BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Finite elements: numerical analysis and implementation

Date: ??

Time: ??

Time Allowed: 2 Hours

This paper has *4 Questions*.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted.
- Each question carries equal weight.
- Calculators may not be used.

1. What is the choice of the geometric decomposition (allocation of nodal variables to cell and vertex entities) that leads to the maximum possible global continuity of finite element spaces defined on the interval $[0, L]$ constructed from the following one-dimensional elements (K, P, N). Justify your answer.

(a) $K = [a, b]$, P is linear polynomials, $N = (N_1, N_2)$ where $N_1[u] = u((a + b)/2)$, $N_2[u] = u'((a + b)/2)$.

[6 marks]

(b) $K = [a, b]$, P is quadratic polynomials, $N = (N_1, N_2, N_3)$ where $N_1[u] = u(a)$, $N_2[u] = u(b)$, $N_3[u] = \int_a^b u \, \mathrm{d}\, x$.

[6 marks]

(c) $K = [a, b]$, P is quadratic polynomials, $N = (N_1, N_2, N_3)$ where $N_1[u] = u'(a)$, $N_2[u] = u'(b)$, $N_3[u] = u((a + b)/2)$.

[7 marks]

2. (a) Consider the finite element $(K, \mathcal{P}, \mathcal{N})$, with

* K is a non-degenerate triangle,
* $\mathcal{P}$ is the space of polynomials on K of degree ≤ 1.
* $\mathcal{N} = (N_1, N_2, N_3)$, where

$$N_i(u) = \int_{f_i} u \, \mathrm{d}\, x,$$

where (f_1, f_2, f_3) are the edges of K, with f_1 joining vertices 1 and 2, f_2 joining vertices 2 and 3, and f_3 joining vertices 3 and 1.

Show that $\mathcal{N}$ determines $\mathcal{P}$.

[10 marks]

(b) Now consider the finite element $(K, \mathcal{P}, \mathcal{N})$, with

* K is a non-degenerate triangle,
* $\mathcal{P}$ is the space of polynomials on K of degree ≤ 2.
* $\mathcal{N} = (N_{1,1}, N_{1,2}, N_{2,1}, N_{2,2}, N_{3,1}, N_{3,2})$, where

$$N_{i,j}(u) = \int_{f_i} \phi_{i,j} u \, \mathrm{d}\, x,$$

where the edge test functions $\phi_{i,j}$ define a basis for linear functions restricted to f_i such that $\phi_{i,1} = 1$ on vertex 1 and 0 on vertex 2, $\phi_{i,2} = 1$ on vertex 2 and 0 on vertex 1, *etc.*

Show that $\mathcal{N}$ does not determine $\mathcal{P}$.

[10 marks]

3. (a) Let b be a continuous, coercive bilinear form on V, and F be a continuous linear form on V. Let $u \in V$ solve the linear variational problem

$$b(u, v) = F(v) \quad \forall v \in V.$$

Let V_h be a finite dimensional subspace of V, and let $u_h \in V$ solve the Galerkin approximation

$$b(u_h, v) = F(v) \quad \forall v \in V_h.$$

Show that

$$b(u - u_h, v) = 0, \quad \forall v \in V_h.$$

[4 marks]

(b) Hence, show that

$$\|u - u_h\|_V \leq \frac{M}{\gamma} \min_{v \in V_h} \|u - v\|_V,$$

where γ and M are the coercivity and continuity constants for b respectively.

[4 marks]

(c) Consider the variational problem of finding $u \in H^1([0, 1])$ such that

$$\int_0^1 vu + v'u' \,\mathrm{d}\,x = \int_0^1 vx \,\mathrm{d}\,x + v(1) - v(0), \quad \forall v \in H^1([0, 1]).$$

After dividing the interval $[0, 1]$ into N equispaced cells and forming a $P1$ C^0 finite element space V_N, the error $\|u - u_h\|_{H^1} = 0$ for any $N > 0$.
Explain why this is expected.

[6 marks]

(d) Let $\mathring{H}^1([0, 1])$ be the subspace of $H^1([0, 1])$ such that $u(0) = 0$. Consider the variational problem of finding $u \in \mathring{H}^1([0, 1])$ with

$$\int_0^1 v'u' \,\mathrm{d}\,x = \int_0^{1/2} v \,\mathrm{d}\,x, \quad \forall v \in \mathring{H}([0, 1]).$$

The interval $[0, 1]$ is divided into $3N$ equispaced cells (where N is a positive integer). After forming a $P1$ C^0 finite element space V_N, the error $\|u - u_h\|_{H^1}$ is found not to converge to zero. Explain why this is expected?

[6 marks]

4. The inhomogeneous Helmholtz equation in two dimensions is given by

$$\alpha(x)u - \nabla^2 u = f, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\partial\Omega$ is the boundary of the problem domain Ω, and $\alpha(x)$ is a $C^\infty(\Omega)$ function with bounds $1 \leq \alpha(x) \leq 2$ for all $z \in \Omega$.

(a) Write down a variational formulation for this problem, in the form

$$a(u, v) = F(v), \quad \forall v \in H^1(\Omega),$$

and show that if u solves the variational formulation, and $u \in H^2(\Omega)$ then u solves (1) in an appropriate sense.

[6 marks]

(b) Show that $a(\cdot, \cdot)$ is continuous and coercive.

[6 marks]

(c) Hence, show that the linear Lagrange finite element approximation satisfies

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}.$$

for $C > 0$, independent of u. (You may make use of the approximation theory estimate

$$\|u - I_h u\|_{H^1(\Omega)} \leq \hat{C}h\|u\|_{H^2(\Omega)}.$$

for $\hat{C} > 0$, independent of u, where I_h is the nodal interpolation operator $I_h : H^2(\Omega) \to V_h$, where V_h is the finite element space with mesh parameter h, and any other results from lectures.)

[8 marks]