16-745 Optimal Control Lecture 5

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January 30, 2025

1 Last Time

1. Minimization with equality constraints

2 Today

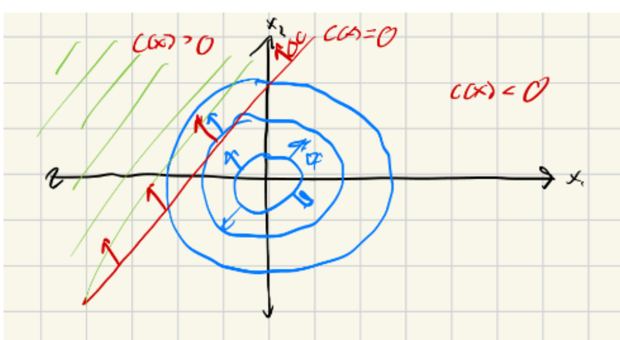
- Inequality constraints

3 Inequality Constrained Minimization

$$\begin{aligned} &\min_x f(x) \\ s.t. \quad &c(x) \geq 0 \end{aligned}$$

- KKT conditions:

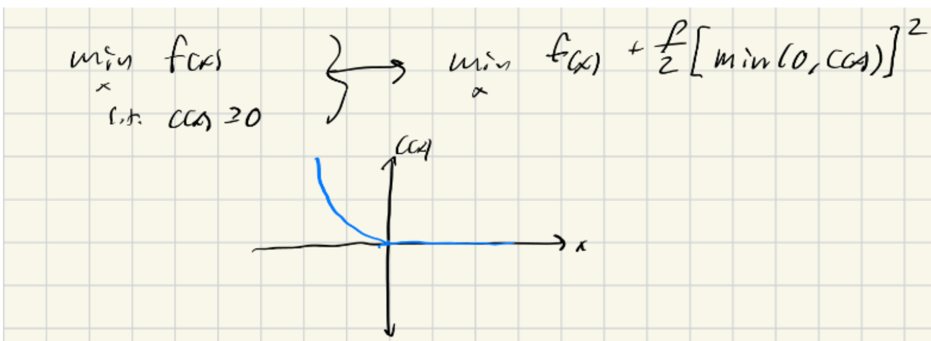
$$\begin{aligned} \nabla f - \left(\frac{\partial c}{\partial x}\right)^T \lambda = 0 \quad &\text{(stationarity)} \\ c(x) \geq 0 \quad &\text{(primal feasibility)} \\ \lambda \geq 0 \quad &\text{(deal feasibilty)} \\ \lambda \circ c(x) = 0 \quad &\text{(complementarity)} \end{aligned}$$



- Unlike equality case, we can't directly solve KKT conditions with Newton's method
- Lots of solution methods:

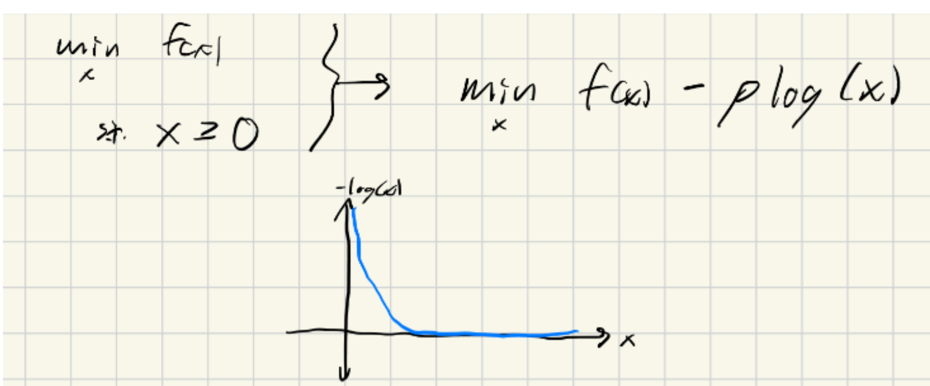
– Active Set:
  * Switch equality constraints on/off and solve inequality constrained problem
  * Works well if you can guess active set well
– Penalty Method
  * Replace constraints with cost terms that penalize violation:

$$\min_x f(x)$$
$$s.t. \quad c(x) \geq 0$$
$$\min_x f(x) + \frac{\rho}{2}[min(0, c(x)]^2$$



  * Easy to implement
  * Has issues with ill-conditioning (have to crank $\rho \to \infty$
  * Can't solve to high accuracy
  * Popular fix: estimate λ from penalty at each iteration $\to$ converge with finite ρ. Called "Augmented Lagrangian" (Also closely related ADMM)
– Interior Point/Barrier Methods
  * Replace inequalities with barrier function in objective

$$\min_x f(x)$$
$$s.t. \quad x \geq 0$$
$$\min_x f(x) - \rho \log(x)$$



* Gold standard for convex problems
* Fast convergence with Newton
* Strong theoretical properties
* Used in IPOPT

3.1 Primal Dual Interior Point Method

$$\min f(x)$$
$$s.t. \quad x \geq 0$$
$$\min_x f(x) - \rho \log(x)$$
$$\frac{\partial f}{\partial x} - \frac{\rho}{x} = 0$$

- This "primal" FON condition blows up as $x \to 0$
- we can fix this with the "primal-dual trick"
- Introduce new variable $\lambda = \frac{\rho}{x} \Rightarrow x\lambda = \rho$

$$\Rightarrow \nabla f - \lambda = 0$$
$$x\lambda = \rho \quad \text{relaxed complementarity from KKT}$$

- Converges to exact KKT solution as $\rho \to 0$. We lower gradually as solver converges from $\rho \approx 1$ to $\rho \approx 10^{-8}$
- Note we still need to enforce $x \geq 0$ and $\lambda \geq 0$ (with linesearch)
- We will use another approach ' Log-domain interior point methods for convex optimization"

3.2 Log-Domain Interior Point Method

- More general case:

$$\min_x f(x)$$
$$s.t. \quad c(x) \geq 0$$

- Simplify by introducing a "slack variable"

$$\min_{x,s} f(x)$$
$$s.t. \quad c(x) - s = 0$$
$$s \geq 0$$
$$\min_{x,s} \quad f(x) - \rho \log(s)$$
$$s.t. \quad c(x) - s = 0$$

- Lagrangian:

$$L(x, s, \lambda) = f(x) - \rho \log(s) - \lambda^T (c(x) - s)$$

- FON (first order necessary) Conditions:

$$\nabla_x L = \nabla f - \left(\frac{\partial c}{\partial x}\right)^T \lambda = 0$$
$$\nabla_s L = \frac{-\rho}{s} + \lambda = 0 \Rightarrow s \circ \lambda = \rho \quad \text{relaxed complementarity}$$
$$\nabla_\lambda L = s - c(x) = 0$$

- To ensure $s \geq 0$ and $\lambda \geq 0$, introduce change of variables:

$$s = \sqrt{\rho}\mathrm{e}^{\sigma} \Rightarrow \lambda = \sqrt{\rho}\mathrm{e}^{-\sigma}$$

- Now (relaxed) complementarity is always satisfied
- Plug back into FON conditions:

$$\nabla f - \left(\frac{\partial c}{\partial x}\right)^T \sqrt{\rho}\mathrm{e}^{-\sigma} = 0$$
$$c(x) - \sqrt{\rho}\mathrm{e}^{\sigma} = 0$$

- We can solve these with (Gauss) Newton:

$$\begin{bmatrix} H & \sqrt{\rho}C^T\mathrm{e}^{-\sigma} \\ C & -\sqrt{\rho}\mathrm{e}^{\sigma} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\sigma \end{bmatrix} = \begin{bmatrix} -\nabla f + C^T\sqrt{\rho}\mathrm{e}^{-\sigma} \\ -c(x) + \sqrt{\rho}\mathrm{e}^{\sigma} \end{bmatrix}$$

- Example: Quadratic Program

$$\min_x \frac{1}{2}x^T Q x + q^T x, \quad Q > 0$$
$$s.t. \quad Ax \geq 0$$
$$Cx = D$$

- Super usefel in control
- Can be solved very fast ($\sim$kHz)