16-745 Optimal Control Lecture 19

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Last Time

- Iterative Learning Control

Today

- Stochastic Optimal Control

Stochastic Control

- So far we have assumed we know the system's state perfectly
- What happens when all we have are noisy measurements of quantities related to the state?

$$\underbrace{y}_{\text{measurements}} = \underbrace{g(x,v))}_{\text{"measurement model"}}$$

where v is noise

Deterministic		Stochastic
x	$\rightarrow$	$p(x\|y) \leftarrow$ PDF if the state conditioned on measurements

Stochastic Optimal control problem

$$\min_{u} \quad E[J(x,u)]$$

- In principle, can solve with DP
- Very hard in general

Linear Quadratic Gaussian (LQG)

- Special case we can solve in closed form:

<u>L</u>inear Dynamics
<u>Q</u>uadratic Cost
<u>G</u>aussian Noise

- Dynamics

$$x_{k+1} = Ax_k + Bu_k + w_k \leftarrow \text{"process noise"}$$
$$y_k = Cx_k + v_k \leftarrow \text{"Measurement noise"}$$

$$w_k \sim N(0, W) \qquad\qquad v_k \sim N(0, V)$$

where $\sim$ is "drawn from", N is Normal Gaussian distribution, 0 is mean, and W, V is covariance

Multivariate Gaussian

$$p(x) = \frac{1}{\sqrt{(2\pi)^k det(S)}} exp(-\frac{1}{2}(x-\mu)^T S^{-1}(x-\mu))$$
$$\text{mean: } \mu = \hat{x} = E[x] \in \mathbb{R}^n$$
$$\text{covariance: } S = E[(x-\mu)(x-\mu)^T] \in S_{tt}^n$$
$$E[f(x)] = \int_{\text{(all space)}} f(x)p(x)dx$$
$$\text{"Uncorrelated"} \Rightarrow E[(x-\hat{x})(y-\hat{y})^T] = 0$$

Where S is the covariance matrix,

- Cost function

$$J = E\left[\frac{1}{2}x_N^T Q x_N + \frac{1}{2}\sum_{k=1}^{N-1}(x_k^T Q x_k + u_k^T R u_k)\right]$$

- D.P. Recursion

$$V_N(x) = \frac{1}{2}E[x_N^T Q x_N] = \frac{1}{2}E[x_N^T P_N x_N]$$
$$\begin{aligned} V_{N-1}(x) = \min_u E\Big[&\frac{1}{2}x_{N-1}^T Q x_{N-1} + \frac{1}{2}u_{N-1}^T R u_{N-1} + \frac{1}{2}(Ax_{N-1} + Bu_{N-1} \\ &+ w_{N-1})^T P_N (Ax_{N-1} + Bu_{N-1} + w_{N-1})\Big] \\ = \min_u &\underbrace{E\left[\frac{1}{2}x_{N-1}^T Q x_{N-1} + \frac{1}{2}u_{N-1}^T R u_{N-1} + (Ax_{N-1} + Bu_{n-1})^T P_N (Ax_{N-1} + Bu_{N-1})\right]}_{\text{Standard LQR}} \\ &\underbrace{+ E\left[\underbrace{(Ax_{N-1} + Bu_{N-1})^T}_{0} w_{N-1} + w_{N-1}^T P_N \underbrace{(Ax_{N-1} + Bu_{N-1})}_{0} + \underbrace{w_{N-1} P_N w_{N-1}}_{Constant!}\right]}_{\text{Noise Terms}} \end{aligned}$$

Noise sample drawn at time K has nothing to do with state (or control) at time k. x_k depends on w_{k-1} (and all past w) but not on w_k or future w

$\Rightarrow$ Uncorrelated $\Rightarrow$ cross-correlation is zero

$\Rightarrow$ Noise terms have no impact on the controller design! (you just get a higher cost)

“Certainty-Equivalence Principle”

- The optimal LQG controller is just LQR with x replaced by E[x]

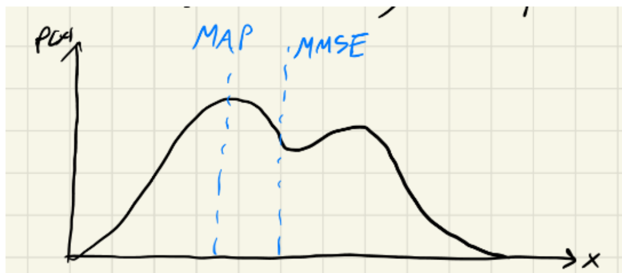
“Separation Principle”

- For LQG we can design an optimal feedback controller and an optimal estimator separately and then hook them together. The resulting feedback policy is optimal.

Neither of these holds in general but are still frequently used in practice to design sub-optimal policies

Optimal state Estimation:

- What should I try to optimize?



- Maximum a-posterior: (MAP):

$$argmax \quad p(x|y)$$

- Minimum mean-squared error(MMSE):

$$argmin \quad E[(x-\hat{x})^T(x-\hat{x})]$$

- These are the same for a Gaussian!