VI.4.3 Wigner–Eckart theorem

Consider an irreducible tensor operator $\hat{T}^{(j)}_m$. According to the *Wigner–Eckart*[(ae)] *theorem*, its matrix elements in the basis of the common eigenvectors of the generator $\hat{\mathscr{J}}_z$ and the Casimir operator $\hat{\vec{\mathscr{J}}}^2$ take the form

$$\boxed{\langle j'', m'' | \hat{T}^{(j)}_m | j', m' \rangle = C^{j'';m''}_{j,j';m,m'} \langle j'' \| \hat{T}^{(j)} \| j' \rangle,} \tag{VI.70}$$

where $C^{j'';m''}_{j,j';m,m'}$ is the Clebsch–Gordan coefficient defined by Eqs. (VI.52) while $\langle j'' \| \hat{T}^{(j)} \| j' \rangle$, known as the *reduced matrix element*, is now independent of m, m', and m''.

Viewing the matrix representation of $\hat{T}^{(j)}_m$ in the basis of eigenvectors $\{|j', m'\rangle\}$, it means that the $(2j'+1) \times (2j''+1)$ block associated to fixed values j' and j'' is proportional to the matrix of the same size consisting of the Clebsch–Gordan coefficients $C^{j'';m''}_{j,j';m,m'}$.

Remark: Throughout this chapter, we only consider transformations under SO(3) (and SU(2)), i.e. the tensor operators introduced in Sec. VI.4.2 are characterized by their commutation relations with generators of a representation of SO(3), while the Clebsch–Gordan coefficients are those of SO(3). The whole construction, including the Wigner–Eckart theorem, can be generalized to other groups as e.g. the special unitary groups SU(n).

Applications

In physical applications, the Wigner–Eckart theorem in particular leads to *selection rules* for the transitions induced by an operator involving an irreducible tensor operator.
Consider for instance an atomic system invariant under rotations, whose energy eigenstates have definite values of the orbital and magnetic quantum numbers ℓ and $m \in \{-\ell, \dots, \ell\}$. When the system is in an external (classical) electric field $\vec{E}$, its Hamilton operator is perturbed by a term which in the dipolar approximation takes the form $\hat{W} = -\hat{\vec{P}} \cdot \vec{E}$, where

$$\hat{\vec{P}} = \sum_i q_i \hat{\vec{r}}_i$$

is the electric dipole operator, which depends on the charges and positions of the particles (nucleus and electrons) constituting the system. This perturbation can induce a transition between an initial state $|\mathrm{i}\rangle$ and a final state $|\mathrm{f}\rangle$, with a rate proportional (to first order in perturbation theory) to the squared amplitude $|\langle f | \hat{W} | i \rangle|^2$.

Expressing the scalar product $\hat{\vec{P}} \cdot \vec{E}$ through the standard components $\hat{P}^{(1)}_m$ with $m = -1, 0, 1$ of the dipole operator, one sees that the computation of the transition rate involves that of the amplitude $\langle f | \hat{P}^{(1)}_m | i \rangle$. Invoking the Wigner–Eckart theorem, the latter is proportional to a Clebsch–Gordan coefficient:

$$\langle f | \hat{P}^{(1)}_m | i \rangle \propto C^{\ell_\mathrm{f};m_\mathrm{f}}_{1,\ell_\mathrm{i};m,m_\mathrm{i}}$$

where $\ell_\mathrm{i}, m_\mathrm{i}$ resp. $\ell_\mathrm{f}, m_\mathrm{f}$ are the quantum numbers of the initial resp. final state. Using the selection rules (VI.53) obeyed by the Clebsch–Gordan coefficients, one deduces the following conditions on these quantum numbers to ensure a non-vanishing matrix element:

$$|\ell_\mathrm{i} - \ell_\mathrm{f}| \leq 1 \quad , \quad |m_\mathrm{i} - m_\mathrm{f}| \leq 1 \quad , \quad \ell_\mathrm{i} + \ell_\mathrm{f} \geq 1.$$

The first two conditions mean that a dipolar transition can at most change the angular momentum by one unit (of $\hbar$), and the third one, that the initial and final states cannot be both s-states (with $\ell = 0$).

[(ae)] C. Eckart, 1902–1973

As second example, consider the case of a spin-j' system whose precise "polarization", i.e. value of m', is unknown — and where every single polarization is equally probable —, so that in the calculation of the expectation value of an observable $\hat{A}$, one needs to average over the different polarization states, i.e.

$$\langle \hat{A} \rangle = \frac{1}{2j'+1} \sum_{m'=-j'}^{j'} \langle j', m' | \hat{A} | j', m' \rangle.$$

In the case of a spin-j tensor observable $\hat{T}_m^{(j)}$, the Wigner–Eckart theorem (VI.70) gives

$$\langle \hat{T}_m^{(j)} \rangle = \frac{1}{2j'+1} \sum_{m'=-j'}^{j'} \langle j', m' | \hat{T}_m^{(j)} | j', m' \rangle = \frac{1}{2j'+1} \sum_{m'=-j'}^{j'} C_{j,j';m,m'}^{j';m'} \langle j' \| \hat{T}^{(j)} \| j' \rangle,$$

where the reduced matrix element $\langle j' \| \hat{T}^{(j)} \| j' \rangle$ can actually be taken out of the sum. Now, one can show that the sum of the coefficients $C_{j,j';m,m'}^{j';m'}$ over all allowed m' values actually vanishes unless $j = 0$ and $m = 0$ — in which case it trivially equals $2j' + 1$, yielding

$$\langle \hat{T}_m^{(j)} \rangle = \delta_{j0}\, \delta_{m0}\, \langle j' \| \hat{T}^{(j)} \| j' \rangle.$$

That is, the expectation value of an irreducible tensor operator in an "unpolarized state" vanishes when the operator is not scalar.