16-745 Optimal Control Lecture 6

Reid Graves

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1 Last Time

- Inequality Constraints
- Interior-Point Methods

2 Today

- Revisit regularization and line search
- Control History
- Deterministic Optimal Control

3 Duality Regularization

- Given:

$$\begin{aligned} &\min_x f(x) \\ s.t. \quad &c(x) = 0 \end{aligned}$$

- We can theoretically turn this into:

$$\min_x f(x) + P_\infty(c(x)), \quad P_\infty(x) = \begin{cases} 0, c(x) = 0 \\ +\infty, c(x) \neq 0 \end{cases}$$

- Practically terrible, but we can get the same effect by solving:

$$\min_x \max_\lambda f(x) + \lambda^T c(x)$$

- Whenever $c(x) \neq 0$, inner problem blows up
- Similar for inequalities:

$$\begin{aligned} &\min_x f(x) \\ s.t. \quad &c(x) \geq 0 \\ &\min_x f(x) + P_\infty^-(c(x)) \\ &P_\infty^- = \begin{cases} 0, c(x) \geq 0 \\ +\infty, c(x) < 0 \end{cases} \\ \Rightarrow &\min_x \max_{\lambda \geq 0} f(x) - \lambda^T c(x) \end{aligned}$$

- Whenever $c(x) < 0$, inner problem blows up
- Interpretation: KKT conditions define a saddle point in (x, λ)
- KKT system should have $\dim(x)$ positive eigenvalues, and $\dim(\lambda)$ negative eigenvalues at an optimum (called "quasi-definite")

$$\begin{bmatrix} H + \beta I & C^T \\ C & -\beta I \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} -\nabla_x L \\ -c(x) \end{bmatrix} \quad \beta > 0$$

- Example:
  - Still overshoot $\Rightarrow$ need line search

4 Merit Function

- How do we do a line search on a root finding problem?

$$\text{find} \quad x^* \quad s.t. \quad c(x^*) = 0$$

- Define scalar "merit function" $P(x)$ that measures distance to the solution
- Standard Choices:
  - $P(x) = \frac{1}{2} c(x)^T c(x) = \frac{1}{2} |c(x)|_2^2$
  - $P(x) = |c(x)|_1$ (any norm works)
- Now just do Armijo on $P(x)$:

$$\begin{aligned}
&\alpha = 1 \\
&\quad \text{while} \quad P(x + \alpha \Delta x > P(x) + b\alpha \nabla P(x)^T \Delta x \\
&\qquad \alpha \leftarrow \theta \alpha \\
&\quad \text{end} \\
&x \leftarrow x + \alpha \Delta x
\end{aligned}$$

- How about constrained optimization?

$$\begin{aligned}
&\min_x f(x) \\
s.t. \quad &c(\infty) \geq 0, d(x) = 0 \\
&L(x, \lambda, \mu) = f(x) - \lambda^T c(x) + \mu^T d(x)
\end{aligned}$$

- Lots of options for merit functions:

$$P(x, \lambda, \mu) = \frac{1}{2} |r_{kkt}(x, \lambda, \mu)|_2^2$$

$$\text{KKT residual} \quad \begin{bmatrix} \nabla_x L \\ \min(0, c(x)) \\ d(x) \end{bmatrix}$$

$$P(x, \lambda, \mu) = f(x) + P \left\| \begin{bmatrix} min(0, C(x) \\ d(x) \end{bmatrix} \right\|_1$$