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FORMULAE FOR CAUSE-DELETED LIFE TABLES

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SUMMARY

The medical and demographic literature contain two popular formulae for the construction of cause-deleted life table l_x values, one developed for applications in follow-up studies and the other for cross-sectional studies. It turns out that for both types of applications one of the two formulae is virtually always of better quality than the other. A formula developed by Keyfitz and Frauenthal[5] is usually better than either.

KEY WORDS Life tables Cause-deletion Follow-up studies Mortality analysis

> In composing this Memoir, . . . I was above all concerned to display in a single table the two conditions of mankind, the one as it actually is and the other as it would be if we were able to rid the whole human race of smallpox. I had in mind that the comparison of these two conditions would explain the difference and the contrast between them better than the most ample commentary; but I had in mind, too, the difficulty of the enterprise and the defective nature of the Bills of Mortality, which do not give the age of those carried off by smallpox and were bound to be a serious obstacle to my purpose. I could see immediately that to carry out such a design demands two items of elementary information: what is the risk, at various ages, of being caught by smallpox, for those who have not already had it, and what is the risk, for those who are attached by it, of dying of it?
>
> DANIEL BERNOULLI (1760)[1]

INTRODUCTION

A problem that confronted actuaries in the century after the invention of the life table was that of competing risks: the effect on death rates for other causes of death when one cause is eliminated or reduced. Daniel Bernoulli[1] derived a solution in 1760 for the effects of eliminating smallpox. He used mathematical formulae to approximate incidence and mortality distributions, but his results did not apply generally. Resolution of the problem came in the 20th century. Two unrelated formulae were introduced in the period before 1950, and a third was added in 1975. This paper explains the relationship of the three formulae to the basic life table and reports the results of tests of their relative quality.

The construction of life tables proceeds either from a count of persons observed from the onset of exposure to an event risk and a count of the events that subsequently occur to them, or from a count of survivors observed part way through an interval of exposure and an event count for the

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complete interval. In the former case, which typically arises in medical follow-up studies and from the use of retrospective questions in single round surveys, the life table event rate for the interval $(a, a+n)$ is simply

$$ {}_n\hat{q}_a = {}_nD_a/N_a \tag{1} $$

where N_a is the population or sample observed throughout the interval and ${}_nD_a$ is the number of events in the interval occurring to it. From the ${}_n\hat{q}_a$ estimates, survival at a particular duration x is

$$ \hat{l}_x = \prod_{a=0}^{x-n} (1 - {}_n\hat{q}_a) $$

on setting $l_0 = 1$. Familiar examples of rates of this type include survival rates after the onset or diagnosis of life-threatening illnesses, continuation rates for various contraceptive methods, and in administrative applications such as the turnover rates of medical staff or failure rates of equipment.

A common but usually unnecessary convention in table construction, the 'actuarial' refinement, adds events occurring to persons observed for only part of the interval to the numerator of (1) and adds half their number (equivalent to their exposure time) to the denominator. The resulting ${}_n\hat{q}_a$ estimate is trivially more precise than (1) since it derives from a larger number of observations, and usually trivially biased, since it imposes a parametric distribution on the survival function for the part of the interval not observed. The method is of interest here because it has given rise to one of two popular estimating formulae that we will examine for the construction of cause-deleted life tables.

When the denominator of the ${}_n\hat{q}_a$ estimator is person years of exposure, or a sample or census of survivors in the age category at approximately mid-interval (${}_nN_a$), we cannot use expression (1) as written. A familiar substitution replaces its N_a term by $\frac{1}{n}\,{}_nN_a + \frac{1}{2}\,{}_nD_a$ which adds half the interval deaths to the average unit population ($\frac{1}{n}\,{}_nN_a$) to reconstruct the population initially exposed to the event risk. With the change the estimator becomes

$$ {}_n\hat{q}_a = {}_nD_a/({}_nN_a/n + \tfrac{1}{2}\,{}_nD_a) \tag{2a} $$

Another solution, of interest for cause-deleted tables, is to consider the observed death rate or incidence density ${}_nM_a = {}_nD_a/{}_nN_a$ equivalent to the event *hazard*, or the risk of its occurrence at any time. The probability that the event will not occur in an interval of length n, assuming the risk to be constant, is

$$ 1 - {}_n\hat{q}_a = {}_n\hat{p}_a = \mathrm{e}^{-n\,{}_nM_a} \tag{2b} $$

(A similar expression measures the erosion in value of a stock of money over an n-year period at a constant rate of inflation M. In demography, a familiar example of this approach is in the estimation of national life tables from vital statistics reports of annual deaths and from midyear population estimates.) Usually, one adds additional terms to the exponential to increase its accuracy, as in Reference 2.

CAUSE-DELETED LIFE TABLES

In the construction of life tables that recognize more than one type of terminating event, as in mortality tables by cause of death, the investigator may want to measure the difference that would (hypothetically) occur in survival rates with elimination of one or more causes. The solution is not simply to omit certain causes of death or termination from the raw tables since, as particular causes cease in their impact, exposure time to other causes increases, and influences their rates. Moreover,

it is rarely possible wholly to disentangle competing event risks. For deaths, the U.S. follows ICD guidelines in distinguishing between immediate and underlying causes rather than to assign a single cause where more than one is indicated; for a contraceptive such as the IUD, expulsions and removals tend to be correlated; and so forth. The reader may consult Manton and Poss[3] for elaboration on this point.

Lacking an ideal solution, most researchers have followed Greville[4] in decomposing the probability of surviving through an interval as the product of the probabilities of surviving for each of the causes separately, with an apportionment factor equal to the number of events of each type. Thus, if ${}_nD_{a,i}$ is the number of events of the ith type, the probability of avoiding termination by that cause will be

$$ {}_n\hat{p}_{a,i} = {}_n\hat{p}_a^{\,{}_nD_{a,i}/{}_nD_a} \tag{3a} $$

When ${}_n\hat{p}_a$ is calculated by expression (2b), expression (3a) simplifies to

$$ {}_n\hat{p}_{a,i} = e^{-n\,{}_nD_{a,i}/{}_nN_a} = e^{-n\,{}_nM_{a,i}} \tag{3b} $$

Expression (3b) has the property that ${}_n\hat{p}_a = \prod_i {}_n\hat{p}_{a,i}$, as it should: the chance of surviving through the interval must equal the chance of surviving all event risks. (For the Keyfitz–Frauenthal and the actuarial formulae introduced below the equality is only approximate and a correction term is occasionally warranted.)

The cause-deleted survival rate for cause i at age x, $\hat{l}_{x,i}$ is found as the product of the ${}_n\hat{p}_{a,i}$ rates in the same way that $\hat{l}_x$ is found as the product of the ${}_n\hat{p}_a$ terms.

Keyfitz and Frauenthal[5] have introduced a generalization of Greville's formula that adjusts the cause-deleted survival rates to take into account the rates of change of mortality and population size across age groups. Under linear assumptions the estimator is

$$ {}_n\hat{p}_{a,i} = {}_n\hat{p}_a^{\,R} $$

$$ R = \frac{{}_nD_{a,i}}{{}_nD_a}\left\{1 + \frac{1}{48}\ln\left[\frac{{}_nN_{a+n}}{{}_nN_{a-n}}\right]\ln\left[\frac{{}_nD_{a+n}\,{}_nD_{a-n,i}}{{}_nD_{a-n}\,{}_nD_{a+n,i}}\right]\right\} \tag{4a} $$

Both the Greville and Keyfitz–Frauenthal formulae interpret the observed event rates as hazard rates, following (2b). The Greville formula is also applicable for ${}_np_a$ rates calculated by (1), but formula (4a) is not as it uses central population terms (${}_nN_{a-n}$, ${}_nN_{a+n}$). One can estimate these using the approximate equality of (1) and (2a), from which ${}_nN_a = n(N_a - \frac{1}{2}\,{}_nD_a)$. Substitution of the right-hand terms into (4a) yields the follow-up estimator

$$ {}_n\hat{p}_{a,i} = {}_n\hat{p}_a^{\,R}, $$

$$ R = \frac{{}_nD_{a,i}}{{}_nD_a}\left\{1 + \frac{1}{48}\ln\left[\frac{N_{a+n} - \frac{1}{2}\,{}_nD_{a+n}}{N_{a-n} - \frac{1}{2}\,{}_nD_{a-n}}\right]\ln\left[\frac{{}_nD_{a+n}\,{}_nD_{a-n,i}}{{}_nD_{a-n}\,{}_nD_{a+n,i}}\right]\right\} \tag{4b} $$

Besides (4a), Keyfitz and Frauenthal introduce the simpler estimator, which we will also test,

$$ {}_n\hat{p}_{a,i} = {}_n\hat{p}_a^{\,R} $$

$$ R = \frac{{}_nD_{a,i}}{{}_nD_a}\left[1 + \frac{1}{24}\ln\left(\frac{{}_nD_{a+n,i} + {}_nD_{a-n,i}}{{}_nD_{a,i}} - \frac{{}_nD_{a+n} + {}_nD_{a-n}}{{}_nD_a}\right)\right] \tag{5} $$

A different solution to cause-deletion is suggested by the actuarial method and commonly used with it. This is to drop from the numerator of (1) those classes of events not of interest, and from the denominator to drop part of their number (usually $n/2$, commensurate with the exposure times the

individuals would have to the risks under investigation if their terminations occurred uniformly over the interval). Distinguishing terminations for causes i and $-i$, the cause-deleted termination rate for i and an initial population N_a would be

$$ {}_n\hat{q}_{a,i} = {}_nD_{a,i}/(N_a - \tfrac{1}{2}{}_nD_{a,-i}) \tag{6} $$

and, for the central population

$$ {}_n\hat{q}_{a,i} = {}_nD_{a,i}/({}_nN_a/n + \tfrac{1}{2}{}_nD_{a,-i}) \tag{7} $$

TESTING THE FORMULAE

Originally, expression (3a) was introduced in the context of cross-sectional analysis and (6) in connection with medical follow-up studies. Their distinct origins and usages probably account for the absence of any comparative tests of the two formulae in the demographic literature. Only limited testing has appeared for (4a) and (5) because of their recency.

Testing the formulae is not difficult. What is required is a data set with terms N_a, ${}_nN_a$, ${}_nD_{a,i}$, ${}_nD_{a,-i}$ of sufficient fineness (we will use national mortality information in time units of $n = \frac{1}{4}$ year) to permit calculation of cause-deleted life tables that will differ only trivially under the various formulae. Recalculation of the tables by each formula but with use of wider age intervals (e.g. $n = 5$ years) permits assessment of the closeness of the new ${}_n\hat{q}_{a,i}$ or $\hat{l}_{x,i}$ terms to those of the $\frac{1}{4}$ year series. In each case the best formula will be that which most nearly replicates the initial series. Formulae for abridged life tables are routinely tested in the same manner.

For the tests, I have used 1930 and 1970 U.S. male life tables for single year ages (from References 7 and 8, respectively). For 1930 I distinguish deaths for cardiovascular diseases, accidental and violent deaths other than motor vehicle fatalities, and all remaining causes[8] and for 1970 neoplasms, cardiovascular diseases, and all remaining causes.[7]

I graduated the death series from 5-year to 1-year intervals using centre panel Sprague multipliers.[9] I then graduated life table l_x and ${}_1L_x$ terms, and numbers of deaths by cause to $\frac{1}{2}$ year and subsequently $\frac{1}{4}$ year intervals by the conventional 5th order polynomial fitting:

$$ a_{n+1/2} = [3(a_{n-2} + a_{n+3}) - 25(a_{n-1} + a_{n+2}) + 150(a_n + a_{n+1})]/256 $$

Both the Sprague multipliers and 5th order polynomial were tested by fitting them to model linear and geometric survival distributions to confirm that they preserve known underlying survival patterns with high accuracy.

After interpolation, I rescaled deaths in each interval $(x, x+\frac{1}{4})$ to sum to the difference ${}_nd_x = l_x - l_{x+1/4}$. Equating l_x with N_x, ${}_{1/4}L_x$ with ${}_{1/4}N_x$ and the scaled death terms ${}_{1/4}d_{x,i}$ with ${}_{1/4}D_{x,i}$ produce the terms necessary for construction of the initial life tables using (1) and (2a) with cause-deletion by formula (3a). Since the central population has been constructed from the life table, in (2a) it is stationary. Cross-sectional tables for non-stationary populations are generated by scaling the terms in ${}_{1/4}N_x$ and ${}_{1/4}D_{x,i}$ either geometrically to represent stable populations or in proportion to the age distribution of an observed population. All life tables considered begin at age 5, since the formulae are not suitable for graduating infant mortality.

Table I displays cause-deleted survival rates and errors for the follow-up life table, using the 1970 U.S. male life table and deaths. Rates in this and Table II appear for ages 50, 65, 75, 80 and 85; under all formulae the largest absolute errors occur at one or another of these ages. The 5-year actuarial $\hat{l}_x$ estimates were calculated using (6), the power estimates use (1) and (3a), and the Keyfitz–Frauenthal rates use (1) and (4b) or (5). For the actuarial estimates, the table includes errors for both unadjusted and adjusted rates: the unadjusted rates are those produced by (6); the

Table I. Errors of cause-deleted actuarial, power and Keyfitz–Frauenthal $l_{x,i}$ estimates for 3 causes of death, at 1970 U.S. male survival rates: follow-up life table

Age	Proportion surviving $(l_{x,i}/l_{5,i})$	Error of: Actuarial expression (6) Unadjusted	Actuarial expression (6) Adjusted	Power expression (3)	Keyfitz–Frauenthal expression (5)	Keyfitz–Frauenthal expression (4b)
(*a*) *Cardiovascular diseases*						
50	0·96875	0·00002	0·00016	0·00001	−0·00017	0·00000
65	0·83512	0·00017	0·00116	0·00007	−0·00024	−0·00002
75	0·62448	0·00104	0·00385	0·00026	−0·00009	−0·00002
80	0·47448	0·00240	0·00680	0·00050	0·00022	0·00008
85	0·31111	0·00448	0·01050	0·00073	0·00075	0·00007
(*b*) *Neoplasms*						
50	0·98475	0·00006	0·00020	0·00001	−0·00008	0·00000
65	0·92346	0·00095	0·00205	0·00001	0·00002	−0·00001
75	0·83316	0·00398	0·00774	−0·00017	0·00002	0·00000
80	0·77277	0·00755	0·01475	−0·00051	−0·00031	0·00003
85	0·70712	0·01322	0·02696	−0·00128	−0·00135	−0·00003
(*c*) *Other*						
50	0·92636	−0·00021	−0·00007	−0·00002	0·00024	0·00000
65	0·85630	−0·00207	−0·00105	−0·00009	0·00023	−0·00001
75	0·76563	−0·00830	−0·00491	−0·00015	0·00009	0·00001
80	0·69785	−0·01652	−0·01023	−0·00027	−0·00004	0·00001
85	0·61458	−0·03098	−0·01985	−0·00034	−0·00031	0·00003

adjusted rates are constructed by scaling the actuarial ${}_n\hat{p}_{a,i}$ estimates so that their product equals the observed ${}_n\hat{p}_a$. (The equality always holds for the power estimates, and holds to 4 significant digits for Keyfitz–Frauenthal, but for the actuarial estimates it sometimes holds to as few as 2 digits at the oldest ages.) Errors in ${}_n\hat{p}_x$ estimates are similar to those for associated $\hat{l}_x$ terms.

For all three causes of death the actuarial formula performs rather poorly. The quality of the adjusted rates is no better than that of the unadjusted ones. By age 85 errors as large as 2–3 per cent occur in two of the three series (neoplasms, other), as compared with maximum errors under 0·2 per cent for power and Keyfitz–Frauenthal (5) estimates and under 0·01 per cent for Keyfitz–Frauenthal (4b) estimates; 1930 results are similar.

Differences between the power estimates and Keyfitz–Frauenthal (5) were small, with neither set of rates clearly superior to the other. For the 1930 U.S. male life table and deaths the power estimates are modestly better (tables not shown). Pending additional evidence, there apparently is no reason for using (5) in preference to (3).

A very clear difference is seen, however, between the power estimates and Keyfitz–Frauenthal (4b). Whereas the error of the power formula reaches −0·00128 in one instance, that of formula (4b) remains below ±0·0001 at all ages. For the 1930 life table its greatest error is 0·00039, as compared to 0·00088 for the power estimates, and 0·00116 for Keyfitz–Frauenthal (5). These errors are for cardiovascular deaths at age 85. The actuarial error for the same cause of death and age is 0·0086; its maximum error (0·035) occurs in another part of the table.

Table II shows cause-deleted survival rates for the cross-sectional data, again using the 1970 U.S. male life table and deaths. The rates are for cardiovascular mortality as it would appear in a stationary population and in stable populations with annual growth rates of −1 per cent and +3

Table II. Errors of cause-deleted actuarial, power and Keyfitz–Frauenthal $l_{x,i}$ estimates for cardiovascular mortality omitting other causes of death, at 1970 U.S. male survival rates: cross-sectional life table for stable populations with intrinsic growth rates of $r = -1$ per cent, 0 and $+3$ per cent annually

		Error of:					
	Proportion surviving	Actuarial expression (7)		Power expression (3)		Keyfitz–Frauenthal expression (4a)	
Age	$(l_{x,i}/l_{5,i})$	Unadjusted	Adjusted	Unadjusted	Adjusted	Unadjusted	Adjusted
(*a*) $r = 0{\cdot}00$							
50	0·96875	0·00005	−0·00004	0·00001	0·00002	0·00000	0·00001
65	0·83512	0·00053	−0·00042	0·00006	0·00009	−0·00002	0·00001
75	0·62448	0·00174	−0·00115	0·00025	0·00027	−0·00002	0·00000
80	0·47448	0·00262	−0·00161	0·00057	0·00048	0·00008	−0·00003
85	0·31111	0·00278	−0·00189	0·00077	0·00070	0·00005	−0·00002
(*b*) $r = -0{\cdot}01$							
50	0·96875	−0·00004	0·00005	−0·00002	−0·00002	0·00000	0·00000
65	0·83512	0·00021	−0·00014	0·00001	0·00001	−0·00001	0·00002
75	0·62448	0·00118	−0·00079	0·00017	0·00018	−0·00002	0·00000
80	0·47448	0·00196	−0·00125	0·00047	0·00038	0·00007	−0·00004
85	0·31111	0·00210	−0·00159	0·00069	0·00060	0·00007	−0·00003
(*c*) $r = +0{\cdot}03$							
50	0·96875	0·00030	−0·00031	0·00010	0·00013	−0·00001	0·00002
65	0·83512	0·00147	−0·00118	0·00022	0·00027	−0·00003	0·00003
75	0·62448	0·00342	−0·00225	0·00047	0·00053	−0·00005	0·00001
80	0·47448	0·00461	−0·00271	0·00081	0·00079	0·00004	0·00000
85	0·31111	0·00480	−0·00280	0·00093	0·00107	−0·00006	0·00005

per cent. (The rates tested varied from -3 per cent to $+3$ per cent.) Calculation of the actuarial estimates was by formula (7), the Greville series uses (2a) and (3a), and the Keyfitz–Frauenthal series uses (2a) and (4a). Since ${}_n\hat{p}_a$ estimates from central rates are approximate, both unadjusted and adjusted rates appear for each estimating function. The actuarial estimate requires greater adjustment than the power or Keyfitz–Frauenthal estimates.

As before, in Table II the actuarial cause-deleted rates are least satisfactory, although their performance is substantially better than in the follow-up rates of Table I. In all of the tests with central data the errors for the actuarial method have remained under 1 per cent, and are greater for high growth rates ($r = 0{\cdot}03$) than for small or negative rates.

The power estimates are again intermediate in quality between the actuarial and Keyfitz–Frauenthl rates, their errors being rarely as large as 1/5 of the actuarial errors. Like the actuarial estimates, they increase with the rate of population growth.

As under formula (4b), Keyfitz–Frauenthal rates using (4a) are highly precise, with errors largely confined to the fifth significant digit for all growth rates between -3 and $+3$ per cent. Errors for 1930 data reach 0·0004, about $\frac{1}{4}$ as great as for the power series in that year, and again much below those of the actuarial method.

Since the Keyfitz–Frauenthal estimates incorporate the population term $\ln({}_nN_{x+n}/{}_nN_{x-n})$, I also tested them on U.S. and French male age distributions for 1970 and projected to 2020, and for Swedish males from 1975 to 2025. Figure 1 shows the initial age distributions for the 3 populations. In each projection, new entrants to the population were assigned a stationary age structure. The life tables and distributions of deaths by cause used in the projections were those for 1970 U.S. males.

The largest error of any projection was $-0{\cdot}00109$ for cardiovascular deaths at age 80, France 1995. Table III displays some of the French results. In other years and other series,

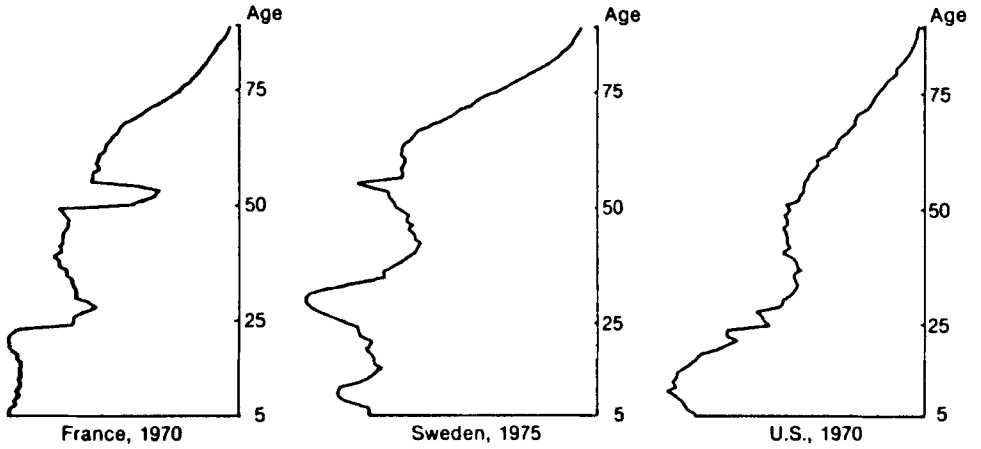

Figure 1. Relative age distributions of French, Swedish and U.S. males ages 5–89 in proportion to males at age 5

Table III. Errors of cause-deleted actuarial, power and Keyfitz–Frauenthal $l_{x,i}$ estimates for cardiovascular mortality omitting other causes of death, at 1970 U.S. male survival rates: France 1970, 1995 and 2020 *

		Error of:					
		Actuarial expression (7)		Power expression (3)		Keyfitz–Frauenthal expression (4a)	
Age	Proportion surviving ($l_{x,i}/l_{5,i}$)	Unadjusted	Adjusted	Unadjusted	Adjusted	Unadjusted	Adjusted
(*a*) 1970							
50	0·96875	0·00000	0·00004	−0·00014	0·00025	−0·00021	0·00019
65	0·83512	0·00036	−0·00023	0·00011	0·00011	0·00004	0·00003
75	0·62448	0·00257	−0·00150	0·00071	0·00018	0·00037	−0·00016
80	0·47448	0·00333	−0·00190	0·00060	0·00060	0·00002	0·00002
85	0·31111	0·00337	−0·00212	0·00096	0·00073	0·00017	−0·00006
(*b*) 1995							
50	0·96875	0·00025	−0·00019	0·00015	−0·00006	0·00011	−0·00009
65	0·83512	0·00008	−0·00011	0·00004	0·00001	0·00000	−0·00003
75	0·62448	0·00015	−0·00012	−0·00075	0·00060	−0·00094	0·00041
80	0·47448	−0·00057	−0·00025	−0·00088	0·00068	0·00109	0·00047
85	0·31111	−0·00210	−0·00017	0·00024	0·00008	0·00020	0·00004
(*c*) 2020							
50	0·96875	0·00000	0·00000	0·00000	0·00000	0·00000	0·00000
65	0·83512	0·00009	−0·00010	0·00006	−0·00004	0·00005	−0·00005
75	0·62448	0·00101	−0·00067	0·00073	−0·00023	0·00063	−0·00033
80	0·47448	0·00069	−0·00074	0·00057	0·00000	0·00040	−0·00017
85	0·31111	−0·00169	−0·00038	0·00048	−0·00012	0·00041	−0·00019

* 1970 French population estimated from Reference 10, T. 2, p. 34.

Keyfitz–Frauenthal (4a) errors were between those of the power series and Keyfitz–Frauenthal (4a) in Table II, converging toward the Keyfitz–Frauenthal series as age distributions stabilized. The errors of the power formula for the same years are marginally to substantially larger, and those of the actuarial are usually so. (The 1995 French series was the only one for any country or year in which the actuarial formula was competitive with other formulae.) Given the sharpness of age fluctuations in two of the populations, we may consider the results supportive of the Keyfitz–Frauenthal formula.

CONCLUSIONS

Our findings suggest mainly that one need not use the 'actuarial' approach to estimate cause-deleted rates. For both follow-up and cross-sectional rates it produces estimates that are consistently of lower quality than those of the simpler power formula (3a) whenever survival rates are much below 90 per cent. The actuarial rates compared with rates estimated by other formulae also require greater adjustment after the fact to bring them into line with the overall survival rate ${}_n\hat{p}_a$. Our findings also recommend against use of one of the Keyfitz–Frauenthal formulae (5) as it is more complex than the power formula but offers no evident improvement in the quality of rate estimates.

The power formula (3a) and Keyfitz–Frauenthal formula (4a) appear well-suited to data sets of both types. The two appear to have equal quality when survival is high ($l_x > 0{\cdot}75$). At lower survival rates and in cross-sectional analyses with stationary or approximately stable populations Keyfitz–Frauenthal estimates are uniformly superior. We recommend them particularly when one wishes very high accuracy. In the tests Keyfitz–Frauenthal estimates were also at least marginally better than power estimates for real populations that departed markedly from stability, although the closeness of some series suggests that power estimates might occasionally be better. A wise option with distorted age structures is to use 1-year in place of 5-year age intervals, for which (3a) suffices. We find the power formula preferable for small populations and sample data, including follow-up data where stochastic fluctuations in ${}_nN_x$ or ${}_nD_x$ may compromise the precision of the Keyfitz–Frauenthal estimates.

REFERENCES

1. Bernoulli, Daniel. 'Essai d'une nouvelle analyse de la mortalité causée par la petite vérole et les avantages de l'inoculation pour la prévenir', *Histoire de l'Académie Royale des Sciences*, Année 1760, pp. 1–45.
2. Reed, Lowell J. and Merrell, Margaret. A short method for constructing an abridged life table', *American Journal of Hygiene*, **30**, 33–41 (1939).
3. Manton, Kenneth G. and Poss, Susan S. 'Effects of dependency among causes of death for cause elimination life table strategies', *Demography*, **16**, 313–327 (1979).
4. Greville, T. N. E. 'Mortality tables analyzed by cause of death', *Record of the American Institute of Actuaries*, **37**, 283–294 (1948).
5. Keyfitz, Nathan and Frauenthal, James. 'An improved life table method', *Biometrics*, **31**, 889–899 (1975).
6. Dublin, Louis I. and Lotka, Alfred J. *Length of Life*, Ronald Press, New York, 1936.
7. National Center for Health Statistics. *Vital Statistics of the United States* 1970, Vol. II, Pt. A. Rockville, MD, 1974.
8. Preston, Samuel, Keyfitz, Nathan and Schoen, Robert, *Causes of Death: Life Tables for National Populations*, Seminar Press, New York, 1972.
9. Shryock, Henry S. and Siegel, Jacob S. 'Interpolation of point data', in Shryock, H. S. and Siegel, J. S., *The Methods and Materials of Demography*, U.S. Government Printing Office, Washington, D.C., 1971, pp. 681–689.
10. Henry, Louis. *Démographie: Analyse et modèles*, Larousse, Paris, 1972.