16-745 Optimal Control Lecture 18

Reid Graves

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Last Time

- Stochastic Optimal Control
- LQG

Today

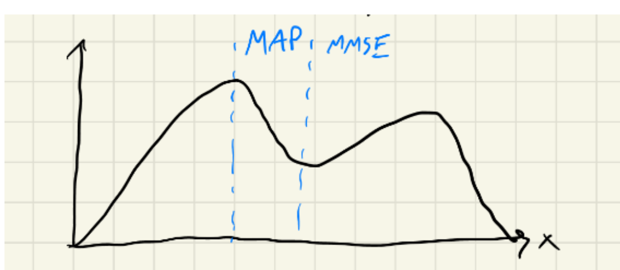
- Optimal Estimation
- Finish LQG
- Duality

1 From Last time

- "Certainty Equivalence"
- "Separation Principle"
- Frequently applied to nonlinear systems in practice

2 Optimal State Estimation

- What should I optimize



- Maximum a-posterior: (MAP)

$$\text{argmax} \quad \underbrace{p(x|y)}_{\text{probability of state given measurements}}$$

- Minimum mean squared error (MMSE)

  $\arg\min_{\hat{x}} \quad E[(x-\hat{x})^T(x-\hat{x})]$ "Least squares" or "minimum variance"

  $E[tr((x-\hat{x})^T(x-\hat{x}))] = E[tr((x-\hat{x})(x-\hat{x})^T)]$ Changed from inner product to outer product (Covariance

  $= tr(E[(x-\hat{x})(x-\hat{x})^T]) = tr(\Sigma)$

- These are the same for a Gaussian!

3 Kalman Filter

- Recursive linear MMSE estimator
- AsSigmae an estimate of the state that includes all measurements up to the current time:

$$\hat{x}_{k|k} = E[x_k|y_{1:k}]$$

- AsSigmae we also know the error covariance:

$$\Sigma_{k|k} = E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T]$$

- We want to update $\hat{x}$ and Σ to include a new measurement at t_{k+1}
- the KF can be broken into 2 steps

3.1 Prediction

$$\begin{aligned}
\hat{x}_{k+1|k} &= E[Ax_k + Bu_k + w_k|y_{1:k}] = A\hat{x}_{k|k} + Bu_k \\
\Sigma_{k+1|k} &= E[(x_{k+1} - \hat{x}_{k+1:k})(\dots)^T] \\
&= E[(Ax_k + Bu_k + w_k - A\hat{x}_{k|k} - Bu_k)(\dots)^T] \\
&= A\,\underbrace{E[(x_k - \hat{x}_{k|k})(\dots)^T]}_{\Sigma_{k|k}}\,A^T + \underbrace{E[w_k w_k^T]}_{W} \\
&= A\Sigma_{k|k}A^T + W \quad (x_k \text{ and } w_k \text{ are encorrelated})
\end{aligned}$$

3.2 Measurement Update

- Define "innovation"

$$\begin{aligned}
z_{k+1} &= y_{k+1} - C\hat{x}_{k+1|k} \\
&= Cx_{k+1} + v_{k+1} - C\hat{x}_{k+1|k}
\end{aligned}$$

- Innovation Covariance

$$\begin{aligned}
S_{k+1} &= E[z_{k+1}z_{k+1}^T] \\
&= E[(Cx_{k+1} + v_{k+1} - C\hat{x}_{k+1|k})(\dots)^T] \\
&* \ v_{k+1} \text{ and } x_{k+1} \text{ are uncorrelated} \\
&\Rightarrow S_{k+1} = C\,\underbrace{E[(x_{k+1} - \hat{x}_{k+1|k})(\dots)^T]}_{\Sigma_{k+1|k}}\,C^T + \underbrace{E[v_{k+1}v_{k+1}^T]}_{V} \\
&= C\Sigma_{k+1|k}C^T + V
\end{aligned}$$

- Innovation is the error signal we feed back into the estimator
- State Update:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \underbrace{L_{k+1}}_{1} z_{k+1}$$

1 : "Kalman Gain"

- Covariance Update

$$\begin{aligned}
\Sigma_{k+1|k+1} &= E[(x_{k+1} - \hat{x}_{k+1})(\dots)^T] \\
&= E[(x_{k+1} - \hat{x}_{k+1|k} - L_{k+1}((x_{k+1} + v_{k+1} - (\hat{x}_{k+1|k}))(\dots)^T] \\
&\ast v_{k+1} \text{ and } x_{k+1} \text{ are uncorrelated} \\
&= \underbrace{(I - L_{k+1}C)\Sigma_{k+1|k}(I - L_{k+1}C)^T + L_{k+1}VL_{k+1}^T}_{\text{“Joseph Form”}}
\end{aligned}$$

- Kalman Gain

$$\begin{aligned}
&MMSE \Rightarrow \text{minimize } E[(x_{k+1} - \hat{x}_{k+1|k+1})^T(\dots)] \\
&E[(x_{k+1} - \hat{x}_{k+1|k+1})^T(\dots)] = tr\left(\Sigma_{k+1|k+1}\right)
\end{aligned}$$

$$\Rightarrow\text{set } \frac{\partial tr(\Sigma_{k+1|k+1})}{\partial L_{k+1}} = 0 \quad (\text{ and solve for } L_{k+1})$$

$$\boxed{L_{k+1=\Sigma_{k+1|k}C^T S_{k+1}^{-1}}}$$

4 Kalman Filter Algorithm Summary

1. Start with $\hat{x}_{0|0}, \Sigma_{0|0}, W, V$
2. Predict:

$$\begin{aligned}
\hat{x}_{x+1|k} &= A\hat{x}_{k|k} + Bu_k \\
\Sigma_{k+1_k} &= A\Sigma_{k|k}A^T + W
\end{aligned}$$

3. Calculate Innovation + Covariance:

$$\begin{aligned}
z_{k+1} &= y_{k+1} - C\hat{x}_{k+1|k} \\
S_{k+1} &= C\Sigma_{k+1|k}C^T + V
\end{aligned}$$

4. Calculate Kalman Gain:

$$L_{k+1} = \Sigma_{k+1|k}C^T S_{k+1}^{-1}$$

5. Update

$$\begin{aligned}
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + L_{k+1}Z_{k+1} \\
\Sigma_{k+1|k+1} &= (I - L_{k+1}C)\Sigma_{k+1|k}(I - L_{k+1}C)^T + L_{k+1}VL_{k+1}^T
\end{aligned}$$

6. GoTo 2

5 How do we apply this to nonlinear systems?

- Extended KF: Linearize abount $\hat{x}$ and proceed as in standard KF
- Many other generalizations

6 Duality and Trajectory Optimization

- MMSE estmation problem is equivalent to the following optimal control problem:

$$\min_{x_{1:N}, w_{1:N}} \sum_{k=1}^{N-1} \frac{1}{2} \underbrace{(y_k - g(x_k))^T V^{-1} (y_k - g(x_k)}_{\text{state cost}} + \underbrace{\frac{1}{2} w_k^T W^{-1} w_k}_{\text{control cost}}$$

$$s.t. \quad x_{k+1} = f(x_k) + w_k \leftarrow \text{controls}$$

$$g(x_k): \ \text{Measurement model}$$

- If $f(x) = Ax$ and $g(x) = Cx$, this is an LQR problem