Notes on Initial Algebras

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Contents

Chapter 1

Preliminaries

1.1 Limits and colimits

Through this section let $\mathscr{A}, \mathscr{D}$ be categories. We recall what limits and colimits are, some significant examples (particularly products and coproducts) and some of their basic properties. We also introduce some related notation. For detailed proofs see [Bor08, Chapter 2].

Definition 1.1.1. Given a functor $F\colon \mathscr{D} \to \mathscr{A}$ a **cone** on F is an object $C \in \mathscr{A}$ and a family of arrows $(\pi_D\colon C \to FD)_{D\in\mathscr{D}}$ such that for every arrow $d\colon D_1 \to D_2$ of $\mathscr{D}$ we have

cone

$$Fd \circ \pi_{D_1} = \pi_{D_2}. \tag{1.1.1}$$

The arrows π_D are called **projections** of the cone, the object C the **vertex**.

projection, vertex

Definition 1.1.2. Given a functor $F\colon \mathscr{D} \to \mathscr{A}$ a **limit** of F is a cone $(L, (\pi_D)_{D\in\mathscr{D}})$ such that for every other cone $(C, (\rho_D)_{D\in\mathscr{D}})$ there is a unique arrow $m\colon C \to L$ such that

limit

$$\rho_D = \pi_D \circ m \quad \text{for every } D \in \mathscr{D}.$$

Proposition 1.1.3. When a functor F has a limit that limit is unique (up to isomorphism).

The following proposition is a way of proving equality of two arrows into a product. It is most useful when working in abstract categories where arrows are not (generally) functions.

Proposition 1.1.4. Let $(L, (\pi_D)_{D\in\mathscr{D}})$ be the limit of F and consider two arrows $f, g\colon C \to L$. If for every $D \in \mathscr{D}, \pi_D \circ f = \pi_D \circ g$ then $f = g$.

The definitions of cone and of limit are dualized to yield those of cocone and colimit.

Definition 1.1.5. Given a functor $F\colon \mathscr{D} \to \mathscr{A}$ a **cocone** on F is an object $C \in \mathscr{A}$ and a family of arrows $(\sigma_D\colon FD \to C)_{D\in\mathscr{D}}$ such that for every arrow $d\colon D_1 \to D_2$ of $\mathscr{D}$ we have

cocone

$$\sigma_{D_2} \circ Fd = \sigma_{D_1}. \tag{1.1.2}$$

The arrows σ_D are called **coprojections** of the cocone.

coprojection

Definition 1.1.6. Given a functor $F\colon \mathscr{D} \to \mathscr{A}$ a **colimit** of F is a cocone $(L, (\sigma_D)_{D\in\mathscr{D}})$ such that for every other cocone $(C, (\tau_D)_{D\in\mathscr{D}})$ there is a unique arrow $m\colon L \to C$ such that

colimit

$$\tau_D = m \circ \sigma_D \quad \text{for every } D \in \mathscr{D}.$$

Propositions 1.1.3 and 1.1.4 are dualized as follows.

Proposition 1.1.7. When a functor F has a colimit that colimit is unique (up to isomorphism).

Proposition 1.1.8. Let $(L, (\sigma_D)_{D\in\mathscr{D}})$ be the colimit of F and consider two arrows $f, g\colon L \to C$. If for every $D \in \mathscr{D}, f \circ \sigma_D = g \circ \sigma_D$ then $f = g$.

Products and coproducts

We now turn to two particular classes of (co)limits: products and coproducts. Recall that a discrete category is a category that has no non-identity arrow.

Definition 1.1.9. Given a functor $F\colon \mathscr{D} \to \mathscr{A}$ where $\mathscr{D}$ is some discrete category a limit of F is called a **product** while a colimit a **coproduct**. product, coproduct

Notation 1.1.10. Notice that to give a functor from a discrete category to $\mathscr{A}$ is equivalent to picking an element of $\mathscr{A}$ for every element of $\mathscr{D}$. We thus speak of the "product of A and B" for $A, B \in \mathscr{A}$ without explicit reference to any functor. Moreover we denote the product of A and B in $\mathscr{A}$, when it exists, by $A \times B$. Similarly we speak of the coproduct of two elements of $\mathscr{A}$ and denote it by $A + B$ when it exists.

Proposition 1.1.11. In a category, when the interested (co)products exists, we have that

1. $A \times B$ is isomorphic to $B \times A$;
2. $A + B$ is isomorphic to $B + A$;
3. $(A \times B) \times C$ is isomorphic to $A \times (B \times C)$;
4. $(A + B) + C$ is isomorphic to $A + (B + C)$.

In light of this proposition we write $A \times B \times C$ and $A + B + C$ with no parenthesis and similarly for any finite number of factors or addenda.

Notation 1.1.12. When we take the (co)product of an infinite number of objects of $\mathscr{A}$ we use the notation $\prod_{i \in I} A_i$ (for products) and $\coprod_{i \in I} A_i$ (for coproducts). Notice that the order of the factors/addenda does not matter.

Notation 1.1.13. Let $(L = \prod_{i \in I} A_i, (\pi_i)_{i \in I})$ be a product in $\mathscr{A}$. Then by Proposition 1.1.4 there is a one-to-one correspondance between arrows $f\colon C \to L$ and cones $(C, (\rho_i\colon C \to A_i)_{i \in I})$. Moreover any collection of arrows $(\rho_i\colon C \to A_i)_{i \in I}$ satisfies condition (1.1.1) so it is a cone. This means that to give an arrow $f\colon C \to L$ is equivalent to giving a family of functions $(\rho_i\colon C \to A_i)_{i \in I}$ so we write

$$f = \langle \rho_1, \rho_2, \rho_3, \ldots \rangle; \tag{1.1.3}$$

having chosen a convenient ordering on I.

By a dual argument we obtain that arrows $f\colon L \to C$ out of a coproduct $L = \coprod_{i \in I} A_i$ are in one-to-one correspondence with families of arrows $(\tau_i\colon A_i \to C)$ and write

$$f = [\tau_1, \tau_2, \tau_3, \ldots] \tag{1.1.4}$$

for a convenient ordering on I.

Notation 1.1.14. Let A, B, C, D be objects of $\mathscr{A}$ such that $A \times B, C \times D$ exist and let $f\colon A \to C, g\colon B \to D$ be arrows. With reference to the following diagram we denote $\langle f \circ \pi_1, g \circ \pi_2 \rangle$ by $f \times g$. Dually if $A + B, C + D$ exist we denote $[\sigma_1 \circ f, \sigma_2 \circ g]$ by $f + g$. $f \times g, f + g$

$$
\begin{tikzcd}
A \arrow[r, "f"] & C \\
A \times B \arrow[u, "\pi_1"] \arrow[r, dashed, "f\times g"] \arrow[d, "\pi_2"'] & C \times D \arrow[u] \arrow[d] \\
B \arrow[r, "g"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & C \arrow[d, "\sigma_1"] \\
A + B \arrow[r, dashed, "f+g"] & C + D \\
B \arrow[u] \arrow[r, "g"] & D \arrow[u, "\sigma_2"']
\end{tikzcd}
$$

The next proposition shows how the notation introduced above for products interacts with the composition. It is most useful for performing calculations; we will use it without explicit reference particularly in the proof of the Primitive Recursion Theorem 2.3.1.

Proposition 1.1.15. Consider the following two diagrams.

$$
\begin{tikzcd}
 & A \arrow[r, "g_1"] & D \\
C \arrow[ur, "f_1"] \arrow[r, "\langle f_1, f_2\rangle"] \arrow[dr, "f_2"'] & A \times B \arrow[u] \arrow[r, "g_1\times g_2"] \arrow[d] & D \times E \arrow[u] \arrow[d] \\
 & B \arrow[r, "g_2"] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A \\
D \arrow[r, "g"] & C \arrow[ur, "f_1"] \arrow[r, "\langle f_1, f_2\rangle"] \arrow[dr, "f_2"'] & A \times B \arrow[u] \arrow[d] \\
 & & B
\end{tikzcd}
$$

We have that

1. $(g_1 \times g_2) \circ (f_1, f_2) = (g_1 \circ f_1, g_2 \circ f_2)$;
2. $(f_1, f_2) \circ g = (f_1 \circ g, f_2 \circ g)$.

Initial and terminal objects

Definition 1.1.16. An **initial object** for a category $\mathscr{A}$ is an object $0 \in \mathscr{A}$ such that for every other object $A \in \mathscr{A}$ there is exactly one arrow from 0 to $\mathscr{A}$. Dually a **terminal object** for $\mathscr{A}$ is an object $1 \in \mathscr{A}$ such that for every other object $A \in \mathscr{A}$ there is exactly one arrow from A to 1.

initial object

terminal object

Remark 1.1.17. Initial objects, when they exist, are colimits of the functor from the empty category into $\mathscr{A}$; as such they are all isomorphic and we speak of *the* initial object. Dually terminal objects are limits of the functor from the empty category and are unique as well.

(Co)limit-preserving functors

Definition 1.1.18. A functor $G\colon \mathscr{A} \to \mathscr{B}$ **preserve limits** if, for every small category $\mathscr{D}$ and functor $F\colon \mathscr{D} \to \mathscr{A}$, when the limit of F exists the limit of $G \circ F$ is obtained by applying G to the limit cone. That is: if $(L, (\pi_D)_{D\in\mathscr{D}})$ is the limit of F then $(GL, (G\pi_D)_{D\in\mathscr{D}})$ is the limit of $G \circ F$. As expected G is said to preserve colimits when, for every small category $\mathscr{D}$ and functor $F\colon \mathscr{D} \to \mathscr{A}$, when the colimit of F exists then the colimit of $G \circ F$ is the image through G of the cocone.

limit preservation

By limiting the definition above to a specific small category $\mathscr{D}$ we obtain the notion of a functor that preserves $\mathscr{D}$-(co)limits. We will be interested in functors that preserve ω-(co)limits where ω is the set of natural numbers with the usual ordering regarded as a category.

1.2 Other notations

Notation 1.2.1. When we are working with a category $\mathscr{A}$ that has enough structure [1] we will often define endofunctors $F\colon \mathscr{A} \to \mathscr{A}$ by their action on objects only such as

$$FX = X \times X + 1.$$

This means that the functor F sends objects A to $A \times A + 1$ (where 1 is the terminal object of $\mathscr{A}$) and arrows f to $f \times f + \mathsf{id}_1$.

[1] i.e. that has all the (co)limits needed for our functor to work.

Chapter 2

Algebras for endofunctors

2.1 F-algebras

Though this section fix an arbitrary category $\mathscr{A}$ that we call the **base category** and an endofunctor $F\colon \mathscr{A} \to \mathscr{A}$.

Definition 2.1.1. An **algebra** for F (or an F**-algebra**) is a pair (A, α) where A is an object of $\mathscr{A}$ (that we call the **carrier** or **base object**) and α is an arrow of type $FA \to A$ (that we call the **structure** of the algebra). F-algebra

As one should expect F-algebras form a category of their own.

Definition 2.1.2. Given F-algebras (A, α) and (B, β) a **morphism** between them is an arrow f of $\mathscr{A}$ such that

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Fh"] & A \arrow[d, "f"] \\ FB \arrow[r, "\beta"] & B \end{tikzcd}$$

commutes.

Remark 2.1.3. Given morphisms of algebras $f\colon (A, \alpha) \to (B, \beta)$ and $g\colon (B, \beta) \to (C, \gamma)$ their composition is exactly $g \circ f$; one can easily check that $g \circ f$ makes the diagram above commute using that f and g are morphisms of F-algebras and that F is a functor. The identity on an algebra (A, α) is exactly id_A.

We call the category of F-algebras and morphisms of F-algebras $\mathsf{Alg}F$. We will explore some of the properties of this category in Section **??**. $\mathsf{Alg}F$

Example 2.1.4. In Set let 1 be the terminal object $\{*\}$ and consider the functor

$$FX = X + 1$$

To give an arrow $\alpha\colon A+1 \to A$ is equivalent to giving two functions $\alpha_0\colon 1 \to A, \alpha_1\colon A \to A$ because α is an arrow out of a coproduct. We also write $\alpha = [\alpha_1, \alpha_0]$. So an algebra for F is set A with a constant α_0 and a unary operation α_1.

A morphism of F-algebras is a function f such that

$$\begin{tikzcd} A + 1 \arrow[r, "{[\alpha_1,\alpha_0]}"] \arrow[d, "f+\mathsf{id}_1"] & A \arrow[d, "f"] \\ B + 1 \arrow[r, "{[\beta_1,\beta_0]}"] & B \end{tikzcd}$$

commutes. This translates, using Proposition 1.1.8, into the following two conditions

$$f \circ \alpha_1 = \beta_1 \circ f, \quad f \circ \alpha_0 = \beta_0.$$

Which together give that f preserves the constant and commutes with the unary operation of the algebras.

Example 2.1.5. Similarly algebras for the endofunctor $FX = X \times X + 1$ are sets equipped with a constant and a binary operation. Morphisms of such algebras preserve the constant and commute with the operation. Indeed if $(A, a, \otimes_A)$ and $(B, b, \otimes_B)$ are two F-algebras a morphism $f\colon A \to B$ is a function such that

1. $fa = b$;
2. $f(x \otimes_A y) = f(x) \otimes_B f(y)$ for all $x, y \in A$.

This is reminiscent of structures such as groups and monoids.

Example 2.1.6. Let Σ_1 be a set; then algebras for $FX = \Sigma_1 \times X$ are sets with a unary operation for every element $\sigma \in \Sigma_1$. Indeed let A be an F-algebra and let $\alpha\colon \Sigma_1 \times A \to A$ be its structure; for a fixed $\sigma \in \Sigma_1$ we have that $\alpha(\sigma, -)$ is a unary operation on A. Morphisms of F-algebras in this case commute with all such unary operations.

Examples 2.1.4, 2.1.5, 2.1.6 can be generalized with the introduction of polynomial functors over Set. This captures the idea of a Σ-algebra from universal algebra and provides a generalization to other base categories with sufficient structure.

Definition 2.1.7. A **signature** Σ is a collection of sets $(\Sigma_n)_{n<\omega}$ where Σ_n is called the set of n-ary symbols. The symbols in Σ_0 are also called constant symbols (or symbols for constants). signature

Definition 2.1.8. Given a signature Σ a Σ**-algebra** is a set A together with interpretations for every symbol of Σ. That is: to every $\sigma \in \Sigma_0$ we associate a fixed element of A and to every $\sigma \in \Sigma_n$ an n-ary function on A. We denote the interpretation of σ in A by σ^A. Σ-algebra

Given two Σ-algebras A and B a **morphism** of Σ-algebras is a function $f\colon A \to B$ such that

$$f\sigma^A = \sigma^B \tag{2.1.1}$$

for every $\sigma \in \Sigma_0$ and

$$f(\sigma^A(x_1, \ldots, x_n)) = \sigma^B(f(x_1), \ldots, f(x_n)) \tag{2.1.2}$$

for every $\sigma \in \Sigma_n$ and $x_1, \ldots, x_n \in A$.

We will now observe how Σ-algebras can be realized as algebras for specific endofunctors. To every signature we associate a **polynomial functor** $H_\Sigma\colon \mathsf{Set} \to \mathsf{Set}$ that operates as follows on objects X and arrows f. polynomial functor

$$\begin{aligned} H_\Sigma X &= \coprod_{n<\omega} \Sigma_n \times X^n \\ H_\Sigma f &= \coprod_{n<\omega} \mathsf{id}_{\Sigma_n} \times \underbrace{f \times \ldots \times f}_{n \text{ times}} \end{aligned}$$

Then if A is a Σ-algebra we can define functions $\alpha_n\colon \Sigma_n \times A^n \to A$ naturally as

$$\alpha_n(\sigma, x_1, \ldots, x_n) \coloneqq \sigma^A(x_1, \ldots, x_n) \tag{2.1.3}$$

and then $\alpha := [\alpha_0, \alpha_1, \ldots]$ makes A into a H_Σ-algebra. Conversely if we have a H_Σ-algebra (A, α) we can define an interpretation of every symbol by reading 2.1.3 the other way around.

For morphisms consider the following square that we know commutes if f is a morphism of H_Σ-algebras.

$$\begin{tikzcd} \coprod_{n<\omega} \Sigma_n \times A^n \arrow[r, "\alpha"] \arrow[d, "H_\Sigma f"] & A \arrow[d, "f"] \\ \coprod_{n<\omega} \Sigma_n \times B^n \arrow[r, "\beta"] & B \end{tikzcd}$$

Since the diagonal of the square is an arrow out of a coproduct we know from Proposition 1.1.8 that $f \circ \alpha$ and $\beta \circ H_\Sigma f$ are equal when composed with the coprojections of the cocone and this can happen if and only if conditions 2.1.1 and 2.1.2 hold.

We now turn to continous algebras.

Definition 2.1.9. A **complete partial order** (or a **CPO**) is a poset $(A, \sqsubseteq)$ in which all ω-chains

complete partial order

$$a_1 \sqsubseteq a_2 \sqsubseteq a_3 \sqsubseteq \ldots$$

have a join i.e. a least upper bound. A function $f : (A, \sqsubseteq) \to (B, \sqsubseteq)$ is **continous** if

continous function

1. it is monotone i.e. $a \sqsubseteq b$ in A implies $f(a) \sqsubseteq f(b)$ in B;
2. if $\overline{a}$ is the join of $a_1 \sqsubseteq a_2 \sqsubseteq \ldots$ in A then $f(\overline{a})$ is the join of $f(a_1) \sqsubseteq f(a_2) \sqsubseteq \ldots$ in B.

Note that by 1 any chain in A is preserved by f and since B is a CPO it must have a join; condition 2 forces that join to be the image of the join of the original chain under f.

Remark 2.1.10. CPOs and continous functions form a category CPO. In CPO (co)products are formed as in Set and have, for products, the pointwise order and, for coproducts, each component keeps its own ordering and elements in different components are never comparable. Moreover CPO has a terminal object 1 that is the singleton $\{*\}$ with the only possible partial order on it.

Example 2.1.11. Consider the functor $FX = X + 1$ on CPO. Algebras for F are CPOs with a constant and a continous unary operation. Similarly algebras for the functor $FX = X \times X + 1$ are CPOs with a constant and a continous binary operation. The continuity of the operation (which we denote here as $\otimes$) in this case gives us that if

$$a_1 \sqsubseteq a_2 \sqsubseteq a_3 \sqsubseteq \ldots \quad \text{and} \quad b_1 \sqsubseteq b_2 \sqsubseteq b_3 \sqsubseteq \ldots$$

are chains with joins respectively $\overline{a}$ and $\overline{b}$ then $\overline{a} \otimes \overline{b}$ is the join of

$$a_1 \otimes b_1 \sqsubseteq a_2 \otimes b_2 \sqsubseteq a_3 \otimes b_3 \sqsubseteq \ldots$$

Example 2.1.12. Now let $FX = X_\perp$ be the functor that adds to a CPO a bottom element (i.e. an element $\perp$ such that $\perp \sqsubseteq x$ for every x in the poset) and that sends a continous function $f\colon X \to Y$ to $f_\perp\colon X_\perp \to Y_\perp$ defined as

$$f_\perp(x) = \begin{cases} f(x) & \text{if } x \neq \perp \\ \perp & \text{if } x = \perp \end{cases}.$$

An algebra (A, α) for F has a unary continous function $\alpha_1\colon A \to A$ and a constant $\alpha_\perp$ such that $\alpha_\perp \sqsubseteq \alpha(a)$ for every $a \in A$ by monotony of α. Comparing this to Example 2.1.11 we now have a condition on our constant.

Finally let $\mathsf{CPO}_\perp$ be the category of CPOs with a bottom element and continous functions that preserve the bottom i.e. such that $f\perp = \perp$. We call such functions **strict continous functions**.

strict continous function

Remark 2.1.13. In $\mathsf{CPO}_\perp$ products work as in CPO (the bottom element is naturally the one whose components are all $\perp$) but the construction of coproducts mush be changed. Indeed coproducts in $\mathsf{CPO}_\perp$ are formed as in CPO but the bottoms of all the components are unified into a single element. The initial object is still the singleton as it clearly has a bottom.

Remark 2.1.14. Notice that the functor $FX = X + 1$ on $\mathsf{CPO}_\perp$ is the identity functor so its algebras are CPOs with a bottom and a strict continous unary operation. If we want a constant on top of the unary operation we can consider the functor $FX = X_\perp + 1_\perp$, $Ff = f_\perp + \mathrm{id}_{1_\perp}$. Indeed if (A, α) is an F-algebra then $\alpha = \left(\begin{smallmatrix} \alpha_1 & \alpha_0 \end{smallmatrix} \right)$ with $\alpha_1\colon A_\perp \to A$ and $\alpha_0\colon 1_\perp \to A$ are strict continous functions. Indeed α_0 picks an element of A: we need the domain to be $1_\perp$ because otherwise we would be forced to pick the bottom of A as the function is strict. Similarly α_1 gives a continous unary operation on A (not necessarily a strict one!) as adding a new bottom to A "frees" the first one from being preserved.

2.2 Initial Algebras

Definition 2.2.1. An **initial algebra** for an endofunctor F is an initial object in $\mathsf{Alg}F$. When an initial algebra exists we denote it by $(\mu F, \iota)$.

initial algebra

Remark 2.2.2. If 0 is the initial object of $\mathscr{A}$ and F preserves initial objects then $(0, \mathrm{id}_0)$ is the initial algebra for F.

We will now look at some non-trivial examples of initial algebras.

Example 2.2.3. Consider the functor $FX = X + 1$ on Set. The initial algebra for F is $(\mathbb{N}, \iota)$ where $\iota = [\iota_1, \iota_0]$ with $\iota_0 = 0$ and ι_1 the successor function on the naturals. Indeed let (B, β) be another algebra with $\beta = [\beta_1, \beta_0]$ and consider the function $f\colon \mathbb{N} \to B$ defined as

$$\begin{cases} f(0) & = \beta_0 \\ f(n+1) & = \beta_1(f(n)) \end{cases}. \tag{2.2.1}$$

This is a morphism of algebras:

1. $f(0) = \beta_0$;

2. $(f \circ \iota_1)(n) = f(n+1) = (\beta_1 \circ f)(n)$ for all $n \in \mathbb{N}$ so $f \circ i_1 = \beta_1 \circ f$;

and clearly unique because any other morphism must satisfy 1 and 2, thus a simple induction gives us that is must be f.

Example 2.2.4. Let $\mathcal{P}_\mathsf{f} \colon \mathsf{Set} \to \mathsf{Set}$ be the finite power-set functor. This functor sends a set X to $\mathcal{P}_\mathsf{f}X = \{Z \subseteq X \colon Z \text{ is finite}\}$ and a function $f \colon X \to Y$ to the function $\mathcal{P}_\mathsf{f}f$ that sends a finite subset of X to its (obviously finite) f-image in Y. Let V_ω be the set of hereditarily finite sets. That is: the set containing the empty set and all finite sets whose elements are hereditarily finite sets as well. We will see in Section ?? that (V_ω, id) is the initial algebra of $\mathcal{P}_\mathsf{f}$.

finite power-set functor

Example 2.2.5. Let $\mathscr{A}$ be a category with countable coproducts. If $A \in \mathscr{A}$ we write $\mathbb{N} \bullet A$ for the coproduct of A with itself countably many times. We shall now consider the functor $FX = X + A$ for which we prove the initial algebra to be $\mathbb{N} \bullet A$.

First we need an algebra structure ι on $\mathbb{N} \bullet A$. If $\mathsf{in}_k \colon A \to \mathbb{N} \bullet A$ is the k-th coprojection then we set

$$\iota = [\alpha_1, \mathsf{in}_0] \colon (\mathbb{N} \bullet A) + A \to \mathbb{N} \bullet A \tag{2.2.2}$$

where α_1 is obtained from the universal property of the coproduct applied to the cone $(\mathsf{in}_k)_{k \geq 1}$. We thus have that the following triangles commute for every $k \in \mathbb{N}$

$$\begin{tikzcd} \mathbb{N} \bullet A \arrow[rr, "\alpha_1"] & & \mathbb{N} \bullet A \\ & A \arrow[ul, "\mathsf{in}_k"] \arrow[ur, "\mathsf{in}_{k+1}"'] & \end{tikzcd} .$$

Now let (B, β) with $\beta = [\beta_1, \beta_0]$ be an algebra and f a morphism from $(\mathbb{N} \bullet A, \iota)$. We have that the following square commutes.

$$\begin{tikzcd} \mathbb{N} \bullet A + A \arrow[r, "{[\alpha_1, \mathsf{in}_0]}"] \arrow[d, "f + \mathsf{id}"] & \mathbb{N} \bullet A \arrow[d, "f"] \\ B + A \arrow[r, "{[\beta_1, \beta_0]}"] & B \end{tikzcd}$$

So any such f must be such that

1. $f \circ \mathsf{in}_0 = \beta_0$;

2. $f \circ \alpha_1 \circ \mathsf{in}_k = f \circ \mathsf{in}_{k+1} = \beta_1 \circ f \circ \mathsf{in}_k$ for all $k \in \mathbb{N}$.

This gives us that $[\beta_0, \beta_1 \circ \beta_0, \beta_1 \circ \beta_1 \circ \beta_0, \ldots]$ is the unique morphism from $(\mathbb{N} \bullet A, \iota)$ to (B, β) which proves our claim.

Finally we describe initial algebras for polynomial functors in two equivalent ways: using closed terms and using trees.

Definition 2.2.6. Let Σ be a signature. A **closed term** for Σ is a string t of symbols of Σ such that: closed term

1. t is a constant symbol;
2. t is of the form $\sigma(t_1, \ldots, t_n)$ where $\sigma \in \Sigma_n$ and $t_1, \ldots, t_n$ are closed terms.

Remark 2.2.7. Let μH_Σ be the set of all closed terms; this is naturally a Σ-algebra:

1. if σ is a contant symbol $\sigma^{\mu H_\Sigma}$ is σ itself but regarded as a term;
2. if σ is an n-ary symbol then it defines an n-ary operation on μH_Σ that takes terms $t_1, \ldots, t_n$ to the term $\sigma(t_1, \ldots, t_n)$.

Remark 2.2.8. Let A be a Σ-algebra and t a closed term. We can define an evaluation function $\mathsf{ev}\colon \mu H_\Sigma \to A$ as follows:

1. if t is a constant symbol σ then $\mathsf{ev}(t) = \sigma^A$;
2. if t is of the form $\sigma(t_1, \ldots, t_n)$ then $\mathsf{ev}(t) = \sigma^A(\mathsf{ev}(t_1), \ldots, \mathsf{ev}(t_n))$.

Clearly ev is a morphism and moreover it is unique so μH_Σ is the initial algebra for H_Σ.

Definition 2.2.9. Given a signature Σ a **Σ-tree** is an ordered tree where every node of k children is labelled by a k-ary symbol of Σ.

Remark 2.2.10. Every n-ary symbol of Σ defines an n-ary operation on Σ-trees which takes n trees to the tree obtained by connecting each root to a new node labelled by σ, which becomes the root of a new tree. We call this operation **tree-tupling**. Note that the order of the trees matters. tree-tupling

Proposition 2.2.11. The initial algebra μH_Σ is the algebra of finite Σ-trees with tree-tupling.

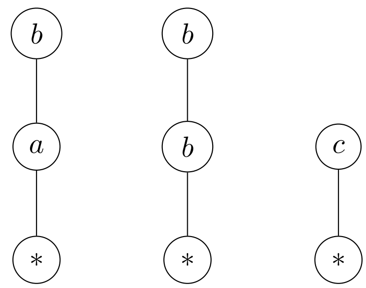
Proof. Let T be the algebra of finite Σ-trees with tree-tupling; we shall find an isomorphism $f\colon \mu H_\Sigma \to T$ of algebras. Define f by structural recursion as follows

1. if t is a constant term σ let $f(t)$ be the tree formed by a single node labelled by σ;
2. if t is a term of the form $\sigma(t_1, \ldots, t_n)$ let $f(\sigma)$ be the tree obtained by tree-tupling $f(t_1), \ldots, f(t_n)$ with the new root labelled by σ.

This function is a morphism by definition and has an inverse, defined similarly by structural recursion, which is again a morphism. □

Example 2.2.12. By the discussion above the initial algebra for the functor $FX = X \times X + 1$ on Set is the algebra of all finite binary trees.

Example 2.2.13. Consider the functor $FX = B \times X + 1$ on Set with $B \in$ Set. Algebras for F are sets with a constant and $|B|$ unary operations. By the discussion above we know that μF is the algebra of finite trees for the signature $\Sigma = (\Sigma_0 = \{*\}, \Sigma_1 = B)$ so its elements are "linear" trees such as



with $a, b, c \in B$. We immediately deduce that μF can also be realized as the set of words over B.

Remark 2.2.14. Notice the importance of constant symbols. Indeed if a polynomial functor on Set has no constant then it preserves the initial object 0 so by Remark 2.2.2 the initial algebra is trivial. Equivalently one can observe that, without constant symbols, the set of closed terms is empty.

Example 2.2.15. Consider $FX = X_\perp$ on CPO$_\perp$. Let $\mathbb{N}^\top$ be the set of the natural numbers with an added topmost element ∞ ordered naturally. Notice that $(\mathbb{N}^\top)_\perp$ is isomorphic (as an order) to $\mathbb{N}^\top$ and consider then the successor function $s\colon (\mathbb{N}^\top)_\perp \to \mathbb{N}^\top$ with $s(\infty) = \infty$. We claim this is the initial algebra for F.

Let (A, α) be another F-algebra and $f\colon (\mathbb{N}^\top, s) \to (A, \alpha)$ a morphism. Because f is an arrow of CPO$_\perp$ it must be strict so $f(0) = \perp$ and because it is a morphism we must have $f(s(n)) = \alpha(f(n))$; this defines f inductively on $\mathbb{N}$. Finally we recall that f must be continous (as an arrow of CPO$_\perp$) so

$$f(\infty) = f\left(\bigvee_{n<\omega} n\right) = \bigvee_{n<\omega} (f(n)).$$

This implies that there is really a unique morphism from $(\mathbb{N}^\top, s)$ to (A, α) so $(\mathbb{N}^\top, s)$ is initial.

TODO: functor $FX = (X \times X)_\perp + 1_\perp$.

We conclude this section with the a classic lemma of Lambeck's which gives necessary conditions for functors to have an initial algebra.

Definition 2.2.16. A **fixed point** of an endofunctor F is an element $A \in \mathscr{A}$ that is isomorphic to FA.

Lemma 2.2.17 (Lambeck's Lemma). An initial algebra for F is always a fixed point.

Proof. Let $(\mu F, \iota)$ be the initial algebra of F. Notice that $(F(\mu F), F\iota)$ is also an algebra and so there is a unique morphism $f\colon (\mu F, \iota) \to (F(\mu F), F\iota)$. We have that the following diagram commutes

$$\begin{tikzcd}
F(\mu F) \arrow[r, "\iota"] \arrow[d, "Ff"] & \mu F \arrow[d, "f"] \\
F(F(\mu F)) \arrow[r, "F\iota"] \arrow[d, "F\iota"] & F(\mu F) \arrow[d, "\iota"] \\
F(\mu F) \arrow[r, "\iota"] & \mu F
\end{tikzcd}$$

so $\iota \circ f$ is an endomorphism of algebras on the initial algebra; hence it is $\mathsf{id}_{\mu F}$. Now

$$f \circ \iota = F\iota \circ Ff = F(\iota \circ f) = F(\mathsf{id}_{\mu F}) = \mathsf{id}_{F(\mu F)}$$

so ι is an isomorphism. This shows that μF is a fixed point of F. □

Remark 2.2.18. We know from Cantor's Theorem that there is no surjection from a given set X into its power set $\mathcal{P}X$ so the functor $\mathcal{P}$ on Set has no fixed point hence no initial algebra.

Lambeck's lemma can be seen as a generalization of the following order-theoretic lemma.

Definition 2.2.19. Let $(P, \sqsubseteq)$ be a poset and $f\colon P \to P$ a monotone function. An element $x \in P$ is a **pre-fixed point** of f if $f(x) \sqsubseteq x$.

Lemma 2.2.20. Given a poset $(P, \sqsubseteq)$ and a monotone function $f\colon P \to P$ let $A = \{x \in P\colon f(x) \sqsubseteq x\}$ be the set of pre-fixed points of f. If $\overline{a}$ is the meet of A then $\overline{a}$ is the least fixed point of f.

Proof. If $x \in A$ then $\overline{a} \sqsubseteq x$ and $f(\overline{a}) \sqsubseteq f(x) \sqsubseteq x$ follows from monotony of f and definition of A. This shows that $f(\overline{a})$ is a lower bound for A so $f(\overline{a}) \sqsubseteq \overline{a}$. Now by applying f again we obtain $f(f(\overline{a})) \sqsubseteq f(\overline{a})$ which gives us $f(\overline{a}) \in A$ and thus $\overline{a} \sqsubseteq f(\overline{a})$. Moreover $\overline{a}$ must be the least fixed point of f since all fixed points are pre-fixed points. □

This lemma then gives us the following well-known theorem for free.

Theorem 2.2.21 (Knaster-Tarski Theorem). Let $(L, \sqsubseteq)$ be a complete lattice. Then every monotone function of L has a fixed point.

Proof. Let f be a monotone function on L and A be the set of pre-fixed points of f. It must have a meet $\overline{a}$ because the lattice is complete so by the Lemma we have a fixed point. □

The generalization of order theoretic results such as Lemma 2.2.20 will be a recurrent theme. Indeed we can see orders as categories in the usual way, order-preserving functions as functors and pre-fixed points as algebras for these functors.

2.3 Recursion and Induction

> TODO: recursively specified morphisms, example, catamorphism?

Theorem 2.3.1 (Primitive Recursion). Assume the base category $\mathscr{A}$ to have finite products and let F be an endofunctor with an initial algebra μF. Then for every $\alpha\colon F(A \times \mu F) \to A$ there is a unique $h\colon \mu F \to A$ such that the following square commutes.

$$\begin{tikzcd} F(\mu F) \arrow[r, "\iota"] \arrow[d, "F\langle h, \mathsf{id}_{\mu F}\rangle"] & \mu F \arrow[d, "h"] \\ F(A \times \mu F) \arrow[r, "\alpha"] & A \end{tikzcd} \tag{2.3.1}$$

Proof. Let π_1, π_2 be the projections of the product $A \times \mu F$ and consider the arrow

$$\overline{\alpha}\colon F(A \times \mu F) \xrightarrow{\langle \mathsf{id}_{F(A\times\mu F)}, F\pi_2\rangle} F(A \times \mu F) \times F(\mu F) \xrightarrow{\alpha\times\iota} A \times \mu F.$$

This gives an algebra structure on $A \times \mu F$ so let $\overline{h}\colon \mu F \to A \times \mu F$ be the unique algebra homomorphism gi en by the initiality of μF.

$$\begin{tikzcd} F(\mu F) \arrow[rrr, "\iota"] \arrow[d, "F\overline{h}"] & & \arrow[d, phantom, "(1)"] & \mu F \arrow[d, "\overline{h}"] \\ F(A \times \mu F) \arrow[r, "\langle \mathsf{id}_{F(A\times\mu F)}, F\pi_2\rangle"] \arrow[d, "F\pi_2"] & F(A \times \mu F) \times F(\mu F) \arrow[rr, "\alpha\times\iota"] \arrow[dl, "\pi_2"] & \arrow[d, phantom, "(2)"] & A \times \mu F \arrow[d, "\pi_2"] \\ F(\mu F) \arrow[rrr, "\iota"] & & & \mu F \end{tikzcd} \tag{2.3.2}$$

(3)

The outer square in 2.3.2 commutes because (1), (2) and (3) do. Indeed (1) commutes because $\overline{h}$ is an algebra homomorphism, (2) by Notation 1.1.14 and (3) because of Notation 1.1.13. By functoriality of F we then have that $\pi_2 \circ \overline{h}$ is an endomorphism on the initial algebra thus $\pi_2 \circ \overline{h} = \mathsf{id}_{\mu F}$.

Now set $h \coloneqq \pi_1 \circ \overline{h}$ so $\overline{h} = \langle h, \mathsf{id}_{\mu F}\rangle$. Extending (1) by π_1 we obtain the following diagram that we know commutes.

$$\begin{tikzcd} F(\mu F) \arrow[rr, "\iota"] \arrow[d, "F\overline{h}=F\langle h,\mathsf{id}_{\mu F}\rangle"] & & \mu F \arrow[d, "\overline{h}"] \arrow[dr, "h"] & \\ F(A \times \mu F) \arrow[r, "\langle \mathsf{id}_{F(A\times\mu F)}, F\pi_2\rangle"] & F(A \times \mu F) \times F(\mu F) \arrow[r, "\alpha\times\iota"] & A \times \mu F \arrow[r, "\pi_1"] & A \end{tikzcd}$$

But notice that

$$\pi \circ (\alpha \times \iota) \circ \langle \mathsf{id}_{F(A\times\mu F)}, F\pi_2\rangle = \pi_1 \circ \langle \alpha, \iota \circ F\pi_2\rangle = \alpha.$$

so we have 2.3.1.

For uniqueness consider $h\colon \mu F \to A$ homomorphism of algebras such that 2.3.1 commutes. We claim that $\overline{h} = \langle h, \mathsf{id}_{\mu F}\rangle$ so that $h = \pi_1 \circ \overline{h}$; proving uniqueness. In order to do it we show that $\langle h, \mathsf{id}_{\mu F}\rangle$ is an algebra homomorphism and conclude it must be $\overline{h}$ by initiality of μF. Indeed we have

$$
\begin{aligned}
\pi_1 \circ (\alpha \times \iota) \circ \langle \mathsf{id}_{F(A\times\mu F)}, F\pi_2\rangle \circ F\langle h, \mathsf{id}_{\mu F}\rangle &= \alpha \circ \mathsf{id}_{F(A\times\mu F)} \circ F\langle h, \mathsf{id}_{\mu F}\rangle \\
&= h \circ \iota \\
&= \pi_1 \circ \langle h, \mathsf{id}_{\mu F}\rangle \circ \iota;
\end{aligned}
$$

$$
\begin{aligned}
\pi_2 \circ (\alpha \times \iota) \circ \langle \mathsf{id}_{F(A\times\mu F)}, F\pi_2\rangle \circ F\langle h, \mathsf{id}_{\mu F}\rangle &= \iota \circ F\pi_2 \circ F\langle h, \mathsf{id}_{\mu F}\rangle \\
&= \iota \circ F(\pi_2 \circ \langle h, \mathsf{id}_{\mu F}\rangle) \\
&= \iota \circ F\mathsf{id}_{\mu F} \\
&= \iota \circ \mathsf{id}_{F(\mu F)} \\
&= \iota \\
&= \pi_2 \circ \langle h, \mathsf{id}_{\mu F}\rangle \circ \iota;
\end{aligned}
$$

so we conclude by Proposition 1.1.4. $\square$

Chapter 3

Initial algebras from finitary iteration

3.1 Adámek's Theorem

We begin this section by recalling Kleene's Fixed Point Theorem and its proof with the goal of generalizing it to the categorical setting.

Theorem 3.1.1 (Kleene's Fixed Point Theorem). Let A be a CPO with bottom $\perp$; then every continous function $F\colon A \to A$ has a least fixed point $\mu F = \sup_{n<\omega} F^n(\perp)$[1].

Proof. Consider the ω-chain $\perp \leq F(\perp) \leq F^2(\perp) \leq \dots$ and let μF be its join. By continuity $F(\mu F) = \bigvee_{n<\omega} F(F^n(\perp))$ but we know that $\bigvee_{n<\omega} F^n(\perp) = \bigvee_{n<\omega} F^{n+1}(\perp)$ so $\mu F = F(\mu F)$, a fixed point.

Now let $F(x) \leq x$ be a pre-fixed point. As $\perp \leq x$ we have $F(\perp) \leq F(x) \leq x$ and by induction we obtain that $F^n(\perp) \leq x$ for all $n < \omega$ so x. But this shows that x is an upper bound so $\mu F \leq x$ by definition and, finally, μF must be the least fixed point because it is less than any pre-fixed point and fixed points are trivially pre-fixed points. □

Unsurprisingly we shall replace CPOs with bottom by categories with an initial obejct and endofunctions by

> TODO: write something decent here.

Definition 3.1.2. The **initial-algebra ω-chain** of an endofunctor F is the diagram

initial-algebra ω-chain

$$0 \xrightarrow{\;!\;} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} F^3 0 \xrightarrow{F^3!} \cdots \tag{3.1.1}$$

where we denote by ! the unique arrow out of the initial object. This diagram can be realized as a functor from ω regarded as a category.

Remark 3.1.3. Let (A, α) be an F-algebra. Then it induces a canonical cocone $(\alpha_n\colon F^n 0 \to A)_{n<\omega}$ on (3.1.1) by

$$\begin{aligned} \alpha_0 &= !; \\ \alpha_{n+1} &= FF^n 0 \xrightarrow{F\alpha_n} FA \xrightarrow{\alpha} A. \end{aligned}$$

To check that the α_ns are a cocone we prove $\alpha_n = \alpha_{n+1} \circ F^n!$ by induction on n. When $n = 0$ the condition becomes $! = \alpha_1 \circ !$ which is true because both sides are arrows out of the initial object. Now suppose the condition holds for $n-1$:

$$\begin{aligned} \alpha_{n+1} \circ F^n! &= \alpha \circ F\alpha_n \circ F^k! && \text{definition of } \alpha_{n+1} \\ &= \alpha \circ F(\alpha_n \circ F^{n-1}!) && \text{functoriality of } F \\ &= \alpha \circ F\alpha_{n-1} && \text{inductive hypothesis} \\ &= \alpha_n && \text{definition of } \alpha_n. \end{aligned}$$

[1] We write F^n for $\underbrace{F \dots F}_{n \text{ times}}$ and F^0 is the identity function. We use a similar notation for functors.

Remark 3.1.4. If we apply F to the inital-algebra ω-chain we obtain

$$F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} F^3 0 \xrightarrow{F^3!} \cdots . \tag{3.1.2}$$

This ω-chain has the same colimit as the original one. Indeed since the first element of (3.1.1) is the initial object there is an obvious one-to-one correspondence between cocones on (3.1.1) and cocones on the new chain above; and the same factorization morphisms work.

> TODO: do we need the remark above?

We are ready to state and prove the main theorem of this chapter.

Theorem 3.1.5 (Adámek). let $\mathscr{A}$ be a category with an initial object 0 and colimits of ω-chains and F an endofunctor that preserves ω-colimits. Then F has an initial algebra $\mu F = \operatorname{colim}_{n<\omega} F^n 0$.

Proof. Let $(\mu F, (c_n \colon F^n 0 \to \mu F)_{n<\omega})$ be the colimit of the initial-algebra ω-chain. Since F preserves ω-colimits we have that $(F(\mu F), (Fc_n \colon F^{n+1} \to F(\mu F))_{n<\omega})$ is the colimit of (3.1.2) but $(c_{n+1} \colon F^{n+1}0 \to \mu F)_{n<\omega}$ is a cocone on (3.1.2) so there is a unique arrow ι from $F(\mu F)$ to μF such that

$$\iota \circ Fc_n = c_{n+1} \quad \text{for all } n < \omega. \tag{3.1.3}$$

This gives an F-algebra structure to μF that we now check is initial.

Let (A, α) be a generic F-algebra. We have the induced cocone $(\alpha_n \colon F^n 0 \to \mu F)_{n<\omega}$ as in Remark 3.1.3 and this gives a unique arrow $h \colon \mu F \to A$ such that $h \circ c_n = \alpha_n$ for $n < \omega$. We shall now prove that h is a morphism of F-algebras and that it is unique.

To prove that $h \circ \iota = \alpha \circ Fh$ we check that both arrows are factorization morphisms for the cocone $(\alpha_{n+1} \colon F^{n+1}0 \to A)_{n<\omega}$. In practice we shall check that

$$h \circ \iota \circ Fc_n = \alpha_{n+1} = \alpha \circ Fh \circ Fc_n$$

for all $n < \omega$. For the first equality notice that $h \circ \iota \circ Fc_n = h \circ c_{n+1} = \alpha_{n+1}$. For the second equality consider the following diagram

$$\begin{tikzcd} F^{n+1}0 \arrow[r, "\alpha_{n+1}"] \arrow[d, "Fc_n"'] \arrow[dr, "F\alpha_n"] & A \\ F(\mu F) \arrow[r, "Fh"] & FA \arrow[u, "\alpha"'] \end{tikzcd} .$$

The upper-right triangle commutes by definition of α_{n+1} (see Remark 3.1.3). The lower-left triangle commute because h is the factorization arrow and by functoriality of F. Then, as the whole square commutes, $\alpha \circ Fh \circ Fc_n = \alpha_{n+1}$. This proves that $h \colon \mu F \to A$ is a homomorphism of F-algebras.

To prove uniqueness suppose there is an arrow $k \colon \mu F \to A$ such that it is also an F-algebra homomorphism i.e. $k \circ \iota = \alpha \circ Fk$. We will show that $k \circ c_n = \alpha_n$ for all

$n < \omega$ and conclude by uniqueness of h. Working by induction, if $n = 0$ then $k \circ c_0 = \alpha_0$ is trivial because both arrows are out of the initial object. Now suppose that $k \circ c_n = \alpha_n$:

$$\begin{aligned} k \circ c_{n+1} &= k \circ \iota \circ Fc_n && (3.1.3) \\ &= \alpha \circ Fk \circ Fc_n && k \text{ homomorphism} \\ &= \alpha \circ F(k \circ c_n) && F \text{ functor} \\ &= \alpha \circ F\alpha_n && \text{inductive hypothesis} \\ &= \alpha_{n+1} && \text{definition of } \alpha_{n+1}. \end{aligned}$$

This concludes the proof. □

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