16-745 Optimal Control Lecture 1

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January 21, 2025

1 Crash Course in Dynamics

1.1 Continuous Time dynamics

- The most generic/general for a smooth system: Smooth meaning that there is not contact/impact such as a foot hitting the ground:

  $$\dot{x} = f(x, u)$$

  Where f is the dynamics, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $\dot{x}$ is the time derivative of the state.

- For a mechanical system:

  $$x = \begin{bmatrix} q \\ v \end{bmatrix}$$

  Where q is the configuration/pose, and is not always a vector, but can be something weirder than that, such as orientation in a rigid body. Attitude is not a vector, so roll and pitch are not a vector, but a weird group thing or manifold- will cover later. v is the velocity, and is always a vector. So this is what the state vector is: Everything you need to specify the initial conditions of the system.

1.1.1 Pendulum Example

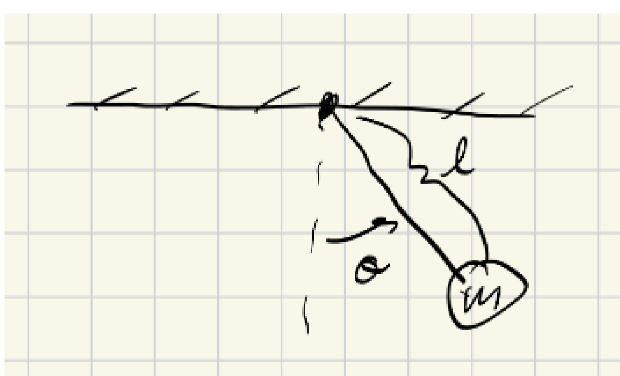

Figure 1: pendulum example

Dynamics:

$$ml^2\ddot{\theta} + myl\sin(\theta) = \tau$$

$$q = \theta, \quad v = \ddot{\theta}, \quad u = \tau$$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \frac{-q}{l}\sin(\theta) + \frac{1}{ml^2}u \end{bmatrix}$$

$q \in S^1$ (circle), $x \in S^1 \times \mathbb{R}$

1.2 Control-Affine Systems

Still non-linear systems in general have term:

$$\dot{x} = f_0 x + B(x)u$$

Ax + B is affine- linear + constant. If you freeze x, then you have control affine system. Where $f_0(x)$ is drift, and $B(x)u$ is input jacobian.

- Most systems can be put in this form. For the pendulum:

$$f_0(x) = \begin{bmatrix} \dot{\theta} \\ \frac{-q}{l}\sin(\theta) \end{bmatrix}$$

$$B(x) = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

- If B is full rank, can pretty do anything you want with the system

1.3 Manipulator Dynamics

$$M(q)\dot{v} + C(q, v) = B(q)u + F$$

M term is mass matrix, C term is dynamic bias (coriolis + gravity), B is input jacobian, and F is external forces

$$\dot{q} = G(q)v$$

Call this equation the velocity kinematics. Can easily turn into:

$$\dot{x} = f(x, u) = \begin{bmatrix} G(q)v \\ M^{-1}(q)(B(q)u + F - C) \end{bmatrix}$$

- Pendulum

$$M(q) = ml^2, \quad (q, v) = mgl\sin(\theta), B = I, G = I)$$

- All mechanical systems can be written this way
- This is just a way of re-writing the Euler-Lagrange equation for:

$$L = \frac{1}{2}V^T M(q)V - U(q)$$

  Where first term is kinetic energy, second is potential energy. M has some requirements, it corresponds to a mass written in joint space coordinates, so it is in quadratic form, and needs to be symmetric-positive-definite, i.e. must be invertable.

1.4 Linear Systems

$$\dot{x} = A(t)x + B(t)u$$

- Called "time invariant" if $A(t) = A$, $B(t) = B$
- Called "time varying" otherwise
- Super important in control
- We often approximate non-linear systems with linear ones locally.

$$\dot{x} = f(x, u)$$
$$A = \frac{\partial f}{\partial x} \quad B = \frac{\partial f}{\partial u}$$