

On the Stability Properties of Quadruped Creeping Gaits*

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Communicated by Richard Bellman

ABSTRACT

While the total number of theoretically possible quadruped gaits is quite large, only six gaits have the property that they can be executed while keeping at least three feet on the ground at all times. These gaits, called creeping gaits, seem to be well suited for low-speed locomotion since they permit a quadruped to remain statically stable during most of a locomotion cycle. A mathematical analysis shows, however, that for only three of the six creeping gaits it is possible to place the feet of an animal or machine so that it is statically stable at all times. Furthermore, among these three, there exists a unique optimum gait that maximizes static stability. This gait corresponds to the normal quadruped crawl favored by most animals for very low-speed locomotion.

INTRODUCTION

The discrete aspects of legged locomotion have drawn the attention of a number of investigators, who have all been interested in the problem of identifying and classifying the natural and theoretically possible patterns of footfalls or gaits used by terrestrial animals for locomotion [1-4]. More recently, improvements in technology have led to the development of artificial legged locomotion systems and have raised certain questions relating to the control and stability of such machines [5-8]. This article is directed toward the formulation and solution of a particular stability problem that arises in connection with both natural and artificial quadrupeds.

* This research was supported by the U.S. Air Force Office of Scientific Research under grant AF-AFOSR-67-1018A.

Specifically, the question to be resolved is the following. Among all of the theoretically possible quadruped gaits, which ones have the property that the feet in contact with the ground can be placed so that the quadruped is statically stable at all times? These gaits are of particular importance for both animals and machines since they represent modes of locomotion that can be employed for arbitrarily slow motion.

The analysis to follow makes use of an extension of the finite state model introduced in [3] to include the basic kinematic parameters associated with the geometrical aspects of a quadruped machine or animal. The principal result obtained is that only three quadruped gaits are suitable for very low-speed locomotion and that one of these possesses stability properties superior to the other two. This gait is the typical quadruped *crawl* recognized by Muybridge [2] as being the normal gait employed by both natural quadrupeds and human beings for very low-speed four-legged locomotion.

SUPPORT PATTERNS AND STATIC STABILITY

Due to the high degree of complexity of natural and artificial locomotion systems based upon articulated limbs, it is necessary to make certain simplifying assumptions in order to obtain some concrete results relating to stability. For the purposes of this paper, only steady-state constant speed locomotion in a straight line over a horizontal plane supporting surface with the legs of the system cycling periodically in both space and time will be considered. The legs of the systems to be investigated will be idealized in the respect that the foot of each supporting leg will be assumed to touch the supporting surface at a single point called the *contact point*. Finally, it will be required that the force exerted by any leg at its contact point be directed into the supporting surface and that no moments be applied by a leg to the support plane; that is, locomotion with grasping feet will not be treated. These assumptions and constraints are formalized in the following definition, which imposes some additional requirements. All of the results to follow are obtained with respect to this mathematical model.

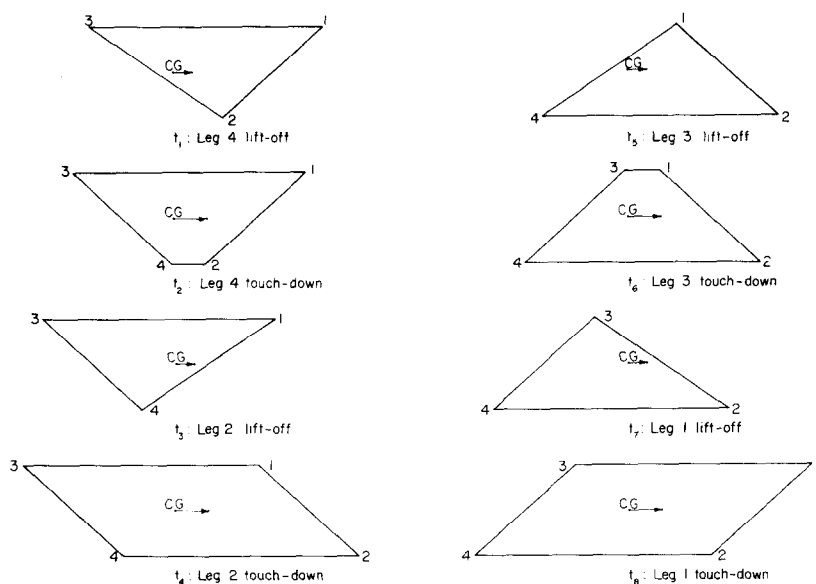
Definition 1. An ideal legged locomotion machine is a rigid body to which are attached a specified number n of massless legs. The length of each leg is arbitrarily controllable. Each leg contacts the supporting surface at a point and can exert an arbitrary force directed into this

surface. Arbitrary moments can be applied to the body by any leg subject only to the constraint that no moment be applied to the supporting surface at any leg contact point.

Given the contact point for each supporting leg of an ideal legged locomotion machine, a *support pattern* [1] for any phase of a gait is defined as follows.

Definition 2. The *support pattern* associated with any phase of a given gait of an ideal legged locomotion machine is the minimum area convex point set in the support plane such that all of the leg contact points are contained.

From this definition it is clear that when m feet of a machine are in contact with its support plane, the support pattern is a polygon of not



DUTY FACTOR: $\frac{11}{12}$

PHASE RELATIONSHIPS: $\phi_2 = \frac{1}{2}$, $\phi_3 = \frac{3}{4}$, $\phi_4 = \frac{1}{4}$

FIG. 1. Support patterns for a nonsingular regular quadruped crawl. Arrows indicate total motion of center of gravity during any phase.

more than m sides. Figure 1 shows the support patterns associated with the previously mentioned crawl gait [2, 3]. This gait alternates three and four feet on the ground so its support patterns are alternately triangles and quadrilaterals.

Support patterns are related to gait stability for ideal legged locomotion machines by the following definition and theorem.

Definition 3. An ideal legged locomotion machine is *statically stable at time t* if all legs in contact with the support plane at the given time remain in contact with that plane when all legs of the machine are fixed at their locations at time t and the translational and rotational velocities of the resulting rigid body are simultaneously reduced to zero.

THEOREM 1. *An ideal legged locomotion machine supported by a stationary horizontal plane surface is statically stable at time t if and only if the vertical projection of the center of gravity of the machine onto the supporting surface lies within its support pattern at the given time.*

Proof. If a machine is not statically stable at time t then, due to the rigid body assumption and the character of the legs of an ideal legged locomotion machine, there must exist an overturning moment resulting solely from gravitational acceleration. Elementary mechanical considerations show that such a moment exists if and only if it is possible to draw a line in the support plane that passes through at least one leg contact point and that separates the vertical projection of the center of gravity onto that plane from the contact points of all other legs. But the set of all such lines is just the set of all tangents to the support pattern at time t . Consequently, since all contact points are interior to the support pattern and this pattern is a convex point set, a machine is statically stable at time t if and only if the vertical projection of the center of gravity is also interior to the support pattern.

CREEPING GAITS AND STABILITY MARGIN

While Theorem 1 provides a necessary and sufficient condition for the static stability of an ideal machine, a real machine will be subjected to various types of disturbances and consequently may not remain upright even though the conditions of this theorem are satisfied at all times. It is therefore desirable to introduce a notion of relative stability

that provides some indication of the ability of a machine to resist disturbing influences. The following definition provides such a measure.

Definition 4. The magnitude of the *static stability margin at time t* for an arbitrary support pattern is equal to the shortest distance from the vertical projection of the center of gravity to any point on the boundary of the support pattern. If the pattern is statically stable, the stability margin is positive. Otherwise, it is negative.

This definition leads to the following theorem.

THEOREM 2. For an ideal legged locomotion machine, the minimum number of legs required to achieve a gait with a strictly positive static stability margin at all times is equal to three. If the time required to transfer a leg contact point to a new position is greater than zero, the number is four.

Proof. If the stability margin is greater than zero, then the area of the support pattern must also be greater than zero. This is possible only if the support pattern has at least three sides, implying at least three feet on the ground. If no time is required for movement of a leg contact point from one position to another, then successive overlapping triangles can be utilized to contain a moving center of gravity at all times. However, if the leg transfer time is greater than zero, whenever any leg is lifted from the supporting surface, three other legs must be in contact with that surface, implying at least four legs altogether. The sufficiency of four legs is established by the support patterns for a crawl shown in Fig. 1.

For quadrupeds, it is evident that all gaits that satisfy Theorem 2 are of the following type.

Definition 5 (Tomović [9]). A gait of an n -legged machine is a *creeeping gait* if every support pattern involves at least $n - 1$ contact points.

Creeeping gaits may be either *singular* or *nonsingular*, depending on whether or not simultaneous lifting and placing of two or more feet ever occurs [3]. Since any singular gait can be obtained as the limit of some nonsingular gait [3], the analysis to follow will concentrate on nonsingular gaits.

While it can be shown that there exist a total of 5040 theoretically possible nonsingular quadruped gaits [3], only a small number of these qualify as creeping gaits. The next theorem establishes the number of such gaits and identifies them.

THEOREM 3. *There are exactly six nonsingular quadruped creeping gaits.*

Proof. A nonsingular quadruped gait always involves exactly eight phases since no two feet are ever lifted or placed simultaneously [3]. Since a creeping gait never has more than one foot off of the supporting surface, if it is nonsingular, its support patterns must involve alternately three and four feet on the ground. Thus, if the feet are numbered 1, 2, 3, 4, as shown in Fig. 1, the succession of supporting feet is uniquely determined by a permutation of these numbers that specifies the order in which feet are placed; for example, the crawl of Fig. 1 corresponds to the permutation 1432. Since the first number in such a permutation is arbitrary, it can always be chosen to be equal to 1. With this choice, the five other possible permutations are 1423, 1342, 1324, 1234, 1243. Each of these is illustrated in Fig. 2.

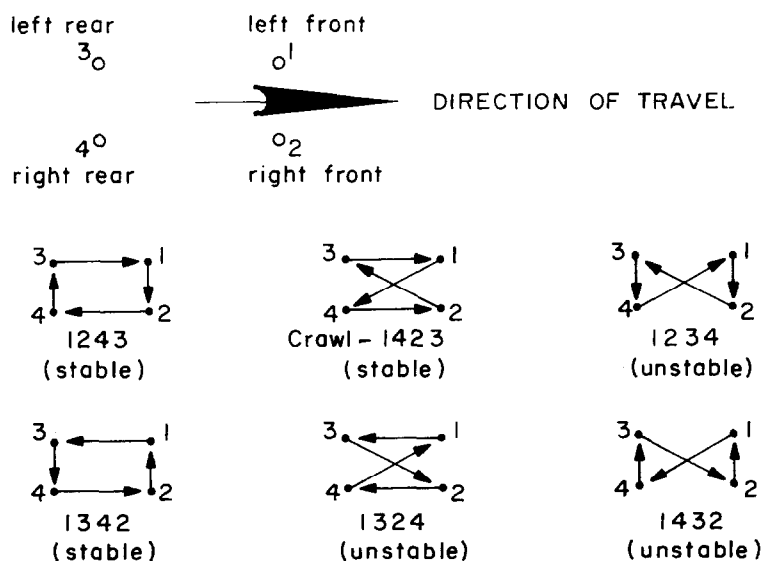


FIG. 2. Quadruped creeping gaits. (a) Labeling of feet; (b) sequence of foot lifting and placing.

A KINEMATIC GAIT MODEL

In [3], a $(2n - 1)$ -parameter model for any gait of an n -legged locomotion machine is introduced. This model, called a "gait formula," completely describes the sequential characteristics of a gait, but omits all of its spatial properties. The following definitions extend the idea of a gait formula to include certain kinematic aspects of locomotion.

Definition 6 (Hildebrand [1]). The *stride length* of a gait is the distance λ by which the body of a locomotion machine is translated during any complete leg cycle.

Definition 7. The *dimensionless foot position* (x_i, y_i) for leg i of a legged locomotion machine is a pair of coordinate values that specifies the position of the contact point of any supporting leg. The origin of the xy coordinate system is the center of gravity of the machine. The x coordinate axis is aligned with the direction of motion with positive x directed forward. The y coordinate axis is normal to the x axis and is oriented so that it is positive on the left side of the machine. The scale of the x and y coordinate axis is chosen so that $\lambda = 1$.

Definition 8. The *dimensionless initial foot position* (γ_i, δ_i) is the value for the pair x_i, y_i that exists at the time leg i first contacts the supporting surface during any locomotion cycle.

The gait formula defined in [3] involves two types of parameters. The first of these is the *duty factor* β_i , which expresses the fraction of time that leg i is in contact with the supporting surface during one complete cycle of locomotion. The second is the relative phase ϕ_i , which indicates the amount by which the motion of leg i lags the motion of leg 1 expressed as a fraction of the time required to complete one cycle of locomotion. These parameters together with the initial foot position variables are combined in the following definition.

Definition 9. A kinematic gait formula k for an n -legged locomotion machine is the $(4n - 1)$ -tuple

$$k = (\beta_1, \beta_2, \dots, \beta_n, \gamma_1, \gamma_2, \dots, \gamma_n, \delta_1, \delta_2, \dots, \delta_n, \phi_2, \phi_3, \dots, \phi_n). \quad (1)$$

The following theorem establishes the extent to which a kinematic gait formula determines the motion of a locomotion machine.

THEOREM 4. *A kinematic gait formula completely specifies the position of the supporting feet and the vertical projection of the center of gravity of an ideal legged locomotion machine in both time and space up to a multiplicative factor of τ in time and λ in space.*

Proof. If t' stands for a time variable measured in arbitrary units and τ is the *cycle time* [3] required for one complete locomotion cycle measured in the same units, then a dimensionless time variable can be defined as $t = t'/\tau$. If $t = 0$ is chosen to correspond to the time when leg 1 first touches the supporting surface, then, since x_i is a spatially normalized variable,

$$x_1(t) = \gamma_1 - t, \quad 0 \leq t \leq \beta_1. \quad (2)$$

Since leg 1 does not touch the ground during $\beta_1 < t < 1$, $x_1(t)$ is undefined for this period of time. Since t is normalized, the period of $x_1(t)$ is equal to 1 so (2) determines $x_1(t)$ for all t such that leg 1 is in a supporting phase. For any other leg i ,

$$x_i(t) = \gamma_i - (t - \phi_i), \quad \phi_i \leq t \leq \phi_i + \beta_i, \quad (3)$$

and $x_i(t)$ is undefined for $\phi_i + \beta_i < t < \phi_i + 1$. Again, since the period of leg i is also equal to 1, (3) determines the dimensionless leg position for any supporting leg.

Since the motion of the machine is entirely along the x axis and the leg positions are required to be periodic in space, $y_i = \delta_i$ whenever x_i is defined. Thus, k determines (x_i, y_i) for all supporting legs for all t . Moreover, if (x_0, y_0) is the dimensionless location of the vertical projection of the center of gravity of the machine relative to its initial position on the support plane, then

$$(x_0, y_0) = (t, 0), \quad (4)$$

so the center of gravity location is also determined. Finally, multiplication of any dimensionless pair (x_i, y_i) by λ produces positions in the units of λ .

OPTIMUM KINEMATIC GAIT FORMULAS

The crawl gait shown in Fig. 1 is described by the kinematic gait formula

$$k = \left(\frac{11}{12}, \frac{11}{12}, \frac{11}{12}, \frac{11}{12}, 1, 1, 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{4} \right). \quad (5)$$

While examination of this figure shows that this gait possesses a positive stability margin at all times, it is possible that some other kinematic gait formula would produce a more stable crawl. It is useful, therefore, to attempt to find *optimum* gait formulas for all six of the creeping gaits of Fig. 2. The following definitions make this notion precise.

Definition 10. The *dimensionless longitudinal stability margin at time t* for an arbitrary support pattern associated with a kinematic gait formula k is denoted by $s(t, k)$ and is equal in magnitude to the shortest distance along the x coordinate axis from the center of gravity of a locomotion machine to an edge of its support pattern. If the pattern is statically stable, the stability margin is positive. Otherwise, it is negative.

Definition 11. A kinematic gait formula k^* for a given gait matrix [3] G is *longitudinally optimum* if

$$\max_{k \in K_G} \min_{t \in [0, 1]} s(t, k) = \min_{t \in [0, 1]} s(t, k^*) = s^*(G) \quad (6)$$

where K_G is the set of all gait formulas that imply G [3]. The function $s^*(G)$ is the *optimum dimensionless longitudinal stability margin* for G .

The connection between $s^*(G)$ and static stability for quadruped creeping gaits is established by the following theorem.

THEOREM 5. *Let G be the gait matrix for a quadruped creeping gait and let K_G be the set of all kinematic gait formulas that imply G . Then there exists a $k \in K_G$ such that the machine associated with k possesses a strictly positive static stability margin at all times if and only if $s^*(G) > 0$. Moreover, when such a k exists, it satisfies the constraints $\delta_1 > 0$, $\delta_3 > 0$, $\delta_2 < 0$, $\delta_4 < 0$.*

Proof. Suppose the condition on the δ variables is satisfied. That is, the machine has two feet on each side of its centerline. In this case,

the projection of its center of gravity is in the interior of its support pattern and it has a strictly positive static stability margin if and only if its longitudinal stability margin is strictly positive. But this is satisfied at all times if and only if $s^*(G) > 0$. Now suppose the condition on the δ variables is not satisfied. Then there must exist some phase of the gait such that the center of gravity of the machine lies either on the right or left boundary of the support pattern or entirely outside of it. Consequently, independently of $s^*(G)$, such a gait will fail to have a strictly positive static stability margin at some time in any complete locomotion cycle. Thus, k exists if and only if $s^*(G) > 0$ and $\delta_1 > 0$, $\delta_3 > 0$, $\delta_2 < 0$, $\delta_4 < 0$.

OPTIMIZATION OF THE CRAWL GAIT

Examination of Fig. 1 shows that, because of the forward motion of the center of gravity of a locomotion machine, the minimum value of the longitudinal stability margin associated with any support pattern of any gait occurs either at the moment the pattern is established or at the end of the interval in which the pattern exists. In addition, if the minimum occurs at the beginning of the interval, it is associated with a line at the rear of the support pattern, whereas if it occurs at the end, it is associated with a line at the front of the pattern. Thus, since a total of eight support patterns comprise a nonsingular crawl gait, there are eight critical times at which the minimum longitudinal stability margin may occur. These are all indicated in the following set of eight linear equations, which establish the position of the critical feet at the times indicated.

t_1 , Leg 4 lift-off:

$$x_3 = \gamma_3 + \phi_3 - \phi_4 - \beta_4; \quad (7)$$

$$x_2 = \gamma_2 + \phi_2 - \phi_4 - \beta_4. \quad (8)$$

t_2 , Leg 4 touch-down:

$$x_1 = \gamma_1 - \phi_4; \quad (9)$$

$$x_2 = \gamma_2 + \phi_2 - \phi_4 - 1. \quad (10)$$

t_3 , Leg 2 lift-off:

$$x_3 = \gamma_3 + \phi_3 - \phi_2 - \beta_2; \quad (11)$$

$$x_4 = \gamma_4 + \phi_4 - \phi_2 - \beta_2 + 1. \quad (12)$$

t_4 , Leg 2 touch-down:

$$x_1 = \gamma_1 - \phi_2; \quad (13)$$

$$x_4 = \gamma_4 + \phi_4 - \phi_2. \quad (14)$$

t_5 , Leg 3 lift-off:

$$x_1 = \gamma_1 - \phi_3 - \beta_3 + 1; \quad (15)$$

$$x_4 = \gamma_4 + \phi_4 - \phi_3 - \beta_3 + 1. \quad (16)$$

t_6 , Leg 3 touch-down:

$$x_1 = \gamma_1 - \phi_3; \quad (17)$$

$$x_2 = \gamma_2 + \phi_2 - \phi_3. \quad (18)$$

t_7 , Leg 1 lift-off:

$$x_3 = \gamma_3 + \phi_3 - \beta_1; \quad (19)$$

$$x_4 = \gamma_4 + \phi_4 - \beta_1. \quad (20)$$

t_8 , Leg 1 touch-down:

$$x_2 = \gamma_2 + \phi_2 - 1; \quad (21)$$

$$x_3 = \gamma_3 + \phi_3 - 1. \quad (22)$$

In order to calculate the longitudinal stability margin at each of the foregoing times, it is necessary to assume some values for the y coordinate of each leg. Since all natural quadrupeds possess right-left symmetry, it will be convenient to make this assumption here. In addition, to simplify the analysis still further, it will also be assumed that the spacing between the front legs in the y direction is the same as for the rear legs. Thus, the class of gait formulas K_G to be considered satisfy the constraint

$$\delta_1 = \delta_3 = -\delta_2 = -\delta_4 = \delta > 0. \quad (23)$$

With this constraint, the longitudinal stability margin is independent of δ . In particular, for the crawl gait, from Fig. 1 and Eqs. (7) and (8), at $t = t_1$, the horizontal distance from the center of gravity to the rear of the support pattern is just

$$f_1(k) = \frac{-x_2 - x_3}{2} = \frac{1}{2} \{-\gamma_2 - \gamma_3 - \phi_2 - \phi_3 - 2\phi_4 + 2\beta_4\}. \quad (24)$$

At $t = t_2$, the distance to the front of the support pattern is

$$f_2(k) = \frac{x_1 + x_2}{2} = \frac{1}{2} \{\gamma_1 + \gamma_2 + \phi_2 - 2\phi_4 - 1\}. \quad (25)$$

Consequently

$$\min_{t \in [t_1, t_2]} x(t, k) = \min_{i \in \{1, 2\}} f_i(k). \quad (26)$$

Continuing this analysis for the other support patterns, and associating $f_i(k)$ with conditions at $t = t_i$, from Eqs. (11) through (22), the appropriate functions are

$$f_3(k) = \frac{-x_3 - x_4}{2} = \frac{1}{2} \{-\gamma_3 - \gamma_4 - \phi_3 - \phi_4 + 2\phi_2 + 2\beta_2 - 1\}; \quad (27)$$

$$f_4(k) = \frac{x_1 + x_4}{2} = \frac{1}{2} \{\gamma_1 + \gamma_4 + \phi_4 - 2\phi_2\}; \quad (28)$$

$$f_5(k) = \frac{-x_1 - x_4}{2} = \frac{1}{2} \{-\gamma_1 - \gamma_4 - \phi_4 + 2\phi_3 + 2\beta_3 - 2\}; \quad (29)$$

$$f_6(k) = \frac{x_1 + x_2}{2} = \frac{1}{2} \{\gamma_1 + \gamma_2 + \phi_2 - 2\phi_3\}; \quad (30)$$

$$f_7(k) = \frac{-x_3 - x_4}{2} = \frac{1}{2} \{-\gamma_3 - \gamma_4 - \phi_3 - \phi_4 + 2\beta_3\}; \quad (31)$$

$$f_8(k) = \frac{x_2 + x_3}{2} = \frac{1}{2} \{\gamma_2 + \gamma_3 + \phi_2 + \phi_3 - 2\}. \quad (32)$$

Finally, by analogy to (26)

$$\min_{t \in [0, 1]} s(t, k) = \min_{i \in \{1, 2, \dots, 8\}} f_i(k) = f_0(k) \quad (33)$$

and k^* is found by maximizing the *criterion function* $f_0(k)$ over K_G .

Referring again to Fig. 1, it is evident that increasing the fraction of time β_i that leg i spends in contact with the supporting surface never reduces the static stability margin of a gait. Thus, for any gait, $f_0(k)$ is nondecreasing in β_i and at least one k^* will involve $\beta_i = 1$ for all i . But this is impossible for a nonsingular gait [3]. To avoid this difficulty, the further restriction will be placed on K_G that

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta; \quad (34)$$

that is, only *regular* gaits [3] will be considered. The criterion function $f_0(k)$ will then be optimized for a fixed value of β .

Even after the parameters β_i are fixed according to (34), $f_0(k)$ remains a nonlinear function of seven variables to be minimized subject to the inequalities

$$\phi_4 \geq 1 - \beta, \quad (35)$$

$$\phi_2 \geq \phi_4 + (1 - \beta), \quad (36)$$

$$\beta \geq \phi_3 \geq \phi_2 + (1 - \beta), \quad (37)$$

which are dictated by the sequence of support patterns involved in a crawl. Problems of this type fall within the domain of nonlinear programming [10] and are typically very difficult to solve. In this particular case, however, a solution has been found by means of relatively straightforward iterative procedure. Rather than recording the details of this calculation, the results are presented as a theorem.

THEOREM 6. *If $\delta_1 = \delta_3 = -\delta_2 = -\delta_4 = \delta$, then for $1 > \beta > \frac{3}{4}$, any longitudinally optimum kinematic gait formula for a regular crawl gait must be of the form*

$$k = \left(\beta, \beta, \beta, \beta, a + \frac{\beta}{2}, a + \frac{\beta}{2}, -a + \frac{\beta}{2}, \right. \\ \left. -a + \frac{\beta}{2}, \delta, -\delta, \delta, -\delta, \frac{1}{2}, \beta, \beta - \frac{1}{2} \right) = k^* \quad (38)$$

where

$$a \geq \frac{\beta}{2}, \quad \delta > 0. \quad (39)$$

Furthermore, such a crawl is a singular crawl with the gait matrix [3]

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (40)$$

The longitudinal stability margin for this gait is given by

$$s^*(G) = \beta - \frac{3}{4}. \quad (41)$$

Proof. First of all, it is clear that the constraints imposed by (35), (36), and (37) are satisfied. Then, substituting k^* into (24), (25), and (27) through (32),

$$f_1(k^*) = \beta - \frac{3}{4}, \quad (42)$$

$$f_2(k^*) = a - \frac{\beta}{2} + \frac{1}{4}, \quad (43)$$

$$f_3(k^*) = a - \frac{\beta}{2} + \frac{1}{4}, \quad (44)$$

$$f_4(k^*) = \beta - \frac{3}{4}, \quad (45)$$

$$f_5(k^*) = \beta - \frac{3}{4}, \quad (46)$$

$$f_6(k^*) = a - \frac{\beta}{2} + \frac{1}{4}, \quad (47)$$

$$f_7(k^*) = a - \frac{\beta}{2} + \frac{1}{4}, \quad (48)$$

$$f_8(k^*) = \beta - \frac{3}{4}. \quad (49)$$

Thus, the criterion function $f_0(k^*)$ is determined either by f_1 , f_4 , f_5 , and f_8 or by f_2 , f_3 , f_6 , and f_7 . To determine which of these cases applies, it is necessary to consider the allowed range of values for the arbitrary constant a . A lower bound on a can be obtained from the requirements

$$x_1(t) \geq x_3(t) \quad (50)$$

and

$$x_2(t) \geq x_4(t) \quad (51)$$

imposed by the necessity of avoiding interference between front and rear feet. From Fig. 1 it can be seen that the minimum value for $x_1(t) - x_3(t)$ occurs at $t = t_6$, at which time

$$x_1(t) - x_3(t) = \gamma_1 - \phi_3 - \gamma_3 = 2a - \beta \geq 0, \quad (52)$$

so

$$a \geq \frac{\beta}{2} \quad (53)$$

and for $i = 2, 3, 6, 7$

$$f_i(k^*) \geq \frac{1}{4}. \quad (54)$$

Solution of (51) yields identical results. On the other hand, for $i = 1, 4, 5, 8$, since $\beta < 1$,

$$f_i(k^*) < \frac{1}{4}. \quad (55)$$

It follows, therefore, that the longitudinal stability margin is determined by f_1 , f_4 , f_5 , and f_8 for all allowed values of a .

Since (42), (45), (46), and (49) all yield the same values for the k given by (38), this value is optimum unless there exists some vector Δk such that

$$f_0(k^* + \Delta k) > f_0(k^*). \quad (56)$$

This in turn is possible if and only if f_1 , f_4 , f_5 , and f_8 can all be increased simultaneously. To determine whether or not this is possible, from (24), (28), (29), and (32), substitution of $k^* + \Delta k$ for k^* results in the relationships

$$\Delta f_1 = \frac{1}{2}\{-\Delta\gamma_2 - \Delta\gamma_3 - \Delta\phi_2 - \Delta\phi_3 + 2\Delta\phi_4\}, \quad (57)$$

$$\Delta f_4 = \frac{1}{2}\{\Delta\gamma_1 + \Delta\gamma_4 + \Delta\phi_4 - 2\Delta\phi_2\}, \quad (58)$$

$$\Delta f_5 = \frac{1}{2}\{-\Delta\gamma_1 - \Delta\gamma_4 - \Delta\phi_4 + 2\Delta\phi_3\}, \quad (59)$$

$$\Delta f_8 = \frac{1}{2}\{\Delta\gamma_2 + \Delta\gamma_3 + \Delta\phi_2 + \Delta\phi_3\}, \quad (60)$$

where

$$\Delta f_i = f_i(k^* + \Delta k) - f_i(k^*). \quad (61)$$

Since $\phi_3 = \beta$ in (39), the left-hand side of (38) is satisfied as an equality. Consequently it is necessary in (57) through (60) that

$$\Delta\phi_3 \leq 0. \quad (62)$$

Likewise, since $\phi_2 = \frac{1}{2}$ and $\phi_4 = \beta - \frac{1}{2}$ in (38), (36) is also satisfied as an equality with the consequence that

$$\Delta\phi_2 = \Delta\phi_4 + \varepsilon, \quad \varepsilon \geq 0, \quad (63)$$

must also be satisfied in the evaluation of (57) through (60).

Substituting (63) into (57) through (60) and requiring that $\Delta f_i > 0$, $i = 1, 4, 5, 8$, results in the expressions

$$-\Delta\gamma_2 - \Delta\gamma_3 - \Delta\phi_3 + \Delta\phi_4 - \varepsilon > 0, \quad (64)$$

$$\Delta\gamma_1 - \Delta\gamma_4 + \Delta\phi_3 - 2\varepsilon > 0, \quad (65)$$

$$-\Delta\gamma_1 - \Delta\gamma_4 + 2\Delta\phi_3 - \Delta\phi_4 > 0, \quad (66)$$

$$\Delta\gamma_2 + \Delta\gamma_3 + \Delta\phi_3 + \Delta\phi_4 + \varepsilon > 0. \quad (67)$$

Addition of these four equations produces the relation

$$2\Delta\phi_3 - 2\varepsilon > 0 \quad (68)$$

or

$$\Delta\phi_3 > \varepsilon \geq 0. \quad (69)$$

But this is a contradiction since (62) requires $\Delta\phi_3 \leq 0$. Consequently, no Δk exists that simultaneously increases f_1, f_4, f_5 , and f_8 so (38) is optimum. The G matrix associated with (38) can be constructed from the algorithm given in [3].

To show the uniqueness of (38), it is sufficient to equate (64) through (67) to zero. In that case, (69) becomes

$$\Delta\phi_3 = \varepsilon \geq 0. \quad (70)$$

In view of (62), this implies

$$\Delta\phi_3 = \varepsilon = 0, \quad (71)$$

which in turn requires that $\Delta\phi_4$ and $\Delta\phi_2$ also equal zero in order for (64) through (67) to be satisfied as equalities. Thus there is no $\Delta k \neq 0$ such that $f_0(k^* + \Delta k) = f_0(k^*)$ and (38) is therefore unique.

OPTIMIZATION OF OTHER CREEPING GAITS

In [5], equations analogous to (24) through (32) are developed for each of the five other creeping gaits. These equations can be subjected to an optimization process in the same way as the crawl gait was treated. The results of such an optimization are summarized in Table I [11], which provides the basis for the following theorem.

TABLE I

OPTIMUM DIMENSIONLESS LONGITUDINAL
STABILITY MARGINS FOR REGULAR
QUADRUPED CREEPING GAITS

Footfall sequence	Stability margin
1423	$\beta - \frac{3}{4}$
1342	$\frac{3}{2}\beta - \frac{5}{4}$
1243	$\frac{3}{2}\beta - \frac{5}{4}$
1432	$\beta - 1$
1324	$\beta - 1$
1234	$\beta - 1$

β = duty factor.

THEOREM 7. *For any specified value of the duty factor β in the range $\frac{3}{4} < \beta < 1$, the optimum dimensionless longitudinal stability margin for a regular crawl is greater than for any other regular gait. Moreover, there are only two other nonsingular creeping gaits, 1342 and 1243, which possess kinematic gait formulas with a strictly positive value for their minimum longitudinal stability margin for any $\beta < 1$.*

Proof. Table I proves the first part of the theorem and the second part with respect to regular gaits. For irregular gaits, it is sufficient to note that since the minimum longitudinal stability margin is nondecreasing in β_i , if

$$\beta_m = \max_i \beta_i, \quad (72)$$

then the regular gait with $\beta = \beta_m$ will be at least as stable as the associated irregular gait. It follows, therefore, that since the stability margin of regular kinematic gait formulas for the gaits 1324, 1432, and 1234 is bounded by

$$s^*(G) = \beta - 1 < 0 \quad (73)$$

this bound also applies to the irregular gaits.

DISCUSSION OF RESULTS

The information contained in Table I is in dimensionless form. In absolute form, in any units, since $s^*(G)$ is normalized to λ , the denormalized optimum stability margin is given by

$$x_s = \lambda s^*(G). \quad (74)$$

This shows that it is desirable to make λ as large as possible consistent with satisfying the requirement for no interference between legs expressed by (52). Since the variable a in this expression is normalized to λ while β is not, if x_a is the denormalized value of a , (52) becomes

$$2x_a - \lambda\beta \geq 0 \quad (75)$$

so the maximum value for λ is

$$\lambda_m = \frac{2x_a}{\beta}. \quad (76)$$

The corresponding value for a is

$$a_m = \frac{x_a}{\lambda_m} = \frac{\beta}{2} \quad (77)$$

so from (38), the unique regular crawl gait that maximizes the denormalized minimum longitudinal stability margin is defined by

$$k = (\beta, \beta, \beta, \beta, \beta, \beta, 0, 0, \delta, -\delta, \delta, -\delta, \frac{1}{2}, \beta, \beta - \frac{1}{2}), \quad (78)$$

which involves only the two parameters δ and β . This gait is illustrated by Fig. 3.

Examination of Muybridge's photographs [2] shows that Fig. 3 corresponds closely both in space and time to the normal slow quadruped

walk. The only important exception is that since (78) requires that each rear foot be placed in the footprint of the foot ahead of it at precisely the time the front foot is lifted, living quadrupeds employ $\beta < \frac{3}{4}$ so that the front foot is lifted a short time before the rear foot is placed in order to provide the necessary clearance between feet. The rear foot is sometimes

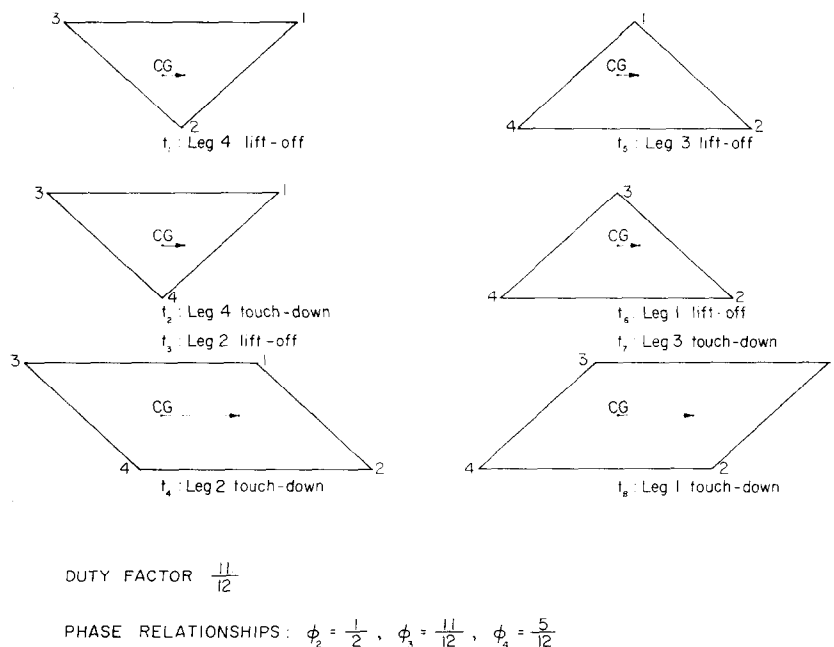


FIG. 3. Support patterns for the optimal regular quadruped crawl. Arrows indicate total motion of the center of gravity during a phase.

placed somewhat ahead of the front foot footprint to compensate for the early lifting of the front foot [1]. Apparently, for moderately low-speed locomotion, this solution to the foot interference problem is more satisfactory for the majority of living quadrupeds than the alternate solution of shortening the leg stroke of the optimum crawl defined by (78). It appears that a true crawl is generally used only for grazing and for other very low-speed activities [1].

Hildebrand [12] has observed that, apparently, no living quadruped uses the gait 1324 (gait number 5 in his classification scheme) and has conjectured that this might be due to the poor stability properties of this gait. The analysis presented here confirms this hypothesis. It seems

likely that the gaits 1432 and 1234 are also unused for the same reason. The authors are unaware of any observations of the gaits 1342 and 1243 in use by a living quadruped even though these gaits are stable for a sufficiently large duty factor. It may well be that the optimality of the crawl has caused it to be adopted as the unique gait utilized by natural quadrupeds for very low-speed locomotion.

SUMMARY AND CONCLUSIONS

The results obtained in this paper show that there exists a unique solution to the problem of choosing a gait to optimize the stability of low-speed quadruped locomotion. This gait corresponds essentially to the low-speed gait preferred by most natural quadrupeds. The analysis excludes certain types of locomotion due to assumptions on the geometry of the machine or animal. In particular, the machine is not allowed to drag a stabilizing tail as kangaroos and some legged reptiles do [13], nor is it permitted to overlap its feet as would be possible if the front pair of legs were spaced more widely or more narrowly than the rear pair.

While the conclusions reached in this paper are obtained with respect to an idealized model, the model appears to be sufficiently close to reality to permit qualitative extrapolation to real locomotion systems. Obviously, such characteristics of real systems as the nonzero mass of legs could be included in a more complicated analysis. Such analysis should probably be carried out for particular machines whenever it is important that the greatest possible degree of static stability be achieved.

The type of analysis carried out here could be readily extended to a study of insect gaits. It would be interesting to determine if insects also utilize optimally stable modes of locomotion. It appears that stability analysis of biped gaits requires the use of dynamic models that are necessarily more complicated than the kinematic model defined in this paper. The same is true of higher-speed quadruped gaits, which typically contain statically unstable phases [1].

REFERENCES

- 1 M. Hildebrand, Symmetrical gaits of horses, *Science* **150**(1965), 701-708.
- 2 E. Muybridge, *Animals in motion*. Dover, New York, 1957.
- 3 R. B. McGhee, Some finite state aspects of legged locomotion, *Math. Biosci.* **2**(1968), 67-84.
- 4 D. M. Wilson, Insect walking, *Ann. Rev. Entomol.* **2**(1966), 103-122.

Mathematical Biosciences **3**, 331-351 (1968)

- 5 A. A. Frank, *Automatic control of legged locomotion machines*. PhD dissertation, University of Southern California, May, 1968.
- 6 R. S. Mosher and R. A. Liston, The development of a quadruped walking machine, *ASME* (paper 67, Tran-34), August, 1967.
- 7 R. B. McGhee, Finite state control of quadruped locomotion, *Simulation* 9(1967), 135–140.
- 8 R. Tomović, On the synthesis of self-moving automata, *Automation and Remote Control* 26(1965), 298–307 (English trans.).
- 9 R. Tomović, A general theoretical model of creeping displacement, *Cybernetica* 4(1961).
- 10 G. Hadley, *Nonlinear and dynamic programming*. Addison-Wesley, Reading, Massachusetts, 1964.
- 11 R. B. McGhee and A. A. Frank, Optimum quadruped creeping gaits, USCEE Rept. 294, University of Southern California (July, 1968).
- 12 M. Hildebrand, Analysis of the symmetrical gaits of tetrapods, *Fol. Biotheoret.* VI(1966), 1–22.
- 13 R. Schaeffer, Evolution of the tarsus in amphibians and reptiles, *Bull. Amer. Mus. Nat. History* (November, 1944), pp. 106–121.