Time Derivative of a Vector

[by Foo Piew Tan]

Let the vector $\mathbf{A}$ be defined as:

$$\mathbf{A} = A\mathbf{e}_r \tag{1}$$

where A is the magnitude of the vector, and $\mathbf{e}_r$ is the unit vector in the radial direction. The unit vector $\mathbf{e}_r$ originates from the origin of the global coordinate system, i.e., $(0, 0, 0)$. Any changes in the magnitude of $\mathbf{A}$ occur along the direction of $\mathbf{e}_r$, which defines the radial direction.

The time derivative of the vector $\mathbf{A}$ is given by:

$$\frac{d}{dt}\mathbf{A} = \frac{dA}{dt}\mathbf{e}_r + A\frac{d}{dt}\mathbf{e}_r \tag{2}$$

This expression indicates that over time, the vector $\mathbf{A}$ may change in both its magnitude A and/or its direction, represented by the unit vector $\mathbf{e}_r$. A change in direction is characterised by a variation in the orientation of $\mathbf{e}_r$. Since $\mathbf{e}_r$ is anchored at the origin and only its endpoint moves, this change in direction can be described by the angular variation θ, which measures the difference between the previous and current orientations of $\mathbf{e}_r$.

We can express the unit vector $\mathbf{e}_r$ in terms of the angle θ as follows:

$$\mathbf{e}_r = \cos\theta\,\mathbf{e}_x + \sin\theta\,\mathbf{e}_y \tag{3}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors along the horizontal and vertical axes, respectively, in the Cartesian coordinate system.

It is important to note that changing the orientation angle θ does not affect the magnitude of the vector. This implies that the direction of angular change, represented by a new unit vector $\mathbf{e}_\theta$, must be perpendicular to $\mathbf{e}_r$. Mathematically, this orthogonality condition is expressed as $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$. This condition can be satisfied if:

$$\mathbf{e}_\theta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{e}_r = -\sin\theta\,\mathbf{e}_x + \cos\theta\,\mathbf{e}_y \tag{4}$$

Here, $\mathbf{e}_\theta$ represents the direction of angular motion and is perpendicular to $\mathbf{e}_r$, as required.

Now, we observe that the time derivative of $\mathbf{e}_r$ can be calculated as:

$$\frac{d}{dt}\mathbf{e}_r = \frac{d\cos\theta}{d\theta}\frac{d\theta}{dt}\mathbf{e}_x + \frac{d\sin\theta}{d\theta}\frac{d\theta}{dt}\mathbf{e}_y \tag{5}$$

Simplifying this expression, we get:

$$\frac{d}{dt}\mathbf{e}_r = \frac{d\theta}{dt}(-\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y) = \frac{d\theta}{dt}\mathbf{e}_\theta \tag{6}$$

Thus, the time derivative of the vector $\mathbf{A}$ can be rewritten as:

$$\frac{d}{dt}\mathbf{A} = \frac{dA}{dt}\mathbf{e}_r + A\frac{d\theta}{dt}\mathbf{e}_\theta \tag{7}$$

This final expression clearly shows that the rate of change of $\mathbf{A}$ has two components: the rate of change of the magnitude A in the radial direction $\mathbf{e}_r$, and the rate of change of the angular position θ in the direction perpendicular to $\mathbf{e}_r$, represented by $\mathbf{e}_\theta$.