1. (a) Let $(K, \mathcal{P}, \mathcal{N})$ be such that:

1. K is a (non-degenerate) triangle,
2. $\mathcal{P}$ is the space of polynomials of degree ≤ 1.
3. $\mathcal{N}$ is the set of nodal variables with

$$N_1(u) = \int_K u \,\mathrm{d}\, x, \quad N_2(u) = \frac{\partial u}{\partial x}, \quad N_3(u) = \frac{\partial u}{\partial y}.$$

Find the nodal basis for $\mathcal{P}$ corresponding to $\mathcal{N}$ by expanding in the monomial basis

$$\psi_1(x) = 1, \quad \psi_2(x) = x, \quad \psi_3(x) = y.$$

(Hint: to make calculation easier, you may make use of the fact that matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{pmatrix}$$

form a group.)

Solution: *We expand* $\phi_i(x) = \sum_{j=1}^{3} a_{ij}\psi_j(x)$. *Then,*

$$\delta_{ik} = N_k(\phi_i) = \sum_{j=1}^{3} a_{ij} N_k(\psi_j) = (AV)_{ik},$$

where A *is the matrix with coefficients* a_{ij}, *and* V *is the matrix with coefficients* $v_{ij} = N_j(\psi_i)$, *given by*

$$V = \begin{pmatrix} |K| & 0 & 0 \\ \bar{x}|K| & 1 & 0 \\ \bar{y}|K| & 0 & 1 \end{pmatrix}, \quad |K| = \int_K \mathrm{d}\, x,\ \bar{f} = \frac{\int_K f(x)\,\mathrm{d}\, x}{|K|}.$$

Using the hint, we write

$$V^{-1} = \begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{pmatrix},$$

so

$$V^{-1}V = \begin{pmatrix} a|K| & 0 & 0 \\ |K|(b+\bar{x}) & 1 & 0 \\ |K|(c+\bar{y}) & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies V^{-1} = \begin{pmatrix} \frac{1}{|K|} & 0 & 0 \\ -\bar{x} & 1 & 0 \\ -\bar{y} & 0 & 1 \end{pmatrix},$$

hence we obtain the basis

$$\phi_1(x) = \frac{1}{|K|},\ \phi_2(x) = x - \bar{x},\ \phi_3(x) = y - \bar{y}.$$

(b) Use this calculation to explain why $\mathcal{N}$ determines $\mathcal{P}$.

Solution: *The functions* $\{\phi_i\}_{i=1}^{3}$ *satisfies the requirements of the dual basis and are written as an invertible transformation from the monomial basis, hence they are linearly independent and span* $\mathcal{P}$.

(c) Comment on the relative sizes of the basis functions as the triangle area goes to zero, and suggest a rescaling of the nodal variables that remedies this.

Solution: *As $|K|$ goes to zero, $\phi_1 \to \infty$ whilst ϕ_2 and ϕ_3 stay finite. This can be fixed by replacing N_1 with $\tilde{N}_1 = \bar{u}$, in which case V becomes*

$$V = \begin{pmatrix} 1 & 0 & 0 \\ \bar{x} & 1 & 0 \\ \bar{y} & 0 & 1 \end{pmatrix},$$

so

$$\psi_1(x) = 1,\ \psi_2(x) = x - \bar{x},\ \psi_3(x) = y - \bar{y}.$$

2. We consider the finite element $(K, \mathcal{P}, \mathcal{N})$ where

1. K is a (non-degenerate) triangle,
2. $\mathcal{P}$ is the space $(P_1)^2$ of vector-valued polynomials (i.e. each vector component is in P_1).
3. Elements of $\mathcal{N}$ are dual functions that return the normal component of vector fields at the end of each edge (2 evaluations per edge, one at each end, and 3 edges, makes 6 dual functions in total).

The geometric decomposition of $(K, \mathcal{P}, \mathcal{N})$ is defined by associating each dual basis function with the edge where the normal component is evaluated.
We consider the finite element space V defined on a triangulation $\mathcal{T}$ of a polygonal domain Ω, constructed from the element above, so that dual basis evaluations agree for triangles on either side of each interior edge.

(a) Show that the weak divergence $\nabla_w \cdot u$ exists for $u \in V$, defined by

$$\int_\Omega \phi \nabla_w \cdot u \,\mathrm{d}\,x = -\int_\Omega \nabla\phi \cdot u \,\mathrm{d}\,x, \quad \forall \phi \in C_0^\infty(\Omega).$$

Solution: *First we note that the normal components of $u \in V$ agree on both sides of each interior edge, since $u.n$ restricted to the edge is a linear scalar function. Since the values of $u.n$ on each side agree at two points, the difference $u^+.n^+ + u^- \cdot n^-$ is a linear function that vanishes at two points, and hence is zero.*
We define

$$\nabla_w \cdot u|_K = \nabla \cdot u|_K,$$

for all triangles $K \in \mathcal{T}$. Then

$$\begin{aligned}\int_\Omega \phi \nabla_w \cdot u \,\mathrm{d}\,x &= \sum_K \int_K \phi \nabla \cdot u \,\mathrm{d}\,x, \\ &= \sum_K \left(-\int_K \nabla\phi \cdot u \,\mathrm{d}\,x + \int_{\partial K} \phi n \cdot u \,\mathrm{d}\,S \right), \\ &= -\int_\Omega \nabla\phi \cdot u \,\mathrm{d}\,x,\end{aligned}$$

as required, since the normal components agree.

(b) Develop a variational formulation for the problem

$$u - c\nabla(\nabla \cdot u) = f, \text{ for } x \in \Omega, \quad u.n = 0 \text{ on } \partial\Omega,$$

using the finite element space V, for c a positive constant. Develop an inner product that gives coercivity and continuity for the corresponding bilinear form with respect to the corresponding normed space.
Solution: *Taking inner product with test function w, integration by parts and application of the boundary condition gives*

$$a(u, v) = F(v), \forall v \in V, \quad a(u, v) = \int_\Omega u.v + c\nabla \cdot u \nabla \cdot v \,\mathrm{d}\,x, \quad F(v) = \int_\Omega f \cdot v \,\mathrm{d}\,x.$$

$a(u, v)$ is symmetric, bilinear, positive-definite in L^2, thus we may use it as an inner product. a therefore has continuity and coercivity constants equal to 1 with respect to the corresponding norm.

(c) Show that $u \in V$ does not have a weak curl $\nabla_w^\perp \cdot u$ in general, where

$$\int_\Omega \Phi \cdot \nabla_w^\perp \cdot u \,\mathrm{d}\,x = \int_\Omega \nabla^\perp \Phi \cdot u \,\mathrm{d}\,x, \quad \forall \Phi \in C_0^\infty(\Omega),$$

where $\nabla^\perp = (-\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$. [Hint: show this by counter-example. First choose a function $u \in V$ that you think does not have a weak curl. Then consider a suitable limit of smooth test functions that contradicts the above definition of a weak curl.]

Solution: *Let f_1 be an interior edge joining two triangles e_1 and e_2, and let f_2 be one of the other edges of e_2. Take $u.n = 1$ on f_2, and $u.n = 0$ on all other edges. Then, on the e_1 side of f_1, $u.t = 0$, where t is the unit tangent to f_1, oriented so that $n_1^\perp = t$, where n_1 is the unit normal pointing from e_1 to e_2. On the e_2 side, $u.t \neq 0$ (because otherwise $u.n = 0$ at the f_1 end of f_2). Define $g = u.t|_{e_2}$ on f_1. Then, pick a sequence of C_0^∞ test functions ϕ_k such that*

$$\phi_k|_{f_2} \to 1,\ \phi_k|_{e_1} \to 0\, \phi_k|_{e_2} \to 0,$$

and $\phi_k \to 0$ in all triangles other than e_1, e_2. We have

$$\begin{aligned}\int_\Omega \nabla^\perp \Phi \cdot u \,\mathrm{d}\,x &= \sum_K \int_K \nabla^\perp \Phi \cdot u \,\mathrm{d}\,x, \\ &= \sum_K \left(-\int_K \nabla^\perp \Phi \cdot u \,\mathrm{d}\,x + \int_{\partial K} \Phi n^\perp \cdot u \,\mathrm{d}\,S \right), \\ &\to \int_f g\Phi \,\mathrm{d}\,S \neq 0.\end{aligned}$$

On the other hand, if $\nabla_w^\perp u$ exists, then

$$0 \neq \int_f g\Phi \,\mathrm{d}\,S = \int_\Omega \Phi \nabla_w^\perp \cdot u \,\mathrm{d}\,S = \sum_K \int_\Omega \Phi|_K \nabla_w^\perp \cdot u \,\mathrm{d}\,S \to 0,$$

which is a contradiction.

3. We consider the following boundary value problem in one dimension.

$$-u'' + (2 + \sin(x))u = f(x), \quad u(0) = 0,\ u'(1) = 1.$$

(a) Construct a formulation of this problem describing a weak solution u in $H^1([0,1])$.
Solution: *Multiplication by test function v satisfying $v(0) = 0$, and integration by parts, using the boundary condition gives*

$$\int_0^1 v'u' + (2 + \sin(x))vu \,\mathrm{d}\,x = \int_0^1 vf \,\mathrm{d}\,x + v(1), \quad \forall v \in H_0^1([0,1]),$$

where $H_0^1([0,1])$ is the subspace of $H^1([0,1])$ such that $v(0) = 0$.

(b) Show that the corresponding bilinear form is continuous and coercive in $H^1([0,1])$, and compute the continuity and coercivity constants.
Solution: *Continuity:*

$$\begin{aligned} a(u,v) &\leq \|v'\|_{L^2(\Omega)}\|u'\|_{L^2(\Omega)} + 3\|v\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}, \\ &\leq 3\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}. \end{aligned}$$

Constant is 3.
Coercivity:

$$\begin{aligned} a(u,u) &= \int_0^1 (u')^2 + (2 + \sin(x))u^2 \,\mathrm{d}\,x, \\ &\geq \int_0^1 (u')^2 + u^2 \,\mathrm{d}\,x = \|u\|^2_{H^1(\Omega)}. \end{aligned}$$

Constant is 1.

(c) What is the required property of f for a unique solution u to exist?
Solution: *The RHS functional is*

$$L[v] = \int_0^1 fv \,\mathrm{d}\,x + v(1).$$

We have

$$|L[v]| \leq \|f\|_{L^2([0,1])}\|v\|_{L^2([0,1])} + C\|v\|_{H^1([0,1])} \leq (\|f\|_{L^2([0,1])} + C)\|v\|_{H^1([0,1])},$$

where C is the trace constant for $H^1([0,1])$. Hence we need $f \in L^2([0,1])$ for L to be bounded/continuous and hence the conditions of Lax-Migram to be satisfied.

(d) Describe the piecewise linear C^0 finite element discretisation of this equation with mesh vertices $[x_0 = 0, x_1, x_2, \ldots, x_n, x_{n+1} = 1]$.
Solution: *Let V_h be the finite element space of continuous, piecewise linear functions defined on this mesh, with subspace $\mathring{V}_h$ containing only the functions that satisfy $v_h(0) = 0$. The Galerkin finite element discretisation seeks $u_h \in \mathring{V}_h$ such that*

$$a(u_h, v) = L[v], \quad \forall v \in \mathring{V}_h.$$

(e) Given an arbitrary basis of the finite element space V_h, show that the resulting matrix A is symmetric ($A^T = A$) and positive definite, i.e. $x^T Ax > 0$ for all x with $\|x\| > 0$.

Solution: *Given a basis* $\{\phi_i\}_{i=1}^n$, *A is defined by*

$$A_{ij} = \int_0^1 \phi_i' \phi_j' + (2 + \sin(x))\phi_i \phi_j \,\mathrm{d}\, x.$$

A is symmetric, since exchanging i and j returns the same answer.
A is PD since if we take

$$u = \sum_{i=1}^n x_i \phi_i,$$

then

$$\begin{aligned} x^T Ax = \sum_{ij} x_i A_{ij} x_j &= \int_0^1 \sum_i x_i \phi_i' \sum_j x_j \phi_j' + (2 + \sin(x)) \sum_i x_i \phi_i \sum_j x_j \phi_j \,\mathrm{d}\, x, \\ &= \int_0^1 (u')^2 + (2 + sin(x))u^2 \,\mathrm{d}\, x, \end{aligned}$$

which is positive due to the coercivity result.

(f) Show that the numerical solution u_h satisfies $\|u - u_h\|_{H^1([0,1])} = \mathcal{O}(h)$ as $h \to 0$. [You may quote any properties of the interpolation operator $\mathcal{I}_h$ without proof, but must show the other steps.]

Solution: *Since* $\mathring{V}_h \in H^1([0, 1])$, *we have*

$$a(v, u - u_h) = 0, \quad \forall v \in \mathring{V}_h.$$

Then, taking $v \in \mathring{V}_h$,

$$\begin{aligned} \|u - u_h\|^2_{H^1([0,1])} &\le a(u - u_h, u - u_h), \\ &= \underbrace{a(u - u_h, v - u_h)}_{=0} + a(u - u_h, u - v), \\ &\le 3\|u - u_h\|_{H^1} \|u - v\|_{H^1}. \end{aligned}$$

Dividing by $\|u - u_h\|_{H^1}$ *and taking* $v = \mathcal{I}_h u$ *gives*

$$\|u - u_h\|_{H^1} \le 3\|u - \mathcal{I}_h u\|_{H^1} \le 3Ch\|u\|_{H^2},$$

where we quoted the property of $\mathcal{I}_h$ *in the last inequality.*

4. Consider the finite element $(K, \mathcal{P}, \mathcal{N})$, with

1. K is a non-degenerate triangle,
2. $\mathcal{P}$ is the space of polynomials on K of degree ≤ 2.
3. $\mathcal{N} = (N_{1,1}, N_{1,2}, N_{2,1}, N_{2,2}, N_{3,1}, N_{3,2})$, where

$$N_{i,j}(u) = \int_{f_i} \phi_{i,j} u \, \mathrm{d}\, x,$$

where (f_1, f_2, f_3) are the edges of K, with f_1 joining vertices 1 and 2, f_2 joining vertices 2 and 3, and f_3 joining vertices 3 and 1. The edge test functions $\phi_{i,j}$ define a basis for linear functions restricted to f_i such that $\phi_{1,1} = 1$ on vertex 1 and 0 on vertex 2, *etc.*

(a) Show that $\mathcal{N}$ determines $\mathcal{P}$.

Solution: *It suffices to show that if $u \in P$, then $N_{i,j}(u) = 0$ for all $i, j \implies u = 0$. Under this assumption, we may expand $u|_{f_1} = u_1\phi_{1,1} + u_2\phi_{1,2}$, hence*

$$\int_{f_i} u^2 \, \mathrm{d}\, x = 0, \implies u|_{f_1} = 0.$$

Similarly we obtain $u|_{f_2} = 0$, $u|_{f_3} = 0$. If $u|_{f_1} = 0$, then $u(x) = L_1(x)Q_1(x)$ where L_1 is a non-degenerate linear polynomial with $L_1|_{f_1} = 0$. Since $L_1|_{f_2} \neq 0$, we conclude that $Q_1(x)|_{f_2} = 0$, so that $Q_1(x) = cL_2(x)$, where c is a constant, and $L_2|_{f_2} = 0$. Finally since $L_2|_{f_3} \neq 0$, we conclude that $c = 0$, hence the result.

(b) We take a geometric decomposition such that $N_{i,j}$ is associated with f_i, $i = 1, 2, 3$, $j = 1, 2$. What is the continuity of the corresponding finite element space V defined on a triangulation $\mathcal{T}$ of a polygonal domain Ω? Explain your answer.

Solution: *There exist discontinuous functions in V, so functions are not even in $C^0(\Omega)$. To see this, pick two neighbouring triangles e_1, e_2 with common face f_1, and another face f_2 in e_1. We pick $u = 1$ on f_2 and $u = 0$ when restricted to all other faces. This function is equal to zero on the e_1 side of f_1, and is a linear function going from 1 to 0 on the e_2 side, hence discontinuous.*

(c) We assume that $u \in H^k(\Omega)$ for some non-negative integer k. What is the minimum value of k such that the global interpolator $\mathcal{I}_h u$ is well-defined? Explain your answer.

Solution: *We require that $u|_f \in L^2(f)$ for each edge in $\mathcal{T}$. This is guaranteed if $u \in H^1(\Omega)$ via the trace theorem. On the other hand $u \in L^2(\Omega)$ is not enough to guarantee that $u|_f \in L^2(f)$. So $k = 1$.*