \mathbb{N}$ be the function that assigns 0 to the root-only tree and, given a tree t obtained by tree-tupling of t_1 and t_2, $h(t) = 1 + \max\{h(t_1), h(t_2)\}$. We now show that h is recursively specified according to our definition.

We need an algebra structure on $\mathbb{N}$ i.e. an arrow $\alpha\colon \mathbb{N}\times\mathbb{N}+1 \to \mathbb{N}$. Since α is an arrow out of a coproduct we have $\alpha = [\alpha_1, \alpha_0]$ so we can let

$$\begin{aligned} \alpha_0 &= 0; \\ \alpha_1(n,m) &= 1 + \max\{n,m\}. \end{aligned}$$

Indeed if ι_1 is tree-tupling and ι_0 is the root-only tree the following square trivially commutes thus h is recursively specified.

$$\begin{tikzcd} T\times T+1 \arrow[r, "{[\iota_1,\iota_0]}"] \arrow[d, "h\times h+\mathrm{id}_1"] & T \arrow[d, "h"] \\ \mathbb{N}\times\mathbb{N}+1 \arrow[r, "{[\alpha_1,\alpha_0]}"] & \mathbb{N} \end{tikzcd}$$

The following theorem is a generalization of the classic recursion with parameters theorem.

Theorem 2.3.3 (Primitive Recursion). Assume the base category $\mathscr{A}$ to have finite products and let F be an endofunctor with an initial algebra μF. Then for every $\alpha\colon F(A \times$

$\mu F) \to A$ there is a unique $h\colon \mu F \to A$ such that the following square commutes.

$$\begin{tikzcd}
F(\mu F) \arrow[r, "\iota"] \arrow[d, "F\langle h, \mathrm{id}_{\mu F}\rangle"] & \mu F \arrow[d, "h"] \\
F(A \times \mu F) \arrow[r, "\alpha"] & A
\end{tikzcd} \tag{2.3.1}$$

Proof. Let π_1, π_2 be the projections of the product $A \times \mu F$ and consider the arrow

$$\overline{\alpha}\colon F(A \times \mu F) \xrightarrow{\langle \mathrm{id}_{F(A\times\mu F)}, F\pi_2\rangle} F(A \times \mu F) \times F(\mu F) \xrightarrow{\alpha\times\iota} A \times \mu F.$$

This gives an algebra structure on $A \times \mu F$ so let $\overline{h}\colon \mu F \to A \times \mu F$ be the unique algebra homomorphism given by the initiality of μF.

$$\begin{tikzcd}
F(\mu F) \arrow[rr, "\iota"] \arrow[d, "F\overline{h}"] \arrow[drr, phantom, "(1)"] & & \mu F \arrow[d, "\overline{h}"] \\
F(A \times \mu F) \arrow[r, "\langle \mathrm{id}_{F(A\times\mu F)}, F\pi_2\rangle"] \arrow[r, phantom, shift right=3, "(3)"'] \arrow[d, "F\pi_2"] & F(A \times \mu F) \times F(\mu F) \arrow[r, "\alpha\times\iota"] \arrow[dl, "\pi_2"] \arrow[dr, phantom, "(2)"] & A \times \mu F \arrow[d, "\pi_2"] \\
F(\mu F) \arrow[rr, "\iota"] & & \mu F
\end{tikzcd} \tag{2.3.2}$$

The outer square in (2.3.2) commutes because (1), (2) and (3) do. Indeed (1) commutes because $\overline{h}$ is an algebra homomorphism, (2) by Notation 1.1.14 and (3) because of Notation 1.1.13. By functoriality of F we then have that $\pi_2 \circ \overline{h}$ is an endomorphism on the initial algebra thus $\pi_2 \circ \overline{h} = \mathrm{id}_{\mu F}$.

Now set $h \coloneqq \pi_1 \circ \overline{h}$ so $\overline{h} = \langle h, \mathrm{id}_{\mu F}\rangle$. Extending (1) by π_1 we obtain the following diagram that we know commutes.

$$\begin{tikzcd}
F(\mu F) \arrow[rr, "\iota"] \arrow[d, "F\overline{h} = F\langle h, \mathrm{id}_{\mu F}\rangle"] & & \mu F \arrow[d, "\overline{h}"] \arrow[dr, "h"] & \\
F(A \times \mu F) \arrow[r, "\langle \mathrm{id}_{F(A\times\mu F)}, F\pi_2\rangle"] & F(A \times \mu F) \times F(\mu F) \arrow[r, "\alpha\times\iota"] & A \times \mu F \arrow[r, "\pi_1"] & A
\end{tikzcd}$$

But notice that

$$\pi_1 \circ (\alpha \times \iota) \circ \langle \mathrm{id}_{F(A\times\mu F)}, F\pi_2\rangle = \pi_1 \circ \langle \alpha, \iota \circ F\pi_2\rangle = \alpha.$$

so we have (2.3.1).

To prove uniqueness consider $h\colon \mu F \to A$ homomorphism of algebras such that (2.3.1) commutes. We claim that $\overline{h} = \langle h, \mathrm{id}_{\mu F}\rangle$ so that $h = \pi_1 \circ \overline{h}$; thus uniqueness. In order to do it we show that $\langle h, \mathrm{id}_{\mu F}\rangle$ is an algebra homomorphism and conclude it must be $\overline{h}$ by initiality of μF. Indeed we have

$$\begin{aligned}
\pi_1 \circ (\alpha \times \iota) \circ \langle \mathsf{id}_{F(A\times\mu F)}, F\pi_2\rangle \circ F\langle h, \mathsf{id}_{\mu F}\rangle &= \alpha \circ \mathsf{id}_{F(A\times\mu F)} \circ F\langle h, \mathsf{id}_{\mu F}\rangle \\
&= h \circ \iota \\
&= \pi_1 \circ \langle h, \mathsf{id}_{\mu F}\rangle \circ \iota;
\end{aligned}$$

$$\begin{aligned}
\pi_2 \circ (\alpha \times \iota) \circ \langle \mathsf{id}_{F(A\times\mu F)}, F\pi_2\rangle \circ F\langle h, \mathsf{id}_{\mu F}\rangle &= \iota \circ F\pi_2 \circ F\langle h, \mathsf{id}_{\mu F}\rangle \\
&= \iota \circ F(\pi_2 \circ \langle h, \mathsf{id}_{\mu F}\rangle) \\
&= \iota \circ F\mathsf{id}_{\mu F} \\
&= \iota \circ \mathsf{id}_{F(\mu F)} \\
&= \iota \\
&= \pi_2 \circ \langle h, \mathsf{id}_{\mu F}\rangle \circ \iota;
\end{aligned}$$

so we conclude by Proposition 1.1.4. □

Recall the classic recursion with parametes theorem. We shall prove it using Theorem 2.3.3; for a classical proof see [And19, Theorem 18.1].

Theorem 2.3.4 (Recursion with parameters). Let A be a set, $a \in A$ and $f\colon A \times \mathbb{N} \to A$ be a function. Then there is a function $g\colon \mathbb{N} \to A$ such that

$$\begin{cases} g(0) = a \\ g(n+1) = f(g(n), n) \end{cases}.$$

Proof. Consider the functor $FX = X + 1$ on Set and form the arrow

$$\alpha\colon F(A \times \mu F) = A \times \mathbb{N} + 1 \xrightarrow{[f,a]} A$$

where we abuse notation and let $a : 1 \to A$ be the function that picks $a \in A$. Now by Theorem 2.3.3 there is a function $g\colon \mathbb{N} \to A$ such that (2.3.1) commutes. This gives us the diagram

$$\begin{tikzcd}
\mathbb{N} + 1 \arrow[r, "{[\iota_1,\iota_0]}"] \arrow[d, "{\langle g,\mathsf{id}_{\mathbb{N}}\rangle + \mathsf{id}_1}"'] & \mathbb{N} \arrow[d, "g"] \\
A \times \mathbb{N} + 1 \arrow[r, "{[f,a]}"] & A
\end{tikzcd} \tag{2.3.3}$$

that yields the conditions

1. $g \circ \iota_0 = a \circ \mathsf{id}_1$;
2. $g \circ \iota_1 = f \circ \langle g, \mathsf{id}_{\mathbb{N}}\rangle$.

Now 1 gives us $g(0) = a$ and 2 that

$$g(n+1) = (g \circ \iota_1)(n) = (f \circ \langle g, \mathsf{id}_{\mathbb{N}}\rangle)(n) = f(g(n), n) \quad \text{for all } n < \omega.$$

So g has the required properties. □

The induction principle, that if a subset of $\mathbb{N}$ contains 0 and is closed under the successor operation then it is $\mathbb{N}$, can also be generalized to the more general setting of F-algebras. Recall that a subobject in a category is represented by a monomorphism. We call subalgebras subobjects in $\mathsf{Alg}F$.

Theorem 2.3.5 (Induction Principle). Let F be an endofunctor and μF its initial algebra. Then every $m\colon (B,\beta) \to (\mu F, \iota)$ monic is an isomorphism.

Proof. By initiality there is a unique homomorphism of F-algebras $h\colon (\mu F, \iota) \to (B, \beta)$. Now $m \circ h$ is an endomorphism of the initial object of $\mathsf{Alg}F$ so $m \circ h = \mathsf{id}_{\mu F}$, but notice that we have m monic and split-epi thus an isomorphism. $\square$

Chapter 3

Initial algebras from finitary iteration

3.1 Adámek's Theorem

We begin this section by recalling Kleene's Fixed Point Theorem and its proof with the goal of generalizing it to the categorical setting.

Theorem 3.1.1 (Kleene's Fixed Point Theorem). Let A be a CPO with bottom $\perp$; then every continous function $F\colon A \to A$ has a least fixed point $\mu F = \sup_{n<\omega} F^n(\perp)$[1].

Proof. Consider the ω-chain $\perp \leq F(\perp) \leq F^2(\perp) \leq \dots$ and let μF be its join. By continuity $F(\mu F) = \bigvee_{n<\omega} F(F^n(\perp))$ but we know that $\bigvee_{n<\omega} F^n(\perp) = \bigvee_{n<\omega} F^{n+1}(\perp)$ so $\mu F = F(\mu F)$, a fixed point.

Now let $F(x) \leq x$ be a pre-fixed point. As $\perp \leq x$ we have $F(\perp) \leq F(x) \leq x$ and by induction we obtain that $F^n(\perp) \leq x$ for all $n < \omega$. But this shows that x is an upper bound so $\mu F \leq x$ by definition and, finally, μF must be the least fixed point because it is less than any pre-fixed point and fixed points are trivially pre-fixed points. □

Unsurprisingly we shall replace CPOs with bottom by categories with an initial obejct and endofunctions by endofunctors. The ω-chain used in the proof of Theorem 3.1.1 becomes a diagram and the continuity condition become a colimit-preservation one.

Definition 3.1.2. The **initial-algebra ω-chain** of an endofunctor F is the diagram

initial-algebra ω-chain

$$0 \xrightarrow{\;!\;} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} F^3 0 \xrightarrow{F^3!} \cdots \tag{3.1.1}$$

where we denote by ! the unique arrow out of the initial object. This diagram can be realized as a functor from ω regarded as a category.

Remark 3.1.3. Let (A, α) be an F-algebra. Then it induces a canonical cocone $(\alpha_n\colon F^n 0 \to A)_{n<\omega}$ on (3.1.1) by

$$\begin{aligned} \alpha_0 &= \,! \\ \alpha_{n+1} &= FF^n 0 \xrightarrow{F\alpha_n} FA \xrightarrow{\alpha} A. \end{aligned}$$

To check that the α_n's are a cocone we prove $\alpha_n = \alpha_{n+1} \circ F^n!$ by induction on n. When $n = 0$ the condition becomes $! = \alpha_1 \circ !$ which is true because both sides are arrows out of the initial object. Now suppose the condition holds for $n - 1$:

$$\begin{aligned} \alpha_{n+1} \circ F^n! &= \alpha \circ F\alpha_n \circ F^k! && \text{definition of } \alpha_{n+1} \\ &= \alpha \circ F(\alpha_n \circ F^{n-1}!) && \text{functoriality of } F \\ &= \alpha \circ F\alpha_{n-1} && \text{inductive hypothesis} \\ &= \alpha_n && \text{definition of } \alpha_n. \end{aligned}$$

Remark 3.1.4. If we apply F to the inital-algebra ω-chain we obtain

$$F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} F^3 0 \xrightarrow{F^3!} \cdots . \tag{3.1.2}$$

[1] We write F^n for $\underbrace{F \dots F}_{n \text{ times}}$ and F^0 is the identity function. We use a similar notation for functors.

This ω-chain has the same colimit as the original one. Indeed since the first element of (3.1.1) is the initial object there is an obvious one-to-one correspondence between cocones on (3.1.1) and cocones on the new chain above; and the same factorization morphisms work.

We are ready to state and prove the main theorem of this chapter.

Theorem 3.1.5 (Adámek). Let $\mathscr{A}$ be a category with an initial object 0 and colimits of ω-chains and F an endofunctor that preserves ω-colimits. Then F has an initial algebra $\mu F = \operatorname{colim}_{n<\omega} F^n 0$.

Proof. Let $(\mu F, (c_n : F^n 0 \to \mu F)_{n<\omega})$ be the colimit of the initial-algebra ω-chain. Since F preserves ω-colimits we have that $(F(\mu F), (Fc_n : F^{n+1} \to F(\mu F))_{n<\omega})$ is the colimit of (3.1.2) but $(c_{n+1} : F^{n+1}0 \to \mu F)_{n<\omega}$ is a cocone on (3.1.2) so there is a unique arrow ι from $F(\mu F)$ to μF such that

$$\iota \circ Fc_n = c_{n+1} \quad \text{for all } n < \omega. \tag{3.1.3}$$

This gives an F-algebra structure to μF that we now check is initial.

Let (A, α) be a generic F-algebra. We have the induced cocone $(\alpha_n : F^n 0 \to \mu F)_{n<\omega}$ as in Remark 3.1.3 and this gives a unique arrow $h : \mu F \to A$ such that $h \circ c_n = \alpha_n$ for $n < \omega$. We shall now prove that h is a morphism of F-algebras and that it is unique.

To prove that $h \circ \iota = \alpha \circ Fh$ we check that both arrows are factorization morphisms for the cocone $(\alpha_{n+1} : F^{n+1}0 \to A)_{n<\omega}$. In practice we shall check that

$$h \circ \iota \circ Fc_n = \alpha_{n+1} = \alpha \circ Fh \circ Fc_n$$

for all $n < \omega$. For the first equality notice that $h \circ \iota \circ Fc_n = h \circ c_{n+1} = \alpha_{n+1}$. For the second equality consider the following diagram

$$\begin{tikzcd}
F^{n+1}0 \arrow[r, "\alpha_{n+1}"] \arrow[d, "Fc_n"'] \arrow[dr, "F\alpha_n"] & A \\
F(\mu F) \arrow[r, "Fh"] & FA \arrow[u, "\alpha"']
\end{tikzcd} .$$

The upper-right triangle commutes by definition of α_{n+1} (see Remark 3.1.3). The lower-left triangle commute because h is the factorization arrow and by functoriality of F. Then, as the whole square commutes, $\alpha \circ Fh \circ Fc_n = \alpha_{n+1}$. This proves that $h : \mu F \to A$ is a homomorphism of F-algebras.

To prove uniqueness suppose there is an arrow $k : \mu F \to A$ such that it is also an F-algebra homomorphism i.e. $k \circ \iota = \alpha \circ Fk$. We will show that $k \circ c_n = \alpha_n$ for all $n < \omega$ and conclude by uniqueness of h. Working by induction, if $n = 0$ then $k \circ c_0 = \alpha_0$

is trivial because both arrows are out of the initial object. Now suppose that $k \circ c_n = \alpha_n$:

$$\begin{aligned} k \circ c_{n+1} &= k \circ \iota \circ F c_n && (3.1.3) \\ &= \alpha \circ F k \circ F c_n && k \text{ homomorphism} \\ &= \alpha \circ F(k \circ c_n) && F \text{ functor} \\ &= \alpha \circ F \alpha_n && \text{inductive hypothesis} \\ &= \alpha_{n+1} && \text{definition of } \alpha_{n+1}. \end{aligned}$$

This concludes the proof. □

Remark 3.1.6. Looking at the proof of Theorem 3.1.5 we notice that it isn't really necessary for $\mathscr{A}$ to have colimits of *all* ω-chains or for F to preserve *all* of them as we care for these properties only when applied to the initial-algebra ω-chain. However in practice it may be more convenient to check the general properties rather than the particular case.

We will now look at some examples of functors to which one can apply Adámek's Theorem. We urge the reader to notice that the Theorem does not only guarantee the existence of an initial algebra, but also that such algebra is a colimit. By looking at the diagram in question (the initial-algebra ω-chain) we can often build an intuition of what the initial algebra is "internally".

Remark 3.1.7. We shall prove that in a category $\mathscr{A}$ with binary coproducts, a terminal object 1 and colimits of ω-chains, the functor $FX = X + 1$ preserves colimits of ω-chains.

Consider an ω-chain

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \xrightarrow{a_3} \cdots \tag{3.1.4}$$

and let $(A, \alpha_i : A_i \to A)_{i<\omega}$ be its colimit. We want to prove that $(A + 1, \alpha_i + \mathsf{id}_1)_{i<\omega}$ is the colimit of

$$A_0 + 1 \xrightarrow{a_0 + \mathsf{id}_1} A_1 + 1 \xrightarrow{a_1 + \mathsf{id}_1} A_2 + 1 \xrightarrow{a_2 + \mathsf{id}_1} A_3 + 1 \xrightarrow{a_3 + \mathsf{id}_1} \cdots . \tag{3.1.5}$$

By functoriality of F we have immediately that it is a cocone on (3.1.5) so we only need to show it is universal. Let $(B, \beta_i : A_i + 1 \to B)_{i<\omega}$ be a cocone on (3.1.5) and note that for all $i < \omega, \beta_i = [\beta_i^1, \beta_i^2]$ with $\beta_i^1 : A_i \to B$ and $\beta_i^2 : 1 \to B$. By composition with the coprojections of the $A_i + 1$'s we obtain that

1. $(B, \beta_i^1)_{i<\omega}$ is a cocone on (3.1.4);
2. $(B, \beta_i^2)_{i<\omega}$ is a cocone on

$$1 \xrightarrow{\mathsf{id}_1} 1 \xrightarrow{\mathsf{id}_1} 1 \xrightarrow{\mathsf{id}_1} 1 \xrightarrow{\mathsf{id}_1} \cdots . \tag{3.1.6}$$

Now (3.1.4) has colimit $(A, \alpha_i)_{i<\omega}$ and (3.1.6) has colimit $(1, \gamma_i = \mathsf{id}_1)_{i<\omega}$ so we obtain unique arrows $m^1 : A \to B$ and $m^2 : 1 \to B$ such that

$$m^1 \circ \alpha_i = \beta_i^1 \quad \text{and} \quad m^2 \circ \mathsf{id}_1 = \beta_i^2 \quad \text{for all } i < \omega.$$

Finally we set $m = [m^1, m^2]$ and this is a factorization arrow for $(B, \beta_i)_{i<\omega}$ as the following calculation shows

$$\begin{aligned} m \circ (\alpha_i + \mathsf{id}_1) &= [m^1, m^2] \circ (\alpha_i + \mathsf{id}_1) \\ &= [m^1 \circ \alpha_i, m^2 \circ \mathsf{id}_1] \\ &= [\beta_i^1, \beta_i^2] \\ &= \beta_i. \end{aligned}$$

We only need to show that such $m : A + 1 \to B$ is unique. Let $n : A + 1 \to B$ be such that $n \circ (\alpha_i + \mathsf{id}_1) = \beta_i$ for all $i < \omega$. Since n is an arrow out of a coproduct we have $n = [n^1, n^2]$ with $n^1 : A \to B, n^2 : 1 \to B$. Then from the factorization condition we obtain $[n^1 \circ \alpha_i, n^2 \circ \mathsf{id}_1] = [\beta_i^1, \beta_i^2]$ and, by composing with the coprojects, $n^1 \circ \alpha_i = \beta_i^1$ and $n^2 \circ \mathsf{id}_1 = \beta_i^2$ for all $i < \omega$. But now we must have $n^1 = m^1$ and $n^2 = m^2$ because m^1, m^2 are unique. Finally $n = m$ and F preserves colimits of ω-chains.

By a similar argument one can show that all polynomial functors preserve colimits of ω-chains.

Example 3.1.8. In a category with binary coproducts and an initial object 1 the initial algebra for $FX = X + 1$, if it exists, is called the **natural numbers object** (or **NNO**) of the category. Notice that a NNO, being an initial algebra, is unique so the use of "the" is justified. Explicitly a NNO is a pair $(N, \iota : N + 1 \to N)$ such that for every other pair $(A, \alpha : A + 1 \to A)$ there is a unique $h : N \to A$ such that

natural numbers object

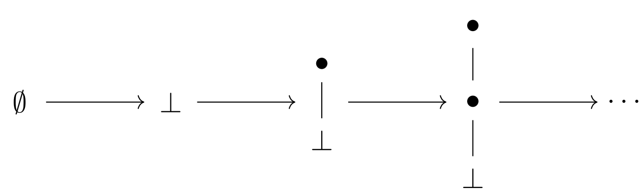
$$\begin{tikzcd} N + 1 \arrow[r, "\iota"] \arrow[d, "h+\mathsf{id}_1"] & N \arrow[d, "h"] \\ A + 1 \arrow[r, "\alpha"] & A \end{tikzcd}$$

commutes. By Adámek's Theorem the NNO is the colimit of the ω-chain

$$0 \longrightarrow 1 \longrightarrow 1+1 \longrightarrow 1+1+1 \longrightarrow \cdots .$$

So if $\mathscr{A} = \mathsf{Set}$ the chain above becomes the sequence of natural numbers, where each $n \in \mathbb{N}$ is regarded as the set $\{0, \ldots, n-1\}$, and N will thus be exactly $\mathbb{N}$; hence the name "natural numbers object".

Example 3.1.9. On Pos, the category of partially ordered sets and order-preserving functions, consider the functor $FX = X_\perp$. Its initial-algebra ω-chain is



So the n-th element of the ω-chain can be thought of as the set $n = \{0, \ldots, n-1\}$ with the natural order upon it while the connecting morphisms are inclusions. The colimit is $\mathbb{N}$ with the standard ordering.

Definition 3.1.10. A function $f : M \to M'$ between metric spaces $(M, d_M), (M', d_{M'})$ is **non-expanding** if $d_{M'}(f(x), f(y)) \leq d_M(x, y)$ for all $x, y \in M$. A metric space (M, d_M) is 1-bounded if $d_M(x, y) \leq 1$ for all $x, y \in M$. We denote by MS the category of 1-bounded metric spaces and non-expanding maps .

non-expanding map

MS

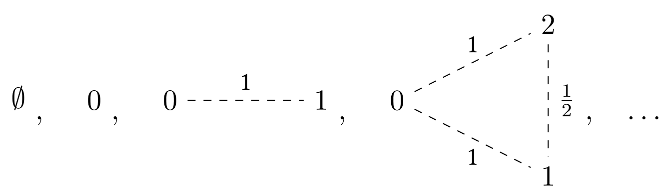
Remark 3.1.11. Coproducts in MS are disjoint unions where each summand retains its metric and points in different summands have distance 1. The product of $M, M' \in \mathsf{MS}$ is the cartesian product $M \times M'$ endowed with the maximum metric

$$d_{M\times M'}((x, x'), (y, y')) = \max\{d_M(x, y), d_{M'}(x', y')\}.$$

Moreover MS has a terminal object $1 = \{*\}$ i.e. the singleton with its unique metric.

Example 3.1.12. We can now consider $FX = X + 1$ on MS. The objects of the initial-algebra ω-chain are again the natural numbers $n = \{0, \dots, n-1\}$, but this time regarded as discrete spaces (that is: spaces where all points have distance 1) and the connecting arrows are the inclusions. Thus the colimit, and initial algebra of F, is $\mathbb{N}$ endowed with the discrete metric.

Now consider the functor $FX = \frac{1}{2}X + 1$ where we denote by $\frac{1}{2}X$ the space obtained by scaling the metric of X by a factor of $\frac{1}{2}$. This functor preserves colimits of ω-chains by Remark 3.1.7, mutatis mutandis. Its initial-algebra ω-chain in the following sequence of spaces



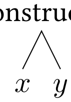
where 0 always denotes the new point added by the functor. The metric on $F^n 0$ is

$$d(i, j) = \begin{cases} 0 & \text{if } i = j \\ 2^{-\min\{i,j\}} & \text{otherwise} \end{cases}$$

and the connecting morphisms are (non-expanding) inclusions. We then obtain that μF is again $\mathbb{N}$, but this time endowed with the metric d above.

In the previous examples we have seen how Adámek's Theorem allows us to look at "finite pieces" of the initial algebra. The next one shows that we can use it also to understand how the structure on the initial algebra works.

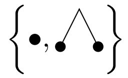
Example 3.1.13. Consider $FX = X \times X + 1$ on Set. This functor preserves colimits of ω-chains and we know from Example 2.2.12 that μF is the algebra of all finite binary trees. We will now apply Adámek's Theorem to obtain a "constructive" reason for why this is so.

If we denote the pair $(x, y) \in X \times X$ by the tree  we have that

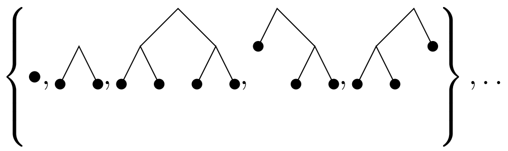
$$F^0 0 = \emptyset,$$

$$F^1 0 = \emptyset \times \emptyset + 1 = 1 = \{\bullet\},$$

$$F^2 0 = 1 \times 1 + 1 =$$



$$F^3 0 = (1 \times 1 + 1) \times (1 \times 1 + 1) + 1 =$$



so $F^n 0$ is the set of binary trees of height less than n. The connecting arrows $F^n 0 \to F^{n+1} 0$ are inclusions and so the colimit of the chain, and thus the initial algebra, is the union $\bigcup_{n<\omega} F^n 0$ i.e. the set of all finite binary trees.

Recall (again from Example 2.2.12) that the structure $\iota : \mu F \times \mu F + 1 \to \mu F$ is given by tree-tupling. This information too can be obtained by Adámek's Theorem. Recall from the proof that, if $(\mu F, c_n : F^n 0 \to \mu F)_{i<\omega}$ is the colimit of the initial-algebra ω-chain, then ι is the unique arrow such that

$$c_n = \iota \circ F c_{n-1} \quad \text{for all } 0 < n < \omega \tag{3.1.7}$$

and, moreover, the c_n's are inclusions. Now fix $0 < k < \omega$ and consider a tree $\overset{\wedge}{t_1 \; t_2}$ in $F^k 0$ that we also regard as the pair (t_1, t_2) for $t_1, t_2 \in F^{k-1} 0$. We have

$$\begin{aligned} c_k\Big(\overset{\wedge}{t_1 \; t_2}\Big) &= \overset{\wedge}{t_1 \; t_2} \in \mu F \\ \iota \circ F c_{k-1}((t_1, t_2)) &= \iota\Big(\underbrace{(t_1, t_2)}_{\in \mu F \times \mu F + 1}\Big) \end{aligned}$$

so by (3.1.7) ι is indeed tree-tupling.

Example 3.1.14. For a finite signature Σ the associated polynomial functor H_Σ preserves colimits of ω-chains so we can build μF, the set of finite Σ-trees, as the colimit of

$$0 \longrightarrow H_\Sigma 0 \longrightarrow H_\Sigma^2 0 \longrightarrow H_\Sigma^3 0 \longrightarrow \cdots.$$

Now $H_\Sigma^n 0$ can be regarded as the set of finite Σ-trees of height less than n while the connecting arrows of the chain are inclusions; thus $\mu F = \bigcup_{n<\omega} H_\Sigma^n 0$ i.e. μF is the set of all finite Σ-trees. By an argument similar to that of Example 3.1.13 one can show that the structure $\iota : H_\Sigma \mu F \to \mu F$ is really tree-tupling, as we know from Proposition 2.2.11.

Example 3.1.15. The finite power-set functor $\mathcal{P}_\mathsf{f} \colon \mathsf{Set} \to \mathsf{Set}$ preserves colimits of ω-chains so it has an initial algebra $\mu\mathcal{P}_\mathsf{f}$ that is the colimit of the following ω-chain

$$\emptyset \longrightarrow \{\emptyset\} \longrightarrow \{\emptyset, \{\emptyset\}\} \longrightarrow \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \longrightarrow \cdots$$

where the arrows are inclusions. We thus obtain that $\mu F = \bigcup_{n<\omega} \mathcal{P}_\mathsf{f}^n 0 = V_\omega$; moreover notice that $\mathcal{P}_\mathsf{f} V_\omega$ is precisely V_ω. Now let $(V_\omega, c_n : V_n \to V_\omega)_{i<\omega}$ be the colimit cocone; we know that $c_n = \iota \circ \mathcal{P}_\mathsf{f} c_{n-1}$ for all $0 < n < \omega$. Fix $0 < n < \omega$ and consider $\{A_1, \ldots, A_k\} \in V_n$; we have

$$\mathcal{P}_\mathsf{f} c_{n-1}(\{A_1, \ldots, A_k\}) = \{c_{n-1}(A_1), \ldots, c_{n-1}(A_k)\} = \{A_1, \ldots, A_k\}$$

because $c_{n-1} : V_{n-1} \to V_\omega$ is an inclusion. Then

$$\iota \circ \mathcal{P}_{\mathrm{f}} c_{n-1}(\{A_1, \dots, A_k\}) = c_n(\{A_1, \dots, A_k\}) \Rightarrow \iota(\{A_1, \dots, A_k\}) = \{A_1, \dots, A_k\}$$

so we obtain $\iota = \mathsf{id}_{V_\omega}$.

3.2 Algebraically Complete Categories

Definition 3.2.1. A category $\mathscr{A}$ is **algebraically complete** if every endofunctor on it has an initial algebra.

algebraically complete category

Remark 3.2.2. By the Knaster-Tarski Theorem (see Theorem 2.2.21) every complete lattice is an algebraically complete category.

The following Theorem shows that, even for categories with very little structure (the existence of products is enough), being algebraically complete is a very strong condition. The theorem was first proven by Adámek and Koubek in 1979; we present a neat proof by Freyd, published in 1991.

Theorem 3.2.3. Let $\mathscr{A}$ be an algebraically complete category with products. Then $\mathscr{A}$ is a preorder.

Proof. Towards a contradiction suppose $\mathscr{A}$ is not a preorder i.e. there is some hom-set $\mathscr{A}(A, B)$ that has at least two elements. Consider the endofunctor

$$F : \mathscr{A} \xrightarrow{\mathscr{A}(A,-)} \mathsf{Set} \xrightarrow{\mathsf{Set}(-,2)} \mathsf{Set}^{\mathrm{op}} \xrightarrow{B^{(-)}} \mathscr{A}$$

where $B^{(-)} : \mathsf{Set}^{\mathrm{op}} \to \mathscr{A}$ is the functor that takes a set M to the object

$$\underbrace{B \times \dots \times B}_{|M| \text{ times}}.$$

We will write $FD = B^{S(D)}$ for $S(D) = 2^{\mathscr{A}(A,D)}$. Note that F has no fixed point. Indeed assume $D \cong FD$ then we have

$$\begin{aligned} \mathscr{A}(A, D) &\cong \mathscr{A}(A, FD) && \text{functors preserve isomorphisms} \\ &\cong \mathscr{A}(A, B^{S(D)}) \\ &\cong \mathscr{A}(A, B)^{S(D)} && \text{hom-functor preserves limits} \end{aligned}$$

but the cardinality of the right hand-side is at least $2^{S(D)} = 2^{2^{\mathscr{A}(A,D)}}$ that it impossible by Cantor's Theorem. Hence, by Lambeck's Lemma (see Lemma 2.2.17), $\mathscr{A}$ is not algebraically complete; contradiction. □

Chapter 4

Initial algebras from transfinite iteration

4.1 Colimits of chains

Through this chapter we regard ordinals $j \in \mathrm{Ord}$ as linear orders and thus as categories. A j-chain in some category $\mathscr{A}$ is a functor $C : j \to \mathscr{A}$; we denote the elements of the chain by C_i, for $i \in j$, and its arrows by $c_{i,i'} : C_i \to C_{i'}$ for $i \leq i' < j$. We say that $\mathscr{A}$ has colimits of chains if for all $j \in \mathrm{Ord}$ every j-chain has a colimit; this includes the existence of an initial object (as the colimit of the empty chain) but does not include colimits of Ord-chains. j-chain

Lemma 4.1.1. Let j be an ordinal and $C : j \to \mathscr{A}$ a j-chain into a category $\mathscr{A}$ with colimits of chains. Now let $j' \subseteq j$ be cofinal in j i.e. such that for every $i \in j$ there is some $i' \in j'$ such that $i \leq i'$ and denote by C' the restriction of C to j'. Then C and C' have the same colimit.

Proof. Notice that there is a canonical bijection between cocones on C and cocones on C'. Given a cocone $(A, \alpha_i : C_i \to A)_{i\in j}$ on C throwing away all the indexes not in j' yields $(A, \alpha_i : C_i \to A)_{i\in j'}$: a cocone on C'. Conversely given $(A, \alpha_i : C_i \to A)_{i\in j'}$ we need to define $\alpha_i : C_i \to A$ for all $i \in j \setminus j'$; but there's only one way to do so. Indeed by cofinality let $i' \in j'$ be such that $i \leq i'$; then by the cocone condition we must set $\alpha_i = \alpha_{i'} \circ c_{i,i'}$ so $(A, \alpha_i : C_i \to A)_{i\in j'}$ can be extended to the whole chain in only one way.

Let $(L, \sigma_i : C_i \to L)_{i\in j}$ be the colimit of C; then $(L, \sigma_i : C_i \to L)_{i\in j'}$ is the colimit of C'. Indeed let $(A, \alpha_i : C_i \to A)_{i\in j'}$ be a cocone on C' that we extend to a cocone on C as explained above. Now, since L is the colimit, let $m\colon L \to A$ be the unique factorization arrow such that

$$m \circ \sigma_i = \alpha_i \quad \text{for all } i \in j.$$

Particularly for all $i \in j'$ so m is a factorization of $(A, \alpha_i : C_i \to A)_{i\in j'}$ through $(L, \sigma_i : C_i \to L)_{i\in j'}$. To prove it is unique let $n\colon L \to A$ be such that $n \circ \sigma_i = \alpha_i$ for all $i \in j'$; we will show that n actually works for all $i \in j$ and thus must be m by uniqueness. Let $i \in j \setminus j'$ and $i' \in j'$ such that $i \leq i'$ by cofinality, then we have

$$n \circ \sigma_i = n \circ \sigma_{i'} \circ c_{i,i'} = \alpha_{i'} \circ c_{i,i'} = \alpha_i.$$

For the converse let $(L, \sigma_i : C_i \to L)_{i\in j'}$ be the colimit of C'. One can prove that its unique extension $(L, \sigma_i : C_i \to L)_{i\in j}$ is the colimit of C by an argument similar to the one above. Finally we have that C and C' have the same colimit (modulo restriction/extension) and, moreover, the factorization arrows are the same. □

Thanks to this lemma we can safely calculate colimits of chains, or apply Proposition 1.1.8, on convenient cofinal subchains. This will allow us to "skip" limit ordinals because given j limit the subset $\{i \in j : i \text{ successor}\}$ is cofinal in j.

4.2 Transfinite iteration

The main result of this section is again a generalization to the categorical setting of an order-theoretic result; this time from Zermeleo. Let P be a chain-complete poset and f :

$P \to P$ an order-preserving map. We can define an Ord-indexed sequence by transfinite recursion as follows

$$
\begin{aligned}
f^0(\bot) &= \bot && \text{base case;}\\
f^{j+1}(\bot) &= f(f^j(\bot)) && \text{for all ordinals } j;\\
f^j(\bot) &= \bigvee_{i<j} f^i(\bot) && \text{for limit ordinals } j.
\end{aligned}
$$

This sequence is clearly a chain.

Theorem 4.2.1 (Zermeleo). Let P be a chain-complete poset. Then every order-preserving $f : P \to P$ has a lest fixed point μf and $\mu f = f^j(\bot)$ for some $j \in \mathsf{Ord}$.

Proof. By Hartog's Lemma (see [Gol96, Theorem 8.18] for a proof) there is an ordinal i such that there is no injection $i \rightarrowtail P$. Thus there are $j < k < i$ such that $f^j(\bot) = f^k(\bot)$ and we have $f^j(\bot) \le f^{j+1}(\bot) \le f^k(\bot)$ because $(f^j(\bot))_{j\in\mathsf{Ord}}$ is a chain. So $f(f^j(\bot)) = f^{j+1}(\bot) = f^j(\bot)$ hence $f^j(\bot)$ is a fixed point.

Now let $x = f(x)$ be another fixed point. By transfinite induction we can prove $x \ge f^i(\bot)$ for all $i \in \mathsf{Ord}$ (similarly to what we did in Theorem 3.1.1) thus when $i = j$ we obtain $x \ge f^j(\bot)$. This proves that $f^j(\bot)$ is indeed the least fixed point of f. $\square$

In order to generalize this theorem and its proof we begin by defining a trasfinite version of the initial-algebra ω-chain. This will be a functor $W : \mathsf{Ord} \to \mathscr{A}$ into a category $\mathscr{A}$ with colimits of chains. It is defined by transfinite recursion as follows.

$$
\begin{aligned}
W_0 &= 0 && \text{base case;}\\
W_{j+1} &= FW_j && \text{for all ordinals } j;\\
W_j &= \operatorname*{colim}_{i<j} W_i && \text{for all limit ordinals } j.
\end{aligned}
$$

As for arrows $w_{0,1}$ is the unique arrow $0 \to W_1 = F0$ and $w_{i+1,j+1} = Fw_{i,j}$ for all $i, j \in \mathsf{Ord}$. When j is limit we have $W_j = \operatorname{colim}_{i<j} W_i$ so we let $\{w_{i,j}\}_{i<j}$ be the arrows forming the colimit cocone. We are only missing $w_{j,j+1}$ for j limit. Such arrows are obtained using the universal property of W_j. First notice that we can leave out the initial object 0 and all the limit ordinals without changing the colimit i.e.

$$
W_j = \operatorname*{colim}_{i<j} W_i = \operatorname*{colim}_{i\,<\,j\text{ successor}} W_i
$$

by Lemma 4.1.1. Now $(Fw_{i,j} : FW_i = W_{i+1} \to FW_j = W_{j+1})_{i<j}$ is a cocone on this reduced diagram. Thus we obtain a unique factorization arrow $W_j \to W_{j+1}$ that we declare to be our $w_{j,j+1}$ so the definition of our transfinite chain can continue. Notice also that, being the factorization arrow, $w_{j,j+1}$ is such that

$$
w_{j,j+1} \circ w_{i+1,j} = Fw_{i,j} \quad \text{for all } i < j. \tag{4.2.1}
$$

We now generalize Remark 3.1.3 to the transfinite case.

Remark 4.2.2. Let (A, α) be an F-algebra; then it induces a cocone $(\alpha_i : W_i \to A)_{i \in \mathsf{Ord}}$ on W as follows.

$$\begin{aligned} \alpha_0 &= ! && \text{base case;} \\ \alpha_{i+1} &= \alpha \circ F\alpha_i && \text{for all ordinals i;} \end{aligned}$$

for limit ordinals j we define $\alpha_j : W_j \to A$ to be the unique factorization arrow such that $\alpha_j \circ w_{i,j} = \alpha_i$ for all $i < j$. In order for this construction to work we need to prove that, at stage j, $(\alpha_i : W_i \to A)_{i<j}$ is a cocone i.e. that

$$\alpha_i \circ w_{k,i} = \alpha_k \tag{4.2.2}$$

for all $k < i < j$. Assuming that all α_i's for $i < j$ have been constructed we have that:

1. for $w_{0,1}$ condition (4.2.2) is trivial;
2. for $w_{k,i}$ with i limit it is trivial by construction of α_i;
3. for $w_{i,i+1}$ with i limit we can prove (4.2.2) holds by showing

   $$\alpha_{i+1} \circ w_{i,i+1} \circ w_{k,i} = \alpha_i \circ w_{k,i} = \alpha_k$$

   for all $k < i$ and conclude by uniqueness of α_i. It is enough to prove the relation for all $k < i$ successors by Lemma 4.1.1 and indeed we have

   $$\begin{aligned} \alpha_{i+1} \circ w_{i,i+1} \circ w_{k,i} &= \alpha \circ F\alpha_i \circ w_{i,i+1} \circ w_{k,i} && \text{definition of } \alpha_{i+1} \\ &= \alpha \circ F\alpha_i \circ Fw_{k-1,i} && (4.2.1) \\ &= \alpha \circ F(\alpha_i \circ w_{k-1,i}) && \text{functoriality of } F \\ &= \alpha \circ F\alpha_{k-1} && \text{definition of } \alpha_i \\ &= \alpha_k && \text{definition of } \alpha_k. \end{aligned}$$

4. for $w_{i,i+1}$ with i non-limit we proceed inductively like in Remark 3.1.3; the base case is provided by item 1 if i is a natural number and by item 3 otherwise.

Finally we conclude that $(\alpha_i : W_i \to A)_{i \in \mathsf{Ord}}$ is well defined and a cocone on W since, as we just proved, for every j limit $(\alpha_i : W_i \to A)_{i<j}$ is a cocone.

Definition 4.2.3. We say that the inital-algebra chain **converges** in λ steps if $w_{\lambda,\lambda+1}$ is an isomorphism and that it converges in **exactly** λ steps when λ is the least such ordinal.

convergence of the initial-algebra chain

We are now ready to state and prove the transfinite version of Adámek's Theorem.

Theorem 4.2.4. Let $\mathscr{A}$ be a category with colimits of chains and F an endofunctor. If the initial-algebra chain of F converges in j steps then W_j is the initial algebra μF with structure given by

$$(w_{j,j+1})^{-1} : W_{j+1} = FW_j \to W_j.$$

Proof. We only need to prove that W_j with the algebra structure given above is initial. To this end let (A, α) be an F-algebra and let h be the arrow $\alpha_j : W_j \to A$ of the cocone induced by (A, α) in the sense of Remark 4.2.2. We shall prove that h is the unique F-algebra homomorphism from W_j to A.

$$\begin{tikzcd}
W_{j+1} = FW_j \arrow[r, "(w_{j,j+1})^{-1}"] \arrow[d, "Fh = F\alpha_j"'] \arrow[dr, "\alpha_{j+1}"] & W_j \arrow[d, "h = \alpha_j"] \\
FA \arrow[r, "\alpha"] & A
\end{tikzcd}$$

In the above diagram the lower left triangle commute by construction of α_{j+1} and the upper right one because the α_i's form a cocone. Thus the outer square commutes i.e. h is an homomorphism.

To prove that h is unique assume that $k : W_j \to A$ is another homomorphism. We will show by transfinite induction that $h \circ w_{i,j} = k \circ w_{i,j}$ for all $i \leq j$ and then conclude by choosing $i = j$. Notice that $h \circ w_{i,j} = \alpha_j \circ w_{i,j} = \alpha_i$ so we are really interested in proving

$$\alpha_i = k \circ w_{i,j} \tag{4.2.3}$$

for all $i \leq j$. If $i = 0$ then (4.2.3) is obvious because both sides are arrows out of the initial object. If we have $\alpha_i = k \circ w_{i,j}$ then

$$\begin{aligned}
\alpha_{i+1} &= \alpha \circ F\alpha_i && \text{definition of } \alpha_i \\
&= \alpha \circ F(k \circ w_{i,j}) && \text{inductive hypothesis} \\
&= \alpha \circ Fk \circ w_{i+1,j+1} && \text{fuctoriality of } F \\
&= k \circ (w_{j,j+1})^{-1} \circ w_{i+1,j+1} && k \text{ is an homomorphism} \\
&= k \circ w_{i+1,j}.
\end{aligned}$$

Finally if i is limit we know α_i is the unique arrow such that $\alpha_i \circ w_{s,i} = \alpha_s$ for all $s < i$; so if we show that $k \circ w_{i,j}$ does the same then we have (4.2.3). Indeed for $s < i$ we have

$$\begin{aligned}
k \circ w_{i,j} \circ w_{s,i} &= k \circ w_{s,j} \\
&= h \circ w_{s,j} && \text{inductive hypothesis since } s < i \\
&= \alpha_j \circ w_{s,j} \\
&= \alpha_s && \text{the } \alpha_i\text{'s are a cocone.}
\end{aligned}$$

This concludes the proof. □

Corollary 4.2.5. Let $\mathscr{A}$ be a category with colimits of chains and F an endofuctor. If F preserves colimits of λ-chains for some limit ordinal λ then the initial algebra chain of F converges in λ steps. Therefore we have $\mu F = W_\lambda$.

Proof. Since $W_\lambda = \mathrm{colim}_{i<\lambda} W_i$ and F preservers colimits of λ-chains we have that $FW_\lambda = \mathrm{colim}_{i<\lambda} FW_i$ with colimit cocone $(Fw_{i,\lambda} = w_{i+1,\lambda+1} : W_{i+1} \to W_{\lambda+1} = FW_\lambda)_{i<\lambda}$. We thus obtain a unique morphism $\iota : W_{\lambda+1} = FW_\lambda \to W_\lambda$ such that $\iota \circ Fw_{i,\lambda} = w_{i+1,\lambda}$ for all $i < \lambda$ (similarly to the proof of Theorem 3.1.5). Combining this universal property with that of $w_{\lambda,\lambda+1}$ (see 4.2.1) we obtain the following two identities for all $i < \lambda$.

$$\iota \circ Fw_{i,\lambda} = w_{i+1,\lambda} \implies \iota \circ w_{\lambda,\lambda+1} \circ w_{i+1,\lambda} = \mathsf{id}_{W_\lambda} \circ w_{i+1,\lambda}$$
$$w_{\lambda,\lambda+1} \circ w_{i+1,\lambda} = Fw_{i,\lambda} \implies w_{\lambda,\lambda+1} \circ \iota \circ Fw_{i,\lambda} = \mathsf{id}_{W_{\lambda+1}} \circ Fw_{i,\lambda}$$

And, by Proposition 1.1.8, $\iota \circ w_{\lambda,\lambda+1} = \mathsf{id}_{W_\lambda}$ and $w_{\lambda,\lambda+1} \circ \iota = \mathsf{id}_{W_{\lambda+1}}$ so the inital-algebra chain converges in λ steps. We conclude by Theorem 4.2.4. □

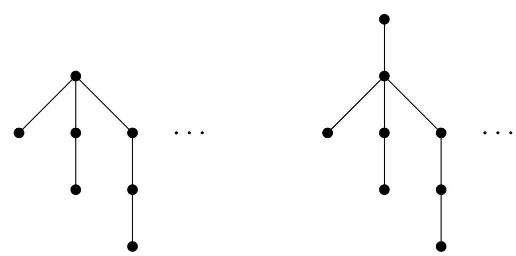
Clearly for $\lambda = \omega$ we obtain Adámek's Theorem. We now give an example of a functor that has initial-algebra convergent in more than ω steps; showing that Theorem 4.2.4 is more powerful than Adámek's Theorem.

well-founded tree

Example 4.2.6. Trees with no infinite paths are called **well-founded**. When a tree t is well-founded we define its **height** by

$$h(t) = \sup\{h(t') + 1 : t' \quad \text{maximal proper subtree of } t\}.$$

Thus finite trees (that are always well-founded) have height equal to the maximum length of a root-to-leaf path. For other trees (such as the ones below) the height is an infinite ordinal.

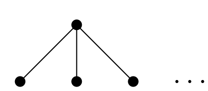


ω $\omega + 1$

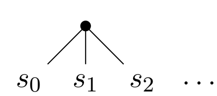
Consider now the functor $FX = X^\omega + 1$ on Set. Its initial-algebra chain is given by

$$W_i = \{\text{all countably branching trees } t \text{ such that } h(t) < i\}$$

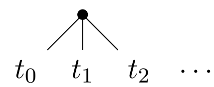
and the connecting arrows are inclusions. Indeed $W_0 = \emptyset$, $W_1 = \{t_0\}$ where t_0 is the root-only tree (the unique tree of height 0), $W_2 = W_1^\omega + 1$ contains the root-only tree, of height 0, and the unique countably branching tree



of height 1. Generally $W_{i+1} = (W_i)^\omega + 1$ gives us that the elements of W_{i+1} are the root-only tree and all trees



where $s_n \in W_n$ for every $n < \omega$. Since the connecting maps $w_{i,i+1}$ are inclusions at limit steps we take unions so, for example, W_ω is the set of all countably branching trees of height less than ω i.e. all countably branching trees of finite heightd. However the chain does not stop at ω because one can take a sequence $t_0, t_1, t_2, \dots$ of trees with heights cofinal in ω; then the tree



(4.2.4)

has height ω i.e. it is an element of $W_{\omega+1} \setminus W_\omega$. Indeed for every countable ordinal $i < \omega_1$, by the same argument, one has that $W_i \subset W_{i+1}$. The chain then converges at the first uncountable ordinal ω_1 because, given a sequence of trees $t_0, t_1, t_2, \dots$ such that $t_n \in W_{i_n}$ for $i_n < \omega_1$ then the tree (4.2.4) has height $\sup\{i_0, i_1, i_2, \dots\}$ that is a countable ordinal, and thus $t \in W_{\omega_1}$. So the initial-algebra μF is the set of all well-founded countable branching trees.

4.3 Smooth monomorphisms and a converse to Lambeck's Lemma

Are all initial algebras obtainable by iteration i.e. by the convergence of the initial-algebra chain?

We recall what subobjects are.

Definition 4.3.1. For an object A in a category $\mathscr{A}$ a **subobject** is represented by a monomorphism $m : B \to A$. If we have two monomorphisms $m_B : B \to A, m_C : C \to A$ we write $m_B \leq m_C$ if there is some $k : B \to C$ such that $m_B = m_C \circ k$ (notice that this k too is mono). When we have $m_B \leq m_C \leq m_B$ the k above is an isomorphism and we say m_B and m_C represent the same subobject. Thus we can consider the collection of equivalence classes of subobjects of A that is partially ordered by $\leq$. If $\mathscr{A}$ is such that every object $A \in \mathscr{A}$ has only a set of subobjects then $\mathscr{A}$ is called **well-powered** .

subobject

well-powered category

We can restrict ourselves to a class $\mathcal{M}$ of monomorphism and speak of $\mathcal{M}$-subobjects i.e. subobjects represented by mono in $\mathcal{M}$. When a given object A has only a set of $\mathcal{M}$-subobjects we write $\mathrm{Sub}_{\mathcal{M}}(A, \leq)$ for the poset of $\mathcal{M}$-subobjects of A. However we do not require the monomorphism that witness $\leq$ to be in $\mathcal{M}$.

From here onwards let $\mathcal{M}$ denote a class of monomorphisms (of some generic category $\mathscr{A}$ or otherwise clear from the context) containing all isomorphisms and closed under composition.

Definition 4.3.2. We say that an object A has **smooth $\mathcal{M}$-subobjects** if $\mathrm{Sub}_{\mathcal{M}}(A, \leq)$ is a chain-complete poset (particularly it is a set and not a proper class) where joins of chains are given by colimits of the corresponding chain of subobjects. We say that the class $\mathcal{M}$ itself is **smooth** if every object has smooth $\mathcal{M}$-subobjects. When the class of all monomorphisms of a given category is smooth we say that the category has **smooth monomorphisms**.

smooth $\mathcal{M}$-subobjects

smooth class of monomorphisms

Remark 4.3.3. To be more precise consider an ordinal λ and $m_i : A_i \to A$ for $i < \lambda$ be a chain of $\mathcal{M}$-subobjects i.e. we have arrows $a_{i,j} : A_i \to A_j$ (that need not be in $\mathcal{M}$) that witness $m_i \leq m_j$ for $i \leq j < \lambda$. By definition if A has smooth $\mathcal{M}$-subobjects then there is an $\mathcal{M}$-subobject $m : C \to A$ that is the join of our chain in $\mathrm{Sub}_{\mathcal{M}}(A, \leq)$. Particularly we have that $m_i \leq m$ for all $i < \lambda$ so there are monomorphisms $c_i : A_i \to C$ such that $m \circ c_i = m_i$ for all $i < \lambda$. For A to really have smooth $\mathcal{M}$-subobjects it is moreover necessary that these $c_i : A_i \to C$ form a colimit cocone.

Remark 4.3.4. Given an object A with smooth $\mathcal{M}$-subobjects then $\mathrm{Sub}_{\mathcal{M}}(A, \leq)$ must have a join of the empty chain. Such join has to be the colimit of the empty chain in the category, thus the category must have an initial object 0. Moreover the unique arrow $0 \to A$ must be in $\mathcal{M}$. We conclude that for a category to have a smooth class of mono $\mathcal{M}$ it is necessary for it to have an initial object and all arrows out of such initial object must be in $\mathcal{M}$.

Example 4.3.5. Set has smooth monomorphisms.

Example 4.3.6. The category of monoids has smooth monomorphisms. Indeed it is well-powered, so $\mathrm{Sub}(M, \leq)$ is always a set, and we know that, given a chain of submonoids of a given monoid M, the union of the chain (i.e. its colimit) is again a submonoid of M and clearly the join of the chain in $\mathrm{Sub}(M, \leq)$. Moreover the trivial monoid $\{*\}$ is the initial object of the category.

Similarly the categories of groups and vector spaces have smooth monomorphisms.

Example 4.3.7. On the other hand monomorphisms are not smooth in the category of rings. Indeed its initial object is $\mathbb{Z}$ and there are non-monomorphisms with domain $\mathbb{Z}$ (e.g. $\mathbb{Z} \to \mathbb{Z}_n$ for some $n \in \mathbb{N}$) so by Remark 4.3.3 the category of rings does not have smooth monomorphisms. One can also notice that having smooth monomorphisms implies that every object of the category must have the initial object as a subobject; this clearly does not happen with rings (again think of $\mathbb{Z}_n$).

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