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Solution to the Bargmann–Wigner equation for a half-integral spin*

Huang Shi-Zhong(黄时中)[a)b)], Ruan Tu-Nan(阮图南)[a)b)†],
Wu Ning(吴　宁)[a)c)], and Zheng Zhi-Peng(郑志鹏)[a)c)]

[a)] *China Center of Advanced Science and Technology (World Laboratory), Beijing* 100080, *China*
[b)] *Department of Modern Physics, University of Science and Technology of China, Hefei* 230027, *China*
[c)] *Institute of High Energy Physics, Chinese Academy of Sciences, Beijing* 100039, *China*

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A rigorous method to solve the Bargmann–Wigner equation for an arbitrary half-integral spin is presented and explicit relativistic wavefunctions for an arbitrary half-integral spin are deduced.

Keywords: half-integral spin, Bargmann-Wigner equation, rigorous solution
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1. Introduction

The field theories that describe spin 0, 1/2 and 1 fields have been well established and they provide powerful tools in many areas of application. An extension of these to the theory that could describe fields of an arbitrary half-integral spin is of crucial importance both in theory and in applications. Theoretically, this theory is a foundation for the complete Fermi–Dirac statistics. In applications, the analysis of amplitudes in high-energy processes relies on explicit high-spin relativistic wavefunctions.[1−3]

Based on the work of Dirac and Fierz,[4−6] Bargmann and Wigner[7] found a set of general equations for an arbitrary spin more than half a century ago, to our knowledge. However, except for the simple cases of spin 1/2, 1 and 3/2, this set of equations has not been solved rigorously (see, for example, Ref.[8]). Recently, we have proposed a rigorous method to solve the Bargmann–Wigner (B–W) equation for spin 5/2.[9] In this paper, we generalize this method to the more general cases, that is to solve the Bargmann–Wigner equation for an arbitrary half-integral spin and to deduce correspondingly the explicit relativistic wavefunctions, and thus to constitute a complete theoretical system.

2. Bargmann–Wigner equation for an arbitrary half-integral spin

The original form of the B–W equation for an arbitrary half-integral spin $n+1/2$ reads[7]

$$(\partial\!\!\!/ + m)_{\alpha_1\alpha_1'}\,\Psi_{\alpha_1'\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = 0, \tag{1a}$$

$$(\partial\!\!\!/ + m)_{\beta_1\beta_1'}\,\Psi_{\alpha_1\beta_1'\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = 0, \tag{1b}$$

$$(\partial\!\!\!/ + m)_{\alpha_2\alpha_2'}\,\Psi_{\alpha_1\beta_1\alpha_2'\beta_2\cdots\alpha_n\beta_n\rho}(x) = 0, \tag{1c}$$

$$(\partial\!\!\!/ + m)_{\beta_2\beta_2'}\,\Psi_{\alpha_1\beta_1\alpha_2\beta_2'\cdots\alpha_n\beta_n\rho}(x) = 0, \tag{1d}$$

$$\cdots\cdots$$

$$(\partial\!\!\!/ + m)_{\alpha_n\alpha_n'}\,\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n'\beta_n\rho}(x) = 0, \tag{1e}$$

$$(\partial\!\!\!/ + m)_{\beta_n\beta_n'}\,\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n'\rho}(x) = 0, \tag{1f}$$

$$(\partial\!\!\!/ + m)_{\rho\rho'}\,\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho'}(x) = 0, \tag{1g}$$

where $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ is a completely symmetric multispinor of rank $2n+1$. In order to solve this set of equations in coordinate representation, we first transform them into a form more easily solved by generalizing the procedure used to deal with the B–W equation for spin 5/2.[9]

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†Author to whom any correspondence should be addressed.

On the one hand, by utilizing Dirac equations (1a) and (1b), (1c) and (1d),..., (1e) and (1f), and taking into account the symmetry requirement of $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ in the indices $\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n$, $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ is expanded as

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = \prod_{j=1}^{n}(\mathrm{i}m\gamma_{\nu_j}C+\Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j}\,\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x), \quad (2)$$

where $C=\gamma_2\gamma_4$ is the charge conjugation matrix, $\gamma_\nu C$ and $\Sigma_{\mu\nu}C$ are the symmetric matrices, and $\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)$ is the tensor-spinor satisfying the following equations

$$(\Box-m^2)\,\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)=0, \quad (3a)$$

$$\partial_{\nu_i}\,\Psi_\rho^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x)=0 \quad (i=1,2,\cdots,n), \quad (3b)$$

$$\Psi_\rho^{\cdots\nu\nu\cdots}(x)=0, \quad (3c)$$

$$\Psi_\rho^{\cdots\nu_i\nu_{i+1}\cdots}(x)=\Psi_\rho^{\cdots\nu_{i+1}\nu_i\cdots}(x) \quad (i=1,2,\cdots,n-1). \quad (3d)$$

On the other hand, $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ satisfies the Dirac equation (1g) concerning the index ρ. Substituting Eq.(2) into Eq.(1g), we have

$$\prod_{j=1}^{n}(\mathrm{i}m\gamma_{\nu_j}C+\Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j} \times(\not\partial+m)_{\rho\rho'}\,\Psi_{\rho'}^{\nu_1\nu_2\cdots\nu_n}(x)=0.$$

Because of the independence of matrices $\gamma_\nu C$ and $\Sigma_{\mu\nu}C$, this equation leads to (omitting the spinor index ρ)

$$(\not\partial+m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=0. \quad (3e)$$

Furthermore, the right-hand side of Eq.(2) is symmetric with indices $\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n$. In order to make sure that $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ is completely symmetric with all the $2n+1$ indices $\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho$, we further require that the right-hand side of Eq.(2) is also symmetric with indices $\alpha_n\beta_n\rho$. A condition for this requirement is that the contraction of the part of $(\mathrm{i}m\gamma_{\nu_n}C+\Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n}\,\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)$ in equation (2) with the three independent antisymmetric Dirac matrices C^{-1}, $C^{-1}\gamma_5$ and $C^{-1}\gamma_5\gamma_\lambda$ with respect to the indices β_n and ρ vanish, namely

$$(\mathrm{i}m\gamma_{\nu_n}C+\Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n} \times\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)(C^{-1})_{\beta_n\rho}=0, \quad (4a)$$

$$(\mathrm{i}m\gamma_{\nu_n}C+\Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n} \times\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)(C^{-1}\gamma_5)_{\beta_n\rho}=0, \quad (4b)$$

$$(\mathrm{i}m\gamma_{\nu_n}C+\Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n} \times\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\beta_n\rho}=0. \quad (4c)$$

Expanding Eqs.(4) and using Eqs.(3a)–(3d), we find (omitting the spinor index ρ)

$$\gamma_{\nu_n}(\not\partial+m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=0, \quad (5a)$$

$$-2m\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x)+\gamma_{\nu_n}(\not\partial+m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=0, \quad (5b)$$

$$(\not\partial+m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=0. \quad (5c)$$

Both Eqs.(5a) and (5c) are equivalent to Eq.(3e), and the last term of Eq.(5b) vanishes while the first term gives

$$\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x)=0. \quad (3f)$$

Combining all the above results, we obtain the wave equations for spin $n+1/2$

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = \prod_{j=1}^{n}(\mathrm{i}m\gamma_{\nu_j}C+\Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j}\,\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x), \quad (6)$$

where $\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)$ is a rank n tensor-spinor satisfying the following equations (the spinor index ρ is suppressed)

$$(\Box-m^2)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=0, \quad (7a)$$

$$\partial_{\nu_i}\,\Psi^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x)=0 \quad (i=1,2,\cdots,n), \quad (7b)$$

$$\Psi^{\cdots\nu\cdots\nu\cdots}(x)=0, \quad (7c)$$

$$\Psi^{\cdots\nu_i\nu_{i+1}\cdots}(x)=\Psi^{\cdots\nu_{i+1}\nu_i\cdots}(x) \quad (i=1,2,\cdots,n-1), \quad (7d)$$

$$(\not\partial+m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=0, \quad (7e)$$

$$\gamma_\nu\,\Psi^{\nu\nu_2\nu_3\cdots\nu_n}(x)=0. \quad (7f)$$

3. The Lagrangian density

The Lagrangian density for fields with arbitrary half-integral spin could be expressed as

$$\begin{aligned}L(x)=&-\overline{\Psi}^{\mu\nu_2\cdots\nu_n}(x)(\not\partial+m)\,\Psi^{\mu\nu_2\cdots\nu_n}(x)\\&+\frac{1}{3}\overline{\Psi}^{\mu\nu_2\cdots\nu_n}(x)(\gamma_\mu\partial_\nu+\gamma_\nu\partial_\mu)\,\Psi^{\nu\nu_2\cdots\nu_n}(x)\\&-\frac{1}{3}\overline{\Psi}^{\mu\nu_2\cdots\nu_n}(x)\gamma_\mu(\not\partial-m)\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x),\end{aligned} \quad (8)$$

with $\overline{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x)=g_{\nu_1\mu_1}g_{\nu_2\mu_2}\cdots g_{\nu_n\mu_n}\gamma_2(\Psi^{\mu_1\mu_2\cdots\mu_n}(x))^+$, $\Psi^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x)$ a symmetric tensor-spinor satisfying the condition $\Psi^{\cdots\nu\cdots\nu\cdots}(x)=0$. Substituting Eq.(8) into the Euler–Lagrange equations

$$\partial_\mu\frac{\partial L}{\partial(\partial_\mu\overline{\Psi}^{\nu\nu_2\cdots\nu_n})}=\frac{\partial L}{\partial\overline{\Psi}^{\nu\nu_2\cdots\nu_n}},$$

we have

$$-(\not{\partial}+m)\,\Psi^{\mu\nu_2\cdots\nu_n}(x)+\frac{1}{3}\gamma_\mu A + \frac{1}{3}\partial_\mu B-\frac{1}{3}\gamma_\mu(\not{\partial}-m)B=0, \tag{9}$$

where

$$A=\partial_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x), B=\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x). \tag{10}$$

Multiplying Eq.(9) by γ_μ gives

$$2A=mB. \tag{11}$$

With ∂_μ acting on Eq.(9) and with the aid of Eq.(11), we have

$$A=0. \tag{12}$$

From Eqs.(9)–(12), the following field equations could be reproduced

$$(\Box-m^2)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=0,\quad \partial_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x)=0,$$

$$(\not{\partial}+m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=0,\quad \gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x)=0,$$

which are in agreement with Eqs.(7).

4. Solution to the B–W equation for spin $n+1/2$

We now generalize the method[9] used to solve the B–W equation for spin 5/2 to the more general cases for spin $n+1/2$. We begin with $\Psi^{\nu_1\nu_2\cdots\nu_n}(x)$ which is expanded into plane waves

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{\mathrm{i}px}\,\Psi^{\nu_1\nu_2\cdots\nu_n}(p). \tag{13}$$

Substituting Eq.(13) into Eq.(7a) yields

$$(p^2+m^2)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(p)=0. \tag{14}$$

By virtue of $x\delta(x)=0$, $\Psi^{\nu_1\nu_2\cdots\nu_n}(p)$ in Eq.(14) could be written as

$$\begin{aligned}&\Psi^{\nu_1\nu_2\cdots\nu_n}(p)\\ =&\delta(p^2+m^2)B^{\nu_1\nu_2\cdots\nu_n}(p)\\ =&\frac{1}{2E}[\delta(E-p_0)+\delta(E+p_0)]B^{\nu_1\nu_2\cdots\nu_n}(p),\end{aligned} \tag{15}$$

where $E=\sqrt{\boldsymbol{p}^2+m^2}$. Inserting Eq.(15) into Eq.(13) and integrating over p_0 gives

$$\begin{aligned}&\Psi^{\nu_1\nu_2\cdots\nu_n}(x)\\ =&\int\frac{\mathrm{d}^3p}{(2\pi)^4}\frac{1}{2E}[\mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{r}-\mathrm{i}Et}B^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p},E)\\ &+\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{r}+\mathrm{i}Et}B^{\nu_1\nu_2\cdots\nu_n}(-\boldsymbol{p},-E)],\end{aligned} \tag{16}$$

or in a discrete form with a simplified notation

$$\begin{aligned}&\Psi^{\nu_1\nu_2\cdots\nu_n}(x)\\ =&\frac{1}{\sqrt{V}}\sum_{\vec{p}}[a^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}+b^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px})],\end{aligned} \tag{17}$$

where $a^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ and $b^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ correspond to the positive and negative solutions respectively, and V is the normalized volume. Substituting Eq.(17) into Eqs.(7b)–(7d), we obtain equations in momentum representation

$$p_{\nu_i}a^{\nu_1\nu_2\nu_i\cdots\nu_n}(\boldsymbol{p})=0,$$

$$p_{\nu_i}b^{\nu_1\nu_2\nu_i\cdots\nu_n}(\boldsymbol{p})=0\quad(i=1,2,\cdots,n), \tag{18a}$$

$$a^{\cdots\nu\nu\cdots}(\boldsymbol{p})=0,\quad b^{\cdots\nu\nu\cdots}(\boldsymbol{p})=0, \tag{18b}$$

$$a^{\cdots\nu_i\nu_{i+1}\cdots}(\boldsymbol{p})=a^{\cdots\nu_{i+1}\nu_i\cdots}(\boldsymbol{p}),$$

$$b^{\cdots\nu_i\nu_{i+1}\cdots}(\boldsymbol{p})=b^{\cdots\nu_{i+1}\nu_i\cdots}(\boldsymbol{p}). \tag{18c}$$

Utilizing $p_\nu e^\nu_\lambda(\boldsymbol{p})=0$, the solution to Eq.(18a) could be expressed as

$$a^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=e^{\nu_1}_{\lambda_1}(\boldsymbol{p})e^{\nu_2}_{\lambda_2}(\boldsymbol{p})\cdots e^{\nu_n}_{\lambda_n}(\boldsymbol{p})a_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$$

$$(\lambda_i=1,0,-1), \tag{19a}$$

$$b^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=\bar{e}^{\nu_1}_{\lambda_1}(\boldsymbol{p})\bar{e}^{\nu_2}_{\lambda_2}(\boldsymbol{p})\cdots\bar{e}^{\nu_n}_{\lambda_n}(\boldsymbol{p})b^{+}_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p}), \tag{19b}$$

where $a_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ and $b^{+}_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ need to be further determined. $e^\nu_\lambda(\boldsymbol{p})$ are the eigenstates of the helicity operator $\boldsymbol{S}\cdot\dfrac{\boldsymbol{p}}{|\boldsymbol{p}|}$ with eigenvalues $\lambda=1,0,-1$, and their explicit form has been given previously,[9] while

$$\bar{e}^\nu_\lambda(\boldsymbol{p})=g_{\nu\mu}(e^\nu_\lambda(\boldsymbol{p}))^*,\quad g_{\nu\mu}=\mathrm{diag}\{1,1,1,-1\}. \tag{20}$$

We use Eqs.(18b) and (18c) to determine $a_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ and $b^{+}_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ in Eq.(19) in a step-by-step way. For the simple case of $n=2$, the method used to determine $a_{\lambda_1\lambda_2}(\boldsymbol{p})$ and $b^{+}_{\lambda_1\lambda_2}(\boldsymbol{p})$ has been presented in Ref.[9], in what follows we further give the procedure that determines $a_{\lambda_1\lambda_2\lambda_3}(\boldsymbol{p})$ and $b^{+}_{\lambda_1\lambda_2\lambda_3}(\boldsymbol{p})$ for the case of $n=3$ and we then give directly, as a generalization, the result of $a_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ and $b^{+}_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$. For the case of $n=3$, Eqs.(18b) and (18c) could be expressed explicitly as

$$\begin{cases}a^{\nu\nu\nu_3}(\boldsymbol{p})=0,\quad b^{\nu\nu\nu_3}(\boldsymbol{p})=0,\\ a^{\nu_1\nu_2\nu_3}(\boldsymbol{p})=a^{\nu_2\nu_1\nu_3}(\boldsymbol{p}),\quad b^{\nu_1\nu_2\nu_3}(\boldsymbol{p})=b^{\nu_2\nu_1\nu_3}(\boldsymbol{p}),\end{cases} \tag{21a}$$

$$a^{\nu_1\nu\nu}(\boldsymbol{p})=0,\quad b^{\nu_1\nu\nu}(\boldsymbol{p})=0, \tag{21b}$$

$$a^{\nu_1\nu_2\nu_3}(\boldsymbol{p})=a^{\nu_1\nu_3\nu_2}(\boldsymbol{p}),\quad b^{\nu_1\nu_2\nu_3}(\boldsymbol{p})=b^{\nu_1\nu_3\nu_2}(\boldsymbol{p}). \tag{21c}$$

Using the result for the case of $n = 2$ (see Ref.[9]), we find from Eq.(21a)

$$a^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = e^{\nu_1\nu_2}_{\lambda_{12}}(\boldsymbol{p})e^{\nu_3}_{\lambda_3}(\boldsymbol{p})a_{\lambda_{12}\lambda_3}(\boldsymbol{p}),$$
$$b^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = \bar{e}^{\nu_1\nu_2}_{\lambda_{12}}(\boldsymbol{p})\bar{e}^{\nu_3}_{\lambda_3}(\boldsymbol{p})b^{+}_{\lambda_{12}\lambda_3}(\boldsymbol{p}), \tag{22}$$

where

$$e^{\nu_1\nu_2}_{\lambda_{12}}(\boldsymbol{p}) = \sum_{\lambda_1\lambda_2} e^{\nu_1}_{\lambda_1}(\boldsymbol{p})e^{\nu_2}_{\lambda_2}(\boldsymbol{p})\langle 1,\lambda_1;1,\lambda_2|1,1,2,\lambda_{12}\rangle$$
$$(\lambda_{12} = 2,1,0,-1,-2), \tag{23a}$$

$$\bar{e}^{\nu_1\nu_2}_{\lambda_{12}}(\boldsymbol{p}) = \sum_{\lambda_1\lambda_2} \bar{e}^{\nu_1}_{\lambda_1}(\boldsymbol{p})\bar{e}^{\nu_2}_{\lambda_2}(\boldsymbol{p})\langle 1,\lambda_1;1,\lambda_2|1,1,2,\lambda_{12}\rangle. \tag{23b}$$

Substituting Eq.(22) into Eq.(21b) yields

$$e^{\nu_1\nu}_{\lambda_{12}}(\boldsymbol{p})e^{\nu}_{\lambda_3}(\boldsymbol{p})a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = 0,$$
$$e^{\nu_1\nu}_{\lambda_{12}}(\boldsymbol{p})e^{\nu}_{\lambda_3}(\boldsymbol{p})b^{+}_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = 0. \tag{24}$$

By virtue of (see Ref.(9))

$$e^{\nu}_{\lambda}(\boldsymbol{p}) = L^{\nu\nu'}e^{\nu'}_{\lambda}(0), \quad L^{\nu\nu_1}L^{\nu\nu_2} = \delta_{\nu_1\nu_2},$$

Eq.(24) can be rewritten as

$$e^{\nu_1\nu}_{\lambda_{12}}(0)e^{\nu}_{\lambda_3}(0)a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = 0, \tag{25a}$$

$$e^{\nu_1\nu}_{\lambda_{12}}(0)e^{\nu}_{\lambda_3}(0)b^{+}_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = 0. \tag{25b}$$

Now we focus our attention on the solution to Eq.(25a), and the solution to Eq.(25b) will be obtained in the same way. Equation (25a) indicates that $a_{\lambda_{12}\lambda_3}(\boldsymbol{p})$ is related to two magnetic quantum numbers λ_{12} and λ_3 ($\lambda_{12} = 2,1,0,-1,-2, \lambda_3 = 1,0,-1$). Recalling the Clebsch–Gordon coefficients for coupling two spin angular momenta with spin 2 and spin 1 respectively, a general candidate for $a_{\lambda_{12}\lambda_3}(\boldsymbol{p})$ is

$$\begin{aligned} a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = &\sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle a_{3m}(\boldsymbol{p}) \\ &+ \sum_{m'} \langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle a_{2m'}(\boldsymbol{p}) \\ &+ \sum_{m''} \langle 2,\lambda_{12};1,\lambda_3|2,1,1,m''\rangle a_{1m''}(\boldsymbol{p}) \end{aligned}$$
$$(m = 3,2,1,0,-1,-2,-3;$$
$$m' = 2,1,0-1,-2; \quad m'' = 1,0,-1). \tag{26}$$

Letting

$$e^{\nu_1\nu_2\nu_3}_{3m}(0) = \sum_{\lambda_{12},\lambda_3} e^{\nu_1\nu_2}_{\lambda_{12}}(0)e^{\nu_3}_{\lambda_3}(0)\langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle, \tag{27a}$$

$$e^{\nu_2\nu_2\nu_3}_{2m'}(0) = \sum_{\lambda_{12},\lambda_3} e^{\nu_1\nu_2}_{\lambda_{12}}(0)e^{\nu_3}_{\lambda_3}(0)\langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle, \tag{27b}$$

$$e^{\nu_2\nu_2\nu_3}_{1m''}(0) = \sum_{\lambda_{12},\lambda_3} e^{\nu_1\nu_2}_{\lambda_{12}}(0)e^{\nu_3}_{\lambda_3}(0)\langle 2,\lambda_{12};1,\lambda_3|2,1,1,m''\rangle, \tag{27c}$$

then Eq.(25a) takes the form

$$e^{\nu_1\nu\nu}_{3m}(0)a_{3m}(\boldsymbol{p}) + e^{\nu_1\nu\nu}_{2m'}(0)a_{2m'}(\boldsymbol{p}) + e^{\nu_1\nu\nu}_{1m''}(0)a_{1m''}(\boldsymbol{p}) = 0. \tag{28}$$

With the help of the explicit form of Eq.(27) and the following sum relations

$$\begin{aligned} &e^{\nu}_{1}(0)e^{\nu}_{1}(0) = e^{\nu}_{-1}(0)e^{\nu}_{-1}(0) \\ =&e^{\nu}_{0}(0)e^{\nu}_{1}(0) = e^{\nu}_{1}(0)e^{\nu}_{0}(0) \\ =&e^{\nu}_{-1}(0)e^{\nu}_{0}(0) = e^{\nu}_{0}(0)e^{\nu}_{-1}(0) = 0, \end{aligned}$$

$$e^{\nu}_{1}(0)e^{\nu}_{-1}(0) = e^{\nu}_{-1}(0)e^{\nu}_{1}(0) = -1,$$
$$e^{\nu}_{0}(0)e^{\nu}_{0}(0) = 1,$$

we find

$$e^{\nu_1\nu\nu}_{3m}(0) = 0 \quad (m = 3,2,1,0,-1,-2,-3),$$
$$e^{\nu_1\nu\nu}_{2m'}(0) = 0 \quad (m' = 2,1,0,-1,-2), \tag{29a}$$

$$e^{\nu_1\nu\nu}_{1m''}(0) = -\sqrt{\frac{5}{3}}e^{\nu_1}_{m''}(0) \neq 0 \quad (m'' = 1,0,-1). \tag{29b}$$

Substituting Eq.(29) into Eq.(28) gives

$$a_{1m''}(\boldsymbol{p}) = 0. \tag{30}$$

Thus Eq.(26) is simplified to

$$\begin{aligned} a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = &\sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle a_{3m}(\boldsymbol{p}) \\ &+ \sum_{m'} \langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle a_{2m'}(\boldsymbol{p}); \end{aligned} \tag{31a}$$

and, similarly, Eq.(25b) leads to

$$\begin{aligned} b^{+}_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = &\sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle b^{+}_{3m}(\boldsymbol{p}) \\ &+ \sum_{m'} \langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle b^{+}_{2m'}(\boldsymbol{p}). \end{aligned} \tag{31b}$$

On the other hand, the symmetric condition (21c) requires

$$a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = a_{\lambda_3\lambda_{12}}(\boldsymbol{p}), \quad b^+_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = b^+_{\lambda_3\lambda_{12}}(\boldsymbol{p}). \quad (32)$$

However

$$\langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle = \langle 1,\lambda_3;2,\lambda_{12}|1,2,3,m\rangle, \quad (33a)$$

$$\langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle = -\langle 1,\lambda_3;2,\lambda_{12}|1,2,2,m'\rangle. \quad (33b)$$

Substituting Eq.(31) into Eq.(32) and with the aid of Eq.(33), we have

$$a_{2m'}(\boldsymbol{p}) = 0, \quad b^+_{2m'}(\boldsymbol{p}) = 0. \quad (34)$$

Therefore Eq.(31) is simplified to

$$a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = \sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle a_{3m}(\boldsymbol{p}), \quad (35a)$$

$$b^+_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = \sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle b^+_{3m}(\boldsymbol{p}), \quad (35b)$$

and Eq.(22) becomes correspondingly

$$a^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = e^{\nu_1\nu_2\nu_3}_{3m}(\boldsymbol{p}) a_{3m}(\boldsymbol{p}),$$
$$b^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = \overline{e}^{\nu_1\nu_2\nu_3}_{3m}(\boldsymbol{p}) b^+_{3m}(\boldsymbol{p}), \quad (36a)$$

with

$$e^{\nu_1\nu_2\nu_3}_{3m}(\boldsymbol{p}) = \sum_{\lambda_1\lambda_2\lambda_3} e^{\nu_1}_{\lambda_1}(\boldsymbol{p}) e^{\nu_2}_{\lambda_2}(\boldsymbol{p}) e^{\nu_3}_{\lambda_3}(\boldsymbol{p}) \times \langle 1,\lambda_1;1,\lambda_2|1,1,2,\lambda_{12}\rangle \times \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle, \quad (36b)$$

$$\overline{e}^{\nu_1\nu_2\nu_3}_{3m}(\boldsymbol{p}) = g_{\nu_1\mu_1} g_{\nu_2\mu_2} g_{\nu_3\mu_3} (e^{\mu_1\mu_2\mu_3}_{3m}(\boldsymbol{p}))^*$$
$$(m = 3,2,1,0,-1,-2,-3), \quad (36c),$$

or (using the Wigner formula for the Clebsch–Gordon coefficients and omitting the index 3)

$$e^{\nu_1\nu_2\nu_3}_{m}(\boldsymbol{p}) = \sum_{\lambda_1\lambda_2\lambda_3=-1}^{1} e^{\nu_1}_{\lambda_1}(\boldsymbol{p}) e^{\nu_2}_{\lambda_2}(\boldsymbol{p}) e^{\nu_3}_{\lambda_3}(\boldsymbol{p}) \delta(\lambda_1+\lambda_2+\lambda_3, m) \times \sqrt{\frac{(3+m)!(3-m)!}{90\prod_{i=1}^{3}(1+\lambda_i)!(1-\lambda_i)!}}. \quad (36d)$$

Generalizing the above procedure, $a_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ and $b^+_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ in Eq.(19) can be determined in a step-by-step way. When the final result is inserted in Eq.(17) we have

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} [e^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p}) a_M(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}px} + \overline{e}^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p}) b^+_M(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}px})], \quad (37)$$

with $\qquad M = 0, \pm 1, \pm 2, \cdots, \pm n,$

$$e^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p}) = \sum_{\lambda_i=-1}^{1} \prod_{i=1}^{n} e^{\nu_i}_{\lambda_i}(\boldsymbol{p}) \times \prod_{i=1}^{n-1} \langle i, \lambda_1+\lambda_2+\cdots\lambda_i; 1,\lambda_{i+1}|i,1,i+1,\lambda_1+\lambda_2+\cdots\lambda_{i+1}\rangle$$
$$= \sum_{\lambda_1\lambda_2\cdots\lambda_n=-1}^{1} \delta(\lambda_1+\lambda_2+\cdots\lambda_n, M) \times \sqrt{\frac{2^n(n+M)!(n-M)!}{(2n)!\prod_{i=1}^{n}(1+\lambda_i)!(1-\lambda_i)!}} \prod_{i=1}^{n} e^{\nu_i}_{\lambda_i}(\boldsymbol{p}). \quad (38)$$

Substituting Eq.(37) into Eq.(7e) yields

$$(\mathrm{i}\not{p}+m)a_M(\boldsymbol{p}) = 0 \quad (-\mathrm{i}\not{p}+m)b^+_M(\boldsymbol{p}) = 0. \quad (39)$$

These are the Dirac equations for spin 1/2 and their solutions are (see Ref.[9])

$$a_M(\boldsymbol{p}) = u_r(\boldsymbol{p}) a_{M,r}(\boldsymbol{p}),$$
$$b^+_M(\boldsymbol{p}) = \nu_r(\boldsymbol{p}) b^+_{M,r}(\boldsymbol{p}) \quad \left(r = \frac{1}{2}, -\frac{1}{2}\right). \quad (40)$$

Inserting Eq.(40) into Eq.(37) gives

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} [e^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p}) u_r(\boldsymbol{p}) a_{M,r}(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}px} + \overline{e}^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p}) \nu_r(\boldsymbol{p}) b^+_{M,r}(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}px})]. \quad (41)$$

With this expression, Eq.(7f) becomes

$$\gamma_\nu e^{\nu\nu_2\cdots\nu_n}_M(\boldsymbol{p}) u_r(\boldsymbol{p}) a_{M,r}(\boldsymbol{p}) = 0, \quad (42a)$$

$$\gamma_\nu \overline{e}^{\nu\nu_2\cdots\nu_n}_M(\boldsymbol{p}) \nu_r(\boldsymbol{p}) b^+_{M,r}(\boldsymbol{p}) = 0. \quad (42b)$$

Hence Eq.(42a) can be rewritten as (see Ref.[9])

$$\gamma_\nu e^{\nu\nu_2\cdots\nu_n}_M(0) u_r(0) a_{M,r}(\boldsymbol{p}) = 0, \quad (43)$$

where $a_{M,r}(\boldsymbol{p})$ is related to two magnetic quantum numbers M and r $\left(M = 0, \pm 1, \pm 2, \cdots, \pm n; r = \frac{1}{2}, -\frac{1}{2}\right)$. Recalling the Clebsch–Gordon coefficients for coupling two spin angular momenta with spin n and spin 1/2 respectively, a general candidate for $a_{M,r}(\boldsymbol{p})$ is

$$\begin{aligned} &a_{M,r}(\boldsymbol{p}) \\ &=\sum_m \left\langle n,M;\frac{1}{2},r|n,\frac{1}{2},n+\frac{1}{2},m\right\rangle a_{n+\frac{1}{2},m}(\boldsymbol{p}) \\ &\quad+\sum_{m'} \left\langle n,M;\frac{1}{2},r|n,\frac{1}{2},n-\frac{1}{2},m'\right\rangle a_{n-\frac{1}{2},m'}(\boldsymbol{p}) \\ &\left(m=\pm\frac{1}{2},\pm\frac{3}{2},\cdots,\pm\left(n+\frac{1}{2}\right);\right. \\ &\left. m'=\pm\frac{1}{2},\pm\frac{3}{2},\cdots,\pm\left(n-\frac{1}{2}\right)\right). \end{aligned} \tag{44}$$

Letting

$$U^{\nu_1\nu_2\cdots\nu_n}_{n+\frac{1}{2},m}(\boldsymbol{p}) = \sum_{M,r} e^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p})u_r(\boldsymbol{p}) \times \left\langle n,M;\frac{1}{2},r|n,\frac{1}{2},n+\frac{1}{2},m\right\rangle, \tag{45a}$$

$$U^{\nu_1\nu_2\cdots\nu_n}_{n-\frac{1}{2},m'}(\boldsymbol{p}) = \sum_{M,r} e^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p})u_r(\boldsymbol{p}) \times \left\langle n,M;\frac{1}{2},r|n,\frac{1}{2},n-\frac{1}{2},m'\right\rangle, \tag{45b}$$

then Eq.(43) takes the form

$$\gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n+\frac{1}{2},m}(0)a_{n+\frac{1}{2},m}(\boldsymbol{p}) + \gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n-\frac{1}{2},m'}(0)a_{n-\frac{1}{2},m}(\boldsymbol{p}) = 0. \tag{46}$$

However

$$\gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n+\frac{1}{2},m}(0) = 0 \quad \left(m=\pm\frac{1}{2},\pm\frac{3}{2},\cdots,\pm\left(n+\frac{1}{2}\right)\right), \tag{47a}$$

$$\gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n-\frac{1}{2},m'}(0) = \mathrm{i}\gamma_5\sqrt{\frac{2n+1}{n}}U^{\nu_2\cdots\nu_n}_{n-\frac{1}{2},m'}(0) \neq 0 \quad \left(m'=\pm\frac{1}{2},\pm\frac{3}{2},\cdots,\pm\left(n-\frac{1}{2}\right)\right). \tag{47b}$$

Substituting Eq.(47) into Eq.(46) yields

$$a_{n-\frac{1}{2},m'}(\boldsymbol{p}) = 0, \tag{48}$$

and Eq.(44) becomes $\left(\text{omit the index } n+\frac{1}{2} \text{ in } a_{n+\frac{1}{2},m}(\boldsymbol{p})\right)$

$$a_{M,r}(\boldsymbol{p}) = \sum_m \left\langle n,M;\frac{1}{2},r|n,\frac{1}{2},n+\frac{1}{2},m\right\rangle a_m(\boldsymbol{p}). \tag{49a}$$

Similarly, Eq.(42b) leads to

$$b^+_{M,r}(\boldsymbol{p}) = \sum_m \left\langle n,M;\frac{1}{2},r|n,\frac{1}{2},n+\frac{1}{2},m\right\rangle b^+_m(\boldsymbol{p}). \tag{49b}$$

Inserting Eq.(49) into Eq.(41) gives the final result

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = \frac{1}{\sqrt{V}}\sum_{\vec{p}}[a_m(\boldsymbol{p})U^{\nu_1\nu_2\cdots\nu_n}_m(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px} + b^+_m(\boldsymbol{p})V^{\nu_1\nu_2\cdots\nu_n}_m(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px})], \tag{50}$$

with

$$U^{\nu_1\nu_2\cdots\nu_n}_m = \sum_{\lambda_1,\lambda_2,\cdots\lambda_n=-1}^{1}\ \sum_{r=-1/2}^{1/2}\ \prod_{i=1}^{n} e^{\nu_i}_{\lambda_i}(\boldsymbol{p})u_r(\boldsymbol{p})\delta(\lambda_1+\lambda_2+\cdots\lambda_n+r,m) \times\sqrt{\frac{2^n\left(n+\frac{1}{2}+m\right)!\left(n+\frac{1}{2}-m\right)!}{(2n+1)!\prod_{i=1}^{n}(1+\lambda_i)!(1-\lambda_i)!\left(\frac{1}{2}+r\right)!\left(\frac{1}{2}-r\right)!}}, \tag{51}$$

$$V^{\nu_1\nu_2\cdots\nu_n}_m = \sum_{\lambda_1,\lambda_2,\cdots\lambda_n=-1}^{1}\ \sum_{r=-1/2}^{1/2}\ \prod_{i=1}^{n} \bar{e}^{\nu_i}_{\lambda_i}(\boldsymbol{p})\nu_r(\boldsymbol{p})\delta(\lambda_1+\lambda_2+\cdots\lambda_n+r,m) \times\sqrt{\frac{2^n\left(n+\frac{1}{2}+m\right)!\left(n+\frac{1}{2}-m\right)!}{(2n+1)!\prod_{i=1}^{n}(1+\lambda_i)!(1-\lambda_i)!\left(\frac{1}{2}+r\right)!\left(\frac{1}{2}-r\right)!}}, \tag{52}$$

where the Wigner formula for the Clebsch–Gordon coefficients has been used in the last step. By utilizing the normalization and orthogonality properties of $e_{\lambda}^{\nu}(\boldsymbol{p})$, $u_r(\boldsymbol{p})$, $\nu_r(\boldsymbol{p})$ and the properties of the Clebsch–Gordon coefficients, it can be shown that both the positive and negative solutions possess normalization and orthogonality properties.

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