Introduction to Chiral Perturbation Theory

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Abstract

This article provides a pedagogical introduction to the basic concepts of chiral perturbation theory and is designed as a text for a two-semester course on that topic. Chapter 1 serves as a general introduction to the empirical and theoretical foundations which led to the development of chiral perturbation theory. Chapter 2 deals with QCD and its global symmetries in the chiral limit; the concept of Green functions and Ward identities reflecting the underlying chiral symmetry is elaborated. In Chap. 3 the idea of a spontaneous breakdown of a global symmetry is discussed and its consequences in terms of the Goldstone theorem are demonstrated. Chapter 4 deals with mesonic chiral perturbation theory and the principles entering the construction of the chiral Lagrangian are outlined. Various examples with increasing chiral orders and complexity are given. Finally, in Chap. 5 the methods are extended to include the interaction between Goldstone bosons and baryons in the single-baryon sector, with the main emphasis put on the heavy-baryon formulation. At the end, the method of infrared regularization in the relativistic framework is discussed.

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Chapter 1

Introduction

1.1 Scope and Aim of the Review

The present review has evolved from two courses I have taught several times at the Johannes Gutenberg-Universität, Mainz. The first course was an introduction to chiral perturbation theory (ChPT) which only covered the purely mesonic sector of the theory. In the second course the methods were extended to also include baryons. I have tried to preserve the spirit of those lectures in this article in the sense that it is meant to be a *pedagogical introduction* to the basic concepts of chiral perturbation theory. By this I do not mean that the material covered is trivial, but that rather I have deliberately also worked out those pieces which by the "experts" are considered as well known. In particular, I have often included intermediate steps in derivations in order to facilitate the understanding of the origin of the final results. My intention was to keep a balance between mathematical rigor and illustrations by means of (numerous) simple examples.

This article addresses both experimentalists and theorists! Ideally, it would help a graduate student interested in theoretical physics getting started in the field of chiral perturbation theory. However, it is also written for an experimental graduate student with the purpose of conveying some ideas why the experiment she/he is performing is important for our theoretical understanding of the strong interactions. My precedent in this context is the review by A. W. Thomas [Tho 84] which appeared in this series many years ago and served for me as an introduction to the cloudy bag model.

Finally, this article clearly is not intended to be a comprehensive overview of the numerous results which have been obtained over the past two decades. For obvious reasons, I would, right at the beginning, like to apologize to all the researchers who have made important contributions to the field that have not been mentioned in this work. Readers interested in the present status of applications are referred to lecture notes and review articles [Leu 92, Bij 93, Mei 93, Leu 95, Ber+ 95b, Raf 95, Pic 95, Eck 95, Man 96, Pic 99, Bur 00] as well as conference proceedings [BH 95, Ber+ 98a, Ber+ 02a].

The present article is organized as follows. Chapter 1 contains a general introduction to the empirical and theoretical foundations which led to the development of chiral perturbation theory. Many of the technical aspects mentioned in the introduction will be treated in great detail later on. Chapter 2 deals with QCD and its global symmetries in the chiral limit and the concept of Green functions and Ward identities reflecting the underlying chiral symmetry is elaborated. In Chap. 3 the idea of a spontaneous breakdown of a global symmetry is discussed and its consequences in terms of the Goldstone theorem are demonstrated. Chapter 4 deals with mesonic chiral perturbation theory and the principles entering the construction of the chiral Lagrangian are outlined. Various examples with increasing chiral orders and complexity are given. Finally, in Chap. 5 the methods are extended to include the interaction between Goldstone bosons and baryons in the single-baryon sector with the main emphasis put on the heavy-baryon formulation. At the end, the method of infrared regularization in the relativistic framework is discussed. Some technical details and simple illustrations are relegated to the Appendices.

1.2 Introduction to Chiral Symmetry and Its Application to Mesons and Single Baryons

In the 1950's a description of the strong interactions in the framework of quantum field theory seemed to fail due to an ever increasing number of observed hadrons as well as a coupling constant which was too large to allow for a sensible application of perturbation theory [Gro 99]. The rich spectrum of hadrons together with their finite sizes (i.e., non-point-like behavior showing up, e.g., in elastic electron-proton scattering through the existence of form factors) were the first hints pointing to a substructure in terms of more fundamental constituents. A calculation of the anomalous magnetic moments of protons and neutrons in the framework of a pseudoscalar pion-nucleon interaction gave rise to values which were far off the empirical ones (see, e.g., [BH 55]). On the other hand, a simple quark model analysis [Beg+ 64, Mor 65] gave a prediction $-3/2$ for the ratio μ_p/μ_n which is very close to the empirical value of -1.46. Nevertheless, the existence of quarks was hotly debated for a long time, since these elementary building blocks, in contrast to the constituents of atomic or nuclear physics, could not be isolated as free particles, no matter what amount of energy was supplied to, say, the proton. Until the early 1970's it was common to talk about "fictitious" constituents allowing for a simplified group-theoretical classification of the hadron spectrum [Gel 64a, Zwe 64], which, however, could not be interpreted as dynamical degrees of freedom in the context of quantum field theory.

In our present understanding, hadrons are highly complex objects built from more fundamental degrees of freedom. These are on the one hand matter fields with spin 1/2 (quarks) and on the other hand massless spin-1 fields (gluons) mediating the strong interactions. Many empirical results of medium- and high-energy physics [Alt 89] such as, e.g., deep-inelastic lepton-hadron scattering, hadron production in electron-positron annihilation, and lepton-pair production in Drell-Yan processes may successfully be described using perturbative methods in the framework of an SU(3) gauge theory, which is referred to as quantum chromodynamics (QCD) [GW 73a, Wei 73, Fri+ 73]. Of particular importance in this context is the concept of asymptotic freedom [GW 73a, GW 73b, Pol 73], referring to the fact that the coupling strength decreases for increasing momentum transfer Q^2, providing an explanation of (approximate) Bjorken scaling in deep inelastic scattering and allowing more generally for a perturbative approach at high energies. Sometimes perturbative QCD is used as a synonym for asymptotic freedom. In Refs. [CG 73, Zee 73] it was shown that Yang-Mills theories, i.e., gauge theories based on non-Abelian Lie groups, provide the only possibility for asymptotically free theories in four dimensions. At present, QCD is compatible with all empirical phenomena of the strong interactions in the asymptotic region. However, one should also keep in mind that many phenomena cannot be treated by perturbation theory. For example, simple (static) properties of hadrons cannot yet be described by *ab initio* calculations from QCD and this remains one of the largest challenges in theoretical particle physics [Gro 99]. In this context it is interesting to note that, of the three open problems of QCD at the quantum level, namely, the "gap problem," "quark confinement," and (spontaneous) "chiral symmetry breaking" [JW 00], the Yang-Mills existence of a mass gap has been chosen as one of the Millennium Prize Problems [CMI] of the Clay Mathematics Institute. From a physical point of view this problem relates to the fact that nuclear forces are strong and short-ranged.

One distinguishes among six quark flavors u (up), d (down), s (strange), c (charm), b (bottom), and t (top), each of which coming in three different color degrees of freedom and transforming as a triplet under the fundamental representation of color SU(3). The interaction between the quarks and the eight gauge bosons does not depend on flavor, i.e., gluons themselves are flavor neutral. On the other hand, due to the non-Abelian character of the group SU(3), also gluons carry "color charges" such that the QCD Lagrangian contains gluon self interactions involving vertices with three and four gluon fields. As a result, the structure of QCD is much more complicated than that of Quantum Electrodynamics (QED) which is based on a local, Abelian U(1) invariance. However, it is exactly the non-Abelian nature of the theory which provides an anti-screening due to gluons that prevails over the screening due to $q\bar{q}$ pairs, leading to an asymptotically free theory [Nie 81]. Since neither quarks nor gluons have been observed as free, asymptotic states, one assumes that any observable hadron must be in a so-called color singlet state, i.e., a physically observable state is invariant under SU(3) color transformations. The strong increase of the running coupling for large distances possibly provides a mechanism for color confinement [GW 73b]. In the framework of lattice QCD this can be shown in the so-called strong coupling limit [Wil 74]. However, one has to keep in mind that the continuum limit of lattice gauge theory is approached for a weak coupling and a mathematical proof for color confinement is still missing [JW 00].

There still exists no *analytical* method for the description of QCD at large distances, i.e., at low energies. For example, how the asymptotically observed hadrons, including their rich resonance spectrum, are created from QCD dynamics is still insufficiently understood.[1] This is one of the reasons why, for many practical purposes, one makes use of phenomenological, more-or-less QCD-inspired, models of hadrons (see, e.g., [Tho 84, HS 86, MZ 86, ZB 86, GW 86, Bha 88, Mos 89, Gia 90, VW 91, Don+ 92, TW 01]).

Besides the *local* SU(3) color symmetry, QCD exhibits further *global* symmetries. For example, in a strong interaction process, a given quark cannot change its flavor, and if quark-antiquark pairs are created or annihilated

[1] For a prediction of hadron masses in the framework of lattice QCD see, e.g., Refs. [But+ 94, Ali+ 02].

during the interaction, these pairs must be flavor neutral. In other words, for each flavor the difference in the number of quarks and antiquarks (flavor number) is a constant of the motion. This symmetry originates in a global invariance under a direct product of U(1) transformations for each quark flavor and is an exact symmetry of QCD independent of the value of the quark masses. Other symmetries are more or less satisfied. It is well known that the hadron spectrum may be organized in terms of (approximately) degenerate basis states carrying irreducible representations of isospin SU(2). Neglecting electromagnetic effects, such a symmetry in QCD results from equal u- and d-quark (current) masses. The extension including the s quark leads to the famous flavor SU(3) symmetry [GN 64] which, however, is already significantly broken due to the larger s-quark mass.

The masses of the three light quarks u, d, and s are small in comparison with the masses of "typical" light hadrons such as, e.g., the ρ meson (770 MeV) or the proton (938 MeV). On the other hand, the eight lightest pseudoscalar mesons are distinguished by their comparatively small masses.[2] Within the pseudoscalar octet, the isospin triplet of pions has a significantly smaller mass (135 MeV) than the mesons containing strange quarks. One finds a relatively large mass gap of about 630 MeV between the isospin triplets of the pseudoscalar and the vector mesons, with the gap between the corresponding multiplets involving strange mesons being somewhat smaller.

In the limit in which the masses of the light quarks go to zero, the left-handed and right-handed quark fields are decoupled from each other in the QCD Lagrangian. At the "classical" level QCD then exhibits a global $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ symmetry. However, at the quantum level (including loops) the singlet axial-vector current develops an anomaly [Adl 69, AB 69, Bar 69, BJ 69, Adl 70] such that the difference in left-handed and right-handed quark numbers is not a constant of the motion. In other words, in the so-called chiral limit, the QCD-Hamiltonian has a $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ symmetry.

Naturally the question arises, whether the hadron spectrum is, at least approximately, in accordance with such a symmetry of the Hamiltonian. The $\mathrm{U}(1)_V$ symmetry is connected to baryon number conservation, where quarks and antiquarks are assigned the baryon numbers $B = 1/3$ and $B = -1/3$, respectively. Mesons and baryons differ by their respective baryon numbers $B = 0$ and $B = 1$. Since baryon number is additive, a nucleus containing A

[2] They are not considered as "typical" hadrons due to their special role as the (approximate) Goldstone bosons of spontaneous chiral symmetry breaking.

nucleons has baryon number $B = A$.

On the other hand, the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry is not even approximately realized by the low-energy spectrum. If one constructs from the 16 generators of the group $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ the linear combinations $Q_V^a = Q_R^a + Q_L^a$ and $Q_A^a = Q_R^a - Q_L^a$, $a = 1, \cdots, 8$, the generators Q_V^a form a Lie algebra corresponding to a $\mathrm{SU}(3)_V$ subgroup H of G. It was shown in Ref. [VW 84] that, in the chiral limit, the ground state is necessarily invariant under the group H, i.e., the eight generators Q_V^a annihilate the ground state. The symmetry with respect to H is said to be realized in the so-called Wigner-Weyl mode. As a consequence of Coleman's theorem [Col 66], the symmetry pattern of the spectrum follows the symmetry of the ground state. Applying one of the axial generators Q_A^a to an arbitrary state of a given multiplet of well-defined parity, one would obtain a degenerate state of opposite parity, since Q_A^a has negative parity and, by definition, commutes with the Hamiltonian in the chiral limit. However, due to Coleman's theorem such a conclusion tacitly assumes that the ground state is annihilated by the Q_A^a. Since such a parity doubling is not observed in the spectrum one reaches the conclusion that the Q_A^a do *not* annihilate the ground state. In other words, the ground state is not invariant under the full symmetry group of the Hamiltonian, a situation which is referred to as spontaneous symmetry breaking or the Nambu-Goldstone realization of a symmetry [Nam 60, NJ 61a, NJ 61b]. As a consequence of the Goldstone theorem [Gol 61, Gol+ 62], each generator which commutes with the Hamiltonian but does not annihilate the ground state is associated with a massless Goldstone boson, whose properties are tightly connected with the generator in question.

The eight generators Q_A^a have negative parity, baryon number zero, and transform as an octet under the subgroup $\mathrm{SU}(3)_V$ leaving the vacuum invariant. Thus one expects the same properties of the Goldstone bosons, and the light pseudoscalar octet (π, K, η) qualifies as candidates for these Goldstone bosons. The finite masses of the physical multiplet are interpreted as a consequence of the explicit symmetry breaking due to the finite u-, d-, and s-quark masses in the QCD Lagrangian [GL 82].

Of course, the above (global) symmetry considerations were long known before the formulation of QCD. In the 1960's they were the cornerstones of the description of low-energy interactions of hadrons in the framework of various techniques, such as the current-algebra approach in combination with the hypothesis of a partially conserved axial-vector current (PCAC) [Gel 64b, AD 68, Trei+ 72, Alf+ 73], the application of phenomenological Lagrangians

[Wei 67, Sch 67, Wei 68, Col+ 69, Cal+ 69, GG 69], and perturbation theory about a symmetry realized in the Nambu-Goldstone mode [Das 69, DW 69, LP 71, Pag 75]. All these methods were equivalent in the sense that they produced the same results for "soft" pions [DW 69] .

Although QCD is widely accepted as the fundamental gauge theory underlying the strong interactions, we still lack the analytical tools for *ab initio* descriptions of low-energy properties and processes. However, new techniques have been developed to extend the results of the current-algebra days and *systematically* explore corrections to the soft-pion predictions based on symmetry properties of QCD Green functions. The method is called chiral perturbation theory (ChPT) [Wei 79, GL 84, GL 85a] and describes the dynamics of Goldstone bosons in the framework of an effective field theory. Although one returns to a field theory in terms of non-elementary hadrons, there is an important distinction between the early quantum field theories of the strong interactions and the new approach in the sense that, now, one is dealing with a so-called *effective* field theory. Such a theory allows for a *perturbative* treatment in terms of a momentum—as opposed to a coupling-constant—expansion.

The starting point is a theorem of Weinberg stating that a perturbative description in terms of the most general effective Lagrangian containing all possible terms compatible with assumed symmetry principles yields the most general S matrix consistent with the fundamental principles of quantum field theory and the assumed symmetry principles [Wei 79]. The proof of the theorem relies on Lorentz invariance and the absence of anomalies [Leu 94, HW 94] and starts from the observation that the Ward identities satisfied by the Green functions of the symmetry currents are equivalent to an invariance of the generating functional under *local* transformations [Leu 94]. For that reason, one considers a *locally* invariant, effective Lagrangian although the symmetries of the underlying theory may originate in a global symmetry. If the Ward identities contain anomalies, they show up as a modification of the generating functional, which can explicitly be incorporated through the Wess-Zumino-Witten construction [WZ 71, Wit 83]. All other terms of the effective Lagrangian remain locally invariant.

In the present case, the assumed symmetry is the $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R\times\mathrm{U}(1)_V$ symmetry of the QCD Hamilton operator in the chiral limit, in combination with a restricted $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$ symmetry of the ground state. For center-of-mass energies below the ρ-meson mass, the only asymptotic states which can explicitly be produced are the Goldstone bosons. For the description of processes in this energy regime one organizes the most general, chirally invariant Lagrangian for the pseudoscalar meson octet in an expansion in terms of momenta and quark masses. Such an ansatz is naturally suggested by the fact that the interactions of Goldstone bosons are known to vanish in the zero-energy limit. Since the effective Lagrangian by construction contains an infinite number of interaction terms, one needs for any practical purpose an organization scheme allowing one to compare the importance of, say, two given diagrams. To that end, for a given diagram, one analyzes its behavior under a linear rescaling of the *external* momenta, $p_i \rightarrow tp_i$, and a quadratic rescaling of the light quark masses, $m_i \rightarrow t^2 m_i$. Applying Weinberg's power counting scheme [Wei 79], one finds that any given diagram behaves as t^D, where $D \geq 2$ is determined by the structure of the vertices and the topology of the diagram in question. For a given D, Weinberg's formula unambiguously determines to which order in the momentum and quark mass expansion the Lagrangian needs to be known. Furthermore, the number of loops is restricted to be smaller than or equal to $D/2-1$, i.e., Weinberg's power counting establishes a relation between the momentum expansion and the loop expansion.[3]

Effective field theories are not renormalizable in the usual sense. However, this is no longer regarded as a serious problem, since by means of Weinberg's counting scheme the infinities arising from loops may be identified order by order in the momentum expansion and then absorbed in a renormalization of the coefficients of the most general Lagrangian. Thus, in any arbitrary order the results are finite. Of course, there is a price to pay: the rapid increase in the number of possible terms as the order increases. Practical applications will hence be restricted to low orders.

The lowest-order mesonic Lagrangian, $\mathcal{L}_2$, is given by the nonlinear σ model coupled to external fields [GL 84, GL 85a]. It contains two free parameters: the pion-decay constant and the scalar quark condensate, both in the chiral limit. The specific values are not determined by chiral symmetry and must, ultimately, be explained from QCD dynamics. When calculating processes in the phenomenological approximation to $\mathcal{L}_2$, i.e., considering only tree-level diagrams, one reproduces the results of current algebra [Wei 79]. Since tree-level diagrams involving vertices derived from a Hermitian Lagrangian are always real, one has to go beyond the tree level in order not

[3]The counting refers to ordinary chiral perturbation theory in the mesonic sector, where D is an even number.

to violate the unitarity of the S matrix. A calculation of one-loop diagrams with $\mathcal{L}_2$, on the one hand, leads to infinities which are not of the original type, but also contributes to a perturbative restoration of unitarity. Due to Weinberg's power counting, the divergent terms are of order $\mathcal{O}(p^4)$ and can thus be compensated by means of a renormalization of the most general Lagrangian at $\mathcal{O}(p^4)$.

The most general, effective Lagrangian at $\mathcal{O}(p^4)$ was first constructed by Gasser and Leutwyler [GL 85a] and contains 10 physical low-energy constants as well as two additional terms containing only external fields. Out of the ten physically relevant structures, eight are required for the renormalization of the infinities due to the one-loop diagrams involving $\mathcal{L}_2$. The finite parts of the constants represent free parameters, reflecting our ignorance regarding the underlying theory, namely QCD, in this order of the momentum expansion. These parameters may be fixed phenomenologically by comparison with experimental data [GL 84, GL 85a, Bij+ 95a]. There are also theoretical approaches for estimating the low-energy constants in the framework of QCD-inspired models [ER 86, Esp+ 90, Ebe+ 93, Bij+ 93], meson-resonance saturation [Eck+ 89a, Eck+ 89b, Don+ 89, KN 01, Leu 01b] and lattice QCD [MC 94, Gol 02].

Without external fields (i.e., pure QCD) or including electromagnetic processes only, the effective Lagrangians $\mathcal{L}_2$ and $\mathcal{L}_4$ have an additional symmetry: they contain interaction terms involving exclusively an even number of Goldstone bosons. This property is often referred to as normal or even *intrinsic* parity, but is obviously not a symmetry of QCD, because it would exclude reactions of the type $\pi^0 \to \gamma\gamma$ or $K^+K^- \to \pi^+\pi^-\pi^0$. In Ref. [Wit 83], Witten discussed how to remove this symmetry from the effective Lagrangian and essentially re-derived the Wess-Zumino anomalous effective action which describes the chiral anomaly [WZ 71]. The corresponding Lagrangian, which is of $\mathcal{O}(p^4)$, cannot be written as a standard local effective Lagrangian in terms of the usual chiral matrix U but can be expressed directly in terms of the Goldstone boson fields. In particular, for the above case, by construction it contains interaction terms with an odd number of Goldstone bosons (odd intrinsic parity). In contrast to the Lagrangian of Gasser and Leutwyler, the Wess-Zumino-Witten (WZW) effective action does not contain any free parameter apart from the number of colors. The excellent description of the neutral pion decay $\pi^0 \to \gamma\gamma$ for $N_c = 3$ is regarded as a key evidence for the existence of *three* color degrees of freedom.

Chiral perturbation theory to $\mathcal{O}(p^4)$ has become a well-established method for describing the low-energy interactions of the pseudoscalar octet. For an overview of its many successful applications the interested reader is referred to Refs. [Leu 92, Bij 93, Mei 93, Leu 95, BH 95, Ber+ 95b, Raf 95, Pic 95, Eck 95, Man 96, Ber+ 98a, Pic 99, Bur 00, Ber+ 02a]. In general, due to the relatively large mass of the s quark, the convergence in the SU(3) sector is somewhat slower as compared with the SU(2) version. Nevertheless, ChPT in the SU(3) sector has significantly contributed to our understanding of previously open questions. A prime example is the decay rate of $\eta \to \pi\pi\pi$ which current algebra predicts to be much too small. In Ref. [GL 85c] it was shown that one-loop corrections substantially increase the theoretical value and remove the previous discrepancy between theory and experiment.

For obvious reasons, the question of convergence of the method is of utmost importance. The so-called chiral symmetry breaking scale $\Lambda_{\rm CSB}$ is the dimensional parameter which characterizes the convergence of the momentum expansion [MG 84, Geo 84]. A "naive" dimensional analysis of loop diagrams suggests that this scale is given by $\Lambda_{\rm CSB} \approx 4\pi F_0$, where $F_0 \approx 93\,\text{MeV}$ denotes the pion-decay constant in the chiral limit and the factor 4π originates from a geometric factor in the calculation of loop diagrams in four dimensions. A second dimensional scale is provided by the masses of the lightest excitations which have been "integrated out" as explicit dynamical degrees of freedom of the theory—in the present case, typically the lightest vector mesons. In a phenomenological approach the exchange of such particles leads to a propagator of the type $(q^2-M^2)^{-1} \approx -M^{-2}(1+q^2/M^2+\cdots)$, where the expansion only converges for $|q^2| < M^2$. The corresponding scale is approximately of the same size as $4\pi F_0$. Assuming a reasonable behavior of the coefficients of the momentum expansion leads to the expectation that ChPT converges for center-of-mass energies sufficiently below the ρ-meson mass. Of course, the validity of such a statement depends on the specific process under consideration and the quantum numbers of the intermediate states.

Clearly, for a given process, it would be desirable to have an idea about the size of the next-to-leading-order corrections. In the odd-intrinsic-parity sector such a calculation is at least of order $\mathcal{O}(p^6)$, because the WZW action itself is already of order $\mathcal{O}(p^4)$. Thus, according to Weinberg's power counting, one-loop diagrams involving exactly one WZW vertex and an arbitrary number of $\mathcal{L}_2$ vertices result in corrections of $\mathcal{O}(p^6)$. Several authors have shown that quantum corrections to the Wess-Zumino-Witten classical action do not renormalize the coefficient of the Wess-Zumino-Witten term

[DW 89, Iss 90, Bij+ 90, AA 91, Ebe 01, Bij+ 02]. Furthermore, the one-loop counter terms lead to conventional chirally invariant structures at $\mathcal{O}(p^6)$ [DW 89, Iss 90, Bij+ 90, AA 91, Ebe 01, Bij+ 02]. There have been several attempts to construct the most general odd-intrinsic-parity Lagrangian at $\mathcal{O}(p^6)$ and only recently two independent calculations have found the same number of 23 independent structures in the SU(3) sector [Ebe+ 02, Bij+ 02]. For an overview of the application of ChPT to anomalous processes, the interested reader is referred to Ref. [Bij 93].

In general, next-to-leading-order corrections to processes in the even-intrinsic-parity sector are of $\mathcal{O}(p^4)$. However, there are also processes which receive their leading-order contributions at $\mathcal{O}(p^4)$. In particular, the reactions $\gamma\gamma \to \pi^0\pi^0$ [MP 91, DH 93, Bel+ 94, Kne+ 94, BB 95, Bel+ 96] and $\eta \to \pi^0\gamma\gamma$ [Ame+ 92, BB 95, Ko 95, Bel+ 96, Jet 96] have received considerable attention, because the predictions at $\mathcal{O}(p^4)$ [BC 88, Don+ 88] and [Ame+ 92], respectively, were in disagreement with experimental results ([Mar+ 90] and [Gro+ 00], respectively). In the case of $\gamma\gamma \to \pi^0\pi^0$ loop corrections at $\mathcal{O}(p^6)$ lead to a considerably improved description, with the result only little sensitive to the tree-level diagrams at $\mathcal{O}(p^6)$ [Bel+ 94]. The opposite picture emerges for the decay $\eta \to \pi^0\gamma\gamma$, where the tree-level diagrams at $\mathcal{O}(p^6)$ play an important role.

A second class of $\mathcal{O}(p^6)$ calculations includes processes which already receive contributions at $\mathcal{O}(p^2)$ such as $\pi\pi$ scattering [Bij+ 96] or $\gamma\gamma \to \pi^+\pi^-$ [Bür 96]. Here, $\mathcal{O}(p^6)$ calculations may be viewed as precision tests of ChPT. The first process is of fundamental importance because it provides information on the mechanism of spontaneous symmetry breaking in QCD [Bij+ 96]. The second reaction is of particular interest because an old current-algebra low-energy theorem [Ter 72] relates the electromagnetic polarizabilities $\bar{\alpha}$ and $\bar{\beta}$ of the charged pion at $\mathcal{O}(p^4)$ to radiative pion decay $\pi^+ \to e^+\nu_e\gamma$. Corrections at $\mathcal{O}(p^6)$ were shown to be 12% and 24% of the $\mathcal{O}(p^4)$ values for $\bar{\alpha}$ and $\bar{\beta}$, respectively [Bür 96]. On the other hand, experimental results for the polarizabilities scatter substantially and still have large uncertainties (see, e.g., Ref. [Unk+ 02]) and new experimental data are clearly needed to test the accuracy of the chiral predictions.

In the SU(3) sector, the first construction of the most general even-intrinsic parity Lagrangian at $\mathcal{O}(p^6)$ was performed in Ref. [FS 96]. Although it was later shown that the original list of terms contained redundant structures [Bij+ 99], even the final number of 90 + 4 free parameters is very large, such that, in contrast to the Lagrangian $\mathcal{L}_4$ of Gasser and Leutwyler, it seems unlikely that all parameters can be fixed through comparison with experimental data. However, chiral symmetry relates different processes to each other, such that groups of interaction terms may be connected with each other and through comparison with experiment the consistency conditions of chiral symmetry may be tested. Furthermore, the same theoretical methods which have been applied to predict the coefficients of $\mathcal{O}(p^4)$ may be extended to the next order [Bel+ 95] which, however, involves much more work.

Chiral perturbation theory has proven to be highly successful in the mesonic sector and, for obvious reasons, one would like to have a generalization including the interaction of Goldstone bosons with baryons. The group-theoretical foundations for a nonlinear realization of chiral symmetry were developed in Refs. [Wei 68, Col+ 69, Cal+ 69], which also included the coupling of Goldstone bosons to other isospin or, for the more general case, SU(N)-flavor multiplets. Numerous low-energy theorems involving the pion-nucleon interaction and its SU(3) extension were derived in the 1960's by use of current-algebra methods and PCAC. However, a *systematic* study of chiral corrections to the low-energy theorems has only become possible when the methods of mesonic ChPT were extended to processes with one external nucleon line [Gas+ 88]. The situation turned out to be more involved than in the pure mesonic sector because the loops have a more complicated structure due to the nucleon mass which, in contrast to the Goldstone boson masses, does not vanish in the chiral limit. This introduces a third scale into the problem beyond the pion decay constant and the scalar quark condensate. In particular, it was shown that the relativistic formulation, at first sight, does not provide such a simple connection between the chiral expansion and the loop expansion as in the mesonic sector [Gas+ 88], i.e., higher-order loop diagrams also contribute to lower orders in the chiral expansion of a physical quantity. This observation was taken as evidence for a breakdown of power counting in the relativistic formulation. Subsequently, techniques borrowed from heavy-quark physics were applied to the baryon sector [JM 91, Ber+ 92b], providing a heavy-baryon formulation of ChPT (HBChPT), where the Lagrangian is not only expanded in the number of derivatives and quark masses but also in powers of inverse nucleon masses. The technique is very similar to the Foldy-Wouthuysen method [FW 50].

There have been many successful applications of HBChPT to "traditional" current-algebra processes such as pion photoproduction [Ber+ 92a, Ber+ 96a] and radiative pion capture [Fea+ 00], pion electroproduction [Ber+ 92c, Ber+ 94, Ber+ 95a, Ber+ 00], pion-nucleon scattering [Ber+ 95c, Moj 98, Fet+ 98, FM 01], to name just a few (for an extensive overview, see Ref. [Ber+ 95b]). In all these cases, ChPT has allowed one to either systematically calculate corrections to the old current-algebra results or to obtain new predictions which are beyond the scope of the old techniques. Other applications include the calculation of static properties such as masses [Jen 92, Ber+ 93a, LL 94, BM 97, MB 99] and various form factors of baryons [Ber+ 92c, Ber+ 96b, Fea+ 97, Kub+ 99]. The role of the pionic degrees of freedom has been extensively discussed for real Compton scattering off the nucleon in terms of the electromagnetic polarizabilities [Ber+ 92d, Ber+ 93b, Gel+ 00, MB 00, Kum+ 00, GH 02]. The new frontier of virtual Compton scattering off the nucleon [Gui+ 95, Dre+ 97, Roc+ 00] has also been addressed in the framework of ChPT [Hem+ 97a, Hem+ 97b, Hem+ 00, Lvo+ 01]. As in the mesonic sector, the most general chiral Lagrangian in the single-baryon sector is needed which, due to the spin degree of freedom, is more complicated [Gas+ 88, Kra 90, EM 96, Fet+ 01].

In the baryonic sector, the $\Delta(1232)$ resonance plays a prominent role because its excitation energy is only about two times the pion mass and its (almost) 100 % branching ratio to the decay mode $N\pi$. In Ref. [Hem+ 97c], the formalism of the so-called small scale expansion was developed, which also treats the nucleon-delta mass splitting as a "small" quantity like the pion mass. Subsequently, the formalism was applied to Compton scattering [Hem+ 98], baryon form factors [Ber+ 98b], the $N\Delta$ transition [Gel+ 99] and virtual Compton scattering [Hem+ 00].

While the heavy-baryon formulation provided a useful low-energy expansion scheme, it was realized in the context of the isovector spectral function entering the calculation of the nucleon electromagnetic form factor that the corresponding perturbation series fails to converge in part of the low-energy region [Ber+ 96b]. Various methods have been suggested to generate a power counting which is also valid for the relativistic approach and which respects the singularity structure of Green functions [Tan 96, ET 98, BL 99, GJ 99, Geg+ 99, Lut 00, LK 02]. The so-called "infrared regularization" of Ref. [BL 99] decomposes one-loop diagrams into singular and regular parts. The singular parts satisfy power counting, whereas the regular parts can be absorbed into local counter terms of the Lagrangian. This technique solves the power counting problem of relativistic baryon chiral perturbation theory at the one-loop level and has already been applied to the calculation of baryon masses in SU(3) ChPT [ET 00], of form factors [KM 01a, Zhu+ 01, KM 01b], pion-nucleon scattering [BL 01] as well as the generalized Gerasimov-Drell-

Hearn sum rule [Ber+ 02b]. At present, the procedure has not yet been generalized to higher-order loop diagrams. In Ref. [Geg+ 99] another approach, based on choosing appropriate renormalization conditions, was proposed, leading to the correct analyticity structure and a consistent power counting, which can also be extended to higher loops.

Finally, the techniques of effective field theory have also been applied to the nucleon-nucleon interaction (see, e.g., Refs. [Wei 91, Ord+ 96, Kai+ 97, Kol 99, Epe+ 00, Bea+ 02, Fin+ 02]). Clearly this is a very important topic in its own right but is beyond the scope of the present work.

Chapter 2

QCD and Chiral Symmetry

Chiral perturbation theory (ChPT) provides a systematic framework for investigating strong-interaction processes at *low* energies, as opposed to a perturbative treatment of quantum chromodynamics (QCD) at high momentum transfers in terms of the "running coupling constant." The basis of ChPT is the global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ symmetry of the QCD Lagrangian in the limit of massless u, d, and s quarks. This symmetry is assumed to be spontaneously broken down to $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$ giving rise to eight massless Goldstone bosons. In this chapter we will describe in detail one of the foundations of ChPT, namely the symmetries of QCD and their consequences in terms of QCD Green functions.

2.1 Some Remarks on SU(3)

The group SU(3) plays an important role in the context of strong interactions, because on the one hand it is the gauge group of QCD and, on the other hand, flavor SU(3) is approximately realized as a global symmetry of the hadron spectrum (Eightfold Way [Nee 61, Gel 62, GN 64]), so that the observed (low-mass) hadrons can be organized in approximately degenerate multiplets fitting the dimensionalities of irreducible representations of SU(3). Finally, as will be discussed later in this chapter, the direct product $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ is the chiral-symmetry group of QCD for vanishing u-, d-, and s-quark masses. Thus, it is appropriate to first recall a few basic properties of SU(3) and its Lie algebra su(3) [BT 84, O'Ra 86, Jon 90].

The group SU(3) is defined as the set of all unitary, unimodular, 3×3

matrices U, i.e. $U^\dagger U = 1$,[1] and $\det(U) = 1$. In mathematical terms, SU(3) is an eight-parameter, simply connected, compact Lie group. This implies that any group element can be parameterized by a set of eight independent real parameters $\Theta = (\Theta_1, \cdots, \Theta_8)$ varying over a continuous range. The Lie-group property refers to the fact that the group multiplication of two elements $U(\Theta)$ and $U(\Psi)$ is expressed in terms of eight *analytic* functions $\Phi_i(\Theta;\Psi)$, i.e. $U(\Theta)U(\Psi) = U(\Phi)$, where $\Phi = \Phi(\Theta;\Psi)$. It is simply connected because every element can be connected to the identity by a continuous path in the parameter space and compactness requires the parameters to be confined in a finite volume. Finally, for compact Lie groups, every finite-dimensional representation is equivalent to a unitary one and can be decomposed into a direct sum of irreducible representations (Clebsch-Gordan series).

Elements of SU(3) are conveniently written in terms of the exponential representation[2]

$$U(\Theta) = \exp\left(-i\sum_{a=1}^{8}\Theta_a\frac{\lambda_a}{2}\right), \tag{2.1}$$

with Θ_a real numbers, and where the eight linearly independent matrices λ_a are the so-called Gell-Mann matrices, satisfying

$$\begin{aligned}
\frac{\lambda_a}{2} &= i\frac{\partial U}{\partial\Theta_a}(0,\cdots,0), &(2.2)\\
\lambda_a &= \lambda_a^\dagger, &(2.3)\\
\mathrm{Tr}(\lambda_a\lambda_b) &= 2\delta_{ab}, &(2.4)\\
\mathrm{Tr}(\lambda_a) &= 0. &(2.5)
\end{aligned}$$

An explicit representation of the Gell-Mann matrices is given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

[1]In this report we often adopt the convention that 1 stands for the unit matrix in n dimensions. It should be clear from the respective context which dimensionality actually applies.

[2]In our notation, the indices denoting group parameters and generators will appear as subscripts or superscripts depending on what is notationally convenient. We do not distinguish between upper and lower indices, i.e., we abandon the methods of tensor analysis.

abc	123	147	156	246	257	345	367	458	678
f_{abc}	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{3}$

Table 2.1: Totally antisymmetric non-vanishing structure constants of SU(3).

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{2.6}$$

The set $\{i\lambda_a\}$ constitutes a basis of the Lie algebra su(3) of SU(3), i.e., the set of all complex traceless skew Hermitian 3×3 matrices. The Lie product is then defined in terms of ordinary matrix multiplication as the commutator of two elements of su(3). Such a definition naturally satisfies the Lie properties of anti-commutativity

$$[A, B] = -[B, A] \tag{2.7}$$

as well as the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{2.8}$$

In accordance with Eqs. (2.1) and (2.2), elements of su(3) can be interpreted as tangent vectors in the identity of SU(3).

The structure of the Lie group is encoded in the commutation relations of the Gell-Mann matrices,

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2}, \tag{2.9}$$

where the totally antisymmetric real structure constants f_{abc} are obtained from Eq. (2.4) as

$$f_{abc} = \frac{1}{4i} \mathrm{Tr}([\lambda_a, \lambda_b]\lambda_c). \tag{2.10}$$

The independent non-vanishing values are explicitly summarized in the scheme of Table 2.1. Roughly speaking, these structure constants are a measure of the non-commutativity of the group SU(3).

The anticommutation relations read

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} + 2 d_{abc} \lambda_c, \tag{2.11}$$

abc	118	146	157	228	247	256	338	344
d_{abc}	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$
abc	355	366	377	448	558	668	778	888
d_{abc}	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$

Table 2.2: Totally symmetric non-vanishing d symbols of SU(3).

where the totally symmetric d_{abc} are given by

$$d_{abc} = \frac{1}{4}\mathrm{Tr}(\{\lambda_a, \lambda_b\}\lambda_c), \tag{2.12}$$

and are summarized in Table 2.2. Clearly, the anticommutator of two Gell-Mann matrices is not necessarily a Gell-Mann matrix. For example, the square of a (nontrivial) skew-Hermitian matrix is not skew Hermitian.

Moreover, it is convenient to introduce as a ninth matrix

$$\lambda_0 = \sqrt{2/3}\,\mathrm{diag}(1,1,1),$$

such that Eqs. (2.3) and (2.4) are still satisfied by the nine matrices λ_a. In particular, the set $\{i\lambda_a | a = 0, \cdots, 8\}$ constitutes a basis of the Lie algebra u(3) of U(3), i.e., the set of all complex skew Hermitian 3×3 matrices. Finally, an *arbitrary* 3×3 matrix M can then be written as

$$M = \sum_{a=0}^{8} \lambda_a M_a, \tag{2.13}$$

where M_a are complex numbers given by

$$M_a = \frac{1}{2}\mathrm{Tr}(\lambda_a M).$$

2.2 The QCD Lagrangian

The gauge principle has proven to be a tremendously successful method in elementary particle physics to generate interactions between matter fields through the exchange of massless gauge bosons (for a detailed account see, e.g., [AL 73, O'Ra 86]). The best-known example is, of course, quantum electrodynamics (QED) which is obtained from promoting the global U(1) symmetry of the Lagrangian describing a free electron,[3]

$$\Psi \mapsto \exp(-i\Theta)\Psi : \mathcal{L}_{\text{free}} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\Psi \mapsto \mathcal{L}_{\text{free}}, \tag{2.14}$$

to a local symmetry. In this process the parameter $0 \leq \Theta \leq 2\pi$ describing an element of U(1) is allowed to vary smoothly in space-time, $\Theta \to \Theta(x)$, which is referred to as gauging the U(1) group. To keep the invariance of the Lagrangian under local transformations one introduces a four-potential $\mathcal{A}_{\mu}$ into the theory which transforms under the gauge transformation $\mathcal{A}_{\mu} \mapsto \mathcal{A}_{\mu} - \partial_{\mu}\Theta/e$. The method is referred to as gauging the Lagrangian with respect to U(1):

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}[i\gamma^{\mu}(\partial_{\mu} - ie\mathcal{A}_{\mu}) - m]\Psi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}, \tag{2.15}$$

where $\mathcal{F}_{\mu\nu} = \partial_{\mu}\mathcal{A}_{\nu} - \partial_{\nu}\mathcal{A}_{\mu}$.[4] The covariant derivative of Ψ,

$$D_{\mu}\Psi \equiv (\partial_{\mu} - ie\mathcal{A}_{\mu})\Psi,$$

is defined such that under a so-called gauge transformation of the second kind

$$\Psi(x) \mapsto \exp[-i\Theta(x)]\Psi(x), \quad \mathcal{A}_{\mu}(x) \mapsto \mathcal{A}_{\mu}(x) - \partial_{\mu}\Theta(x)/e, \tag{2.16}$$

it transforms in the same way as Ψ itself:

$$D_{\mu}\Psi(x) \mapsto \exp[-i\Theta(x)]D_{\mu}\Psi(x). \tag{2.17}$$

In Eq. (2.15), the term containing the squared field strength makes the gauge potential a dynamical degree of freedom as opposed to a pure external field. A mass term $M^2\mathcal{A}^2/2$ is not included since it would violate gauge invariance and thus the gauge principle requires massless gauge bosons.[5] In the present case we identify the $\mathcal{A}_{\mu}$ with the electromagnetic four-potential and $\mathcal{F}_{\mu\nu}$ with the field strength tensor containing the electric and magnetic fields. The gauge principle has (naturally) generated the interaction of the electromagnetic field with matter. If the underlying gauge group is non-Abelian, the

[3]We use the standard representation for the Dirac matrices (see, e.g., Ref. [BD 64a]).

[4] We use natural units, i.e., $\hbar = c = 1, e > 0$, and $\alpha = e^2/4\pi \approx 1/137$.

[5]Masses of gauge fields can be induced through a spontaneous breakdown of the gauge symmetry.

gauge principle associates an independent gauge field with each independent continuous parameter of the gauge group.

QCD is the gauge theory of the strong interactions [GW 73a, Wei 73, Fri+ 73] with color SU(3) as the underlying gauge group.[6] The matter fields of QCD are the so-called quarks which are spin-1/2 fermions, with six different flavors in addition to their three possible colors (see Table 2.3). Since quarks have not been observed as asymptotically free states, the meaning of quark masses and their numerical values are tightly connected with the method by which they are extracted from hadronic properties (see Ref. [Man 00] for a thorough discussion). Regarding the so-called current-quark-mass values of the light quarks, one should view the quark mass terms merely as symmetry breaking parameters with their magnitude providing a measure for the extent to which chiral symmetry is broken [Sch 96]. For example, *ratios* of the light quark masses can be inferred from the masses of the light pseudoscalar octet (see Ref. [Leu 96]). Comparing the proton mass, $m_p = 938$ MeV, with the sum of two up and one down current-quark masses (see Table 2.3),

$$m_p \gg 2m_u + m_d, \tag{2.18}$$

shows that an interpretation of the proton mass in terms of current-quark mass parameters must be very different from, say, the situation in the hydrogen atom, where the mass is essentially given by the sum of the electron and proton masses, corrected by a small amount of binding energy.

The QCD Lagrangian obtained from the gauge principle reads [MP 78, Alt 82]

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f=\substack{u,d,s,\\ c,b,t}} \bar{q}_f (i \not{D} - m_f) q_f - \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}^{\mu\nu}_a. \tag{2.19}$$

For each quark flavor f the quark field q_f consists of a color triplet (subscripts r, g, and b standing for "red," "green," and "blue"),

$$q_f = \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix}, \tag{2.20}$$

[6] Historically, the color degree of freedom was introduced into the quark model to account for the Pauli principle in the description of baryons as three-quark states [Gre 64, HN 65].

flavor	u	d	s
charge [e]	2/3	$-1/3$	$-1/3$
mass [MeV]	5.1 ± 0.9 [Leu 96]	9.3 ± 1.4 [Leu 96]	175 ± 25 [Bij+ 95b]
flavor	c	b	t
charge [e]	2/3	$-1/3$	2/3
mass [GeV]	$1.15 - 1.35$ [Man 00]	$4.0 - 4.4$ [Man 00]	$174.3 \pm 3.2 \pm 4.0$ [Man 00]

Table 2.3: Quark flavors and their charges and masses. The absolute magnitude of m_s is determined using QCD sum rules. The result is given for the $\overline{\mathrm{MS}}$ running mass at scale $\mu = 1$ GeV. The light quark masses are obtained from the mass ratios found using chiral perturbation theory, using the strange quark mass as input. The heavy-quark masses m_c and m_b are determined by the charmonium and D masses, and the bottomium and B masses, respectively. The top quark mass m_t results from the measurement of lepton + jets and dilepton + jets channels in the DØ and CDF experiments at Fermilab.

which transforms under a gauge transformation $g(x)$ described by the set of parameters $\Theta(x) = [\Theta_1(x), \cdots, \Theta_8(x)]$ according to[7]

$$q_f \mapsto q_f' = \exp\left[-i \sum_{a=1}^{8} \Theta_a(x) \frac{\lambda_a^C}{2}\right] q_f = U[g(x)] q_f. \tag{2.21}$$

Technically speaking, each quark field q_f transforms according to the fundamental representation of color SU(3). Because SU(3) is an eight-parameter group, the covariant derivative of Eq. (2.19) contains eight independent gauge potentials $\mathcal{A}_{\mu,a}$,

$$D_\mu \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix} = \partial_\mu \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix} - ig \sum_{a=1}^{8} \frac{\lambda_a^C}{2} \mathcal{A}_{\mu,a} \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix}. \tag{2.22}$$

We note that the interaction between quarks and gluons is independent of the quark flavors. Demanding gauge invariance of $\mathcal{L}_{\mathrm{QCD}}$ imposes the following

[7]For the sake of clarity, the Gell-Mann matrices contain a superscript C, indicating the action in color space.

transformation property of the gauge fields

$$\frac{\lambda_a^C}{2}\mathcal{A}_{\mu,a}(x) \mapsto U[g(x)]\frac{\lambda_a^C}{2}\mathcal{A}_{\mu,a}(x)U^\dagger[g(x)] - \frac{i}{g}\partial_\mu U[g(x)]U^\dagger[g(x)]. \tag{2.23}$$

Again, with this requirement the covariant derivative $D_\mu q_f$ transforms as q_f, i.e. $D_\mu q \mapsto D'_\mu q' = U(g)D_\mu q$. Under a gauge transformation of the first kind, i.e., a global SU(3) transformation, the second term on the right-hand side of Eq. (2.23) would vanish and the gauge fields would transform according to the adjoint representation.

So far we have only considered the matter-field part of $\mathcal{L}_{\rm QCD}$ including its interaction with the gauge fields. Equation (2.19) also contains the generalization of the field strength tensor to the non-Abelian case,

$$\mathcal{G}_{\mu\nu,a} = \partial_\mu\mathcal{A}_{\nu,a} - \partial_\nu\mathcal{A}_{\mu,a} + gf_{abc}\mathcal{A}_{\mu,b}\mathcal{A}_{\nu,c}, \tag{2.24}$$

with the SU(3) structure constants given in Table 2.1 and a summation over repeated indices implied. Given Eq. (2.23) the field strength tensor transforms under SU(3) as

$$\mathcal{G}_{\mu\nu} \equiv \frac{\lambda_a^C}{2}\mathcal{G}_{\mu\nu,a} \mapsto U[g(x)]\mathcal{G}_{\mu\nu}U^\dagger[g(x)]. \tag{2.25}$$

Using Eq. (2.4) the purely gluonic part of $\mathcal{L}_{\rm QCD}$ can be written as

$$-\frac{1}{2}\mathrm{Tr}_C(\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}),$$

which, using the cyclic property of traces, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, together with $UU^\dagger = 1$, is easily seen to be invariant under the transformation of Eq. (2.25).

In contradistinction to the Abelian case of QED, the squared field strength tensor gives rise to gauge-field self interactions involving vertices with three and four gauge fields of strength g and g^2, respectively. Such interaction terms are characteristic of non-Abelian gauge theories and make them much more complicated than Abelian theories.

From the point of view of gauge invariance the strong-interaction Lagrangian could also involve a term of the type

$$\mathcal{L}_\theta = \frac{g^2\bar{\theta}}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}\sum_{a=1}^{8}\mathcal{G}^a_{\mu\nu}\mathcal{G}^a_{\rho\sigma}, \tag{2.26}$$

where $\epsilon_{\mu\nu\rho\sigma}$ denotes the totally antisymmetric Levi-Civita tensor.[8] The so-called θ term of Eq. (2.26) implies an explicit P and CP violation of the strong interactions which, for example, would give rise to an electric dipole moment of the neutron (for an upper limit, see Ref. [Har+ 99]). The present empirical information indicates that the θ term is small and, in the following, we will omit Eq. (2.26) from our discussion and refer the interested reader to Refs. [PR 91, Bor 00, KL 00].

2.3 Accidental, Global Symmetries of $\mathcal{L}_{\rm QCD}$

2.3.1 Light and Heavy Quarks

The six quark flavors are commonly divided into the three light quarks u, d, and s and the three heavy flavors c, b, and t,

$$\left(\begin{array}{r} m_u = 0.005\,\mathrm{GeV} \\ m_d = 0.009\,\mathrm{GeV} \\ m_s = 0.175\,\mathrm{GeV} \end{array}\right) \ll 1\,\mathrm{GeV} \leq \left(\begin{array}{r} m_c = (1.15-1.35)\,\mathrm{GeV} \\ m_b = (4.0-4.4)\,\mathrm{GeV} \\ m_t = 174\,\mathrm{GeV} \end{array}\right), \tag{2.27}$$

where the scale of 1 GeV is associated with the masses of the lightest hadrons containing light quarks, e.g., m_ρ= 770 MeV, which are not Goldstone bosons resulting from spontaneous symmetry breaking. The scale associated with spontaneous symmetry breaking, $4\pi F_\pi \approx 1170$ MeV, is of the same order of magnitude [Pag 75, MG 84, Geo 84].

The masses of the lightest meson and baryon containing a charmed quark, $D^+ = c\bar{d}$ and $\Lambda_c^+ = udc$, are $(1869.4 \pm 0.5)\,\mathrm{MeV}$ and $(2284.9 \pm 0.6)\,\mathrm{MeV}$, respectively [Gro+ 00]. The threshold center-of-mass energy to produce, say, a D^+D^- pair in e^+e^- collisions is approximately 3.74 GeV, and thus way beyond the low-energy regime which we are interested in. In the following, we will approximate the full QCD Lagrangian by its light-flavor version, i.e., we will ignore effects due to (virtual) heavy quark-antiquark pairs $h\bar{h}$. In particular, Eq. (2.18) suggests that the Lagrangian $\mathcal{L}^0_{\rm QCD}$, containing only the light-flavor quarks in the so-called chiral limit $m_u, m_d, m_s \to 0$, might be

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$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ is an even permutation of } \{0,1,2,3\} \\ -1 & \text{if } \{\mu,\nu,\rho,\sigma\} \text{ is an odd permutation of } \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

a good starting point in the discussion of low-energy QCD:

$$\mathcal{L}^0_{\mathrm{QCD}} = \sum_{l=u,d,s} \bar{q}_l i \not{D}\, q_l - \frac{1}{4}\mathcal{G}_{\mu\nu,a}\mathcal{G}^{\mu\nu}_a. \tag{2.28}$$

We repeat that the covariant derivative $\not{D}\, q_l$ acts on color and Dirac indices only, but is independent of flavor.

2.3.2 Left-Handed and Right-Handed Quark Fields

In order to fully exhibit the global symmetries of Eq. (2.28), we consider the chirality matrix $\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\{\gamma^\mu, \gamma_5\} = 0$, $\gamma_5^2 = 1$,[9] and introduce projection operators

$$P_R = \frac{1}{2}(1+\gamma_5) = P_R^\dagger, \quad P_L = \frac{1}{2}(1-\gamma_5) = P_L^\dagger, \tag{2.29}$$

where the indices R and L refer to right-handed and left-handed, respectively, as will become more clear below. Obviously, the 4×4 matrices P_R and P_L satisfy a completeness relation,

$$P_R + P_L = 1, \tag{2.30}$$

are idempotent, i.e.,

$$P_R^2 = P_R, \quad P_L^2 = P_L, \tag{2.31}$$

and respect the orthogonality relations

$$P_R P_L = P_L P_R = 0. \tag{2.32}$$

The combined properties of Eqs. (2.30) – (2.32) guarantee that P_R and P_L are indeed projection operators which project from the Dirac field variable q to its chiral components q_R and q_L,

$$q_R = P_R q, \quad q_L = P_L q. \tag{2.33}$$

We recall in this context that a chiral (field) variable is one which under parity is transformed into neither the original variable nor its negative [Dou 90].[10] Under parity, the quark field is transformed into its parity conjugate,

$$P : q(\vec{x}, t) \mapsto \gamma_0 q(-\vec{x}, t),$$

[9] Unless stated otherwise, we use the convention of Ref. [BD 64a].

[10] In case of fields, a transformation of the argument $\vec{x} \to -\vec{x}$ is implied.

and hence

$$q_R(\vec{x},t) = P_R\, q(\vec{x},t) \mapsto P_R\gamma_0 q(-\vec{x},t) = \gamma_0 P_L q(-\vec{x},t) \neq \pm q_R(-\vec{x},t),$$

and similarly for q_L.[11]

The terminology right-handed and left-handed fields can easily be visualized in terms of the solution to the free Dirac equation. For that purpose, let us consider an extreme relativistic positive-energy solution with three-momentum $\vec{p}$,[12]

$$u(\vec{p},\pm) = \sqrt{E+M}\begin{pmatrix} \chi_\pm \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+M}\chi_\pm \end{pmatrix} \overset{E \gg M}{\rightarrow} \sqrt{E}\begin{pmatrix} \chi_\pm \\ \pm\chi_\pm \end{pmatrix} = u_\pm(\vec{p}),$$

where we assume that the spin in the rest frame is either parallel or antiparallel to the direction of momentum

$$\vec{\sigma}\cdot\hat{p}\chi_\pm = \pm\chi_\pm.$$

In the standard representation of Dirac matrices we find

$$P_R = \frac{1}{2}\begin{pmatrix} 1_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 1_{2\times2} \end{pmatrix}, \quad P_L = \frac{1}{2}\begin{pmatrix} 1_{2\times2} & -1_{2\times2} \\ -1_{2\times2} & 1_{2\times2} \end{pmatrix},$$

such that

$$P_R u_+ = \sqrt{E}\frac{1}{2}\begin{pmatrix} 1_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 1_{2\times2} \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_+ \end{pmatrix} = \sqrt{E}\begin{pmatrix} \chi_+ \\ \chi_+ \end{pmatrix} = u_+,$$

and similarly

$$P_L u_+ = 0, \quad P_R u_- = 0, \quad P_L u_- = u_-.$$

In the extreme relativistic limit (or better, in the zero-mass limit), the operators P_R and P_R project to the positive and negative helicity eigenstates, i.e., in this limit chirality equals helicity.

Our goal is to analyze the symmetry of the QCD Lagrangian with respect to independent global transformations of the left- and right-handed fields. In

[11] Note that in the above sense, also q is a chiral variable. However, the assignment of handedness does not have such an intuitive meaning as in the case of q_L and q_R.

[12] Here we adopt a covariant normalization of the spinors, $u^{(\alpha)\dagger}(\vec{p})u^{(\beta)}(\vec{p}) = 2E\delta_{\alpha\beta}$, etc.

order to decompose the 16 quadratic forms into their respective projections to right- and left-handed fields, we make use of [Gas 89]

$$\bar{q}\Gamma_i q = \begin{cases} \bar{q}_R\Gamma_1 q_R + \bar{q}_L\Gamma_1 q_L & \text{for} \quad \Gamma_1 \in \{\gamma^\mu, \gamma^\mu\gamma_5\} \\ \bar{q}_R\Gamma_2 q_L + \bar{q}_L\Gamma_2 q_R & \text{for} \quad \Gamma_2 \in \{1, \gamma_5, \sigma^{\mu\nu}\} \end{cases}, \tag{2.34}$$

where $\bar{q}_R = \bar{q}P_L$ and $\bar{q}_L = \bar{q}P_R$. Equation (2.34) is easily proven by inserting the completeness relation of Eq. (2.30) both to the left and the right of Γ_i,

$$\bar{q}\Gamma_i q = \bar{q}(P_R + P_L)\Gamma_i(P_R + P_L)q,$$

and by noting $\{\Gamma_1, \gamma_5\} = 0$ and $[\Gamma_2, \gamma_5] = 0$. Together with the orthogonality relations of Eq. (2.32) we then obtain

$$P_R\Gamma_1 P_R = \Gamma_1 P_L P_R = 0,$$

and similarly

$$P_L\Gamma_1 P_L = 0, \quad P_R\Gamma_2 P_L = 0, \quad P_L\Gamma_2 P_R = 0.$$

We stress that the validity of Eq. (2.34) is general and does not refer to "massless" quark fields.

We now apply Eq. (2.34) to the term containing the contraction of the covariant derivative with γ^μ. This quadratic quark form decouples into the sum of two terms which connect only left-handed with left-handed and right-handed with right-handed quark fields. The QCD Lagrangian in the chiral limit can then be written as

$$\mathcal{L}^0_{\text{QCD}} = \sum_{l=u,d,s} (\bar{q}_{R,l} i \not{D}\, q_{R,l} + \bar{q}_{L,l} i \not{D}\, q_{L,l}) - \frac{1}{4}\mathcal{G}_{\mu\nu,a}\mathcal{G}^{\mu\nu}_a. \tag{2.35}$$

Due to the flavor independence of the covariant derivative $\mathcal{L}^0_{\text{QCD}}$ is invariant under

$$\begin{aligned} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} &\mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i\sum_{a=1}^{8}\Theta^L_a\frac{\lambda_a}{2}\right)e^{-i\Theta^L}\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}, \\ \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} &\mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i\sum_{a=1}^{8}\Theta^R_a\frac{\lambda_a}{2}\right)e^{-i\Theta^R}\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}, \end{aligned} \tag{2.36}$$

where U_L and U_R are independent unitary 3×3 matrices. Note that the Gell-Mann matrices act in flavor space.

$\mathcal{L}^0_{\mathrm{QCD}}$ is said to have a classical *global* $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ symmetry. Applying Noether's theorem (see, for example, [Hil 51, Alf+ 73]) from such an invariance one would expect a total of $2 \times (8+1) = 18$ conserved currents.

2.3.3 Noether's Theorem

In order to identify the conserved currents associated with this invariance, we briefly recall the method of Ref. [GL 60] and consider the variation of Eq. (2.35) under a *local* infinitesimal transformation.[13] For simplicity we consider only internal symmetries. To that end we start with a Lagrangian $\mathcal{L}$ depending on n independent fields Φ_i and their first partial derivatives,

$$\mathcal{L} = \mathcal{L}(\Phi_i, \partial_\mu \Phi_i), \tag{2.37}$$

from which one obtains n equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0, \quad i = 1, \cdots, n. \tag{2.38}$$

For each of the r generators of infinitesimal transformations representing the underlying symmetry group, we consider a *local* infinitesimal transformation of the fields [GL 60],[14]

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) + \delta\Phi_i(x) = \Phi_i(x) - i\epsilon_a(x) F_i^a[\Phi_j(x)], \tag{2.39}$$

and obtain, neglecting terms of order ϵ^2, as the variation of the Lagrangian,

$$\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}(\Phi_i', \partial_\mu \Phi_i') - \mathcal{L}(\Phi_i, \partial_\mu \Phi_i) \\
&= \frac{\partial \mathcal{L}}{\partial \Phi_i} \delta\Phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \partial_\mu \delta\Phi_i \\
&= \epsilon_a(x) \left(-i \frac{\partial \mathcal{L}}{\partial \Phi_i} F_i^a - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \partial_\mu F_i^a \right) + \partial_\mu \epsilon_a(x) \left(-i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} F_i^a \right) \\
&\equiv \epsilon_a(x) \partial_\mu J^{\mu,a} + \partial_\mu \epsilon_a(x) J^{\mu,a}.
\end{aligned} \tag{2.40}$$

[13] By exponentiating elements of the Lie algebra $\mathrm{u}(N)$ any element of $\mathrm{U}(N)$ can be obtained.

[14] Note that the transformation need not be realized linearly on the fields.

According to this equation we define for each infinitesimal transformation a four-current density as

$$J^{\mu,a} = -i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}F_i^a. \tag{2.41}$$

By calculating the divergence $\partial_\mu J^{\mu,a}$ of Eq. (2.41)

$$\begin{aligned}
\partial_\mu J^{\mu,a} &= -i\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\right)F_i^a - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu F_i^a \\
&= -i\frac{\partial\mathcal{L}}{\partial\Phi_i}F_i^a - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu F_i^a,
\end{aligned}$$

where we made use of the equations of motion, Eq. (2.38), we explicitly verify the consistency with the definition of $\partial_\mu J^{\mu,a}$ according to Eq. (2.40). From Eq. (2.40) it is straightforward to obtain the four-currents as well as their divergences as

$$J^{\mu,a} = \frac{\partial\delta\mathcal{L}}{\partial\partial_\mu\epsilon_a}, \tag{2.42}$$

$$\partial_\mu J^{\mu,a} = \frac{\partial\delta\mathcal{L}}{\partial\epsilon_a}. \tag{2.43}$$

For a conserved current, $\partial_\mu J^{\mu,a} = 0$, the charge

$$Q^a(t) = \int d^3x J_0^a(\vec{x},t) \tag{2.44}$$

is time independent, i.e., a constant of the motion, which is shown in the standard fashion by applying the divergence theorem for an infinite volume with appropriate boundary conditions for $R \to \infty$.

So far we have discussed Noether's theorem on the classical level, implying that the charges $Q^a(t)$ can have any continuous real value. However, we also need to discuss the implications of a transition to a quantum theory. After canonical quantization, the fields Φ_i and their conjugate momenta $\Pi_i = \partial\mathcal{L}/\partial(\partial_0\Phi_i)$ are considered as linear operators acting on a Hilbert space which, in the Heisenberg picture, are subject to the equal-time commutation relations

$$\begin{aligned}
[\Phi_i(\vec{x},t), \Pi_j(\vec{y},t)] &= i\delta^3(\vec{x}-\vec{y})\delta_{ij}, \\
[\Phi_i(\vec{x},t), \Phi_j(\vec{y},t)] &= 0, \\
[\Pi_i(\vec{x},t), \Pi_j(\vec{y},t)] &= 0.
\end{aligned} \tag{2.45}$$

As a special case of Eq. (2.39) let us consider infinitesimal transformations which are *linear* in the fields,

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) - i\epsilon_a(x) t^a_{ij} \Phi_j(x), \tag{2.46}$$

where the t^a_{ij} are constants generating a mixing of the fields. From Eq. (2.41) we then obtain[15]

$$\begin{aligned} J^{\mu,a}(x) &= -it^a_{ij}\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\Phi_j, & (2.47)\\ Q^a(t) &= -i\int d^3x\, \Pi_i(x) t^a_{ij}\Phi_j(x), & (2.48)\end{aligned}$$

where $J^{\mu,a}(x)$ and $Q^a(t)$ are now operators. In order to interpret the charge operators $Q^a(t)$, let us make use of the equal-time commutation relations, Eqs. (2.45), and calculate their commutators with the field operators,

$$\begin{aligned} [Q^a(t), \Phi_k(\vec{y}, t)] &= -it^a_{ij}\int d^3x\, [\Pi_i(\vec{x}, t)\Phi_j(\vec{x}, t), \Phi_k(\vec{y}, t)] \\ &= -t^a_{kj}\Phi_j(\vec{y}, t). & (2.49)\end{aligned}$$

Note that we did not require the charge operators to be time independent. On the other hand, for the transformation behavior of the Hilbert space associated with a global infinitesimal transformation, we make an ansatz in terms of an infinitesimal unitary transformation[16]

$$|\alpha'\rangle = [1 + i\epsilon_a G^a(t)]|\alpha\rangle, \tag{2.50}$$

with Hermitian operators G^a. Demanding

$$\langle\beta|A|\alpha\rangle = \langle\beta'|A'|\alpha'\rangle \quad \forall\, |\alpha\rangle, |\beta\rangle, \epsilon_a, \tag{2.51}$$

in combination with Eq. (2.46) yields the condition

$$\begin{aligned} \langle\beta|\Phi_i(x)|\alpha\rangle &= \langle\beta'|\Phi_i'(x)|\alpha'\rangle \\ &= \langle\beta|[1 - i\epsilon_a G^a(t)][\Phi_i(x) - i\epsilon_b t^b_{ij}\Phi_j(x)][1 + i\epsilon_c G^c(t)]|\alpha\rangle.\end{aligned}$$

[15] Normal ordering symbols are suppressed.

[16] We have chosen to have the fields (field operators) rotate actively and thus must transform the states of Hilbert space in the opposite direction.

By comparing the terms linear in ϵ_a on both sides,

$$0 = -i\epsilon_a[G^a(t), \Phi_i(x)] \underbrace{-i\epsilon_a t^a_{ij}\Phi_j(x)}_{i\epsilon_a[Q^a(t),\Phi_i(x)]}, \tag{2.52}$$

we see that the infinitesimal generators acting on the states of Hilbert space which are associated with the transformation of the fields are identical with the charge operators $Q^a(t)$ of Eq. (2.48).

Finally, evaluating the commutation relations for the case of several generators,

$$[Q^a(t), Q^b(t)] \;=\; -i(t^a_{ij}t^b_{jk} - t^b_{ij}t^a_{jk})\int d^3x\, \Pi_i(\vec{x},t)\Phi_k(\vec{x},t), \tag{2.53}$$

we find the right-hand side of Eq. (2.53) to be again proportional to a charge operator, if

$$t^a_{ij}t^b_{jk} - t^b_{ij}t^a_{jk} = iC_{abc}t^c_{ik}, \tag{2.54}$$

i.e., in that case the charge operators $Q^a(t)$ form a Lie algebra

$$[Q^a(t), Q^b(t)] = iC_{abc}Q^c(t) \tag{2.55}$$

with structure constants C_{abc}. The quantization of the charges (as opposed to continuous values in the classical case) can be understood in analogy to the algebraic construction of the angular momentum eigenvalues in quantum mechanics starting from the su(2) algebra. Of course, for conserved currents, the charge operators are time independent, i.e., they commute with the Hamilton operator of the system.

From now on we assume the validity of Eq. (2.54) and interpret the constants t^a_{ij} as the entries in the ith row and jth column of an $n\times n$ matrix T^a,

$$T^a = \begin{pmatrix} t^a_{11} & \cdots & t^a_{1n} \\ \vdots & & \vdots \\ t^a_{n1} & \cdots & t^a_{nn} \end{pmatrix}.$$

Because of Eq. (2.54), these matrices form an n-dimensional representation of a Lie algebra,

$$[T^a, T^b] = iC_{abc}T^c.$$

The infinitesimal, linear transformations of the fields Φ_i may then be written in a compact form,

$$\begin{pmatrix} \Phi_1(x) \\ \vdots \\ \Phi_n(x) \end{pmatrix} = \Phi(x) \mapsto \Phi'(x) = (1 - i\epsilon_a T^a)\Phi(x). \tag{2.56}$$

In general, through an appropriate unitary transformation, the matrices T_a may be decomposed into their irreducible components, i.e., brought into block-diagonal form, such that only fields belonging to the same multiplet transform into each other under the symmetry group.

2.3.4 Global Symmetry Currents of the Light Quark Sector

The method of Ref. [GL 60] can now easily be applied to the QCD Lagrangian by calculating the variation under the infinitesimal, local form of Eqs. (2.36),

$$\delta\mathcal{L}^0_{\mathrm{QCD}} = \bar{q}_R \left(\sum_{a=1}^{8} \partial_\mu \Theta^R_a \frac{\lambda_a}{2} + \partial_\mu \Theta^R \right) \gamma^\mu q_R + \bar{q}_L \left(\sum_{a=1}^{8} \partial_\mu \Theta^L_a \frac{\lambda_a}{2} + \partial_\mu \Theta^L \right) \gamma^\mu q_L, \tag{2.57}$$

from which, by virtue of Eqs. (2.42) and (2.43), one obtains the currents associated with the transformations of the left-handed or right-handed quarks

$$\begin{aligned} L^{\mu,a} &= \bar{q}_L \gamma^\mu \frac{\lambda^a}{2} q_L, \quad \partial_\mu L^{\mu,a} = 0, \\ R^{\mu,a} &= \bar{q}_R \gamma^\mu \frac{\lambda^a}{2} q_R, \quad \partial_\mu R^{\mu,a} = 0. \end{aligned} \tag{2.58}$$

The eight currents $L^{\mu,a}$ transform under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ as an $(8,1)$ multiplet, i.e., as octet and singlet under transformations of the left- and right-handed fields, respectively. Similarly, the right-handed currents transform as a $(1,8)$ multiplet under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$. Instead of these chiral currents one often uses linear combinations,

$$V^{\mu,a} = R^{\mu,a} + L^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q, \tag{2.59}$$

$$A^{\mu,a} = R^{\mu,a} - L^{\mu,a} = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda^a}{2} q, \tag{2.60}$$

transforming under parity as vector and axial-vector current densities, respectively,

$$P : V^{\mu,a}(\vec{x},t) \mapsto V_\mu^a(-\vec{x},t), \tag{2.61}$$
$$P : A^{\mu,a}(\vec{x},t) \mapsto -A_\mu^a(-\vec{x},t). \tag{2.62}$$

From Eqs. (2.42) and (2.43) one also obtains a conserved singlet vector current resulting from a transformation of all left-handed and right-handed quark fields by the *same* phase,

$$V^\mu = \bar{q}_R\gamma^\mu q_R + \bar{q}_L\gamma^\mu q_L = \bar{q}\gamma^\mu q, \quad \partial_\mu V^\mu = 0. \tag{2.63}$$

The singlet axial-vector current,

$$\begin{aligned} A^\mu &= \bar{q}_R\gamma^\mu q_R - \bar{q}_L\gamma^\mu q_L = \bar{q}P_L\gamma^\mu P_R q - \bar{q}P_R\gamma^\mu P_L q \\ &= \bar{q}\gamma^\mu P_R q - \bar{q}\gamma^\mu P_L q = \bar{q}\gamma^\mu\gamma_5 q, \end{aligned} \tag{2.64}$$

originates from a transformation of all left-handed quark fields with one phase and all right-handed with the *opposite* phase. However, such a singlet axial-vector current is only conserved on the *classical* level. This symmetry is not preserved by quantization and there will be extra terms, referred to as anomalies [Adl 69, AB 69, Bar 69, BJ 69, Adl 70], resulting in[17]

$$\partial_\mu A^\mu = \frac{3g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathcal{G}_a^{\mu\nu}\mathcal{G}_a^{\rho\sigma}, \quad \epsilon_{0123} = 1, \tag{2.65}$$

where the factor of 3 originates from the number of flavors.

2.3.5 The Chiral Algebra

The invariance of $\mathcal{L}^0_{\mathrm{QCD}}$ under global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ transformations implies that also the QCD Hamilton operator in the chiral limit, H^0_{QCD}, exhibits a global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ symmetry. As usual, the "charge operators" are defined as the space integrals of the charge densities,

$$Q_L^a(t) = \int d^3x\, q_L^\dagger(\vec{x},t)\frac{\lambda^a}{2}q_L(\vec{x},t), \quad a = 1,\cdots,8, \tag{2.66}$$
$$Q_R^a(t) = \int d^3x\, q_R^\dagger(\vec{x},t)\frac{\lambda^a}{2}q_R(\vec{x},t), \quad a = 1,\cdots,8, \tag{2.67}$$
$$Q_V(t) = \int d^3x \left[q_L^\dagger(\vec{x},t)q_L(\vec{x},t) + q_R^\dagger(\vec{x},t)q_R(\vec{x},t)\right]. \tag{2.68}$$

[17]In the large N_c (number of colors) limit of Ref. [Hoo 74] the singlet axial-vector current is conserved, because the strong coupling constant behaves as $g^2 \sim N_c^{-1}$.

For conserved symmetry currents, these operators are time independent, i.e., they commute with the Hamiltonian,

$$[Q^a_L, H^0_{\rm QCD}] = [Q^a_R, H^0_{\rm QCD}] = [Q_V, H^0_{\rm QCD}] = 0. \tag{2.69}$$

The commutation relations of the charge operators with each other are obtained by using the equal-time commutation relations of the quark fields in the Heisenberg picture,

$$\begin{aligned}
\{q_{\alpha,r}(\vec{x},t), q^\dagger_{\beta,s}(\vec{y},t)\} &= \delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta}\delta_{rs}, & (2.70)\\
\{q_{\alpha,r}(\vec{x},t), q_{\beta,s}(\vec{y},t)\} &= 0, & (2.71)\\
\{q^\dagger_{\alpha,r}(\vec{x},t), q^\dagger_{\beta,s}(\vec{y},t)\} &= 0, & (2.72)
\end{aligned}$$

where α and β are Dirac indices and r and s flavor indices, respectively.[18] The equal-time commutator of two quadratic quark forms is of the type

$$\begin{aligned}
&[q^\dagger(\vec{x},t)\Gamma_1 F_1 q(\vec{x},t), q^\dagger(\vec{y},t)\Gamma_2 F_2 q(\vec{y},t)] = \\
&\quad \Gamma_{1,\alpha\beta}\Gamma_{2,\gamma\delta}F_{1,rs}F_{2,tu}[q^\dagger_{\alpha,r}(\vec{x},t)q_{\beta,s}(\vec{x},t), q^\dagger_{\gamma,t}(\vec{y},t)q_{\delta,u}(\vec{y},t)], \qquad (2.73)
\end{aligned}$$

where Γ_i and F_i are 4×4 Dirac matrices and 3×3 flavor matrices, respectively. Using

$$[ab, cd] = a\{b,c\}d - ac\{b,d\} + \{a,c\}db - c\{a,d\}b, \tag{2.74}$$

we express the commutator of Fermi fields in terms of anticommutators and make use of the equal-time commutation relations of Eqs. (2.70) – (2.72) to obtain

$$\begin{aligned}
&[q^\dagger_{\alpha,r}(\vec{x},t)q_{\beta,s}(\vec{x},t), q^\dagger_{\gamma,t}(\vec{y},t)q_{\delta,u}(\vec{y},t)] = \\
&\quad q^\dagger_{\alpha,r}(\vec{x},t)q_{\delta,u}(\vec{y},t)\delta^3(\vec{x}-\vec{y})\delta_{\beta\gamma}\delta_{st} - q^\dagger_{\gamma,t}(\vec{y},t)q_{\beta,s}(\vec{x},t)\delta^3(\vec{x}-\vec{y})\delta_{\alpha\delta}\delta_{ru}.
\end{aligned}$$

With this result Eq. (2.73) reads

$$\begin{aligned}
&[q^\dagger(\vec{x},t)\Gamma_1 F_1 q(\vec{x},t), q^\dagger(\vec{y},t)\Gamma_2 F_2 q(\vec{y},t)] = \\
&\quad \delta^3(\vec{x}-\vec{y})\left[q^\dagger(\vec{x},t)\Gamma_1\Gamma_2 F_1 F_2 q(\vec{y},t) - q^\dagger(\vec{y},t)\Gamma_2\Gamma_1 F_2 F_1 q(\vec{x},t)\right]. \qquad (2.75)
\end{aligned}$$

After inserting appropriate projectors $P_{L/R}$, Eq. (2.75) is easily applied to the charge operators of Eqs. (2.66), (2.67), and (2.68), showing that these

[18] Strictly speaking, we should also include the color indices. However, since we are only discussing color-neutral quadratic forms a summation over such indices is always implied, with the net effect that one can completely omit them from the discussion.

operators indeed satisfy the commutation relations corresponding to the Lie algebra of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$,

$$
\begin{aligned}
[Q_L^a, Q_L^b] &= i f_{abc} Q_L^c, && (2.76)\\
[Q_R^a, Q_R^b] &= i f_{abc} Q_R^c, && (2.77)\\
[Q_L^a, Q_R^b] &= 0, && (2.78)\\
[Q_L^a, Q_V] &= [Q_R^a, Q_V] = 0. && (2.79)
\end{aligned}
$$

It should be stressed that, even without being able to explicitly solve the equation of motion of the quark fields entering the charge operators of Eqs. (2.66) - (2.68), we know from the equal-time commutation relations and the symmetry of the Lagrangian that these charge operators are the generators of infinitesimal transformations of the Hilbert space associated with H^0_{QCD}. Furthermore, their commutation relations with a given operator specify the transformation behavior of the operator in question under the group $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$.

2.3.6 Chiral Symmetry Breaking Due to Quark Masses

The finite u-, d-, and s-quark masses in the QCD Lagrangian result in explicit divergences of the symmetry currents. As a consequence, the charge operators are, in general, no longer time independent. However, as first pointed out by Gell-Mann, the equal-time-commutation relations still play an important role even if the symmetry is explicitly broken [Gel 62]. As will be discussed later on in more detail, the symmetry currents will give rise to chiral Ward identities relating various QCD Green functions to each other. Equation (2.43) allows one to discuss the divergences in the presence of quark masses. To that end, let us consider the quark-mass matrix of the three light quarks and project it on the nine λ matrices of Eq. (2.13),

$$
\begin{aligned}
M &= \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}\\
&= \frac{m_u + m_d + m_s}{\sqrt{6}} \lambda_0 + \frac{(m_u + m_d)/2 - m_s}{\sqrt{3}} \lambda_8 + \frac{m_u - m_d}{2} \lambda_3. \quad (2.80)
\end{aligned}
$$

In particular, applying Eq. (2.34) we see that the quark mass term mixes left- and right-handed fields,

$$
\mathcal{L}_M = -\bar{q} M q = -(\bar{q}_R M q_L + \bar{q}_L M q_R). \quad (2.81)
$$

The symmetry-breaking term transforms under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ as a member of a $(3,3^*)+(3^*,3)$ representation, i.e.,

$$\bar{q}_{R,i} M_{ij} q_{L,j} + \bar{q}_{L,i} M_{ij} q_{R,j} \mapsto U_{L,jk} U^*_{R,il} \bar{q}_{R,l} M_{ij} q_{L,k} + (L \leftrightarrow R),$$

where $(U_L, U_R) \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$. Such symmetry-breaking *patterns* were already discussed in the pre-QCD era in Refs. [GW 68, Gel+ 68].

From $\mathcal{L}_M$ one obtains as the variation $\delta\mathcal{L}_M$ under the transformations of Eqs. (2.36),

$$\begin{aligned} \delta\mathcal{L}_M = & -i \left[\sum_{a=1}^{8} \Theta_a^R \left(\bar{q}_R \frac{\lambda_a}{2} M q_L - \bar{q}_L M \frac{\lambda_a}{2} q_R \right) + \Theta^R \left(\bar{q}_R M q_L - \bar{q}_L M q_R \right) \right. \\ & \left. + \sum_{a=1}^{8} \Theta_a^L \left(\bar{q}_L \frac{\lambda_a}{2} M q_R - \bar{q}_R M \frac{\lambda_a}{2} q_L \right) + \Theta^L \left(\bar{q}_L M q_R - \bar{q}_R M q_L \right) \right], \end{aligned} \tag{2.82}$$

which results in the following divergences,

$$\begin{aligned} \partial_\mu L^{\mu,a} &= \frac{\partial \delta\mathcal{L}_M}{\partial \Theta_a^L} = -i \left(\bar{q}_L \frac{\lambda_a}{2} M q_R - \bar{q}_R M \frac{\lambda_a}{2} q_L \right), \\ \partial_\mu R^{\mu,a} &= \frac{\partial \delta\mathcal{L}_M}{\partial \Theta_a^R} = -i \left(\bar{q}_R \frac{\lambda_a}{2} M q_L - \bar{q}_L M \frac{\lambda_a}{2} q_R \right), \\ \partial_\mu L^{\mu} &= \frac{\partial \delta\mathcal{L}_M}{\partial \Theta^L} = -i \left(\bar{q}_L M q_R - \bar{q}_R M q_L \right), \\ \partial_\mu R^{\mu} &= \frac{\partial \delta\mathcal{L}_M}{\partial \Theta^R} = -i \left(\bar{q}_R M q_L - \bar{q}_L M q_R \right). \end{aligned} \tag{2.83}$$

The anomaly has not yet been considered. Applying Eq. (2.34) to the case of the vector currents and inserting projection operators as in the derivation of Eq. (2.64) for the axial-vector current, the corresponding divergences read

$$\begin{aligned} \partial_\mu V^{\mu,a} &= i\bar{q}[M, \frac{\lambda_a}{2}]q, \\ \partial_\mu A^{\mu,a} &= i \left(\bar{q}_L \{ \frac{\lambda_a}{2}, M \} q_R - \bar{q}_R \{ \frac{\lambda_a}{2}, M \} q_L \right) = i\bar{q}\{ \frac{\lambda_a}{2}, M \} \gamma_5 q, \\ \partial_\mu V^{\mu} &= 0, \\ \partial_\mu A^{\mu} &= 2i\bar{q}M\gamma_5 q + \frac{3g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathcal{G}_a^{\mu\nu} \mathcal{G}_a^{\rho\sigma}, \quad \epsilon_{0123} = 1, \end{aligned} \tag{2.84}$$

where the axial anomaly has also been taken into account. We are now in the position to summarize the various (approximate) symmetries of the strong interactions in combination with the corresponding currents and their divergences.

- In the limit of massless quarks, the sixteen currents $L^{\mu,a}$ and $R^{\mu,a}$ or, alternatively, $V^{\mu,a}$ and $A^{\mu,a}$ are conserved. The same is true for the singlet vector current V^{μ}, whereas the singlet axial-vector current A^{μ} has an anomaly.

- For any value of quark masses, the individual flavor currents $\bar{u}\gamma^{\mu}u$, $\bar{d}\gamma^{\mu}d$, and $\bar{s}\gamma^{\mu}s$ are always conserved in strong interactions reflecting the flavor independence of the strong coupling and the diagonality of the quark mass matrix. Of course, the singlet vector current V^{μ}, being the sum of the three flavor currents, is always conserved.

- In addition to the anomaly, the singlet axial-vector current has an explicit divergence due to the quark masses.

- For equal quark masses, $m_u = m_d = m_s$, the eight vector currents $V^{\mu,a}$ are conserved, because $[\lambda_a, 1] = 0$. Such a scenario is the origin of the SU(3) symmetry originally proposed by Gell-Mann and Ne'eman [GN 64]. The eight axial currents $A^{\mu,a}$ are not conserved. The divergences of the octet axial-vector currents of Eq. (2.84) are proportional to pseudoscalar quadratic forms. This can be interpreted as the microscopic origin of the PCAC relation (partially conserved axial-vector current) which states that the divergences of the axial-vector currents are proportional to renormalized field operators representing the lowest lying pseudoscalar octet (for a comprehensive discussion of the meaning of PCAC see Refs. [Gel 64b, AD 68, Trei+ 72, Alf+ 73]).

- Various symmetry-breaking patterns are discussed in great detail in Ref. [Pag 75].

2.4 Green Functions and Chiral Ward Identities

2.4.1 Chiral Green Functions

For conserved currents, the spatial integrals of the charge densities are time independent, i.e., in a quantized theory the corresponding charge operators commute with the Hamilton operator. These operators are generators of infinitesimal transformations on the Hilbert space of the theory. The mass eigenstates should organize themselves in degenerate multiplets with dimensionalities corresponding to irreducible representations of the Lie group in question.[19] Which irreducible representations ultimately appear, and what the actual energy eigenvalues are, is determined by the dynamics of the Hamiltonian. For example, SU(2) isospin symmetry of the strong interactions reflects itself in degenerate SU(2) multiplets such as the nucleon doublet, the pion triplet and so on. Ultimately, the actual masses of the nucleon and the pion should follow from QCD (for a prediction of hadron masses in lattice QCD see, e.g., Refs. [But+ 94, Ali+ 02]).

It is also well-known that symmetries imply relations between S-matrix elements. For example, applying the Wigner-Eckart theorem to pion-nucleon scattering, assuming the strong-interaction Hamiltonian to be an isoscalar, it is sufficient to consider two isospin amplitudes describing transitions between states of total isospin $I = 1/2$ or $I = 3/2$ (see, for example, [EW 88]). All the dynamical information is contained in these isospin amplitudes and the results for physical processes can be expressed in terms of these amplitudes together with geometrical coefficients, namely, the Clebsch-Gordan coefficients.

In quantum field theory, the objects of interest are the Green functions which are vacuum expectation values of time-ordered products.[20] Pictorially, these Green functions can be understood as vertices and are related to physical scattering amplitudes through the Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism [Leh+ 55]. Symmetries provide strong constraints not only for scattering amplitudes, i.e. their transforma-

[19] Here we assume that the dynamical system described by the Hamiltonian does not lead to a spontaneous symmetry breakdown. We will come back to this point later.

[20] Later on, we will also refer to matrix elements of time-ordered products between states other than the vacuum as Green functions.

tion behavior, but, more generally speaking, also for Green functions and, in particular, *among* Green functions. The famous example in this context is, of course, the Ward identity of QED associated with U(1) gauge invariance [War 50],

$$\Gamma^\mu(p,p) = -\frac{\partial}{\partial p_\mu}\Sigma(p), \tag{2.85}$$

which relates the electromagnetic vertex of an electron at zero momentum transfer, $\gamma^\mu + \Gamma^\mu(p,p)$, to the electron self energy, $\Sigma(p)$.

Such symmetry relations can be extended to non-vanishing momentum transfer and also to more complicated groups and are referred to as Ward-Fradkin-Takahashi identities [War 50, Fra 55, Tak 57] (or Ward identities for short). Furthermore, even if a symmetry is broken, i.e., the infinitesimal generators are time dependent, conditions related to the symmetry breaking terms can still be obtained using equal-time commutation relations [Gel 62].

At first, we are interested in time-ordered products of color-neutral, Hermitian quadratic forms involving the light quark fields evaluated between the vacuum of QCD. Using the LSZ reduction formalism [Leh+ 55, IZ 80] such Green functions can be related to physical processes involving mesons as well as their interactions with the electroweak gauge fields of the Standard Model. The interpretation depends on the transformation properties and quantum numbers of the quadratic forms, determining for which mesons they may serve as an interpolating field. In addition to the vector and axial-vector currents of Eqs. (2.59), (2.60), and (2.63) we want to investigate scalar and pseudoscalar densities,[21]

$$S_a(x) = \bar{q}(x)\lambda_a q(x), \quad P_a(x) = i\bar{q}(x)\gamma_5\lambda_a q(x), \quad a = 0, \cdots, 8, \tag{2.86}$$

which enter, for example, in Eqs. (2.84) as the divergences of the vector and axial-vector currents for nonzero quark masses. Whenever it is more convenient, we will also use

$$S(x) = \bar{q}(x)q(x), \quad P(x) = i\bar{q}(x)\gamma_5 q(x), \tag{2.87}$$

instead of S_0 and P_0.

Later on, we will also consider similar time-ordered products evaluated between a single nucleon in the initial and final states in addition to the

[21] The singlet axial-vector current involves an anomaly such that the Green functions involving this current operator are related to Green functions containing the contraction of the gluon field-strength tensor with its dual.

vacuum Green functions. This will allow us to discuss properties of the nucleon as well as dynamical processes involving a single nucleon.

Generally speaking, a chiral Ward identity relates the divergence of a Green function containing at least one factor of $V^{\mu,a}$ or $A^{\mu,a}$ [see Eqs. (2.59) and (2.60)] to some linear combination of other Green functions. The terminology *chiral* refers to the underlying $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ group. To make this statement more precise, let us consider as a simple example the two-point Green function involving an axial-vector current and a pseudoscalar density,[22]

$$\begin{aligned} G^{\mu,ab}_{AP}(x,y) &= \langle 0|T[A^\mu_a(x)P_b(y)]|0\rangle \\ &= \Theta(x_0-y_0)\langle 0|A^\mu_a(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)A^\mu_a(x)|0\rangle, \end{aligned} \tag{2.88}$$

and evaluate the divergence

$$\begin{aligned} &\partial^x_\mu G^{\mu,ab}_{AP}(x,y) \\ &= \delta(x_0-y_0)\langle 0|A^a_0(x)P_b(y)|0\rangle - \delta(x_0-y_0)\langle 0|P_b(y)A^a_0(x)|0\rangle \\ &\quad +\Theta(x_0-y_0)\langle 0|\partial^x_\mu A^\mu_a(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)\partial^x_\mu A^\mu_a(x)|0\rangle \\ &= \delta(x_0-y_0)\langle 0|[A^a_0(x),P_b(y)]|0\rangle + \langle 0|T[\partial^x_\mu A^\mu_a(x)P_b(y)]|0\rangle, \end{aligned}$$

where we made use of $\partial^x_\mu\Theta(x_0-y_0)=\delta(x_0-y_0)g_{0\mu}=-\partial^x_\mu\Theta(y_0-x_0)$. This simple example already shows the main features of (chiral) Ward identities. From the differentiation of the theta functions one obtains equal-time commutators between a charge density and the remaining quadratic forms. The results of such commutators are a reflection of the underlying symmetry, as will be shown below. As a second term, one obtains the divergence of the current operator in question. If the symmetry is perfect, such terms vanish identically. For example, this is always true for the electromagnetic case with its U(1) symmetry. If the symmetry is only approximate, an additional term involving the symmetry breaking appears. For a soft breaking such a divergence can be treated as a perturbation.

Via induction, the generalization of the above simple example to an $(n+1)$-point Green function is symbolically of the form

$$\partial^x_\mu\langle 0|T\{J^\mu(x)A_1(x_1)\cdots A_n(x_n)\}|0\rangle =$$

[22] The time ordering of n points $x_1,\cdots,x_n$ gives rise to $n!$ distinct orderings, each involving products of $n-1$ theta functions.

$$
\begin{aligned}
&\langle 0|T\{[\partial_\mu^x J^\mu(x)]A_1(x_1)\cdots A_n(x_n)\}|0\rangle\\
&+\delta(x^0-x_1^0)\langle 0|T\{[J_0(x),A_1(x_1)]A_2(x_2)\cdots A_n(x_n)\}|0\rangle\\
&+\delta(x^0-x_2^0)\langle 0|T\{A_1(x_1)[J_0(x),A_2(x_2)]\cdots A_n(x_n)\}|0\rangle\\
&+\cdots+\delta(x^0-x_n^0)\langle 0|T\{A_1(x_1)\cdots[J_0(x),A_n(x_n)]\}|0\rangle, \qquad (2.89)
\end{aligned}
$$

where J^μ stands generically for any of the Noether currents.

2.4.2 The Algebra of Currents

In the above example, we have seen that chiral Ward identities depend on the equal-time commutation relations of the *charge densities* of the symmetry currents with the relevant quadratic quark forms. Unfortunately, a naive application of Eq. (2.75) may lead to erroneous results. Let us illustrate this by means of a simplified example, the equal-time commutator of the time and space components of the ordinary electromagnetic current in QED. A naive use of the canonical commutation relations leads to

$$
\begin{aligned}
[J_0(\vec{x},t),J_i(\vec{y},t)] &= [\Psi^\dagger(\vec{x},t)\Psi(\vec{x},t),\Psi^\dagger(\vec{y},t)\gamma_0\gamma_i\Psi(\vec{y},t)]\\
&= \delta^3(\vec{x}-\vec{y})\Psi^\dagger(\vec{x},t)[1,\gamma_0\gamma_i]\Psi(\vec{x},t)=0, \qquad (2.90)
\end{aligned}
$$

where we made use of the delta function to evaluate the fields at $\vec{x}=\vec{y}$. It was noticed a long time ago by Schwinger that this result cannot be true [Sch 59]. In order to see this, consider the commutator

$$
[J_0(\vec{x},t),\vec{\nabla}_y\cdot\vec{J}(\vec{y},t)]=-[J_0(\vec{x},t),\partial_t J_0(\vec{y},t)],
$$

where we made use of current conservation, $\partial_\mu J^\mu=0$. If Eq. (2.90) were true, one would necessarily also have

$$
0=[J_0(\vec{x},t),\partial_t J_0(\vec{y},t)],
$$

which we evaluate for $\vec{x}=\vec{y}$ between the ground state,

$$
\begin{aligned}
0 &= \langle 0|[J_0(\vec{x},t),\partial_t J_0(\vec{x},t)]|0\rangle\\
&= \sum_n\Big(\langle 0|J_0(\vec{x},t)|n\rangle\langle n|\partial_t J_0(\vec{x},t)|0\rangle-\langle 0|\partial_t J_0(\vec{x},t)|n\rangle\langle n|J_0(\vec{x},t)|0\rangle\Big)\\
&= 2i\sum_n(E_n-E_0)|\langle 0|J_0(\vec{x},t)|n\rangle|^2.
\end{aligned}
$$

Here, we inserted a complete set of states and made use of

$$\partial_t J_0(\vec{x},t) = i[H, J_0(\vec{x},t)].$$

Since every individual term in the sum is non-negative, one would need $\langle 0|J_0(\vec{x},t)|n\rangle = 0$ for any intermediate state which is obviously unphysical. The solution is that the starting point, Eq. (2.90), is not true. The corrected version of Eq. (2.90) picks up an additional, so-called Schwinger term containing a derivative of the delta function.

Quite generally, by evaluating commutation relations with the component Θ^{00} of the energy-momentum tensor one can show that the equal-time commutation relation between a charge density and a current density can be determined up to one derivative of the δ function [Jac 72],

$$[J_0^a(\vec{x},0), J_i^b(\vec{y},0)] = iC_{abc}J_i^c(\vec{x},0)\delta^3(\vec{x}-\vec{y}) + S_{ij}^{ab}(\vec{y},0)\partial^j\delta^3(\vec{x}-\vec{y}), \quad (2.91)$$

where the Schwinger term possesses the symmetry

$$S_{ij}^{ab}(\vec{y},0) = S_{ji}^{ba}(\vec{y},0),$$

and C_{abc} denote the structure constants of the group in question.

However, in our above derivation of the chiral Ward identity, we also made use of the *naive* time-ordered product (T) as opposed to the *covariant* one (T^*) which, typically, differ by another non-covariant term which is called a seagull. Feynman's conjecture [Jac 72] states that there is a cancelation between Schwinger terms and seagull terms such that a Ward identity obtained by using the naive T product and by simultaneously omitting Schwinger terms ultimately yields the correct result to be satisfied by the Green function (involving the covariant T^* product). Although this will not be true in general, a sufficient condition for it to happen is that the time component algebra of the full theory remains the same as the one derived canonically and does not possess a Schwinger term. For a detailed discussion, the interested reader is referred to Ref. [Jac 72].

Keeping the above discussion in mind, the complete list of equal-time commutation relations, omitting Schwinger terms, reads

$$\begin{aligned}
[V_0^a(\vec{x},t), V_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} V_c^\mu(\vec{x},t), \\
[V_0^a(\vec{x},t), V^\mu(\vec{y},t)] &= 0, \\
[V_0^a(\vec{x},t), A_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} A_c^\mu(\vec{x},t),
\end{aligned}$$

$$
\begin{aligned}
[V_0^a(\vec{x},t), S_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} S_c(\vec{x},t), \quad b=1,\cdots,8,\\
[V_0^a(\vec{x},t), S_0(\vec{y},t)] &= 0,\\
[V_0^a(\vec{x},t), P_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} P_c(\vec{x},t), \quad b=1,\cdots,8,\\
[V_0^a(\vec{x},t), P_0(\vec{y},t)] &= 0,\\
[A_0^a(\vec{x},t), V_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} A_c^\mu(\vec{x},t),\\
[A_0^a(\vec{x},t), V^\mu(\vec{y},t)] &= 0,\\
[A_0^a(\vec{x},t), A_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} V_c^\mu(\vec{x},t),\\
[A_0^a(\vec{x},t), S_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} P_c(\vec{x},t), \quad b=1,\cdots,8,\\
[A_0^a(\vec{x},t), S_0(\vec{y},t)] &= 0,\\
[A_0^a(\vec{x},t), P_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} S_c(\vec{x},t), \quad b=1,\cdots,8,\\
[A_0^a(\vec{x},t), P_0(\vec{y},t)] &= 0.
\end{aligned}
\tag{2.92}
$$

2.4.3 Two Simple Examples

We now return to our specific example, namely, the divergence of Eq. (2.88). Inserting the results of Eqs. (2.84) and (2.92) one obtains

$$
\begin{aligned}
\partial_\mu^x G_{AP}^{\mu,ab}(x,y) &= \delta^4(x-y) i f_{abc} \langle 0|S_c(x)|0\rangle\\
&\quad + i\langle 0|T[\bar{q}(x)\{\frac{\lambda_a}{2}, M\}\gamma_5 q(x) P_b(y)]|0\rangle.
\end{aligned}
\tag{2.93}
$$

The second term on the right-hand side of Eq. (2.93) can be re-expressed using Eq. (2.80) and the anti-commutation relations of Eq. (2.11) in combination with the d coefficients of Table 2.2 (no summation over a implied),

$$
\begin{aligned}
& i\bar{q}(x)\{\frac{\lambda_a}{2}, M\}\gamma_5 q(x) =\\
&\quad \left[\frac{1}{3}(m_u+m_d+m_s) + \frac{1}{\sqrt{3}}\left(\frac{m_u+m_d}{2} - m_s\right) d_{aa8}\right] P_a(x)\\
&\quad + \left[\sqrt{\frac{1}{6}}(m_u-m_d)\delta_{a3} + \frac{\sqrt{2}}{3}\left(\frac{m_u+m_d}{2} - m_s\right)\delta_{a8}\right] P_0(x)\\
&\quad + \frac{m_u-m_d}{2}\sum_{c=1}^{8} d_{a3c} P_c(x).
\end{aligned}
$$

Equation (2.93) serves to illustrate two distinct features of chiral Ward identities. The first term of Eq. (2.93) originates in the algebra of currents and thus represents a consequence of the *transformation properties* of the quadratic quark forms entering the Green function. In general, depending on whether the appropriate equal-time commutation relation of Eq. (2.92) vanishes or not, the resulting term in the divergence of an n-point Green function vanishes or is proportional to an $(n-1)$-point Green function. In our specific example, the divergence of the Green function involving the axial-vector current and the pseudoscalar density is related to the so-called scalar quark condensate which will be discussed in more detail in Sec. 4.1.2. The second term of Eq. (2.93) is due to an explicit symmetry breaking resulting from the quark masses. This shows the second property of chiral Ward identities, namely, symmetry breaking terms give rise to another n-point Green function. To summarize, chiral Ward identities incorporate both transformation properties of quadratic quark forms as well as symmetry breaking patterns.

As another well-known and simple example, let us briefly consider, for the two-flavor case, the nucleon matrix element of the axial-vector current operator[23]

$$\langle N(p_f)|A_\mu^i(x)|N(p_i)\rangle = \langle N(p_f)|\bar{q}(x)\gamma_\mu\gamma_5\frac{\tau_i}{2}q(x)|N(p_i)\rangle. \tag{2.94}$$

This matrix element serves as an illustration of chiral Ward identities which are taken between one-nucleon states instead of the vacuum. According to Eq. (2.84), the divergence of Eq. (2.94) is related to the pseudoscalar density evaluated between one-nucleon states. Of course, in the chiral limit $M = 0$ and the axial-vector current is conserved.

2.4.4 QCD in the Presence of External Fields and the Generating Functional

Here, we want to consider the consequences of Eqs. (2.92) for the Green functions of QCD (in particular, at low energies). In principle, using the techniques of the last section, for each Green function one can *explicitly* work out the chiral Ward identity which, however, becomes more and more tedious as the number n of quark quadratic forms increases. However, there exists an elegant way of formally combining all Green functions in a generating functional. The (infinite) set of *all* chiral Ward identities is encoded as an

[23]This matrix element will be dealt with in Sec. 5.3.1.

invariance property of that functional. To see this, one has to consider a coupling to external c-number fields such that through functional methods one can, in principle, obtain all Green functions from a generating functional. The rationale behind this approach is that, in the absence of anomalies, the Ward identities obeyed by the Green functions are equivalent to an invariance of the generating functional under a *local* transformation of the external fields [Leu 94]. The use of local transformations allows one to also consider divergences of Green functions. For an illustration of this statement, the reader is referred to Appendix A.

Following the procedure of Gasser and Leutwyler [GL 84, GL 85a], we introduce into the Lagrangian of QCD the couplings of the nine vector currents and the eight axial-vector currents as well as the scalar and pseudoscalar quark densities to external c-number fields $v^\mu(x)$, $v^\mu_{(s)}$, $a^\mu(x)$, $s(x)$, and $p(x)$,

$$\mathcal{L} = \mathcal{L}^0_{\rm QCD} + \mathcal{L}_{\rm ext} = \mathcal{L}^0_{\rm QCD} + \bar{q}\gamma_\mu\left(v^\mu + \frac{1}{3}v^\mu_{(s)} + \gamma_5 a^\mu\right)q - \bar{q}(s - i\gamma_5 p)q. \quad (2.95)$$

The external fields are color-neutral, Hermitian 3×3 matrices, where the matrix character, with respect to the (suppressed) flavor indices u, d, and s of the quark fields, is[24]

$$v^\mu = \sum_{a=1}^{8}\frac{\lambda_a}{2}v^\mu_a, \quad a^\mu = \sum_{a=1}^{8}\frac{\lambda_a}{2}a^\mu_a, \quad s = \sum_{a=0}^{8}\lambda_a s_a, \quad p = \sum_{a=0}^{8}\lambda_a p_a. \quad (2.96)$$

The ordinary three flavor QCD Lagrangian is recovered by setting $v^\mu = v^\mu_{(s)} = a^\mu = p = 0$ and $s = \mathrm{diag}(m_u, m_d, m_s)$ in Eq. (2.95).

If one defines the generating functional[25]

$$\exp(iZ[v,a,s,p]) = \langle 0|T\exp\left[i\int d^4x\mathcal{L}_{\rm ext}(x)\right]|0\rangle, \quad (2.97)$$

then any Green function consisting of the time-ordered product of color-neutral, Hermitian quadratic forms can be obtained from Eq. (2.97) through

[24] As in Refs. [GL 84, GL 85a], we omit the coupling to the singlet axial-vector current which has an anomaly, but include a singlet vector current $v^\mu_{(s)}$ which is of some physical relevance in the two-flavor sector.

[25] Many books on quantum field theory such as Refs. [IZ 80, Col 84, Ryd 85, Riv 87] reserve the symbol $Z[v,a,s,p]$ for the generating functional of all Green functions as opposed to the argument of the exponential which denotes the generating functional of connected Green functions.

a functional derivative with respect to the external fields. The quark fields are operators in the Heisenberg picture and have to satisfy the equation of motion and the canonical anti-commutation relations. The actual value of the generating functional for a given configuration of external fields v, a, s, and p reflects the dynamics generated by the QCD Lagrangian. The generating functional is related to the vacuum-to-vacuum transition amplitude in the presence of external fields [GL 84, GL 85a],

$$\exp[iZ(v,a,s,p)] = \langle 0_{\rm out}|0_{\rm in}\rangle_{v,a,s,p}, \tag{2.98}$$

where the dynamics is determined by the Lagrangian of Eq. (2.95).

For example,[26] the $\bar{u}u$ component of the scalar quark condensate in the chiral limit, $\langle 0|\bar{u}u|0\rangle_0$, is given by

$$\langle 0|\bar{u}(x)u(x)|0\rangle_0 = \frac{i}{2}\left[\sqrt{\frac{2}{3}}\frac{\delta}{\delta s_0(x)} + \frac{\delta}{\delta s_3(x)} + \frac{1}{\sqrt{3}}\frac{\delta}{\delta s_8(x)}\right]\exp(iZ[v,a,s,p])\bigg|_{v=a=s=p=0}, \tag{2.99}$$

where we made use of Eq. (2.13). Note that both the quark field operators and the ground state are considered in the chiral limit, which is denoted by the subscript 0.

As another example, let us consider the two-point function of the axial-vector currents of Eq. (2.60) of the "real world," i.e., for $s=\mathrm{diag}(m_u,m_d,m_s)$, and the "true vacuum" $|0\rangle$,

$$\langle 0|T[A^a_\mu(x)A^b_\nu(0)]|0\rangle = (-i)^2\frac{\delta^2}{\delta a^\mu_a(x)\delta a^\nu_b(0)}\exp(iZ[v,a,s,p])\bigg|_{v=a=p=0,s=\mathrm{diag}(m_u,m_d,m_s)}. \tag{2.100}$$

Requiring the total Lagrangian of Eq. (2.95) to be Hermitian and invariant under P, C, and T leads to constraints on the transformation behavior of

[26]In order to obtain Green functions from the generating functional the simple rule

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x-y)$$

is extremely useful. Furthermore, the functional derivative satisfies properties similar to the ordinary differentiation, namely linearity, the product and chain rules.

Γ	1	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$\gamma_0\Gamma\gamma_0$	1	γ_μ	$\sigma_{\mu\nu}$	$-\gamma_5$	$-\gamma_\mu\gamma_5$

Table 2.4: Transformation properties of the Dirac matrices Γ under parity.

the external fields. In fact, it is sufficient to consider P and C, only, because T is then automatically incorporated owing to the CPT theorem.

Under parity, the quark fields transform as

$$q_f(\vec{x},t) \overset{P}{\mapsto} \gamma^0 q_f(-\vec{x},t), \tag{2.101}$$

and the requirement of parity conservation,

$$\mathcal{L}(\vec{x},t) \overset{P}{\mapsto} \mathcal{L}(-\vec{x},t), \tag{2.102}$$

leads, using the results of Table 2.4, to the following constraints for the external fields,

$$v^\mu \overset{P}{\mapsto} v_\mu, \quad v^\mu_{(s)} \overset{P}{\mapsto} v^{(s)}_\mu, \quad a^\mu \overset{P}{\mapsto} -a_\mu, \quad s \overset{P}{\mapsto} s, \quad p \overset{P}{\mapsto} -p. \tag{2.103}$$

In Eq. (2.103) it is understood that the arguments change from $(\vec{x},t)$ to $(-\vec{x},t)$.

Similarly, under charge conjugation the quark fields transform as

$$q_{\alpha,f} \overset{C}{\mapsto} C_{\alpha\beta}\bar{q}_{\beta,f}, \quad \bar{q}_{\alpha,f} \overset{C}{\mapsto} -q_{\beta,f}C^{-1}_{\beta\alpha}, \tag{2.104}$$

where the subscripts α and β are Dirac spinor indices, $C = i\gamma^2\gamma^0 = -C^{-1} = -C^\dagger = -C^T$ is the usual charge conjugation matrix in the convention of Ref. [BD 64a] and f refers to flavor. Using Eq. (2.104) in combination with Table 2.5 it is straightforward to show that invariance of $\mathcal{L}_{\text{ext}}$ under charge conjugation requires the transformation properties[27]

$$v_\mu \xrightarrow{C} -v^T_\mu, \quad v^{(s)}_\mu \xrightarrow{C} -v^{(s)T}_\mu, \quad a_\mu \xrightarrow{C} a^T_\mu, \quad s,p \xrightarrow{C} s^T, p^T, \tag{2.105}$$

where the transposition refers to the flavor space.

[27] In deriving these results we need to make use of $q_{\gamma,f}\bar{q}_{\delta,f'} = -\bar{q}_{\delta,f'}q_{\gamma,f}$, since the quark fields are anti-commuting field operators.

Γ	1	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$-C\Gamma^T C$	1	$-\gamma^\mu$	$-\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$

Table 2.5: Transformation properties of the Dirac matrices Γ under charge conjugation.

Finally, we need to discuss the requirements to be met by the external fields under local $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R\times\mathrm{U}(1)_V$ transformations. In a first step, we write Eq. (2.95) in terms of the left- and right-handed quark fields. Besides the properties of Eqs. (2.30) - (2.32) we make use of the auxiliary formulae

$$\gamma_5 P_R = P_R\gamma_5 = P_R, \quad \gamma_5 P_L = P_L\gamma_5 = -P_L,$$

and

$$\gamma^\mu P_R = P_L\gamma^\mu, \quad \gamma^\mu P_L = P_R\gamma^\mu,$$

to obtain

$$\begin{aligned}\bar q\gamma^\mu\left(v_\mu+\frac{1}{3}v^{(s)}_\mu+\gamma_5 a_\mu\right)q &= \frac{1}{2}\bar q\gamma^\mu\left[r_\mu+l_\mu+\frac{2}{3}v^{(s)}_\mu+\gamma_5(r_\mu-l_\mu)\right]q\\ &= \bar q_R\gamma^\mu\left(r_\mu+\frac{1}{3}v^{(s)}_\mu\right)q_R+\bar q_L\gamma^\mu\left(l_\mu+\frac{1}{3}v^{(s)}_\mu\right)q_L,\end{aligned}$$

where

$$v_\mu=\frac{1}{2}(r_\mu+l_\mu), \quad a_\mu=\frac{1}{2}(r_\mu-l_\mu). \tag{2.106}$$

Similarly, we rewrite the second part containing the external scalar and pseudoscalar fields,

$$\bar q(s-i\gamma_5 p)q = \bar q_L(s-ip)q_R+\bar q_R(s+ip)q_L,$$

yielding for the Lagrangian of Eq. (2.95)

$$\begin{aligned}\mathcal{L} &= \mathcal{L}^0_{\mathrm{QCD}}+\bar q_L\gamma^\mu\left(l_\mu+\frac{1}{3}v^{(s)}_\mu\right)q_L+\bar q_R\gamma^\mu\left(r_\mu+\frac{1}{3}v^{(s)}_\mu\right)q_R\\ &\quad -\bar q_R(s+ip)q_L-\bar q_L(s-ip)q_R.\end{aligned} \tag{2.107}$$

Equation (2.107) remains invariant under *local* transformations

$$\begin{aligned}q_R &\mapsto \exp\left(-i\frac{\Theta(x)}{3}\right)V_R(x)q_R,\\ q_L &\mapsto \exp\left(-i\frac{\Theta(x)}{3}\right)V_L(x)q_L,\end{aligned} \tag{2.108}$$

where $V_R(x)$ and $V_L(x)$ are independent space-time-dependent SU(3) matrices, provided the external fields are subject to the transformations

$$
\begin{aligned}
r_\mu &\mapsto V_R r_\mu V_R^\dagger + iV_R\partial_\mu V_R^\dagger, \\
l_\mu &\mapsto V_L l_\mu V_L^\dagger + iV_L\partial_\mu V_L^\dagger, \\
v_\mu^{(s)} &\mapsto v_\mu^{(s)} - \partial_\mu\Theta, \\
s+ip &\mapsto V_R(s+ip)V_L^\dagger, \\
s-ip &\mapsto V_L(s-ip)V_R^\dagger. \qquad (2.109)
\end{aligned}
$$

The derivative terms in Eq. (2.109) serve the same purpose as in the construction of gauge theories, i.e., they cancel analogous terms originating from the kinetic part of the quark Lagrangian.

There is another, yet, more practical aspect of the local invariance, namely: such a procedure allows one to also discuss a coupling to external gauge fields in the transition to the effective theory to be discussed later. For example, we have seen in Sec. 2.2 that a coupling of the electromagnetic field to point-like fundamental particles results from gauging a U(1) symmetry. Here, the corresponding U(1) group is to be understood as a subgroup of a local $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$. Another example deals with the interaction of the light quarks with the charged and neutral gauge bosons of the weak interactions.

Let us consider both examples explicitly. The coupling of quarks to an external electromagnetic field $\mathcal{A}_\mu$ is given by

$$r_\mu = l_\mu = -eQ\mathcal{A}_\mu, \qquad (2.110)$$

where $Q = \mathrm{diag}(2/3, -1/3, -1/3)$ is the quark charge matrix:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{ext}} &= -e\mathcal{A}_\mu(\bar{q}_L Q\gamma^\mu q_L + \bar{q}_R Q\gamma^\mu q_R) \\
&= -e\mathcal{A}_\mu \bar{q}Q\gamma^\mu q \\
&= -e\mathcal{A}_\mu\left(\frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s\right) \\
&= -e\mathcal{A}_\mu J^\mu.
\end{aligned}
$$

On the other hand, if one considers only the two-flavor version of ChPT one has to insert for the external fields

$$r_\mu = l_\mu = -e\frac{\tau_3}{2}\mathcal{A}_\mu, \quad v_\mu^{(s)} = -\frac{e}{2}\mathcal{A}_\mu. \qquad (2.111)$$

In the description of semileptonic interactions such as $\pi^- \to \mu^- \bar{\nu}_\mu$, $\pi^- \to \pi^0 e^- \bar{\nu}_e$, or neutron decay $n \to p e^- \bar{\nu}_e$ one needs the interaction of quarks with the massive charged weak bosons $\mathcal{W}^\pm_\mu = (\mathcal{W}_{1\mu} \mp i\mathcal{W}_{2\mu})/\sqrt{2}$,

$$r_\mu = 0, \quad l_\mu = -\frac{g}{\sqrt{2}}(\mathcal{W}^+_\mu T_+ + h.c.), \tag{2.112}$$

where $h.c.$ refers to the Hermitian conjugate and

$$T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, V_{ij} denote the elements of the Cabibbo-Kobayashi-Maskawa quark-mixing matrix describing the transformation between the mass eigenstates of QCD and the weak eigenstates [Gro+ 00],

$$|V_{ud}| = 0.9735 \pm 0.0008, \quad |V_{us}| = 0.2196 \pm 0.0023.$$

At lowest order in perturbation theory, the Fermi constant is related to the gauge coupling g and the W mass as

$$G_F = \sqrt{2}\frac{g^2}{8M_W^2} = 1.16639(1) \times 10^{-5}\,\mathrm{GeV}^{-2}.$$

Making use of

$$\begin{aligned}
\bar{q}_L \gamma^\mu \mathcal{W}^+_\mu T_+ q_L &= \mathcal{W}^+_\mu (\bar{u}\ \bar{d}\ \bar{s}) P_R \gamma^\mu \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P_L \begin{pmatrix} u \\ d \\ s \end{pmatrix} \\
&= \mathcal{W}^+_\mu (\bar{u}\ \bar{d}\ \bar{s}) \gamma^\mu \frac{1}{2}(1-\gamma_5) \begin{pmatrix} V_{ud} d + V_{us} s \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{1}{2}\mathcal{W}^+_\mu [V_{ud} \bar{u}\gamma^\mu(1-\gamma_5) d + V_{us}\bar{u}\gamma^\mu(1-\gamma_5)s],
\end{aligned}$$

we see that inserting Eq. (2.112) into Eq. (2.107) leads to the standard charged-current weak interaction in the light quark sector,

$$\mathcal{L}_{\text{ext}} = -\frac{g}{2\sqrt{2}}\left\{\mathcal{W}^+_\mu [V_{ud}\bar{u}\gamma^\mu(1-\gamma_5)d + V_{us}\bar{u}\gamma^\mu(1-\gamma_5)s] + h.c.\right\}.$$

The situation is slightly different for the neutral weak interaction. Here, the SU(3) version requires a coupling to the singlet axial-vector current which, because of the anomaly of Eq. (2.65), we have dropped from our discussion. On the other hand, in the SU(2) version the axial-vector current part is traceless and we have

$$\begin{aligned} r_\mu &= e\tan(\theta_W)\frac{\tau_3}{2}\mathcal{Z}_\mu, \\ l_\mu &= -\frac{g}{\cos(\theta_W)}\frac{\tau_3}{2}\mathcal{Z}_\mu + e\tan(\theta_W)\frac{\tau_3}{2}\mathcal{Z}_\mu, \\ v_\mu^{(s)} &= \frac{e\tan(\theta_W)}{2}\mathcal{Z}_\mu, \end{aligned} \tag{2.113}$$

where θ_W is the weak angle. With these external fields, we obtain the standard weak neutral-current interaction [Gro+ 00]

$$\begin{aligned} \mathcal{L}_{\text{ext}} &= -\frac{g}{2\cos(\theta_W)}\mathcal{Z}_\mu\left(\bar{u}\gamma^\mu\left\{\left[\frac{1}{2}-\frac{4}{3}\sin^2(\theta_W)\right]-\frac{1}{2}\gamma_5\right\}u\right. \\ &\quad \left.+\bar{d}\gamma^\mu\left\{\left[-\frac{1}{2}+\frac{2}{3}\sin^2(\theta_W)\right]+\frac{1}{2}\gamma_5\right\}d\right), \end{aligned}$$

where we made use of $e = g\sin(\theta_W)$.

2.4.5 PCAC in the Presence of an External Electromagnetic Field

Finally, the technique of coupling the QCD Lagrangian to external fields also allows us to determine the current divergences for rigid external fields, i.e., which are *not* simultaneously transformed. For the sake of simplicity we restrict ourselves to the SU(2) sector. (The generalization to the SU(3) case is straightforward.) If the external fields are not simultaneously transformed and one considers a *global* chiral transformation only, the divergences of the currents read [see Eq. (2.43)]

$$\partial_\mu V_i^\mu = i\bar{q}\gamma^\mu[\frac{\tau_i}{2}, v_\mu]q + i\bar{q}\gamma^\mu\gamma_5[\frac{\tau_i}{2}, a_\mu]q - i\bar{q}[\frac{\tau_i}{2}, s]q - \bar{q}\gamma_5[\frac{\tau_i}{2}, p]q, \tag{2.114}$$

$$\partial_\mu A_i^\mu = i\bar{q}\gamma^\mu\gamma_5[\frac{\tau_i}{2}, v_\mu]q + i\bar{q}\gamma^\mu[\frac{\tau_i}{2}, a_\mu]q + i\bar{q}\gamma_5\{\frac{\tau_i}{2}, s\}q + \bar{q}\{\frac{\tau_i}{2}, p\}q. \tag{2.115}$$

As an example, let us consider the QCD Lagrangian for a finite light quark mass m_q in combination with a coupling to an external electromagnetic field $\mathcal{A}_\mu$ [see Eq. (2.111), $a_\mu = 0 = p$]. In this case the expressions for the divergence of the vector and axial-vector currents, respectively, read

$$\partial_\mu V_i^\mu = -\epsilon_{3ij} e \mathcal{A}_\mu \bar{q}\gamma^\mu \frac{\tau_j}{2} q = -\epsilon_{3ij} e \mathcal{A}_\mu V_j^\mu, \tag{2.116}$$

$$\partial_\mu A_i^\mu = m_q P_i - e\mathcal{A}_\mu \epsilon_{3ij} A_j^\mu + \delta_{i3} \frac{e^2 N_c}{96\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\mu\nu} \mathcal{F}^{\rho\sigma}, \tag{2.117}$$

where we have introduced the isovector pseudoscalar density

$$P_i = i\bar{q}\gamma_5 \tau_i q, \tag{2.118}$$

and $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ is the electromagnetic field strength tensor. The third component of the axial-vector current, A_3^μ, has an anomaly [Adl 69, AB 69, Bar 69, BJ 69, Adl 70] which is related to the decay $\pi^0 \to \gamma\gamma$. We emphasize the formal similarity of Eq. (2.117) to the (pre-QCD) PCAC relation obtained by Adler through the inclusion of the electromagnetic interactions with minimal electromagnetic coupling (see the Appendix of Ref. [Adl 65]).[28] Since in QCD the quarks are taken as truly elementary, their interaction with an (external) electromagnetic field is of such a minimal type.

[28] In Adler's version, the right-hand side of Eq. (2.117) contains a renormalized field operator creating and destroying pions instead of $m_q P_i$. From a modern point of view, the combination $m_q P_i/(M_\pi^2 F_\pi)$ serves as an interpolating pion field (see Sec. 4.6.2). Furthermore, the anomaly term is not yet present in Ref. [Adl 65].

Chapter 3

Spontaneous Symmetry Breaking and the Goldstone Theorem

So far we have concentrated on the chiral symmetry of the QCD Hamiltonian and the *explicit* symmetry breaking through the quark masses. We have discussed the importance of chiral symmetry for the properties of Green functions with particular emphasis on the relations *among* different Green functions as expressed through the chiral Ward identities. Now it is time to address a second aspect which, for the low-energy structure of QCD, is equally important, namely, the concept of *spontaneous* symmetry breaking. A (continuous) symmetry is said to be spontaneously broken or hidden, if the ground state of the system is no longer invariant under the full symmetry group of the Hamiltonian. In this chapter we will first illustrate this by means of a discrete symmetry and then turn to the case of a spontaneously broken continuous global symmetry.

3.1 Degenerate Ground States

Before discussing the case of a *continuous* symmetry, we will first have a look at a field theory with a *discrete* internal symmetry. This will allow us to distinguish between two possibilities: a dynamical system with a unique ground state or a system with a finite number of distinct degenerate ground states. In particular, we will see how, for the second case, an infinitesimal

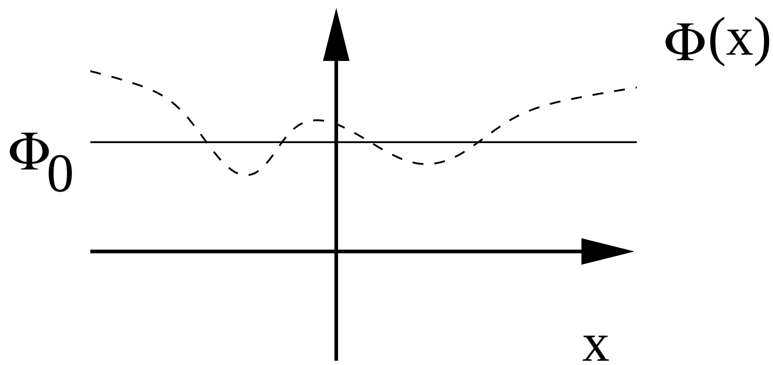


Figure 3.1: Solid line: constant field configuration Φ_0 minimizing the potential; dashed line: arbitrary field configuration $\Phi(x)$.

perturbation selects a particular vacuum state.

To that end we consider the Lagrangian of a real scalar field $\Phi(x)$ [Geo 84]

$$\mathcal{L}(\Phi, \partial_\mu \Phi) = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{m^2}{2}\Phi^2 - \frac{\lambda}{4}\Phi^4, \tag{3.1}$$

which is invariant under the discrete transformation $R : \Phi \rightarrow -\Phi$. The corresponding classical energy density reads

$$\mathcal{H} = \Pi\dot{\Phi} - \mathcal{L} = \frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}(\vec{\nabla}\Phi)^2 + \underbrace{\frac{m^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4}_{\mathcal{V}(\Phi)}, \tag{3.2}$$

where one chooses $\lambda > 0$ so that $\mathcal{H}$ is bounded from below. The field Φ_0 which minimizes the Hamilton density $\mathcal{H}$ must be constant and uniform since in that case the first two terms take everywhere their minimum values of zero. It must also minimize the potential since $\mathcal{V}(\Phi(x)) \geq \mathcal{V}(\Phi_0)$ (see Fig. 3.1), from which we obtain the condition

$$\mathcal{V}'(\Phi) = \Phi(m^2 + \lambda\Phi^2) = 0.$$

We now distinguish two different cases:

- $m^2 > 0$ (see Fig. 3.2): In this case the potential $\mathcal{V}$ has its minimum for $\Phi = 0$. In the quantized theory we associate a unique ground state $|0\rangle$

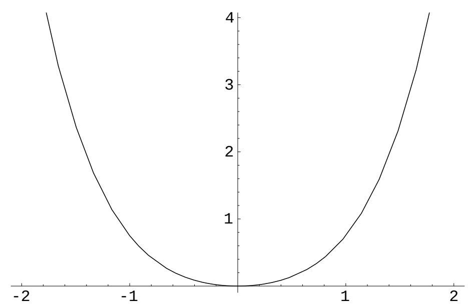


Figure 3.2: $\mathcal{V}(x) = x^2/2 + x^4/4$ (Wigner-Weyl mode).

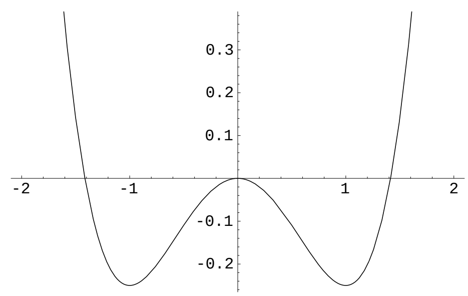


Figure 3.3: $\mathcal{V}(x) = -x^2/2 + x^4/4$ (Nambu-Goldstone mode).

with this minimum. Later on, in the case of a continuous symmetry, this situation will be referred to as the Wigner-Weyl realization of the symmetry.

- $m^2 < 0$ (see Fig. 3.3): Now the potential exhibits two distinct minima. (In the continuous symmetry case this will be referred to as the Nambu-Goldstone realization of the symmetry.)

We will concentrate on the second situation, because this is the one which we would like to generalize to a continuous symmetry and which ultimately leads to the appearance of Goldstone bosons. In the present case, $\mathcal{V}(\Phi)$ has a maximum for $\Phi = 0$ and *two* minima for

$$\Phi_\pm = \pm\sqrt{\frac{-m^2}{\lambda}} \equiv \pm\Phi_0. \tag{3.3}$$

As will be explained below, the quantized theory develops two degenerate vacua $|0,+\rangle$ and $|0,-\rangle$ which are distinguished through their vacuum expectation values of the field $\Phi(x)$:[1]

$$\begin{aligned}
\langle 0,+|\Phi(x)|0,+\rangle &= \langle 0,+|e^{iP\cdot x}\Phi(0)e^{-iP\cdot x}|0,+\rangle = \langle 0,+|\Phi(0)|0,+\rangle \equiv \Phi_0, \\
\langle 0,-|\Phi(x)|0,-\rangle &= -\Phi_0.
\end{aligned} \tag{3.4}$$

We made use of translational invariance, $\Phi(x) = e^{iP\cdot x}\Phi(0)e^{-iP\cdot x}$, and the fact that the ground state is an eigenstate of energy and momentum. We associate with the transformation $R : \Phi \mapsto \Phi' = -\Phi$ a unitary operator $\mathcal{R}$ acting on the Hilbert space of our model, with the properties

$$\mathcal{R}^2 = 1, \quad \mathcal{R} = \mathcal{R}^{-1} = \mathcal{R}^\dagger.$$

In accord with Eq. (3.4) the action of the operator $\mathcal{R}$ on the ground states is given by

$$\mathcal{R}|0,\pm\rangle = |0,\mp\rangle. \tag{3.5}$$

For the moment we select one of the two expectation values and expand the field with respect to $\pm\Phi_0$:[2]

$$\begin{aligned}
\Phi &= \pm\Phi_0 + \Phi', \\
\partial_\mu\Phi &= \partial_\mu\Phi'.
\end{aligned} \tag{3.6}$$

A short calculation yields

$$\mathcal{V}(\Phi) = \tilde{\mathcal{V}}(\Phi') = -\frac{\lambda}{4}\Phi_0^4 + \frac{1}{2}(-2m^2)\Phi'^2 \pm \lambda\Phi_0\Phi'^3 + \frac{\lambda}{4}\Phi'^4,$$

such that the Lagrangian in terms of the shifted dynamical variable reads

$$\mathcal{L}'(\Phi',\partial_\mu\Phi') = \frac{1}{2}\partial_\mu\Phi'\partial^\mu\Phi' - \frac{1}{2}(-2m^2)\Phi'^2 \mp \lambda\Phi_0\Phi'^3 - \frac{\lambda}{4}\Phi'^4 + \frac{\lambda}{4}\Phi_0^4. \tag{3.7}$$

In terms of the new dynamical variable Φ', the symmetry R is no longer manifest, i.e., it is hidden. Selecting one of the ground states has led to a

[1] The case of a quantum field theory with an infinite volume V has to be distinguished from, say, a nonrelativistic particle in a one-dimensional potential of a shape similar to the function of Fig. 3.3. For example, in the case of a symmetric double-well potential, the solutions with positive parity have always lower energy eigenvalues than those with negative parity (see, e.g., Ref. [Gre 85]).

[2] The field Φ' instead of Φ is assumed to vanish at infinity.

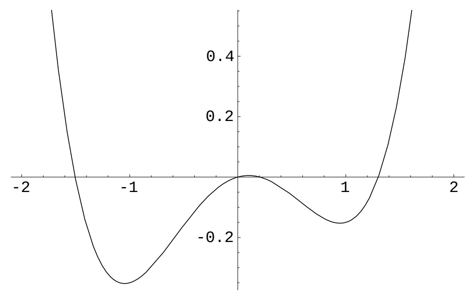


Figure 3.4: Potential with a small odd component: $\mathcal{V}(x) = x/10 - x^2/2 + x^4/4$.

spontaneous symmetry breaking which is always related to the existence of several degenerate vacua.

At this stage it is not clear why the quantum mechanical ground state should be one or the other of $|0,\pm\rangle$ and not a superposition of both. For example, the linear combination

$$\frac{1}{\sqrt{2}}\left(|0,+\rangle + |0,-\rangle\right)$$

is invariant under $\mathcal{R}$ as is the original Lagrangian of Eq. (3.1). However, this superposition is not stable against any infinitesimal external perturbation which is odd in Φ (see Fig. 3.4),

$$\mathcal{R}(\epsilon H')\mathcal{R}^\dagger = -\epsilon H'.$$

Any such perturbation will drive the ground state into the vicinity of either $|0,+\rangle$ or $|0,-\rangle$ rather than $\frac{1}{\sqrt{2}}(|0,+\rangle \pm |0,-\rangle)$. This can easily be seen in the framework of perturbation theory for degenerate states. Consider

$$|1\rangle = \frac{1}{\sqrt{2}}(|0,+\rangle + |0,-\rangle), \quad |2\rangle = \frac{1}{\sqrt{2}}(|0,+\rangle - |0,-\rangle),$$

such that

$$\mathcal{R}|1\rangle = |1\rangle \quad \mathcal{R}|2\rangle = -|2\rangle.$$

The condition for the energy eigenvalues of the ground state, $E = E^{(0)} + \epsilon E^{(1)} + \cdots$, to first order in ϵ results from

$$\det\begin{pmatrix} \langle 1|H'|1\rangle - E^{(1)} & \langle 1|H'|2\rangle \\ \langle 2|H'|1\rangle & \langle 2|H'|2\rangle - E^{(1)} \end{pmatrix} = 0.$$

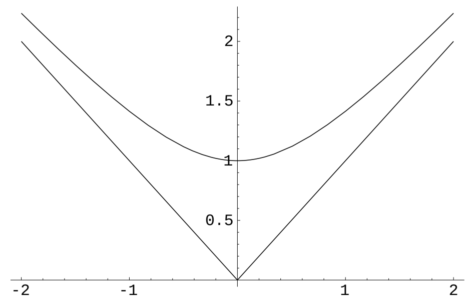


Figure 3.5:
Dispersion relation $E = \sqrt{1+\vec{p}^{\,2}}$ and asymptote $E = |\vec{p}\,|$.

Due to the symmetry properties of Eq. (3.5), we obtain

$$\langle 1|H'|1\rangle = \langle 1|\mathcal{R}^{-1}\mathcal{R}H'\mathcal{R}^{-1}\mathcal{R}|1\rangle = \langle 1| - H'|1\rangle = 0$$

and similarly $\langle 2|H'|2\rangle = 0$. Setting $\langle 1|H'|2\rangle = a > 0$, which can always be achieved by multiplication of one of the two states by an appropriate phase, one finds

$$\langle 2|H'|1\rangle \stackrel{H'=H'^\dagger}{=} \langle 1|H'|2\rangle^* = a^* = a = \langle 1|H'|2\rangle,$$

resulting in

$$\det\begin{pmatrix} -E^{(1)} & a \\ a & -E^{(1)} \end{pmatrix} = {E^{(1)}}^2 - a^2 \stackrel{!}{=} 0, \quad \Rightarrow \quad E^{(1)}_{1/2} = \pm a.$$

In other words, the degeneracy has been lifted and we get for the energy eigenvalues

$$E_{1/2} = E^{(0)} \pm \epsilon a + \cdots. \tag{3.8}$$

The corresponding eigenstates of zeroth order in ϵ are $|0,+\rangle$ and $|0,-\rangle$, respectively. We thus conclude that an arbitrarily small external perturbation which is odd with respect to R will push the ground state to either $|0,+\rangle$ or $|0,-\rangle$.

In the above discussion, we have tacitly assumed that the Hamiltonian and the field $\Phi(x)$ can simultaneously be diagonalized in the vacuum sector, i.e. $\langle 0,+|0,-\rangle = 0$. Following Ref. [Wei 96], we will justify this assumption which will also be crucial for the continuous case to be discussed later.

For an infinite volume, a general vacuum state $|v\rangle$ is defined as a state with momentum eigenvalue $\vec{0}$,

$$\vec{P}|v\rangle = \vec{0},$$

where $\vec{0}$ is a *discrete* eigenvalue as opposed to an eigenvalue of single- or many-particle states for which $\vec{p} = 0$ is an element of a continuous spectrum (see Fig. 3.5). We deal with the situation of several degenerate ground states which will be denoted by $|u\rangle$, $|v\rangle$, *etc*[3] and start from the identity

$$0 = \langle u|[H, \Phi(x)]|v\rangle \ \forall\ x, \tag{3.9}$$

from which we obtain for $t = 0$

$$\int d^3y \langle u|\mathcal{H}(\vec{y},0)\Phi(\vec{x},0)|v\rangle \ = \ \int d^3y \langle u|\Phi(\vec{x},0)\mathcal{H}(\vec{y},0)|v\rangle. \tag{3.10}$$

Let us consider the left-hand side,

$$\int d^3y \langle u|\mathcal{H}(\vec{y},0)\Phi(\vec{x},0)|v\rangle = \sum_w \langle u|H|w\rangle\langle w|\Phi(0)|v\rangle$$
$$+ \int d^3y \int d^3p \sum_n \langle u|\mathcal{H}(\vec{y},0)|n,\vec{p}\rangle\langle n,\vec{p}|\Phi(0)|v\rangle e^{-i\vec{p}\cdot\vec{x}},$$

where we inserted a complete set of states which we split into the vacuum contribution and the rest, and made use of translational invariance. We now define

$$f_n(\vec{y},\vec{p}) = \langle u|\mathcal{H}(\vec{y},0)|n,\vec{p}\rangle\langle n,\vec{p}|\Phi(0)|v\rangle$$

and assume f_n to be reasonably behaved such that one can apply the lemma of Riemann and Lebesgue,

$$\lim_{|\vec{x}|\to\infty} \int d^3p f(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} = 0.$$

At this point the assumption of an infinite volume, $|\vec{x}| \to \infty$, is crucial. Repeating the argument for the right-hand side and taking the limit $|\vec{x}| \to \infty$, only the vacuum contributions survive in Eq. (3.10) and we obtain

$$\sum_w \langle u|H|w\rangle\langle w|\Phi(0)|v\rangle = \sum_w \langle u|\Phi(0)|w\rangle\langle w|H|v\rangle$$

[3]For continuous symmetry groups one may have a non-countably infinite number of ground states.

for arbitrary ground states $|u\rangle$ and $|v\rangle$. In other words, the matrices $(H_{uv}) \equiv (\langle u|H|v\rangle)$ and $(\Phi_{uv}) \equiv (\langle u|\Phi(0)|v\rangle)$ commute and can be diagonalized simultaneously. Choosing an appropriate basis, one can write

$$\langle u|\Phi(0)|v\rangle = \delta_{uv}v, \quad v \in R,$$

where v denotes the expectation value of Φ in the state $|v\rangle$.

In the above example, the ground states $|0,+\rangle$ and $|0,-\rangle$ with vacuum expectation values $\pm\Phi_0$ are thus indeed orthogonal and satisfy

$$\langle 0,+|H|0,-\rangle = \langle 0,-|H|0,+\rangle = 0.$$

3.2 Spontaneous Breakdown of a Global, Continuous, Non-Abelian Symmetry

We now extend the discussion to a system with a continuous, non-Abelian symmetry such as SO(3). To that end, we consider the Lagrangian

$$\begin{aligned}\mathcal{L}(\vec{\Phi}, \partial_\mu\vec{\Phi}) &= \mathcal{L}(\Phi_1, \Phi_2, \Phi_3, \partial_\mu\Phi_1, \partial_\mu\Phi_2, \partial_\mu\Phi_3) \\ &= \frac{1}{2}\partial_\mu\Phi_i\partial^\mu\Phi_i - \frac{m^2}{2}\Phi_i\Phi_i - \frac{\lambda}{4}(\Phi_i\Phi_i)^2, \end{aligned} \qquad (3.11)$$

where $m^2 < 0$, $\lambda > 0$, with Hermitian fields Φ_i. The Lagrangian of Eq. (3.11) is invariant under a global "isospin" rotation,[4]

$$g \in \mathrm{SO}(3): \ \Phi_i \to \Phi_i' = D_{ij}(g)\Phi_j = (e^{-i\alpha_k T_k})_{ij}\Phi_j. \qquad (3.12)$$

For the Φ_i' to also be Hermitian, the Hermitian T_k must be purely imaginary and thus antisymmetric. The iT_k provide the basis of a representation of the so(3) Lie algebra and satisfy the commutation relations $[T_i, T_j] = i\epsilon_{ijk}T_k$. We will use the representation with the matrix elements given by $t^i_{jk} = -i\epsilon_{ijk}$. As in Sec. 3.1, we now look for a minimum of the potential which does not depend on x and find

$$|\vec{\Phi}_{\min}| = \sqrt{\frac{-m^2}{\lambda}} \equiv v, \quad |\vec{\Phi}| = \sqrt{\Phi_1^2 + \Phi_2^2 + \Phi_3^2}. \qquad (3.13)$$

[4]Of course, the Lagrangian is invariant under the full group O(3) which can be decomposed into its two components: the proper rotations connected to the identity, SO(3), and the rotation-reflections. For our purposes it is sufficient to discuss SO(3).

Since $\vec{\Phi}_{\rm min}$ can point in any direction in isospin space we now have a non-countably infinite number of degenerate vacua. In analogy to the discussion of the last section, any infinitesimal external perturbation which is not invariant under SO(3) will select a particular direction which, by an appropriate orientation of the internal coordinate frame, we denote as the 3 direction,

$$\vec{\Phi}_{\rm min} = v\hat{e}_3. \tag{3.14}$$

Clearly, $\vec{\Phi}_{\rm min}$ of Eq. (3.14) is *not* invariant under the full group $G = \mathrm{SO}(3)$ since rotations about the 1 and 2 axis change $\vec{\Phi}_{\rm min}$.[5] To be specific, if

$$\vec{\Phi}_{\rm min} = v \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right),$$

we obtain

$$T_1\vec{\Phi}_{\rm min} = v \left(\begin{array}{r} 0 \\ -i \\ 0 \end{array} \right), \quad T_2\vec{\Phi}_{\rm min} = v \left(\begin{array}{c} i \\ 0 \\ 0 \end{array} \right), \quad T_3\vec{\Phi}_{\rm min} = 0. \tag{3.15}$$

Note that the set of transformations which do not leave $\vec{\Phi}_{\rm min}$ invariant does *not* form a group, because it does not contain the identity. On the other hand, $\vec{\Phi}_{\rm min}$ is invariant under a subgroup H of G, namely, the rotations about the 3 axis:

$$h \in H: \quad \vec{\Phi}' = D(h)\vec{\Phi} = e^{-i\alpha_3 T_3}\vec{\Phi}, \quad D(h)\vec{\Phi}_{\rm min} = \vec{\Phi}_{\rm min}. \tag{3.16}$$

In analogy to Eq. (3.6), we expand Φ_3 with respect to v,

$$\Phi_3 = v + \eta, \tag{3.17}$$

where $\eta(x)$ is a new field replacing $\Phi_3(x)$, and obtain the new expression for the potential

$$\tilde{\mathcal{V}} = \frac{1}{2}(-2m^2)\eta^2 + \lambda v\eta(\Phi_1^2 + \Phi_2^2 + \eta^2) + \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2 + \eta^2)^2 - \frac{\lambda}{4}v^4. \tag{3.18}$$

[5]We say, somewhat loosely, that T_1 and T_2 do not annihilate the ground state or, equivalently, finite group elements generated by T_1 and T_2 do not leave the ground state invariant. This should become clearer later on.

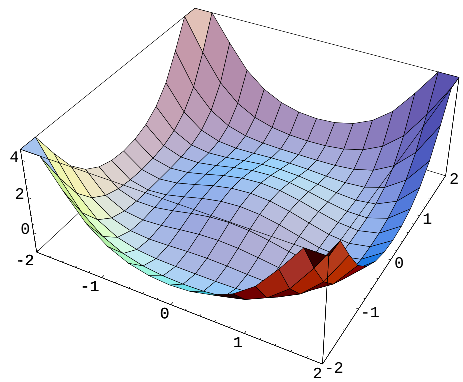


Figure 3.6:
Two-dimensional rotationally invariant potential:
$\mathcal{V}(x, y) = -(x^2 + y^2) + \frac{(x^2+y^2)^2}{4}$.

Upon inspection of the terms quadratic in the fields, one finds after spontaneous symmetry breaking two massless Goldstone bosons and one massive boson:

$$\begin{aligned} m^2_{\Phi_1} = m^2_{\Phi_2} &= 0, \\ m^2_\eta &= -2m^2. \end{aligned} \tag{3.19}$$

The model-independent feature of the above example is given by the fact that for each of the two generators T_1 and T_2 which do not annihilate the ground state one obtains a *massless* Goldstone boson. By means of a two-dimensional simplification (see the "Mexican hat" potential shown in Fig. 3.6) the mechanism at hand can easily be visualized. Infinitesimal variations orthogonal to the circle of the minimum of the potential generate quadratic terms, i.e., "restoring forces linear in the displacement," whereas tangential variations experience restoring forces only of higher orders.

Now let us generalize the model to the case of an arbitrary compact Lie group G of order n_G resulting in n_G infinitesimal generators.[6] Once again, we start from a Lagrangian of the form [Gol+ 62]

$$\mathcal{L}(\vec{\Phi}, \partial_\mu \vec{\Phi}) = \frac{1}{2} \partial_\mu \vec{\Phi} \cdot \partial^\mu \vec{\Phi} - \mathcal{V}(\vec{\Phi}), \tag{3.20}$$

[6]The restriction to compact groups allows for a complete decomposition into finite-dimensional irreducible unitary representations.

where $\vec{\Phi}$ is a multiplet of scalar (or pseudoscalar) Hermitian fields. The Lagrangian $\mathcal{L}$ and thus also $\mathcal{V}(\vec{\Phi})$ are supposed to be globally invariant under G, where the infinitesimal transformations of the fields are given by

$$g \in G: \quad \Phi_i \to \Phi_i + \delta\Phi_i, \quad \delta\Phi_i = -i\epsilon_a t^a_{ij}\Phi_j. \tag{3.21}$$

The Hermitian representation matrices $T^a = (t^a_{ij})$ are again antisymmetric and purely imaginary. We now assume that, by choosing an appropriate form of $\mathcal{V}$, the Lagrangian generates a spontaneous symmetry breaking resulting in a ground state with a vacuum expectation value $\vec{\Phi}_{\rm min} = \langle\vec{\Phi}\rangle$ which is invariant under a continuous subgroup H of G. We expand $\mathcal{V}(\vec{\Phi})$ with respect to $\vec{\Phi}_{\rm min}$, $|\vec{\Phi}_{\rm min}| = v$, i.e., $\vec{\Phi} = \vec{\Phi}_{\rm min} + \vec{\chi}$,

$$\mathcal{V}(\vec{\Phi}) = \mathcal{V}(\vec{\Phi}_{\rm min}) + \underbrace{\frac{\partial\mathcal{V}(\vec{\Phi}_{\rm min})}{\partial\Phi_i}}_{0}\chi_i + \frac{1}{2}\underbrace{\frac{\partial^2\mathcal{V}(\vec{\Phi}_{\rm min})}{\partial\Phi_i\partial\Phi_j}}_{m^2_{ij}}\chi_i\chi_j + \cdots. \tag{3.22}$$

The matrix $M^2 = (m^2_{ij})$ must be symmetric and, since one is expanding around a minimum, positive semidefinite, i.e.,

$$\sum_{i,j} m^2_{ij}x_i x_j \geq 0 \quad \forall \quad \vec{x}. \tag{3.23}$$

In that case, all eigenvalues of M^2 are nonnegative. Making use of the invariance of $\mathcal{V}$ under the symmetry group G,

$$\begin{aligned}\mathcal{V}(\vec{\Phi}_{\rm min}) &= \mathcal{V}(D(g)\vec{\Phi}_{\rm min}) = \mathcal{V}(\vec{\Phi}_{\rm min} + \delta\vec{\Phi}_{\rm min}) \\ &\overset{(3.22)}{=} \mathcal{V}(\vec{\Phi}_{\rm min}) + \frac{1}{2}m^2_{ij}\delta\Phi_{\rm min,i}\delta\Phi_{\rm min,j} + \cdots,\end{aligned} \tag{3.24}$$

one obtains, by comparing coefficients,

$$m^2_{ij}\delta\Phi_{\rm min,i}\delta\Phi_{\rm min,j} = 0. \tag{3.25}$$

Differentiating Eq. (3.25) with respect to $\delta\Phi_{\rm min,k}$ and using $m^2_{ij} = m^2_{ji}$ results in the matrix equation

$$M^2\delta\vec{\Phi}_{\rm min} = \vec{0}. \tag{3.26}$$

Inserting the variations of Eq. (3.21) for arbitrary ϵ_a, $\delta\vec{\Phi}_{\rm min} = -i\epsilon_a T^a\vec{\Phi}_{\rm min}$, we conclude

$$M^2 T^a\vec{\Phi}_{\rm min} = \vec{0}. \tag{3.27}$$

The solutions of Eq. (3.27) can be classified into two categories:

1. T^a, $a = 1, \cdots, n_H$, is a representation of an element of the Lie algebra belonging to the subgroup H of G, leaving the selected ground state invariant. In that case one has

$$T^a \vec{\Phi}_{\rm min} = \vec{0}, \quad a = 1, \cdots, n_H,$$

such that Eq. (3.27) is automatically satisfied without any knowledge of M^2.

2. T^a, $a = n_H + 1, \cdots, n_G$, is *not* a representation of an element of the Lie algebra belonging to the subgroup H. In that case $T^a \vec{\Phi}_{\rm min} \neq \vec{0}$, and $T^a \vec{\Phi}_{\rm min}$ is an eigenvector of M^2 with eigenvalue 0. To each such eigenvector corresponds a massless Goldstone boson. In particular, the different $T^a \vec{\Phi}_{\rm min} \neq \vec{0}$ are linearly independent, resulting in $n_G - n_H$ independent Goldstone bosons. (If they were not linearly independent, there would exist a nontrivial linear combination

$$\vec{0} = \sum_{a=n_H+1}^{n_G} c_a (T^a \vec{\Phi}_{\rm min}) = \underbrace{\left(\sum_{a=n_H+1}^{n_G} c_a T^a \right)}_{:= T} \vec{\Phi}_{\rm min},$$

such that T is an element of the Lie algebra of H in contradiction to our assumption.)

Let us check these results by reconsidering the example of Eq. (3.11). In that case $n_G = 3$ and $n_H = 1$, generating 2 Goldstone bosons [see Eq. (3.19)].

We conclude this section with two remarks. First, the number of Goldstone bosons is determined by the structure of the symmetry groups. Let G denote the symmetry group of the Lagrangian, with n_G generators and H the subgroup with n_H generators which leaves the ground state after spontaneous symmetry breaking invariant. For each generator which does not annihilate the vacuum one obtains a massless Goldstone boson, i.e., the total number of Goldstone bosons equals $n_G - n_H$. Second, the Lagrangians used in *motivating* the phenomenon of a spontaneous symmetry breakdown are typically constructed in such a fashion that the degeneracy of the ground states is built into the potential at the classical level (the prototype being the "Mexican hat" potential of Fig. 3.6). As in the above case, it is then argued that an *elementary* Hermitian field of a multiplet transforming non-trivially under the symmetry group G acquires a vacuum expectation value signaling a spontaneous symmetry breakdown. However, there also exist theories such as QCD where one cannot infer from inspection of the Lagrangian whether the theory exhibits spontaneous symmetry breaking. Rather, the criterion for spontaneous symmetry breaking is a non-vanishing vacuum expectation value of some Hermitian operator, not an elementary field, which is generated through the dynamics of the underlying theory. In particular, we will see that the quantities developing a vacuum expectation value may also be local Hermitian operators composed of more fundamental degrees of freedom of the theory. Such a possibility was already emphasized in the derivation of Goldstone's theorem in Ref. [Gol+ 62].

3.3 Goldstone's Theorem

By means of the above example, we motivate another approach to Goldstone's theorem without delving into all the subtleties of a quantum field-theoretical approach [Ber 74]. Given a Hamilton operator with a global symmetry group $G = \mathrm{SO}(3)$, let $\vec{\Phi}(x) = (\Phi_1(x), \Phi_2(x), \Phi_3(x))$ denote a triplet of local Hermitian operators transforming as a vector under G,

$$\begin{aligned} g \in G: \qquad \vec{\Phi}(x) \mapsto \vec{\Phi}'(x) &= e^{i\sum_{k=1}^{3}\alpha_k Q_k}\vec{\Phi}(x)e^{-i\sum_{l=1}^{3}\alpha_l Q_l} \\ &= e^{-i\sum_{k=1}^{3}\alpha_k T_k}\vec{\Phi}(x) \neq \vec{\Phi}(x), \end{aligned} \tag{3.28}$$

where the Q_i are the generators of the SO(3) transformations on the Hilbert space satisfying $[Q_i, Q_j] = i\epsilon_{ijk}Q_k$ and the $T_i = (t^i_{jk})$ are the matrices of the three dimensional representation satisfying $t^i_{jk} = -i\epsilon_{ijk}$. We assume that one component of the multiplet acquires a non-vanishing vacuum expectation value:

$$\langle 0|\Phi_1(x)|0\rangle = \langle 0|\Phi_2(x)|0\rangle = 0, \quad \langle 0|\Phi_3(x)|0\rangle = v \neq 0. \tag{3.29}$$

Then the two generators Q_1 and Q_2 do not annihilate the ground state, and to each such generator corresponds a massless Goldstone boson.

In order to prove these two statements let us expand Eq. (3.28) to first order in the α_k:

$$\vec{\Phi}' = \vec{\Phi} + i\sum_{k=1}^{3}\alpha_k[Q_k, \vec{\Phi}] = (1 - i\sum_{k=1}^{3}\alpha_k T_k)\vec{\Phi} = \vec{\Phi} + \vec{\alpha} \times \vec{\Phi}.$$

Comparing the terms linear in the α_k

$$i[\alpha_k Q_k, \Phi_l] = \epsilon_{lkm}\alpha_k \Phi_m$$

and noting that all three α_k can be chosen independently, we obtain

$$i[Q_k, \Phi_l] = -\epsilon_{klm}\Phi_m,$$

which, of course, simply expresses the fact that the field operators Φ_i transform as a vector. Using $\epsilon_{klm}\epsilon_{kln} = 2\delta_{mn}$, we find

$$-\frac{i}{2}\epsilon_{kln}[Q_k, \Phi_l] = \delta_{mn}\Phi_m = \Phi_n.$$

In particular,

$$\Phi_3 = -\frac{i}{2}([Q_1, \Phi_2] - [Q_2, \Phi_1]), \tag{3.30}$$

with cyclic permutations for the other two cases.

In order to prove that Q_1 and Q_2 do not annihilate the ground state, let us consider Eq. (3.28) for $\vec{\alpha} = (0, \pi/2, 0)$,

$$\begin{aligned} e^{-i\frac{\pi}{2}T_2}\vec{\Phi} &= \begin{pmatrix} \cos(\pi/2) & 0 & \sin(\pi/2) \\ 0 & 1 & 0 \\ -\sin(\pi/2) & 0 & \cos(\pi/2) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \begin{pmatrix} \Phi_3 \\ \Phi_2 \\ -\Phi_1 \end{pmatrix} \\ &= e^{i\frac{\pi}{2}Q_2} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} e^{-i\frac{\pi}{2}Q_2}. \end{aligned}$$

From the first row we obtain

$$\Phi_3 = e^{i\frac{\pi}{2}Q_2}\Phi_1 e^{-i\frac{\pi}{2}Q_2}.$$

Taking the vacuum expectation value

$$v = \langle 0|e^{i\frac{\pi}{2}Q_2}\Phi_1 e^{-i\frac{\pi}{2}Q_2}|0\rangle$$

and using Eq. (3.29) clearly $Q_2|0\rangle \neq 0$, since otherwise the exponential operator could be replaced by unity and the right-hand side would vanish. A similar argument shows $Q_1|0\rangle \neq 0$.

At this point let us make two remarks. The “states” $Q_{1(2)}|0\rangle$ cannot be normalized. In a more rigorous derivation one makes use of integrals of the form

$$\int d^3x \langle 0|[J^{0,b}(\vec{x},t),\Phi_c(0)]|0\rangle,$$

and first determines the commutator before evaluating the integral [Ber 74]. Some derivations of Goldstone's theorem right away start by assuming $Q_{1(2)}|0\rangle \neq 0$. However, for the discussion of spontaneous symmetry breaking in the framework of QCD it is advantageous to establish the connection between the existence of Goldstone bosons and a non-vanishing expectation value.

Let us now turn to the existence of Goldstone bosons, taking the vacuum expectation value of Eq. (3.30)

$$0 \neq v = \langle 0|\Phi_3(0)|0\rangle = -\frac{i}{2}\langle 0|\left([Q_1,\Phi_2(0)] - [Q_2,\Phi_1(0)]\right)|0\rangle \equiv -\frac{i}{2}(A-B).$$

We will first show $A = -B$. To that end we perform a rotation of the fields as well as the generators by $\pi/2$ about the 3 axis [see Eq. (3.28) with $\vec{\alpha} = (0,0,\pi/2)$]:

$$e^{-i\frac{\pi}{2}T_3}\vec{\Phi} \quad = \quad \begin{pmatrix} -\Phi_2 \\ \Phi_1 \\ \Phi_3 \end{pmatrix} = e^{i\frac{\pi}{2}Q_3} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} e^{-i\frac{\pi}{2}Q_3},$$

and analogously for the charge operators

$$\begin{pmatrix} -Q_2 \\ Q_1 \\ Q_3 \end{pmatrix} = e^{i\frac{\pi}{2}Q_3} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} e^{-i\frac{\pi}{2}Q_3}.$$

We thus obtain

$$\begin{aligned} B = \langle 0|[Q_2,\Phi_1(0)]|0\rangle \quad &= \quad \langle 0|\Big(e^{i\frac{\pi}{2}Q_3}(-Q_1)\underbrace{e^{-i\frac{\pi}{2}Q_3}e^{i\frac{\pi}{2}Q_3}}_{1}\Phi_2(0)e^{-i\frac{\pi}{2}Q_3} \\ &\quad -e^{i\frac{\pi}{2}Q_3}\Phi_2(0)e^{-i\frac{\pi}{2}Q_3}e^{i\frac{\pi}{2}Q_3}(-Q_1)e^{-i\frac{\pi}{2}Q_3}\Big)|0\rangle \\ &= \quad -\langle 0|[Q_1,\Phi_2(0)]|0\rangle = -A, \end{aligned}$$

where we made use of $Q_3|0\rangle = 0$, i.e., the vacuum is invariant under rotations about the 3 axis. In other words, the non-vanishing vacuum expectation value

v can also be written as

$$\begin{aligned} 0 \neq v &= \langle 0|\Phi_3(0)|0\rangle = -i\langle 0|[Q_1, \Phi_2(0)]|0\rangle \\ &= -i\int d^3x \langle 0|[J_0^1(\vec{x},t), \Phi_2(0)]|0\rangle. \end{aligned} \tag{3.31}$$

We insert a complete set of states $1 = \sum\!\!\!\!\!\!\int_n |n\rangle\langle n|$ into the commutator[7]

$$v = -i \sum\!\!\!\!\!\!\!\int_n \int d^3x \left(\langle 0|J_0^1(\vec{x},t)|n\rangle\langle n|\Phi_2(0)|0\rangle - \langle 0|\Phi_2(0)|n\rangle\langle n|J_0^1(\vec{x},t)|0\rangle\right),$$

and make use of translational invariance

$$\begin{aligned} &= -i \sum\!\!\!\!\!\!\!\int_n \int d^3x \left(e^{-iP_n\cdot x}\langle 0|J_0^1(0)|n\rangle\langle n|\Phi_2(0)|0\rangle - \cdots\right) \\ &= -i \sum\!\!\!\!\!\!\!\int_n (2\pi)^3\delta^3(\vec{P}_n)\left(e^{-iE_nt}\langle 0|J_0^1(0)|n\rangle\langle n|\Phi_2(0)|0\rangle \right. \\ &\quad \left. -e^{iE_nt}\langle 0|\Phi_2(0)|n\rangle\langle n|J_0^1(0)|0\rangle\right). \end{aligned}$$

Integration with respect to the momentum of the inserted intermediate states yields an expression of the form

$$= -i(2\pi)^3 \sum_n{}' \left(e^{-iE_nt}\cdots - e^{iE_nt}\cdots\right),$$

where the prime indicates that only states with $\vec{P} = 0$ need to be considered. Due to the Hermiticity of the symmetry current operators $J^{\mu,a}$ as well as the Φ_l, we have

$$c_n := \langle 0|J_0^1(0)|n\rangle\langle n|\Phi_2(0)|0\rangle = \langle n|J_0^1(0)|0\rangle^*\langle 0|\Phi_2(0)|n\rangle^*,$$

such that

$$v = -i(2\pi)^3 \sum_n{}' \left(c_n e^{-iE_nt} - c_n^* e^{iE_nt}\right). \tag{3.32}$$

From Eq. (3.32) we draw the following conclusions.

[7] The abbreviation $\sum\!\!\!\!\!\!\int_n |n\rangle\langle n|$ includes an integral over the total momentum $\vec{p}$ as well as all other quantum numbers necessary to fully specify the states.

1. Due to our assumption of a non-vanishing vacuum expectation value v, there must exist states $|n\rangle$ for which both $\langle 0|J^0_{1(2)}(0)|n\rangle$ and $\langle n|\Phi_{1(2)}(0)|0\rangle$ do not vanish. The vacuum itself cannot contribute to Eq. (3.32) because $\langle 0|\Phi_{1(2)}(0)|0\rangle = 0$.

2. States with $E_n > 0$ contribute (φ_n is the phase of c_n)

$$\begin{aligned}\frac{1}{i}\left(c_n e^{-iE_nt} - c_n^* e^{iE_nt}\right) &= \frac{1}{i}|c_n|\left(e^{i\varphi_n}e^{-iE_nt} - e^{-i\varphi_n}e^{iE_nt}\right)\\ &= 2|c_n|\sin(\varphi_n - E_nt)\end{aligned}$$

   to the sum. However, v is time-independent and therefore the sum over states with $(E_n > 0, \vec{0})$ must vanish.

3. The right-hand side of Eq. (3.32) must therefore contain the contribution from states with zero energy as well as zero momentum thus zero mass. These zero-mass states are the Goldstone bosons.

3.4 Explicit Symmetry Breaking: A First Look

Finally, let us illustrate the consequences of adding to our Lagrangian of Eq. (3.11) a small perturbation which *explicitly* breaks the symmetry. To that end, we modify the potential of Eq. (3.11) by adding a term $a\Phi_3$,

$$\mathcal{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2 + a\Phi_3, \qquad (3.33)$$

where $m^2 < 0$, $\lambda > 0$, and $a > 0$, with Hermitian fields Φ_i. Clearly, the potential no longer has the original O(3) symmetry but is only invariant under O(2). The conditions for the new minimum, obtained from $\vec{\nabla}_\Phi \mathcal{V} = 0$, read

$$\Phi_1 = \Phi_2 = 0, \quad \lambda\Phi_3^3 + m^2\Phi_3 + a = 0.$$

Let us solve the cubic equation for Φ_3 using the perturbative ansatz

$$\langle\Phi_3\rangle = \Phi_3^{(0)} + a\Phi_3^{(1)} + \mathcal{O}(a^2), \qquad (3.34)$$

from which we obtain

$$\Phi_3^{(0)} = \pm\sqrt{-\frac{m^2}{\lambda}}, \quad \Phi_3^{(1)} = \frac{1}{2m^2}.$$

Of course, $\Phi_3^{(0)}$ corresponds to our result without explicit perturbation. The condition for a *minimum* [see Eq. (3.23)] excludes $\Phi_3^{(0)} = +\sqrt{-\frac{m^2}{\lambda}}$. Expanding the potential with $\Phi_3 = \langle\Phi_3\rangle + \eta$ we obtain, after a short calculation, for the masses

$$\begin{aligned} m_{\Phi_1}^2 = m_{\Phi_2}^2 &= a\sqrt{\frac{\lambda}{-m^2}}, \\ m_\eta^2 &= -2m^2 + 3a\sqrt{\frac{\lambda}{-m^2}}. \end{aligned} \tag{3.35}$$

The important feature here is that the original Goldstone bosons of Eq. (3.19) are now massive. The squared masses are proportional to the symmetry breaking parameter a. Calculating *quantum* corrections to observables in terms of Goldstone-boson loop diagrams will generate corrections which are non-analytic in the symmetry breaking parameter such as $a\ln(a)$ [LP 71]. Such so-called chiral logarithms originate from the mass terms in the Goldstone boson propagators entering the calculation of loop integrals. We will come back to this point in Chapter 4 when we discuss the masses of the pseudoscalar octet in terms of the quark masses which, in QCD, represent the analogue to the parameter a in the above example.

Chapter 4

Chiral Perturbation Theory for Mesons

Chiral perturbation theory provides a systematic method for discussing the consequences of the global flavor symmetries of QCD at low energies by means of an *effective field theory*. The effective Lagrangian is expressed in terms of those hadronic degrees of freedom which, at low energies, show up as observable asymptotic states. At very low energies these are just the members of the pseudoscalar octet (π, K, η) which are regarded as the Goldstone bosons of the *spontaneous* breaking of the chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry down to $\mathrm{SU}(3)_V$. The non-vanishing masses of the light pseudoscalars in the "real" world are related to the explicit symmetry breaking in QCD due to the light quark masses.

We will first consider the indications for a spontaneous breakdown of chiral symmetry in QCD and then, in quite general terms, discuss the transformation properties of Goldstone bosons under the symmetry groups of the Lagrangian and the ground state, respectively. This will lead us to the concept of a nonlinear realization of a symmetry. After introducing the lowest-order effective Lagrangian relevant to the spontaneous breakdown from $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ to $\mathrm{SU}(3)_V$, we will illustrate how Weinberg's power counting scheme allows for a systematic classification of Feynman diagrams in the so-called momentum expansion. We will then outline the principles entering the construction of the effective Lagrangian and discuss how, at lowest order, the results of current algebra are reproduced. After presenting the Lagrangian of Gasser and Leutwyler and the Wess-Zumino-Witten action we will discuss some applications at chiral order $\mathcal{O}(p^4)$. We will conclude the presentation of the mesonic sector with referring to some selected examples at $\mathcal{O}(p^6)$.

4.1 Spontaneous Symmetry Breaking in QCD

While the toy model of Sec. 3.2 by construction led to a spontaneous symmetry breaking, it is not fully understood theoretically why QCD should exhibit this phenomenon [JW 00]. We will first motivate why experimental input, the hadron spectrum of the "real" world, indicates that spontaneous symmetry breaking happens in QCD. Secondly, we will show that a non-vanishing singlet scalar quark condensate is a sufficient condition for a spontaneous symmetry breaking in QCD.

4.1.1 The Hadron Spectrum

We saw in Sec. 2.3 that the QCD Lagrangian possesses a $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ symmetry in the chiral limit in which the light quark masses vanish. From symmetry considerations involving the Hamiltonian H^0_{QCD} only, one would naively expect that hadrons organize themselves into approximately degenerate multiplets fitting the dimensionalities of irreducible representations of the group $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$. The $\mathrm{U}(1)_V$ symmetry results in baryon number conservation[1] and leads to a classification of hadrons into mesons ($B = 0$) and baryons ($B = 1$). The linear combinations $Q^a_V = Q^a_R + Q^a_L$ and $Q^a_A = Q^a_R - Q^a_L$ of the left- and right-handed charge operators commute with H^0_{QCD}, have opposite parity, and thus for any state of positive parity one would expect the existence of a degenerate state of negative parity (parity doubling) which can be seen as follows. Let $|i,+\rangle$ denote an eigenstate of H^0_{QCD} with eigenvalue E_i,

$$H^0_{\mathrm{QCD}}|i,+\rangle = E_i|i,+\rangle,$$

having positive parity,

$$P|i,+\rangle = +|i,+\rangle,$$

[1] See Ref. [Gro+ 00] for empirical limits on nucleon decay as well as baryon-number violating Z and τ decays.

such as, e.g., a member of the ground state baryon octet (in the chiral limit). Defining $|\phi\rangle = Q_A^a|i,+\rangle$, because of $[H^0_{\rm QCD}, Q_A^a] = 0$, we have

$$H^0_{\rm QCD}|\phi\rangle = H^0_{\rm QCD}Q_A^a|i,+\rangle = Q_A^a H^0_{\rm QCD}|i,+\rangle = E_i Q_A^a|i,+\rangle = E_i|\phi\rangle,$$

i.e, the new state $|\phi\rangle$ is also an eigenstate of $H^0_{\rm QCD}$ with the same eigenvalue E_i but of opposite parity:

$$P|\phi\rangle = PQ_A^a P^{-1}P|i,+\rangle = -Q_A^a(+|i,+\rangle) = -|\phi\rangle.$$

The state $|\phi\rangle$ can be expanded in terms of the members of the multiplet with negative parity,

$$|\phi\rangle = Q_A^a|i,+\rangle = -t^a_{ij}|j,-\rangle.$$

However, the low-energy spectrum of baryons does not contain a degenerate baryon octet of negative parity. Naturally the question arises whether the above chain of arguments is incomplete. Indeed, we have tacitly assumed that the ground state of QCD is annihilated by Q_A^a.

Let $a_i^\dagger$ symbolically denote an operator which creates quanta with the quantum numbers of the state $|i,+\rangle$, whereas $b_j^\dagger$ creates degenerate quanta of opposite parity. Let us assume the states $|i,+\rangle$ and $|j,-\rangle$ to be members of a basis of an irreducible representation of $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$. In analogy to Eq. (2.49), we assume that under $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ the creation operators are related by

$$[Q_A^a, a_i^\dagger] = -t^a_{ij}b_j^\dagger.$$

The usual chain of arguments then works as

$$Q_A^a|i,+\rangle \;=\; Q_A^a a_i^\dagger|0\rangle = \Big([Q_A^a, a_i^\dagger] + a_i^\dagger \underbrace{Q_A^a}_{\hookrightarrow 0}\Big)|0\rangle = -t^a_{ij}b_j^\dagger|0\rangle. \qquad (4.1)$$

However, if the ground state is *not* annihilated by Q_A^a, the reasoning of Eq. (4.1) does no longer apply.

Two empirical facts about the hadron spectrum suggest that a spontaneous symmetry breaking happens in the chiral limit of QCD. First, SU(3) instead of $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ is approximately realized as a symmetry of the hadrons. Second, the octet of the pseudoscalar mesons is special in the sense that the masses of its members are small in comparison with the corresponding 1^- vector mesons. They are candidates for the Goldstone bosons of a spontaneous symmetry breaking.

In order to understand the origin of the SU(3) symmetry let us consider the vector charges $Q_V^a = Q_R^a + Q_L^a$ [see Eq. (2.59)]. They satisfy the commutation relations of an SU(3) Lie algebra [see Eqs. (2.76) - (2.78)],

$$[Q_R^a + Q_L^a, Q_R^b + Q_L^b] = [Q_R^a, Q_R^b] + [Q_L^a, Q_L^b] = if_{abc}Q_R^c + if_{abc}Q_L^c = if_{abc}Q_V^c. \tag{4.2}$$

In Ref. [VW 84] it was shown that, in the chiral limit, the ground state is necessarily invariant under $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$, i.e., the eight vector charges Q_V^a as well as the baryon number operator[2] $Q_V/3$ annihilate the ground state,

$$Q_V^a|0\rangle = Q_V|0\rangle = 0. \tag{4.3}$$

If the vacuum is invariant under $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$, then so is the Hamiltonian [Col 66] (but not vice versa). Moreover, the invariance of the ground state *and* the Hamiltonian implies that the physical states of the spectrum of H^0_{QCD} can be organized according to irreducible representations of $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$. The index V (for vector) indicates that the generators result from integrals of the zeroth component of vector current operators and thus transform with a positive sign under parity.

Let us now turn to the linear combinations $Q_A^a = Q_R^a - Q_L^a$ satisfying the commutation relations [see Eqs. (2.76) - (2.78)]

$$\begin{aligned}
[Q_A^a, Q_A^b] &= [Q_R^a - Q_L^a, Q_R^b - Q_L^b] = [Q_R^a, Q_R^b] + [Q_L^a, Q_L^b] \\
&= if_{abc}Q_R^c + if_{abc}Q_L^c = if_{abc}Q_V^c, \\
[Q_V^a, Q_A^b] &= [Q_R^a + Q_L^a, Q_R^b - Q_L^b] = [Q_R^a, Q_R^b] - [Q_L^a, Q_L^b] \\
&= if_{abc}Q_R^c - if_{abc}Q_L^c = if_{abc}Q_A^c.
\end{aligned} \tag{4.4}$$

Note that these charge operators do *not* form a closed algebra, i.e., the commutator of two axial charge operators is not again an axial charge operator. Since the parity doubling is not observed for the low-lying states, one assumes that the Q_A^a do *not* annihilate the ground state,

$$Q_A^a|0\rangle \neq 0, \tag{4.5}$$

i.e., the ground state of QCD is not invariant under "axial" transformations. According to Goldstone's theorem [Nam 60, NJ 61a, NJ 61b, Gol 61, Gol+ 62], to each axial generator Q_A^a, which does not annihilate the ground

[2] Recall that each quark is assigned a baryon number 1/3.

state, corresponds a massless Goldstone boson field $\phi^a(x)$ with spin 0, whose symmetry properties are tightly connected to the generator in question. The Goldstone bosons have the same transformation behavior under parity,

$$\phi^a(\vec{x},t) \overset{P}{\mapsto} -\phi^a(-\vec{x},t), \tag{4.6}$$

i.e., they are pseudoscalars, and transform under the subgroup $H = \mathrm{SU(3)}_V$, which leaves the vacuum invariant, as an octet [see Eq. (4.4)]:

$$[Q_V^a, \phi^b(x)] = if_{abc}\phi^c(x). \tag{4.7}$$

In the present case, $G = \mathrm{SU(3)}_L \times \mathrm{SU(3)}_R$ with $n_G = 16$ and $H = \mathrm{SU(3)}_V$ with $n_H = 8$ and we expect eight Goldstone bosons.

4.1.2 The Scalar Quark Condensate $\langle\bar{q}q\rangle$

In the following, we will show that a non-vanishing scalar quark condensate in the chiral limit is a sufficient (but not a necessary) condition for a spontaneous symmetry breaking in QCD.[3] The subsequent discussion will parallel that of the toy model in Sec. 3.3 after replacement of the elementary fields Φ_i by appropriate composite Hermitian operators of QCD.

Let us first recall the definition of the nine scalar and pseudoscalar quark densities:

$$\begin{aligned} S_a(y) &= \bar{q}(y)\lambda_a q(y), \quad a = 0,\cdots,8, & (4.8)\\ P_a(y) &= i\bar{q}(y)\gamma_5\lambda_a q(y), \quad a = 0,\cdots,8. & (4.9) \end{aligned}$$

The equal-time commutation relation of two quark operators of the form $A_i(x) = q^\dagger(x)\hat{A}_i q(x)$, where $\hat{A}_i$ symbolically denotes Dirac- and flavor matrices and a summation over color indices is implied, can compactly be written as [see Eq. (2.75)]

$$[A_1(\vec{x},t), A_2(\vec{y},t)] = \delta^3(\vec{x}-\vec{y})q^\dagger(x)[\hat{A}_1,\hat{A}_2]q(x). \tag{4.10}$$

With the definition

$$Q_V^a(t) = \int d^3x q^\dagger(\vec{x},t)\frac{\lambda^a}{2}q(\vec{x},t),$$

[3] In this Section all physical quantities such as the ground state, the quark operators etc. are considered in the chiral limit.

and using

$$
\begin{aligned}
[\frac{\lambda_a}{2},\gamma_0\lambda_0] &= 0,\\
[\frac{\lambda_a}{2},\gamma_0\lambda_b] &= \gamma_0 i f_{abc}\lambda_c,
\end{aligned}
$$

we see, after integration of Eq. (4.10) over $\vec{x}$, that the scalar quark densities of Eq. (4.8) transform under $\mathrm{SU}(3)_V$ as a singlet and as an octet, respectively,

$$
\begin{aligned}
[Q_V^a(t),S_0(y)] &= 0,\quad a=1,\cdots,8, && (4.11)\\
[Q_V^a(t),S_b(y)] &= i\sum_{c=1}^{8} f_{abc}S_c(y),\quad a,b=1,\cdots,8, && (4.12)
\end{aligned}
$$

with analogous results for the pseudoscalar quark densities. In the $\mathrm{SU}(3)_V$ limit and, of course, also in the even more restrictive chiral limit, the charge operators in Eqs. (4.11) and (4.12) are actually time independent.[4] Using the relation

$$\sum_{a,b=1}^{8} f_{abc}f_{abd}=3\delta_{cd} \tag{4.13}$$

for the structure constants of SU(3), we re-express the octet components of the scalar quark densities as

$$S_a(y)=-\frac{i}{3}\sum_{b,c=1}^{8} f_{abc}[Q_V^b(t),S_c(y)], \tag{4.14}$$

which represents the analogue of Eq. (3.30) in the discussion of Goldstone's theorem.

In the chiral limit the ground state is necessarily invariant under $\mathrm{SU}(3)_V$ [VW 84], i.e., $Q_V^a|0\rangle=0$, and we obtain from Eq. (4.14)

$$\langle 0|S_a(y)|0\rangle=\langle 0|S_a(0)|0\rangle\equiv\langle S_a\rangle=0,\quad a=1,\cdots,8, \tag{4.15}$$

where we made use of translational invariance of the ground state. In other words, the octet components of the scalar quark condensate *must* vanish in the chiral limit. From Eq. (4.15), we obtain for $a=3$

$$\langle\bar{u}u\rangle-\langle\bar{d}d\rangle=0,$$

[4] The commutation relations also remain valid for *equal* times if the symmetry is explicitly broken.

i.e. $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle$ and for $a = 8$

$$\langle\bar{u}u\rangle + \langle\bar{d}d\rangle - 2\langle\bar{s}s\rangle = 0,$$

i.e. $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle = \langle\bar{s}s\rangle$.

Because of Eq. (4.11) a similar argument cannot be used for the singlet condensate, and if we assume a non-vanishing singlet scalar quark condensate in the chiral limit, we thus find using Eq. (4.15)

$$0 \neq \langle\bar{q}q\rangle = \langle\bar{u}u + \bar{d}d + \bar{s}s\rangle = 3\langle\bar{u}u\rangle = 3\langle\bar{d}d\rangle = 3\langle\bar{s}s\rangle. \tag{4.16}$$

Finally, we make use of (no summation implied!)

$$(i)^2[\gamma_5\frac{\lambda_a}{2}, \gamma_0\gamma_5\lambda_a] = \lambda_a^2\gamma_0$$

in combination with

$$\begin{aligned}
\lambda_1^2 = \lambda_2^2 = \lambda_3^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4^2 = \lambda_5^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\lambda_6^2 = \lambda_7^2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\lambda_8^2 &= \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\end{aligned}$$

to obtain

$$i[Q_a^A(t), P_a(y)] = \begin{cases} \bar{u}u + \bar{d}d, & a = 1, 2, 3 \\ \bar{u}u + \bar{s}s, & a = 4, 5 \\ \bar{d}d + \bar{s}s, & a = 6, 7 \\ \frac{1}{3}(\bar{u}u + \bar{d}d + 4\bar{s}s), & a = 8 \end{cases} \tag{4.17}$$

where we have suppressed the y dependence on the right-hand side. We evaluate Eq. (4.17) for a ground state which is invariant under $\mathrm{SU}(3)_V$, assuming a non-vanishing singlet scalar quark condensate,

$$\langle 0|i[Q_a^A(t), P_a(y)]|0\rangle = \frac{2}{3}\langle\bar{q}q\rangle, \quad a = 1, \cdots, 8, \tag{4.18}$$

where, because of translational invariance, the right-hand side is independent of y. Inserting a complete set of states into the commutator of Eq. (4.18) yields, in complete analogy to Sec. 3.3 [see the discussion following Eq. (3.31)] that both the pseudoscalar density $P_a(y)$ as well as the axial charge operators Q_A^a must have a non-vanishing matrix element between the vacuum and massless one particle states $|\phi^b\rangle$. In particular, because of Lorentz covariance, the matrix element of the axial-vector current operator between the vacuum and these massless states, appropriately normalized, can be written as

$$\langle 0|A_\mu^a(0)|\phi^b(p)\rangle = ip_\mu F_0 \delta^{ab}, \tag{4.19}$$

where $F_0 \approx 93$ MeV denotes the "decay" constant of the Goldstone bosons in the chiral limit. Assuming $Q_A^a|0\rangle \neq 0$, a non-zero value of F_0 is a necessary and sufficient criterion for spontaneous chiral symmetry breaking. On the other hand, because of Eq. (4.18) a non-vanishing scalar quark condensate $\langle \bar{q}q\rangle$ is a sufficient (but not a necessary) condition for a spontaneous symmetry breakdown in QCD.

Table 4.1 contains a summary of the patterns of spontaneous symmetry breaking as discussed in Sec. 3.3, the generalization of Sec. 3.2 to the so-called O(N) linear sigma model, and QCD.

4.2 Transformation Properties of the Goldstone Bosons

The purpose of this section is to discuss the transformation properties of the field variables describing the Goldstone bosons [Wei 68, Col+ 69, Cal+ 69, Bal+ 91, Leu 92]. We will need the concept of a *nonlinear realization* of a group in addition to a *representation* of a group which one usually encounters in Physics. We will first discuss a few general group-theoretical properties before specializing to QCD.

4.2.1 General Considerations

Let us consider a physical system with a Hamilton operator $\hat{H}$ which is invariant under a compact Lie group G. Furthermore we assume the ground state of the system to be invariant under only a subgroup H of G, giving rise to $n = n_G - n_H$ Goldstone bosons. Each of these Goldstone bosons will be

	Sec. 3.3	O(N) linear sigma model	QCD
Symmetry group G of the Lagrangian density	O(3)	O(N)	$\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$
Number of generators n_G	3	$N(N-1)/2$	16
Symmetry group H of the ground state	O(2)	O($N-1$)	$\mathrm{SU}(3)_V$
Number of generators n_H	1	$(N-1)(N-2)/2$	8
Number of Goldstone bosons $n_G - n_H$	2	$N-1$	8
Multiplet of Goldstone boson fields	$(\Phi_1(x), \Phi_2(x))$	$(\Phi_1(x), \cdots, \Phi_{N-1}(x))$	$i\bar{q}(x)\gamma_5\lambda_a q(x)$
Vacuum expectation value	$v = \langle\Phi_3\rangle$	$v = \langle\Phi_N\rangle$	$v = \langle\bar{q}q\rangle$

Table 4.1: Comparison of spontaneous symmetry breaking.

described by an independent field ϕ_i which is a continuous real function on Minkowski space M^4.[5] We collect these fields in an n-component vector Φ and define the vector space

$$M_1 \equiv \{\Phi : M^4 \to R^n | \phi_i : M^4 \to R \text{ continuous}\}. \tag{4.20}$$

Our aim is to find a mapping φ which uniquely associates with each pair $(g, \Phi) \in G \times M_1$ an element $\varphi(g, \Phi) \in M_1$ with the following properties:

$$\varphi(e, \Phi) = \Phi \;\forall\; \Phi \in M_1,\; e \text{ identity of } G, \tag{4.21}$$

$$\varphi(g_1, \varphi(g_2, \Phi)) = \varphi(g_1 g_2, \Phi) \;\forall\; g_1, g_2 \in G, \forall\, \Phi \in M_1. \tag{4.22}$$

Such a mapping defines an *operation* of the group G on M_1. The second condition is the so-called group-homomorphism property [BT 84, O'Ra 86, Jon 90]. The mapping will, in general, *not* define a *representation* of the group G, because we do not require the mapping to be linear, i.e., $\varphi(g, \lambda\Phi) \neq \lambda\varphi(g, \Phi)$.

Let $\Phi = 0$ denote the "origin" of M_1 [Leu 92] which, in a theory containing Goldstone bosons only, loosely speaking corresponds to the ground state configuration. Since the ground state is supposed to be invariant under the subgroup H we require the mapping φ to be such that all elements $h \in H$ map the origin onto itself. In this context the subgroup H is also known as the little group of $\Phi = 0$. Given that such a mapping indeed exists, we need to verify for infinite groups that (see Chap. 2.4 of [Jon 90]):

1. H is not empty, because the identity e maps the origin onto itself.

2. If h_1 and h_2 are elements satisfying $\varphi(h_1, 0) = \varphi(h_2, 0) = 0$, so does $\varphi(h_1 h_2, 0) = \varphi(h_1, \varphi(h_2, 0)) = \varphi(h_1, 0) = 0$, i.e., because of the homomorphism property also the product $h_1 h_2 \in H$.

3. For $h \in H$ we have

$$\varphi(h^{-1}, 0) = \varphi(h^{-1}, \varphi(h, 0)) = \varphi(h^{-1}h, 0) = \varphi(e, 0).$$

i.e., $h^{-1} \in H$.

[5] Depending on the equations of motion, we will require more restrictive properties of the functions ϕ_i.

Following Ref. [Leu 92] we will establish a connection between the Goldstone boson fields and the set of all left cosets $\{gH|g \in G\}$ which is also referred to as the quotient G/H. For a subgroup H of G the set $gH = \{gh|h \in H\}$ defines the left coset of g (with an analogous definition for the right coset) which is one element of G/H.[6] For our purposes we need the property that cosets either completely overlap or are completely disjoint (see, e.g., [Jon 90]), i.e, the quotient is a set whose elements themselves are sets of group elements, and these sets are completely disjoint.

Let us first show that for all elements of a given coset, φ maps the origin onto the same vector in R^n:

$$\varphi(gh, 0) = \varphi(g, \varphi(h, 0)) = \varphi(g, 0) \;\forall\; g \in G \,\text{and}\, h \in H.$$

Secondly, the mapping is injective with respect to the cosets, which can be proven as follows. Consider two elements g and g' of G where $g' \notin gH$. We need to show $\varphi(g, 0) \neq \varphi(g', 0)$. Let us assume $\varphi(g, 0) = \varphi(g', 0)$:

$$0 = \varphi(e, 0) = \varphi(g^{-1}g, 0) = \varphi(g^{-1}, \varphi(g, 0)) = \varphi(g^{-1}, \varphi(g', 0)) = \varphi(g^{-1}g', 0).$$

However, this implies $g^{-1}g' \in H$ or $g' \in gH$ in contradiction to the assumption. Thus $\varphi(g, 0) = \varphi(g', 0)$ cannot be true. In other words, the mapping can be inverted on the image of $\varphi(g, 0)$. The conclusion is that there exists an *isomorphic mapping* between the quotient G/H and the Goldstone boson fields.[7]

Now let us discuss the transformation behavior of the Goldstone boson fields under an arbitrary $g \in G$ in terms of the isomorphism established above. To each Φ corresponds a coset $\tilde{g}H$ with appropriate $\tilde{g}$. Let $f = \tilde{g}h \in \tilde{g}H$ denote a representative of this coset such that

$$\Phi = \varphi(f, 0) = \varphi(\tilde{g}h, 0).$$

Now apply the mapping $\varphi(g)$ to Φ:

$$\varphi(g, \Phi) = \varphi(g, \varphi(\tilde{g}h, 0)) = \varphi(g\tilde{g}h, 0) = \varphi(f', 0) = \Phi', \quad f' \in g(\tilde{g}H).$$

[6] An invariant subgroup has the additional property that the left and right cosets coincide for each g which allows for a definition of the factor group G/H in terms of the complex product. However, here we do not need this property.

[7] Of course, the Goldstone boson fields are not constant vectors in R^n but functions on Minkowski space [see Eq. (4.20)]. This is accomplished by allowing the cosets gH to also depend on x.

In other words, in order to obtain the transformed Φ' from a given Φ we simply need to multiply the left coset $\tilde{g}H$ representing Φ by g in order to obtain the new left coset representing Φ'. This procedure uniquely determines the transformation behavior of the Goldstone bosons up to an appropriate choice of variables parameterizing the elements of the quotient G/H.

4.2.2 Application to QCD

Now let us apply the above general considerations to the specific case relevant to QCD and consider the group $G = \mathrm{SU}(N) \times \mathrm{SU}(N) = \{(L, R)|L \in \mathrm{SU}(N), R \in \mathrm{SU}(N)\}$ and $H = \{(V, V)|V \in \mathrm{SU}(N)\}$ which is isomorphic to $\mathrm{SU}(N)$. Let $\tilde{g} = (\tilde{L}, \tilde{R}) \in G$. We may uniquely characterize the left coset of $\tilde{g}$, $\tilde{g}H = \{(\tilde{L}V, \tilde{R}V)|V \in \mathrm{SU}(N)\}$, through the $\mathrm{SU}(N)$ matrix $U = \tilde{R}\tilde{L}^\dagger$ [Bal+ 91],

$$(\tilde{L}V, \tilde{R}V) = (\tilde{L}V, \tilde{R}\tilde{L}^\dagger\tilde{L}V) = (1, \tilde{R}\tilde{L}^\dagger)\underbrace{(\tilde{L}V, \tilde{L}V)}_{\in H}, \quad \text{i.e.} \quad \tilde{g}H = (1, \tilde{R}\tilde{L}^\dagger)H,$$

if we follow the convention that we choose the representative of the coset such that the unit matrix stands in its first argument. According to the above derivation, U is isomorphic to a Φ. The transformation behavior of U under $g = (L, R) \in G$ is obtained by multiplication in the left coset:

$$g\tilde{g}H = (L, R\tilde{R}\tilde{L}^\dagger)H = (1, R\tilde{R}\tilde{L}^\dagger L^\dagger)(L, L)H = (1, R(\tilde{R}\tilde{L}^\dagger)L^\dagger)H,$$

i.e.

$$U = \tilde{R}\tilde{L}^\dagger \mapsto U' = R(\tilde{R}\tilde{L}^\dagger)L^\dagger = RUL^\dagger. \tag{4.23}$$

As mentioned above, we finally need to introduce an x dependence so that

$$U(x) \mapsto RU(x)L^\dagger. \tag{4.24}$$

Let us now restrict ourselves to the physically relevant cases of $N = 2$ and $N = 3$ and define

$$M_1 \equiv \begin{cases} \{\Phi : M^4 \to R^3|\phi_i : M^4 \to R \text{ continuous}\} \text{ for } N = 2, \\ \{\Phi : M^4 \to R^8|\phi_i : M^4 \to R \text{ continuous}\} \text{ for } N = 3. \end{cases}$$

Furthermore let $\tilde{\mathcal{H}}(N)$ denote the set of all Hermitian and traceless $N \times N$ matrices,

$$\tilde{\mathcal{H}}(N) \equiv \{A \in \mathrm{gl}(N, C)|A^\dagger = A \wedge \mathrm{Tr}(A) = 0\},$$

which under addition of matrices defines a real vector space. We define a second set $M_2 \equiv \{\phi : M^4 \rightarrow \tilde{\mathcal{H}}(N) | \phi \text{ continuous}\}$, where the entries are continuous functions. For $N = 2$ the elements of M_1 and M_2 are related to each other according to

$$\phi(x) \;=\; \sum_{i=1}^{3} \tau_i \phi_i(x) = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix},$$

where the τ_i are the usual Pauli matrices and $\phi_i(x) = \frac{1}{2}\mathrm{Tr}[\tau_i \phi(x)]$. Analogously for $N = 3$,

$$\begin{aligned} \phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) \;&=\; \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix} \\ &\equiv\; \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}, \end{aligned}$$

with the Gell-Mann matrices λ_a and $\phi_a(x) = \frac{1}{2}\mathrm{Tr}[\lambda_a \phi(x)]$. Again, M_2 forms a real vector space. Let us finally define

$$M_3 \equiv \left\{ U : M^4 \rightarrow \mathrm{SU}(N) | U(x) = \exp\left(i\frac{\phi(x)}{F_0} \right), \phi \in M_2 \right\}.$$

At this point it is important to note that M_3 does not define a vector space because the sum of two $\mathrm{SU}(N)$ matrices is not an $\mathrm{SU}(N)$ matrix.

We are now in the position to discuss the so-called nonlinear realization of $\mathrm{SU}(N) \times \mathrm{SU}(N)$ on M_3. The homomorphism

$$\varphi : G \times M_3 \rightarrow M_3 \quad \text{with} \quad \varphi[(L, R), U](x) \equiv RU(x)L^\dagger,$$

defines an operation of G on M_3, because

1. $RUL^\dagger \in M_3$, since $U \in M_3$ and $R, L^\dagger \in \mathrm{SU}(N)$.

2. $\varphi[(1_{N\times N}, 1_{N\times N}), U](x) = 1_{N\times N} U(x) 1_{N\times N} = U(x)$.

3. Let $g_i = (L_i, R_i) \in G$ and thus $g_1 g_2 = (L_1 L_2, R_1 R_2) \in G$.

$$\begin{aligned}\varphi[g_1, \varphi[g_2, U]](x) &= \varphi[g_1, (R_2 U L_2^\dagger)](x) = R_1 R_2 U(x) L_2^\dagger L_1^\dagger, \\ \varphi[g_1 g_2, U](x) &= R_1 R_2 U(x)(L_1 L_2)^\dagger = R_1 R_2 U(x) L_2^\dagger L_1^\dagger.\end{aligned}$$

The mapping φ is called a nonlinear realization, because M_3 is *not* a vector space.

The origin $\phi(x) = 0$, i.e. $U_0 = 1$, denotes the ground state of the system. Under transformations of the subgroup $H = \{(V,V)|V \in \mathrm{SU}(N)\}$ corresponding to rotating both left- and right-handed quark fields in QCD by the same V, the ground state remains invariant,

$$\varphi[g = (V,V), U_0] = V U_0 V^\dagger = V V^\dagger = 1 = U_0.$$

On the other hand, under "axial transformations," i.e. rotating the left-handed quarks by A and the right-handed quarks by $A^\dagger$, the ground state does *not* remain invariant,

$$\varphi[g = (A, A^\dagger), U_0] = A^\dagger U_0 A^\dagger = A^\dagger A^\dagger \neq U_0,$$

which, of course, is consistent with the assumed spontaneous symmetry breakdown.

Let us finally discuss the transformation behavior of $\phi(x)$ under the subgroup $H = \{(V,V)|V \in \mathrm{SU}(N)\}$. Expanding

$$U = 1 + i\frac{\phi}{F_0} - \frac{\phi^2}{2F_0^2} + \cdots,$$

we immediately see that the realization restricted to the subgroup H,

$$1+i\frac{\phi}{F_0}-\frac{\phi^2}{2F_0^2}+\cdots \mapsto V(1+i\frac{\phi}{F_0}-\frac{\phi^2}{2F_0^2}+\cdots)V^\dagger = 1+i\frac{V\phi V^\dagger}{F_0}-\frac{V\phi V^\dagger V\phi V^\dagger}{2F_0^2}+\cdots, \tag{4.25}$$

defines a linear representation on $M_2 \ni \phi \mapsto V\phi V^\dagger \in M_2$, because

$$\begin{aligned}&(V\phi V^\dagger)^\dagger = V\phi V^\dagger, \quad \mathrm{Tr}(V\phi V^\dagger) = \mathrm{Tr}(\phi) = 0, \\ &V_1(V_2 \phi V_2^\dagger)V_1^\dagger = (V_1 V_2)\phi(V_1 V_2)^\dagger.\end{aligned}$$

Let us consider the SU(3) case and parameterize

$$V = \exp\left(-i\Theta_a^V \frac{\lambda_a}{2}\right),$$

from which we obtain, by comparing both sides of Eq. (4.25),

$$\phi = \lambda_b\phi_b \overset{h \in \mathrm{SU}(3)_V}{\mapsto} V\phi V^\dagger = \phi - i\Theta_a^V \underbrace{[\frac{\lambda_a}{2}, \phi_b\lambda_b]}_{\phi_b i f_{abc}\lambda_c} + \cdots = \phi + f_{abc}\Theta_a^V\phi_b\lambda_c + \cdots. \tag{4.26}$$

However, this corresponds exactly to the adjoint representation, i.e., in SU(3) the fields ϕ_a transforms as an octet which is also consistent with the transformation behavior we discussed in Eq. (4.7):

$$\begin{aligned} e^{i\Theta_a^V Q_V^a}\lambda_b\phi_b e^{-i\Theta_a^V Q_V^a} &= \lambda_b\phi_b + i\Theta_a^V\lambda_b \underbrace{[Q_V^a, \phi_b]}_{i f_{abc}\phi_c} + \cdots \\ &= \phi + f_{abc}\Theta_a^V\phi_b\lambda_c + \cdots. \end{aligned} \tag{4.27}$$

For group elements of G of the form $(A, A^\dagger)$ one may proceed in a completely analogous fashion. However, one finds that the fields ϕ_a do *not* have a simple transformation behavior under these group elements. In other words, the commutation relations of the fields with the *axial* charges are complicated nonlinear functions of the fields [Wei 68].

4.3 The Lowest-Order Effective Lagrangian

Our goal is the construction of the most general theory describing the dynamics of the Goldstone bosons associated with the spontaneous symmetry breakdown in QCD. In the chiral limit, we want the effective Lagrangian to be invariant under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$. It should contain exactly eight pseudoscalar degrees of freedom transforming as an octet under the subgroup $H = \mathrm{SU}(3)_V$. Moreover, taking account of spontaneous symmetry breaking, the ground state should only be invariant under $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$.

Following the discussion of Sec. 4.2.2 we collect the dynamical variables in the SU(3) matrix $U(x)$,

$$\begin{aligned} U(x) &= \exp\left(i\frac{\phi(x)}{F_0}\right), \\ \phi(x) &= \sum_{a=1}^{8}\lambda_a\phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \end{aligned} \tag{4.28}$$

The most general, chirally invariant, effective Lagrangian density with the minimal number of derivatives reads

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4}\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right), \tag{4.29}$$

where $F_0 \approx 93$ MeV is a free parameter which later on will be related to the pion decay $\pi^+ \to \mu^+\nu_\mu$ (see Sec. 4.6.1).

First of all, the Lagrangian is invariant under the *global* $\text{SU}(3)_L \times \text{SU}(3)_R$ transformations of Eq. (4.23):

$$\begin{aligned}
U &\mapsto RUL^\dagger, \\
\partial_\mu U &\mapsto \partial_\mu(RUL^\dagger) = \underbrace{\partial_\mu R}_{0} UL^\dagger + R\partial_\mu UL^\dagger + RU\underbrace{\partial_\mu L^\dagger}_{0} = R\partial_\mu UL^\dagger, \\
U^\dagger &\mapsto LU^\dagger R^\dagger, \\
\partial_\mu U^\dagger &\mapsto L\partial_\mu U^\dagger R^\dagger,
\end{aligned}$$

because

$$\mathcal{L}_{\text{eff}} \mapsto \frac{F_0^2}{4}\text{Tr}\Big(R\partial_\mu U \underbrace{L^\dagger L}_{1} \partial^\mu U^\dagger R^\dagger\Big) = \frac{F_0^2}{4}\text{Tr}\Big(\underbrace{R^\dagger R}_{1} \partial_\mu U \partial^\mu U^\dagger\Big) = \mathcal{L}_{\text{eff}},$$

where we made use of the trace property $\text{Tr}(AB) = \text{Tr}(BA)$. The global $\text{U}(1)_V$ invariance is trivially satisfied, because the Goldstone bosons have baryon number zero, thus transforming as $\phi \mapsto \phi$ under $\text{U}(1)_V$ which also implies $U \mapsto U$.

The substitution $\phi_a(\vec{x},t) \mapsto -\phi_a(\vec{x},t)$ or, equivalently, $U(\vec{x},t) \mapsto U^\dagger(\vec{x},t)$ provides a simple method of testing, whether an expression is of so-called even or odd *intrinsic* parity,[8] i.e., even or odd in the number of Goldstone boson fields. For example, it is easy to show, using the trace property, that the Lagrangian of Eq. (4.29) is even.

The purpose of the multiplicative constant $F_0^2/4$ in Eq. (4.29) is to generate the standard form of the kinetic term $\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a$, which can be seen by expanding the exponential $U = 1 + i\phi/F_0 + \cdots$, $\partial_\mu U = i\partial_\mu\phi/F_0 + \cdots$, resulting in

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4}\text{Tr}\left[\frac{i\partial_\mu\phi}{F_0}\left(-\frac{i\partial^\mu\phi}{F_0}\right)\right] + \cdots = \frac{1}{4}\text{Tr}(\lambda_a\partial_\mu\phi_a\lambda_b\partial^\mu\phi_b) + \cdots$$

[8] Since the Goldstone bosons are pseudoscalars, a true parity transformation is given by $\phi_a(\vec{x},t) \mapsto -\phi_a(-\vec{x},t)$ or, equivalently, $U(\vec{x},t) \mapsto U^\dagger(-\vec{x},t)$.

$$= \frac{1}{4}\partial_\mu\phi_a\partial^\mu\phi_b\mathrm{Tr}(\lambda_a\lambda_b) + \cdots = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \mathcal{L}_{\mathrm{int}},$$

where we made use of $\mathrm{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. In particular, since there are no other terms containing only two fields ($\mathcal{L}_{\mathrm{int}}$ starts with interaction terms containing at least four Goldstone bosons) the eight fields ϕ_a describe eight independent *massless* particles.[9]

A term of the type $\mathrm{Tr}[(\partial_\mu\partial^\mu U)U^\dagger]$ may be re-expressed as[10]

$$\mathrm{Tr}[(\partial_\mu\partial^\mu U)U^\dagger] = \partial_\mu[\mathrm{Tr}(\partial^\mu U U^\dagger)] - \mathrm{Tr}(\partial^\mu U\partial_\mu U^\dagger),$$

i.e., up to a total derivative it is proportional to the Lagrangian of Eq. (4.29). However, in the present context, total derivatives do not have a dynamical significance, i.e. they leave the equations of motion unchanged and can thus be dropped. The product of two invariant traces is excluded at lowest order, because $\mathrm{Tr}(\partial_\mu U U^\dagger) = 0$. Let us prove the general SU(N) case by considering an SU(N)-valued field

$$U = \exp\left(i\frac{\Lambda_a\phi_a(x)}{F_0}\right),$$

with $N^2 - 1$ Hermitian, traceless matrices Λ_a and real fields $\phi_a(x)$. Defining $\Phi = \Lambda_a\phi_a/F_0$, we expand the exponential

$$U = 1 + i\Phi + \frac{1}{2}(i\Phi)^2 + \frac{1}{3!}(i\Phi)^3 + \cdots$$

and consider the derivative[11]

$$\partial_\mu U = i\partial_\mu\Phi + \frac{1}{2}(i\partial_\mu\Phi i\Phi + i\Phi i\partial_\mu\Phi) + \frac{1}{3!}[i\partial_\mu\Phi(i\Phi)^2 + i\Phi i\partial_\mu\Phi i\Phi + (i\Phi)^2 i\partial_\mu\Phi] + \cdots.$$

We then find

$$\begin{aligned}
\mathrm{Tr}(\partial_\mu U U^\dagger) &= \mathrm{Tr}[i\partial_\mu\Phi U^\dagger + \frac{1}{2}(i\partial_\mu\Phi i\Phi + i\Phi i\partial_\mu\Phi)U^\dagger + \cdots] \\
&= \mathrm{Tr}[i\partial_\mu\Phi U^\dagger + i\partial_\mu\Phi i\Phi U^\dagger + \frac{1}{2}i\partial_\mu\Phi(i\Phi)^2 U^\dagger + \cdots] \\
&= \mathrm{Tr}(i\partial_\mu\Phi \underbrace{UU^\dagger}_{1}) = \mathrm{Tr}(i\partial_\mu\Phi) = i\partial_\mu\phi_a \underbrace{\mathrm{Tr}(\Lambda_a)}_{0} = 0, \quad (4.30)
\end{aligned}$$

[9] At this stage, this is only a tree-level argument. We will see in Sec. 4.9.1 that the Goldstone bosons remain massless in the chiral limit even when loop corrections have been included.

[10] In the present case $\mathrm{Tr}(\partial^\mu U U^\dagger) = 0$.

[11] Φ and $\partial_\mu\Phi$ are matrices which, in general, do not commute.

where we made use of $[\Phi, U^\dagger] = 0$.

Let us turn to the vector and axial-vector currents associated with the global $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry of the effective Lagrangian of Eq. (4.29). To that end, we parameterize

$$
\begin{aligned}
L &= \exp\left(-i\Theta_a^L \frac{\lambda_a}{2}\right), && (4.31)\\
R &= \exp\left(-i\Theta_a^R \frac{\lambda_a}{2}\right). && (4.32)
\end{aligned}
$$

In order to construct $J_L^{\mu,a}$, set $\Theta_a^R = 0$ and choose $\Theta_a^L = \Theta_a^L(x)$ (see Sec. 2.3.3). Then, to first order in Θ_a^L,

$$
\begin{aligned}
U &\mapsto U' = RUL^\dagger = U\left(1 + i\Theta_a^L \frac{\lambda_a}{2}\right),\\
U^\dagger &\mapsto U'^\dagger = \left(1 - i\Theta_a^L \frac{\lambda_a}{2}\right) U^\dagger,\\
\partial_\mu U &\mapsto \partial_\mu U' = \partial_\mu U \left(1 + i\Theta_a^L \frac{\lambda_a}{2}\right) + U i\partial_\mu \Theta_a^L \frac{\lambda_a}{2},\\
\partial_\mu U^\dagger &\mapsto \partial_\mu U'^\dagger = \left(1 - i\Theta_a^L \frac{\lambda_a}{2}\right) \partial_\mu U^\dagger - i\partial_\mu \Theta_a^L \frac{\lambda_a}{2} U^\dagger, \qquad (4.33)
\end{aligned}
$$

from which we obtain for $\delta\mathcal{L}_{\text{eff}}$:

$$
\begin{aligned}
\delta\mathcal{L}_{\text{eff}} &= \frac{F_0^2}{4} \mathrm{Tr}\left[U i\partial_\mu \Theta_a^L \frac{\lambda_a}{2} \partial^\mu U^\dagger + \partial_\mu U \left(-i\partial^\mu \Theta_a^L \frac{\lambda_a}{2} U^\dagger\right)\right]\\
&= \frac{F_0^2}{4} i\partial_\mu \Theta_a^L \mathrm{Tr}\left[\frac{\lambda_a}{2}(\partial^\mu U^\dagger U - U^\dagger \partial^\mu U)\right]\\
&= \frac{F_0^2}{4} i\partial_\mu \Theta_a^L \mathrm{Tr}\left(\lambda_a \partial^\mu U^\dagger U\right). && (4.34)
\end{aligned}
$$

(In the last step we made use of

$$
\partial^\mu U^\dagger U = -U^\dagger \partial^\mu U,
$$

which follows from differentiating $U^\dagger U = 1$.) We thus obtain for the left currents

$$
J_L^{\mu,a} = \frac{\partial \delta\mathcal{L}_{\text{eff}}}{\partial \partial_\mu \Theta_a^L} = i\frac{F_0^2}{4} \mathrm{Tr}\left(\lambda_a \partial^\mu U^\dagger U\right), \qquad (4.35)
$$

and, completely analogously, choosing $\Theta_a^L = 0$ and $\Theta_a^R = \Theta_a^R(x)$,

$$J_R^{\mu,a} = \frac{\partial \delta \mathcal{L}_{\text{eff}}}{\partial \partial_\mu \Theta_a^R} = -i\frac{F_0^2}{4}\text{Tr}\left(\lambda_a U \partial^\mu U^\dagger\right) \tag{4.36}$$

for the right currents. Combining Eqs. (4.35) and (4.36) the vector and axial-vector currents read

$$\begin{aligned} J_V^{\mu,a} &= J_R^{\mu,a} + J_L^{\mu,a} = -i\frac{F_0^2}{4}\text{Tr}\left(\lambda_a[U, \partial^\mu U^\dagger]\right), & (4.37)\\ J_A^{\mu,a} &= J_R^{\mu,a} - J_L^{\mu,a} = -i\frac{F_0^2}{4}\text{Tr}\left(\lambda_a\{U, \partial^\mu U^\dagger\}\right). & (4.38) \end{aligned}$$

Furthermore, because of the symmetry of $\mathcal{L}_{\text{eff}}$ under $\text{SU}(3)_L \times \text{SU}(3)_R$, both vector and axial-vector currents are conserved. The vector current densities $J_V^{\mu,a}$ of Eq. (4.37) contain only terms with an even number of Goldstone bosons,

$$\begin{aligned} J_V^{\mu,a} &\overset{\phi \mapsto -\phi}{\mapsto} -i\frac{F_0^2}{4}\text{Tr}[\lambda_a(U^\dagger \partial^\mu U - \partial^\mu U U^\dagger)] \\ &= -i\frac{F_0^2}{4}\text{Tr}[\lambda_a(-\partial^\mu U^\dagger U + U\partial^\mu U^\dagger)] = J_V^{\mu,a}. \end{aligned}$$

On the other hand, the expression for the axial-vector currents is *odd* in the number of Goldstone bosons,

$$\begin{aligned} J_A^{\mu,a} &\overset{\phi \mapsto -\phi}{\mapsto} -i\frac{F_0^2}{4}\text{Tr}[\lambda_a(U^\dagger \partial^\mu U + \partial^\mu U U^\dagger)] \\ &= i\frac{F_0^2}{4}\text{Tr}[\lambda_a(\partial^\mu U^\dagger U + U\partial^\mu U^\dagger)] = -J_A^{\mu,a}. \end{aligned}$$

To find the leading term let us expand Eq. (4.38) in the fields,

$$J_A^{\mu,a} = -i\frac{F_0^2}{4}\text{Tr}\left(\lambda_a\left\{1 + \cdots, -i\frac{\lambda_b \partial^\mu \phi_b}{F_0} + \cdots\right\}\right) = -F_0 \partial^\mu \phi_a + \cdots$$

from which we conclude that the axial-vector current has a non-vanishing matrix element when evaluated between the vacuum and a one-Goldstone boson state [see Eq. (4.19)]:

$$\begin{aligned} \langle 0|J_A^{\mu,a}(x)|\phi^b(p)\rangle &= \langle 0| - F_0 \partial^\mu \phi_a(x)|\phi^b(p)\rangle \\ &= -F_0 \partial^\mu \exp(-ip\cdot x)\delta^{ab} = ip^\mu F_0 \exp(-ip\cdot x)\delta^{ab}. \end{aligned}$$

In Sec. 4.6.1 F_0 will be related to the pion-decay constant entering $\pi^+ \to \mu^+\nu_\mu$.

So far we have assumed a perfect $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry. However, in Sec. 3.4 we saw, by means of a simple example, how an explicit symmetry breaking may lead to finite masses of the Goldstone bosons. As has been discussed in Sec. 2.3.6, the quark mass term of QCD results in such an explicit symmetry breaking,

$$\mathcal{L}_M = -\bar{q}_R M q_L - \bar{q}_L M^\dagger q_R, \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{4.39}$$

In order to incorporate the consequences of Eq. (4.39) into the effective-Lagrangian framework, one makes use of the following argument [Geo 84]: Although M is in reality just a constant matrix and does not transform along with the quark fields, $\mathcal{L}_M$ of Eq. (4.39) *would be* invariant *if* M transformed as

$$M \mapsto RML^\dagger. \tag{4.40}$$

One then constructs the most general Lagrangian $\mathcal{L}(U, M)$ which is invariant under Eqs. (4.24) and (4.40) and expands this function in powers of M. At lowest order in M one obtains

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2}\mathrm{Tr}(MU^\dagger + UM^\dagger), \tag{4.41}$$

where the subscript s.b. refers to symmetry breaking. In order to interpret the new parameter B_0 let us consider the energy density of the ground state ($U = U_0 = 1$),

$$\langle \mathcal{H}_{\text{eff}} \rangle = -F_0^2 B_0 (m_u + m_d + m_s), \tag{4.42}$$

and compare its derivative with respect to (any of) the light quark masses m_q with the corresponding quantity in QCD,

$$\left.\frac{\partial \langle 0|\mathcal{H}_{\text{QCD}}|0\rangle}{\partial m_q}\right|_{m_u=m_d=m_s=0} = \frac{1}{3}\langle 0|\bar{q}q|0\rangle_0 = \frac{1}{3}\langle \bar{q}q\rangle,$$

where $\langle \bar{q}q\rangle$ is the chiral quark condensate of Eq. (4.16). Within the framework of the lowest-order effective Lagrangian, the constant B_0 is thus related to the chiral quark condensate as

$$3F_0^2 B_0 = -\langle \bar{q}q\rangle. \tag{4.43}$$

Let us add a few remarks.

1. A term $\mathrm{Tr}(M)$ by itself is not invariant.
2. The combination $\mathrm{Tr}(MU^\dagger - UM^\dagger)$ has the wrong behavior under parity $\phi(\vec{x},t) \mapsto -\phi(-\vec{x},t)$, because

$$\begin{aligned}\mathrm{Tr}[MU^\dagger(\vec{x},t) - U(\vec{x},t)M^\dagger] &\stackrel{P}{\mapsto} \mathrm{Tr}[MU(-\vec{x},t) - U^\dagger(-\vec{x},t)M^\dagger] \\ &\stackrel{M=M^\dagger}{=} -\mathrm{Tr}[MU^\dagger(-\vec{x},t) - U(-\vec{x},t)M^\dagger].\end{aligned}$$

3. Because $M = M^\dagger$, $\mathcal{L}_{\text{s.b.}}$ contains only terms even in ϕ.

In order to determine the masses of the Goldstone bosons, we identify the terms of second order in the fields in $\mathcal{L}_{\text{s.b.}}$,

$$\mathcal{L}_{\text{s.b}} = -\frac{B_0}{2}\mathrm{Tr}(\phi^2 M) + \cdots. \tag{4.44}$$

Using Eq. (4.28) we find

$$\begin{aligned}\mathrm{Tr}(\phi^2 M) &= 2(m_u+m_d)\pi^+\pi^- + 2(m_u+m_s)K^+K^- + 2(m_d+m_s)K^0\bar{K}^0 \\ &\quad +(m_u+m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u-m_d)\pi^0\eta + \frac{m_u+m_d+4m_s}{3}\eta^2.\end{aligned}$$

For the sake of simplicity we consider the isospin-symmetric limit $m_u = m_d = m$ so that the $\pi^0\eta$ term vanishes and there is no π^0-η mixing. We then obtain for the masses of the Goldstone bosons, to lowest order in the quark masses,

$$M_\pi^2 = 2B_0 m, \tag{4.45}$$
$$M_K^2 = B_0(m+m_s), \tag{4.46}$$
$$M_\eta^2 = \frac{2}{3}B_0\left(m+2m_s\right). \tag{4.47}$$

These results, in combination with Eq. (4.43), $B_0 = -\langle\bar{q}q\rangle/(3F_0^2)$, correspond relations obtained in Ref. [Gel+ 68] and are referred to as the Gell-Mann, Oakes, and Renner relations. Furthermore, the masses of Eqs. (4.45) - (4.47) satisfy the Gell-Mann-Okubo relation

$$4M_K^2 = 4B_0(m+m_s) = 2B_0(m+2m_s) + 2B_0 m = 3M_\eta^2 + M_\pi^2 \tag{4.48}$$

independent of the value of B_0. Without additional input regarding the numerical value of B_0, Eqs. (4.45) - (4.47) do not allow for an extraction of the absolute values of the quark masses m and m_s, because rescaling $B_0 \to \lambda B_0$ in combination with $m_q \to m_q/\lambda$ leaves the relations invariant. For the ratio of the quark masses one obtains, using the empirical values of the pseudoscalar octet,

$$\begin{aligned} \frac{M_K^2}{M_\pi^2} &= \frac{m+m_s}{2m} \quad \Rightarrow \quad \frac{m_s}{m} = 25.9, \\ \frac{M_\eta^2}{M_\pi^2} &= \frac{2m_s+m}{3m} \quad \Rightarrow \quad \frac{m_s}{m} = 24.3. \end{aligned} \tag{4.49}$$

Let us conclude this section with the following remark. We saw in Sec. 4.1.2 that a non-vanishing quark condensate in the chiral limit is a sufficient but not a necessary condition for a spontaneous chiral symmetry breaking. The effective Lagrangian term of Eq. (4.41) not only results in a shift of the vacuum energy but also in finite Goldstone boson masses.[12] These are related via the parameter B_0 and we recall that it was a symmetry argument which excluded a term $\mathrm{Tr}(M)$ which, at leading order in M, would decouple the vacuum energy shift from the Goldstone boson masses. The scenario underlying $\mathcal{L}_{\text{s.b.}}$ of Eq. (4.41) is similar to that of a Heisenberg ferromagnet [AM 76, Leu 92] which exhibits a spontaneous magnetization $\langle \vec{M} \rangle$, breaking the O(3) symmetry of the Heisenberg Hamiltonian down to O(2). In the present case the analogue of the order parameter $\langle \vec{M} \rangle$ is the quark condensate $\langle \bar{q}q \rangle$. In the case of the ferromagnet, the interaction with an external magnetic field is given by $-\langle \vec{M} \rangle \cdot \vec{H}$, which corresponds to Eq. (4.42), with the quark masses playing the role of the external field $\vec{H}$. However, in principle, it is also possible that B_0 vanishes or is rather small. In such a case the quadratic masses of the Goldstone bosons might be dominated by terms which are nonlinear in the quark masses, i.e., by higher-order terms in the expansion of $\mathcal{L}(U, M)$. Such a scenario is the origin of the so-called generalized chiral perturbation theory [Kne+ 95, Kne+ 96, Ste 98]. The analogue would be an antiferromagnet which shows a spontaneous symmetry breaking but with $\langle \vec{M} \rangle = 0$.

The analysis of recent data on $K^+ \to \pi^+\pi^-e^+\nu_e$ [Pis+ 01] in terms of the isoscalar s-wave scattering length a_0^0 [Col+ 01a] supports the conjecture that the quark condensate is indeed the leading order parameter of the spontaneously broken chiral symmetry. For a recent discussion on the relation

[12] Later on we will also see that the $\pi\pi$ scattering amplitude is effected by $\mathcal{L}_{\text{s.b.}}$.

between the quark condensate and s-wave $\pi\pi$ scattering the interested reader is referred to Ref. [Leu 01a].

4.4 Effective Lagrangians and Weinberg's Power Counting Scheme

An essential prerequisite for the construction of effective field theories is a "theorem" of Weinberg stating that a perturbative description in terms of the most general effective Lagrangian containing all possible terms compatible with assumed symmetry principles yields the most general S matrix consistent with the fundamental principles of quantum field theory and the assumed symmetry principles [Wei 79]. The corresponding effective Lagrangian will contain an infinite number of terms with an infinite number of free parameters. Turning Weinberg's theorem into a practical tool requires two steps: one needs some scheme to organize the effective Lagrangian and a systematic method of assessing the importance of diagrams generated by the interaction terms of this Lagrangian when calculating a physical matrix element.

In the framework of mesonic chiral perturbation theory, the most general chiral Lagrangian describing the dynamics of the Goldstone bosons is organized as a string of terms with an increasing number of derivatives and quark mass terms,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \cdots, \tag{4.50}$$

where the subscripts refer to the order in the momentum and quark mass expansion. The index 2, for example, denotes either two derivatives or one quark mass term. In the context of Feynman rules, derivatives generate four-momenta, whereas the convention of counting quark mass terms as being of the same order as two derivatives originates from Eqs. (4.45) - (4.47) in conjunction with the on-shell condition $p^2 = M^2$. In an analogous fashion, $\mathcal{L}_4$ and $\mathcal{L}_6$ denote more complicated terms of so-called chiral orders $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ with corresponding numbers of derivatives and quark mass terms. With such a counting scheme, the chiral orders in the mesonic sector are always even $[\mathcal{O}(p^{2n})]$ because Lorentz indices of derivatives always have to be contracted with either the metric tensor $g^{\mu\nu}$ or the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ to generate scalars, and the quark mass terms are counted as $\mathcal{O}(p^2)$.

Weinberg's power counting scheme [Wei 79] analyzes the behavior of a given diagram under a linear rescaling of all the *external* momenta, $p_i \mapsto tp_i$, and a quadratic rescaling of the light quark masses, $m_q \mapsto t^2 m_q$, which, in terms of the Goldstone boson masses, corresponds to $M^2 \mapsto t^2 M^2$. The chiral dimension D of a given diagram with amplitude $\mathcal{M}(p_i, m_q)$ is defined by

$$\mathcal{M}(tp_i, t^2 m_q) = t^D \mathcal{M}(p_i, m_q), \tag{4.51}$$

and thus

$$D = 2 + \sum_{n=1}^{\infty} 2(n-1)N_{2n} + 2N_L, \tag{4.52}$$

where N_{2n} denotes the number of vertices originating from $\mathcal{L}_{2n}$, and N_L is the number of independent loops. Clearly, for small enough momenta and masses diagrams with small D, such as $D = 2$ or $D = 4$, should dominate. Of course, the rescaling of Eq. (4.51) must be viewed as a mathematical tool. While external three-momenta can, to a certain extent, be made arbitrarily small, the rescaling of the quark masses is a theoretical instrument only. Note that loop diagrams are always suppressed due to the term $2N_L$ in Eq. (4.52). It may happen, though, that the leading-order tree diagrams vanish and therefore that the lowest-order contribution to a certain process is a one-loop diagram. An example is the reaction $\gamma\gamma \to \pi^0\pi^0$ [BC 88].

In order to prove Eq. (4.52) we start from the usual Feynman rules for evaluating an S-matrix element (see, e.g., Appendix A-4 of Ref. [IZ 80]). Each internal meson line contributes a factor

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - M^2 + i\epsilon} &\overset{(M^2 \mapsto t^2M^2)}{\mapsto} t^{-2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2/t^2 - M^2 + i\epsilon} \\ &\overset{(k = tl)}{=} t^2 \int \frac{d^4l}{(2\pi)^4}\frac{i}{l^2 - M^2 + i\epsilon}.\end{aligned} \tag{4.53}$$

For each vertex, originating from $\mathcal{L}_{2n}$, we obtain symbolically a factor p^{2n} together with a four-momentum conserving delta function resulting in t^{2n} for the vertex factor and t^{-4} for the delta function. At this point one has to take into account the fact that, although Eq. (4.51) refers to a rescaling of *external* momenta, a substitution $k = tl$ for internal momenta as in Eq. (4.53) acts in exactly the same way as a rescaling of external momenta:

$$\begin{aligned}\delta^4(p+k) &\overset{p \mapsto tp,\, k = tl}{\mapsto} t^{-4}\delta^4(p+l), \\ p^{2n-m}k^m &\overset{p \mapsto tp,\, k = tl}{\mapsto} t^{2n}p^{2n-m}l^m,\end{aligned}$$

where p and k denote external and internal momenta, respectively.

So far we have discussed the rules for determining the power D_S referring to the S-matrix element which is related to the invariant amplitude through a four-momentum conserving delta function,

$$S \sim \delta^4(P_f - P_i)\mathcal{M}.$$

The delta function contains external momenta only, and thus re-scales under $p_i \mapsto tp_i$ as t^{-4}, so

$$t^{D_S} = t^{-4}t^D.$$

We thus find as an intermediate result

$$D = 4 + 2N_I + \sum_{n=1}^{\infty} N_{2n}(2n-4), \tag{4.54}$$

where N_I denotes the number of internal lines. The number of independent loops, total number of vertices, and number of internal lines are related by[13]

$$N_L = N_I - (N_V - 1),$$

because each of the N_V vertices generates a delta function. After extracting one overall delta function this yields $N_V - 1$ conditions for the internal momenta. Using $N_V = \sum_n N_{2n}$ we finally obtain from Eq. (4.54)

$$D = 4 + 2(N_L + N_V - 1) + \sum_{n=1}^{\infty} N_{2n}(2n-4) = 2 + 2N_L + \sum_{n=1}^{\infty} N_{2n}(2n-2).$$

By means of a simple example we will illustrate how the mechanism of rescaling actually works. To that end we consider as a toy model of an effective field theory the self interaction of a scalar field,

$$\mathcal{L}_2 = g\Phi^2\partial_\mu\Phi\partial^\mu\Phi, \tag{4.55}$$

where the coupling constant g has the dimension of energy^{-2}.[14] The Feynman

[13]Note that the number of independent momenta is *not* the number of faces or closed circuits that may be drawn on the internal lines of a diagram. This may, for example, be seen using a diagram with the topology of a tetrahedron which has four faces but $N_L = 6 - (4 - 1) = 3$ (see, e.g., Chap. 6-2 of Ref. [IZ 80]).

[14]Recall that the dimensions of a Lagrangian density and a field Φ are energy^4 and energy, respectively.

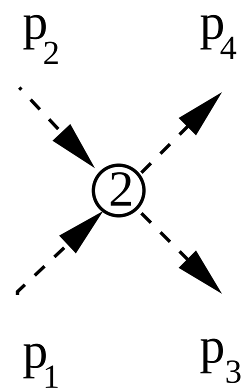


Figure 4.1: Tree-level diagram corresponding to Eq. (4.56).

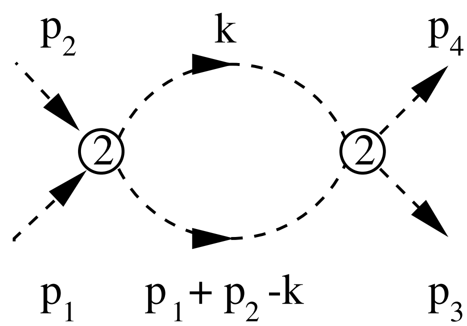


Figure 4.2: Typical one-loop diagram contributing to the scattering of two particles.

rules give the amplitude corresponding to the simple tree diagram of Fig. 4.1 for the scattering of two particles,

$$\begin{aligned}\mathcal{M}(p_1, p_2, ; p_3, p_4) &= 4ig\left[(p_1 + p_2) \cdot (p_3 + p_4) - p_1 \cdot p_2 - p_3 \cdot p_4\right] \\ &\overset{p_i \mapsto tp_i}{\mapsto} t^2 \mathcal{M}(p_1, p_2; p_3, p_4). \end{aligned} \tag{4.56}$$

As expected, the behavior under rescaling is in agreement with Eq. (4.52) for $N_L = 0$, $N_2 = 1$, and $N_{2n} = 0$ for all remaining n. Now let us consider a typical loop diagram of Fig. 4.2 contributing to the same process, where the 2 in the interaction blob indicates the $\mathcal{L}_2$ term in the Lagrangian containing two derivatives. Applying the usual Feynman rules, with the vertex of Eq.

(4.56), we obtain

$$\begin{aligned}
\mathcal{M} \quad &= \quad \frac{1}{2}\int\frac{d^4k}{(2\pi)^4} \\
&\quad\ \times 4ig\left[(p_1+p_2-k+k)\cdot(p_3+p_4)-(p_1+p_2-k)\cdot k-p_3\cdot p_4\right] \\
&\quad\ \times\frac{i}{k^2-M^2+i\epsilon} \\
&\quad\ \times\frac{i}{(p_1+p_2-k)^2-M^2+i\epsilon} \\
&\quad\ \times 4ig\left[(p_1+p_2)\cdot(p_3+p_4-k+k)-p_1\cdot p_2-(p_1+p_2-k)\cdot k\right] \\
&= \quad 8g^2\int\frac{d^4k}{(2\pi)^4}\left[(p_1+p_2)\cdot(p_3+p_4)-(p_1+p_2-k)\cdot k-p_3\cdot p_4\right] \\
&\quad\ \times\frac{1}{k^2-M^2+i\epsilon}\frac{1}{(p_1+p_2-k)^2-M^2+i\epsilon} \\
&\quad\ \times\left[(p_1+p_2)\cdot(p_3+p_4)-p_1\cdot p_2-(p_1+p_2-k)\cdot k\right] \\
&\overset{\substack{p_i\mapsto tp_i \\ M^2\mapsto t^2M^2}}{\mapsto} \quad 8g^2\int\frac{d^4k}{(2\pi)^4}\left[(p_1+p_2)\cdot(p_3+p_4)-(p_1+p_2-\frac{k}{t})\cdot\frac{k}{t}-p_3\cdot p_4\right] \\
&\quad\ \times\frac{1}{\frac{k^2}{t^2}-M^2+i\epsilon}\frac{1}{(p_1+p_2-\frac{k}{t})^2-M^2+i\epsilon} \\
&\quad\ \times\left[(p_1+p_2)\cdot(p_3+p_4)-p_1\cdot p_2-(p_1+p_2-\frac{k}{t})\cdot\frac{k}{t}\right] \\
tl \overset{\equiv}{=} k \quad & \quad 8g^2\int\frac{t^4d^4l}{(2\pi)^4}\left[(p_1+p_2)\cdot(p_3+p_4)-(p_1+p_2-l)\cdot l-p_3\cdot p_4\right] \\
&\quad\ \times\frac{1}{l^2-M^2+i\epsilon}\frac{1}{(p_1+p_2-l)^2-M^2+i\epsilon} \\
&\quad\ \times\left[(p_1+p_2)\cdot(p_3+p_4)-p_1\cdot p_2-(p_1+p_2-l)\cdot l\right] \\
&= \quad t^4\mathcal{M},
\end{aligned}\tag{4.57}$$

This agrees with the value $D=4$ given by Eq. (4.52) for $N_L=1$ and $N_2=2$.

For the sake of completeness, let us comment on the symmetry factor 1/2 in Eq. (4.57). When deriving the Feynman rule of Eq. (4.56), we took account of $4!=24$ distinct combinations of contracting four field operators with four external lines. The "product" of two such vertices thus contains 24×24 combinations. However, from each vertex two lines have to be selected as internal lines and there exist 6 possibilities to choose one pair out of 4 field operators to form internal lines. For the two remaining operators one has two possibilities of contracting them with external lines. Finally, the respective pairs of internal lines of the first and second vertices may be contracted in two ways with each other, leaving us with $12\times 12\times 2 = (24\times 24)/2$ combinations.

In the discussion of the loop integral we did not address the question of convergence. This needs to be addressed since applying the substitution $tl = k$ in Eq. (4.57) is well-defined only for convergent integrals. Later on we will regularize the integrals by use of the method of dimensional regularization, introducing a renormalization scale μ which also has to be rescaled linearly. However, at a given chiral order, the sum of all diagrams will, by construction, not depend on the renormalization scale.

Finally, the proof of Weinberg's theorem [Leu 94, HW 94] for chiral perturbation theory is rather technical and lengthy and beyond the scope of this review. In Ref. [Leu 94] it was shown that global symmetry constraints alone do not suffice to fully determine the low-energy structure of the effective Lagrangian. In fact, a determination of the (low-energy) Green functions of QCD off the mass shell, i.e., for momenta which do not correspond to the mass-shell conditions for Goldstone bosons, one needs to study the Ward identities, and therefore the symmetries have to be extended to the local level. One thus considers a *locally* invariant, effective Lagrangian although the symmetries of the underlying theory originate in a global symmetry. If the Ward identities contain anomalies, they show up as a modification of the generating functional, which can explicitly be incorporated through the Wess-Zumino-Witten construction [WZ 71, Wit 83].

4.5 Construction of the Effective Lagrangian

In Sec. 4.3 we have derived the lowest-order effective Lagrangian for a *global* $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry. On the other hand, the Ward identities originating in the global $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ symmetry of QCD are obtained from a *locally* invariant generating functional involving a coupling to external fields (see Sec. 2.4.4 and App. A). Our goal is to approximate the "true" generating functional $Z_{\mathrm{QCD}}[v,a,s,p]$ of Eq. (2.97) by a sequence $Z_{\mathrm{eff}}^{(2)}[v,a,s,p] + Z_{\mathrm{eff}}^{(4)}[v,a,s,p] + \cdots$, where the effective generating functionals are obtained using the effective field theory. Therefore, we need to promote the global symmetry of the effective Lagrangian to a local one and introduce a coupling to the *same* external fields v, a, s, and p as in QCD.

In the following we will outline the principles entering the construction of the effective Lagrangian for a local $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry (see Refs. [FS 96, Bij+ 99, Ebe+ 02] for details).[15] The matrix U transforms as $U \mapsto U' = V_R U V_L^\dagger$, where $V_L(x)$ and $V_R(x)$ are independent space-time-dependent SU(3) matrices. As in the case of gauge theories, we need external fields $l_\mu^a(x)$ and $r_\mu^a(x)$ [see Eqs. (2.96), (2.106), and (2.109) and Table 4.2] corresponding to the parameters $\Theta_a^L(x)$ and $\Theta_a^R(x)$ of $V_L(x)$ and $V_R(x)$, respectively. For any object A transforming as $V_R A V_L^\dagger$ such as, e.g., U we define the covariant derivative $D_\mu A$ as

$$\begin{aligned}
D_\mu A &\equiv \partial_\mu A - i r_\mu A + i A l_\mu \\
&\mapsto \partial_\mu (V_R A V_L^\dagger) - i(V_R r_\mu V_R^\dagger + i V_R \partial_\mu V_R^\dagger) V_R A V_L^\dagger \\
&\quad + i V_R A V_L^\dagger (V_L l_\mu V_L^\dagger + i V_L \partial_\mu V_L^\dagger) \\
&= \partial_\mu V_R A V_L^\dagger + V_R \partial_\mu A V_L^\dagger + V_R A \partial_\mu V_L^\dagger - i V_R r_\mu A V_L^\dagger - \partial_\mu V_R A V_L^\dagger \\
&\quad + i V_R A l_\mu V_L^\dagger - V_R A \partial_\mu V_L^\dagger \\
&= V_R(\partial_\mu A - i r_\mu A + i A l_\mu) V_L^\dagger = V_R (D_\mu A) V_L^\dagger, \qquad (4.58)
\end{aligned}$$

where we made use of $V_R \partial_\mu V_R^\dagger = -\partial_\mu V_R V_R^\dagger$. Again, the defining property for the covariant derivative is that it should transform in the same way as the object it acts on.[16] Since the effective Lagrangian will ultimately contain arbitrarily high powers of derivatives we also need the field strength tensors $f_{\mu\nu}^L$ and $f_{\mu\nu}^R$ corresponding to the gauge fields,

$$f_{\mu\nu}^R \equiv \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \qquad (4.59)$$

$$f_{\mu\nu}^L \equiv \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu]. \qquad (4.60)$$

The field strength tensors are traceless,

$$\mathrm{Tr}(f_{\mu\nu}^L) = \mathrm{Tr}(f_{\mu\nu}^R) = 0, \qquad (4.61)$$

[15] In principle, we could also "gauge" the $\mathrm{U}(1)_V$ symmetry. However, this is primarily of relevance to the SU(2) sector in order to fully incorporate the coupling to the electromagnetic field [see Eq. (2.111)]. Since in SU(3), the quark-charge matrix is traceless, this important case is included in our considerations. For further discussions, see Ref. [Ebe+ 02].

[16] Under certain circumstances it is advantageous to introduce for each object with a well-defined transformation behavior a separate covariant derivative. One may then use a product rule similar to the one of ordinary differentiation [see Eqs. (18) and (19) of Ref. [FS 96]].

Table 4.2: Transformation properties under the group (G), charge conjugation (C), and parity (P). The expressions for adjoint matrices are trivially obtained by taking the Hermitian conjugate of each entry. In the parity transformed expression it is understood that the argument is $(-\vec{x}, t)$ and that partial derivatives ∂_μ act with respect to x and not with respect to the argument of the corresponding function.

element	G	C	P
U	$V_R U V_L^\dagger$	U^T	$U^\dagger$
$D_{\lambda_1} \cdots D_{\lambda_n} U$	$V_R D_{\lambda_1} \cdots D_{\lambda_n} U V_L^\dagger$	$(D_{\lambda_1} \cdots D_{\lambda_n} U)^T$	$(D^{\lambda_1} \cdots D^{\lambda_n} U)^\dagger$
χ	$V_R \chi V_L^\dagger$	χ^T	$\chi^\dagger$
$D_{\lambda_1} \cdots D_{\lambda_n} \chi$	$V_R D_{\lambda_1} \cdots D_{\lambda_n} \chi V_L^\dagger$	$(D_{\lambda_1} \cdots D_{\lambda_n} \chi)^T$	$(D^{\lambda_1} \cdots D^{\lambda_n} \chi)^\dagger$
r_μ	$V_R r_\mu V_R^\dagger + i V_R \partial_\mu V_R^\dagger$	$-l_\mu^T$	l^μ
l_μ	$V_L l_\mu V_L^\dagger + i V_L \partial_\mu V_L^\dagger$	$-r_\mu^T$	r^μ
$f_{\mu\nu}^R$	$V_R f_{\mu\nu}^R V_R^\dagger$	$-(f_{\mu\nu}^L)^T$	$f_L^{\mu\nu}$
$f_{\mu\nu}^L$	$V_L f_{\mu\nu}^L V_L^\dagger$	$-(f_{\mu\nu}^R)^T$	$f_R^{\mu\nu}$

because $\mathrm{Tr}(l_\mu) = \mathrm{Tr}(r_\mu) = 0$ and the trace of any commutator vanishes. Finally, following the convention of Gasser and Leutwyler we introduce the linear combination $\chi \equiv 2B_0(s+ip)$ with the scalar and pseudoscalar external fields of Eq. (2.96), where B_0 is defined in Eq. (4.43). Table 4.2 contains the transformation properties of all building blocks under the group (G), charge conjugation (C), and parity (P).

In the chiral counting scheme of chiral perturbation theory the elements are counted as:

$$U = \mathcal{O}(p^0),\ D_\mu U = \mathcal{O}(p),\ r_\mu, l_\mu = \mathcal{O}(p),\ f_{\mu\nu}^{L/R} = \mathcal{O}(p^2),\ \chi = \mathcal{O}(p^2). \quad (4.62)$$

The external fields r_μ and l_μ count as $\mathcal{O}(p)$ to match $\partial_\mu A$, and χ is of $\mathcal{O}(p^2)$ because of Eqs. (4.45) - (4.47). Any additional covariant derivative counts as $\mathcal{O}(p)$.

The construction of the effective Lagrangian in terms of the building blocks of Eq. (4.62) proceeds as follows.[17] Given objects $A, B, \ldots$, all of

[17] There is a certain freedom in the choice of the elementary building blocks. For example, by a suitable multiplication with U or $U^\dagger$ any building block can be made to transform as

which transform as $A' = V_R A V_L^\dagger$, $B' = V_R B V_L^\dagger$, $\ldots$, one can form invariants by taking the trace of products of the type $AB^\dagger$:

$$\begin{aligned}\mathrm{Tr}(AB^\dagger) &\mapsto \mathrm{Tr}[V_R A V_L^\dagger (V_R B V_L^\dagger)^\dagger] = \mathrm{Tr}(V_R A V_L^\dagger V_L B^\dagger V_R^\dagger) = \mathrm{Tr}(AB^\dagger V_R^\dagger V_R) \\ &= \mathrm{Tr}(AB^\dagger).\end{aligned}$$

The generalization to more terms is obvious and, of course, the product of invariant traces is invariant:

$$\mathrm{Tr}(AB^\dagger CD^\dagger), \quad \mathrm{Tr}(AB^\dagger)\mathrm{Tr}(CD^\dagger), \quad \cdots. \tag{4.63}$$

The complete list of elements up to and including order $\mathcal{O}(p^2)$ transforming as $V_R \cdots V_L^\dagger$ reads

$$U, D_\mu U, D_\mu D_\nu U, \chi, U f_{\mu\nu}^L, f_{\mu\nu}^R U. \tag{4.64}$$

For the invariants up to $\mathcal{O}(p^2)$ we then obtain

$$\begin{aligned}\mathcal{O}(p^0) &: \mathrm{Tr}(UU^\dagger) = 3, \\ \mathcal{O}(p) &: \mathrm{Tr}(D_\mu U U^\dagger) \stackrel{*}{=} -\mathrm{Tr}[U(D_\mu U)^\dagger] \stackrel{*}{=} 0, \\ \mathcal{O}(p^2) &: \mathrm{Tr}(D_\mu D_\nu U U^\dagger) \stackrel{**}{=} -\mathrm{Tr}[D_\nu U (D_\mu U)^\dagger] \stackrel{**}{=} \mathrm{Tr}[U(D_\nu D_\mu U)^\dagger], \\ &\quad \mathrm{Tr}(\chi U^\dagger), \\ &\quad \mathrm{Tr}(U\chi^\dagger), \\ &\quad \mathrm{Tr}(U f_{\mu\nu}^L U^\dagger) = \mathrm{Tr}(f_{\mu\nu}^L) = 0, \\ &\quad \mathrm{Tr}(f_{\mu\nu}^R) = 0.\end{aligned} \tag{4.65}$$

In $*$ we made use of two important properties of the covariant derivative $D_\mu U$:

$$D_\mu U U^\dagger = -U(D_\mu U)^\dagger, \tag{4.66}$$

$$\mathrm{Tr}(D_\mu U U^\dagger) = 0. \tag{4.67}$$

The first relation results from the unitarity of U in combination with the definition of the covariant derivative, Eq. (4.58). Equation (4.67) is shown using $\mathrm{Tr}(r_\mu) = \mathrm{Tr}(l_\mu) = 0$ together with Eq. (4.30), $\mathrm{Tr}(\partial_\mu U U^\dagger) = 0$:

$$\mathrm{Tr}(D_\mu U U^\dagger) = \mathrm{Tr}(\partial_\mu U U^\dagger - i r_\mu U U^\dagger + i U l_\mu U^\dagger) = 0.$$

$V_R \cdots V_R^\dagger$ without changing its chiral order [FS 96]. The present approach most naturally leads to the Lagrangian of Gasser and Leutwyler [GL 85a].

The relations ∗∗ can either be verified by explicit calculation or, more elegantly, using the product rule of Ref. [FS 96] for the covariant derivatives.

Finally, we impose Lorentz invariance, i.e., Lorentz indices have to be contracted, resulting in three candidate terms:

$$\mathrm{Tr}[D_\mu U (D^\mu U)^\dagger], \tag{4.68}$$

$$\mathrm{Tr}(\chi U^\dagger \pm U\chi^\dagger). \tag{4.69}$$

The term in Eq. (4.69) with the minus sign is excluded because it has the wrong sign under parity (see Table 4.2), and we end up with the most general, *locally* invariant, effective Lagrangian at lowest chiral order,[18]

$$\mathcal{L}_2 = \frac{F_0^2}{4}\mathrm{Tr}[D_\mu U (D^\mu U)^\dagger] + \frac{F_0^2}{4}\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger). \tag{4.70}$$

Note that $\mathcal{L}_2$ contains two free parameters: the pion-decay constant F_0 and B_0 of Eq. (4.43) (hidden in the definition of χ).

Let us finally derive the equations of motion associated with the lowest-order Lagrangian. These are important because they can be used to eliminate so-called equation-of-motion terms in the construction of the higher-order Lagrangians [Geo 91, Leu 92, Gro 94, Arz 95, SF 95] by applying field transformations [Chi 61, Kam+ 61]. To that end we need to consider an infinitesimal change of the SU(3) matrix $U(x)$. Since the set of SU(3) matrices forms a group, to each pair of elements U and U' corresponds a unique element $\tilde{U}$, connecting the two via $U' = \tilde{U}U$. Let us parameterize $\tilde{U}$ by means of the Gell-Mann matrices,

$$\tilde{U} = \exp(i\Delta), \quad \Delta = \sum_{a=1}^{8} \lambda_a \Delta_a, \quad \Delta_a \in R, \tag{4.71}$$

and consider small variations of the SU(3) matrix as

$$U'(x) = U(x) + \delta U(x) = \left(1 + i\sum_{a=1}^{8} \Delta_a(x)\lambda_a\right) U(x), \tag{4.72}$$

where the $\Delta_a(x)$ are now real functions. With such an ansatz, the matrix U' satisfies both conditions

$$U'U'^\dagger = 1, \quad \det(U') = 1, \tag{4.73}$$

[18]At $\mathcal{O}(p^2)$ invariance under C does not provide any additional constraints.

up to and including the terms linear in $\Delta_a(x)$.[19] Given the fields at t_1 and t_2, the dynamics is determined by the principle of stationary action. We obtain for the variation of the action

$$\begin{aligned}
\delta S &= \frac{F_0^2}{4}\int_{t_1}^{t_2} dt \int d^3x\, \mathrm{Tr}\left[D_\mu \delta U (D^\mu U)^\dagger + D_\mu U (D^\mu \delta U)^\dagger + \chi \delta U^\dagger + \delta U \chi^\dagger\right] \\
&= \frac{F_0^2}{4}\int_{t_1}^{t_2} dt \int d^3x\, \mathrm{Tr}\left[-\delta U (D_\mu D^\mu U)^\dagger - D_\mu D^\mu U \delta U^\dagger + \chi \delta U^\dagger + \delta U \chi^\dagger\right] \\
&= i\frac{F_0^2}{4}\int_{t_1}^{t_2} dt \int d^3x \sum_{a=1}^{8} \Delta_a(x) \\
&\quad \times \mathrm{Tr}\left\{\lambda_a [D_\mu D^\mu U U^\dagger - U(D_\mu D^\mu U)^\dagger - \chi U^\dagger + U\chi^\dagger]\right\}. \qquad (4.74)
\end{aligned}$$

In the second equation we made use of the standard boundary conditions $\Delta_a(t_1,\vec{x}) = \Delta_a(t_2,\vec{x}) = 0$, the divergence theorem, and the definition of the covariant derivative of Eq. (4.58). The third equality results from $\delta U^\dagger = -U^\dagger \delta U U^\dagger$ and the invariance of the trace with respect to cyclic permutations. The functions $\Delta_a(x)$ may be chosen arbitrarily, and we obtain eight Euler-Lagrange equations

$$\mathrm{Tr}\left\{\lambda_a [D^2 U U^\dagger - U(D^2 U)^\dagger - \chi U^\dagger + U\chi^\dagger]\right\} = 0, \quad a = 1, \cdots, 8. \qquad (4.75)$$

Since any 3×3 matrix A can be written as

$$A = a_0 1_{3\times 3} + \sum_{i=1}^{8} a_i \lambda_i, \quad a_0 = \frac{1}{3}\mathrm{Tr}(A), \quad a_i = \frac{1}{2}\mathrm{Tr}(\lambda_i A), \qquad (4.76)$$

the eight equations of motion of Eq. (4.75) may compactly be written in matrix form[20]

$$\mathcal{O}^{(2)}_{\mathrm{EOM}}(U) \equiv D^2 U U^\dagger - U(D^2U)^\dagger - \chi U^\dagger + U\chi^\dagger + \frac{1}{3}\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger) = 0. \quad (4.77)$$

The additional term involving the trace is included to guarantee that the component proportional to the identity matrix vanishes identically and thus one does not erroneously generate a ninth equation of motion.

[19] Some derivations in the literature neglect the second condition of Eq. (4.73) and thus obtain the wrong equations of motion.

[20] Applying Eq. (4.65) one finds $\mathrm{Tr}[D^2UU^\dagger - U(D^2U)^\dagger] = 0$.

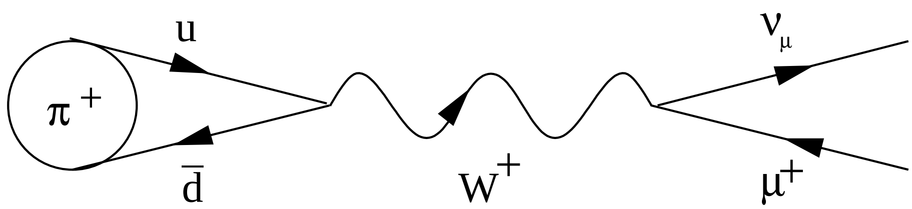


Figure 4.3: Pion decay $\pi^+ \to \mu^+\nu_\mu$.

4.6 Applications at Lowest Order

Let us consider two simple examples at lowest order $D = 2$. According to Eq. (4.52) we only need to consider tree-level diagrams with vertices of $\mathcal{L}_2$.

4.6.1 Pion Decay $\pi^+ \to \mu^+\nu_\mu$

Our first example deals with the weak decay $\pi^+ \to \mu^+\nu_\mu$ which will allow us to relate the free parameter F_0 of $\mathcal{L}_2$ to the pion-decay constant. At the level of the degrees of freedom of the Standard Model, pion decay is described by the annihilation of a u quark and a $\bar{d}$ antiquark, forming the π^+, into a W^+ boson, propagation of the intermediate W^+, and creation of the leptons μ^+ and ν_μ in the final state (see Fig. 4.3). The coupling of the W bosons to the leptons is given by

$$\mathcal{L} = -\frac{g}{2\sqrt{2}}\left[\mathcal{W}^+_\mu \bar{\nu}_\mu \gamma^\mu (1-\gamma_5)\mu + \mathcal{W}^-_\mu \bar{\mu}\gamma^\mu(1-\gamma_5)\nu_\mu\right], \tag{4.78}$$

whereas their interaction with the quarks forming the Goldstone bosons is effectively taken into account by inserting Eq. (2.112) into the Lagrangian of Eq. (4.70). Let us consider the first term of Eq. (4.70) and set $r_\mu = 0$ with, at this point, still arbitrary l_μ. Using $D_\mu U = \partial_\mu U + iUl_\mu$ we find

$$\begin{aligned}
\frac{F_0^2}{4}\mathrm{Tr}[D_\mu U (D^\mu U)^\dagger] &= \frac{F_0^2}{4}\mathrm{Tr}[(\partial_\mu U + iUl_\mu)(\partial^\mu U^\dagger - il^\mu U^\dagger)] \\
&= \cdots + i\frac{F_0^2}{4}\mathrm{Tr}(Ul_\mu\partial^\mu U^\dagger - l^\mu \underbrace{U^\dagger\partial_\mu U}_{-\partial_\mu U^\dagger U}) + \cdots \\
&= i\frac{F_0^2}{2}\mathrm{Tr}(l_\mu \partial^\mu U^\dagger U) + \cdots,
\end{aligned}$$

where only the term linear in l_μ is shown. If we parameterize

$$l_\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} l_\mu^a,$$

the interaction term linear in l_μ reads

$$\mathcal{L}_{\rm int} = \sum_{a=1}^{8} l_\mu^a \left[i\frac{F_0^2}{4}{\rm Tr}(\lambda_a \partial^\mu U^\dagger U) \right] = \sum_{a=1}^{8} l_\mu^a J_L^{\mu,a}, \tag{4.79}$$

where we made use of Eq. (4.35) defining $J_L^{\mu,a}$. Again, we expand $J_L^{\mu,a}$ by using Eq. (4.28) to first order in ϕ,

$$J_L^{\mu,a} = \frac{F_0}{2}\partial^\mu\phi^a + O(\phi^2), \tag{4.80}$$

from which we obtain the matrix element

$$\langle 0|J_L^{\mu,a}(0)|\phi^b(p)\rangle = \frac{F_0}{2}\langle 0|\partial^\mu\phi^a(0)|\phi^b(p)\rangle = -ip^\mu\frac{F_0}{2}\delta^{ab}. \tag{4.81}$$

Inserting l_μ of Eq. (2.112), we find for the interaction term of a single Goldstone boson with a W

$$\mathcal{L}_{W\phi} = \frac{F_0}{2}{\rm Tr}(l_\mu\partial^\mu\phi) = -\frac{g}{\sqrt{2}}\frac{F_0}{2}{\rm Tr}[(\mathcal{W}_\mu^+ T_+ + \mathcal{W}_\mu^- T_-)\partial^\mu\phi].$$

Thus, we need to calculate[21]

$$\begin{aligned}
&{\rm Tr}(T_+\partial^\mu\phi)\\
&= {\rm Tr}\left[\begin{pmatrix} 0 & V_{ud} & V_{us}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\partial^\mu\begin{pmatrix} \pi^0+\frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+\\ \sqrt{2}\pi^- & -\pi^0+\frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0\\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta\end{pmatrix}\right]\\
&= V_{ud}\sqrt{2}\partial^\mu\pi^- + V_{us}\sqrt{2}\partial^\mu K^-,\\
&{\rm Tr}(T_-\partial^\mu\phi)\\
&= {\rm Tr}\left[\begin{pmatrix} 0 & 0 & 0\\ V_{ud} & 0 & 0\\ V_{us} & 0 & 0\end{pmatrix}\partial^\mu\begin{pmatrix} \pi^0+\frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+\\ \sqrt{2}\pi^- & -\pi^0+\frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0\\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta\end{pmatrix}\right]\\
&= V_{ud}\sqrt{2}\partial^\mu\pi^+ + V_{us}\sqrt{2}\partial^\mu K^+.
\end{aligned}$$

[21]Recall that the entries V_{ud} and V_{us} of the Cabibbo-Kobayashi-Maskawa matrix are real.

We then obtain for the interaction term

$$\mathcal{L}_{W\phi} = -g\frac{F_0}{2}[\mathcal{W}_\mu^+(V_{ud}\partial^\mu\pi^- + V_{us}\partial^\mu K^-) + \mathcal{W}_\mu^-(V_{ud}\partial^\mu\pi^+ + V_{us}\partial^\mu K^+)]. \quad (4.82)$$

In combination with the Feynman propagator for W bosons,

$$\frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2}}{k^2 - M_W^2} = \frac{g_{\mu\nu}}{M_W^2} + O(\frac{kk}{M_W^4}), \quad (4.83)$$

the Feynman rule for the invariant amplitude for the weak pion decay reads

$$\begin{aligned} \mathcal{M} &= i\left[-\frac{g}{2\sqrt{2}}\bar{u}_{\nu_\mu}\gamma^\nu(1-\gamma_5)v_{\mu^+}\right]\frac{ig_{\nu\mu}}{M_W^2}i\left[-g\frac{F_0}{2}V_{ud}(-ip^\mu)\right] \\ &= -G_F V_{ud} F_0 \bar{u}_{\nu_\mu}\not{p}(1-\gamma_5)v_{\mu^+}, \end{aligned} \quad (4.84)$$

where p denotes the four-momentum of the pion and

$$G_F = \frac{g^2}{4\sqrt{2}M_W^2} = 1.16639(1)\times 10^{-5}\,\mathrm{GeV}^{-2}$$

is the Fermi constant. The evaluation of the decay rate is a standard textbook exercise and we only quote the final result[22]

$$\frac{1}{\tau} = \frac{G_F^2|V_{ud}|^2}{4\pi}F_0^2 M_\pi m_\mu^2\left(1-\frac{m_\mu^2}{M_\pi^2}\right)^2. \quad (4.85)$$

The constant F_0 is referred to as the pion-decay constant in the chiral limit.[23] It measures the strength of the matrix element of the axial-vector current operator between a one-Goldstone-boson state and the vacuum [see Eq. (4.19)]. Since the interaction of the W boson with the quarks is of the type $l_\mu^a L^{\mu,a} = l_\mu^a(V^{\mu,a} - A^{\mu,a})/2$ [see Eq. (2.112)] and the vector current operator does not contribute to the matrix element between a single pion and the vacuum, pion decay is completely determined by the axial-vector current. The degeneracy of a single constant F_0 in Eq. (4.19) is lifted at $\mathcal{O}(p^4)$ [GL 85a] once SU(3) symmetry breaking is taken into account. The empirical numbers for F_π and F_K are 92.4 MeV and 113 MeV, respectively.[24]

[22]See Chap. 10.14 of Ref. [BD 64a] with the substitution $a/\sqrt{2} \to V_{ud}F_0$ in Eq. (10.140).

[23]Of course, in the chiral limit, the pion is massless and, in such a world, the massive leptons would decay into Goldstone bosons, e.g., $e^- \to \pi^-\nu_e$. However, at $\mathcal{O}(p^2)$, the symmetry breaking term of Eq. (4.41) gives rise to Goldstone-boson masses, whereas the decay constant is not modified at $\mathcal{O}(p^2)$.

[24]In the analysis of Ref. [Gro+ 00] $f_\pi = \sqrt{2}F_\pi$ is used.

4.6.2 Pion-Pion Scattering

Our second example deals with the prototype of a Goldstone boson reaction: $\pi\pi$ scattering. For the sake of simplicity we will restrict ourselves to the SU(2)×SU(2) version of Eq. (4.70). We will contrast two different methods of calculating the scattering amplitude: the "direct" calculation in terms of the Goldstone boson fields of the effective Lagrangian versus the calculation of the QCD Green function in combination with the LSZ reduction formalism. Loosely speaking, the "direct" calculation is somewhat more along the spirit of Weinberg's original paper [Wei 79]: one considers the most general Lagrangian satisfying the general symmetry constraints and calculates S-matrix elements with that Lagrangian. The second method will allow one to also consider QCD Green functions "off shell," i.e., for arbitrary squared invariant momenta. We will discuss under which circumstances the two methods are equivalent and also work out the more general scope of the Green function approach.

For the "direct" calculation we set to zero all external fields except for the quark mass term, $\chi = 2B_0\mathrm{diag}(m_q, m_q) = M_\pi^2 1_{2\times 2}$ [see Eq. (4.45)],

$$\mathcal{L}_2 = \frac{F_0^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{F_0^2 M_\pi^2}{4}\mathrm{Tr}(U^\dagger + U). \tag{4.86}$$

In our general discussion of the transformation behavior of Goldstone bosons at the end of Sec. 4.2.1 we argued that we still have a choice how to represent the variables parameterizing the elements of the set of cosets G/H. In the present case these are elements of SU(2) and we will illustrate this freedom by making use of two different parameterizations of the matrix U [FS 00],[25]

$$U(x) = \frac{1}{F_0}\left[\sigma(x) + i\vec{\tau}\cdot\vec{\pi}(x)\right], \quad \sigma(x) = \sqrt{F_0^2 - \vec{\pi}^{\,2}(x)}, \tag{4.87}$$

$$U(x) = \exp\left[i\frac{\vec{\tau}\cdot\vec{\phi}(x)}{F_0}\right], \tag{4.88}$$

where in both cases the three Hermitian fields π_i and ϕ_i describe pion fields

[25] The first parameterization is popular, because the pion field appears only linearly in the term proportional to the Pauli matrices, leading to a substantial simplification when deriving Feynman rules. It is specific to SU(2) because, in contrast to the general case of SU(N), in SU(2) the totally symmetric d symbols vanish [see Eq. (2.12)]. On the other hand, the exponential parameterization can be used for any N.

transforming as isovectors under $\mathrm{SU}(2)_V$. The fields in the two parameterizations are non-linearly related,

$$\frac{\vec{\pi}}{F_0} = \hat{\phi} \sin\left(\frac{|\vec{\phi}|}{F_0}\right) = \frac{\vec{\phi}}{F_0}\left(1 - \frac{1}{6}\frac{\vec{\phi}^{\,2}}{F_0^2} + \cdots\right). \tag{4.89}$$

This can be interpreted in terms of a change of variables which leaves the free-field part of the Lagrangian unchanged [Chi 61, Kam+ 61]. As a consequence of the equivalence theorem of field theory [Chi 61, Kam+ 61] the result for a physical observable should not depend on the choice of variables.

The substitution $U \leftrightarrow U^\dagger$ corresponding, respectively, to $\vec{\pi} \mapsto -\vec{\pi}$ and $\vec{\phi} \mapsto -\vec{\phi}$ tells us that $\mathcal{L}_2$ generates only interaction terms containing an even number of pion fields. Since there exists no vertex involving 3 Goldstone bosons, $\pi\pi$ scattering must be described by a contact interaction at $\mathcal{O}(p^2)$.

By inserting the expressions for U of Eqs. (4.87) and (4.88) into Eq. (4.86) and collecting those terms containing four pion fields we obtain the interaction Lagrangians

$$\mathcal{L}_2^{4\pi} = \frac{1}{2F_0^2}\partial_\mu\vec{\pi}\cdot\vec{\pi}\partial^\mu\vec{\pi}\cdot\vec{\pi} - \frac{M_\pi^2}{8F_0^2}(\vec{\pi}^{\,2})^2, \tag{4.90}$$

$$\mathcal{L}_2^{4\phi} = \frac{1}{6F_0^2}(\partial_\mu\vec{\phi}\cdot\vec{\phi}\partial^\mu\vec{\phi}\cdot\vec{\phi} - \vec{\phi}^{\,2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}) + \frac{M_\pi^2}{24F_0^2}(\vec{\phi}^{\,2})^2. \tag{4.91}$$

Observe that the two interaction Lagrangians depend differently on the respective pion fields. The corresponding Feynman rules are obtained in the usual fashion by considering all possible ways of contracting pion fields of $i\mathcal{L}_{\mathrm{int}}$ with initial and final pion lines, with the derivatives ∂_μ generating $-ip_\mu$ (ip_μ) for an initial (final) line. For Cartesian isospin indices a, b, c, d the Feynman rules for the scattering process $\pi^a(p_a) + \pi^b(p_b) \to \pi^c(p_c) + \pi^d(p_d)$ as obtained from Eqs. (4.90) and (4.91) read, respectively,

$$\mathcal{M}_2^{4\pi} = i\left[\delta^{ab}\delta^{cd}\frac{s - M_\pi^2}{F_0^2} + \delta^{ac}\delta^{bd}\frac{t - M_\pi^2}{F_0^2} + \delta^{ad}\delta^{bc}\frac{u - M_\pi^2}{F_0^2}\right], \tag{4.92}$$

$$\begin{aligned}\mathcal{M}_2^{4\phi} &= i\left[\delta^{ab}\delta^{cd}\frac{s - M_\pi^2}{F_0^2} + \delta^{ac}\delta^{bd}\frac{t - M_\pi^2}{F_0^2} + \delta^{ad}\delta^{bc}\frac{u - M_\pi^2}{F_0^2}\right] \\ &\quad - \frac{i}{3F_0^2}\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right)\left(\Lambda_a + \Lambda_b + \Lambda_c + \Lambda_d\right),\end{aligned} \tag{4.93}$$

where we introduced $\Lambda_k = p_k^2 - M_\pi^2$ and the usual Mandelstam variables

$$s = (p_a + p_b)^2 = (p_c + p_d)^2,$$

$$t = (p_a - p_c)^2 = (p_d - p_b)^2,$$

$$u = (p_a - p_d)^2 = (p_c - p_b)^2,$$

which are related by $s+t+u = p_a^2+p_b^2+p_c^2+p_d^2$. If the initial and final pions are all on the mass shell, i.e., $\Lambda_k = 0$, the scattering amplitudes are the same, in agreement with the equivalence theorem [Chi 61, Kam+ 61].[26] The on-shell result also agrees with the current-algebra prediction for low-energy $\pi\pi$ scattering [Wei 66]. We will come back to $\pi\pi$ scattering in Sec. 4.10.2 when we also discuss corrections of higher order [Bij+ 96]. On the other hand, if one of the momenta of the external lines is off mass shell, the amplitudes of Eqs. (4.92) and (4.93) differ. In other words, a "direct" calculation gives a unique result independent of the parameterization of U only for the on-shell matrix element.

The second method, developed by Gasser and Leutwyler [GL 84], deals with the Green functions of QCD and their interrelations as expressed in the Ward identities. In particular, these Green functions can, in principle, be calculated for any value of squared momenta even though ChPT is set up only for a low-energy description. For the discussion of $\pi\pi$ scattering one considers the four-point function [GL 84]

$$G^{abcd}_{PPPP}(x_a, x_b, x_c, x_d) \equiv \langle 0|T[P_a(x_a)P_b(x_b)P_c(x_c)P_d(x_d)]|0\rangle \tag{4.94}$$

with the pseudoscalar quark densities of Eq. (4.9).

In order so see that Eq. (4.94) can indeed be related to $\pi\pi$ scattering, let us first investigate the matrix element of the pseudoscalar density evaluated between a single-pion state and the vacuum, which is defined in terms of the coupling constant G_π [GL 84]:

$$\langle 0|P_i(0)|\pi_j(q)\rangle = \delta_{ij} G_\pi. \tag{4.95}$$

At $\mathcal{O}(p^2)$ we determine the coupling of an external pseudoscalar source p to the Goldstone bosons by inserting $\chi = 2B_0 ip$ into the Lagrangian of Eq. (4.70) (see Fig. 4.4),

$$\mathcal{L}_{\text{ext}} = i\frac{F_0^2 B_0}{2}\text{Tr}(pU^\dagger - Up) = \left\{ \begin{array}{l} 2B_0F_0p_i\pi_i, \\ 2B_0F_0p_i\phi_i[1 - \vec{\phi}^{\,2}/(6F_0^2) + \cdots], \end{array} \right. \tag{4.96}$$

[26] For a general proof of the equivalence of S-matrix elements evaluated at tree level (phenomenological approximation), see Sec. 2 of Ref. [Col+ 69].

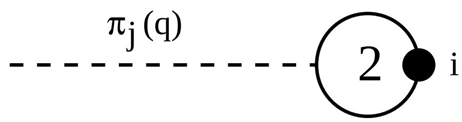


Figure 4.4: Coupling of an external pseudoscalar field p_i (denoted by "• i") to a pion π_j at $\mathcal{O}(p^2)$.

where the first and second lines refer to the parameterizations of Eqs. (4.87) and (4.88), respectively. From Eq. (4.96) we obtain $G_\pi = 2B_0F_0$ independent of the parameterization used which, since the pion is on-shell, is a consequence of the equivalence theorem [Chi 61, Kam+ 61]. As a consistency check, let us verify the PCAC relation of Eq. (2.117) (without an external electromagnetic field) evaluated between a single-pion state and the vacuum. For the axial-vector current matrix element, we found at $\mathcal{O}(p^2)$

$$\langle 0|A_i^\mu(x)|\pi_j(q)\rangle = iq^\mu F_0 e^{-iq\cdot x}\delta_{ij}. \tag{4.97}$$

Taking the divergence we obtain

$$\langle 0|\partial_\mu A_i^\mu(x)|\pi_j(q)\rangle \;=\; iq^\mu F_0\partial_\mu e^{-iq\cdot x}\delta_{ij} = M_\pi^2 F_0 e^{-iq\cdot x}\delta_{ij} = 2m_q B_0 F_0 e^{-iq\cdot x}\delta_{ij},$$

where we made use of Eq. (4.45) for the pion mass. Multiplying Eq. (4.95) by m_q and using $G_\pi = 2B_0F_0$ we explicitly verify the PCAC relation.

Every field $\Phi_i(x)$, which satisfies the relation

$$\langle 0|\Phi_i(x)|\pi_j(q)\rangle = \delta_{ij}e^{-iq\cdot x}, \tag{4.98}$$

can serve as a so-called *interpolating* pion field [Bor 60] in the LSZ reduction formulas [Leh+ 55, IZ 80]. For the case of $\pi^a(p_a) + \pi^b(p_b) \to \pi^c(p_c) + \pi^d(p_d)$ the reduction formula relates the S-matrix element to the Green function of the interpolating field as

$$\begin{aligned} S_{fi} &= i^4 \int d^4x_a \cdots d^4x_d\, e^{-ip_a\cdot x_a}\cdots e^{ip_d\cdot x_d} \\ &\quad \times(\Box_a + M_\pi^2)\cdots(\Box_d + M_\pi^2)\langle 0|T[\Phi_a(x_a)\Phi_b(x_b)\Phi_c(x_c)\Phi_d(x_d)]|0\rangle. \end{aligned}$$

After partial integrations, the Klein-Gordon operators convert into inverse free propagators

$$\begin{aligned} S_{fi} &= (-i)^4(p_a^2 - M_\pi^2)\cdots(p_d^2 - M_\pi^2) \\ &\quad \times \int d^4x_a \cdots d^4x_d\, e^{-ip_a\cdot x_a}\cdots e^{ip_d\cdot x_d}\langle 0|T[\Phi_a(x_a)\Phi_b(x_b)\Phi_c(x_c)\Phi_d(x_d)]|0\rangle. \end{aligned}$$

In the present context, we will use

$$\Phi_i(x) = \frac{P_i(x)}{G_\pi} = \frac{P_i(x)}{2B_0F_0} = \frac{m_q P_i(x)}{M_\pi^2 F_0}, \tag{4.99}$$

which then relates the S-matrix element of $\pi\pi$ scattering to the QCD Green function involving four pseudoscalar densities

$$\begin{aligned} S_{fi} &= \left(\frac{-i}{G_\pi}\right)^4 (p_a^2 - M_\pi^2)\cdots(p_d^2 - M_\pi^2) \\ &\quad\times \int d^4x_a \cdots d^4x_d\, e^{-ip_a\cdot x_a}\cdots e^{ip_d\cdot x_d} G^{abcd}_{PPPP}(x_a, x_b, x_c, x_d). \end{aligned}$$

Using translational invariance, let us define the momentum space Green function as

$$\begin{aligned} &(2\pi)^4\delta^4(p_a + p_b + p_c + p_d) F^{abcd}_{PPPP}(p_a, p_b, p_c, p_d) = \\ &\int d^4x_a d^4x_b d^4x_c d^4x_d\, e^{-ip_a\cdot x_a} e^{-ip_b\cdot x_b} e^{-ip_c\cdot x_c} e^{-ip_d x_d} G^{abcd}_{PPPP}(x_a, x_b, x_c, x_d), \end{aligned} \tag{4.100}$$

where we define all momenta as incoming. The usual relation between the S matrix and the T matrix, $S = I + iT$, implies for the T-matrix element $\langle f|T|i\rangle = (2\pi)^4\delta^4(P_f - P_I)\mathcal{T}_{fi}$ and, finally, for $\mathcal{M} = i\mathcal{T}_{fi}$:

$$\mathcal{M} = \frac{1}{G_\pi^4}\left[\prod_{k=a,b,c,d} \lim_{p_k^2\to M_\pi^2} (p_k^2 - M_\pi^2)\right] F^{abcd}_{PPPP}(p_a, p_b, -p_c, -p_d). \tag{4.101}$$

We will now determine the Green function $F^{abcd}_{PPPP}(p_a, p_b, -p_c, -p_d)$ using the parameterizations of Eqs. (4.87) and (4.88) for U. In the first parameterization we only obtain a linear coupling between the external pseudoscalar field and the pion field [see Eq. (4.96)] so that only the Feynman diagram of Fig. 4.5 contributes

$$F^{abcd}_{PPPP}(p_a, p_b, -p_c, -p_d) = (2B_0F_0)^4 \frac{i}{p_a^2 - M_\pi^2}\cdots\frac{i}{p_d^2 - M_\pi^2}\mathcal{M}_2^{4\pi}, \tag{4.102}$$

where $\mathcal{M}_2^{4\pi}$ is given in Eq. (4.92). The Green function depends on six independent Lorentz scalars which can be chosen as the squared invariant momenta p_k^2 and the three Mandelstam variables s, t, and u satisfying the constraint $s + t + u = \sum_k p_k^2$.

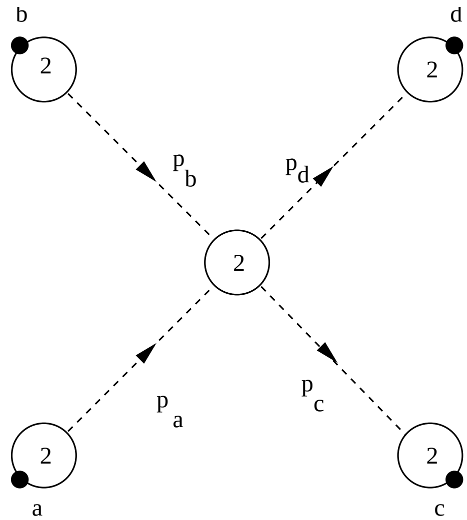


Figure 4.5: Four-point Green function $F^{abcd}_{PPPP}(p_a, p_b, -p_c, -p_d)$ at $\mathcal{O}(p^2)$ in the parameterization of Eq. (4.87). The $\bullet$ denote the pseudoscalar sources which are "removed" from the diagram.

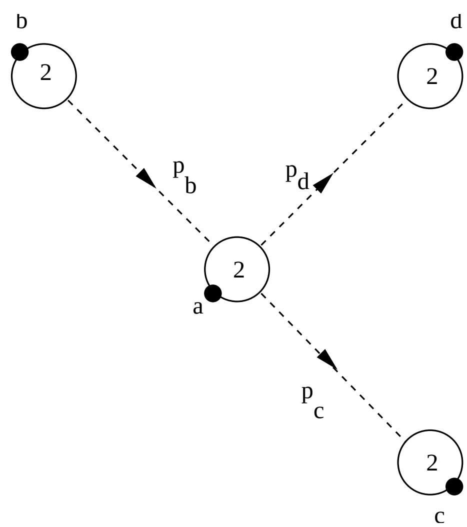


Figure 4.6: Additional contribution to the four-point Green function $F^{abcd}_{PPPP}(p_a, p_b, -p_c, -p_d)$ at $\mathcal{O}(p^2)$ in the parameterization of Eq. (4.88). The remaining three permutations are not shown. The $\bullet$ denote the pseudoscalar sources which are "removed" from the diagram.

Using the second parameterization we will obtain a contribution which is of the same form as Fig. 4.5 but with $\mathcal{M}_2^{4\pi}$ replaced by $\mathcal{M}_2^{4\phi}$ of Eq. (4.93). Clearly, this is not yet the same result as Eq. (4.102) because of the terms proportional to Λ_k in Eq. (4.93). However, in this parameterization the external pseudoscalar field also couples to three pion fields [see Eq. (4.96)], resulting in four additional diagrams of the type shown in Fig. 4.6. For example, the contribution shown in Fig. 4.6 reads

$$
\begin{aligned}
&\Delta_a F^{abcd}_{PPPP}(p_a, p_b, -p_c, -p_d)\\
&= (2B_0F_0)^3 \frac{i}{p_b^2-M_\pi^2}\frac{i}{p_c^2-M_\pi^2}\frac{i}{p_d^2-M_\pi^2}\left(-\frac{2B_0}{3F_0}\right)(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})\\
&= (2B_0F_0)^4 \frac{i}{p_a^2-M_\pi^2}\cdots\frac{i}{p_d^2-M_\pi^2}\frac{i\Lambda_a}{3F_0^2}(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}), \qquad (4.103)
\end{aligned}
$$

where $\Lambda_a=(p_a^2-M_\pi^2)$. In combination with the contribution of the remaining three diagrams, we find a complete cancelation with those terms proportional to Λ_k of Fig. 4.5 (in the second parameterization) and the end result is identical with Eq. (4.102). Finally, using $G_\pi=2B_0F_0$ and inserting the result of Eq. (4.102) into Eq. (4.101) we obtain the same scattering amplitude as in the "direct" calculation of Eqs. (4.92) and (4.93) evaluated for on-shell pions.

This example serves as an illustration that the method of Gasser and Leutwyler generates unique results for the Green functions of QCD for arbitrary four-momenta. There is no ambiguity resulting from the choice of variables used to parameterize the matrix U in the effective Lagrangian. These Green functions can be evaluated for arbitrary (but small) four-momenta. Using the reduction formalism, on-shell matrix elements such as the $\pi\pi$ scattering amplitude can be calculated from the QCD Green functions. The result for the $\pi\pi$ scattering amplitude as derived from Eq. (4.101) agrees with the "direct" calculation of the on-shell matrix elements of Eqs. (4.92) and (4.93). On the other hand, the Feynman rules of Eqs. (4.92) and (4.93), when taken *off shell*, have to be considered as intermediate building blocks only and thus need not be unique.

4.7 The Chiral Lagrangian at Order $\mathcal{O}(p^4)$

Applying the ideas outlined in Sec. 4.5 it is possible to construct the most general Lagrangian at $\mathcal{O}(p^4)$. Here we only quote the result of Ref. [GL 85a]:

$$\begin{aligned}
\mathcal{L}_4 = {} & L_1 \left\{\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\right\}^2 + L_2 \mathrm{Tr}\left[D_\mu U(D_\nu U)^\dagger\right] \mathrm{Tr}\left[D^\mu U(D^\nu U)^\dagger\right] \\
& + L_3 \mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger D_\nu U(D^\nu U)^\dagger\right] + L_4 \mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger\right] \mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right) \\
& + L_5 \mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger(\chi U^\dagger + U\chi^\dagger)\right] + L_6 \left[\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right)\right]^2 \\
& + L_7 \left[\mathrm{Tr}\left(\chi U^\dagger - U\chi^\dagger\right)\right]^2 + L_8 \mathrm{Tr}\left(U\chi^\dagger U\chi^\dagger + \chi U^\dagger \chi U^\dagger\right) \\
& - iL_9 \mathrm{Tr}\left[f^R_{\mu\nu} D^\mu U(D^\nu U)^\dagger + f^L_{\mu\nu}(D^\mu U)^\dagger D^\nu U\right] + L_{10} \mathrm{Tr}\left(U f^L_{\mu\nu} U^\dagger f^{\mu\nu}_R\right) \\
& + H_1 \mathrm{Tr}\left(f^R_{\mu\nu} f^{\mu\nu}_R + f^L_{\mu\nu} f^{\mu\nu}_L\right) + H_2 \mathrm{Tr}\left(\chi\chi^\dagger\right). \qquad (4.104)
\end{aligned}$$

The numerical values of the low-energy coupling constants L_i are not determined by chiral symmetry. In analogy to F_0 and B_0 of $\mathcal{L}_2$ they are parameters containing information on the underlying dynamics and should, in principle, be calculable in terms of the (remaining) parameters of QCD, namely, the heavy-quark masses and the QCD scale Λ_{QCD}. In practice, they parameterize our inability to solve the dynamics of QCD in the nonperturbative regime. So far they have either been fixed using empirical input [GL 84, GL 85a, Bij+ 95a] or theoretically using QCD-inspired models [ER 86, Esp+ 90, Ebe+ 93, Bij+ 93], meson-resonance saturation [Eck+ 89a, Eck+ 89b, Don+ 89, KN 01, Leu 01b], and lattice QCD [MC 94, Gol 02].

From a practical point of view the coefficients are also required for another purpose. When calculating one-loop graphs, using vertices from $\mathcal{L}_2$ of Eq. (4.70), one generates infinities which, according to Weinberg's power counting of Eq. (4.52), are of $\mathcal{O}(p^4)$, i.e., which cannot be absorbed by a renormalization of the coefficients F_0 and B_0. In the framework of dimensional regularization (see App. B) these divergences appear as poles at space-time dimension $n = 4$. In Refs. [GL 84, GL 85a] the poles, together with the relevant counter terms, were given in closed form. To that end, Gasser and Leutwyler made use of the so-called saddle-point method which, in the path-integral approach, allows one to identify the one-loop contribution to the generating functional. The action is expanded around the classical solution and the path integral is performed with respect to the terms quadratic in the fluctuations about the classical solution. The resulting one-loop piece of the generating functional is treated within the dimensional-regularization procedure and the poles are isolated by applying the so-called heat-kernel technique.[27] Except for L_3 and L_7 the low-energy coupling constants L_i and the "contact terms"—i.e., pure external field terms—H_1 and H_2 are required in the renormalization of the one-loop graphs [GL 85a]. Since H_1 and H_2 contain only external fields, they are of no physical relevance [GL 85a].

By construction Eq. (4.104) represents the most general Lagrangian at $\mathcal{O}(p^4)$, and it is thus possible to absorb the one-loop divergences by an appropriate renormalization of the coefficients L_i and H_i [GL 85a]:

$$
\begin{aligned}
L_i &= L_i^r + \frac{\Gamma_i}{32\pi^2} R, \quad i = 1, \cdots, 10, && (4.105)\\
H_i &= H_i^r + \frac{\Delta_i}{32\pi^2} R, \quad i = 1, 2, && (4.106)
\end{aligned}
$$

where R is defined as (see App. B)

$$
R = \frac{2}{n-4} - [\ln(4\pi) - \gamma_E + 1], \tag{4.107}
$$

with n denoting the number of space-time dimensions and $\gamma_E = -\Gamma'(1)$ being Euler's constant. The constants Γ_i and Δ_i are given in Table 4.3. The renormalized coefficients L_i^r depend on the scale μ introduced by dimensional regularization [see Eq. (B.12)] and their values at two different scales μ_1 and μ_2 are related by

$$
L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{16\pi^2} \ln\left(\frac{\mu_1}{\mu_2}\right). \tag{4.108}
$$

We will see that the scale dependence of the coefficients and the finite part of the loop-diagrams compensate each other in such a way that physical observables are scale independent.

We finally discuss the method of using field transformations to eliminate redundant terms in the most general effective Lagrangian [Geo 91, Leu 92, Gro 94, Arz 95, SF 95]. From a "naive" point of view the two structures

$$
\mathrm{Tr}[D^2U(D^2U)^\dagger], \quad \mathrm{Tr}[D^2U\chi^\dagger + \chi(D^2U)^\dagger] \tag{4.109}
$$

would qualify as independent terms of order $\mathcal{O}(p^4)$. Loosely speaking, by using the classical equation of motion of Eq. (4.77) these terms can be shown

[27] Since the whole procedure is rather technical, we will restrict ourselves, by means of the example to be discussed in Sec. 4.9.1, to an explicit verification that the renormalization procedure indeed leads to finite predictions for physical observables.

Table 4.3: Renormalized low-energy coupling constants L_i^r in units of 10^{-3} at the scale $\mu = M_\rho$ [Bij+ 95a]. $\Delta_1 = -1/8$, $\Delta_2 = 5/24$.

Coefficient	Empirical Value	Γ_i
L_1^r	0.4 ± 0.3	$\frac{3}{32}$
L_2^r	1.35 ± 0.3	$\frac{3}{16}$
L_3^r	-3.5 ± 1.1	0
L_4^r	-0.3 ± 0.5	$\frac{1}{8}$
L_5^r	1.4 ± 0.5	$\frac{3}{8}$
L_6^r	-0.2 ± 0.3	$\frac{11}{144}$
L_7^r	-0.4 ± 0.2	0
L_8^r	0.9 ± 0.3	$\frac{5}{48}$
L_9^r	6.9 ± 0.7	$\frac{1}{4}$
L_{10}^r	-5.5 ± 0.7	$-\frac{1}{4}$

to be redundant. We will justify this statement in terms of field transformations. To that end let us consider another SU(3) matrix $U'(x)$ which is related to $U(x)$ by a field transformation of the form

$$U(x) = \exp[iS(x)]U'(x). \tag{4.110}$$

Since both U and U' are SU(3) matrices, $S(x)$ must be a Hermitian traceless 3×3 matrix. We demand that $U'(x)$ satisfies the same symmetry properties as $U(x)$ (see Table 4.2),

$$U' \stackrel{G}{\mapsto} V_R U' V_L^\dagger, \quad U'(\vec{x},t) \stackrel{P}{\mapsto} U'^\dagger(-\vec{x},t), \quad U' \stackrel{C}{\mapsto} U'^T, \tag{4.111}$$

from which we obtain the following conditions for S:

$$S \stackrel{G}{\mapsto} V_R S V_R^\dagger, \quad S(\vec{x},t) \stackrel{P}{\mapsto} -U'^\dagger(-\vec{x},t)S(-\vec{x},t)U'(-\vec{x},t), \quad S \stackrel{C}{\mapsto} (U'^\dagger S U')^T. \tag{4.112}$$

The most general transformation is constructed iteratively in the momentum and quark-mass expansion,

$$U = \exp[iS_2(x)]U^{(1)}(x), \quad U^{(1)}(x) = \exp[iS_4(x)]U^{(2)}(x), \quad \cdots, \tag{4.113}$$

where the matrices S_{2n} are of $\mathcal{O}(p^{2n})$, satisfy the properties of Eq. (4.112), and have to be constructed from the same building blocks as the effective Lagrangian.

To be explicit, let us derive the most general matrix $S_2(x)$. At $\mathcal{O}(p^2)$, the field strength tensors cannot contribute as building blocks because of their antisymmetry under interchange of the Lorentz indices. Imposing the transformation behavior under the group $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$, we obtain a list of five terms

$$D^2U'U'^\dagger, \quad U'(D^2U')^\dagger, \quad D_\mu U'(D^\mu U')^\dagger, \quad \chi U'^\dagger, \quad U'\chi^\dagger. \tag{4.114}$$

Parity eliminates three combinations and we are left with

$$D^2U'U'^\dagger - U'(D^2U')^\dagger, \quad \chi U'^\dagger - U'\chi^\dagger. \tag{4.115}$$

Demanding Hermiticity and a vanishing trace, we end up with two terms at $\mathcal{O}(p^2)$:

$$S_2 = i\alpha_1[D^2U'U'^\dagger - U'(D^2U')^\dagger] + i\alpha_2[\chi U'^\dagger - U'\chi^\dagger - \frac{1}{3}\mathrm{Tr}(\chi U'^\dagger - U'\chi^\dagger)], \tag{4.116}$$

where α_1 and α_2 are real numbers. At $\mathcal{O}(p^2)$, charge conjugation does not provide an additional constraint.

What are the consequences of working with $U'(x)$ instead of $U(x)$? In Sec. 4.6.2 we have already argued, by means of a simple example, that the results for the Green functions are independent of the parameterizations of $U(x)$ of Eqs. (4.87) and (4.88). Expressing $U(x)$ of Eq. (4.110) by using Eq. (4.116) and inserting the result into $\mathcal{L}_2$ of Eq. (4.70), we obtain

$$\mathcal{L}_2(U) = \mathcal{L}_2(U') + \Delta\mathcal{L}_2(U'), \tag{4.117}$$

where $\Delta\mathcal{L}_2$, to leading order in S_2, is given by

$$\Delta\mathcal{L}_2(U') = \frac{F_0^2}{4}\mathrm{Tr}[iS_2\mathcal{O}^{(2)}_{\mathrm{EOM}}(U')] + O(S_2^2). \tag{4.118}$$

The functional form of $\mathcal{O}^{(2)}_{\mathrm{EOM}}$ has been defined in Eq. (4.77). Note, however, that we do *not* assume $\mathcal{O}^{(2)}_{\mathrm{EOM}} = 0$. We have dropped a total derivative, since it does not modify the dynamics. Both S_2 and $\mathcal{O}^{(2)}_{\mathrm{EOM}}$ are of order $\mathcal{O}(p^2)$ so that $\Delta\mathcal{L}_2$ is of order $\mathcal{O}(p^4)$. Of course, higher powers of S_2 in Eq. (4.118) induce additional terms of higher orders in the momentum expansion which we will discuss in a moment.

Through a suitable choice of the parameters α_1 and α_2 it is possible to eliminate two structures at order $\mathcal{O}(p^4)$, i.e., one generates a new Lagrangian with a different functional form which, however, according to the equivalence theorem leads to the same observables [Chi 61, Kam+ 61]. Such a procedure is commonly referred to as using the classical equation of motion to eliminate terms. For example, it is straightforward but tedious to re-express the two structures of Eq. (4.109) through the terms of Gasser and Leutwyler, Eq. (4.104), and the following two terms

$$c_1 \mathrm{Tr}\left([D^2UU^\dagger - U(D^2U)^\dagger]\mathcal{O}^{(2)}_{\mathrm{EOM}}\right) + c_2 \mathrm{Tr}\left((\chi U^\dagger - U\chi^\dagger)\mathcal{O}^{(2)}_{\mathrm{EOM}}\right). \quad (4.119)$$

Choosing $\alpha_1 = 4c_1/F_0^2$ and $\alpha_2 = 4c_2/F_0^2$ in Eq. (4.116), the two terms of Eq. (4.119) and the modification $\Delta\mathcal{L}_2$ of Eq. (4.118) precisely cancel and one is left with the canonical form of Gasser and Leutwyler.

A field redefinition, of course, also leads to modifications of the functional form of the effective Lagrangians of higher orders. However, for S_2 such terms are at least of order $\mathcal{O}(p^6)$ as are the higher-order terms in Eq. (4.118). Thus one proceeds iteratively [SF 95]. Using S_2 one generates the simplest form of $\mathcal{L}_4$. Next one constructs S_4, inserts it again into $\mathcal{L}_2$ to simplify $\mathcal{L}_6$, etc.

From a point of view of *constructing* the simplest Lagrangian at a given order it is sufficient to identify those terms proportional to the classical, i.e. lowest-order, equation of motion and drop them right from the beginning using the argument that, by choosing appropriate generators, they can be transformed away. A completely different situation arises if one tries to express the effective Lagrangian obtained within the framework of a specific model in the canonical form. In such a case it is necessary to explicitly perform the iteration process consistently to a given order and, in particular, take into account the modification of the higher-order coefficients due to the transformation. An explicit example is given in Appendix AII of Ref. [Bel+ 95].

4.8 The Effective Wess-Zumino-Witten Action

The Lagrangians $\mathcal{L}_2$ and $\mathcal{L}_4$ discussed so far exhibit a larger symmetry than the "real world." For example, if we consider the case of "pure" QCD, i.e., no external fields except for $\chi = 2B_0M$ with the quark mass matrix M of Eq. (4.39), the two Lagrangians are invariant under the substitution $\phi(x) \mapsto -\phi(x)$. As discussed in Sec. 4.3 they contain interaction terms with an even number of Goldstone bosons only, i.e., they are of even intrinsic parity, and it would not be possible to describe the reaction $K^+K^- \to \pi^+\pi^-\pi^0$.[28] Analogously, the process $\pi^0 \to \gamma\gamma$ cannot be described by $\mathcal{L}_2$ and $\mathcal{L}_4$ in the presence of external electromagnetic fields.

These observations lead us to a discussion of the effective Wess-Zumino-Witten action [WZ 71, Wit 83]. Whereas normal Ward identities are related to the *invariance* of the generating functional under local transformations of the external fields, the anomalous Ward identities [Adl 69, AB 69, Bar 69, BJ 69, Adl 70], which were first obtained in the framework of renormalized perturbation theory, give a particular form to the *variation* of the generating functional [WZ 71, GL 84]. Wess and Zumino derived consistency or integrability relations which are satisfied by the anomalous Ward identities and then explicitly constructed a functional involving the pseudoscalar octet which satisfies the anomalous Ward identities [WZ 71]. In particular, Wess and Zumino emphasized that their interaction Lagrangians cannot be obtained as part of a chiral invariant Lagrangian.

In the construction of Witten [Wit 83] the simplest term possible which breaks the symmetry of having only an even number of Goldstone bosons at the Lagrangian level is added to the equation of motion of Eq. (4.77) for the case of massless Goldstone bosons without any external fields,[29]

$$\partial_\mu \left(\frac{F_0^2}{2} U \partial^\mu U^\dagger \right) + \lambda \epsilon^{\mu\nu\rho\sigma} U \partial_\mu U^\dagger U \partial_\nu U^\dagger U \partial_\rho U^\dagger U \partial_\sigma U^\dagger = 0, \tag{4.120}$$

where λ is a (purely imaginary) constant. Substituting $U \leftrightarrow U^\dagger$ in Eq. (4.120) and subsequently multiplying from the left by U and from the right by $U^\dagger$, we verify that the two terms transform with opposite relative signs. Recall that a term which is even (odd) in the Lagrangian leads to a term which, in the equation of motion, is odd (even).

However, the action functional corresponding to the new term cannot be written as the four-dimensional integral of a Lagrangian expressed in terms of U and its derivatives. Rather, one has to extend the range of definition of

[28] The ϕ meson can decay into both K^+K^- and $\pi^+\pi^-\pi^0$.

[29] In order to conform with our previous convention of Eq. (4.23), we need to substitute $U_{\mathrm{W}} \to U^\dagger$. Furthermore F_π of Ref. [Wit 83] corresponds to $2F_0$. Finally,

$$\partial^2 U U^\dagger - U \partial^2 U^\dagger = 2\partial_\mu(\partial^\mu U U^\dagger).$$

the fields to a hypothetical fifth dimension,

$$U(y) = \exp\left(i\alpha\frac{\phi(x)}{F_0}\right), \quad y^i = (x^\mu, \alpha),\ i = 0, \cdots, 4,\ 0 \le \alpha \le 1, \tag{4.121}$$

where Minkowski space is defined as the surface of the five-dimensional space for $\alpha = 1$. Let us first quote the result of the effective Wess-Zumino-Witten action in the absence of external fields (denoted by a superscript 0):

$$\begin{aligned} S^0_{\text{ano}} &= nS^0_{\text{WZW}}, & (4.122)\\ S^0_{\text{WZW}} &= -\frac{i}{240\pi^2}\int_0^1 d\alpha \int d^4x \epsilon^{ijklm}\text{Tr}\left(\mathcal{U}^L_i\mathcal{U}^L_j\mathcal{U}^L_k\mathcal{U}^L_l\mathcal{U}^L_m\right), & (4.123)\end{aligned}$$

where the indices $i, \cdots, m$ run from 0 to 4, $y_4 = y^4 = \alpha$, ϵ_{ijklm} is the completely antisymmetric tensor with $\epsilon_{01234} = -\epsilon^{01234} = 1$, and $\mathcal{U}^L_i = U^\dagger \partial U/\partial y^i$. By calculating the variation of the action functional as in Eq. (4.74) we find that the constant λ of Eq. (4.120) and n of Eq. (4.122) are related by $\lambda = in/(48\pi^2)$. Using topological arguments Witten showed that the constant n appearing in Eq. (4.122) must be an integer. Below, n will be identified with the number of colors N_c. Expanding the SU(3) matrix $U(y)$ in terms of the Goldstone boson fields, $U(y) = 1 + i\alpha\phi(x)/F_0 + O(\phi^2)$, one obtains an infinite series of terms, each involving an odd number of Goldstone bosons, i.e., the WZW action S^0_{WZW} is of odd intrinsic parity. For each individual term the α integration can be performed explicitly resulting in an ordinary action in terms of a four-dimensional integral of a local Lagrangian. For example, the term with the smallest number of Goldstone bosons reads

$$\begin{aligned} S^{5\phi}_{\text{WZW}} &= \frac{1}{240\pi^2F_0^5}\int_0^1 d\alpha \int d^4x \epsilon^{ijklm}\text{Tr}[\partial_i(\alpha\phi)\partial_j(\alpha\phi)\partial_k(\alpha\phi)\partial_l(\alpha\phi)\partial_m(\alpha\phi)] \\ &= \frac{1}{240\pi^2F_0^5}\int_0^1 d\alpha \int d^4x \epsilon^{ijklm}\partial_i\text{Tr}[\alpha\phi\partial_j(\alpha\phi)\partial_k(\alpha\phi)\partial_l(\alpha\phi)\partial_m(\alpha\phi)] \\ &= \frac{1}{240\pi^2F_0^5}\int d^4x \epsilon^{\mu\nu\rho\sigma}\text{Tr}(\phi\partial_\mu\phi\partial_\nu\phi\partial_\rho\phi\partial_\sigma\phi). \end{aligned} \tag{4.124}$$

In the last step we made use of the fact that exactly one index can take the value 4. The term involving $i = 4$ has been integrated with respect to α whereas the other four possibilities cancel each other because the ϵ tensor in four dimensions is antisymmetric under a cyclic permutation of the indices whereas the trace is symmetric under a cyclic permutation. In particular, the

WZW action without external fields involves at least five Goldstone bosons [WZ 71].

The connection to the number of colors N_c is established by introducing a coupling to electromagnetism [WZ 71, Wit 83]. In the presence of external fields there will be an additional term in the anomalous action,

$$S_{\text{ano}} = S^0_{\text{ano}} + S^{\text{ext}}_{\text{ano}} = n(S^0_{\text{WZW}} + S^{\text{ext}}_{\text{WZW}}), \tag{4.125}$$

given by [Cho+ 84, PR 85, Man 85, Bij 93]

$$S^{\text{ext}}_{\text{WZW}} = -\frac{i}{48\pi^2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\text{Tr}(Z_{\mu\nu\rho\sigma}) \tag{4.126}$$

with

$$\begin{aligned}
Z_{\mu\nu\rho\sigma} \;=\;& \frac{1}{2}\; Ul_\mu U^\dagger r_\nu U l_\rho U^\dagger r_\sigma \\
&+Ul_\mu l_\nu l_\rho U^\dagger r_\sigma - U^\dagger r_\mu r_\nu r_\rho U l_\sigma \\
&+iU\partial_\mu l_\nu l_\rho U^\dagger r_\sigma - iU^\dagger\partial_\mu r_\nu r_\rho U l_\sigma \\
&+i\partial_\mu r_\nu U l_\rho U^\dagger r_\sigma - i\partial_\mu l_\nu U^\dagger r_\rho U l_\sigma \\
&-i\mathcal{U}^L_\mu l_\nu U^\dagger r_\rho U l_\sigma + i\mathcal{U}^R_\mu r_\nu U l_\rho U^\dagger r_\sigma \\
&-i\mathcal{U}^L_\mu l_\nu l_\rho l_\sigma + i\mathcal{U}^R_\mu r_\nu r_\rho r_\sigma \\
&+\frac{1}{2}\mathcal{U}^L_\mu U^\dagger\partial_\nu r_\rho U l_\sigma - \frac{1}{2}\mathcal{U}^R_\mu U\partial_\nu l_\rho U^\dagger r_\sigma \\
&+\frac{1}{2}\mathcal{U}^L_\mu U^\dagger r_\nu U\partial_\rho l_\sigma - \frac{1}{2}\mathcal{U}^R_\mu U l_\nu U^\dagger\partial_\rho r_\sigma \\
&-\mathcal{U}^L_\mu\mathcal{U}^L_\nu U^\dagger r_\rho U l_\sigma + \mathcal{U}^R_\mu\mathcal{U}^R_\nu U l_\rho U^\dagger r_\sigma \\
&+\mathcal{U}^L_\mu l_\nu\partial_\rho l_\sigma - \mathcal{U}^R_\mu r_\nu\partial_\rho r_\sigma \\
&+\mathcal{U}^L_\mu\partial_\nu l_\rho l_\sigma - \mathcal{U}^R_\mu\partial_\nu r_\rho r_\sigma \\
&+\frac{1}{2}\;\mathcal{U}^L_\mu l_\nu\mathcal{U}^L_\rho l_\sigma - \frac{1}{2}\;\mathcal{U}^R_\mu r_\nu\mathcal{U}^R_\rho r_\sigma \\
&-i\mathcal{U}^L_\mu\mathcal{U}^L_\nu\mathcal{U}^L_\rho l_\sigma + i\mathcal{U}^R_\mu\mathcal{U}^R_\nu\mathcal{U}^R_\rho r_\sigma,
\end{aligned} \tag{4.127}$$

where we defined the abbreviations $\mathcal{U}^L_\mu = U^\dagger\partial_\mu U$ and $\mathcal{U}^R_\mu = U\partial_\mu U^\dagger$. In the commonly used expression [Bij 93], we performed the replacement

$$\begin{aligned}
&\mathcal{U}^L_\mu U^\dagger\partial_\nu r_\rho U l_\sigma - \mathcal{U}^R_\mu U\partial_\nu l_\rho U^\dagger r_\sigma \to \\
&\quad \frac{1}{2}\mathcal{U}^L_\mu U^\dagger\partial_\nu r_\rho U l_\sigma - \frac{1}{2}\mathcal{U}^R_\mu U\partial_\nu l_\rho U^\dagger r_\sigma + \frac{1}{2}\mathcal{U}^L_\mu U^\dagger r_\nu U\partial_\rho l_\sigma - \frac{1}{2}\mathcal{U}^R_\mu U l_\nu U^\dagger\partial_\rho r_\sigma,
\end{aligned}$$

in order to generate a manifestly C invariant and Hermitian action. Without this replacement charge-conjugation invariance and Hermiticity are satisfied up to a total derivative only. As a special case, let us consider a coupling to external electromagnetic fields by inserting

$$r_\mu = l_\mu = -eQ\mathcal{A}_\mu,$$

where Q is the quark charge matrix [see Eq. (2.110)]. The terms involving three and four electromagnetic four-potentials vanish upon contraction with the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$, because their contributions to $Z_{\mu\nu\rho\sigma}$ are symmetric in at least two indices, and we obtain

$$\begin{aligned} n\mathcal{L}^{\text{ext}}_{\text{WZW}} &= -en\mathcal{A}_\mu J^\mu + i\frac{ne^2}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu\mathcal{A}_\rho\mathcal{A}_\sigma \text{Tr}[2Q^2(U\partial_\mu U^\dagger - U^\dagger\partial_\mu U) \\ &\quad -QU^\dagger Q\partial_\mu U + QUQ\partial_\mu U^\dagger]. \end{aligned} \tag{4.128}$$

We note that the current

$$J^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}}{48\pi^2}\text{Tr}(Q\partial_\nu UU^\dagger\partial_\rho UU^\dagger\partial_\sigma UU^\dagger + QU^\dagger\partial_\nu UU^\dagger\partial_\rho UU^\dagger\partial_\sigma U), \quad \epsilon_{0123} = 1, \tag{4.129}$$

by itself is not gauge invariant and the additional terms of Eq. (4.128) are required to obtain a gauge-invariant action.

The identification of the constant n with the number of colors N_c [Wit 83] results from finding in Eq. (4.128) the interaction Lagrangian which is relevant to the decay $\pi^0 \to \gamma\gamma$. Since $U = 1 + i\text{diag}(\pi^0, -\pi^0, 0)/F_0 + \cdots$, Eq. (4.128) contains a piece

$$\mathcal{L}_{\pi^0\gamma\gamma} = -\frac{ne^2}{96\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\frac{\pi^0}{F_0}, \tag{4.130}$$

where we made use of a partial integration to shift the derivative from the pion field onto the electromagnetic four-potential. The corresponding invariant amplitude reads

$$\mathcal{M} = i\frac{ne^2}{12\pi^2 F_0}\epsilon^{\mu\nu\rho\sigma}q_{1\mu}\epsilon^*_{1\nu}q_{2\rho}\epsilon^*_{2\sigma}, \tag{4.131}$$

which agrees with a direct calculation of the anomaly term in terms of u and d quarks with $n = N_c$ colors (see, e.g., Ref. [Vel 94]). After summation over the final photon polarizations and integration over phase space, the decay rate reads

$$\Gamma_{\pi^0\to\gamma\gamma} = \frac{\alpha^2 M_{\pi^0}^3 n^2}{576\pi^3 F_0^2} = 7.6\,\mathrm{eV} \times \left(\frac{n}{3}\right)^2, \tag{4.132}$$

which is in good agreement with the experimental value (7.7 ± 0.6) eV for $n = N_c = 3$ [Gro+ 00].

4.9 Applications at Order $\mathcal{O}(p^4)$

4.9.1 Masses of the Goldstone Bosons

A discussion of the masses at $\mathcal{O}(p^4)$ will allow us to illustrate various properties typical of chiral perturbation theory:

1. The relation between the bare low-energy coupling constants L_i and the renormalized coefficients L_i^r in Eq. (4.105) is such that the divergences of one-loop diagrams are canceled.

2. Similarly, the scale dependence of the coefficients $L_i^r(\mu)$ on the one hand and of the finite contributions of the one-loop diagrams on the other hand leads to scale-independent predictions for physical observables.

3. A perturbation expansion in the explicit symmetry breaking with respect to a symmetry that is realized in the Nambu-Goldstone mode generates corrections which are non-analytic in the symmetry breaking parameter [LP 71], here the quark masses.

Let us consider $\mathcal{L}_2 + \mathcal{L}_4$ for QCD with finite quark masses but in the absence of external fields. We restrict ourselves to the limit of isospin symmetry, i.e., $m_u = m_d = m$. In order to determine the masses we calculate the self energies $\Sigma(p^2)$ of the Goldstone bosons.

The propagator of a (pseudo-) scalar field is defined as the Fourier transform of the two-point Green function:

$$i\Delta(p) = \int d^4x e^{-ip\cdot x}\langle 0|T\left[\Phi_0(x)\Phi_0(0)\right]|0\rangle, \tag{4.133}$$

where the index 0 refers to the fact that we still deal with the bare unrenormalized field—not to be confused with a free field without interaction. At

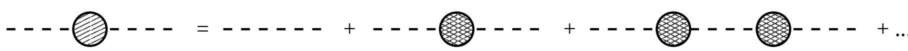


Figure 4.7: Unrenormalized propagator as the sum of irreducible self-energy diagrams. Hatched and cross-hatched "vertices" denote one-particle-reducible and one-particle-irreducible contributions, respectively.

lowest order ($D = 2$) the propagator simply reads

$$i\Delta(p) = \frac{i}{p^2 - M_0^2 + i0^+}, \tag{4.134}$$

where the lowest-order masses M_0 are given in Eqs. (4.45) - (4.47):

$$\begin{aligned} M_{\pi,0}^2 &= 2B_0 m, \\ M_{K,0}^2 &= B_0(m + m_s), \\ M_{\eta,0}^2 &= \frac{2}{3} B_0 \left(m + 2m_s\right). \end{aligned}$$

The loop diagrams with $\mathcal{L}_2$ and the contact diagrams with $\mathcal{L}_4$ result in so-called proper self-energy insertions $-i\Sigma(p^2)$, which may be summed using a geometric series (see Fig. 4.7):

$$\begin{aligned} i\Delta(p) &= \frac{i}{p^2 - M_0^2 + i0^+} + \frac{i}{p^2 - M_0^2 + i0^+}[-i\Sigma(p^2)]\frac{i}{p^2 - M_0^2 + i0^+} + \cdots \\ &= \frac{i}{p^2 - M_0^2 - \Sigma(p^2) + i0^+}. \end{aligned} \tag{4.135}$$

Note that $-i\Sigma(p^2)$ consists of one-particle-irreducible diagrams only, i.e., diagrams which do not fall apart into two separate pieces when cutting an arbitrary internal line. The physical mass, including the interaction, is defined as the position of the pole of Eq. (4.135),

$$M^2 - M_0^2 - \Sigma(M^2) \stackrel{!}{=} 0. \tag{4.136}$$

Let us assume that $\Sigma(p^2)$ can be expanded in a series around $p^2 = \lambda^2$,

$$\Sigma(p^2) = \Sigma(\lambda^2) + (p^2 - \lambda^2)\Sigma'(\lambda^2) + \tilde{\Sigma}(p^2), \tag{4.137}$$

where the remainder $\tilde{\Sigma}(p^2)$ depends on the choice of λ^2 and satisfies $\tilde{\Sigma}(\lambda^2) = \tilde{\Sigma}'(\lambda^2) = 0$. We then obtain for the propagator

$$i\Delta(p) = \frac{i}{p^2 - M_0^2 - \Sigma(\lambda^2) - (p^2 - \lambda^2)\Sigma'(\lambda^2) - \tilde{\Sigma}(p^2) + i0^+}. \qquad (4.138)$$

Taking $\lambda^2 = M^2$ in Eq. (4.138) and applying the condition of Eq. (4.136), the propagator may be written as

$$i\Delta(p) = \frac{i}{(p^2 - M^2)[1 - \Sigma'(M^2)] - \tilde{\Sigma}(p^2) + i0^+} = \frac{iZ_\Phi}{p^2 - M^2 - Z_\Phi \tilde{\Sigma}(p^2) + i0^+},$$

where we have introduced the wave function renormalization constant

$$Z_\Phi = \frac{1}{1 - \Sigma'(M^2)}.$$

Introducing renormalized fields as $\Phi_R = \Phi_0/\sqrt{Z_\Phi}$, the renormalized propagator is given by

$$\begin{aligned} i\Delta_R(p) &= \int d^4x e^{-ip\cdot x} \langle 0|T[\Phi_R(x)\Phi_R(0)]|0\rangle \\ &= \frac{i}{p^2 - M^2 - Z_\Phi \tilde{\Sigma}(p^2) + i0^+}. \end{aligned}$$

In particular, since $\tilde{\Sigma}(M^2) = \tilde{\Sigma}'(M^2) = 0$, in the vicinity of the pole, the renormalized propagator behaves as a free propagator with physical mass M^2.

Let us now turn to the calculation within the framework of ChPT (see, e.g., Ref. [Rud+ 94]). Since $\mathcal{L}_2$ and $\mathcal{L}_4$ without external fields generate vertices with an even number of Goldstone bosons only, the candidate terms at $D = 4$ contributing to the self energy are those shown in Fig. 4.8. For our particular application with exactly two external meson lines, the relevant interaction Lagrangians can be written as [Rud+ 94]

$$\mathcal{L}_{\text{int}} = \mathcal{L}_2^{4\phi} + \mathcal{L}_4^{2\phi}, \qquad (4.139)$$

where $\mathcal{L}_2^{4\phi}$ is given by

$$\mathcal{L}_2^{4\phi} = \frac{1}{24F_0^2}\left\{\text{Tr}([\phi, \partial_\mu\phi]\phi\partial^\mu\phi) + B_0\text{Tr}(M\phi^4)\right\}. \qquad (4.140)$$

Figure 4.8: Self-energy diagrams at $\mathcal{O}(p^4)$. Vertices derived from $\mathcal{L}_{2n}$ are denoted by $2n$ in the interaction blobs.

The terms of $\mathcal{L}_4$ proportional to L_9, L_{10}, H_1, and H_2 do not contribute, because they either contain field-strength tensors or external fields only. Since $\partial_\mu U = O(\phi)$, the L_1, L_2, and L_3 terms of Eq. (4.104) are $O(\phi^4)$ and need not be considered. The only candidates are the L_4 - L_8 terms, of which we consider the L_4 term as an explicit example,[30]

$$\begin{aligned}
&L_4 \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) = \\
&\quad L_4 \frac{2}{F_0^2}[\partial_\mu \eta \partial^\mu \eta + \partial_\mu \pi^0 \partial^\mu \pi^0 + 2\partial_\mu \pi^+ \partial^\mu \pi^- + 2\partial_\mu K^+ \partial^\mu K^- \\
&\quad +2\partial_\mu K^0 \partial^\mu \bar{K}^0 + O(\phi^4)][4B_0(2m+m_s) + O(\phi^2)].
\end{aligned}$$

The remaining terms are treated analogously and we obtain for $\mathcal{L}_4^{2\phi}$

$$\begin{aligned}
\mathcal{L}_4^{2\phi} \quad = \quad & \frac{1}{2}\left(a_\eta \partial_\mu \eta \partial^\mu \eta - b_\eta \eta^2\right) \\
& +\frac{1}{2}\left(a_\pi \partial_\mu \pi^0 \partial^\mu \pi^0 - b_\pi \pi^0 \pi^0\right) \\
& +a_\pi \partial_\mu \pi^+ \partial^\mu \pi^- - b_\pi \pi^+ \pi^- \\
& +a_K \partial_\mu K^+ \partial^\mu K^- - b_K K^+ K^- \\
& +a_K \partial_\mu K^0 \partial^\mu \bar{K}^0 - b_K K^0 \bar{K}^0, \qquad (4.141)
\end{aligned}$$

where the constants a_ϕ and b_ϕ are given by

$$\begin{aligned}
a_\eta &= \frac{16B_0}{F_0^2}\left[(2m+m_s)L_4 + \frac{1}{3}(m+2m_s)L_5\right], \\
b_\eta &= \frac{64B_0^2}{3F_0^2}\left[(2m+m_s)(m+2m_s)L_6 + 2(m-m_s)^2 L_7 + (m^2+2m_s^2)L_8\right],
\end{aligned}$$

[30]For pedagogical reasons, we make use of the physical fields. From a technical point of view, it is often advantageous to work with the Cartesian fields and, at the end of the calculation, express physical processes in terms of the Cartesian components.

$$
\begin{aligned}
a_\pi &= \frac{16B_0}{F_0^2}\left[(2m+m_s)L_4 + mL_5\right], \\
b_\pi &= \frac{64B_0^2}{F_0^2}\left[(2m+m_s)mL_6 + m^2L_8\right], \\
a_K &= \frac{16B_0}{F_0^2}\left[(2m+m_s)L_4 + \frac{1}{2}(m+m_s)L_5\right], \\
b_K &= \frac{32B_0^2}{F_0^2}\left[(2m+m_s)(m+m_s)L_6 + \frac{1}{2}(m+m_s)^2L_8\right].
\end{aligned}
\tag{4.142}
$$

At $\mathcal{O}(p^4)$ the self energies are of the form

$$
\Sigma_\phi(p^2) = A_\phi + B_\phi p^2, \tag{4.143}
$$

where the constants A_ϕ and B_ϕ receive a tree-level contribution from $\mathcal{L}_4$ and a one-loop contribution with a vertex from $\mathcal{L}_2$ (see Fig. 4.8). For the tree-level contribution of $\mathcal{L}_4$ this is easily seen, because the Lagrangians of Eq. (4.141) contain either exactly two derivatives of the fields or no derivatives at all. For example, the contact contribution for the η reads

$$
-i\Sigma_\eta^{\text{contact}}(p^2) = i2\left[\frac{1}{2}a_\eta(ip_\mu)(-ip^\mu) - \frac{1}{2}b_\eta\right] = i(a_\eta p^2 - b_\eta),
$$

where, as usual, $\partial_\mu\phi$ generates $-ip_\mu$ and ip_μ for initial and final lines, respectively, and the factor two takes account of two combinations of contracting the fields with external lines.

For the one-loop contribution the argument is as follows. The Lagrangian $\mathcal{L}_2^{4\phi}$ contains either two derivatives or no derivatives at all which, symbolically, can be written as $\phi\phi\partial\phi\partial\phi$ and ϕ^4, respectively. The first term results in M^2 or p^2, depending on whether the ϕ or the $\partial\phi$ are contracted with the external fields. The "mixed" situation vanishes upon integration. The second term, ϕ^4, does not generate a momentum dependence.

As a specific example, we evaluate the pion-loop contribution to the π^0 self energy (see Fig. 4.9) by applying the Feynman rule of Eq. (4.93) for $a = c = 3$, $p_a = p_c = p$, $b = d = j$, and $p_b = p_d = k$:[31]

$$
\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}i\left[\underbrace{\delta^{3j}\delta^{3j}}_{1}\frac{(p+k)^2 - M_{\pi,0}^2}{F_0^2} + \underbrace{\delta^{33}\delta^{jj}}_{3}\frac{-M_{\pi,0}^2}{F_0^2} + \underbrace{\delta^{3j}\delta^{j3}}_{1}\frac{(p-k)^2 - M_{\pi,0}^2}{F_0^2}\right.
$$

[31] Note that we work in SU(3) and thus with the exponential parameterization of U.

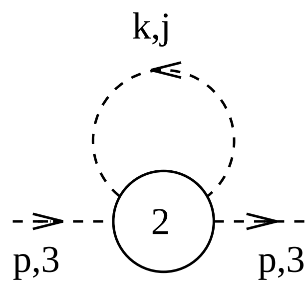


Figure 4.9: Contribution of the pion loops to the π^0 self energy.

$$
\left. -\frac{1}{3F_0^2}\underbrace{(\delta^{3j}\delta^{3j}+\delta^{33}\delta^{jj}+\delta^{3j}\delta^{j3})}_{5}(2p^2+2k^2-4M_{\pi,0}^2)\right]\frac{i}{k^2-M_{\pi,0}^2+i0^+}
$$
$$
=\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{3F_0^2}[-4p^2-4k^2+5M_{\pi,0}^2]\frac{i}{k^2-M_{\pi,0}^2+i0^+}, \qquad (4.144)
$$

where 1/2 is a symmetry factor, as explained in Sec. 4.4. The integral of Eq. (4.144) diverges and we thus consider its extension to n dimensions in order to make use of the dimensional-regularization technique described in App. B. In addition to the loop-integral of Eq. (B.12),

$$
\begin{aligned}
I(M^2,\mu^2,n) &= \mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{i}{k^2-M^2+i0^+}\\
&= \frac{M^2}{16\pi^2}\left[R+\ln\left(\frac{M^2}{\mu^2}\right)\right]+O(n-4),
\end{aligned} \qquad (4.145)
$$

where R is given in Eq. (4.107), we need

$$
\mu^{4-n}i\int\frac{d^nk}{(2\pi)^n}\frac{k^2}{k^2-M^2+i0^+}=\mu^{4-n}i\int\frac{d^nk}{(2\pi)^n}\frac{k^2-M^2+M^2}{k^2-M^2+i0^+},
$$

where we have added $0=-M^2+M^2$ in the numerator. We make use of

$$
\mu^{4-n}i\int\frac{d^nk}{(2\pi)^n}=0
$$

in dimensional regularization (see the discussion at the end of Appendix C.2.2) and obtain

$$\mu^{4-n} i \int \frac{d^n k}{(2\pi)^n} \frac{k^2}{k^2 - M^2 + i0^+} = M^2 I(M^2, \mu^2, n),$$

with $I(M^2, \mu^2, n)$ of Eq. (4.145). The pion-loop contribution to the π^0 self energy is thus

$$\frac{i}{6F_0^2}(-4p^2 + M_{\pi,0}^2) I(M_{\pi,0}^2, \mu^2, n),$$

which is indeed of the type discussed in Eq. (4.143) and diverges as $n \to 4$.

After analyzing all loop contributions and combining them with the contact contributions of Eqs. (4.142), the constants A_ϕ and B_ϕ of Eq. (4.143) are given by

$$\begin{aligned}
A_\pi &= \frac{M_\pi^2}{F_0^2}\Bigg\{ \underbrace{-\frac{1}{6} I(M_\pi^2) - \frac{1}{6} I(M_\eta^2) - \frac{1}{3} I(M_K^2)}_{\text{one-loop contribution}} \\
&\quad \underbrace{+32[(2m + m_s) B_0 L_6 + m B_0 L_8]}_{\text{contact contribution}} \Bigg\}, \\
B_\pi &= \frac{2}{3}\frac{I(M_\pi^2)}{F_0^2} + \frac{1}{3}\frac{I(M_K^2)}{F_0^2} - \frac{16 B_0}{F_0^2}\left[(2m + m_s) L_4 + m L_5\right], \\
A_K &= \frac{M_K^2}{F_0^2}\Bigg\{ \frac{1}{12} I(M_\eta^2) - \frac{1}{4} I(M_\pi^2) - \frac{1}{2} I(M_K^2) \\
&\quad +32\left[(2m + m_s) B_0 L_6 + \frac{1}{2}(m + m_s) B_0 L_8\right] \Bigg\}, \\
B_K &= \frac{1}{4}\frac{I(M_\eta^2)}{F_0^2} + \frac{1}{4}\frac{I(M_\pi^2)}{F_0^2} + \frac{1}{2}\frac{I(M_K^2)}{F_0^2} \\
&\quad -16\frac{B_0}{F_0^2}\left[(2m + m_s) L_4 + \frac{1}{2}(m + m_s) L_5\right], \\
A_\eta &= \frac{M_\eta^2}{F_0^2}\left[-\frac{2}{3} I(M_\eta^2)\right] + \frac{M_\pi^2}{F_0^2}\left[\frac{1}{6} I(M_\eta^2) - \frac{1}{2} I(M_\pi^2) + \frac{1}{3} I(M_K^2)\right] \\
&\quad + \frac{M_\eta^2}{F_0^2}[16 M_\eta^2 L_8 + 32(2m + m_s) B_0 L_6] \\
&\quad + \frac{128}{9}\frac{B_0^2 (m - m_s)^2}{F_0^2}(3L_7 + L_8),
\end{aligned}$$

$$B_\eta = \frac{I(M_K^2)}{F_0^2} - \frac{16}{F_0^2}(2m+m_s)B_0L_4 - 8\frac{M_\eta^2}{F_0^2}L_5, \tag{4.146}$$

where, for simplicity, we have suppressed the dependence on the scale μ and the number of dimensions n in the integrals $I(M^2,\mu^2,n)$ [see Eq. (4.145)]. Furthermore, the squared masses appearing in the loop integrals of Eq. (4.146) are given by the predictions of lowest order, Eqs. (4.45) - (4.47). Finally, the integrals I as well as the bare coefficients L_i (with the exception of L_7) have $1/(n-4)$ poles and finite pieces. In particular, the coefficients A_ϕ and B_ϕ are *not* finite as $n \to 4$.

The masses at $\mathcal{O}(p^4)$ are determined by solving the general equation

$$M^2 = M_0^2 + \Sigma(M^2) \tag{4.147}$$

with the predictions of Eq. (4.143) for the self energies,

$$M^2 = M_0^2 + A + BM^2,$$

where the lowest-order terms, M_0^2, are given in Eqs. (4.45) - (4.47). We then obtain

$$M^2 = \frac{M_0^2 + A}{1-B} = M_0^2(1+B) + A + \mathcal{O}(p^6),$$

because $A = \mathcal{O}(p^4)$ and $\{B, M_0^2\} = \mathcal{O}(p^2)$. Expressing the bare coefficients L_i in Eq. (4.146) in terms of the renormalized coefficients by using Eq. (4.105), the results for the masses of the Goldstone bosons at $\mathcal{O}(p^4)$ read

$$\begin{aligned} M_{\pi,4}^2 &= M_{\pi,2}^2\Bigg\{1 + \frac{M_{\pi,2}^2}{32\pi^2F_0^2}\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right) - \frac{M_{\eta,2}^2}{96\pi^2F_0^2}\ln\left(\frac{M_{\eta,2}^2}{\mu^2}\right) \\ &\quad + \frac{16}{F_0^2}\left[(2m+m_s)B_0(2L_6^r - L_4^r) + mB_0(2L_8^r - L_5^r)\right]\Bigg\}, \qquad (4.148) \\ M_{K,4}^2 &= M_{K,2}^2\Bigg\{1 + \frac{M_{\eta,2}^2}{48\pi^2F_0^2}\ln\left(\frac{M_{\eta,2}^2}{\mu^2}\right) \\ &\quad + \frac{16}{F_0^2}\left[(2m+m_s)B_0(2L_6^r - L_4^r) + \frac{1}{2}(m+m_s)B_0(2L_8^r - L_5^r)\right]\Bigg\}, \\ &\qquad (4.149) \\ M_{\eta,4}^2 &= M_{\eta,2}^2\Bigg[1 + \frac{M_{K,2}^2}{16\pi^2F_0^2}\ln\left(\frac{M_{K,2}^2}{\mu^2}\right) - \frac{M_{\eta,2}^2}{24\pi^2F_0^2}\ln\left(\frac{M_{\eta,2}^2}{\mu^2}\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{16}{F_0^2}(2m+m_s)B_0(2L_6^r-L_4^r)+8\frac{M_{\eta,2}^2}{F_0^2}(2L_8^r-L_5^r)\Bigg]\\
&+M_{\pi,2}^2\left[\frac{M_{\eta,2}^2}{96\pi^2F_0^2}\ln\left(\frac{M_{\eta,2}^2}{\mu^2}\right)-\frac{M_{\pi,2}^2}{32\pi^2F_0^2}\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)\right.\\
&\left.+\frac{M_{K,2}^2}{48\pi^2F_0^2}\ln\left(\frac{M_{K,2}^2}{\mu^2}\right)\right]\\
&+\frac{128}{9}\frac{B_0^2(m-m_s)^2}{F_0^2}(3L_7^r+L_8^r).
\end{aligned}\tag{4.150}
$$

In Eqs. (4.148) - (4.150) we have included the subscripts 2 and 4 in order to indicate from which chiral order the predictions result. First of all, we note that the expressions for the masses are finite. The bare coefficients L_i of the Lagrangian of Gasser and Leutwyler must be infinite in order to cancel the infinities resulting from the divergent loop integrals. Furthermore, at $\mathcal{O}(p^4)$ the masses of the Goldstone bosons vanish, if the quark masses are sent to zero. This is, of course, what we had expected from QCD in the chiral limit but it is comforting to see that the self interaction in $\mathcal{L}_2$ (in the absence of quark masses) does not generate Goldstone boson masses at higher order. At $\mathcal{O}(p^4)$, the squared Goldstone boson masses contain terms which are analytic in the quark masses, namely, of the form m_q^2 multiplied by the renormalized low-energy coupling constants L_i^r. However, there are also non-analytic terms of the type $m_q^2\ln(m_q)$—so-called chiral logarithms—which do not involve new parameters. Such a behavior is an illustration of the mechanism found by Li and Pagels [LP 71], who noticed that a perturbation theory around a symmetry which is realized in the Nambu-Goldstone mode results in both analytic as well as non-analytic expressions in the perturbation. Finally, the scale dependence of the renormalized coefficients L_i^r of Eq. (4.108) is by construction such that it cancels the scale dependence of the chiral logarithms. Thus, physical observables do not depend on the scale μ. Let us verify this statement by differentiating Eqs. (4.148) - (4.150) with respect to μ. Using Eq. (4.108),

$$
L_i^r(\mu)=L_i^r(\mu')+\frac{\Gamma_i}{16\pi^2}\ln\left(\frac{\mu'}{\mu}\right),
$$

we obtain

$$
\frac{dL_i^r(\mu)}{d\mu}=-\frac{\Gamma_i}{16\pi^2\mu}
$$

and, analogously, for the chiral logarithms

$$\frac{d}{d\mu}\ln\left(\frac{M^2}{\mu^2}\right) = 2\frac{d}{d\mu}\left[\ln(M)-\ln(\mu)\right] = -\frac{2}{\mu}.$$

As a specific example, let us differentiate the expression for the pion mass

$$\begin{aligned}
\frac{dM^2_{\pi,4}}{d\mu} &= \frac{M^2_{\pi,2}}{16\pi^2\mu F_0^2}\left\{\frac{M^2_{\pi,2}}{2}(-2)-\frac{M^2_{\eta,2}}{6}(-2)\right.\\
&\quad \left.+16[(2m+m_s)B_0(-2\Gamma_6+\Gamma_4)+mB_0(-2\Gamma_8+\Gamma_5)]\right\}\\
&= \frac{M^2_{\pi,2}}{16\pi^2\mu F_0^2}\left\{-2B_0m+\frac{2}{9}(m+2m_s)B_0\right.\\
&\quad \left.+16\left[(2m+m_s)B_0\underbrace{\left(-2\frac{11}{144}+\frac{1}{8}\right)}_{-\frac{1}{36}}+mB_0\underbrace{\left(-2\frac{5}{48}+\frac{3}{8}\right)}_{\frac{1}{6}}\right]\right\}\\
&= \frac{M^2_{\pi,2}}{16\pi^2\mu F_0^2}\left\{B_0m\left(-2+\frac{2}{9}-\frac{8}{9}+\frac{8}{3}\right)+B_0m_s\left(\frac{4}{9}-\frac{16}{36}\right)\right\}\\
&= 0,
\end{aligned}$$

where we made use of the Γ_i of Table 4.3.

4.9.2 The Electromagnetic Form Factor of the Pion

As a second application at $\mathcal{O}(p^4)$, we discuss the electromagnetic (or vector) form factor of the pion in SU(2)×SU(2) chiral perturbation theory. We will work with two commonly used versions of the $\mathcal{O}(p^4)$ SU(2) × SU(2) mesonic Lagrangian [GL 84, Gas+ 88] which are related by a field transformation (see App. D.1). Furthermore, we will perform the calculation with the two parameterizations for U of Eqs. (4.87) and (4.88). We will thus be able to extend the observations of Sec. 4.6.2 regarding the invariance of physical results under a change of variables to the one-loop level.

According to Eq. (2.111), in the two-flavor sector the coupling to the electromagnetic field $\mathcal{A}_\mu$ contains both isoscalar and isovector terms:

$$\mathcal{L}_{\text{ext}} = \bar{q}\gamma^\mu\left(\frac{1}{3}v_\mu^{(s)}+v_\mu\right)q = -e\mathcal{A}_\mu\bar{q}\gamma^\mu\left(\frac{1}{6}+\frac{\tau_3}{2}\right)q = -e\mathcal{A}_\mu J^\mu,$$

i.e.

$$\begin{aligned} v_\mu &= r_\mu = l_\mu = -e\frac{\tau_3}{2}\mathcal{A}_\mu, && (4.151)\\ v_\mu^{(s)} &= -\frac{e}{2}\mathcal{A}_\mu. && (4.152)\end{aligned}$$

When evaluating the electromagnetic current operator

$$J^\mu = \frac{1}{6}\bar{q}\gamma^\mu q + \bar{q}\gamma^\mu\frac{\tau_3}{2}q$$

between $|\pi^i(p)\rangle$ and $\langle\pi^j(p')|$, the isoscalar first term does not contribute,[32] and the matrix element of the electromagnetic current operator must be of the form[33]

$$\langle\pi^j(p')|J^\mu(0)|\pi^i(p)\rangle = i\epsilon_{3ij}(p'+p)^\mu F(q^2), \quad q = p'-p. \tag{4.153}$$

In other words, we only need to consider Eq. (4.151) which corresponds to a coupling to the third component of the isovector current operator.

In the calculations that follow we make use of the parameterization of Eq. (4.87) for the SU(2) matrix $U(x)$:

$$U(x) = \frac{1}{F_0}\left[\sigma(x) + i\vec{\tau}\cdot\vec{\pi}(x)\right], \quad \sigma(x) = \sqrt{F_0^2 - \vec{\pi}^{\,2}(x)}, \tag{4.154}$$

but we will comment along the way on the features, which would differ when using the parameterization of Eq. (4.88). (The equivalence theorem guarantees that physical observables do not depend on the specific choice of parameterization of U [Chi 61, Kam+ 61].) The covariant derivative of U with the external fields of Eq. (4.151) reads [see Eq. (4.58)]

$$D_\mu U = \partial_\mu U + \frac{i}{2}e\mathcal{A}_\mu[\tau_3, U]$$

and generates, when inserted into the lowest-order Lagrangian of Eq. (4.70), the interaction term

$$\mathcal{L}_2^{\gamma\pi\pi} = -e\epsilon_{3ij}\pi_i\partial^\mu\pi_j\mathcal{A}_\mu. \tag{4.155}$$

[32] The matrix element $\langle\pi^j(p')|\bar{q}\gamma^\mu q|\pi^i(p)\rangle$ must be of the form $\delta^{ij}(p'+p)^\mu f(q^2)$ which results in $(p+p')^\mu f(q^2)$ for the neutral pion ($i=j=3$). On the other hand, under charge conjugation $\bar{q}\gamma^\mu q \mapsto -\bar{q}\gamma^\mu q$ and $|\pi^0\rangle \mapsto |\pi^0\rangle$, and thus $f(q^2) = -f(q^2) = 0$.

[33] A second structure proportional to q^μ vanishes for on-mass-shell pions because of current conservation.

At $\mathcal{O}(p^2)$, therefore, the interaction is that of point-like pions with form factor $F(q^2) = 1$, resulting in the Feynman amplitude (see Fig. 4.10)

$$e\epsilon_{3ij}\epsilon \cdot (p' + p), \tag{4.156}$$

where ϵ denotes the polarization vector of the external real or virtual photon.[34] In particular, using the parameterization of Eq. (4.154), all interaction

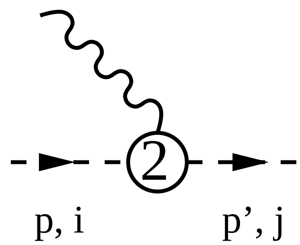


Figure 4.10: Tree-level diagram at $\mathcal{O}(p^2)$.

terms containing one electromagnetic field and $2n$ pions vanish for $n \geq 2$. This is not the case for the exponential parameterization of Eq. (4.88) which generates the more complicated interaction Lagrangian

$$-e\epsilon_{3ij}\phi_i\partial^\mu\phi_j\mathcal{A}_\mu \underbrace{\frac{F_0^2}{|\vec{\phi}|^2}\sin^2\left(\frac{|\vec{\phi}|}{F_0}\right)}_{1-\frac{1}{3}\frac{|\vec{\phi}|^2}{F_0^2}+\cdots} \tag{4.157}$$

which is the same as Eq. (4.155) only at lowest order in the fields.

At $\mathcal{O}(p^4)$ we need to consider a contact term of $\mathcal{L}_4$ (Fig. 4.11) and one-loop diagrams with vertices from $\mathcal{L}_2$ (Figs. 4.12 and 4.13).

We will first work with the Lagrangian of Gasser and Leutwyler [GL 84], Eq. (D.2) of Appendix D.1,[35]

$$\mathcal{L}_4^{\mathrm{GL}} = \cdots + i\frac{l_6}{2}\mathrm{Tr}[f_{\mu\nu}^R D^\mu U(D^\nu U)^\dagger + f_{\mu\nu}^L(D^\mu U)^\dagger D^\nu U] + \cdots, \tag{4.158}$$

[34]For example, in electron scattering reactions often the polarization vector $\epsilon_\mu = e\bar{u}(k_f)\gamma_\mu u(k_i)/q^2$ is used, with four-momentum transfer $q = k_i - k_f$.

[35]The low-energy coupling constants of the SU(2)×SU(2) Lagrangian are denoted by l_i in distinction to the L_i of the SU(3)×SU(3) Lagrangian.

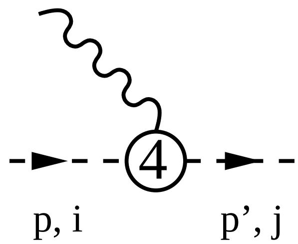


Figure 4.11: Tree-level diagram at $\mathcal{O}(p^4)$.

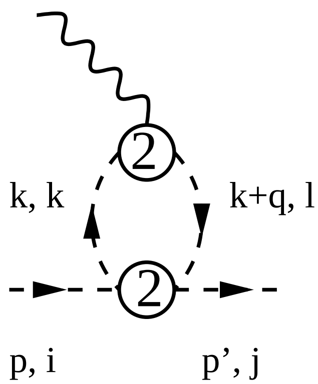


Figure 4.12: One-loop diagram at $\mathcal{O}(p^4)$.

which produces the contact interaction

$$\mathcal{L}_4^{\gamma\pi\pi} = e\frac{l_6}{F_0^2}\epsilon_{3ij}\partial^\mu\pi_i\partial^\nu\pi_j\mathcal{F}_{\mu\nu}, \tag{4.159}$$

resulting in the Feynman amplitude

$$\frac{el_6\epsilon_{3ij}}{F_0^2}\left[-q^2\epsilon\cdot(p'+p)+\epsilon\cdot(p'-p)(p'^2-p^2)\right] \tag{4.160}$$

which vanishes for real photons, $q^2 = q\cdot\epsilon = 0$. The second term vanishes if both pions are on the mass shell, $p^2 = p'^2 = M_\pi^2$, but can be of relevance if the vertex is used as an intermediate building block in a calculation such as virtual Compton scattering off the pion [Unk+ 00]. The Feynman amplitude resulting from Eq. (4.158) is the same for both parameterizations.

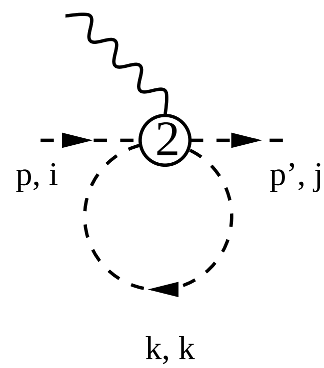


Figure 4.13: One-loop diagram at $\mathcal{O}(p^4)$ contributing in the parameterization of Eq. (4.88) only.

When using the $\mathcal{L}_4$ Lagrangian of Ref. [Gas+ 88], Eq. (D.13), one obtains an additional contact interaction,

$$-2e\frac{l_4 M_{\pi,2}^2}{F_0^2}\epsilon_{3ij}\pi_i\partial^\mu\pi_j\mathcal{A}_\mu, \tag{4.161}$$

where $M_{\pi,2}^2 = 2B_0 m$. For both parameterizations of U we obtain the additional term

$$\frac{2l_4 M_{\pi,2}^2}{F_0^2}e\epsilon_{3ij}\epsilon\cdot(p'+p). \tag{4.162}$$

Let us now turn to the one-loop diagram of Fig. 4.12. The corresponding Feynman amplitude in the parameterization of Eq. (4.154) using the pion-pion vertex of Eq. (4.92) reads

$$\frac{1}{2}\frac{ie\epsilon_{3ij}}{F_0^2}\int\frac{d^4k}{(2\pi)^4}\frac{(2\epsilon\cdot k+\epsilon\cdot q)[(2p+q)\cdot(2k+q)]}{[k^2-M_{\pi,2}^2+i0^+][(k+q)^2-M_{\pi,2}^2+i0^+]}, \tag{4.163}$$

where the 1/2 is a symmetry factor. The integral diverges and its extension to n dimensions is given by

$$\frac{1}{2}\frac{e\epsilon_{3ij}}{F_0^2}\left\{\epsilon\cdot(p'+p)4q^2B_{21}(q^2,M_{\pi,2}^2)\right.$$
$$\left.+\epsilon\cdot q(p'+p)\cdot q[4B_{20}(q^2,M_{\pi,2}^2)+4B_1(q^2,M_{\pi,2}^2)+B_0(q^2,M_{\pi,2}^2)]\right\},$$

where the functions B_0, B_1, B_{20}, and B_{21} are defined in Eqs. (C.7), (C.9), and (C.11) of Appendix C.1. Inserting the results of Eqs. (C.14) and (C.15) the one-loop contribution of Fig. 4.12 finally reads

$$\begin{aligned}
&e\epsilon_{3ij}\left\{\epsilon\cdot(p'+p)\frac{M_{\pi,2}^2}{16\pi^2F_0^2}\left[R+\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)\right]\right.\\
&-\frac{1}{96\pi^2F_0^2}[q^2\epsilon\cdot(p'+p)-\epsilon\cdot q(p'^2-p^2)]\\
&\left.\times\left[R+\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)+\frac{1}{3}+\left(1-\frac{M_{\pi,2}^2}{q^2}\right)J^{(0)}\left(\frac{q^2}{M_{\pi,2}^2}\right)\right]\right\}+O(n-4).
\end{aligned}\tag{4.164}$$

The (infinite) contribution of the first term will be precisely canceled by taking the pion wave function renormalization into account. The second structure is separately gauge invariant and also contains an infinite piece which will be canceled by a corresponding infinite piece of the bare coefficient l_6 [see Eq. (4.160)]. Finally, a calculation of the one-loop diagram of Fig. 4.12 with the exponential parameterization of Eq. (4.88) and the pion-pion vertex of Eq. (4.93) yields exactly the same result as Eq. (4.164).

Using the exponential parameterization there is a $4\pi\gamma$ vertex at $\mathcal{O}(p^4)$ [see Eq. (4.157)] resulting in the additional loop diagram of Fig. 4.13. The corresponding contribution in dimensional regularization,

$$-\frac{5}{3}e\epsilon_{3ij}\epsilon\cdot(p'+p)\frac{M_{\pi,2}^2}{16\pi^2F_0^2}\left[R+\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)\right]+O(n-4),\tag{4.165}$$

also generates an infinite contribution to the vertex at zero four-momentum transfer.

The renormalized vertex is obtained by adding the bare contributions and multiplying the result by a factor $\sqrt{Z_\pi}$ for each external pion line. The wave function renormalization constant Z_π is not an observable and depends on both the parameterization for U and the $\mathcal{O}(p^4)$ Lagrangian. The corresponding results are summarized in Table D.2 of Appendix D.2. We add the bare contributions which were obtained using the parameterization of Eq. (4.154) and the $\mathcal{O}(p^4)$ Lagrangian of Eq. (4.159), Eqs. (4.156), (4.160), and (4.164), and multiply the result by the appropriate wave function renormalization constant [see entry "GL, Eq. (D.15)" of Table D.2],

$$\mathcal{M}_R=e\epsilon_{3ij}\left(\epsilon\cdot(p'+p)\left\{1+\frac{M_{\pi,2}^2}{16\pi^2F_0^2}\left[R+\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)\right]\right\}\right.$$

$$
-\frac{q^2\epsilon\cdot(p'+p)-\epsilon\cdot q(p'^2-p^2)}{F_0^2}
$$

$$
\times\left\{l_6+\frac{1}{96\pi^2}\left[R+\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)+\frac{1}{3}+\left(1-\frac{M_{\pi,2}^2}{q^2}\right)J^{(0)}\left(\frac{q^2}{M_{\pi,2}^2}\right)\right]\right\}\Bigg)
$$

$$
\times\left\{1-\frac{M_{\pi,2}^2}{16\pi^2F_0^2}\left[R+\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)\right]\right\}+O(n-4). \tag{4.166}
$$

The factor $\epsilon\cdot(p'+p)$ counts as $\mathcal{O}(p^2)$, because the external electromagnetic field, represented by the polarization vector ϵ, counts as $\mathcal{O}(p)$ [see Eq. (4.62)]. It is multiplied by

$$
\begin{aligned}
1+I(M_{\pi,2}^2) &= 1+\frac{M_{\pi,2}^2}{16\pi^2}\left[R+\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)\right]+O(n-4)\\
&= 1+\mathcal{O}(p^2).
\end{aligned}
$$

Since the wave function renormalization constant is $Z_\pi=1-I(M_{\pi,2}^2)=1+\mathcal{O}(p^2)$ (see Appendix D.2), it is only the tree-level contribution derived from $\mathcal{L}_2$ which gets modified. The product $[1+I(M_{\pi,2}^2)][1-I(M_{\pi,2}^2)]=1+\mathcal{O}(p^4)$ is such that the renormalized vertex is properly normalized to the charge at $\mathcal{O}(p^4)$. The factor $[q^2\epsilon\cdot(p'+p)-\epsilon\cdot q(p'^2-p^2)]$ is $\mathcal{O}(p^4)$ and thus the apparent infinity R cannot be canceled through the wave function renormalization. Here it is the connection between the bare parameter l_6 and the renormalized parameter $l_6^r(\mu)$, $l_6^r=l_6+R/(96\pi^2)$, which cancels the divergence [see Eq. (9.6) of Ref. [GL 84]]. Moreover, the explicit dependence on the renormalization scale μ cancels with a corresponding scale dependence of the parameter l_6^r. Finally, using the exponential parameterization or the Lagrangian of Ref. [Gas+ 88] results in the same expression as Eq. (4.166). The additional contributions from Eqs. (4.161) and/or (4.165) to the unrenormalized vertex are precisely canceled by modified wave function renormalization constants, resulting in the same renormalized vertex.

On the mass shell $p^2=p'^2=M_\pi^2$, and we obtain for the electromagnetic form factor [GL 84]

$$
F(q^2)=1-l_6^r\frac{q^2}{F_\pi^2}-\frac{1}{6}\frac{q^2}{(4\pi F_\pi)^2}\left[\ln\left(\frac{M_\pi^2}{\mu^2}\right)+\frac{1}{3}+\left(1-4\frac{M_\pi^2}{q^2}\right)J^{(0)}\left(\frac{q^2}{M_\pi^2}\right)\right], \tag{4.167}
$$

where we replaced the $\mathcal{O}(p^2)$ quantities F_0 and $M_{\pi,2}^2$ by their physical values, the error introduced being of order $\mathcal{O}(p^6)$. Given a spherically symmetric charge distribution $eZ\rho(r)$ normalized so that $\int d^3x\rho(r) = 1$, the form factor $F(|\vec{q}\,|)$ in a nonrelativistic framework is given by

$$F(|\vec{q}\,|) = \int d^3x e^{i\vec{q}\cdot\vec{x}}\rho(r) = 4\pi\int_0^\infty dr r^2 j_0(|\vec{q}\,|r)\rho(r) = 1 - \frac{1}{6}|\vec{q}\,|^2\langle r^2\rangle + \cdots,$$

where $\langle r^2\rangle$ denotes the mean square radius.[36] In analogy, the Lorentz-invariant form factor of Eq. (4.167) is expanded for small q^2 as[37]

$$F(q^2) = 1 + \frac{q^2}{6}\langle r^2\rangle + \cdots, \tag{4.168}$$

and the charge radius of the pion is *defined* as

$$\langle r^2\rangle_\pi = 6\,\frac{dF(q^2)}{dq^2}\bigg|_{q^2=0} = -\frac{6}{F_\pi^2}\left\{l_6^r(\mu) + \frac{1}{96\pi^2}\left[1 + \ln\left(\frac{M_\pi^2}{\mu^2}\right)\right]\right\}, \tag{4.169}$$

where we made use of $J^{(0)}(x) = -x/6 + O(x^2)$. Following Ref. [GL 84], we introduce a scale-independent quantity (see Appendix D.1)

$$\bar{l}_6 = -96\pi^2 l_6^r(\mu) - \ln\left(\frac{M_\pi^2}{\mu^2}\right)$$

which can be determined using the empirical information on the charge radius of the pion: $\bar{l}_6 = 16\pi^2F_\pi^2\langle r^2\rangle_\pi + 1$. In a two-loop calculation of the vector form factor [Bij+ 98], higher-order terms in the chiral expansion terms were also taken into account and a fit to several experimental data sets was performed with the result $\bar{l}_6 = 16.0 \pm 0.5 \pm 0.7$, where the last error is of theoretical origin. Once the value of the parameter $\bar{l}_6$ has been determined it can be used to predict other processes such as, e.g., virtual Compton scattering off the pion [Unk+ 00, Unk+ 02].

The results for the electromagnetic form factors of the charged pion, and the charged and neutral kaons in SU(3)×SU(3) chiral perturbation theory at $\mathcal{O}(p^4)$ can be found in Refs. [GL 85b, Rud+ 94]. The calculation is very similar to the SU(2)×SU(2) case and the mean square radii of the charged pions and kaons are dominated by the low-energy parameter L_9^r, whereas the

[36] For neutral particles such as the neutron or the K^0 one has $e\int d^3x\rho(r) = 0$.

[37] Breit-frame kinematics, i.e. $q^2 = -\vec{q}\,^2$, comes closest to the nonrelativistic situation.

one-loop diagrams generate a small contribution only:[38]

$$\begin{aligned}
\langle r^2\rangle_{\pi^+} &= 12\frac{L_9^r}{F_0^2}-\frac{1}{32\pi^2F_0^2}\left[3+2\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)+\ln\left(\frac{M_{K,2}^2}{\mu^2}\right)\right]\\
&= (0.37\pm 0.04+0.07)\,\mathrm{fm}^2=(0.44\pm 0.04)\,\mathrm{fm}^2,\\
\langle r^2\rangle_{K^+} &= 12\frac{L_9^r}{F_0^2}-\frac{1}{32\pi^2F_0^2}\left[3+2\ln\left(\frac{M_{K,2}^2}{\mu^2}\right)+\ln\left(\frac{M_{\pi,2}^2}{\mu^2}\right)\right]\\
&= (0.37\pm 0.04+0.03)\,\mathrm{fm}^2=(0.40\pm 0.04)\,\mathrm{fm}^2. \qquad (4.170)
\end{aligned}$$

In Ref. [GL 85b] the empirical value $\langle r^2\rangle_\pi=(0.439\pm0.030)\,\mathrm{fm}^2$ of [Dal+ 82] was used to fix L_9^r.[39] The result for the mean square radius of the charged kaon is then a prediction which has to be compared with the empirical values $\langle r^2\rangle_{K^-}=(0.28\pm 0.05)\,\mathrm{fm}^2$ of [Dal+ 80] and $\langle r^2\rangle_{K^-}=(0.34\pm 0.05)\,\mathrm{fm}^2$ of [Ame+ 86a]. In Ref. [BT 02] the empirical data on the charged pion and kaon form factors were analyzed at two-loop order and the low-energy constant including the p^6 terms was determined as $L_9^r(\mu=770\,\mathrm{MeV})=(5.93\pm 0.43)\times 10^{-3}$.

At $\mathcal{O}(p^4)$ the form factor of the K^0 receives one-loop contributions only and thus is predicted in terms of the pion-decay constant and the Goldstone boson masses. The mean square radius is given by

$$\langle r^2\rangle_{K^0}=-\frac{1}{32\pi^2F_0^2}\ln\left(\frac{M_K^2}{M_\pi^2}\right)=-0.037\,\mathrm{fm}^2 \qquad (4.171)$$

which has to be compared with the empirical value $\langle r^2\rangle_{K^0}=(-0.054\pm 0.026)\,\mathrm{fm}^2$ [Mol+ 78]. For a two-loop analysis of the neutral-kaon form factor, see Refs. [PS 97, PS 01].

Since the neutral pion and the eta are their own antiparticles, their electromagnetic vertices vanish because of charge conjugation symmetry as noted in footnote 32 for the case of the π^0.

---

[38] The numerical values in Ref. [GL 85b] were obtained with $M_\pi=135$ MeV, $M_K=495$ MeV, $F_0\approx F_\pi=93.3$ MeV, $\mu=M_\rho=770$ MeV, and $L_9^r(M_\rho)=(6.9\pm0.7)\cdot 10^{-3}$.

[39] A more recent value is given by $\langle r^2\rangle_\pi=(0.439\pm0.008)\,\mathrm{fm}^2$ [Ame+ 86b]. Also (model-dependent) results have been obtained from pion-electroproduction experiments [Lie+ 99, Vol+ 01].

4.10 Chiral Perturbation Theory at $\mathcal{O}(p^6)$

Mesonic chiral perturbation theory at order $\mathcal{O}(p^4)$ has led to a host of successful applications and may be considered as a full-grown and mature area of low-energy particle physics. In this section we will briefly touch upon its extension to $\mathcal{O}(p^6)$ [Iss 90, AA 91, FS 96, Bij+ 99, Ebe 01, Ebe+ 02, Bij+ 02], which naturally divides into the even- and odd-intrinsic-parity sectors. Calculations in the even-intrinsic-parity sector start at $\mathcal{O}(p^2)$ and two-loop calculations at $\mathcal{O}(p^6)$ are thus of next-to-next-to-leading order (NNLO). NNLO calculations at $\mathcal{O}(p^6)$ have been performed for $\gamma\gamma \to \pi^0\pi^0$ [Bel+ 94], vector two-point functions [GK 95, Mal 96, DK 00, Amo+ 00a] and axial-vector two-point functions [Amo+ 00a], $\pi\pi$ scattering [Bij+ 96], $\gamma\gamma \to \pi^+\pi^-$ [Bür 96], $\tau \to \pi\pi\nu_\tau$ [Col+ 96], $\pi \to l\nu\gamma$ [BT 97], Sirlin's combination of SU(3) form factors [PS 97], scalar and electromagnetic form factors of the pion [Bij+ 98], the $K \to \pi\pi l\nu$ (K_{l4}) form factors [Amo+ 00b], the electromagnetic form factor of the K^0 [PS 01], the $K \to \pi l\nu$ (K_{l3}) form factors [PS 02], and the electromagnetic form factors of pions and kaons in SU(3)$\times$SU(3) ChPT [BT 02]. Further applications deal with more technical aspects such as the evaluation of specific two-loop integrals [PT 96, GS 99] and the renormalization of the even-intrinsic-parity Lagrangian at $\mathcal{O}(p^6)$ [Bij+ 00].

The odd-intrinsic-parity sector starts at $\mathcal{O}(p^4)$ with the anomalous WZW action, as discussed in Sec. 4.8. In this sector next-to-leading-order (NLO), i.e. one-loop calculations, are of $\mathcal{O}(p^6)$. It has been known for some time that quantum corrections to the WZW classical action do not renormalize the coefficient of the WZW term [DW 89, Iss 90, Bij+ 90, AA 91, Ebe 01, Bij+ 02]. The counter terms needed to renormalize the one-loop singularities at $\mathcal{O}(p^6)$ are of a conventional chirally invariant structure. The inclusion of the photon as a dynamical degree of freedom in the odd-intrinsic-parity sector has been discussed in Ref. [AM 02]. For an overview of applications in the odd-intrinsic-parity sector, we refer to Ref. [Bij 93]. A two-loop calculation at $\mathcal{O}(p^8)$ for $\gamma\pi \to \pi\pi$ was performed in Ref. [Han 01].

Here, we will mainly be concerned with some aspects of the construction of the most general mesonic chiral Lagrangian at $\mathcal{O}(p^6)$ and discuss as an application a two-loop calculation for the s-wave $\pi\pi$ scattering lengths [Bij+ 96].

4.10.1 The Mesonic Chiral Lagrangian at Order $\mathcal{O}(p^6)$

The rapid increase in the number of free parameters when going from $\mathcal{L}_2$ to $\mathcal{L}_4$ naturally leads to the expectation of a very large number of chirally invariant structures at $\mathcal{O}(p^6)$. One of the problems with the construction of effective Lagrangians at higher orders is that it is far too easy to think of terms satisfying the necessary criteria of Lorentz invariance and invariance under the discrete symmetries as well as chiral transformations. To our knowledge there is neither a general formula, even at $\mathcal{O}(p^4)$, for determining the number of independent structures to expect nor an algorithm to decide whether a set of given structures is independent or not. Experience has shown that for almost any sector of higher-order effective chiral Lagrangians the number of terms found to be independent has gone down with time (see Refs. [Iss 90, AA 91, FS 96, Ebe+ 02, Bij+ 02] for the odd-intrinsic-parity sector, [FS 96, Bij+ 99] for the even-intrinsic-parity sector, and [EM 96, Fet+ 01] for the heavy-baryon πN Lagrangian, respectively). For that reason, it is important to define a strategy for obtaining all of the independent terms without generating a lot of extraneous terms which have to be eliminated by hand. In the following, we will outline the main principles entering the construction of the $\mathcal{O}(p^6)$ Lagrangian and refer the reader to Refs. [FS 96, Bij+ 99, Ebe+ 02, Bij+ 02] for more details.

The effective Lagrangian is constructed from the elements U, $U^\dagger$, χ, $\chi^\dagger$, and the field strength tensors $f^L_{\mu\nu}$ and $f^R_{\mu\nu}$ (see Sec. 4.5, in particular, Table 4.2). The external fields l_μ and r_μ only appear in the field strength tensors or the covariant derivatives which we define as

$$\begin{aligned}
A \stackrel{G}{\mapsto} V_R A V_L^\dagger: &\qquad D_\mu A \equiv \partial_\mu A - i r_\mu A + i A l_\mu, \quad \text{e.g., } U, \chi,\\
B \stackrel{G}{\mapsto} V_L B V_R^\dagger: &\qquad D_\mu B \equiv \partial_\mu B + i B r_\mu - i l_\mu B, \quad \text{e.g., } U^\dagger, \chi^\dagger,\\
C \stackrel{G}{\mapsto} V_R C V_R^\dagger: &\qquad D_\mu C \equiv \partial_\mu C - i r_\mu C + i C r_\mu, \quad \text{e.g., } f^R_{\mu\nu},\\
D \stackrel{G}{\mapsto} V_L D V_L^\dagger: &\qquad D_\mu D \equiv \partial_\mu D - i l_\mu D + i D l_\mu, \quad \text{e.g., } f^L_{\mu\nu},\\
E \stackrel{G}{\mapsto} E: &\qquad D_\mu E \equiv \partial_\mu E, \quad \text{e.g., } \mathrm{Tr}(\chi\chi^\dagger).
\end{aligned} \tag{4.172}$$

In other words, the covariant derivative knows about the transformation property under $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ of the object it acts on and adjusts itself accordingly. With such a convention a product rule analogous to that for ordinary derivatives holds. Given the product $Z = XY$ where X, Y, Z have, according to Eq. (4.172), well-defined but not necessarily the same transformation behavior, the product rule

$$D_\mu Z = D_\mu(XY) = (D_\mu X)Y + X(D_\mu Y) \tag{4.173}$$

applies, which can be easily verified using the definitions of Eq. (4.172). This product rule is valuable as an intermediate step in a number of the derivations of various relations.

In order to avoid unnecessary and tedious repetitions during the process of construction, one would like to perform as many manipulations as possible on a formal level without explicit reference to the specific building blocks. It is thus convenient to handle the external field terms χ, $f^R_{\mu\nu}$, and $f^L_{\mu\nu}$ in the same way. To that end we define

$$G^{\mu\nu} \equiv f_R^{\mu\nu} U + U f_L^{\mu\nu}, \quad H^{\mu\nu} \equiv f_R^{\mu\nu} U - U f_L^{\mu\nu}, \tag{4.174}$$

and introduce $\chi^{\mu\nu}$ as a common abbreviation for any of the building blocks χ ($\equiv \chi^{\mu\mu}$), $G^{\mu\nu}$ and $H^{\mu\nu}$ ($\mu \neq \nu$). With these definitions, we have only two basic building blocks U, $\chi^{\mu\nu}$, covariant derivatives acting on them and the respective adjoints. Due to the product rule of Eq. (4.173) it is not necessary to consider derivatives acting on products of these basic terms. All building blocks then transform as U (or $U^\dagger$). In terms of the momentum expansion, U is of order 1, $\chi^{\mu\nu}$ of order p^2 and each covariant derivative D_μ of order p.

Up to this point we have treated a building block and its adjoint on a different footing which we will now remedy by defining the Hermitian and anti-Hermitian combinations

$$(A)_\pm = u^\dagger A u^\dagger \pm u A^\dagger u, \tag{4.175}$$

where A is taken as $\chi^{\mu\nu}$ or $D_\mu U$, or as some number of covariant derivatives acting on $\chi^{\mu\nu}$ or $D_\mu U$.[40] Here u is defined as the square root of U, i.e., $u^2 \equiv U$. In order to discuss the transformation behavior of Eq. (4.175), we define the SU(3)-valued function $K(V_L, V_R, U)$, referred to as the compensator field [Eck 95], through [Gas+ 88]

$$u(x) \mapsto u'(x) = \sqrt{V_R U V_L^\dagger} \equiv V_R u K^{-1}(V_L, V_R, U), \tag{4.176}$$

[40] In Ref. [FS 96] the building blocks $[A]_\pm \equiv \frac{1}{2}(AU^\dagger \pm UA^\dagger)$ transforming as $V_R \cdots V_R^\dagger$ were used. The notation of Eq. (4.175) has some advantages when implementing the total-derivative procedure to be discussed below. Moreover, it is more closely related to the conventions used in the baryonic sector (see Sec. 5.1).

from which one obtains

$$K(V_L, V_R, U) = u'^{-1} V_R u = \sqrt{V_R U V_L^\dagger}^{-1} V_R \sqrt{U}. \tag{4.177}$$

From a group-theoretical point of view, K defines a nonlinear realization of SU(3)×SU(3) [Gas+ 88], because[41]

$$\begin{aligned}
& K(V_{L1}, V_{R1}, V_{R2} U V_{L2}^\dagger) K(V_{L2}, V_{R2}, U) \\
= & \sqrt{V_{R1}(V_{R2} U V_{L2}^\dagger) V_{L1}^\dagger}^{-1} V_{R1} \sqrt{V_{R2} U V_{L2}^\dagger} \sqrt{V_{R2} U V_{L2}^\dagger}^{-1} V_{R2} \sqrt{U} \\
= & \sqrt{V_{R1} V_{R2} U (V_{L1} V_{L2})^\dagger}^{-1} V_{R1} V_{R2} \sqrt{U} \\
= & K((V_{L1} V_{L2}), (V_{R1} V_{R2}), U).
\end{aligned} \tag{4.178}$$

It is important to note that the first K has the transformed $U' = V_{R2} U V_{L2}^\dagger$ as its argument. With these definitions, the building blocks $(A)_\pm$ transform as

$$(A)_\pm \mapsto K(A)_\pm K^\dagger. \tag{4.179}$$

The corresponding covariant derivative is defined as[42]

$$\nabla_\mu (A)_\pm \equiv \partial_\mu (A)_\pm + [\Gamma_\mu, (A)_\pm], \tag{4.180}$$

where Γ_μ is the so-called connection [Eck 95], and is given by

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger, \partial_\mu u\right] - \frac{i}{2} u^\dagger r_\mu u - \frac{i}{2} u l_\mu u^\dagger. \tag{4.181}$$

As usual, the covariant derivative transforms in the same way as the object it acts on,

$$\nabla_\mu (A)_\pm \stackrel{G}{\mapsto} K \nabla_\mu (A)_\pm K^\dagger.$$

Invariants under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ are constructed by forming products of objects, each transforming as $K \cdots K^\dagger$, and then taking the trace. For example, consider the trace

$$\mathrm{Tr}[(A_1)_\pm \cdots (A_n)_\pm] \stackrel{G}{\mapsto} \mathrm{Tr}[K(A_1)_\pm K^\dagger \cdots K(A_n)_\pm K^\dagger] = \mathrm{Tr}[(A_1)_\pm \cdots (A_n)_\pm], \tag{4.182}$$

[41] K does not define an operation of SU(3)×SU(3) on SU(3), because $K(1,1,U) = 1 \neq U \,\forall\, U$ (see Sec. 4.2.1).

[42] From an aesthetical point of view it would have been more satisfactory to introduce the covariant derivative as $\nabla_\mu (A)_\pm \equiv \partial_\mu (A)_\pm - i[\Gamma_\mu, (A)_\pm]$ to generate a closer formal correspondence to Eqs. (4.172). However, we follow the standard convention used in the literature.

where we made use of $K^\dagger K = 1$ and the invariance of the trace under cyclic permutations. Obviously, products of such traces are also invariant. Basically, the construction of the most general Lagrangian then proceeds by forming products of elements $(A)_\pm$, where A is either $\chi^{\mu\nu}$ or $D^\mu U$ or covariant derivatives of these objects, taking appropriate traces, and forming Lorentz scalars by contracting the Lorentz indices with the metric tensor $g^{\mu\nu}$ or the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$.

After having chosen the set of building blocks and their transformation behavior, we define a strategy concerning the order of constructing invariants at $\mathcal{O}(p^6)$. As shown in Refs. [FS 96, Ebe+ 02], by applying the product rule, it is sufficient to restrict oneself to $(D^m U)_-$, $(D^n G)_+$, $(D^n H)_+$, and $(D^n \chi)_\pm$ (m, n integer with $m > 0, n \geq 0$), because the other combinations $(D^m U)_+$, $(D^n G)_-$, and $(D^n H)_-$ can either be expressed in terms of these or vanish. One then immediately finds that all possible terms at $\mathcal{O}(p^6)$ can either include no, one, two, or three $(D^n \chi_{\mu\nu})_\pm$ blocks which naturally defines four distinct levels to be considered:

1. terms with six D_μ's;
2. terms with four D_μ's and one $\chi^{\mu\nu}$;
3. terms with two D_μ's and two $\chi^{\mu\nu}$'s;
4. terms with three $\chi^{\mu\nu}$'s.

We always try to get rid of terms as high in the hierarchy (with the most D_μ's) as possible. In particular, with this strategy one ensures that the number of terms is minimal also for the special case in which all external fields are set equal to zero.[43] The motivation for such an approach is that at each level there exist relations which allow one to eliminate structures, as long as one keeps *all* terms at the lower levels of the hierarchy. To be specific, when considering multiple covariant derivatives, just one general (i.e., non-contracted) index combination is actually independent in the sense of the

[43] In that sense, the final $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ set given in Ref. [Bij+ 99] for the even-intrinsic-parity sector is not minimal when setting all external fields to zero. In that case the structures Y_1 and Y_4 are not independent and can be eliminated. However, if one replaces these two terms by the terms (120) and (139) of Ref. [FS 96], then the entire set will remain independent whether or not there are external fields and both structures, (120) and (139), vanish explicitly when the fields are set to zero.

hierarchy defined above. For double derivatives this statement reads

$$(D_\mu D_\nu A)_\pm = (D_\nu D_\mu A)_\pm + \frac{i}{4}[(A)_\pm, (G_{\mu\nu})_+] - \frac{i}{4}\{(A)_\mp, (H_{\mu\nu})_+\}, \quad (4.183)$$

i.e., it is sufficient to only keep the block $(D_\mu D_\nu A)_\pm$ as long as one considers all terms lower in the hierarchy.

The construction then proceeds as follows. First of all, write down all conceivable Lorentz-invariant structures satisfying P and C invariance, Hermiticity, and chiral order p^6 in terms of the basic building blocks defined above. Then collect as many relations as possible among these structures and use these to eliminate structures. The relations can follow from any of the following mechanisms:

(1) partial integration;

(2) equation-of-motion argument;

(3) epsilon relations;

(4) Bianchi identities;

(5) trace relations.

Let us illustrate the meaning of each of the above items by selected examples.

The partial-integration or total-derivative argument refers to the fact that a total derivative in the Lagrangian density does not change the equation of motion. One thus generates relations of the following type

$$\begin{aligned}
&\underbrace{\partial_\mu \mathrm{Tr}[(A_1)_\pm \cdots (A_m)_\pm]}_{\text{tot. der.}} + \underbrace{\mathrm{Tr}\{[\Gamma_\mu, (A_1)_\pm \cdots (A_m)_\pm]\}}_{0} \\
&= \mathrm{Tr}\{\nabla_\mu[(A_1)_\pm \cdots (A_m)_\pm]\} \\
&= \mathrm{Tr}[\nabla_\mu(A_1)_\pm \cdots (A_m)_\pm] + \cdots + \mathrm{Tr}[(A_1)_\pm \cdots \nabla_\mu(A_m)_\pm], \quad (4.184)
\end{aligned}$$

where we made use of Eq. (4.180) and the product rule. This derivative shifting procedure is also valid for multiple traces. At this stage we note the advantage of working with the basic building blocks of Eq. (4.175) in comparison with those of Ref. [FS 96] due to the relatively simple connection between the covariant derivative ∇_μ outside the block brackets and the covariant derivative D_μ inside when $(A)_\pm$ are used:

$$\nabla_\mu(A)_\pm = (D_\mu A)_\pm - \frac{1}{4}\{(D_\mu U)_-, (A)_\mp\}. \quad (4.185)$$

From a technical point of view Eq. (4.185) is important because it helps avoid extremely tedious algebraic manipulations one had to perform in the old framework of Ref. [FS 96]. The combination of shifting derivatives back and forth and interchanging indices of multiple derivatives is referred to as index exchange. In Ref. [FS 96] not all total-derivative terms were properly identified which has led to subsequent reductions in the number of terms in both the even-intrinsic-parity sector [Bij+ 99] and the odd-intrinsic-parity sector [Ebe+ 02, Bij+ 02].

The equation-of-motion argument makes use of the invariance of physical observables under field transformations, as discussed in Sec. 4.7. The aim is to collect as many terms as possible containing a factor $(D^2U)_-$. Such terms can be supplemented by corresponding $(\chi)_-$ terms lower in the hierarchy to generate an equation-of-motion term which can be eliminated by a field redefinition.

The epsilon relations refer to the odd-intrinsic-parity sector with the basic idea being as follows. Consider a structure with six Lorentz indices transforming under parity as a Lorentz pseudotensor, i.e., $Q_{\lambda\mu\nu\rho\sigma\tau}(\vec{x},t) \mapsto -Q^{\lambda\mu\nu\rho\sigma\tau}(-\vec{x},t)$. In order to form a Lorentz scalar, one needs to contract two indices pairwise and the remaining four with the totally antisymmetric tensor $\epsilon^{\alpha\beta\gamma\delta}$ in four dimensions. Suppose $Q_{\lambda\mu\nu\rho\sigma\tau}(\vec{x},t)$ is neither symmetric nor antisymmetric under the exchange of any pair of indices. Naively one would then expect $5+4+\cdots+1=15$ independent contractions. However, such a counting does not take the totally antisymmetric nature of the epsilon tensor into account [AA 91], from which one obtains, for the above case of no symmetry in the indices, five additional conditions [AA 91, FS 96]. These additional identities have not been considered in the pioneering construction of Ref. [Iss 90].

In general, $Q_{\lambda\mu\nu\rho\sigma\tau}(\vec{x},t)$ has some symmetry in its indices, and not all five epsilon relations are independent. For example, using the transformation behavior of Table 4.2 it is easy to verify that

$$\mathrm{Tr}\{(G_{\lambda\mu})_+[(G_{\nu\rho})_+(H_{\sigma\tau})_+ + (H_{\sigma\tau})_+(G_{\nu\rho})_+]\}$$

is an example for a pseudotensor which is invariant under G (and Hermitian). Its symmetries are given by

$$Q_{\lambda\mu\nu\rho\sigma\tau} = -Q_{\mu\lambda\nu\rho\sigma\tau} = -Q_{\lambda\mu\rho\nu\sigma\tau} = -Q_{\lambda\mu\nu\rho\tau\sigma} = Q_{\nu\rho\lambda\mu\sigma\tau},$$

from which one would naively end up with a single combination

$$\mathrm{Tr}\{(G_{\mu\nu})_+[(G_{\lambda\alpha})_+(H^{\lambda}{}_{\beta})_+ + (H^{\lambda}{}_{\beta})_+(G_{\lambda\alpha})_+]\}\epsilon^{\mu\nu\alpha\beta}$$

which, however, vanishes due to the epsilon relation.

The Bianchi identities refer to certain relations among covariant derivatives of field-strength tensors. Starting from the Jacobi identity

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0, \tag{4.186}$$

we consider the linear combination

$$\begin{aligned}
D_\mu f^R_{\nu\rho}+D_\nu f^R_{\rho\mu}+D_\rho f^R_{\mu\nu} &\equiv \sum_{\text{c.p.}\{\mu,\nu,\rho\}} D_\mu f^R_{\nu\rho} = \sum_{\text{c.p.}\{\mu,\nu,\rho\}} (\partial_\mu f^R_{\nu\rho} - i[r_\mu, f^R_{\nu\rho}]) \\
&= \sum_{\text{c.p.}\{\mu,\nu,\rho\}} \Big(\partial_\mu\partial_\nu r_\rho - \partial_\mu\partial_\rho r_\nu - i[\partial_\mu r_\nu, r_\rho] \\
&- i[r_\nu,\partial_\mu r_\rho] - i[r_\mu, \partial_\nu r_\rho - \partial_\rho r_\nu] - [r_\mu,[r_\nu,r_\rho]]\Big) \\
&= 0,
\end{aligned} \tag{4.187}$$

where use of the Schwarz theorem, $\partial_\mu\partial_\nu\cdots=\partial_\nu\partial_\mu\cdots$, relabeling of indices, and the Jacobi identity, Eq. (4.186), has been made. Observe that the cyclic permutations of the indices μ,ν, and ρ has been denoted by c.p.$\{\mu,\nu,\rho\}$. The same arguments hold for the independent field strength tensor $f^L_{\mu\nu}$, and we can summarize the constraints as

$$\sum_{\text{c.p.}\{\mu,\nu,\rho\}} D_\mu f^{L/R}_{\nu\rho}=0, \tag{4.188}$$

which, because of their similarity to an analogous equation for the Riemann-Christoffel curvature tensor in general relativity, are referred to as the Bianchi identities (see, e.g., Refs. [Ryd 85, Wei 96]). Equation (4.188) does not require that $f^{R/L}_{\mu\nu}$ satisfy any equations of motion. In terms of the building blocks $(D_\mu U)_-$, $(G_{\mu\nu})_+$, and $(H_{\mu\nu})_+$ the Bianchi identities read

$$\sum_{\text{c.p.}\{\mu,\nu,\rho\}} (D_\mu G_{\nu\rho})_+ = -\frac{1}{4}\sum_{\text{c.p.}\{\mu,\nu,\rho\}} [(D_\mu U)_-,(H_{\nu\rho})_+], \tag{4.189}$$

$$\sum_{\text{c.p.}\{\mu,\nu,\rho\}} (D_\mu H_{\nu\rho})_+ = -\frac{1}{4}\sum_{\text{c.p.}\{\mu,\nu,\rho\}} [(D_\mu U)_-,(G_{\nu\rho})_+]. \tag{4.190}$$

Given the definition of Eq. (4.174), the Bianchi identities can be used to generate relations among the building blocks in terms of structures kept in lower orders of the hierarchy defined above. In Ref. [FS 96] each term of the sum on the left-hand side of Eqs. (4.189) and (4.190) was treated as an independent element so that the final list of supposedly independent structures contained redundant elements.

Finally, the trace relations refer to the fact that the construction of invariants uses traces and products of traces. One is thus particularly interested in finding any relations among those traces. We know from the Cayley-Hamilton theorem that any $n \times n$ matrix A is a solution of its associated characteristical polynomial χ_A. For $n = 2$ this statement reads

$$\begin{aligned} 0 = \chi_A(A) &= A^2 - \mathrm{Tr}(A)A + \det(A)1_{2\times 2} \\ &= A^2 - \mathrm{Tr}(A)A + \frac{1}{2}\{[\mathrm{Tr}(A)]^2 - \mathrm{Tr}(A^2)\}1_{2\times 2}. \end{aligned} \tag{4.191}$$

Setting $A = A_1 + A_2$ in (4.191) and making use of $\chi_{A_1}(A_1) = 0 = \chi_{A_2}(A_2)$ one ends up with the matrix equation

$$\begin{aligned} 0 = F_2(A_1, A_2) \equiv\ & \{A_1, A_2\} - \mathrm{Tr}(A_1)A_2 - \mathrm{Tr}(A_2)A_1 \\ & +\mathrm{Tr}(A_1)\mathrm{Tr}(A_2)1_{2\times 2} - \mathrm{Tr}(A_1A_2)1_{2\times 2} \end{aligned} \tag{4.192}$$

which is the central piece of information needed to derive the trace relations in the SU(2)×SU(2) sector. The analogous $n = 3$ equation is slightly more complex

$$\begin{aligned} 0 &= F_3(A_1, A_2, A_3) \\ &\equiv A_1\{A_2, A_3\} + A_2\{A_3, A_1\} + A_3\{A_1, A_2\} \\ &\quad -\mathrm{Tr}(A_1)\{A_2, A_3\} - \mathrm{Tr}(A_2)\{A_3, A_1\} - \mathrm{Tr}(A_3)\{A_1, A_2\} \\ &\quad +\mathrm{Tr}(A_1)\mathrm{Tr}(A_2)A_3 + \mathrm{Tr}(A_2)\mathrm{Tr}(A_3)A_1 + \mathrm{Tr}(A_3)\mathrm{Tr}(A_1)A_2 \\ &\quad -\mathrm{Tr}(A_1A_2)A_3 - \mathrm{Tr}(A_3A_1)A_2 - \mathrm{Tr}(A_2A_3)A_1 \\ &\quad -\mathrm{Tr}(A_1A_2A_3)1_{3\times 3} - \mathrm{Tr}(A_1A_3A_2)1_{3\times 3} \\ &\quad +\mathrm{Tr}(A_1A_2)\mathrm{Tr}(A_3)1_{3\times 3} + \mathrm{Tr}(A_3A_1)\mathrm{Tr}(A_2)1_{3\times 3} + \mathrm{Tr}(A_2A_3)\mathrm{Tr}(A_1)1_{3\times 3} \\ &\quad -\mathrm{Tr}(A_1)\mathrm{Tr}(A_2)\mathrm{Tr}(A_3)1_{3\times 3}. \end{aligned} \tag{4.193}$$

We can now derive trace relations by simply multiplying Eq. (4.192) or Eq. (4.193) with another arbitrary matrix of the same dimension and finally taking the trace of the whole construction, i.e.,

$$0 = \mathrm{Tr}[F_2(A_1, A_2)A_3], \tag{4.194}$$

$$0 = \mathrm{Tr}[F_3(A_1, A_2, A_3)A_4]. \tag{4.195}$$

Note that A_i may be any $n \times n$ matrix, even a string of our basic building blocks. For example, Eq. (D.7) of Appendix D.1, is identical to Eq. (4.194).

In principle, the ideas developed above apply to the general $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ case and only at the end it is necessary to specify the number of flavors N_f. The reduction to the cases $N_f = 2$ and $N_f = 3$ is achieved in terms of the trace relations summarized in Eqs. (4.194) and (4.195). Although we have never come across a trace relation that could not be obtained in the manner explained above, we are not aware of a general proof showing that any kind of trace relation must be related to the Caley-Hamilton theorem.

In the even-intrinsic-parity sector the Lagrangian has 112 in principle measurable + 3 contact terms for the general $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ case, 90 + 4 for the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ case, and 53 + 4 for the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ case [Bij+ 99]. The contact terms refer to structures which can be expressed in terms of only external fields such as the H_1 and H_2 terms of the $\mathcal{L}_4$ Lagrangian of Eq. (4.104). The reduction in the number of terms in comparison with the 111 $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ structures of Ref. [FS 96] is due to a more complete application of the partial-integration relations, the use of additional trace relations, and the use of four relations due to the Bianchi identities which were not taken into account in Ref. [FS 96]. The odd-intrinsic-parity sector was reconsidered in Refs. [Ebe+ 02, Bij+ 02]. Both analyses found 24 $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, 23 $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$, and 5 $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ terms. Moreover, 8 additional terms due to the extension of the chiral group to $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times U(1)_V$ were found, which are of some relevance when considering the electromagnetic interaction for the two-flavor case. In comparison to Ref. [FS 96], the new analysis of Ref. [Ebe+ 02] could eliminate two structures via partial integration, 6 via Bianchi identities and one by a trace relation.

It is unlikely that the coefficients of all the terms of $\mathcal{L}_6$ will be determined from experiment. However, usually a much smaller subset actually contributes to most simple processes, and it is possible to get information on some of the corresponding coefficients.

4.10.2 Elastic Pion-Pion Scattering at $\mathcal{O}(p^6)$

Elastic pion-pion scattering represents a nice example of the success of mesonic chiral perturbation theory. A complete analytical calculation at two-loop order was performed in Ref. [Bij+ 96].

Let us consider the T-matrix element of the scattering process $\pi^a(p_a) +$

$\pi^b(p_b) \to \pi^c(p_c) + \pi^d(p_d)$,

$$T^{ab;cd}(p_a, p_b; p_c, p_d) = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,s,u) + \delta^{ad}\delta^{bc}A(u,t,s), \tag{4.196}$$

where $s = (p_a + p_b)^2$, $t = (p_a - p_c)^2$, and $u = (p_a - p_d)^2$ denote the usual Mandelstam variables, the indices $a, \cdots, d$ refer to the Cartesian isospin components, and the function A satisfies $A(s,t,u) = A(s,u,t)$ [Wei 66]. Since the pions form an isospin triplet, the two isovectors of both the initial and final states may be coupled to $I = 0, 1, 2$. For $m_u = m_d = m$ the strong interactions are invariant under isospin transformations, implying that scattering matrix elements can be decomposed as

$$\langle I', I_3'|T|I, I_3\rangle = T^I \delta_{II'}\delta_{I_3 I_3'}. \tag{4.197}$$

For the case of $\pi\pi$ scattering the three isospin amplitudes are given in terms of the invariant amplitude A of Eq. (4.196) by [GL 84]

$$\begin{aligned} T^{I=0} &= 3A(s,t,u) + A(t,u,s) + A(u,s,t), \\ T^{I=1} &= A(t,u,s) - A(u,s,t), \\ T^{I=2} &= A(t,u,s) + A(u,s,t). \end{aligned} \tag{4.198}$$

For example, the physical $\pi^+\pi^+$ scattering process is described by $T^{I=2}$. Other physical processes are obtained using the appropriate Clebsch-Gordan coefficients. Evaluating the T matrices at threshold, one obtains the s-wave $\pi\pi$-scattering lengths[44]

$$T^{I=0}|_{\text{thr}} = 32\pi a_0^0, \quad T^{I=2}|_{\text{thr}} = 32\pi a_0^2, \tag{4.199}$$

where the subscript 0 refers to s wave and the superscript to the isospin. ($T^{I=1}|_{\text{thr}}$ vanishes because of Bose symmetry). The current-algebra prediction of Ref. [Wei 66] is identical with the lowest-order result obtained from Eqs. (4.92) or (4.93),

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.156, \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045, \tag{4.200}$$

where we replaced F_0 by F_π and made use of the numerical values $F_\pi = 93.2$ MeV and $M_\pi = 139.57$ MeV of Ref. [Bij+ 96]. In order to obtain the results of Eq. (4.200), use has been made of $s_{\text{thr}} = 4M_\pi^2$ and $t_{\text{thr}} = u_{\text{thr}} = 0$.

[44]The definition differs by a factor of $(-M_\pi)$ [GL 84] from the conventional definition of scattering lengths in the effective range expansion (see, e.g., Ref. [Pre 62]).

The predictions for the s-wave scattering lengths at $\mathcal{O}(p^6)$ read [Bij+ 96]

$$
\begin{aligned}
a_0^0 &= \overbrace{0.156}^{\mathcal{O}(p^2)} + \overbrace{\underbrace{0.039}_{\mathrm{L}} + \underbrace{0.005}_{\mathrm{anal.}}}^{\mathcal{O}(p^4):\ +28\%} + \overbrace{\underbrace{0.013}_{k_i} + \underbrace{0.003}_{\mathrm{L}} + \underbrace{0.001}_{\mathrm{anal.}}}^{\mathcal{O}(p^6):\ +8.5\%} = \overbrace{0.217}^{\mathrm{total}}, \\
a_0^0 - a_0^2 &= \overbrace{0.201}^{\mathcal{O}(p^2)} + \overbrace{\underbrace{0.036}_{\mathrm{L}} + \underbrace{0.006}_{\mathrm{anal.}}}^{\mathcal{O}(p^4):\ +21\%} + \overbrace{\underbrace{0.012}_{k_i} + \underbrace{0.003}_{\mathrm{L}} + \underbrace{0.001}_{\mathrm{anal.}}}^{\mathcal{O}(p^6):\ +6.6\%} = \overbrace{0.258}^{\mathrm{total}}.
\end{aligned}
$$

The corrections at $\mathcal{O}(p^4)$ consist of a dominant part from the chiral logarithms (L) of the one-loop diagrams and a less important analytical contribution (anal.) resulting from the one-loop diagrams as well as the tree graphs of $\mathcal{L}_4$. The total corrections at $\mathcal{O}(p^4)$ amount to 28% and 21% of the $\mathcal{O}(p^2)$ predictions, respectively. At $\mathcal{O}(p^6)$ one obtains two-loop corrections, one-loop corrections, and $\mathcal{L}_6$ tree-level contributions. Once again, the loop corrections (k_i, involving double chiral logarithms, and L) are more important than the analytical contributions. The influence of $\mathcal{L}_6$ was estimated via scalar- and vector-meson exchange and found to be very small.

The empirical results for the $\pi\pi$ s-wave scattering lengths are, so far, obtained from the K_{e4} decay $K^+ \to \pi^+\pi^-e^+\nu_e$ and the $\pi^\pm p \to \pi^\pm\pi^+ n$ reactions. In the former case, the connection with low-energy $\pi\pi$ scattering stems from a partial-wave analysis of the form factors relevant for the K_{e4} decay in terms of $\pi\pi$ angular momentum eigenstates. In the low-energy regime the phases of these form factors are related by (a generalization of) Watson's theorem [Wat 54] to the corresponding phases of $I = 0$ s-wave and $I = 1$ p-wave elastic scattering [Col+ 01a]. Using a dispersion-theory approach in terms of the Roy equations [Roy 71, Ana+ 01], the most recent analysis of $K^+ \to \pi^+\pi^-e^+\nu_e$ [Pis+ 01] has obtained

$$a_0^0 = 0.228 \pm 0.012 \pm 0.003. \tag{4.201}$$

This result has to be compared with older determinations [Ros+ 77, FP 77, Nag+ 79]

$$a_0^0 = 0.26 \pm 0.05, \quad a_0^2 = -0.028 \pm 0.012, \tag{4.202}$$

and the more recent one from $\pi^\pm p \to \pi^\pm\pi^+ n$ [Ker+ 98]

$$a_0^0 = 0.204 \pm 0.014\,(stat.) \pm 0.008\,(syst.), \tag{4.203}$$

which makes use of an extrapolation to the pion pole to extract the $\pi\pi$ amplitude.

In particular, when analyzing the data of Ref. [Pis+ 01] in combination with the Roy equations, an upper limit $|\bar{l}_3| \leq 16$ was obtained in Ref. [Col+ 01a] for the scale-independent low-energy coupling constant which is related to l_3 of the SU(2)×SU(2) Lagrangian of Gasser and Leutwyler (see Appendix D.1). The great interest generated by this result is to be understood in the context of the pion mass at $\mathcal{O}(p^4)$ [see Eq. (D.19) of App. D.2],

$$M_\pi^2 = M^2 - \frac{\bar{l}_3}{32\pi^2 F_0^2} M^4 + \mathcal{O}(M^6), \tag{4.204}$$

where $M^2 = (m_u + m_d)B_0$. Recall that the constant B_0 is related to the scalar quark condensate in the chiral limit [see Eq. (4.43)] and that a nonvanishing quark condensate is a sufficient criterion for spontaneous chiral symmetry breakdown in QCD (see Sec. 4.1.2). If the expansion of M_π^2 in powers of the quark masses is dominated by the linear term in Eq. (4.204), the result is often referred to as the Gell-Mann-Oakes-Renner relation [Gel+ 68]. If the terms of order m^2 were comparable or even larger than the linear terms, a different power counting or bookkeeping in ChPT would be required [Kne+ 95, Kne+ 96, Ste 98]. The estimate $|\bar{l}_3| \leq 16$ implies that the Gell-Mann-Oakes-Renner relation [Gel+ 68] is indeed a decent starting point, because the contribution of the second term of Eq. (4.204) to the pion mass is approximately given by

$$-\frac{\bar{l}_3 M_\pi^2}{64\pi^2 F_\pi^2} M_\pi = -0.054 M_\pi \text{ for } \bar{l}_3 = 16,$$

i.e., more than 94 % of the pion mass must stem from the quark condensate [Col+ 01a].

Chapter 5

Chiral Perturbation Theory for Baryons

So far we have considered the purely mesonic sector involving the interaction of Goldstone bosons with each other and with the external fields. However, ChPT can be extended to also describe the dynamics of baryons at low energies. Here we will concentrate on matrix elements with a single baryon in the initial and final states. With such matrix elements we can, e.g, describe static properties such as masses or magnetic moments, form factors, or, finally, more complicated processes, such as pion-nucleon scattering, Compton scattering, pion photoproduction etc. Technically speaking, we are interested in the baryon-to-baryon transition amplitude in the presence of external fields (as opposed to the vacuum-to-vacuum transition amplitude of Sec. 2.4.4) [Gas+ 88, Kra 90],

$$\mathcal{F}(\vec{p}\,', \vec{p}; v, a, s, p) = \langle \vec{p}\,'; \text{out}|\vec{p}; \text{in}\rangle^{\text{c}}_{v,a,s,p}, \quad \vec{p} \neq \vec{p}\,', \tag{5.1}$$

determined by the Lagrangian of Eq. (2.95),

$$\mathcal{L} = \mathcal{L}^0_{\text{QCD}} + \mathcal{L}_{\text{ext}} = \mathcal{L}^0_{\text{QCD}} + \bar{q}\gamma_\mu(v^\mu + \frac{1}{3}v^\mu_{(s)} + \gamma_5 a^\mu)q - \bar{q}(s - i\gamma_5 p)q. \tag{5.2}$$

In Eq. (5.1) $|\vec{p}; \text{in}\rangle$ and $|\vec{p}\,'; \text{out}\rangle$ denote asymptotic one-baryon in- and out-states, i.e., states which in the remote past and distant future behave as free one-particle states of momentum $\vec{p}$ and $\vec{p}\,'$, respectively. The functional $\mathcal{F}$ consists of connected diagrams only (superscript c). For example, the matrix elements of the vector and axial-vector currents between one-baryon states are given by [Kra 90]

$$\langle \vec{p}\,'|V^{\mu,a}(x)|\vec{p}\rangle = \frac{\delta}{i\delta v^a_\mu(x)}\mathcal{F}(\vec{p}\,',\vec{p};v,a,s,p)\Big|_{v=0,a=0,s=M,p=0}, \quad (5.3)$$

$$\langle \vec{p}\,'|A^{\mu,a}(x)|\vec{p}\rangle = \frac{\delta}{i\delta a^a_\mu(x)}\mathcal{F}(\vec{p}\,',\vec{p};v,a,s,p)\Big|_{v=0,a=0,s=M,p=0}, \quad (5.4)$$

where $M = \mathrm{diag}(m_u, m_d, m_s)$ denotes the quark-mass matrix and

$$V^{\mu,a}(x) = \bar{q}(x)\gamma^\mu\frac{\lambda^a}{2}q(x), \quad A^{\mu,a}(x) = \bar{q}(x)\gamma^\mu\gamma_5\frac{\lambda^a}{2}q(x).$$

As in the mesonic sector the method of calculating the Green functions associated with the functional of Eq. (5.1) consists of an effective Lagrangian-approach in combination with an appropriate power counting. Specific matrix elements will be calculated applying the Gell-Mann and Low formula of perturbation theory [GL 51]. The group-theoretical foundations of constructing phenomenological Lagrangians in the presence of spontaneous symmetry breaking have been developed in Refs. [Wei 68, Col+ 69, Cal+ 69]. The fields entering the Lagrangian are assumed to transform under irreducible representations of the subgroup H which leaves the vacuum invariant whereas the symmetry group G of the Hamiltonian or Lagrangian is nonlinearly realized (for the transformation behavior of the Goldstone bosons, see Sec. 4.2).

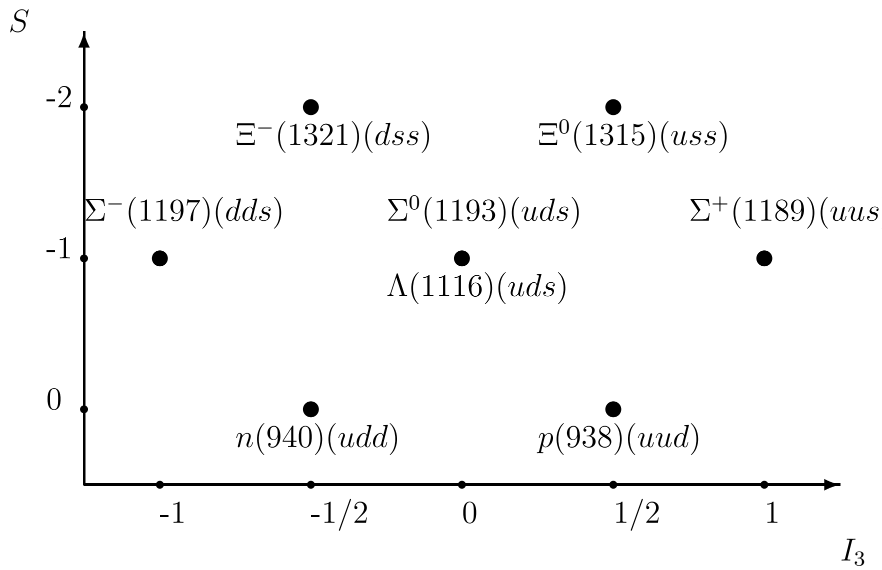
5.1 Transformation Properties of the Fields

Our aim is a description of the interaction of baryons with the Goldstone bosons as well as the external fields at low energies. To that end we need to specify the transformation properties of the dynamical fields entering the Lagrangian. Our discussion follows Refs. [Geo 84, Gas+ 88].

To be specific, we consider the octet of the $\frac{1}{2}^+$ baryons (see Fig. 5.1). With each member of the octet we associate a complex, four-component Dirac field which we arrange in a traceless 3×3 matrix B,

$$B = \sum_{a=1}^{8}\lambda_a B_a = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}, \quad (5.5)$$

Figure 5.1: The baryon octet in an (I_3, S) diagram. We have included the masses in MeV as well as the quark content.



where we have suppressed the dependence on x. For later use, we have to keep in mind that each entry of Eq. (5.5) is a Dirac field, but for the purpose of discussing the transformation properties under global flavor SU(3) this can be ignored, because these transformations act on each of the four components in the same way. In contrast to the mesonic case of Eq. (4.28), where we collected the fields of the Goldstone octet in a Hermitian traceless matrix ϕ, the B_a of the spin 1/2-case are not real (Hermitian), i.e., $B \neq B^\dagger$. Now let us define the set

$$M \equiv \{B(x)|B(x)\,\text{complex, traceless } 3\times 3 \text{ matrix}\} \tag{5.6}$$

which under the addition of matrices is a complex vector space. The following homomorphism is a representation of the abstract group $H = \mathrm{SU}(3)_V$ on the vector space M [see also Eq. (4.25)]:

$$\begin{aligned} &\varphi : H \to \varphi(H), \quad V \mapsto \varphi(V) \quad \text{where} \quad \varphi(V) : M \to M, \\ &B(x) \mapsto B'(x) = \varphi(V)B(x) \equiv VB(x)V^\dagger. \end{aligned} \tag{5.7}$$

First of all, $B'(x)$ is again an element of M, because $\mathrm{Tr}[B'(x)] = \mathrm{Tr}[VB(x)V^\dagger] = \mathrm{Tr}[B(x)] = 0$. Equation (5.7) satisfies the homomorphism property

$$\begin{aligned}\varphi(V_1)\varphi(V_2)B(x) &= \varphi(V_1)V_2B(x)V_2^\dagger = V_1V_2B(x)V_2^\dagger V_1^\dagger = (V_1V_2)B(x)(V_1V_2)^\dagger \\ &= \varphi(V_1V_2)B(x)\end{aligned}$$

and is indeed a *representation* of SU(3), because

$$\begin{aligned}\varphi(V)[\lambda_1B_1(x)+\lambda_2B_2(x)] &= V[\lambda_1B_1(x)+\lambda_2B_2(x)]V^\dagger \\ &= \lambda_1VB_1(x)V^\dagger+\lambda_2VB_2(x)V^\dagger \\ &= \lambda_1\varphi(V)B_1(x)+\lambda_2\varphi(V)B_2(x).\end{aligned}$$

Equation (5.7) is just the familiar statement that B transforms as an octet under (the adjoint representation of) $\mathrm{SU}(3)_V$.[1]

Let us now turn to various representations and realizations of the group $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$. We consider two explicit examples and refer the interested reader to Ref. [Geo 84] for more details. In analogy to the discussion of the quark fields in QCD, we may introduce left- and right-handed components of the baryon fields [see Eq. (2.33)]:

$$B_1 = P_LB_1 + P_RB_1 = B_L + B_R. \tag{5.8}$$

We define the set $M_1 \equiv \{(B_L(x), B_R(x))\}$ which under the addition of matrices is a complex vector space. The following homomorphism is a representation of the abstract group $G = \mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ on M_1:

$$(B_L, B_R) \mapsto (B_L', B_R') \equiv (LB_LL^\dagger, RB_RR^\dagger), \tag{5.9}$$

where we have suppressed the x dependence. The proof proceeds in complete analogy to that of Eq. (5.7).

As a second example, consider the set $M_2 \equiv \{B_2(x)\}$ with the homomorphism

$$B_2 \mapsto B_2' \equiv LB_2L^\dagger, \tag{5.10}$$

i.e. the transformation behavior is independent of R. The mapping defines a representation of the group $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$, although the transformation behavior is drastically different from the first example. However, the

[1] Technically speaking the adjoint representation is faithful (one-to-one) modulo the center Z of SU(3) which is defined as the set of all elements commuting with all elements of SU(3) and is given by $Z = \{1_{3\times3}, \exp(2\pi i/3)1_{3\times3}, \exp(4\pi i/3)1_{3\times3}\}$ [O'Ra 86].

important feature which both mappings have in common is that under the subgroup $H = \{(V,V)|V \in \mathrm{SU}(3)\}$ of G both fields B_i transform as an octet:

$$\begin{aligned} B_1 = B_L + B_R &\overset{H}{\mapsto} VB_LV^\dagger + VB_RV^\dagger = VB_1V^\dagger, \\ B_2 &\overset{H}{\mapsto} VB_2V^\dagger. \end{aligned}$$

We will now show how in a theory also containing Goldstone bosons the various realizations may be connected to each other using field redefinitions. The procedure is actually very similar to Sec. 4.10.1, where we discussed how, by an appropriate multiplication with U or $U^\dagger$, all building blocks of the mesonic effective Lagrangian could be made to transform in the same way. Here we consider the second example, with the fields B_2 of Eq. (5.10) and U of Eq. (4.28) transforming as

$$B_2 \mapsto LB_2L^\dagger, \quad U \mapsto RUL^\dagger,$$

and define new baryon fields by

$$\tilde{B} \equiv UB_2,$$

so that the new pair $(\tilde{B}, U)$ transforms as

$$\tilde{B} \mapsto RUL^\dagger LBL^\dagger = R\tilde{B}L^\dagger, \quad U \mapsto RUL^\dagger.$$

Note in particular that $\tilde{B}$ still transforms as an octet under the subgroup $H = \mathrm{SU}(3)_V$.

Given that physical observable are invariant under field transformations we may choose a description of baryons that is maximally convenient for the construction of the effective Lagrangian [Geo 84] and which is commonly used in chiral perturbation theory. We start with $G = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and consider the case of $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ later. Let

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix} \tag{5.11}$$

denote the nucleon field with two four-component Dirac fields for the proton and the neutron and U the SU(2) matrix containing the pion fields. We have already seen in Sec. 4.2.2 that the mapping $U \mapsto RUL^\dagger$ defines a nonlinear realization of G. We denote the square root of U by u, $u^2(x) = U(x)$, and define the SU(2)-valued function $K(L,R,U)$ by [see Eqs. (4.176) and (4.177)]

$$u(x) \mapsto u'(x) = \sqrt{RUL^\dagger} \equiv RuK^{-1}(L,R,U), \tag{5.12}$$

i.e.

$$K(L,R,U) = u'^{-1}Ru = \sqrt{RUL^\dagger}^{-1} R\sqrt{U}.$$

The following homomorphism defines an operation of G on the set $\{(U,\Psi)\}$ in terms of a nonlinear realization:

$$\varphi(g) : \begin{pmatrix} U \\ \Psi \end{pmatrix} \mapsto \begin{pmatrix} U' \\ \Psi' \end{pmatrix} = \begin{pmatrix} RUL^\dagger \\ K(L,R,U)\Psi \end{pmatrix}, \tag{5.13}$$

because the identity leaves (U,Ψ) invariant and [see Sec. 4.2.2 and Eq. (4.178)]

$$\begin{aligned}
\varphi(g_1)\varphi(g_2)\begin{pmatrix} U \\ \Psi \end{pmatrix} &= \varphi(g_1)\begin{pmatrix} R_2UL_2^\dagger \\ K(L_2,R_2,U)\Psi \end{pmatrix} \\
&= \begin{pmatrix} R_1R_2UL_2^\dagger L_1^\dagger \\ K(L_1,R_1,R_2UL_2^\dagger)K(L_2,R_2,U)\Psi \end{pmatrix} \\
&= \begin{pmatrix} R_1R_2U(L_1L_2)^\dagger \\ K(L_1L_2,R_1R_2,U)\Psi \end{pmatrix} \\
&= \varphi(g_1g_2)\begin{pmatrix} U \\ \Psi \end{pmatrix}.
\end{aligned}$$

Note that for a general group element $g = (L,R)$ the transformation behavior of Ψ depends on U. For the special case of an isospin transformation, $R = L = V$, one obtains $u' = VuV^\dagger$, because

$$U' = u'^2 = VuV^\dagger VuV^\dagger = Vu^2V^\dagger = VUV^\dagger.$$

Comparing with Eq. (5.12) yields $K^{-1}(V,V,U) = V^\dagger$ or $K(V,V,U) = V$, i.e., Ψ transforms linearly as an isospin doublet under the isospin subgroup $\mathrm{SU}(2)_V$ of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. A general feature here is that the transformation behavior under the subgroup which leaves the ground state invariant is independent of U. Moreover, as already discussed in Sec. 4.2.2, the Goldstone bosons ϕ transform according to the adjoint representation of $\mathrm{SU}(2)_V$, i.e., as an isospin triplet.

For the case $G = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ one uses the nonlinear realization

$$\varphi(g) : \begin{pmatrix} U \\ B \end{pmatrix} \mapsto \begin{pmatrix} U' \\ B' \end{pmatrix} = \begin{pmatrix} RUL^\dagger \\ K(L,R,U)BK^\dagger(L,R,U) \end{pmatrix}, \tag{5.14}$$

where K is defined completely analogously to Eq. (5.12) after inserting the corresponding SU(3) matrices.

5.2 Lowest-Order Effective Baryonic Lagrangian

Given the dynamical fields of Eqs. (5.13) and (5.14) and their transformation properties, we will now discuss the most general effective baryonic Lagrangian at lowest order. As in the vacuum sector, chiral symmetry provides constraints among the single-baryon Green functions contained in the functional of Eq. (5.1). Analogous to the mesonic sector, these Ward identities will be satisfied if the Green functions are calculated from the most general effective Lagrangian coupled to external fields with a *local* invariance under the chiral group (see Appendix A).

Let us start with the construction of the πN effective Lagrangian $\mathcal{L}_{\pi N}^{(1)}$ which we demand to have a *local* $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$ symmetry. The transformation behavior of the external fields is given in Eq. (2.109), whereas the nucleon doublet and U transform as

$$\left(\begin{array}{c} U(x) \\ \Psi(x) \end{array}\right) \mapsto \left(\begin{array}{c} V_R(x)U(x)V_L^\dagger(x) \\ \exp[-i\Theta(x)]K[V_L(x),V_R(x),U(x)]\Psi(x) \end{array}\right). \tag{5.15}$$

The local character of the transformation implies that we need to introduce a covariant derivative $D_\mu\Psi$ with the usual property that it transforms in the same way as Ψ [compare with Eq. (2.17) for the case of QED]:

$$D_\mu\Psi(x) \mapsto [D_\mu\Psi(x)]' \stackrel{!}{=} \exp[-i\Theta(x)]K[V_L(x),V_R(x),U(x)]D_\mu\Psi(x). \tag{5.16}$$

Since K not only depends on V_L and V_R but also on U, we may expect the covariant derivative to contain u and $u^\dagger$ and their derivatives. In fact, the connection of Eq. (4.181) (recall $\partial_\mu u u^\dagger = -u\partial_\mu u^\dagger$),

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\right], \tag{5.17}$$

is also an integral part of the covariant derivative of the nucleon doublet:

$$D_\mu\Psi = (\partial_\mu + \Gamma_\mu - iv_\mu^{(s)})\Psi. \tag{5.18}$$

What needs to be shown is

$$D'_\mu\Psi' = [\partial_\mu + \Gamma'_\mu - i(v_\mu^{(s)} - \partial_\mu\Theta)]\exp(-i\Theta)K\Psi = \exp(-i\Theta)K(\partial_\mu + \Gamma_\mu - iv_\mu^{(s)})\Psi. \tag{5.19}$$

To that end, we make use of the product rule,

$$\partial_\mu[\exp(-i\Theta)K\Psi] = -i\partial_\mu\Theta\exp(-i\Theta)K\Psi + \exp(-i\Theta)\partial_\mu K\Psi + \exp(-i\Theta)K\partial_\mu\Psi,$$

in Eq. (5.19) and multiply by $\exp(i\Theta)$, reducing it to

$$\partial_\mu K = K\Gamma_\mu - \Gamma'_\mu K.$$

Using Eq. (5.12),

$$K = u'^\dagger V_R u = \underbrace{u'u'^\dagger}_{1} u'^\dagger V_R u = u'U'^\dagger V_R u = u'V_L \underbrace{U^\dagger}_{u^\dagger u^\dagger} \underbrace{V_R^\dagger V_R}_{1} u = u'V_L u^\dagger,$$

we find

$$\begin{aligned}
2(K\Gamma_\mu - \Gamma'_\mu K) &= K\left[u^\dagger(\partial_\mu - ir_\mu)u\right] - \left[u'^\dagger(\partial_\mu - iV_R r_\mu V_R^\dagger + V_R\partial_\mu V_R^\dagger)u'\right]K \\
&\quad +(R\to L, u\leftrightarrow u^\dagger, u'\leftrightarrow u'^\dagger) \\
&= u'^\dagger V_R(\partial_\mu u - ir_\mu u) - u'^\dagger\partial_\mu u' \underbrace{K}_{u'^\dagger V_R u} \\
&\quad +iu'^\dagger V_R r_\mu \underbrace{V_R^\dagger u' K}_{u} - u'^\dagger V_R \partial_\mu V_R^\dagger \underbrace{u'K}_{V_R u} \\
&\quad +(R\to L, u\leftrightarrow u^\dagger, u'\leftrightarrow u'^\dagger) \\
&= u'^\dagger V_R\partial_\mu u - iu'^\dagger V_R r_\mu u - \underbrace{u'^\dagger\partial_\mu u' u'^\dagger}_{-\partial_\mu u'^\dagger} V_R u \\
&\quad +iu'^\dagger V_R r_\mu u - u'^\dagger \underbrace{V_R\partial_\mu V_R^\dagger V_R}_{-\partial_\mu V_R} u \\
&\quad +(R\to L, u\leftrightarrow u^\dagger, u'\leftrightarrow u'^\dagger) \\
&= u'^\dagger V_R\partial_\mu u + \partial_\mu u'^\dagger V_R u + u'^\dagger\partial_\mu V_R u \\
&\quad +(R\to L, u\leftrightarrow u^\dagger, u'\leftrightarrow u'^\dagger) \\
&= \partial_\mu(u'^\dagger V_R u + u'V_L u^\dagger) = 2\partial_\mu K,
\end{aligned}$$

i.e., the covariant derivative defined in Eq. (5.18) indeed satisfies the condition of Eq. (5.16). At $\mathcal{O}(p)$ there exists another Hermitian building block, the so-called vielbein [Eck 95],[2]

$$u_\mu \equiv i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right], \tag{5.20}$$

which under parity transforms as an axial vector:

$$u_\mu \overset{P}{\mapsto} i\left[u(\partial^\mu - il^\mu)u^\dagger - u^\dagger(\partial^\mu - ir^\mu)u\right] = -u^\mu.$$

[2]The relation with the notation of Sec. 4.10.1 is given by $(D_\mu U)_- = -2iu_\mu$ [Ebe+ 02].

The transformation behavior under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$ is given by

$$u_\mu \mapsto K u_\mu K^\dagger,$$

which is shown using [see Eq. (5.12)]

$$u' = V_R u K^\dagger = K u V_L^\dagger$$

and the corresponding adjoints. We obtain

$$\begin{aligned}
u_\mu \quad \mapsto \quad & i[u'^\dagger(\partial_\mu - iV_R r_\mu V_R^\dagger + V_R \partial_\mu V_R^\dagger)u' \\
& -u'(\partial_\mu - iV_L l_\mu V_L^\dagger + V_L \partial_\mu V_L^\dagger)u'^\dagger] \\
= \quad & i[K u^\dagger V_R^\dagger(\partial_\mu - iV_R r_\mu V_R^\dagger + V_R \partial_\mu V_R^\dagger)V_R u K^\dagger \\
& -K u V_L^\dagger(\partial_\mu - iV_L l_\mu V_L^\dagger + V_L \partial_\mu V_L^\dagger)V_L u^\dagger K^\dagger] \\
= \quad & i[K u^\dagger V_R^\dagger \partial_\mu V_R u K^\dagger + K u^\dagger \partial_\mu u K^\dagger + K \partial_\mu K^\dagger \\
& -iK u^\dagger r_\mu u K^\dagger + K u^\dagger \underbrace{\partial_\mu V_R^\dagger V_R}_{-V_R^\dagger \partial_\mu V_R} u K^\dagger \\
& -K u V_L^\dagger \partial_\mu V_L u^\dagger K^\dagger - K u \partial_\mu u^\dagger K^\dagger - K \partial_\mu K^\dagger \\
& +iK u l_\mu u^\dagger K^\dagger - K u \underbrace{\partial_\mu V_L^\dagger V_L}_{-V_L^\dagger \partial_\mu V_L} u^\dagger K^\dagger] \\
= \quad & iK[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger]K^\dagger \\
= \quad & K u_\mu K^\dagger.
\end{aligned}$$

The most general effective πN Lagrangian describing processes with a single nucleon in the initial and final states is then of the type $\bar{\Psi}\widehat{O}\Psi$, where $\widehat{O}$ is an operator acting in Dirac and flavor space, transforming under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$ as $K\widehat{O}K^\dagger$. As in the mesonic sector, the Lagrangian must be a Hermitian Lorentz scalar which is even under the discrete symmetries C, P, and T.

The most general such Lagrangian with the smallest number of derivatives is given by [Gas+ 88][3]

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi}\left(i\not{D} - \overset{\circ}{m}_N + \frac{\overset{\circ}{g}_A}{2}\gamma^\mu \gamma_5 u_\mu\right)\Psi. \tag{5.21}$$

[3]The power counting will be discussed below.

It contains two parameters not determined by chiral symmetry: the nucleon mass $\overset{\circ}{m}_N$ and the axial-vector coupling constant $\overset{\circ}{g}_A$, both taken in the chiral limit (denoted by $\circ$). The overall normalization of the Lagrangian is chosen such that in the case of no external fields and no pion fields it reduces to that of a free nucleon of mass $\overset{\circ}{m}_N$.

Since the nucleon mass m_N does not vanish in the chiral limit, the zeroth component ∂^0 of the partial derivative acting on the nucleon field does not produce a "small" quantity. We thus have to address the new features of chiral power counting in the baryonic sector. The counting of the external fields as well as of covariant derivatives acting on the mesonic fields remains the same as in mesonic chiral perturbation theory [see Eq. (4.62) of Sec. 4.5]. On the other hand, the counting of bilinears $\bar{\Psi}\Gamma\Psi$ is probably easiest understood by investigating the matrix elements of positive-energy plane-wave solutions to the free Dirac equation in the Dirac representation:

$$\psi^{(+)}(\vec{x},t) = \exp(-ip_N\cdot x)\sqrt{E_N+m_N}\left(\begin{array}{c}\chi\\ \frac{\vec{\sigma}\cdot\vec{p}_N}{E_N+m_N}\chi\end{array}\right), \tag{5.22}$$

where χ denotes a two-component Pauli spinor and $p_N^\mu = (E_N, \vec{p}_N)$ with $E_N = \sqrt{\vec{p}_N^{\,2} + m_N^2}$. In the low-energy limit, i.e. for nonrelativistic kinematics, the lower (small) component is suppressed as $|\vec{p}_N|/m_N$ in comparison with the upper (large) component. For the analysis of the bilinears it is convenient to divide the 16 Dirac matrices into even and odd ones, $\mathcal{E} = \{1, \gamma_0, \gamma_5\gamma_i, \sigma_{ij}\}$ and $\mathcal{O} = \{\gamma_5, \gamma_5\gamma_0, \gamma_i, \sigma_{i0}\}$ [FW 50, Fea+ 94], respectively, where odd matrices couple large and small components but not large with large, whereas even matrices do not. Finally, $i\partial^\mu$ acting on the nucleon solution produces p_N^μ which we write symbolically as $p_N^\mu = (m_N, \vec{0}) + (E_N - m_N, \vec{p}_N)$ where we count the second term as $\mathcal{O}(p)$, i.e., as a small quantity. We are now in the position to summarize the chiral counting scheme for the (new) elements of baryon chiral perturbation theory [Kra 90]:

$$\begin{aligned}&\Psi, \bar{\Psi} = \mathcal{O}(p^0),\ D_\mu\Psi = \mathcal{O}(p^0),\ (i\not{D} - \overset{\circ}{m}_N)\Psi = \mathcal{O}(p),\\ &1, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu} = \mathcal{O}(p^0),\ \gamma_5 = \mathcal{O}(p),\end{aligned} \tag{5.23}$$

where the order given is the minimal one. For example, γ_μ has both an $\mathcal{O}(p^0)$ piece, γ_0, as well as an $\mathcal{O}(p)$ piece, γ_i. A rigorous nonrelativistic reduction may be achieved in the framework of the Foldy-Wouthuysen method [FW 50] or the heavy-baryon approach [JM 91, Ber+ 92b] which will be discussed later (for a pedagogical introduction see Ref. [Hol 97]).

The construction of the $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ Lagrangian proceeds similarly except for the fact that the baryon fields are contained in the 3×3 matrix of Eq. (5.5) transforming as $KBK^\dagger$. As in the mesonic sector, the building blocks are written as products transforming as $K \cdots K^\dagger$ with a trace taken at the end. The lowest-order Lagrangian reads [Geo 84, Kra 90]

$$\mathcal{L}_{MB}^{(1)} = \mathrm{Tr}\left[\bar{B}\left(i\not{D} - M_0\right)B\right] - \frac{D}{2}\mathrm{Tr}\left(\bar{B}\gamma^\mu\gamma_5\{u_\mu, B\}\right) - \frac{F}{2}\mathrm{Tr}\left(\bar{B}\gamma^\mu\gamma_5[u_\mu, B]\right), \tag{5.24}$$

where M_0 denotes the mass of the baryon octet in the chiral limit. The covariant derivative of B is defined as

$$D_\mu B = \partial_\mu B + [\Gamma_\mu, B], \tag{5.25}$$

with Γ_μ of Eq. (5.17) [for $\mathrm{SU}(3)_L \times\ \mathrm{SU}(3)_R$]. The constants D and F may be determined by fitting the semi-leptonic decays $B \to B' + e^- + \bar{\nu}_e$ at tree level [Bor 99]:

$$D = 0.80, \quad F = 0.50. \tag{5.26}$$

5.3 Applications at Tree Level

5.3.1 Goldberger-Treiman Relation and the Axial-Vector Current Matrix Element

We have seen in Sec. 2.3.6 that the quark masses in QCD give rise to a nonvanishing divergence of the axial-vector current operator [see Eq. (2.84)]. Here we will discuss the implications for the matrix elements of the pseudoscalar density and of the axial-vector current evaluated between single-nucleon states in terms of the lowest-order Lagrangians of Eqs. (4.70) and (5.21). In particular, we will see that the Ward identity

$$\langle N(p')|\partial_\mu A_i^\mu(0)|N(p)\rangle = \langle N(p')|m_q P_i(0)|N(p)\rangle, \tag{5.27}$$

where $m_q = m_u = m_d$, is satisfied.

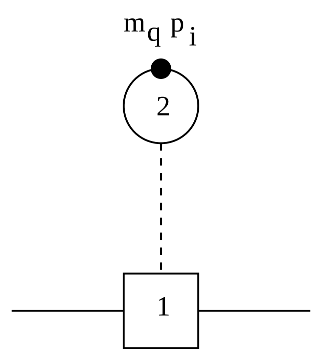
The nucleon matrix element of the pseudoscalar density can be parameterized as

$$m_q\langle N(p')|P_i(0)|N(p)\rangle = \frac{M_\pi^2 F_\pi}{M_\pi^2 - t} G_{\pi N}(t) i\bar{u}(p')\gamma_5\tau_i u(p), \tag{5.28}$$

where $t = (p' - p)^2$. Equation (5.28) *defines* the form factor $G_{\pi N}(t)$ in terms of the QCD operator $m_q P_i(x)$. As we have seen in the discussion of $\pi\pi$ scattering of Sec. 4.6.2, the operator $m_q P_i(x)/(M_\pi^2 F_\pi)$ serves as an interpolating pion field [see Eq. (4.99)], and thus $G_{\pi N}(t)$ is also referred to as the pion-nucleon form factor (for this specific choice of the interpolating pion field). The pion-nucleon coupling constant $g_{\pi N}$ is defined through $G_{\pi N}(t)$ evaluated at $t = M_\pi^2$.

The Lagrangian $\mathcal{L}_{\pi N}^{(1)}$ of Eq. (5.21) does not generate a direct coupling of an external pseudoscalar field $p_i(x)$ to the nucleon, i.e., it does not contain any terms involving χ or $\chi^\dagger$. At lowest order in the chiral expansion, the matrix element of the pseudoscalar density is therefore given in terms of the diagram of Fig. 5.2, i.e., the pseudoscalar source produces a pion which propagates and is then absorbed by the nucleon. The coupling of a pseudoscalar

Figure 5.2: Lowest-order contribution to the single-nucleon matrix element of the pseudoscalar density. Mesonic and baryonic vertices are denoted by a circle and a box, respectively, with the numbers 2 and 1 referring to the chiral order of $\mathcal{L}_2$ and $\mathcal{L}_{\pi N}^{(1)}$.



field to the pion in the framework of $\mathcal{L}_2$ has already been discussed in Eq. (4.96),

$$\mathcal{L}_{\text{ext}} = i\frac{F_0^2 B_0}{2}\text{Tr}(pU^\dagger - Up) = 2B_0 F_0 p_i \phi_i + \cdots. \tag{5.29}$$

When working with the nonlinear realization of Eq. (5.13) it is convenient to use the exponential parameterization of Eq. (4.88),

$$U(x) = \exp\left[i\frac{\vec{\tau}\cdot\vec{\phi}(x)}{F_0}\right],$$

because in that case the square root is simply given by

$$u(x) = \exp\left[i\frac{\vec{\tau}\cdot\vec{\phi}(x)}{2F_0}\right].$$

According to Fig. 5.2, we need to identify the interaction term of a nucleon with a single pion. In the absence of external fields the vielbein of Eq. (5.20) is odd in the pion fields,

$$u_\mu = i\left[u^\dagger\partial_\mu u - u\partial_\mu u^\dagger\right] \overset{\phi^a\mapsto-\phi^a}{\mapsto} i\left[u\partial_\mu u^\dagger - u^\dagger\partial_\mu u\right] = -u_\mu. \tag{5.30}$$

Expanding u and $u^\dagger$ as

$$u = 1 + i\frac{\vec{\tau}\cdot\vec{\phi}}{2F_0} + O(\phi^2), \quad u^\dagger = 1 - i\frac{\vec{\tau}\cdot\vec{\phi}}{2F_0} + O(\phi^2), \tag{5.31}$$

we obtain

$$u_\mu = -\frac{\vec{\tau}\cdot\partial_\mu\vec{\phi}}{F_0} + O(\phi^3), \tag{5.32}$$

which, when inserted into $\mathcal{L}_{\pi N}^{(1)}$ of Eq. (5.21), generates the following interaction Lagrangian:

$$\mathcal{L}_{\mathrm{int}} = -\frac{1}{2}\frac{\overset{\circ}{g}_A}{F_0}\bar{\Psi}\gamma^\mu\gamma_5\underbrace{\vec{\tau}\cdot\partial_\mu\vec{\phi}}_{\tau^b\partial_\mu\phi^b}\Psi. \tag{5.33}$$

(Note that the sign is opposite to the conventionally used pseudovector pion-nucleon coupling.[4]) The Feynman rule for the vertex of an incoming pion with four-momentum q and Cartesian isospin index a is given by

$$i\left(-\frac{1}{2}\frac{\overset{\circ}{g}_A}{F_0}\right)\gamma^\mu\gamma_5\tau^b\delta^{ba}(-iq_\mu) = -\frac{1}{2}\frac{\overset{\circ}{g}_A}{F_0}\not{q}\gamma_5\tau^a. \tag{5.34}$$

[4]In fact, also the definition of the pion-nucleon form factor of Eq. (5.28) contains a sign opposite to the standard convention so that, in the end, the Goldberger-Treiman relation emerges with the conventional sign.

On the other hand, the connection of Eq. (5.17) with the external fields set to zero is even in the pion fields,

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right] \stackrel{\phi^a\mapsto-\phi^a}{\mapsto} \frac{1}{2}\left[u\partial_\mu u^\dagger + u^\dagger\partial_\mu u\right] = \Gamma_\mu, \tag{5.35}$$

i.e., it does not contribute to the single-pion vertex.

We now put the individual pieces together and obtain for the diagram of Fig. 5.2

$$\begin{aligned} & m_q 2B_0 F_0 \frac{i}{t-M_\pi^2}\bar{u}(p')\left(-\frac{1}{2}\frac{\overset{\circ}{g}_A}{F_0}\not{q}\gamma_5\tau_i\right)u(p) \\ = \quad & M_\pi^2 F_0 \frac{\overset{\circ}{m}_N\overset{\circ}{g}_A}{F_0}\frac{1}{M_\pi^2-t}\bar{u}(p')\gamma_5 i\tau_i u(p), \end{aligned}$$

where we used $M_\pi^2 = 2B_0 m_q$, and the Dirac equation to show $\bar{u}\not{q}\gamma_5 u = 2\overset{\circ}{m}_N \bar{u}\gamma_5 u$. At $\mathcal{O}(p^2)$ $F_\pi = F_0$ so that, by comparison with Eq. (5.28), we can read off the lowest-order result

$$G_{\pi N}(t) = \frac{\overset{\circ}{m}_N}{F_0}\overset{\circ}{g}_A, \tag{5.36}$$

i.e., at this order the form factor does not depend on t. In general, the pion-nucleon coupling constant is defined at $t = M_\pi^2$ which, in the present case, simply yields

$$g_{\pi N} = G_{\pi N}(M_\pi^2) = \frac{\overset{\circ}{m}_N}{F_0}\overset{\circ}{g}_A. \tag{5.37}$$

Equation (5.37) represents the famous Goldberger-Treiman relation [GT 58a, GT 58b, Nam 60] which establishes a connection between quantities entering weak processes, F_π and g_A (to be discussed below), and a typical strong-interaction quantity, namely the pion-nucleon coupling constant $g_{\pi N}$. The numerical violation of the Goldberger-Treiman relation, as expressed in the so-called Goldberger-Treiman discrepancy

$$\Delta_{\pi N} \equiv 1 - \frac{g_A m_N}{g_{\pi N}F_\pi}, \tag{5.38}$$

is at the percent level,[5] although one has to keep in mind that *all four* physical quantities move from their chiral-limit values $\overset{\circ}{g}_A$ etc. to the empirical ones g_A etc.

[5] Using $m_N = 938.3$ MeV, $g_A = 1.267$, $F_\pi = 92.4$ MeV, and $g_{\pi N} = 13.21$ [Sch+ 01], one obtains $\Delta_{\pi N} = 2.6$ %.

Using Lorentz covariance and isospin symmetry, the matrix element of the axial-vector current between initial and final nucleon states—excluding second-class currents [Wei 58]—can be parameterized as[6]

$$\langle N(p')|A_i^\mu(0)|N(p)\rangle = \bar{u}(p')\left[\gamma^\mu G_A(t) + \frac{(p'-p)^\mu}{2m_N}G_P(t)\right]\gamma_5\frac{\tau_i}{2}u(p), \quad (5.39)$$

where $t=(p'-p)^2$, and $G_A(t)$ and $G_P(t)$ are the axial and induced pseudoscalar form factors, respectively.

At lowest order, an external axial-vector field a_μ^i couples directly to the nucleon as

$$\mathcal{L}_{\rm ext} = \overset{\circ}{g}_A\bar{\Psi}\gamma^\mu\gamma_5\frac{\tau_i}{2}\Psi a_\mu^i + \cdots, \quad (5.40)$$

which is obtained from $\mathcal{L}_{\pi N}^{(1)}$ through $u_\mu = (r_\mu - l_\mu) + \cdots = 2a_\mu + \cdots$. The coupling to the pions is obtained from $\mathcal{L}_2$ with $r_\mu = -l_\mu = a_\mu$,

$$\mathcal{L}_{\rm ext} = -F_0\partial^\mu\phi_i a_\mu^i + \cdots, \quad (5.41)$$

which gives rise to a diagram similar to Fig. 5.2, with $m_q p_i$ replaced by a_i^μ.

The matrix element is thus given by

$$\bar{u}(p')\left\{\overset{\circ}{g}_A\gamma^\mu\gamma_5\frac{\tau_i}{2} + \left[-\frac{1}{2}\frac{\overset{\circ}{g}_A}{F_0}(\not{p}' - \not{p})\gamma_5\tau_i\right]\frac{i}{q^2-M_\pi^2}(-iF_0q^\mu)\right\}u(p),$$

from which we obtain, by applying the Dirac equation,

$$G_A(t) = \overset{\circ}{g}_A, \quad (5.42)$$

$$G_P(t) = -\frac{4\overset{\circ}{m}_N^2\overset{\circ}{g}_A}{t-M_\pi^2}. \quad (5.43)$$

At this order the axial form factor does not yet show a t dependence. The axial-vector coupling constant is defined as $G_A(0)$ which is simply given by

[6]The terminology "first and second classes" refers to the transformation property of strangeness-conserving semi-leptonic weak interactions under $\mathcal{G}$ conjugation [Wei 58] which is the product of charge symmetry and charge conjugation $\mathcal{G}=\mathcal{C}\exp(i\pi I_2)$. A second-class contribution would show up in terms of a third form factor G_T contributing as

$$G_T(t)\bar{u}(p')i\frac{\sigma^{\mu\nu}q_\nu}{2m_N}\gamma_5\frac{\tau_i}{2}u(p).$$

Assuming a perfect $\mathcal{G}$-conjugation symmetry, the form factor G_T vanishes.

$\overset{\circ}{g}_A$. We have thus identified the second new parameter of $\mathcal{L}_{\pi N}^{(1)}$ besides the nucleon mass $\overset{\circ}{m}_N$. The induced pseudoscalar form factor is determined by the pion exchange which is the simplest version of the so-called pion-pole dominance. The $1/(t - M_\pi^2)$ behavior of G_P is not in conflict with the bookkeeping of a calculation at chiral order $\mathcal{O}(p)$, because, according to Eq. (4.62), the external axial-vector field a_μ counts as $\mathcal{O}(p)$, and the definition of the matrix element contains a momentum $(p' - p)^\mu$ and the Dirac matrix γ_5 [see Eq. (5.23)] so that the combined order of all elements is indeed $\mathcal{O}(p)$.

It is straightforward to verify that the form factors of Eqs. (5.36), (5.42), and (5.43) satisfy the relation

$$2m_N G_A(t) + \frac{t}{2m_N} G_P(t) = 2\frac{M_\pi^2 F_\pi}{M_\pi^2 - t} G_{\pi N}(t), \tag{5.44}$$

which is required by the Ward identity of Eq. (5.27) with the parameterizations of Eqs. (5.28) and (5.39) for the matrix elements. In other words, only two of the three form factors G_A, G_P, and $G_{\pi N}$ are independent. Note that this relation is not restricted to small values of t but holds for any t.

In the chiral limit, Eq. (5.27) implies

$$2\,\overset{\circ}{m}_N\,\overset{\circ}{G}_A(t) + \frac{t}{2\,\overset{\circ}{m}_N}\,\overset{\circ}{G}_P(t) = 0, \tag{5.45}$$

which also follows from Eq. (5.42) and Eq. (5.43) for $M_\pi^2 = 0$. Equation (5.45) for $\overset{\circ}{G}_A(0) \neq 0$ requires that in the chiral limit the induced pseudoscalar form factor has a pole in the limit $t \to 0$. The interpretation of this pole is, of course, given in terms of the exchange of a massless Goldstone pion. To understand this in more detail consider the most general contribution of the pion exchange to the axial-vector current matrix element:

$$\langle N(p')|A_i^\mu(0)|N(p)\rangle_\pi = -\frac{2F_\pi(t)g_{\pi N}(t)}{t - M_\pi^2 - \tilde{\Sigma}(t)}\bar{u}(p')q^\mu\gamma_5\frac{\tau_i}{2}u(p),$$

where $\tilde{\Sigma}(M_\pi^2) = \tilde{\Sigma}'(M_\pi^2) = 0$ for the renormalized propagator [see Eq. (4.137)]. The functions $F_\pi(t)$ and $g_{\pi N}(t)$ denote the most general parameterizations for the pion-decay vertex and the pion-nucleon vertex (note that we have *not* specified the interpolating pion field). For general t their values depend on the interpolating field, but for $t = M_\pi^2$ they are identical with the pion-decay constant F_π and the pion-nucleon coupling constant $g_{\pi N}$, respectively. In the chiral limit, $M_\pi^2 \to 0$, we obtain

$$-\frac{2F_0(t)\stackrel{\circ}{g}_{\pi N}(t)}{t-\stackrel{\circ}{\tilde{\Sigma}}(t)} 2\stackrel{\circ}{m}_N \bar{u}(p')\frac{q^\mu}{2\stackrel{\circ}{m}_N}\gamma_5\frac{\tau_i}{2}u(p),$$

where $\stackrel{\circ}{\tilde{\Sigma}}(0)=\stackrel{\circ}{\tilde{\Sigma}}{}'(0)=0$. In other words, the most general contribution of a massless pion to the induced pseudoscalar form factor in the chiral limit is given by

$$\stackrel{\circ}{G}_{P,\pi}(t) = -\frac{4\stackrel{\circ}{m}_N F_0(t)\stackrel{\circ}{g}_{\pi N}(t)}{t-\stackrel{\circ}{\tilde{\Sigma}}(t)}.$$

We divide the pseudoscalar form factor into the pion contribution and the rest. Making use of Eq. (5.45), we consider the limit

$$\begin{aligned}\lim_{t\to 0} t[\stackrel{\circ}{G}_{P,\pi}(t)+\stackrel{\circ}{G}_{P,R}(t)] &= -4\stackrel{\circ}{m}_N F_0\stackrel{\circ}{g}_{\pi N}\\ &\stackrel{!}{=} -4\stackrel{\circ}{m}{}_N^2\stackrel{\circ}{G}_A(0)\end{aligned}$$

from which we obtain the Goldberger-Treiman relation

$$\frac{\stackrel{\circ}{g}_A}{F_0} = \frac{\stackrel{\circ}{g}_{\pi N}}{\stackrel{\circ}{m}_N}.$$

Of course, we have assumed that there is no other massless particle in the theory which could produce a pole in the residual part $\stackrel{\circ}{G}_{P,R}(t)$ as $t \to 0$.

5.3.2 Pion-Nucleon Scattering at Tree Level

As another example, we will consider pion-nucleon scattering and show how the effective Lagrangian of Eq. (5.21) reproduces the Weinberg-Tomozawa predictions for the s-wave scattering lengths [Wei 66, Tom 66]. We will contrast the results with those of a tree-level calculation within pseudoscalar (PS) and pseudovector (PV) pion-nucleon couplings.

Before calculating the πN scattering amplitude within ChPT we introduce a general parameterization of the invariant amplitude $\mathcal{M} = iT$ for the process $\pi^a(q) + N(p) \to \pi^b(q') + N(p')$ [Che+ 57, Bro+ 71]:[7]

$$
\begin{aligned}
T^{ab}(p,q;p',q') &= \bar{u}(p')\Bigg\{ \underbrace{\frac{1}{2}\{\tau^b,\tau^a\}}_{\delta^{ab}} A^+(\nu,\nu_B) + \underbrace{\frac{1}{2}[\tau^b,\tau^a]}_{-i\epsilon_{abc}\tau^c} A^-(\nu,\nu_B) \\
&\quad + \frac{1}{2}(\not q + \not q')\left[\delta^{ab}B^+(\nu,\nu_B) - i\epsilon_{abc}\tau^c B^-(\nu,\nu_B)\right]\Bigg\} u(p),
\end{aligned}
\tag{5.46}
$$

with the two independent scalar kinematical variables

$$
\nu = \frac{s-u}{4m_N} = \frac{(p+p')\cdot q}{2m_N} = \frac{(p+p')\cdot q'}{2m_N}, \tag{5.47}
$$

$$
\nu_B = -\frac{q\cdot q'}{2m_N} = \frac{t-2M_\pi^2}{4m_N}, \tag{5.48}
$$

where $s=(p+q)^2$, $t=(p'-p)^2$, and $u=(p'-q)^2$ are the usual Mandelstam variables satisfying $s+t+u=2m_N^2+2M_\pi^2$. From pion-crossing symmetry $T^{ab}(p,q;p',q') = T^{ba}(p,-q';p',-q)$ we obtain for the crossing behavior of the amplitudes

$$
\begin{aligned}
A^+(-\nu,\nu_B) &= A^+(\nu,\nu_B), \quad A^-(-\nu,\nu_B) = -A^-(\nu,\nu_B),\\
B^+(-\nu,\nu_B) &= -B^+(\nu,\nu_B), \quad B^-(-\nu,\nu_B) = B^-(\nu,\nu_B).
\end{aligned}
$$

As in $\pi\pi$ scattering one often also finds the isospin decomposition as in Eq. (4.197),

$$
\langle I', I_3'|T|I,I_3\rangle = T^I\delta_{II'}\delta_{I_3 I'_3},
$$

where the relation between the two sets is given by [EW 88]

$$
\begin{aligned}
T^{\frac{1}{2}} &= T^+ + 2T^-,\\
T^{\frac{3}{2}} &= T^+ - T^-.
\end{aligned}
\tag{5.49}
$$

Let us turn to the tree-level approximation to the πN scattering amplitude as obtained from $\mathcal{L}_{\pi N}^{(1)}$ of Eq. (5.21). In order to derive the relevant

[7]One also finds the parameterization [BL 01]

$$
T = \bar{u}(p')\left(D - \frac{1}{4m_N}[\not q', \not q]B\right)u(p)
$$

with $D = A + \nu B$, where, for simplicity, we have omitted the isospin indices.

interaction Lagrangians from Eq. (5.21), we reconsider the connection of Eq. (5.17) with the external fields set to zero and obtain

$$\Gamma_\mu = \frac{i}{4F_0^2}\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi}+O(\phi^4). \tag{5.50}$$

The linear pion-nucleon interaction term was already derived in Eq. (5.33) so that we end up with the following interaction Lagrangian:

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}\frac{\overset{\circ}{g}_A}{F_0}\bar{\Psi}\gamma^\mu\gamma_5\tau^b\partial_\mu\phi^b\Psi - \frac{1}{4F_0^2}\bar{\Psi}\gamma^\mu\underbrace{\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi}}_{\epsilon_{cde}\tau^c\phi^d\partial_\mu\phi^e}\Psi. \tag{5.51}$$

The first term is the pseudovector pion-nucleon coupling and the second the contact interaction with two factors of the pion field interacting with the nucleon at a single point. The Feynman rules for the vertices derived from Eq. (5.51) read

- for an incoming pion with four-momentum q and Cartesian isospin index a:

$$-\frac{1}{2}\frac{\overset{\circ}{g}_A}{F_0}\not{q}\gamma_5\tau^a, \tag{5.52}$$

- for an incoming pion with q, a and an outgoing pion with q', b:

$$i\left(-\frac{1}{4F_0^2}\right)\gamma^\mu\epsilon_{cde}\tau^c\left(\delta^{da}\delta^{eb}iq'_\mu+\delta^{db}\delta^{ea}(-iq)_\mu\right)=\frac{\not{q}+\not{q}'}{4F_0^2}\epsilon_{abc}\tau^c. \tag{5.53}$$

The latter gives the contact contribution to $\mathcal{M}$,

$$\mathcal{M}_{\text{cont}} = \bar{u}(p')\frac{\not{q}+\not{q}'}{4F_0^2}\underbrace{\epsilon_{abc}\tau^c}_{i\frac{1}{2}[\tau^b,\tau^a]}u(p) = i\frac{1}{2F_0^2}\bar{u}(p')\frac{1}{2}[\tau^b,\tau^a]\frac{1}{2}(\not{q}+\not{q}')u(p). \tag{5.54}$$

We emphasize that such a term is not present in a conventional calculation with either a pseudoscalar or a pseudovector pion-nucleon interaction. For the s- and u-channel nucleon-pole diagrams the pseudovector vertex appears twice and we obtain

$$\begin{aligned}\mathcal{M}_{s+u} &= i\frac{\overset{\circ}{g}{}_A^2}{4F_0^2}\bar{u}(p')\tau^b\tau^a(-\not{q}')\gamma_5\frac{1}{\not{p}'+\not{q}'-\overset{\circ}{m}_N}\not{q}\gamma_5 u(p)\\ &\quad +i\frac{\overset{\circ}{g}{}_A^2}{4F_0^2}\bar{u}(p')\tau^a\tau^b\not{q}\gamma_5\frac{1}{\not{p}'-\not{q}-\overset{\circ}{m}_N}(-\not{q}')\gamma_5 u(p).\end{aligned} \tag{5.55}$$

The s- and u-channel contributions are related to each other through pion crossing $a \leftrightarrow b$ and $q \leftrightarrow -q'$. In what follows we explicitly calculate only the s channel and make use of pion-crossing symmetry at the end to obtain the u-channel result. Moreover, we perform the manipulations such that the result of pseudoscalar coupling may also be read off. Using the Dirac equation, we rewrite

$$\not{q}\gamma_5 u(p) = (\not{p}'+\not{q}'-\overset{\circ}{m}_N+\overset{\circ}{m}_N-\not{p})\gamma_5 u(p) = (\not{p}'+\not{q}'-\overset{\circ}{m}_N)\gamma_5 u(p)+2\overset{\circ}{m}_N\gamma_5 u(p)$$

and obtain

$$\begin{aligned}\mathcal{M}_s &= i\frac{\overset{\circ}{g}{}_A^2}{4F_0^2}\bar{u}(p')\tau^b\tau^a(-\not{q}')\gamma_5\frac{1}{\not{p}'+\not{q}'-\overset{\circ}{m}_N}\left[(\not{p}'+\not{q}'-\overset{\circ}{m}_N)+2\overset{\circ}{m}_N\right]\gamma_5 u(p)\\ &\overset{\gamma_5^2=1}{=} i\frac{\overset{\circ}{g}{}_A^2}{4F_0^2}\bar{u}(p')\tau^b\tau^a\left[(-\not{q}')+(-\not{q}')\gamma_5\frac{1}{\not{p}'+\not{q}'-\overset{\circ}{m}_N}2\overset{\circ}{m}_N\gamma_5\right]u(p).\end{aligned}$$

We repeat the above procedure

$$\bar{u}(p')\not{q}'\gamma_5 = \bar{u}(p')[-2\overset{\circ}{m}_N\gamma_5-\gamma_5(\not{p}+\not{q}-\overset{\circ}{m}_N)],$$

yielding

$$\mathcal{M}_s = i\frac{\overset{\circ}{g}{}_A^2}{4F_0^2}\bar{u}(p')\tau^b\tau^a[(-\not{q}')+\underbrace{4m_N^2\gamma_5\frac{1}{\not{p}'+\not{q}'-\overset{\circ}{m}_N}\gamma_5}_{\text{PS coupling}}+2\overset{\circ}{m}_N]u(p), \qquad (5.56)$$

where, for the identification of the PS-coupling result, one has to make use of the Goldberger-Treiman relation [GT 58a, GT 58b, Nam 60] (see Sec. 5.3.1)

$$\frac{\overset{\circ}{g}_A}{F_0}=\frac{\overset{\circ}{g}_{\pi N}}{\overset{\circ}{m}_N},$$

where $\overset{\circ}{g}_{\pi N}$ denotes the pion-nucleon coupling constant in the chiral limit. Using

$$s-m_N^2=2m_N(\nu-\nu_B),$$

we find

$$\begin{aligned}
\bar{u}(p')\gamma_5 \frac{1}{\not{p}' + \not{q}' - \overset{\circ}{m}_N}\gamma_5 u(p) &= \bar{u}(p')\gamma_5 \frac{\not{p}' + \not{q}' + \overset{\circ}{m}_N}{(p'+q')^2 - \overset{\circ}{m}_N^{\,2}}\gamma_5 u(p) \\
&= \frac{1}{2\overset{\circ}{m}_N (\nu - \nu_B)}\left[-\frac{1}{2}\bar{u}(p')(\not{q} + \not{q}')u(p)\right],
\end{aligned}$$

where we again made use of the Dirac equation. We finally obtain for the s-channel contribution

$$\mathcal{M}_s = i\frac{\overset{\circ}{g}_A^{\,2}}{4F_0^2}\bar{u}(p')\tau^b\tau^a\left[2\overset{\circ}{m}_N + \frac{1}{2}(\not{q} + \not{q}')\left(-1 - \frac{2\overset{\circ}{m}_N}{\nu - \nu_B}\right)\right]u(p). \tag{5.57}$$

As noted above, the expression for the u channel results from the substitution $a \leftrightarrow b$ and $q \leftrightarrow -q'$

$$\mathcal{M}_u = i\frac{\overset{\circ}{g}_A^{\,2}}{4F_0^2}\bar{u}(p')\tau^a\tau^b\left[2\overset{\circ}{m}_N + \frac{1}{2}(\not{q} + \not{q}')\left(1 - \frac{2\overset{\circ}{m}_N}{\nu + \nu_B}\right)\right]u(p). \tag{5.58}$$

We combine the s- and u-channel contributions using

$$\tau^b\tau^a = \frac{1}{2}\{\tau^b,\tau^a\} + \frac{1}{2}[\tau^b,\tau^a], \quad \tau^a\tau^b = \frac{1}{2}\{\tau^b,\tau^a\} - \frac{1}{2}[\tau^b,\tau^a],$$

and

$$\frac{1}{\nu - \nu_B} \pm \frac{1}{\nu + \nu_B} = \frac{\left\{\begin{array}{c} 2\nu \\ 2\nu_B \end{array}\right\}}{\nu^2 - \nu_B^2}$$

and summarize the contributions to the functions $A^\pm$ and $B^\pm$ of Eq. (5.46) in Table 5.1 [see also Eq. (A.26) of Ref. [Gas+ 88]].

In order to extract the scattering lengths, let us consider threshold kinematics

$$p^\mu = p'^\mu = (m_N, 0), \quad q^\mu = q'^\mu = (M_\pi, 0), \quad \nu|_{\rm thr} = M_\pi, \quad \nu_B|_{\rm thr} = -\frac{M_\pi^2}{2m_N}. \tag{5.59}$$

Since we only work at lowest-order tree level, we replace $\overset{\circ}{m}_N \to m_N$, etc. Together with[8]

$$u(p) \to \sqrt{2m_N}\begin{pmatrix} \chi \\ 0 \end{pmatrix}, \quad \bar{u}(p') \to \sqrt{2m_N}\begin{pmatrix} \chi'^\dagger & 0 \end{pmatrix}$$

[8]Recall that we use the normalization $\bar{u}u = 2m_N$.

Table 5.1: Tree-level contributions to the functions $A^\pm$ and $B^\pm$ of Eq. (5.46). The second column (PS) denotes the result using pseudoscalar pion-nucleon coupling (using the Goldberger-Treiman relation). The sum of the second and third column (PS+ΔPV) represents the result of pseudovector pion-nucleon coupling. The contact term is specific to the chiral approach. The last column, the sum of the second, third, and fourth columns, is the lowest-order ChPT result.

amplitude\origin	PS	ΔPV	contact	sum
A^+	0	$\frac{\overset{\circ}{g}{}_A^2 \overset{\circ}{m}_N}{F_0^2}$	0	$\frac{\overset{\circ}{g}{}_A^2 \overset{\circ}{m}_N}{F_0^2}$
A^-	0	0	0	0
B^+	$-\frac{\overset{\circ}{g}{}_A^2}{F_0^2}\frac{\overset{\circ}{m}_N \nu}{\nu^2-\nu_B^2}$	0	0	$-\frac{\overset{\circ}{g}{}_A^2}{F_0^2}\frac{\overset{\circ}{m}_N \nu}{\nu^2-\nu_B^2}$
B^-	$-\frac{\overset{\circ}{g}{}_A^2}{F_0^2}\frac{\overset{\circ}{m}_N \nu_B}{\nu^2-\nu_B^2}$	$-\frac{\overset{\circ}{g}{}_A^2}{2F_0^2}$	$\frac{1}{2F_0^2}$	$\frac{1-\overset{\circ}{g}{}_A^2}{2F_0^2} - \frac{\overset{\circ}{g}{}_A^2}{F_0^2}\frac{\overset{\circ}{m}_N \nu_B}{\nu^2-\nu_B^2}$

we find for the threshold matrix element

$$T|_{\rm thr} = 2m_N\chi'^\dagger\left[\delta^{ab}\left(A^+ + M_\pi B^+\right) - i\epsilon_{abc}\tau^c\left(A^- + M_\pi B^-\right)\right]_{\rm thr}\chi. \qquad (5.60)$$

Using

$$\left[\nu^2-\nu_B^2\right]_{\rm thr} = M_\pi^2\left(1-\frac{\mu^2}{4}\right), \quad \mu = \frac{M_\pi}{m_N} \approx \frac{1}{7}$$

we obtain

$$\begin{aligned} T|_{\rm thr} &= 2m_N\chi'^\dagger\Bigg[\delta^{ab}\underbrace{\left(\frac{g_A^2 m_N}{F_\pi^2} + \underbrace{M_\pi\left(-\frac{g_A^2}{F_\pi^2}\right)\frac{m_N}{M_\pi}\frac{1}{1-\frac{\mu^2}{4}}}_{\rm PS}\right)}_{\rm ChPT\,=\,PV} \\ &\quad -i\epsilon_{abc}\tau^c M_\pi \underbrace{\left(\frac{1}{2F_\pi^2} \underbrace{-\frac{g_A^2}{2F_\pi^2} \underbrace{-\frac{g_A^2}{F_\pi^2}\left(-\frac{1}{2}\right)\frac{1}{1-\frac{\mu^2}{4}}}_{\rm PS}}_{\rm PV}\right)}_{\rm ChPT}\Bigg]\chi, \qquad (5.61)\end{aligned}$$

where we have indicated the results for the various coupling schemes.

Let us discuss the s-wave scattering lengths resulting from Eq. (5.61). Using the above normalization for the Dirac spinors, the differential cross section in the center-of-mass frame is given by [EW 88]

$$\frac{d\sigma}{d\Omega} = \frac{|\vec{q}\,'|}{|\vec{q}\,|}\left(\frac{1}{8\pi\sqrt{s}}\right)^2 |T|^2, \tag{5.62}$$

which, at threshold, reduces to

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm thr} = \left(\frac{1}{8\pi(m_N+M_\pi)}\right)^2 |T|^2 \stackrel{!}{=} |a|^2. \tag{5.63}$$

The s-wave scattering lengths are defined as[9]

$$a_{0+}^{\pm} = \frac{1}{8\pi(m_N+M_\pi)} T^{\pm}|_{\rm thr} = \frac{1}{4\pi(1+\mu)}\left[A^{\pm}+M_\pi B^{\pm}\right]_{\rm thr}. \tag{5.64}$$

The subscript 0+ refers to the fact that the πN system is in an orbital s wave ($l=0$) with total angular momentum $1/2 = 0+1/2$. Inserting the results of Table 5.1 we obtain[10]

$$a_{0+}^{-} = \frac{M_\pi}{8\pi(1+\mu)F_\pi^2}\left(1+\frac{g_A^2\mu^2}{4}\frac{1}{1-\frac{\mu^2}{4}}\right) = \frac{M_\pi}{8\pi(1+\mu)F_\pi^2}[1+\mathcal{O}(p^2)], \tag{5.65}$$

$$a_{0+}^{+} = -\frac{g_A^2 M_\pi}{16\pi(1+\mu)F_\pi^2}\frac{\mu}{1-\frac{\mu^2}{4}} = \mathcal{O}(p^2), \tag{5.66}$$

where we have also indicated the chiral order. Taking the linear combinations $a^{\frac{1}{2}} = a_{0+}^{+} + 2a_{0+}^{-}$ and $a^{\frac{3}{2}} = a_{0+}^{+} - a_{0+}^{-}$ [see Eq. (5.49)], we see that the results of Eqs. (5.65) and (5.66) indeed satisfy the Weinberg-Tomozawa relation [Wei 66, Tom 66]:[11]

$$a^I = -\frac{M_\pi}{8\pi(1+\mu)F_\pi^2}[I(I+1)-\frac{3}{4}-2]. \tag{5.67}$$

[9] The threshold parameters are defined in terms of a multipole expansion of the πN scattering amplitude [Che+ 57]. The sign convention for the s-wave scattering parameters $a_{0+}^{(\pm)}$ is opposite to the convention of the effective range expansion.

[10] We do not expand the fraction $1/(1+\mu)$, because the μ dependence is not of dynamical origin.

[11] The result, in principle, holds for a general target of isospin T (except for the pion) after replacing $3/4$ by $T(T+1)$ and μ by M_π/M_T.

As in $\pi\pi$ scattering, the scattering lengths vanish in the chiral limit reflecting the fact that the interaction of Goldstone bosons vanishes in the zero-energy limit. The pseudoscalar pion-nucleon interaction produces a scattering length a^+_{0+} proportional to m_N instead of μM_π and is clearly in conflict with the requirements of chiral symmetry. Moreover, the scattering length a^-_{0+} of the pseudoscalar coupling is too large by a factor g_A^2 in comparison with the two-pion contact term of Eq. (5.54) (sometimes also referred to as the Weinberg-Tomozawa term) induced by the nonlinear realization of chiral symmetry. On the other hand, the pseudovector pion-nucleon interaction gives a totally wrong result for a^-_{0+}, because it misses the two-pion contact term of Eq. (5.54).

Using the values

$$\begin{aligned}
&g_A = 1.267, \quad F_\pi = 92.4\,\mathrm{MeV}, \\
&m_N = m_p = 938.3\,\mathrm{MeV}, \quad M_\pi = M_{\pi^+} = 139.6\,\mathrm{MeV},
\end{aligned} \tag{5.68}$$

the numerical results for the scattering lengths are given in Table 5.2. We have included the full results of Eqs. (5.65) and (5.66) and the consistent corresponding prediction at $\mathcal{O}(p)$. The results of heavy-baryon chiral perturbation theory (HBChPT) (see Sec. 5.5) are taken from Ref. [Moj 98]. At $\mathcal{O}(p^3)$ the calculation involves nine low-energy constants of the chiral Lagrangian which have been fit to the extrapolated threshold parameters of the partial wave analysis of Ref. [KP 80], the pion-nucleon σ term and the Goldberger-Treiman discrepancy. Up to and including $\mathcal{O}(p^4)$ the HBChPT calculation contains 14 free parameters [FM 00]. In Ref. [FM 00] the complete one-loop amplitude at $\mathcal{O}(p^4)$ was fit to the phase shifts provided by three different partial wave analyses [Koc 86] [I], [Mat 97] [II], and SP98 of [SAID] [III]. Table 5.2 includes the results for the s-wave scattering lengths obtained from those fits in combination with the empirical values of the three analyses. Finally, the results of the recently proposed manifestly Lorentz-invariant form of baryon ChPT [R(elativistic)BChPT] [BL 99] (see Sec. 5.6) are included up to $\mathcal{O}(p^4)$ [BL 01]. The first entries (a) refer to a dispersive representation of the function $D = A + \nu B$ entering the threshold matrix element [see Eq. (5.64) and recall $\nu_{\mathrm{thr}} = M_\pi$] whereas the second entries (b) involve only the one-loop approximation. Whereas for a^-_{0+} there is no difference, the value for a^+_{0+} differs substantially which has been interpreted as the result of an insufficient approximation of the one-loop representation to allow for an extrapolation from the Cheng-Dashen point [$(\nu = 0, \nu_B = 0)$] to the physical region [BL 01].

The empirical results quoted have been taken from low-energy partial-wave analyses [Koc 86, Mat 97] and recent precision X-ray experiments on pionic hydrogen and deuterium [Sch+ 01].

Table 5.2: s-wave scattering lengths $a_{0+}^{\pm}$.

Scattering length	a_{0+}^{+} [MeV^{-1}]	a_{0+}^{-} [MeV^{-1}]
Tree-level result	-6.80×10^{-5}	$+5.71 \times 10^{-4}$
ChPT $\mathcal{O}(p)$	0	$+5.66 \times 10^{-4}$
HBChPT $\mathcal{O}(p^2)$ [Moj 98]	-1.3×10^{-4}	$+5.5 \times 10^{-4}$
HBChPT $\mathcal{O}(p^3)$ [Moj 98]	$(-7 \pm 9) \times 10^{-5}$	$(+6.7 \pm 1.0) \times 10^{-4}$
HBChPT $\mathcal{O}(p^4)$ [I] [FM 00]	-6.9×10^{-5}	$+6.47 \times 10^{-4}$
HBChPT $\mathcal{O}(p^4)$ [II] [FM 00]	$+3.2 \times 10^{-5}$	$+5.52 \times 10^{-4}$
HBChPT $\mathcal{O}(p^4)$ [III] [FM 00]	$+1.9 \times 10^{-5}$	$+6.21 \times 10^{-4}$
RChPT $\mathcal{O}(p^4)$ (a) [BL 01]	-6.0×10^{-5}	$+6.55 \times 10^{-4}$
RChPT $\mathcal{O}(p^4)$ (b) [BL 01]	-9.4×10^{-5}	$+6.55 \times 10^{-4}$
PS	-1.23×10^{-2}	$+9.14 \times 10^{-4}$
PV	-6.80×10^{-5}	$+5.06 \times 10^{-6}$
Empirical values [Koc 86]	$(-7 \pm 1) \times 10^{-5}$	$(6.6 \pm 0.1) \times 10^{-4}$
Empirical values [Mat 97]	$(2.04 \pm 1.17) \times 10^{-5}$	$(5.71 \pm 0.12) \times 10^{-4}$ $(5.92 \pm 0.11) \times 10^{-4}$
Experiment [Sch+ 01]	$(-2.7 \pm 3.6) \times 10^{-5}$	$(+6.59 \pm 0.30) \times 10^{-4}$

5.4 Examples of Loop Diagrams

In Sec. 4.4 we saw that, in the purely mesonic sector, contributions of n-loop diagrams are at least of order $\mathcal{O}(p^{2n+2})$, i.e., they are suppressed by p^{2n} in comparison with tree-level diagrams. An important ingredient in deriving this result was the fact that we treated the squared pion mass as a small quantity of order p^2. Such an approach is motivated by the observation that the masses of the Goldstone bosons must vanish in the chiral limit. In the framework of ordinary chiral perturbation theory $M_\pi^2 \sim m_q$ [see Eq. (4.45) and the discussion at the end of Sec. 4.10.2] which translates into a momentum expansion of observables at fixed ratio m_q/p^2. On the other hand, there is no reason to believe that the masses of hadrons other than the Goldstone bosons should vanish or become small in the chiral limit. In other words, the nucleon mass entering the pion-nucleon Lagrangian of Eq. (5.21) should—as already anticipated in the discussion following Eq. (5.21)—not be treated as a small quantity of, say, order $\mathcal{O}(p)$.

Naturally the question arises how all this affects the calculation of loop diagrams and the setup of a consistent power counting scheme. We will follow Ref. [Gas+ 88] and consider, for illustrative purposes, two examples: a one-loop contribution to the nucleon mass and a loop diagram contributing to πN scattering.

5.4.1 First Example: One-Loop Correction to the Nucleon Mass

The discussion of the modification of the nucleon mass due to pion loops is very similar to that of Sec. 4.9.1 for the masses of the Goldstone bosons. The lowest-order Feynman propagator of the nucleon, corresponding to the free-field part of $\mathcal{L}_{\pi N}^{(1)}$ of Eq. (5.21),

$$iS_F(p) = \frac{i}{\not p - \overset{\circ}{m}_N + i0^+}, \tag{5.69}$$

is modified by the self energy $\Sigma(p)$ (see for example the one-loop contribution of Fig. 5.3) in a way analogous to the modification of the meson propagator in Eq. (4.135),

$$\frac{i}{\not p - \overset{\circ}{m}_N + i0^+} + \frac{i}{\not p - \overset{\circ}{m}_N + i0^+}[-i\Sigma(p)]\frac{i}{\not p - \overset{\circ}{m}_N + i0^+} + \cdots,$$

resulting in the full (but still unrenormalized) propagator

$$iS(p) = \frac{i}{\not p - \overset{\circ}{m}_N - \Sigma(p) + i0^+}. \tag{5.70}$$

In the absence of external fields (but including the quark mass term), the most general expression for the self energy can be written as

$$\Sigma(p) = -f(p^2)\not p + g(p^2)\,\overset{\circ}{m}_N, \tag{5.71}$$

where f and g are as yet undetermined functions of the invariant p^2. We assume that f and g may be determined in a perturbative (momentum or loop) expansion which, symbolically, we denote by some indicator λ,

$$\begin{aligned} f(p^2,\lambda) &= f_0(p^2)+\lambda f_1(p^2)+\lambda^2 f_2(p^2)+\cdots, \\ g(p^2,\lambda) &= g_0(p^2)+\lambda g_1(p^2)+\lambda^2 g_2(p^2)+\cdots. \end{aligned} \tag{5.72}$$

When switching off the interaction, we would like to recover the lowest-order result of Eq. (5.69), i.e. $\Sigma \to 0$, implying $f_0 = g_0 = 0$. The mass of the nucleon is defined through the position of the pole of the full propagator, i.e., for $\not p = m_N$ we require

$$m_N - \overset{\circ}{m}_N + f(m_N^2) m_N - g(m_N^2) \overset{\circ}{m}_N = 0,$$

from which we obtain

$$m_N = \overset{\circ}{m}_N \frac{1+g(m_N^2)}{1+f(m_N^2)}. \tag{5.73}$$

The perturbative result to first order in λ reads

$$m_N = \overset{\circ}{m}_N \frac{1+\lambda g_1(\overset{\circ}{m}{}_N^2)+\cdots}{1+\lambda f_1(\overset{\circ}{m}{}_N^2)+\cdots} = \overset{\circ}{m}_N \left\{1+\lambda\left[g_1(\overset{\circ}{m}{}_N^2)-f_1(\overset{\circ}{m}{}_N^2)\right]+\cdots\right\}. \tag{5.74}$$

[Note that the argument m_N^2 of the functions f and g also has to be expanded in powers of λ, $m_N^2 = \overset{\circ}{m}{}_N^2 + O(\lambda)$.] The wave function renormalization constant is defined through the residue at $\not p = m_N$,

$$S(p) \to \frac{Z_N}{\not p - m_N + i0^+} \text{ for } \not p \to m_N, \tag{5.75}$$

i.e., the renormalized propagator, defined through $S(p) = Z_N S_R(p)$, has a pole at $\not p = m_N$ with residue 1. Using $(p^2 - m_N^2)^n = (\not p - m_N)^n (\not p + m_N)^n$ and Eq. (5.73) we find that for $\not p \to m_N$

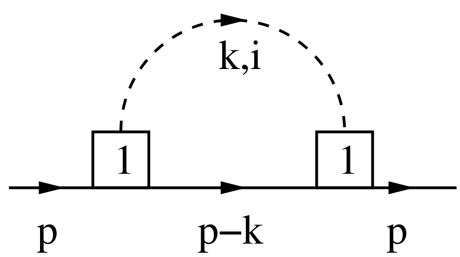
$$\begin{aligned} S(p) &= \frac{1}{\not p[1+f(p^2)] - \overset{\circ}{m}_N [1+g(p^2)]} \\ &= \Big\{\not p[1+f(m_N^2)+(\not p - m_N)(\not p + m_N) f'(m_N^2)+\cdots] \\ &\quad - \overset{\circ}{m}_N [1+g(m_N^2)+(\not p - m_N)(\not p + m_N) g'(m_N^2)+\cdots]\Big\}^{-1} \\ &\to \frac{1}{(\not p - m_N)[1+f(m_N^2)+2m_N^2 f'(m_N^2) - 2\overset{\circ}{m}_N m_N g'(m_N^2)]}, \end{aligned}$$

yielding for the wave function renormalization constant

$$
\begin{aligned}
Z_N &= \frac{1}{1+f(m_N^2)+2m_N^2 f'(m_N^2)-2\overset{\circ}{m}_N\, m_N g'(m_N^2)} \\
&= 1-\lambda\left\{f_1(\overset{\circ}{m}{}_N^2)+2\overset{\circ}{m}{}_N^2\left[f_1'(\overset{\circ}{m}{}_N^2)-g_1'(\overset{\circ}{m}{}_N^2)\right]\right\}+\cdots. \qquad (5.76)
\end{aligned}
$$

With these definitions let us consider the contribution of Fig. 5.3 to the self energy, where, for the sake of simplicity, we perform the calculation in the chiral limit $M_\pi^2=0$. Using the vertex of Eq. (5.34) we obtain the contribution

Figure 5.3: Example of a pion-loop contribution to the nucleon self energy.



of the self energy

$$
\begin{aligned}
-i\overset{\circ}{\Sigma}_{\rm loop}(p) &= \int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{2}\frac{\overset{\circ}{g}_A}{F_0}(-\not k)\gamma_5\tau_i\right]\frac{i}{k^2+i0^+} \\
&\quad\times\frac{i}{\not p-\not k-\overset{\circ}{m}_N+i0^+}\left[-\frac{1}{2}\frac{\overset{\circ}{g}_A}{F_0}\not k\gamma_5\tau_i\right]. \qquad (5.77)
\end{aligned}
$$

Counting powers we see that the integral has a cubic divergence. We make use of (normal) dimensional regularization [Jeg 01], where the integrand is first simplified using[12]

$$
\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu},\qquad g_\mu{}^\mu=n,\quad \{\gamma_\mu,\gamma_5\}=0,\quad \gamma_5^2=1. \qquad (5.78)
$$

[12]For a recent discussion of the problem with γ_5 in dimensional regularization, see Ref. [Jeg 01]. Since we are neither dealing with matrix elements containing anomalies nor considering closed fermion loops, we can safely make use of normal dimensional regularization [Gas+ 88, Jeg 01].

In the standard fashion, we first insert

$$1 = \frac{\not p - \not k + \overset{\circ}{m}_N - i0^+}{\not p - \not k + \overset{\circ}{m}_N - i0^+},$$

simplify the numerator using Eq. (5.78),

$$\not k \gamma_5 (\not p - \not k + \overset{\circ}{m}_N) \not k \gamma_5 = -(\not p + \overset{\circ}{m}_N) k^2 + (p^2 - \overset{\circ}{m}{}_N^2) \not k - [(k-p)^2 - \overset{\circ}{m}{}_N^2] \not k,$$

and obtain, with $\tau_i \tau_i = 3$

$$\begin{aligned}
\overset{\circ}{\Sigma}_{\text{loop}}(p) &= \frac{3\,\overset{\circ}{g}{}_A^2}{4F_0^2} \Bigg\{ -(\not p + \overset{\circ}{m}_N) \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{i}{(k-p)^2 - \overset{\circ}{m}{}_N^2 + i0^+} \\
&\quad + (p^2 - \overset{\circ}{m}{}_N^2) \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{i \not k}{(k^2 + i0^+)[(k-p)^2 - \overset{\circ}{m}{}_N^2 + i0^+]} \\
&\quad - \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{i \not k}{k^2 + i0^+} \Bigg\}. \qquad (5.79)
\end{aligned}$$

Indeed, when discussing the contribution to the nucleon mass [see Eq. (5.74)] we only need to consider the first integral of Eq. (5.79), because the second term does not contribute at $p^2 = \overset{\circ}{m}{}_N^2$ and the third term vanishes in dimensional regularization because the integrand is odd in k. Using Eqs. (C.2) and (C.3) of Appendix C.1.1 with the replacement $M_\pi \to \overset{\circ}{m}_N$ we obtain, in the language of Eq. (5.72),

$$\lambda f_1(\overset{\circ}{m}{}_N^2) = \frac{3\,\overset{\circ}{g}{}_A^2}{4F_0^2} I_N(0), \quad \lambda g_1(\overset{\circ}{m}{}_N^2) = -\frac{3\,\overset{\circ}{g}{}_A^2}{4F_0^2} I_N(0).$$

Applying Eq. (5.74) we find for the nucleon mass including the one-loop contribution of Fig. 5.3 [see Eq. (4.1) of Ref. [Gas+ 88]]

$$m_N = \overset{\circ}{m}_N \left[1 - \frac{3\,\overset{\circ}{g}{}_A^2}{2F_0^2} I_N(0) \right], \qquad (5.80)$$

where

$$\begin{aligned}
I_N(0) &= \frac{\overset{\circ}{m}{}_N^2}{16\pi^2} \left[R + \ln\left(\frac{\overset{\circ}{m}{}_N^2}{\mu^2} \right) \right] + O(n-4), \\
R &= \frac{2}{n-4} - [\ln(4\pi) + \Gamma'(1) + 1].
\end{aligned}$$

The pion loop of Fig. 5.3 generates an (infinite) contribution to the nucleon mass, even in the chiral limit, i.e., the parameter $\overset{\circ}{m}_N$ of $\mathcal{L}_{\pi N}^{(1)}$ needs to be renormalized. The same is true for the second parameter $\overset{\circ}{g}_A$ [Gas+ 88]. This situation is completely different from the mesonic sector, where the two parameters F_0 and B_0 of the lowest-order Lagrangian do not change due to higher-order corrections in the chiral limit. For example, in the SU(2)×SU(2) sector, the pion-decay constant at $\mathcal{O}(p^4)$ is given by [see Eq. (12.2) of Ref. [GL 84]]

$$F_\pi = F_0\left[1+\frac{M^2}{16\pi^2F_0^2}\bar{l}_4+\mathcal{O}(M^4)\right], \tag{5.81}$$

where $M^2 = 2B_0 m_q$, and the scale-independent low-energy parameter $\bar{l}_4$ is defined in Eq. (D.4). Since $F_\pi \to F_0$ in the chiral limit $M^2 \to 0$, the pion-decay constant in the chiral limit is still given by F_0 of $\mathcal{L}_2$. Similarly, in the chiral limit the Goldstone boson masses vanish, not only at $\mathcal{O}(p^2)$ but also at higher orders, as we have seen in Eqs. (4.148) - (4.150).

5.4.2 Second Example: One-Loop Correction to πN Scattering

In the previous section we have seen that the parameters of the lowest-order Lagrangian must be renormalized in the chiral limit. As a second example, we will discuss the πN-scattering loop diagram of Fig. 5.4, which will allow us to draw some further conclusions regarding the differences between the mesonic and baryonic sectors of ChPT.

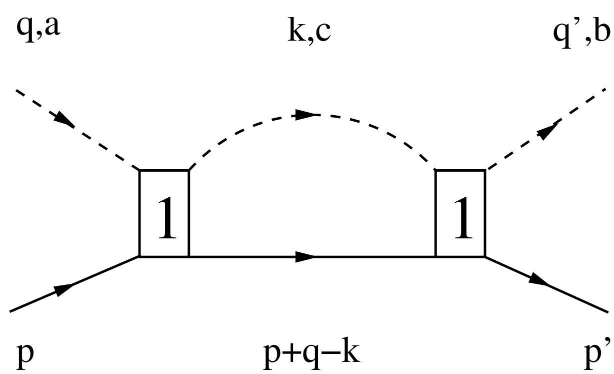
Given the Feynman rule of Eq. (5.53), the contribution of Fig. 5.4 to the invariant amplitude reads

$$\begin{aligned}\mathcal{M}_{\text{loop}} &= \int\frac{d^4k}{(2\pi)^4}\bar{u}(p')\frac{\not{k}+\not{q}'}{4F_0^2}\epsilon_{cbd}\tau^d\frac{i}{\not{p}+\not{q}-\not{k}-\overset{\circ}{m}_N+i0^+}\\ &\quad\times\frac{i}{k^2-M_\pi^2+i0^+}\frac{\not{q}+\not{k}}{4F_0^2}\epsilon_{ace}\tau^e u(p),\end{aligned}$$

where, counting powers, we expect the integral to have a cubic divergence. The isospin structure is given by

$$\epsilon_{cbd}\epsilon_{ace}\tau^d\tau^e = (\delta_{be}\delta_{da}-\delta_{ba}\delta_{de})\tau^d\tau^e = \tau^a\tau^b-\delta^{ba}\underbrace{\tau^d\tau^d}_{3} = -\underbrace{\left(2\delta^{ab}+\frac{1}{2}[\tau^b,\tau^a]\right)}_{\text{isospin}},$$

Figure 5.4: Example of a loop diagram contributing to pion-nucleon scattering.



i.e., the diagram contributes to both $\pm$ isospin amplitudes. We obtain

$$
\begin{aligned}
\mathcal{M}_{\text{loop}} &= \text{isospin}\,\frac{1}{16F_0^4}\int\frac{d^4k}{(2\pi)^4}\underbrace{\bar{u}(p')(\not{k}+\not{q}')(\not{p}+\not{q}-\not{k}+\overset{\circ}{m}_N)(\not{q}+\not{k})u(p)}_{\bar{u}(p')\widehat{O}(k)u(p)}\\
&\quad\times\frac{1}{(p+q-k)^2-\overset{\circ}{m}{}_N^2+i0^+}\,\frac{1}{k^2-M_\pi^2+i0^+}.
\end{aligned}
$$

We will outline the evaluation of the integral using dimensional regularization. To do this, we first combine the denominators using Feynman's trick, Eq. (C.6) of Appendix C.1.2, yielding

$$
\int_0^1 dz\frac{1}{[k^2-2z(p+q)\cdot k+z(s-\overset{\circ}{m}{}_N^2)+(z-1)M_\pi^2+i0^+]^2},
$$

where $s=(p+q)^2$. Shifting the integration variables as $k\to k+z(p+q)$, the amplitude reads

$$
\begin{aligned}
\mathcal{M}_{\text{loop}} &= \text{isospin}\,\frac{1}{16F_0^4}\mu^{4-n}\int_0^1 dz\int\frac{d^nk}{(2\pi)^n}\bar{u}(p')\widehat{O}[k+z(p+q)]u(p)\\
&\quad\times\frac{1}{[k^2+z(s-\overset{\circ}{m}{}_N^2)-z^2s+(z-1)M_\pi^2+i0^+]^2}.
\end{aligned}
$$

For the final conclusions, it is actually sufficient to consider the chiral limit, $M_\pi^2 = 0$, which simplifies the discussion of the loop integral. We define

$$A(z) \equiv sz^2 + (\overset{\circ}{m}{}_N^2 - s)z = z(sz + \overset{\circ}{m}{}_N^2 - s)$$

and will discuss the properties of the function A in more detail below. Note that A is a real, but not necessarily positive, number. The numerator of our integral is of the form

$$\widehat{O}[k + z(p+q)] = \widehat{O}_0 + \widehat{O}_1^\mu k_\mu + \widehat{O}_2^{\mu\nu} k_\mu k_\nu + \widehat{O}_3^{\mu\nu\rho} k_\mu k_\nu k_\rho,$$

generating integrals of the type

$$\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{\{1, k_\mu, k_\mu k_\nu, k_\mu k_\nu k_\rho\}}{(k^2 - A + i0^+)^2}, \tag{5.82}$$

where the integrals with an odd power of integration momenta in the numerator vanish in dimensional regularization, because of an integration over a symmetric interval. (The denominator is even). Let us discuss the scalar integral (numerator 1) of Eq. (5.82).[13] After a Wick rotation [see Eq. (B.3)], one chooses n-dimensional spherical coordinates for the Euclidean integral, and the angular integration is carried out as in Eq. (B.6). The remaining one-dimensional integration can be done using Eq. (B.10), and the result is expanded for small $\epsilon \equiv 4 - n$,

$$\begin{aligned} &\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - A + i0^+)^2} = \\ &\quad \frac{i}{(4\pi)^2} \left[-\frac{2}{n-4} + \ln(4\pi) + \Gamma'(1) - \ln\left(\frac{A - i0^+}{\mu^2}\right) + O(n-4) \right], \end{aligned} \tag{5.83}$$

where $-\Gamma'(1) = \gamma_E = 0.5772\cdots$ is Euler's constant. The infinity as $n \to 4$ must be canceled by some counter term of the effective πN Lagrangian. In order to perform the remaining integration over the Feynman parameter z, we make use of Eq. (B.15),

$$\ln(A - i0^+) = \ln(|A|) - i\pi\Theta(-A) \quad \text{for } A \in R, \tag{5.84}$$

[13]It is straightforward to also determine the second-rank tensor integral of Eq. (5.82) using the methods described in Appendices B and C. Regarding the analyticity properties we are interested in, one does not obtain any new information.

i.e., we need to discuss A as a function of $z \in [0,1]$ (for, in principle, arbitrary s). It is easy to show that A can take negative values in the interval $0 \leq z \leq 1$ only if $s > \overset{\circ}{m}{}_N^2$ in which case $A \leq 0$ for $0 \leq z \leq 1 - \overset{\circ}{m}{}_N^2/s$. In combination with Eq. (5.84) we obtain

$$
\begin{aligned}
\int_0^1 dz \ln\left(\frac{A(z) - i0^+}{\mu^2}\right) &= -i\pi \frac{s - \overset{\circ}{m}{}_N^2}{s} \Theta\left(s - \overset{\circ}{m}{}_N^2\right) \\
&\quad + \int_0^1 dz \ln\left(\frac{|sz^2 + (\overset{\circ}{m}{}_N^2 - s)z|}{\mu^2}\right).
\end{aligned}
$$

The remaining integral can be evaluated using elementary methods, and the final expression is

$$
\begin{aligned}
\int_0^1 dz \ln\left(\frac{A(z) - i0^+}{\mu^2}\right) &= -i\pi \frac{s - \overset{\circ}{m}{}_N^2}{s} \Theta\left(s - \overset{\circ}{m}{}_N^2\right) \\
&\quad + \ln\left(\frac{\overset{\circ}{m}{}_N^2}{\mu^2}\right) - 2 + \frac{s - \overset{\circ}{m}{}_N^2}{s} \ln\left(\frac{|s - \overset{\circ}{m}{}_N^2|}{\overset{\circ}{m}{}_N^2}\right).
\end{aligned}
\tag{5.85}
$$

At this point, we refrain from presenting the final expression of $\mathcal{M}_{\text{loop}}$ in detail, because Eq. (5.85) suffices to point out the difference between one-loop diagrams in the mesonic and the baryonic sectors. To do this, we expand s for small pion four-momenta in the chiral limit about $s_0 = \overset{\circ}{m}{}_N^2$:

$$
\begin{aligned}
\frac{s - \overset{\circ}{m}{}_N^2}{\overset{\circ}{m}{}_N^2} &= \frac{(p+q)^2 - \overset{\circ}{m}{}_N^2}{\overset{\circ}{m}{}_N^2} = \frac{2p \cdot q}{\overset{\circ}{m}{}_N^2} \equiv \alpha, \\
\frac{s - \overset{\circ}{m}{}_N^2}{s} &= \frac{s - \overset{\circ}{m}{}_N^2}{\overset{\circ}{m}{}_N^2 + s - \overset{\circ}{m}{}_N^2} = \frac{\alpha}{1+\alpha} = \alpha - \alpha^2 + \alpha^3 + \cdots.
\end{aligned}
\tag{5.86}
$$

[Note that α is a small quantity of chiral order $\mathcal{O}(p)$.] Taking into account that the two extracted Dirac structures (which we have not displayed) are (at least) of order $\mathcal{O}(p^2)$ [see Eq. (4.3) of Ref. [Gas+ 88]], one can draw the following conclusions [Gas+ 88]:

- The counter term needed to renormalize the contribution of Fig. 5.4 must contain terms which are of order $\mathcal{O}(p^2)$ and $\mathcal{O}(p^3)$.

- The finite part of the loop diagram has a logarithmic singularity of the form $p^3 \ln(p)$.

- Expanding the finite part of the diagram in terms of small external momenta one obtains an infinite series with arbitrary powers of (small) momenta p [see Eq. (5.86)].

In combination with the result of the previous section we see that a loop calculation with the relativistic Lagrangians $\mathcal{L}_{\pi N}^{(1)}$ and $\mathcal{L}_2$ using dimensional regularization leads to rather different properties in the mesonic and baryonic sectors. The example of the nucleon mass shows that loop diagrams may contribute at the same order as the tree diagrams which has to be contrasted with the mesonic sector where, according to the power counting of Eq. (4.52), loops are always suppressed by a factor p^{2N_L}, with N_L denoting the number of independent loops. In particular, with each new order of the loop expansion one has to expect that the low-energy coefficients including those of the lowest-order Lagrangian $\mathcal{L}_{\pi N}^{(1)}$ have to be renormalized. On the other hand, in the mesonic sector a one-loop calculation in the even-intrinsic parity sector leads to a renormalization of the $\mathcal{O}(p^4)$ coefficients (and possibly higher-order coefficients if vertices of higher order are used), a two-loop calculation to a renormalization of the $\mathcal{O}(p^6)$ and so on.

A second difference refers to the orders produced by a loop contribution. In the mesonic sector, a one-loop calculation involving vertices of $\mathcal{L}_2$ produces exclusively an $\mathcal{O}(p^4)$ contribution. We have seen in the πN-scattering example above that in the baryonic sector *all* higher orders are generated, even though, in principle, there is nothing wrong with such a result as long as one can organize and predict the leading order of the corresponding contribution beforehand.

In the next section we will discuss the so-called heavy-baryon formulation of ChPT [JM 91, Ber+ 92b], which provides a framework allowing for a power counting scheme which is very similar to the mesonic sector. One trades the manifestly covariant formulation for the systematic power counting. Moreover, under certain circumstances, the results obtained in HBChPT do not converge in all of the low-energy region. This problem has recently been solved in the framework of the so-called infrared regularization [BL 99] which will be discussed in Sec. 5.6.

5.5 The Heavy-Baryon Formulation

We have already seen in Sec. 5.2 that the baryonic sector introduces another energy scale—the nucleon mass—which does not vanish in the chiral limit. Furthermore, the mass of the nucleon has about the same size as the scale $4\pi F_0$ which appears in the calculation of pion-loop contributions [see, for example, the discussion of πN scattering, where the tree-level contributions of Table 5.1 are $\sim 1/F_0^2$, whereas the one-loop diagram of Fig. 5.4 is $\sim 1/(F_0^2(4\pi F_0)^2)$]. The heavy-baryon formulation of ChPT [JM 91, Ber+ 92b] consists in an expansion (of matrix elements) in terms of

$$\frac{p}{4\pi F_0} \quad \text{and} \quad \frac{p}{\overset{\circ}{m}_N},$$

where p represents a small external momentum. Clearly p cannot simply be the four-momentum of the initial and final nucleons of Eq. (5.1), because the energy components E_i and E_f are not small. Instead, a method has been devised which separates an external nucleon four-momentum into a large piece of the order of the nucleon mass and a small residual component. The approach is similar to the nonrelativistic reduction of Foldy and Wouthuysen [FW 50] which provides a systematic procedure to block-diagonalize a relativistic Hamiltonian in $1/m$ and produce a decoupling of positive- and negative energy states to any desired order in $1/m$. A criterion for the Foldy-Wouthuysen method to work is that the potentials in the Dirac Hamiltonian (corresponding to the interaction with external fields) are small in comparison with the nucleon mass. This may be considered as the analogue of treating external fields as small quantities of order $\mathcal{O}(p)$ (r_μ and l_μ) or $\mathcal{O}(p^2)$ ($f^R_{\mu\nu}$, $f^L_{\mu\nu}$, χ, and $\chi^\dagger$) in ChPT.

As in the previous cases we will discuss the lowest-order Lagrangian in quite some detail. For a discussion of the higher-order Lagrangians, the reader is referred to Refs. [EM 96, Ber+ 97, Fet+ 01].

5.5.1 Nonrelativistic Reduction

Before discussing the heavy-baryon framework let us start with the more familiar nonrelativistic limit of the Dirac equation for a charged particle interacting with an external electromagnetic field. Using this example, we will later be able to develop a better understanding of a peculiarity inherent in the heavy-baryon formulation regarding wave function (re)normalization. Our presentation will closely follow Refs. [Oku 54, Das 94].

Consider the Dirac equation of a point-particle of charge q and mass m interacting with an electromagnetic four-potential[14]

$$i\partial_0\Psi = [\vec{\alpha}\cdot(\vec{p}-q\vec{A})+\beta m+qA_0]\Psi \equiv H\Psi, \tag{5.87}$$

where α_i and β are the usual Dirac matrices

$$\alpha_i = \begin{pmatrix} 0_{2\times 2} & \sigma_i \\ \sigma_i & 0_{2\times 2} \end{pmatrix}, \quad \beta = \begin{pmatrix} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & -1_{2\times 2} \end{pmatrix},$$

and $\vec{p} = \vec{\nabla}/i$ is the momentum operator. For simplicity, we consider the interaction with a static external electric field,

$$\vec{E} = -\vec{\nabla}A_0, \quad \vec{A} = 0.$$

Since we want to describe a nonrelativistic particle-like solution, it is convenient to separate a factor $\exp(-imt)$ from the wave function,[15]

$$\Psi(\vec{x},t) = e^{-imt}\Psi'(\vec{x},t),$$

so that the Dirac equation (after multiplication with e^{+imt}) results in

$$i\partial_0\Psi' = [\vec{\alpha}\cdot\vec{p}+(\beta-1)m+qA_0]\Psi' \equiv H'\Psi'. \tag{5.88}$$

Note that both H and H' are Hermitian operators. In the spirit of the nonrelativistic reduction, we write Ψ' in terms of a pair of two-component spinors Ψ_L and Ψ_S (L for large and S for small)

$$\Psi' = \frac{1}{2}(1+\beta)\Psi' + \frac{1}{2}(1-\beta)\Psi' = \begin{pmatrix} \Psi_L \\ \Psi_S \end{pmatrix}, \tag{5.89}$$

and obtain, after insertion into Eq. (5.88), a set of two coupled partial differential equations

$$(i\partial_0 - qA_0)\Psi_L = \vec{\sigma}\cdot\vec{p}\,\Psi_S, \tag{5.90}$$
$$(i\partial_0 - qA_0 + 2m)\Psi_S = \vec{\sigma}\cdot\vec{p}\,\Psi_L. \tag{5.91}$$

[14] In order to facilitate the comparison with the Foldy-Wouthuysen result below, we make use of the "non-covariant" form of the Dirac equation.

[15] The (second) quantization of the relevant fields will be discussed in Sec. 5.5.4.

The second equation can formally be solved for Ψ_S,

$$\Psi_S = (2m + i\partial_0 - qA_0)^{-1}\vec{\sigma}\cdot\vec{p}\,\Psi_L \equiv A\Psi_L, \tag{5.92}$$

where, for later use, we have introduced the abbreviation A for the operator $(2m + i\partial_0 - qA_0)^{-1}\vec{\sigma}\cdot\vec{p}$. We expand Eq. (5.92) in terms of $1/m$ up to and including order $1/m^2$,

$$\begin{aligned}
A\Psi_L &= \frac{\vec{\sigma}\cdot\vec{p}}{2m}\Psi_L - \frac{i\partial_0 - qA_0}{2m}\frac{\vec{\sigma}\cdot\vec{p}}{2m}\Psi_L + O\left(\frac{1}{m^3}\right)\\
&= \frac{\vec{\sigma}\cdot\vec{p}}{2m}\left(1 - \frac{i\partial_0 - qA_0}{2m}\right)\Psi_L - iq\frac{\vec{\sigma}\cdot\vec{E}}{4m^2}\Psi_L + O\left(\frac{1}{m^3}\right)\\
&= \left(\frac{\vec{\sigma}\cdot\vec{p}}{2m} - iq\frac{\vec{\sigma}\cdot\vec{E}}{4m^2}\right)\Psi_L + O\left(\frac{1}{m^3}\right),
\end{aligned} \tag{5.93}$$

where we made use of the commutation relation $[A_0, \vec{p}] = i(\vec{\nabla}A_0) = -i\vec{E}$ and of $(i\partial_0 - qA_0)\Psi_L = O(1/m)\Psi_L$ [see Eqs. (5.90) and (5.92)]. Inserting this result into the right-hand side of Eq. (5.90) and using $\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B} = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot\vec{A}\times\vec{B}$, we obtain the Schrödinger-type equation

$$(i\partial_0 - qA_0)\Psi_L = \left\{\frac{\vec{p}^{\,2}}{2m} - \frac{q}{4m^2}\left[(\vec{\nabla}\cdot\vec{E}) + \vec{\sigma}\cdot\vec{E}\times\vec{p} + i\vec{E}\cdot\vec{p}\right]\right\}\Psi_L. \tag{5.94}$$

As already noted by Okubo [Oku 54], the last term on the right-hand side of Eq. (5.94) is not Hermitian and, when written as $V = -\vec{d}\cdot\vec{E}$, represents the interaction of an electric field with a (momentum-dependent) imaginary electric dipole moment $\vec{d} = iq\vec{p}/(4m^2)$ [Das 94].[16] As pointed out in Ref. [Das 94], the non-Hermiticity of the Hamilton operator of Eq. (5.94) is a consequence of the procedure for eliminating the small-component spinors. The method can be thought of as applying the transformation

$$S = \begin{pmatrix} 1_{2\times2} & 0_{2\times2} \\ -A & 1_{2\times2} \end{pmatrix} \tag{5.95}$$

to the four-component spinor Ψ' to generate a four-component spinor consisting exclusively of the upper component Ψ_L, and then solving the corresponding transformed Dirac equation. Since S is *not* a unitary operator,

[16] The standard textbook treatment of the nonrelativistic reduction leading to the Pauli equation considers only terms of $1/m$ and thus does not yet generate non-Hermitian terms (see, e.g., Refs. [BD 64a, IZ 80]).

i.e.,

$$\begin{pmatrix} 1_{2\times 2} & -A^\dagger \\ 0_{2\times 2} & 1_{2\times 2} \end{pmatrix} = S^\dagger \neq S^{-1} = \begin{pmatrix} 1_{2\times 2} & 0_{2\times 2} \\ A & 1_{2\times 2} \end{pmatrix},$$

the norm of the original spinor Ψ' and the transformed spinor Ψ_L, in general, will not be the same

$$\int d^3x \Psi'^\dagger \Psi' = \int d^3x (\Psi_L^\dagger \Psi_L + \Psi_S^\dagger \Psi_S) = \int d^3x \Psi_L^\dagger (1 + A^\dagger A)\Psi_L \neq \int d^3x \Psi_L^\dagger \Psi_L. \tag{5.96}$$

Equation (5.96) suggests considering a field redefinition of the form [Oku 54, Das 94]

$$\widetilde{\Psi}_L = (1 + A^\dagger A)^{\frac{1}{2}} \Psi_L, \tag{5.97}$$

so that the new spinor $\widetilde{\Psi}_L$ has the same norm as Ψ'. For the specific Hamiltonian of Eq. (5.88) we have

$$A = \frac{\vec{\sigma}\cdot\vec{p}}{2m} + O\left(\frac{1}{m^2}\right),$$

so that we find[17]

$$\Psi_L = (1 + A^\dagger A)^{-\frac{1}{2}} \widetilde{\Psi}_L = \left[1 - \frac{\vec{p}^{\,2}}{8m^2} + O\left(\frac{1}{m^3}\right)\right] \widetilde{\Psi}_L. \tag{5.98}$$

When inserting Eq. (5.98) into Eq. (5.94), we make use of

$$A_0 \vec{p}^{\,2} \widetilde{\Psi}_L = \vec{p}^{\,2}(A_0 \widetilde{\Psi}_L) - (\vec{\nabla}\cdot\vec{E})\widetilde{\Psi}_L - 2i\vec{E}\cdot\vec{p}\,\widetilde{\Psi}_L$$

and, as above, $(i\partial_0 - qA_0)\widetilde{\Psi}_L = O(1/m)\widetilde{\Psi}_L$, yielding the Schrödinger equation for the two-component spinor $\widetilde{\Psi}_L$, including relativistic corrections up to order $1/m^2$,

$$(i\partial_0 - qA_0)\widetilde{\Psi}_L = \left[\frac{\vec{p}^{\,2}}{2m} - \frac{q}{4m^2}\vec{\sigma}\cdot\vec{E}\times\vec{p} - \frac{q}{8m^2}(\vec{\nabla}\cdot\vec{E})\right]\widetilde{\Psi}_L, \tag{5.99}$$

where the second term, for a central potential, corresponds to the usual spin-orbit interaction and the last term is the so-called Darwin term [BD 64a,

[17]In the framework of plane-wave solutions, Eq. (5.98) already provides a hint that one may have to expect "unconventional normalization factors" when dealing with Feynman rules in the heavy-baryon approach.

IZ 80]. Note that the Hamiltonian here *is* Hermitian, i.e., the imaginary dipole moment has disappeared. Moreover, because of Eqs. (5.96) and (5.97), the spinors are normalized conventionally.

The result of Eq. (5.99) is identical with a nonrelativistic reduction using the Foldy-Wouthuysen method [FW 50] which uses a sequence of unitary transformations to block-diagonalize a relativistic Hamiltonian of the form

$$H = \beta m + \mathcal{O} + \mathcal{E} \tag{5.100}$$

to any desired order in $1/m$. In Eq. (5.100) $\mathcal{O}$ and $\mathcal{E}$ denote the so-called odd and even operators of H, respectively, where odd operators couple large and small components whereas even operators do not. In the present case we have

$$\mathcal{O} = \vec{\alpha}\cdot\vec{p}, \quad \mathcal{E} = qA_0,$$

and after three successive transformations one obtains the block-diagonal Hamiltonian (see, e.g., Refs. [BD 64a, IZ 80, Fea+ 94])

$$\begin{aligned} H_{\mathrm{FW}}^{(3)} &= \beta\left(m+\frac{\vec{p}^{\,2}}{2m}\right)+qA_0-\frac{1}{8m^2}[\vec{\alpha}\cdot\vec{p},[\vec{\alpha}\cdot\vec{p},qA_0]]+O\left(\frac{1}{m^3}\right)\\ &= \beta\left(m+\frac{\vec{p}^{\,2}}{2m}\right)+qA_0-\frac{q}{8m^2}(\vec{\nabla}\cdot\vec{E})-\frac{q}{4m^2}\vec{\Sigma}\cdot\vec{E}\times\vec{p}+O\left(\frac{1}{m^3}\right), \end{aligned} \tag{5.101}$$

where

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0_{2\times 2} \\ 0_{2\times 2} & \vec{\sigma} \end{pmatrix}. \tag{5.102}$$

Restricting ourselves to the upper left block of Eq. (5.101) and noting that in Eq. (5.88) we have already separated the time dependence $\exp(-imt)$ from Ψ, we find that Eqs. (5.99) and (5.101) are indeed identical. In Ref. [Das 94], the equivalence of the two approaches was explicitly shown to order $1/m^5$.

We will see that the heavy-baryon approach proceeds along lines very similar to the nonrelativistic reduction leading from Eq. (5.87) to (5.94). In analogy to Eqs. (5.96) and (5.97) we thus have to be alert to surprises related to the normalization of the relevant wave functions.

5.5.2 Light and Heavy Components

As mentioned above, the idea of the heavy-baryon approach consists of separating the large nucleon mass from the external four-momenta of the nucleons in the initial and final states and, in a sense to be discussed in Sec. 5.5.3 below, eliminating it from the Lagrangian.

The starting point is the relativistic Lagrangian of Eq. (5.21),

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi}\left(i\not{D} - m + \frac{g_A}{2}\not{u}\gamma_5\right)\Psi, \tag{5.103}$$

where the covariant derivative $D_\mu\Psi$ and u_μ are defined in Eqs. (5.18) and (5.20), respectively. The corresponding Euler-Lagrange equation for the nucleon field reads

$$-\partial_\mu \frac{\partial\mathcal{L}_{\pi N}^{(1)}}{\partial\partial_\mu\bar{\Psi}} + \frac{\partial\mathcal{L}_{\pi N}^{(1)}}{\partial\bar{\Psi}} = \left(i\not{D} - m + \frac{g_A}{2}\not{u}\gamma_5\right)\Psi = 0. \tag{5.104}$$

(For notational convenience we replace $\overset{\circ}{m}_N \to m$ and $\overset{\circ}{g}_A \to g_A$ in Secs. 5.5.2 and 5.5.3). For a general four-vector v^μ with the properties $v^2 = 1$ and $v^0 \geq 1$, we define the projection operators[18]

$$P_{v\pm} \equiv \frac{1 \pm \not{v}}{2}, \quad P_{v+} + P_{v-} = 1, \quad P_{v\pm}^2 = P_{v\pm}, \quad P_{v\pm}P_{v\mp} = 0, \tag{5.105}$$

and introduce the so-called velocity-dependent fields $\mathcal{N}_v$ and $\mathcal{H}_v$ as

$$\mathcal{N}_v \equiv e^{imv\cdot x}P_{v+}\Psi, \quad \mathcal{H}_v \equiv e^{imv\cdot x}P_{v-}\Psi, \tag{5.106}$$

so that Ψ can be written as

$$\Psi(x) = e^{-imv\cdot x}\left[\mathcal{N}_v(x) + \mathcal{H}_v(x)\right]. \tag{5.107}$$

The fields $\mathcal{N}_v$ and $\mathcal{H}_v$ satisfy the properties

$$\not{v}\mathcal{N}_v = \mathcal{N}_v, \quad \not{v}\mathcal{H}_v = -\mathcal{H}_v. \tag{5.108}$$

For a particle with four-momentum $p^\mu = (E, \vec{p})$ the particular choice $v^\mu = p^\mu/m$ corresponds to its world velocity which is why v is also referred to as a four-velocity. The fields $\mathcal{N}_v$ and $\mathcal{H}_v$ are often called the light and heavy components of the field Ψ, which will become clearer below.

[18] It may be worthwhile to remember that $P_{v\pm}$ do not define orthogonal projectors in the mathematical sense, because they do not satisfy $P_{v\pm}^\dagger = P_{v\pm}$, with the exception of the special case $v^\mu = (1, 0, 0, 0)$ used in Eq. (5.89).

In order to motivate the ansatz of Eq. (5.107) let us consider a positive-energy plane wave solution to the free Dirac equation with three-momentum $\vec{p}$:

$$\begin{aligned}
\psi_{\vec{p}}^{(+)(\alpha)}(\vec{x},t) &= u^{(\alpha)}(\vec{p})e^{-ip\cdot x}, \\
u^{(\alpha)}(\vec{p}) &= \sqrt{E(\vec{p})+m}\left(\begin{array}{c} \chi^{(\alpha)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E(\vec{p})+m}\chi^{(\alpha)} \end{array}\right),
\end{aligned}$$

where

$$\chi^{(1)} = \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \quad \chi^{(2)} = \left(\begin{array}{c} 0 \\ 1 \end{array}\right),$$

are ordinary two-component Pauli spinors, and $E(\vec{p}) = \sqrt{m^2+\vec{p}^{\,2}}$. We can think of $\psi_{\vec{p}}^{(+)(\alpha)}(\vec{x},t)$ entering the calculation of, say, an S-matrix element through covariant perturbation theory in terms of the matrix element of an in-field $\Psi_{\rm in}(x)$ between the vacuum and a single-nucleon state:

$$\langle 0|\Psi_{\rm in}(x)|N(\vec{p},\alpha),{\rm in}\rangle = u^{(\alpha)}(\vec{p})e^{-ip\cdot x}\chi_N,$$

where χ_N denotes the nucleon isospinor. For the special case $v^\mu = (1,0,0,0) \equiv v_1^\mu$, i.e.

$$P_{v_1+} = \left(\begin{array}{cc} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{array}\right), \quad P_{v_1-} = \left(\begin{array}{cc} 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 1_{2\times 2} \end{array}\right),$$

the components N_{v_1} and H_{v_1} are, up to the modified time dependence, equivalent to the large and small components of the "one-particle wave function"

$$\begin{aligned}
N_{v_1}^{(\alpha)}(x) &= \sqrt{E(\vec{p})+m}\left(\begin{array}{c} \chi^{(\alpha)} \\ 0_{2\times 1} \end{array}\right)e^{-i[E(\vec{p})-m]t+i\vec{p}\cdot\vec{x}}, \\
H_{v_1}^{(\alpha)}(x) &= \sqrt{E(\vec{p})+m}\left(\begin{array}{c} 0_{2\times 1} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E(\vec{p})+m}\chi^{(\alpha)} \end{array}\right)e^{-i[E(\vec{p})-m]t+i\vec{p}\cdot\vec{x}}. \qquad (5.109)
\end{aligned}$$

In other words, for this choice of v the light and heavy components of the positive-energy solutions are closely related to the large and small components of the nonrelativistic reduction discussed in Sec. 5.5.1. Moreover, assuming $|\vec{p}| \ll m$, $\exp[-i(E-m)t]$ varies slowly with time in comparison with $\exp(-iEt)$ of $\psi_{\vec{p}}^{(+)(\alpha)}(\vec{x},t)$, with the result that a time derivative $i\partial/\partial t$ generates a factor $(E-m)$ which is small in comparison with m.

Another choice is $v^\mu = p^\mu/m \equiv v_2^\mu$, in which case P_{v_2+} and P_{v_2-} correspond to the usual projection operators for positive- and negative-energy states

$$P_{v_2\pm} = \Lambda_\pm(p) = \frac{\pm \not p + m}{2m}.$$

For this case we find

$$\begin{aligned} N_{v_2}^{(\alpha)}(x) &= u^{(\alpha)}(\vec{p}), \\ H_{v_2}^{(\alpha)}(x) &= 0, \end{aligned} \tag{5.110}$$

i.e., the x dependence has completely disappeared in N_{v_2} and, due to the projection property $\Lambda_-(p)u^{(\alpha)}(\vec{p}) = 0$, H_{v_2} vanishes identically.

In general, one decomposes the four-momentum p^μ of a low-energy nucleon into mv^μ and a residual momentum k^μ,[19]

$$p^\mu = mv^\mu + k^\mu, \tag{5.111}$$

so that

$$v \cdot k = -\frac{k^2}{2m} \overset{v^\mu=(1,0,0,0)}{=} k_0 = E - m \ll m. \tag{5.112}$$

For v^μ in the vicinity of $(1,0,0,0)$, a partial derivative $i\partial^\mu$ acting on $e^{-ip\cdot x + imv\cdot x}$ produces a small residual momentum k^μ and, in particular,

$$iv \cdot \partial \mapsto v \cdot k \ll m.$$

The actual choice of v^μ is, to some extent, a matter of convenience. For low-energy processes involving a single nucleon in the initial and final states, the four-momentum q^μ transferred in the reaction is defined as $q = p_f - p_i$, and is considered as a small quantity of chiral order $\mathcal{O}(p)$. For $p_i = mv + k_i$ and $p_f = mv + k_f$, where, say, k_i is a small residual momentum in the sense of Eq. (5.112), also $k_f = k_i + q$ is a small four-momentum. The implications on a chiral power-counting scheme will be discussed in Sec. 5.5.8 below.

[19] Of course, the decomposition of Eq. (5.111) alone is not a sufficient criterion for $v \cdot k \ll m$. Taking, for example, $\vec{p} \perp \vec{v}$ one finds $v \cdot k = v \cdot p - m = Ev^0 - m \gg m$ for large v^0.

5.5.3 Lowest-Order Lagrangian

In order to proceed with the construction of the lowest-order heavy-baryon Lagrangian we insert Eq. (5.107) into the EOM of Eq. (5.104),[20]

$$
\begin{aligned}
&\left(i\not{D} - m + \frac{g_A}{2}\not{u}\gamma_5\right) e^{-imv\cdot x}\left(\mathcal{N}_v + \mathcal{H}_v\right) = \\
&\qquad e^{-imv\cdot x}\left(m\not{v} + i\not{D} - m + \frac{g_A}{2}\not{u}\gamma_5\right)\left(\mathcal{N}_v + \mathcal{H}_v\right) = 0,
\end{aligned}
$$

make use of Eq. (5.108), multiply by $e^{imv\cdot x}$, and obtain

$$
\left(i\not{D} + \frac{g_A}{2}\not{u}\gamma_5\right)\mathcal{N}_v + \left(i\not{D} - 2m + \frac{g_A}{2}\not{u}\gamma_5\right)\mathcal{H}_v = 0. \tag{5.113}
$$

In the next step we would like to separate the P_{v+} and the P_{v-} part of the EOM of Eq. (5.113). To that end we make use of the algebra of the gamma matrices to derive

$$
\begin{aligned}
P_{v+}\not{A}\,P_{v+} &= v\cdot A\,P_{v+}, \\
P_{v+}\not{A}P_{v-} &= \not{A}_\perp P_{v-} = P_{v+}\not{A}_\perp, \\
P_{v-}\not{A}\,P_{v-} &= -v\cdot A\,P_{v-}, \\
P_{v-}\not{A}\,P_{v+} &= \not{A}_\perp P_{v+} = P_{v-}\not{A}_\perp, \\
P_{v+}\not{B}\gamma_5 P_{v+} &= \not{B}_\perp\gamma_5 P_{v+}, \\
P_{v+}\not{B}\gamma_5 P_{v-} &= v\cdot B\,\gamma_5 P_{v-} = v\cdot B\,P_{v+}\gamma_5, \\
P_{v-}\not{B}\gamma_5 P_{v-} &= \not{B}_\perp\gamma_5 P_{v-}, \\
P_{v-}\not{B}\gamma_5 P_{v+} &= -v\cdot B\gamma_5 P_{v+} = -v\cdot BP_{v-}\gamma_5,
\end{aligned} \tag{5.114}
$$

where

$$
P_{v\pm} = \frac{1\pm\not{v}}{2}, \quad v^2 = 1, \quad A_\perp^\mu = A^\mu - v\cdot A\,v^\mu, \quad v\cdot A_\perp = 0, \quad \not{A}_\perp = A_\perp^\mu\gamma_\mu.
$$

As an example, let us explicitly show the first relation of Eq. (5.114)

$$
\begin{aligned}
P_{v+}\not{A}\,P_{v+} &= \frac{1}{2}(1+\not{v})\not{A}\,P_{v+} = \frac{1}{2}(\not{A} + \not{v}\,\not{A})P_{v+} = \frac{1}{2}(\not{A} - \not{A}\not{v} + 2v\cdot A)P_{v+} \\
&= (\not{A}\,P_{v-} + v\cdot A)P_{v+} = v\cdot A\,P_{v+}.
\end{aligned}
$$

[20]For a derivation in the framework of the path-integral approach, see Ref. [Man+ 92] and Appendix A of Ref. [Ber+ 92b].

The remaining results of Eq. (5.114) follow analogously. Using Eqs. (5.105) and (5.114) we are now in the position to project onto the P_{v+} and P_{v-} parts of the EOM of Eq. (5.113),

$$\begin{aligned}\left(iv\cdot D+\frac{g_A}{2}\not{u}_\perp\gamma_5\right)\mathcal{N}_v+\left(i\not{D}_\perp+\frac{g_A}{2}v\cdot u\gamma_5\right)\mathcal{H}_v &= 0, &(5.115)\\ \left(i\not{D}_\perp-\frac{g_A}{2}v\cdot u\gamma_5\right)\mathcal{N}_v+\left(-iv\cdot D-2m+\frac{g_A}{2}\not{u}_\perp\gamma_5\right)\mathcal{H}_v &= 0, &(5.116)\end{aligned}$$

which corresponds to Eqs. (5.90) and (5.91) of the nonrelativistic reduction of Sec. 5.5.1. We formally solve Eq. (5.116) for $\mathcal{H}_v$,

$$\mathcal{H}_v=\left(2m+iv\cdot D-\frac{g_A}{2}\not{u}_\perp\gamma_5\right)^{-1}\left(i\not{D}_\perp-\frac{g_A}{2}v\cdot u\gamma_5\right)\mathcal{N}_v, \qquad (5.117)$$

which, similar to the discussion of Sec. 5.5.1, shows that the heavy component $\mathcal{H}_v$ is formally suppressed by powers of $1/m$ relative to the light component $\mathcal{N}_v$.[21] Inserting Eq. (5.117) into Eq. (5.115), the EOM for the light component reads

$$\begin{aligned}&\left(iv\cdot D+\frac{g_A}{2}\not{u}_\perp\gamma_5\right)\mathcal{N}_v+\left(i\not{D}_\perp+\frac{g_A}{2}v\cdot u\gamma_5\right)\\ &\quad\times\left(2m+iv\cdot D-\frac{g_A}{2}\not{u}_\perp\gamma_5\right)^{-1}\left(i\not{D}_\perp-\frac{g_A}{2}v\cdot u\gamma_5\right)\mathcal{N}_v=0, \qquad (5.118)\end{aligned}$$

which represents the analogue of Eq. (5.94). This EOM may be obtained from applying the variational principle to the effective Lagrangian[22]

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \bar{\mathcal{N}}_v\left(iv\cdot D+\frac{g_A}{2}\not{u}_\perp\gamma_5\right)\mathcal{N}_v+\bar{\mathcal{N}}_v\left(i\not{D}_\perp+\frac{g_A}{2}v\cdot u\gamma_5\right)\\ &\quad\times\left(2m+iv\cdot D-\frac{g_A}{2}\not{u}_\perp\gamma_5\right)^{-1}\left(i\not{D}_\perp-\frac{g_A}{2}v\cdot u\gamma_5\right)\mathcal{N}_v. \qquad (5.119)\end{aligned}$$

Note that the second term is suppressed by $1/m$ relative to the first term. Equation (5.119) corresponds to the leading-order result for Eq. (A.10) of

[21] In fact, setting all external fields to zero and dropping the interaction term proportional to g_A, it is easy to verify that the one-particle wave functions indeed satisfy the relation implied by Eq. (5.117).

[22] Replacing $\mathcal{N}_v\to h^{(+)}$ and $\mathcal{H}_v\to h^{(-)}$, omitting all terms containing the chiral vielbein u_μ, and interpreting the covariant derivative as that of QCD, the result of Eq. (5.119) is identical with Eq. (7) of the discussion of heavy quark effective theory in Sec. 2 of Ref. [Bal+ 94].

Ref. [Ber+ 92b] which was obtained in the framework of the path-integral approach, but does not yet represent the final form commonly used in HBChPT.[23] Having the discussion following Eq. (5.95) in mind, in order for the two Lagrangians of Eqs. (5.103) and (5.119) to describe the same observables, we cannot expect both fields Ψ and $\mathcal{N}_v$ to be normalized in the same way. We will come back to this question in Sec. 5.5.4.

To obtain the heavy-baryon Lagrangian we define the spin matrix S_v^μ as[24]

$$S_v^\mu = \frac{i}{2}\gamma_5\sigma^{\mu\nu}v_\nu = -\frac{1}{2}\gamma_5(\gamma^\mu \not{v} - v^\mu), \quad S_v^{\mu\dagger} = \gamma_0 S_v^\mu \gamma_0, \tag{5.120}$$

which, in four dimensions, has the properties

$$v \cdot S_v = 0, \quad \{S_v^\mu, S_v^\nu\} = \frac{1}{2}(v^\mu v^\nu - g^{\mu\nu}), \quad [S_v^\mu, S_v^\nu] = i\epsilon^{\mu\nu\rho\sigma}v_\rho S_\sigma^v. \tag{5.121}$$

Using the properties of Eq. (5.108) together with Eq. (5.121), the 16 combinations $\bar{\mathcal{N}}_v\Gamma\mathcal{N}_v$ may be written as [see Eqs. (9) - (12) of Ref. [JM 91]]

$$\begin{aligned}
(\bar{\mathcal{N}}_v 1_{4\times4}\mathcal{N}_v &= \bar{\mathcal{N}}_v 1_{4\times4}\mathcal{N}_v,) \\
\bar{\mathcal{N}}_v\gamma_5\mathcal{N}_v &= 0, \\
\bar{\mathcal{N}}_v\gamma^\mu\mathcal{N}_v &= v^\mu\bar{\mathcal{N}}_v\mathcal{N}_v,
\end{aligned}$$

[23] In order to be able to invert the operator $\mathcal{C}$ of Ref. [Ber+ 92b], strictly speaking the projection operators P_{v-} should not be included in the definition of $\mathcal{C}$.

[24] For the classification of the irreducible representations of the Poincaré group, one makes use of the so-called Pauli-Lubanski vector

$$W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma,$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor in four indices, $\epsilon_{0123} = 1$, $J^{\mu\nu}$ denotes the generalized angular momentum operator, and P^μ is the four-momentum operator (see, e.g, Refs. [IZ 80, Jon 90]). Both W^2 and P^2 are Lorentz invariant and translationally invariant and are thus used as Casimir operators, where the eigenvalues are denoted by m^2 and $-m^2 s(s+1)$, $s = 0, 1/2, 1, \cdots$. For the massive spin-1/2 case one obtains

$$W_\mu = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\sigma^{\nu\rho}P^\sigma.$$

Using (in four dimensions)

$$\gamma_5\sigma_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma},$$

together with the special choice $\tilde{v}^\mu = P^\mu/m$, one easily finds that the spin matrix is, for this special case, proportional to the Pauli-Lubanski vector, $W^\mu = mS_{\tilde{v}}^\mu$.

$$\begin{aligned}
\bar{\mathcal{N}}_v\gamma^\mu\gamma_5\mathcal{N}_v &= 2\bar{\mathcal{N}}_v S_v^\mu \mathcal{N}_v,\\
\bar{\mathcal{N}}_v\sigma^{\mu\nu}\mathcal{N}_v &= 2\epsilon^{\mu\nu\rho\sigma}v_\rho\bar{\mathcal{N}}_v S_\sigma^v\mathcal{N}_v,\\
\bar{\mathcal{N}}_v\sigma^{\mu\nu}\gamma_5\mathcal{N}_v &= 2i(v^\mu\bar{\mathcal{N}}_v S_v^\nu\mathcal{N}_v - v^\nu\bar{\mathcal{N}}_v S_v^\mu\mathcal{N}_v).
\end{aligned}\tag{5.122}$$

For example,

$$\bar{\mathcal{N}}_v\gamma_5\mathcal{N}_v = \bar{\mathcal{N}}_v\gamma_5\not{v}\,\mathcal{N}_v = -\bar{\mathcal{N}}_v\not{v}\,\gamma_5\mathcal{N}_v = -\bar{\mathcal{N}}\gamma_5\mathcal{N}_v = 0,$$

where we made use of Eq. (5.108). The remaining relations are shown analogously. Eqs. (5.122) result in a nice simplification of the Dirac structures in the heavy-baryon approach, because one ultimately only deals with two groups of 4×4 matrices, the unit matrix and S_v^μ, instead of the original six groups on the left-hand side of Eq. (5.122).

Expanding Eq. (5.119) formally into a series in $1/m$,

$$\mathcal{L}_{\text{eff}} = \bar{\mathcal{N}}_v\left(iv\cdot D + \frac{g_A}{2}\not{u}_\perp\gamma_5\right)\mathcal{N}_v + \sum_{n=1}^{\infty}\frac{1}{(2m)^n}\mathcal{L}_{\text{eff},n},\tag{5.123}$$

and applying Eq. (5.122), the lowest-order term reads

$$\widehat{\mathcal{L}}_{\pi N}^{(1)} = \bar{\mathcal{N}}_v\left[iv\cdot D + g_A S_v\cdot u\right]\mathcal{N}_v,\tag{5.124}$$

where we made use of the fourth relation of Eq. (5.122) and the first relation of Eq. (5.121). (Recall that the second term of Eq. (5.119) is of order $1/m$.) Equation (5.124) represents the lowest-order Lagrangian of heavy-baryon chiral perturbation theory (HBChPT), indicated by the symbol $\widehat{\ }$. When comparing with the relativistic Lagrangian of Eq. (5.103), we see that the nucleon mass has disappeared from the leading-order Lagrangian. It only shows up in higher orders as powers of $1/m$, as will be discussed in Sec. 5.5.7. In the power counting scheme $\widehat{\mathcal{L}}_{\pi N}^{(1)}$ counts as $\mathcal{O}(p)$, because the covariant derivative D_μ and the chiral vielbein u_μ both count as $\mathcal{O}(p)$.

When calculating loop diagrams with the Lagrangian of Eq. (5.124) one will encounter divergences which are treated in the framework of (normal) dimensional regularization [see Eq. (5.78)]. Since the definition of the spin matrix S_v^μ contains γ_5 and the commutator of two such spin matrices, in four dimensions, involves the epsilon tensor, one needs some convention for dealing with products of spin matrices when evaluating integrals in n dimensions. To be on the safe side, we always reduce the gamma matrices using only the rules of Eq. (5.78). Let us consider the following example, which appears in the calculation of the pion-nucleon form factor at the one-loop level:[25]

$$
\begin{aligned}
S^v_\mu S^v_\nu S^\mu_v &= -\frac{1}{8}\gamma_5(\gamma_\mu \not{v} - v_\mu)\gamma_5(\gamma_\nu \not{v} - v_\nu)\gamma_5(\gamma^\mu \not{v} - v^\mu) \\
&= -\frac{1}{8}(n-3)\gamma_5(\gamma_\nu \not{v} - v_\nu) = \frac{n-3}{4} S^v_\nu, \qquad (5.125)
\end{aligned}
$$

where we consistently made use of the anticommutation relations of Eq. (5.78). In contrast, using the anticommutator and commutator of Eq. (5.121), one ends up with $S^v_\nu/4$ which only coincides with Eq. (5.125) for $n = 4$. However, the factor $(n-3)$ needs to be written as $1+(n-4)$ when it is multiplied with a singularity of the form $C/(n-4)$ in order to produce the constant non-divergent term C in the product. Similarly, using the conventions of Eq. (5.78), the squared spin operator in n dimensions reads

$$
S_v^2 = \frac{1-n}{4}. \qquad (5.126)
$$

5.5.4 Normalization of Fields and States

So far we have calculated matrix elements of the relativistic Lagrangian $\mathcal{L}^{(1)}_{\pi N}$ of Eq. (5.21) in the framework of covariant perturbation theory based on the formula of Gell-Mann and Low [GL 51] in combination with Wick's theorem [Wic 50]. Let us recall that, for a generic field $\Phi(x)$ described by the Lagrangian

$$
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}},
$$

the "magic formula of covariant perturbation theory" [Haa 92] allows one to calculate the Green functions

$$
\tau_n(x_1, \cdots, x_n) = \langle \Omega | T[\Phi(x_1) \cdots \Phi(x_n)] | \Omega \rangle \qquad (5.127)
$$

in terms of the generating functional

$$
\begin{aligned}
\mathcal{T}[f] &= \frac{N[f]}{N[0]}, \qquad (5.128) \\
N[f] &= \langle \Omega_0 | T \exp\left\{ i \int d^4x \left[\Phi^0(x) f(x) + \mathcal{L}^0_{\text{int}}(x) \right] \right\} | \Omega_0 \rangle. \qquad (5.129)
\end{aligned}
$$

[25] In evaluating Eq. (5.125), we made use of

$$
\gamma_\mu \not{a} \gamma^\mu = (2-n)\not{a}, \quad \gamma_\mu \not{a} \not{b} \gamma^\mu = 4a\cdot b + (n-4)\not{a}\not{b},
$$

in n dimensions.

While the Green functions of Eq. (5.127) involve the interacting field $\Phi(x)$ and the vacuum Ω of the corresponding interacting theory, the formula of Gell-Mann and Low, in principle, provides an explicit expression for the generating functional in terms of the quantities $\Phi^0(x)$ and Ω_0 defined in the free theory. Note that the Green functions of Eq. (5.127) are expressed in terms of the bare fields and, in the end, still have to be renormalized (see, e.g, Sec. 4.9.1).

Here we want to address the question of how to establish contact between matrix elements evaluated perturbatively using the relativistic Lagrangian of Eq. (5.21), on the one hand, and the heavy-baryon Lagrangian of Eq. (5.119) on the other hand. The presentation will make use of the ideas developed in Refs. [Dug+ 92, Bal+ 94], where this issue was discussed for the case of a heavy-quark effective theory. A different route was taken in Refs. [EM 97, Ste+ 98, KM 99], where the path-integral approach to the generating functional was used to define the wave function renormalization constant (including interaction). Later on we will explicitly see that the two approaches yield identical results at $\mathcal{O}(p^3)$.

For later comparison with the heavy-baryon approach, we first need to collect a few properties of the free Dirac field operator $\Psi^0(x)$ which we decompose in the standard fashion in terms of the solutions of the free Dirac equation[26]

$$\Psi^0(x) = \sum_{\alpha=1}^{2} \int \frac{d^3p}{(2\pi)^3 2E(\vec{p})} \left[b_\alpha(\vec{p}) u^{(\alpha)}(\vec{p}) e^{-ip\cdot x} + d_\alpha^\dagger(\vec{p}) v^{(\alpha)}(\vec{p}) e^{ip\cdot x} \right], \tag{5.130}$$

where $p_0 = E(\vec{p}) = \sqrt{m^2 + \vec{p}^{\,2}}$. The Dirac spinors have the following properties

$$\begin{aligned}
(\not{p} - m) u^{(\alpha)}(\vec{p}) &= 0, \\
(\not{p} + m) v^{(\alpha)}(\vec{p}) &= 0, \\
\bar{u}^{(\alpha)}(\vec{p}) u^{(\beta)}(\vec{p}) &= -\bar{v}^{(\alpha)}(\vec{p}) v^{(\beta)}(\vec{p}) = 2m\delta_{\alpha\beta}, \\
u^{(\alpha)\dagger}(\vec{p}) u^{(\beta)}(\vec{p}) &= v^{(\alpha)\dagger}(\vec{p}) v^{(\beta)}(\vec{p}) = 2E(\vec{p})\delta_{\alpha\beta}, \\
u^{(\alpha)\dagger}(\vec{p}) v^{(\beta)}(-\vec{p}) &= \bar{u}^{(\alpha)}(\vec{p}) v^{(\beta)}(\vec{p}) = 0.
\end{aligned} \tag{5.131}$$

The creation operators $b_\alpha^\dagger$ and $d_\alpha^\dagger$ (annihilation operators b_α and d_α) of par-

[26] For the sake of simplicity, we consider one generic fermion field.

ticles and antiparticles, respectively, satisfy the anticommutation relations

$$\begin{aligned}\{b_\alpha(\vec{p}), b^\dagger_\beta(\vec{p}\,')\} &= (2\pi)^3 2E(\vec{p})\delta^3(\vec{p}-\vec{p}\,')\delta_{\alpha\beta},\\ \{d_\alpha(\vec{p}), d^\dagger_\beta(\vec{p}\,')\} &= (2\pi)^3 2E(\vec{p})\delta^3(\vec{p}-\vec{p}\,')\delta_{\alpha\beta},\end{aligned}$$

where all remaining anticommutators such as, e.g., $\{b_\alpha(\vec{p}), b_\beta(\vec{p}\,')\}$ vanish. With this convention, single-particle states $|\vec{p},\alpha,+\rangle = b^\dagger_\alpha(\vec{p})|0\rangle$ are normalized as

$$\langle\vec{p}\,',\beta,+|\vec{p},\alpha,+\rangle = (2\pi)^3 2E(\vec{p})\delta^3(\vec{p}\,'-\vec{p})\delta_{\alpha\beta}. \tag{5.132}$$

Let us now turn to the heavy-baryon formulation and consider the leading-order term of Eq. (5.124) which we write as

$$\widehat{\mathcal{L}}^{(1)}_{\pi N} = \bar{\mathcal{N}}_v iv\cdot\partial\mathcal{N}_v + \widehat{\mathcal{L}}_{\rm int} \equiv \widehat{\mathcal{L}}_0 + \widehat{\mathcal{L}}_{\rm int} \tag{5.133}$$

for later use in the formula of Gell-Mann and Low. We decompose the solution to the free equation of motion

$$iv\cdot\partial\mathcal{N}^0_v(x) = 0, \quad \not{v}\,\mathcal{N}^0_v = \mathcal{N}^0_v, \tag{5.134}$$

as

$$\mathcal{N}^0_v(x) = \sum_{\alpha=1}^{2}\int\frac{d^3k}{(2\pi)^3 2mv_0}e^{-ik\cdot x}b_{\alpha,v}(\vec{k})u^{(\alpha)}_v, \tag{5.135}$$

where k_0, at leading order, is defined through $v\cdot k = 0$ in order to satisfy Eq. (5.134), i.e.,

$$k_0 = \frac{\vec{v}\cdot\vec{k}}{v_0}, \quad k\cdot x = -\vec{k}\cdot\left(\vec{x}-\vec{v}\,\frac{x_0}{v_0}\right).$$

The spinors are given by

$$u^{(\alpha)}_v = \sqrt{\frac{2m}{v_0}}P_{v+}\left(\begin{array}{c}\chi^{(\alpha)}\\ 0_{2\times 1}\end{array}\right), \quad \bar{u}^{(\alpha)}_v u^{(\beta)}_v = 2m\delta_{\alpha\beta}, \tag{5.136}$$

with $\chi^{(\alpha)}$ ordinary two-component Pauli spinors. Note that, at lowest order in $1/m$, the spinors do not depend on the residual momentum $\vec{k}$. Moreover, for an arbitrary choice of v, the $u^{(\alpha)}_v$ are four-component objects which only for the special case $v^\mu = (1,0,0,0)$ effectively reduce to two-component spinors. As will be shown below, the operators $b_{\alpha,v}(\vec{k})$ and $b^\dagger_{\alpha,v}(\vec{k})$ destroy and create a nucleon (isospin index suppressed) with residual three-momentum $\vec{k}$. They satisfy the anticommutation relations

$$\{b_{\alpha,v}(\vec{k}), b^{\dagger}_{\beta,v}(\vec{k}')\} = 2mv_0(2\pi)^3\delta^3(\vec{k}-\vec{k}\,')\delta_{\alpha\beta}, \tag{5.137}$$

where, as usual, the anticommutator of two annihilation or two creation operators vanishes. Accordingly, the single-particle states are normalized as

$$\langle v, \vec{k}\,', \beta|v, \vec{k}, \alpha\rangle = 2mv_0(2\pi)^3\delta^3(\vec{k}\,'-\vec{k}\,)\delta_{\alpha\beta}. \tag{5.138}$$

Note that the normalization of the states of Eqs. (5.132) and (5.138) coincide only at leading order in $1/m$ (or as $m \to \infty$).

Using Eq. (5.137) it is straightforward to verify that the (free) theory has been quantized "canonically" [Dug+ 92], i.e.

$$\{\Pi^0_v(\vec{x},t), \mathcal{N}^0_v(\vec{y},t)\} = iv_0\{\bar{\mathcal{N}}^0_v(\vec{x},t), \mathcal{N}^0_v(\vec{y},t)\} = i\delta^3(\vec{x}-\vec{y})P_{v+},$$

where $\Pi^0_v = \partial\widehat{\mathcal{L}}_0/\partial(\partial_0\mathcal{N}_v) = iv_0\bar{\mathcal{N}}^0_v$ is the momentum conjugate to $\mathcal{N}^0_v$ and we made use of the completeness relation

$$\sum_{\alpha=1}^{2} u^{(\alpha)}_v \bar{u}^{(\alpha)}_v = 2mP_{v+}. \tag{5.139}$$

Constructing the energy-momentum tensor corresponding to $\widehat{\mathcal{L}}_0$,

$$\Theta^{\mu\nu}_v = \partial^\nu\bar{\mathcal{N}}_v\frac{\partial\widehat{\mathcal{L}}_0}{\partial(\partial_\mu\bar{\mathcal{N}}_v)} + \frac{\partial\widehat{\mathcal{L}}_0}{\partial(\partial_\mu\mathcal{N}_v)}\partial^\nu\mathcal{N}_v - g^{\mu\nu}\widehat{\mathcal{L}}_0 = \bar{\mathcal{N}}^0_v iv^\mu\partial^\nu\mathcal{N}_v,$$

(we made use of the equation of motion) we obtain for the four-momentum operator

$$K^\mu_v = \int d^3x\bar{\mathcal{N}}^0_v iv_0\partial^\mu\mathcal{N}_v = \sum_{\alpha=1}^{2}\int\frac{d^3k}{(2\pi)^3 2mv_0}k^\mu b^{\dagger}_{\alpha,v}(\vec{k})b_{\alpha,v}(\vec{k}). \tag{5.140}$$

Using $[ab,c] = a\{b,c\} - \{a,c\}b$ it is then straightforward to verify the commutation relations

$$[K^\mu_v, b_{\alpha,v}(\vec{k})] = -k^\mu b_{\alpha,v}(\vec{k}), \quad [K^\mu_v, b^{\dagger}_{\alpha,v}(\vec{k})] = k^\mu b^{\dagger}_{\alpha,v}(\vec{k}). \tag{5.141}$$

Eq. (5.141) implies that $b_{\alpha,v}(\vec{k})$ and $b^{\dagger}_{\alpha,v}(\vec{k})$ destroy and create quanta with (residual) four-momentum k^μ and total four-momentum $p^\mu = mv^\mu + k^\mu$.

This can be seen by comparing with the four-momentum operator of the free relativistic theory

$$P^\mu = \int d^3x \bar{\Psi}^0 \gamma^0 i\partial^\mu \Psi^0, \tag{5.142}$$

which, to leading order in $1/m$, is related to Eq. (5.140) by $P^\mu = K_v^\mu + mv^\mu N$, where $N = \int d^3x \Psi^{0\dagger}\Psi^0$ is the number operator [Dug+ 92].

Using the orthogonality relations of Eq. (5.136) we may express the creation and annihilation operators in terms of the fields in the standard way

$$\begin{aligned} b^\dagger_{\alpha,v}(\vec{k}) &= v_0 \int d^3x \bar{\mathcal{N}}_v(x) u_v^{(\alpha)} e^{-ik\cdot x}, \\ b_{\alpha,v}(\vec{k}) &= v_0 \int d^3x e^{ik\cdot x} \bar{u}_v^{(\alpha)} \mathcal{N}_v(x). \end{aligned} \tag{5.143}$$

Eqs. (5.143) are the starting point for the LSZ reduction [Leh+ 55, BD 64b, IZ 80] in the framework of the heavy-baryon approach. We consider the matrix element of Eq. (5.1) for the transition in the presence of external fields v, a, s, and p (we omit spin and isospin labels)

$$\begin{aligned} &\mathcal{F}(\vec{p}\,', \vec{p}; v, a, s, p) = \langle \vec{p}\,'; \mathrm{out} | \vec{p}; \mathrm{in}\rangle^{\mathrm{c}}_{v,a,s,p} \\ &= \sqrt{\frac{E}{mv_0}} \langle \vec{p}\,'; \mathrm{out} | b^\dagger_{v,\mathrm{in}}(\vec{k}) |\Omega\rangle^{\mathrm{c}}_{v,a,s,p} \\ &= \sqrt{\frac{E}{mv_0}} v_0 \int d^3x \langle \vec{p}\,'; \mathrm{out} | \bar{\mathcal{N}}_{v,\mathrm{in}}(x) |\Omega\rangle^{\mathrm{c}}_{v,a,s,p} u_v e^{-ik\cdot x} \\ &= \sqrt{\frac{E}{mv_0}} \lim_{t\to-\infty} v_0 \int d^3x \langle \vec{p}\,'; \mathrm{out} | \frac{\bar{\mathcal{N}}_v(x)}{\sqrt{Z}} |\Omega\rangle^{\mathrm{c}}_{v,a,s,p} u_v e^{-ik\cdot x} \\ &= \cdots \\ &= \left(\frac{-i}{\sqrt{Z}}\right)^2 NN' \int d^4x d^4y \\ &\quad \times e^{ik'\cdot y} \bar{u}_v iv\cdot \overrightarrow{\partial}_y \langle \Omega | T[\mathcal{N}_v(y) \bar{\mathcal{N}}_v(x)] |\Omega\rangle^{\mathrm{c}}_{v,a,s,p} (-iv\cdot \overleftarrow{\partial}_x) u_v e^{-ik\cdot x}. \end{aligned} \tag{5.144}$$

The intermediate steps indicated by $\cdots$ proceed in complete analogy to the usual reduction formula as described in, e.g., Refs. [BD 64b, IZ 80]. In Eq. (5.144), the factors of the type $N = \sqrt{E/(mv_0)}$ are related to the relative normalization of the states [see Eq. (5.132) vs. (5.138)], whereas $\sqrt{Z}$ refers to the wave function renormalization in the framework of the heavy-baryon Lagrangian.

The Green function entering Eq. (5.144) will be calculated perturbatively using the formula of Gell-Mann and Low [GL 51],

$$\langle\Omega|T[\mathcal{N}_v(y)\bar{\mathcal{N}}_v(x)]|\Omega\rangle^{\mathrm{c}}_{v,a,s,p} = \langle\Omega_0|T[\mathcal{N}_v^0(y)\bar{\mathcal{N}}_v^0(x)\exp\left(i\int d^4z\widehat{\mathcal{L}}^{\,0}_{\mathrm{int}}(z)\right)]|\Omega_0\rangle^{\mathrm{c}}, \tag{5.145}$$

where, on the right-hand side, $|\Omega_0\rangle$ denotes the vacuum of the free theory, and the external fields are part of the Lagrangian $\widehat{\mathcal{L}}^{\,0}_{\mathrm{int}}(z)$.[27]

5.5.5 Propagator at Lowest Order

We will now discuss the propagator of the lowest-order Lagrangian both on the "classical level" as well as in the quantized theory of the last section. The lowest-order equation of motion corresponding to Eq. (5.124) reads

$$(iv\cdot D + g_A S_v\cdot u)\mathcal{N}_v = 0, \quad P_{v+}\mathcal{N}_v = \mathcal{N}_v, \tag{5.146}$$

where the second relation implies $P_{v-}\mathcal{N}_v = 0$ [see Eq. (5.108)]. We define the propagator corresponding to Eq. (5.146) through

$$(iv\cdot D + g_A S_v\cdot u)G_v(x,x') = P_{v+}\delta^4(x-x'), \quad P_{v-}G_v(x,x') = 0. \tag{5.147}$$

In order to solve Eq. (5.146) perturbatively, we re-write the equation of motion in the standard form as

$$iv\cdot\partial\mathcal{N}_v(x) = V(x)\mathcal{N}_v(x),$$

where V denotes the interaction term, and search for the unperturbed Green function $G_v^0(x,x')$ satisfying the properties

$$\begin{aligned} iv\cdot\partial G_v^0(x,x') &= \delta^4(x-x')P_{v+}, &(5.148)\\ P_{v-}G_v^0(x,x') &= 0, &(5.149)\\ G_v^0(x,x') &= 0 \text{ for } x_0' > x_0. &(5.150)\end{aligned}$$

In terms of G_v^0 the propagator G_v is then given by

$$G_v(x,x') = G_v^0(x,x') + \int d^4y G_v^0(x,y)V(y)G_v(y,x').$$

[27] Strictly speaking we should also include the mesonic Lagrangian.

Inserting the standard ansatz in terms of a Fourier decomposition

$$G_v^0(x,x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} G_v^0(k) \tag{5.151}$$

into Eq. (5.148),

$$\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} v\cdot k G_v^0(k) = \delta^4(x-x')P_{v+} = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} P_{v+},$$

we obtain by comparing both sides

$$G_v^0(k) = \frac{P_{v+}}{v\cdot k} \text{ for } v\cdot k \neq 0.$$

The boundary condition of Eq. (5.150) may be incorporated by introducing an infinitesimally small imaginary part into the denominator:

$$G_v^0(k) = \frac{P_{v+}}{v\cdot k + i0^+}. \tag{5.152}$$

That this is indeed the correct choice is easily seen by evaluating the integral

$$\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} e^{-ik_0(x_0-x_0')} \frac{1}{k_0 - \frac{\vec{v}\cdot\vec{k}}{v_0} + i0^+} = -i\Theta(x_0-x_0') \exp\left[-i\frac{(x_0-x_0')\vec{v}\cdot\vec{k}}{v_0}\right]$$

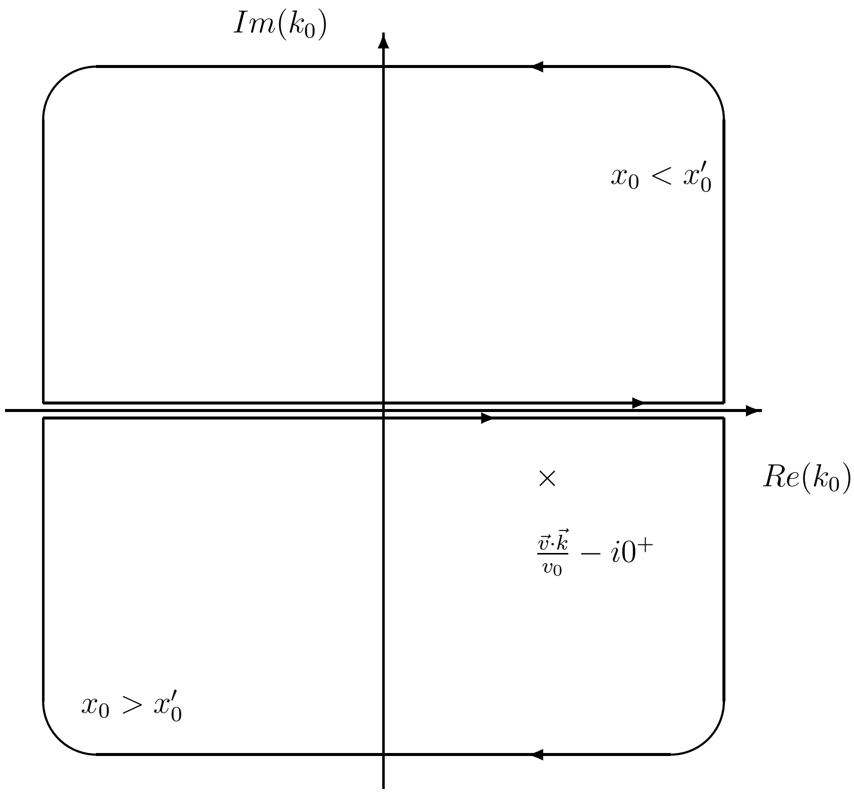
as a contour integral in the complex k_0 plane (see Fig. 5.5). For $x_0 > x_0'$ the contour is closed in the lower half plane and one makes use of the residue theorem. On the other hand, for $x_0 < x_0'$ the contour is closed in the upper half plane and, since the contour does not contain a pole, the integral vanishes. We then obtain

$$\begin{aligned} G_v^0(x,x') &= -i\frac{\Theta(x_0-x_0')}{v_0} \int \frac{d^3k}{(2\pi)^3} \exp\left[i\vec{k}\cdot\left(\vec{x}-\vec{x}\,'-\vec{v}\,\frac{x_0-x_0'}{v_0}\right)\right] P_{v+} \\ &= -i\frac{\Theta(x_0-x_0')}{v_0} \delta^3\left(\vec{x}-\vec{x}\,'-\vec{v}\,\frac{x_0-x_0'}{v_0}\right) P_{v+}. \end{aligned} \tag{5.153}$$

For the special choice $v^\mu = (1,0,0,0) \equiv \tilde{v}^\mu$ the propagator reduces to that of a static source

$$G_{\tilde{v}}^0(x,x') = -i\Theta(x_0-x_0')\delta^3(\vec{x}-\vec{x}\,') \begin{pmatrix} 1_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{pmatrix}.$$

Figure 5.5: Contour integration in the complex k_0 plane.



Finally, it is easy to show that a definition of the propagator in terms of the field operators $\mathcal{N}_v^0$ and $\bar{\mathcal{N}}_v^0$ [Dug+ 92],

$$G_v^0(x,x') = -i\Theta(x_0 - x'_0)\langle\Omega_0|\mathcal{N}_v^0(x)\bar{\mathcal{N}}_v^0(x')|\Omega_0\rangle, \tag{5.154}$$

yields the same result as Eq. (5.153). To that end, one inserts for each of the two operators a sum according to Eq. (5.135), commutes the creation and annihilation operators using Eq. (5.137), applies the completeness relation of Eq. (5.139), and makes use of $v\cdot k = 0$ for the individual Fourier components. Performing the remaining integration over $\vec{k}$ one ends up with Eq. (5.153), i.e., as expected the two methods yield the same result.

5.5.6 Example: πN Scattering at Lowest Order

As a simple example, let us return to πN scattering, but now in the framework of the heavy-baryon Lagrangian of Eq. (5.124). The four-momenta of the initial and final nucleons are written as $p = \overset{\circ}{m}_N\, v + k$ and $p' = \overset{\circ}{m}_N\, v + k'$, respectively, with $v \cdot k = 0 = v \cdot k'$ to leading order in $1/\overset{\circ}{m}_N$. The relevant interaction Lagrangian is obtained in complete analogy to Eq. (5.51),

$$\widehat{\mathcal{L}}^{(1)}_{\rm int} = -\frac{\overset{\circ}{g}_A}{F_0}\bar{\mathcal{N}}_v S^\mu_v \vec{\tau}\cdot\partial_\mu\vec{\phi}\mathcal{N}_v - \frac{1}{4F_0^2}v^\mu\bar{\mathcal{N}}_v\vec{\tau}\cdot\vec{\phi}\times\partial_\mu\vec{\phi}\mathcal{N}_v, \tag{5.155}$$

and the corresponding Feynman rules for the vertices derived from Eq. (5.155) read

- for a single incoming pion with four-momentum q and Cartesian isospin index a:
$$-\frac{\overset{\circ}{g}_A}{F_0} S_v\cdot q\tau^a, \tag{5.156}$$

- for an incoming pion with q, a and an outgoing pion with q', b:
$$\frac{v\cdot(q+q')}{4F_0^2}\epsilon_{abc}\tau^c. \tag{5.157}$$

As in the case of the relativistic calculation of Sec. 5.3.2 the latter gives rise to a contact contribution to $\mathcal{M}^v$

$$\mathcal{M}^v_{\rm cont} \;=\; N'N\bar{u}'_v\frac{v\cdot(q+q')}{4F_0^2}\epsilon_{abc}\tau^c u_v, \tag{5.158}$$

where the spinors are given in Eq. (5.136) and N and N' are the normalization factors appearing in the reduction formula of Eq. (5.144). The result for the direct-channel nucleon pole term reads

$$\mathcal{M}^v_{\rm d} = -i\frac{\overset{\circ}{g}{}^2_A}{F_0^2}N'N\tau^b\tau^a\bar{u}'_v S_v\cdot q'\frac{P_{v+}}{v\cdot(k+q)}S_v\cdot q u_v, \tag{5.159}$$

where, at leading order, we can make use of $v\cdot k = 0$. The crossed channel is obtained from Eq. (5.159) by the replacement $a \leftrightarrow b$ and $q \leftrightarrow -q'$ (pion crossing).

The evaluation of the total matrix element $\mathcal{M}^v = \mathcal{M}^v_{\rm cont} + \mathcal{M}^v_{\rm d} + \mathcal{M}^v_{\rm c}$ is particularly simple for the special choice $v^\mu = (1,0,0,0) \equiv \tilde{v}^\mu$, for which we have

$$P_{\tilde{v}+} = \begin{pmatrix} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{pmatrix}.$$

In that case, the calculation effectively reduces to that of a two-component theory as in the Foldy-Wouthuysen transformation, because the 4×4 matrices of the vertices are multiplied both from the left and the right by $P_{\tilde{v}+}$ originating from either the propagator of Eq. (5.152) or the spinors of Eq. (5.136). To be specific, for a 4×4 matrix Γ of the type

$$\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where each block A, B, C, and D is a 2×2 matrix, one has

$$P_{\tilde{v}+}\Gamma P_{\tilde{v}+} = \begin{pmatrix} A & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{pmatrix}$$

and

$$P_{\tilde{v}+}\Gamma_1 P_{\tilde{v}+}\Gamma_2 P_{\tilde{v}+} = \begin{pmatrix} A_1 A_2 & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{pmatrix}.$$

Moreover, the spin matrix of Eq. (5.120) is very simple for $\tilde{v}$,

$$S^0_{\tilde{v}} = 0, \quad \vec{S}_{\tilde{v}} = \frac{1}{2}\vec{\Sigma}, \tag{5.160}$$

where $\vec{\Sigma}$ has been defined in Eq. (5.102). With this special choice of v the T matrix in the center-of-mass frame reads

$$T = 2\,\overset{\circ}{m}_N\,\chi'^\dagger \left[-i\epsilon_{abc}\tau^c\left(\frac{E_\pi}{2F_0^2} - \frac{\overset{\circ}{g}{}_A^2}{F_0^2}\frac{\vec{q}\cdot\vec{q}\,'}{2E_\pi}\right) + \delta^{ab}\frac{\overset{\circ}{g}{}_A^2}{F_0^2}\left(-\frac{i\vec{\sigma}\cdot\vec{q}\,'\times\vec{q}}{2E_\pi}\right)\right]\chi. \tag{5.161}$$

Performing a nonrelativistic reduction of Eq. (5.46) in the center-of-mass frame,

$$T = 2\,\overset{\circ}{m}_N\,\chi'^\dagger\left[A + \left(E_\pi + \frac{\vec{q}^{\,2} + \vec{q}\,'\cdot\vec{q}}{2\,\overset{\circ}{m}_N}\right)B + i\frac{\vec{\sigma}\cdot\vec{q}\,'\times\vec{q}}{2\,\overset{\circ}{m}_N}B + \cdots\right]\chi, \tag{5.162}$$

and using

$$\begin{aligned}
\frac{1}{\nu-\nu_B}+\frac{1}{\nu+\nu_B} &= \frac{1}{E_\pi}\left[2-\frac{\vec{q}^{\,2}+\vec{q}\cdot\vec{q}\,'}{E_\pi \overset{\circ}{m}_N}+O\left(\frac{1}{\overset{\circ}{m}{}_N^2}\right)\right],\\
\frac{1}{\nu-\nu_B}-\frac{1}{\nu+\nu_B} &= \frac{1}{E_\pi}\left[-\frac{E_\pi}{\overset{\circ}{m}_N}+\frac{\vec{q}\cdot\vec{q}\,'}{E_\pi \overset{\circ}{m}_N}+O\left(\frac{1}{\overset{\circ}{m}{}_N^2}\right)\right]
\end{aligned}$$

in the expansion of A and B of Table 5.1, one verifies that, at leading order in $1/\overset{\circ}{m}_N$, the relativistic Lagrangian of Eq. (5.21) and the heavy-baryon Lagrangian of Eq. (5.124) indeed generate the same πN scattering amplitude. We emphasize that in order to obtain this equivalence of the two approaches an expansion of Eq. (5.162) to $1/\overset{\circ}{m}_N$ is mandatory, because the functions $A^{(+)}$ and $B^{(+)}$ contain terms of leading order $\overset{\circ}{m}_N$. These terms disappear through a cancellation in the final result.[28]

5.5.7 Corrections at First Order in $1/m$

So far we have concentrated on the leading-order, m-independent, heavy-baryon Lagrangian of Eq. (5.124). In comparison with Eq. (5.23), the chiral counting scheme of HBChPT is different, because an ordinary partial derivative acting on a heavy-baryon field $\mathcal{N}_v$ produces a small residual *four*-momentum [see also Eq. (4.62) for the mesonic sector]:

$$\mathcal{N}_v, \bar{\mathcal{N}}_v=\mathcal{O}(p^0),\ D_\mu\mathcal{N}_v=\mathcal{O}(p),\ v_\mu, S_\mu^v, 1_{4\times 4}=\mathcal{O}(p^0). \tag{5.163}$$

In the heavy-baryon approach four-momenta are considered small if their components are small in comparison with either the nucleon mass m_N or the chiral symmetry breaking scale $4\pi F_\pi$, both of which we denote by a common scale $\Lambda \simeq 1$ GeV.[29] It is clear that the Lagrangian of Eq. (5.123) also generates terms of higher order in $1/m$ and, in analogy to the mesonic sector, we also expect additional new chiral structures from the most general chiral Lagrangian at higher orders. Recall that in the baryonic sector the chiral orders increase in units of one, because of the additional possibility of forming

[28] The overall factor $2\overset{\circ}{m}_N$ in Eq. (5.162) is a result of our normalization of the spinors [see Eq. (5.60)].

[29] In reality, the excitation energy of the $\Delta(1232)$ resonance very often provides the limit of convergence of the expansion.

Lorentz invariants by contracting (covariant) derivatives with gamma matrices (see Sec. 5.2). (The relativistic πN Lagrangian at $\mathcal{O}(p^2)$ has (partially) been given in Ref. [Gas+ 88].)

Let us first consider the $1/m$ correction resulting from Eq. (5.123)

$$\begin{aligned} & \frac{1}{2m}\bar{\mathcal{N}}_v\left(i\not{D}_\perp+\frac{g_A}{2}v\cdot u\gamma_5\right)\left(i\not{D}_\perp-\frac{g_A}{2}v\cdot u\gamma_5\right)\mathcal{N}_v \\ = & \frac{1}{2m}\bar{\mathcal{N}}_v\left[-\not{D}_\perp\not{D}_\perp-i\frac{g_A}{2}\not{D}_\perp v\cdot u\gamma_5+i\frac{g_A}{2}v\cdot u\gamma_5\not{D}_\perp-\frac{g_A^2}{4}(v\cdot u)^2\right]\mathcal{N}_v. \end{aligned}$$

We make use of Eqs. (5.122) to identify the relevant replacements in the heavy-baryon bilinears:

$$\begin{aligned} \not{D}_\perp\gamma_5 &= \not{D}\gamma_5-v\cdot D\not{v}\,\gamma_5\mapsto 2D\cdot S_v-2v\cdot D\,\underbrace{v\cdot S_v}_{0}=2D\cdot S_v, \\ \gamma_5\not{D}_\perp &\mapsto -2D\cdot S_v, \\ \not{D}_\perp\not{D}_\perp &= (D_\mu-v\cdot Dv_\mu)(D_\nu-v\cdot Dv_\nu)\underbrace{\gamma^\mu\gamma^\nu}_{g^{\mu\nu}-i\sigma^{\mu\nu}} \\ &= (D^2-v\cdot Dv\cdot D)-i\underbrace{\sigma^{\mu\nu}}_{\mapsto 2\epsilon^{\mu\nu\rho\sigma}v_\rho S^v_\sigma}(D_\mu-v\cdot Dv_\mu)(D_\nu-v\cdot Dv_\nu) \\ &\mapsto D^2-(v\cdot D)^2-i\epsilon^{\mu\nu\rho\sigma}[D_\mu,D_\nu]v_\rho S^v_\sigma \\ &= D^2-(v\cdot D)^2-\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}[u_\mu,u_\nu]v_\rho S^v_\sigma \\ &\quad -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}f^+_{\mu\nu}v_\rho S^v_\sigma-\epsilon^{\mu\nu\rho\sigma}v^{(s)}_{\mu\nu}v_\rho S^v_\sigma, \end{aligned}$$

where the expression for the commutator $[D_\mu, D_\nu]$ of the covariant derivative of Eq. (5.18) is obtained after straightforward algebra. The field-strength tensors are defined as

$$f^\pm_{\mu\nu}=uf^L_{\mu\nu}u^\dagger\pm u^\dagger f^R_{\mu\nu}u,\quad v^{(s)}_{\mu\nu}=\partial_\mu v^{(s)}_\nu-\partial_\nu v^{(s)}_\mu, \tag{5.164}$$

where $f^R_{\mu\nu}$ and $f^L_{\mu\nu}$ are given in Eqs. (4.59) and (4.60), respectively.

Collecting all terms, we finally obtain as the contribution of Eq. (5.123) of order $1/m$ (returning to the notation in terms of expressions in the chiral limit)

$$\widehat{\mathcal{L}}^{(2)}_{\pi N,1/m} = \frac{1}{2\,\overset{\circ}{m}_N}\bar{\mathcal{N}}_v\Bigg[(v\cdot D)^2-D^2-i\,\overset{\circ}{g}_A\,\{S_v\cdot D,v\cdot u\}-\frac{\overset{\circ}{g}{}_A^2}{4}(v\cdot u)^2$$

$$+\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}v_\rho S^v_\sigma\left(iu_\mu u_\nu + f^+_{\mu\nu} + 2v^{(s)}_{\mu\nu}\right)\Bigg]\mathcal{N}_v. \tag{5.165}$$

Applying the counting rules of Eqs. (4.62) and (5.163), we see that Eq. (5.165) is indeed of $\mathcal{O}(p^2)$, where the suppression relative to (5.124) is of the form $p/\stackrel{\circ}{m}_N$.

At $\mathcal{O}(p^2)$ the heavy-baryon Lagrangian $\widehat{\mathcal{L}}^{(2)}_{\pi N}$ contains another contribution which, in analogy to $\widehat{\mathcal{L}}^{(1)}_{\pi N}$ in Sec. 5.5.3, may be obtained as the projection of the relativistic Lagrangian $\mathcal{L}^{(2)}_{\pi N}$ of [Gas+ 88] onto the light components. Here we quote the result in the convention of Ref. [Ber+ 97](except for the c_6 and c_7 terms, where, following Ref. [EM 96], we explicitly separate the traceless and isoscalar terms)[30]

$$\begin{aligned}\widehat{\mathcal{L}}^{(2)}_{\pi N,c_i} \;=\; & \bar{\mathcal{N}}_v\Bigg[c_1\mathrm{Tr}(\chi_+) + c_2(v\cdot u)^2 + c_3 u\cdot u + c_4[S^\mu_v, S^\nu_v]u_\mu u_\nu \\ & + c_5\left[\chi_+ - \frac{1}{2}\mathrm{Tr}(\chi_+)\right] - ic_6[S^\mu_v, S^\nu_v]f^+_{\mu\nu} - ic_7[S^\mu_v, S^\nu_v]v^{(s)}_{\mu\nu}\Bigg]\mathcal{N}_v,\end{aligned} \tag{5.166}$$

where

$$\chi_\pm = u^\dagger\chi u^\dagger \pm u\chi^\dagger u. \tag{5.167}$$

In the parameterization of Eq. (5.166) the constants c_i carry the dimension of an inverse mass and should be of the order of $1/\Lambda$ in order to produce a reasonable convergence of the chiral expansion. (The details of convergence generally depend on the observable in question). The complete heavy-baryon Lagrangian at $\mathcal{O}(p^2)$ is then given by the sum of Eqs. (5.165) and (5.166),

$$\widehat{\mathcal{L}}^{(2)}_{\pi N} = \widehat{\mathcal{L}}^{(2)}_{\pi N,1/m} + \widehat{\mathcal{L}}^{(2)}_{\pi N,c_i}. \tag{5.168}$$

It is worthwhile mentioning that the contribution of Eq. (5.165) to $\widehat{\mathcal{L}}^{(2)}_{\pi N}$ contains chirally invariant structures that are *not* part of Eq. (5.166). Unless such terms can be transformed away by a field transformation (see below) their coefficients are fixed in terms of the parameters of the lowest-order Lagrangian. As stressed by Ecker [Eck 95], these fixed coefficients are a

[30]The nomenclature of Refs. [Gas+ 88] and [Ber+ 92b] differs from the (more or less) standard convention of Eq. (5.166). The constants c_i of Ecker and Mojžiš [EM 96] differ by a factor $1/\stackrel{\circ}{m}_N$ from those of Eq. (5.166).

consequence of the Lorentz covariance of the whole approach. A related issue is the so-called reparameterization invariance, i.e., if a heavy particle of physical four-momentum p is described by, say, $p = mv + k$ with $p^2 = m^2$ and $v^2 = 1$ implying $2mv\cdot k + k^2 = 0$, physical observables should not change under the replacement $(v,k) \to (v+q/m, k-q)$ giving rise to an equivalent parameterization $p = mv' + k'$, if q satisfies $(v+q/m)^2 = 1$ [LM 92]. As a result, some coefficients of terms in the effective Lagrangian which are of different order in the $1/m$ expansion are related. For a detailed discussion, the reader is referred to Refs. [LM 92, Che 93, Fin+ 97].

The seven low-energy constants c_i are determined by comparison with experimental information. For example, if we consider the interaction with an external electromagnetic field [see Eq. (2.111)],

$$r_\mu = l_\mu = -e\frac{\tau_3}{2}\mathcal{A}_\mu, \quad v_\mu^{(s)} = -\frac{e}{2}\mathcal{A}_\mu,$$

we obtain

$$f_{\mu\nu}^+ = -e\tau_3\mathcal{F}_{\mu\nu} + \cdots, \quad v_{\mu\nu}^{(s)} = -\frac{e}{2}\mathcal{F}_{\mu\nu}, \quad \mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu,$$

so that the interaction with the field-strength tensor is given by

$$\widehat{\mathcal{L}}_{\text{int}}^{(2)} = -e\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\mu\nu}v^\rho\bar{\mathcal{N}}_v S_v^\sigma\left[\frac{1}{4\overset{\circ}{m}_N} + \frac{c_7}{2} + \tau_3\left(\frac{1}{4\overset{\circ}{m}_N} + c_6\right)\right]\mathcal{N}_v. \quad (5.169)$$

[We made use of Eq. (5.121).] For the special choice $v^\mu = (1,0,0,0) = \tilde{v}^\mu$ we find [see Eq. (5.160)]

$$\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\mu\nu}\tilde{v}^\rho S_{\tilde{v}}^\sigma = \epsilon_{ijk}\mathcal{F}^{ij}\frac{1}{2}\Sigma^k = -\vec{\Sigma}\cdot\vec{B},$$

and the interaction term reduces to[31]

$$\frac{e}{2\overset{\circ}{m}_N}\bar{\mathcal{N}}_{\tilde{v}}\,\vec{\sigma}\cdot\vec{B}\,\mathcal{N}_{\tilde{v}}\left[\frac{1}{2}\left(1 + 2\overset{\circ}{m}_N\,c_7\right) + \frac{\tau_3}{2}\left(1 + 4\overset{\circ}{m}_N\,c_6\right)\right], \quad (5.170)$$

which describes the interaction Lagrangian of a magnetic field with the magnetic moment of the nucleon. We define the isospin decomposition of the magnetic moment (in units of the nuclear magneton $e/2m_p$) as

$$\mu = \frac{1}{2}\mu^{(s)} + \frac{\tau_3}{2}\mu^{(v)} = \frac{1}{2}(1+\kappa^{(s)}) + \frac{\tau_3}{2}(1+\kappa^{(v)}),$$

[31] Recall that $\mathcal{N}_{\tilde{v}}$ are two-component fields.

where $\kappa^{(s)}$ and $\kappa^{(v)}$ denote the isoscalar and isovector *anomalous* magnetic moments of the nucleon, respectively, with empirical values $\kappa^{(s)} = -0.120$ and $\kappa^{(v)} = 3.706$. A comparison with Eq. (5.170) shows that the constants c_6 and c_7 are related to the *anomalous* magnetic moments of the nucleon in the chiral limit

$$\overset{\circ}{\kappa}{}^{(s)} = 2\, \overset{\circ}{m}_N\, c_7, \quad \overset{\circ}{\kappa}{}^{(v)} = 4\, \overset{\circ}{m}_N\, c_6.$$

The results for $\kappa^{(s)}$ and $\kappa^{(v)}$ up to and including $\mathcal{O}(p^3)$ [Ber+ 95b, Fea+ 97]

$$\begin{aligned}
\kappa^{(s)} &= \overset{\circ}{\kappa}{}^{(s)} + \mathcal{O}(p^4),\\
\kappa^{(v)} &= \overset{\circ}{\kappa}{}^{(v)} - \frac{M_\pi m_N g_A^2}{4\pi F_\pi^2} + \mathcal{O}(p^4),
\end{aligned}$$

are used to express the parameters c_6 and c_7 in terms of physical quantities. Note that the numerical correction of -1.96 [parameters of Eq. (5.68)] to the isovector anomalous magnetic moment is substantial. Differences by factors of about 1.5 were generally observed for the determination of the c_i at $\mathcal{O}(p^2)$ and to one-loop accuracy $\mathcal{O}(p^3)$ [Ber+ 95b, Ber+ 97].

The numerical values of the low-energy constants $c_1, \cdots, c_4$ have been determined in Ref. [Ber+ 97] by performing a best fit to a set of nine pion-nucleon scattering observables at $\mathcal{O}(p^3)$ which do not contain any new low-energy constants from the $\mathcal{O}(p^3)$ Lagrangian. Finally, c_5 was determined in terms of the strong contribution to the neutron-proton mass difference. The results in units of GeV^{-1} are given by (see also Ref. [BM 00])

$$\begin{aligned}
&c_1 = -0.93 \pm 0.10, \quad c_2 = 3.34 \pm 0.20, \quad c_3 = -5.29 \pm 0.25,\\
&c_4 = 3.63 \pm 0.10, \quad c_5 = -0.09 \pm 0.01. \qquad (5.171)
\end{aligned}$$

For a phenomenological interpretation of the low-energy constants in terms of (meson and Δ) resonance exchanges see Ref. [Ber+ 97].

We will see in the next section that the constants c_i are *not* required to compensate divergences of one-loop integrals. Such infinities first appear at $\mathcal{O}(p^3)$.

The Lagrangian of Eq. (5.165) still contains terms of the type $v \cdot D$ which appears in the lowest-order equation of motion of Eq. (5.146). As discussed in detail for the mesonic sector in Sec. 4.7 and Appendix D.1, such terms can be eliminated by appropriate field redefinitions. For example, the field transformation eliminating in Eq. (5.165) the term

$$\frac{1}{2\, \overset{\circ}{m}_N} \bar{\mathcal{N}}_v (v \cdot D)^2 \mathcal{N}_v$$

is given by [EM 96]

$$\mathcal{N}_v = \left[1 + \frac{iv \cdot D}{4 \overset{\circ}{m}_N} - \frac{\overset{\circ}{g}_A\, S_v \cdot u}{4 \overset{\circ}{m}_N}\right] \tilde{\mathcal{N}}_v. \tag{5.172}$$

Inserting Eq. (5.172) into the lowest-order Lagrangian of Eq. (5.124) yields

$$\begin{aligned}
&\bar{\tilde{\mathcal{N}}}_v(iv \cdot D + \overset{\circ}{g}_A\, S_v \cdot u)\tilde{\mathcal{N}}_v - \frac{1}{2 \overset{\circ}{m}_N}\bar{\tilde{\mathcal{N}}}_v(v \cdot D)^2\tilde{\mathcal{N}}_v - \frac{\overset{\circ}{g}{}_A^2}{2 \overset{\circ}{m}_N}\bar{\tilde{\mathcal{N}}} S_v \cdot u S_v \cdot u \tilde{\mathcal{N}}_v \\
&+ \text{total derivative} + O\left(\frac{1}{\overset{\circ}{m}{}_N^2}\right).
\end{aligned} \tag{5.173}$$

The second term cancels the equation-of-motion term, whereas rewriting the last term by using Eq. (5.121),

$$S_v \cdot u S_v \cdot u = \frac{1}{4}[(v \cdot u)^2 - u \cdot u] + \frac{1}{2} i \epsilon^{\mu\nu\rho\sigma} v_\rho S^v_\sigma u_\mu u_\nu,$$

we find that some of the coefficients at $\mathcal{O}(p^2)$ (and at higher orders) are modified. As in the case of the SU(2)×SU(2) mesonic Lagrangian at $\mathcal{O}(p^4)$ (see Appendix D.1) one finds *equivalent* parameterizations of $\widehat{\mathcal{L}}^{(2)}_{\pi N}$ (and also of the higher-order Lagrangians) in the baryonic sector. For the sake of completeness we quote the result of Ecker and Mojžiš [EM 96],

$$\begin{aligned}
\widehat{\mathcal{L}}^{(2)}_{\pi N} = \bar{\mathcal{N}}_v \Bigg\{ &- \frac{1}{2 \overset{\circ}{m}_N}\Big(D^2 + i\, \overset{\circ}{g}_A \{S_v \cdot D, v \cdot u\}\Big) + \frac{a_1}{\overset{\circ}{m}_N}\mathrm{Tr}(u \cdot u) \\
&+ \frac{a_2}{\overset{\circ}{m}_N}\mathrm{Tr}\left[(v \cdot u)^2\right] + \frac{a_3}{\overset{\circ}{m}_N}\mathrm{Tr}(\chi_+) + \frac{a_4}{\overset{\circ}{m}_N}\left[\chi_+ - \frac{1}{2}\mathrm{Tr}(\chi_+)\right] \\
&+ \frac{1}{\overset{\circ}{m}_N}\epsilon^{\mu\nu\rho\sigma} v_\rho S^v_\sigma \left[i a_5 u_\mu u_\nu + a_6 f^+_{\mu\nu} + a_7 v^{(s)}_{\mu\nu}\right] \Bigg\} \mathcal{N}_v,
\end{aligned} \tag{5.174}$$

where the relation to the coefficients c_i of Eq. (5.166) is given by

$$\begin{aligned}
&a_1 = \frac{\overset{\circ}{m}_N c_3}{2} + \frac{\overset{\circ}{g}{}_A^2}{16}, \quad a_2 = \frac{\overset{\circ}{m}_N c_2}{2} - \frac{\overset{\circ}{g}{}_A^2}{8}, \quad a_3 = \overset{\circ}{m}_N c_1, \quad a_4 = \overset{\circ}{m}_N c_5, \\
&a_5 = \overset{\circ}{m}_N c_4 + \frac{1 - \overset{\circ}{g}{}_A^2}{4}, \quad a_6 = \overset{\circ}{m}_N c_6 + \frac{1}{4}, \quad a_7 = \overset{\circ}{m}_N c_7 + \frac{1}{2}.
\end{aligned} \tag{5.175}$$

Of course, the Lagrangians of Eq. (5.168) and (5.174) yield the same results for physical observables, provided their parameters are related by Eq. (5.175). However, they will differ for intermediate mathematical quantities such as vertices or wave function renormalization constants as observed in Ref. [Fea+ 97] for the case of the nucleon wave function renormalization constant. We repeat that the coefficients of the first two terms of Eq. (5.174) are fixed in terms of $\overset{\circ}{m}_N$ and $\overset{\circ}{g}_A$, whereas the constants a_i are free parameters which have to be determined by comparison with experimental information.

5.5.8 The Power Counting Scheme

The power counting scheme of HBChPT may be formulated in close analogy to the mesonic sector (see Sec. 4.4). On the scale of either the nucleon mass m_N or $4\pi F_\pi$ we consider as small external momenta the four-momenta of pions, the four-momenta transferred by external sources, and the residual momenta k^μ of the nucleon appearing in the separation $p^\mu = \overset{\circ}{m}_N v^\mu + k^\mu$. For a given Feynman diagram we introduce

- the number of independent loop momenta N_L,
- the number of internal pion lines I_M,
- the number of pion vertices N_{2n}^M originating from $\mathcal{L}_{2n}$,
- the total number of pion vertices $N_M = \sum_{n=1}^\infty N_{2n}^M$,
- the number of internal nucleon lines I_B,
- the number of baryonic vertices N_n^B originating from $\widehat{\mathcal{L}}_{\pi N}^{(n)}$,
- and the total number of baryonic vertices $N_B = \sum_{n=1}^\infty N_n^B$.

As in the mesonic sector, the internal momenta appearing in the loop integration are not necessarily small. However, via the four-momentum conserving delta functions at the vertices and a substitution of integration variables, the rescaling of the external momenta is transferred to the internal momenta (see Sec. 4.4). The chiral dimension D of a given diagram is then given by [Wei 91, Eck 95]

$$D = 4N_L - 2I_M - I_B + \sum_{n=1}^\infty 2nN_{2n}^M + \sum_{n=1}^\infty nN_n^B. \tag{5.176}$$

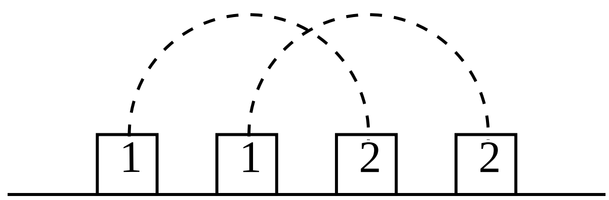
We make use of the topological relation [see, e.g., Eq. (2.130) of Ref. [CL 84]]

$$N_L = I_M + I_B - N_M - N_B + 1 \tag{5.177}$$

to eliminate I_M from Eq. (5.176)

$$D = 2N_L + I_B + 2 + \sum_{n=1}^{\infty} 2(n-1)N_{2n}^M + \sum_{n=1}^{\infty}(n-2)N_n^B. \tag{5.178}$$

Figure 5.6: Two-loop contribution to the nucleon self energy.



For processes containing exactly one nucleon in the initial and final states we have[32] $N_B = I_B + 1$ and we thus obtain

$$D = 2N_L + 1 + \sum_{n=1}^{\infty} 2(n-1)N_{2n}^M + \sum_{n=1}^{\infty}(n-1)N_n^B. \tag{5.179}$$

The power counting is very similar to the mesonic sector. We first observe that $D \geq 1$. Moreover, as already mentioned in Sec. 5.5.7, loops start contributing at $D = 3$. In other words, the low-energy coefficients c_i of $\widehat{\mathcal{L}}_{\pi N}^{(2)}$ are not needed to renormalize infinities from one-loop calculations. Again, we have a connection between the number of loops and the chiral dimension D: $N_L \leq (D-1)/2$. Each additional loop adds two units to the chiral dimension.

As an example, let us consider the two-loop contribution to the nucleon self energy of Fig. 5.6. First of all, the number of independent loops is $N_L = 2$ in agreement with Eq. (5.177) for $I_M = 2$, $I_B = 3$, $N_M = 0$, and $N_B = 4$.

[32]In the heavy-baryon formulation one has no closed fermion loops. In other words, in the single-nucleon sector exactly one fermion line runs through the diagram connecting the initial and final states.

The counting of the chiral dimension is most intuitively performed in the framework of Eq. (5.176), because it associates with each building block a unique term which is easy to remember (+4 for each independent loop, −2 for each internal meson propagator, etc). For $N_{2n}^M = 0$, $N_1^B = 2$, and $N_2^B = 2$ we obtain $D = 8 - 4 - 3 + 0 + 2 + 4 = 7$.

5.5.9 Application at $\mathcal{O}(p^3)$: One-Loop Correction to the Nucleon Mass

As a simple example, we will return to the modification of the nucleon mass through higher-order terms in the heavy-baryon approach. The calculation will proceed along the lines of Ref. [Fea+ 97], where use was made of the Lagrangian of Ecker and Mojžiš [EM 96] [see Eq. (5.174)].

The determination of the physical nucleon mass and the discussion of the wave function renormalization factor will be very similar to Secs. 4.9.1 for the masses of the Goldstone bosons and 5.4.1 for the nucleon mass in the relativistic approach. Let us denote the four-momentum of the nucleon by $p = \overset{\circ}{m}_N\, v + r$, where, since we are interested in the propagator, we must allow the four-momentum to be off the mass shell. The on-shell case is, of course, given by $p^2 = m_N^2$ with m_N denoting the *physical* nucleon mass. Let us stress that, due to the interaction, we must expect the physical mass to be different from the mass $\overset{\circ}{m}_N$ in the chiral limit.

We start from the lowest-order propagator of Eq. (5.152),

$$\frac{P_{v+}}{v \cdot r + i0^+} = \frac{P_{v+}}{v \cdot p - \overset{\circ}{m}_N + i0^+}, \tag{5.180}$$

and first determine its modification in terms of the tree-level contribution of Eq. (5.174) to the self energy.[33] Neglecting isospin-symmetry breaking effects proportional to $m_u - m_d$, we obtain at $\mathcal{O}(p^2)$

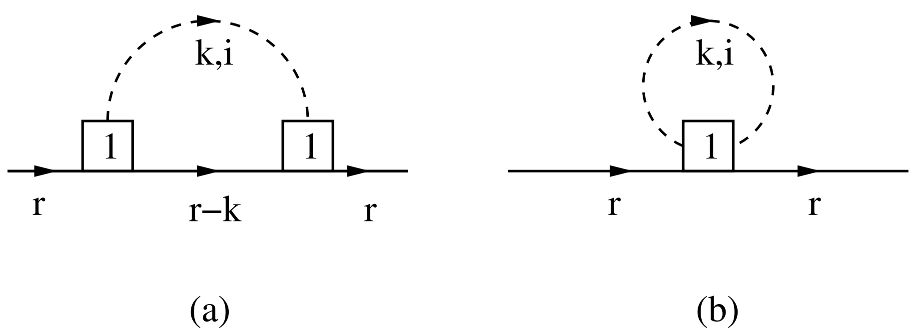
$$\Sigma^{(2)}(p) = -\frac{r^2}{2\,\overset{\circ}{m}_N} - \frac{4 a_3 M^2}{\overset{\circ}{m}_N}, \tag{5.181}$$

where $M^2 = 2B_0 m_q$ denotes the squared pion mass at $\mathcal{O}(p^2)$. The r^2 term comes from the term in $\widehat{\mathcal{L}}_{\pi N}^{(2)}$ proportional to $-\partial^2/2\,\overset{\circ}{m}_N$ which involves no

[33] In the remaining part of this section, we adopt the common practice of leaving out the projector P_{v+} in the propagator and (possibly) in vertices with the understanding that all operators act only in the projected subspace.

pions. In the spirit of the reduction formula of Eq. (5.144), in combination with the formula of Gell-Mann and Low of Eq. (5.145), we choose to include this as part of the interaction rather than part of the free Lagrangian and reserve for the free Lagrangian the $iv\cdot\partial$ term from $\widehat{\mathcal{L}}^{(1)}_{\pi N}$. The term involving a_3 is a contact term coming ultimately from Eq. (5.166), where $a_3 = \overset{\circ}{m}_N c_1$.

Figure 5.7: One-loop contribution to the nucleon self energy at $\mathcal{O}(p^3)$ in the heavy-baryon approach. The diagram (b) vanishes because of its isospin structure.



The heavy-baryon Lagrangian at $\mathcal{O}(p^3)$ [EM 96] does not produce a contact contribution to the self energy, because all the structures contain at least one pion or an external field. Moreover, given the Feynman rule of Eq. (5.157), the second one-loop diagram of Fig. 5.7 (b) vanishes, because $\epsilon_{iij}\tau_j = 0$. In other words, the only contribution at $\mathcal{O}(p^3)$ results from the one-loop diagram of Fig. 5.7 (a). Using the vertex of Eq. (5.156) and the propagator of Eq. (5.152) we obtain for the one-loop contribution at $\mathcal{O}(p^3)$

$$\begin{aligned} -i\Sigma^{(3)}_{\rm loop}(p) &= \int \frac{d^4k}{(2\pi)^4}\left[\frac{\overset{\circ}{g}_A}{F_0}(-S_v\cdot k)\tau_i\right]\frac{i}{v\cdot(r-k)+i0^+} \\ &\quad \times\frac{i}{k^2-M^2+i0^+}\left[\frac{\overset{\circ}{g}_A}{F_0}S_v\cdot k\tau_i\right]. \end{aligned} \tag{5.182}$$

As in the relativistic case of Eq. (5.77), counting powers, we expect the integral to have a cubic divergence. Extending the integral to n dimensions, using $\tau_i\tau_i = 3$, performing the substitution $k \to -k$, and applying Eq. (C.31) of Appendix C.2.2, we obtain the intermediate result

$$\Sigma^{(3)}_{\text{loop}}(p) = 3\frac{\overset{\circ}{g}{}_A^2}{F_0^2} S_v^\mu S_v^\nu [v_\mu v_\nu C_{20}(v\cdot r, M^2) + g_{\mu\nu} C_{21}(v\cdot r, M^2)].$$

Since $S_v \cdot v = 0$ and $S_v^2 = (1-n)/4$ [see Eq. (5.126)], we obtain, applying the first equality of Eq. (C.35),

$$\Sigma^{(3)}_{\text{loop}}(p) = -\frac{3}{4}\frac{\overset{\circ}{g}{}_A^2}{F_0^2}\left\{[M^2 - (v\cdot r)^2] J_{\pi N}(0; v\cdot r) + v\cdot r I_\pi(0)\right\}, \tag{5.183}$$

where the integrals $J_{\pi N}$ and I_π are given in Eqs. (C.27) and (C.2), respectively.

Combining with Eq. (5.181) and using Eq. (C.27) we thus obtain for the (unrenormalized) nucleon self energy at $\mathcal{O}(p^3)$ [Fea+ 97]

$$\begin{aligned}
\Sigma(p) = &-\frac{r^2}{2\overset{\circ}{m}_N} - \frac{4a_3M^2}{\overset{\circ}{m}_N}\\
&-\frac{3\overset{\circ}{g}{}_A^2}{(4\pi F_0)^2}\left(\frac{v\cdot r}{4}\left\{[3M^2 - 2(v\cdot r)^2]\left[R + \ln\left(\frac{M^2}{\mu^2}\right)\right] - \frac{1}{2}\left[M^2 - (v\cdot r)^2\right]\right\}\right.\\
&\left.+[M^2 - (v\cdot r)^2]^{\frac{3}{2}} \arccos\left(-\frac{v\cdot r}{M}\right)\right),
\end{aligned} \tag{5.184}$$

for $(v\cdot r)^2 < M^2$. Clearly, the self-energy contribution generated by the loop diagram of Fig. 5.7 (a) contains a divergent piece proportional to R of Eq. (B.13).

We have chosen to express the self energy as a function of the four-momentum p. In the relativistic case of Eq. (5.71) we needed two scalar functions depending on p^2 to parameterize the self energy. In contrast to the relativistic case, the heavy-baryon self energy of Eq. (5.184) is given by one function depending on two scalar variables for which one can take, say, r^2 and $v\cdot r$ or

$$\eta \equiv v\cdot p - m_N, \quad \xi \equiv (p - m_N v)^2. \tag{5.185}$$

Making use of $r = (m_N - \overset{\circ}{m}_N)v + (p - m_N v)$, the two sets are related by

$$\begin{aligned}
r^2 &= (m_N - \overset{\circ}{m}_N)^2 + 2(m_N - \overset{\circ}{m}_N)\eta + \xi, \qquad (5.186)\\
v\cdot r &= m_N - \overset{\circ}{m}_N + \eta. \qquad (5.187)
\end{aligned}$$

The choice of Eq. (5.185) is convenient for the determination of the physical nucleon mass m_N and renormalization constant Z_N, because, in view of Eq. (5.180), we want the full (but yet unrenormalized) propagator to have a pole at $p = m_N v$ which includes both the mass-shell condition $p^2 = m_N^2$ and $v \cdot p = m_N$. In the vicinity of the pole at $p = m_N v$ the second choice of variables corresponds to terms which are, respectively, first and second order in the (small) distance from the pole. Thus in the following discussion we will use both notations $\Sigma(p)$ and $\Sigma(\eta, \xi)$ for the self energy, where it should be clear from the context which expression applies.

In analogy to the mesonic case discussed in Sec. 4.9.1 the full heavy-baryon propagator is written as [see Eq. (4.138)]

$$
\begin{aligned}
iG_v(p) &= \frac{i}{v \cdot p - \overset{\circ}{m}_N - \Sigma(p)} \\
&= \frac{i}{v \cdot p - \overset{\circ}{m}_N - \Sigma(0,0) - \eta\Sigma'(0,0) - \tilde{\Sigma}(\eta,\xi)} \\
&= \frac{i}{[1 - \Sigma'(0,0)]\left\{\eta - \frac{\tilde{\Sigma}(\eta,\xi)}{[1-\Sigma'(0,0)]}\right\}} \\
&= \frac{iZ_N}{\eta - Z_N\tilde{\Sigma}(\eta,\xi)},
\end{aligned}
\tag{5.188}
$$

where

$$
m_N = \overset{\circ}{m}_N + \Sigma(0,0), \tag{5.189}
$$

$$
Z_N = \frac{1}{1 - \Sigma'(0,0)}. \tag{5.190}
$$

In these equations $\Sigma'(0,0)$ denotes the first partial derivative of $\Sigma(\eta, \xi)$ with respect to η evaluated at $(\eta, \xi) = (0, 0)$,

$$
\Sigma'(0,0) = \left.\frac{\partial\Sigma(\eta,\xi)}{\partial\eta}\right|_{(\eta,\xi)=(0,0)},
$$

and $\tilde{\Sigma}(\eta, \xi)$ is at least of second order in the distance from the pole.

For the evaluation of $\Sigma(0,0)$, $\Sigma'(0,0)$, and $\tilde{\Sigma}(\eta,\xi)$ we need to expand Eq. (5.184). To the order we are working the a_3 term contributes only to $\Sigma(0,0)$ whereas the loop piece contributes to all three. In contrast to the mesonic sector at $\mathcal{O}(p^4)$, $\tilde{\Sigma}$ is not zero in this case. Using Eq. (5.186), we obtain for the r^2 term

$$\frac{r^2}{2\stackrel{\circ}{m}_N} = \frac{(m_N - \stackrel{\circ}{m}_N)^2}{2\stackrel{\circ}{m}_N} + \frac{(m_N - \stackrel{\circ}{m}_N)}{\stackrel{\circ}{m}_N}\eta + \frac{\xi}{2\stackrel{\circ}{m}_N}. \tag{5.191}$$

The first term on the right-hand side contributes to $\Sigma(0,0)$ but is $\mathcal{O}(1/m_N^3)$, since, as we will see, the difference $(m_N - \stackrel{\circ}{m}_N)$ is $\mathcal{O}(1/m_N)$. The second term is $\mathcal{O}(1/m_N^2)$ and will contribute to $\Sigma'(0,0)$. Finally the third term contributes only to $\tilde{\Sigma}$.

Applying Eq. (5.189) we obtain for the physical mass

$$m_N = \stackrel{\circ}{m}_N - \frac{(m_N - \stackrel{\circ}{m}_N)^2}{2\stackrel{\circ}{m}_N} - \frac{4a_3M^2}{\stackrel{\circ}{m}_N} + \Sigma^{(3)}_{\rm loop}(0,0), \tag{5.192}$$

which implies $m_N - \stackrel{\circ}{m}_N = \mathcal{O}(1/m_N)$.[34] We can thus neglect the second term on the right-hand side of Eq. (5.192). The loop contribution is only a function of $v\cdot r$ and thus a function only of η, and, neglecting terms of higher order in $1/m_N$, we may replace $v\cdot r$ by 0, yielding

$$\Sigma^{(3)}_{\rm loop}(0,0) = -\frac{3\stackrel{\circ}{g}{}_A^2 M^3}{(4\pi F_0)^2}\arccos(0).$$

We finally obtain for the physical nucleon mass

$$m_N \simeq \stackrel{\circ}{m}_N \left[1 - \frac{4a_3M_\pi^2}{m_N^2} - \frac{3\pi g_A^2 M_\pi^3}{2m_N(4\pi F_\pi)^2}\right], \tag{5.193}$$

where, in the expression between the brackets, we have replaced all quantities in terms of the physical quantities, because the difference is of higher order in the chiral expansion. In the chiral limit, both the counter-term contribution $\sim M^2 \sim m_q$ and the pion-loop correction $\sim M^3 \sim m_q^{3/2}$ disappear. In other words, in the heavy-baryon framework the situation is again as in the mesonic sector, where the parameters of the lowest-order Lagrangian do not get modified due to higher-order corrections in the chiral limit. The same is actually true for the second parameter $\stackrel{\circ}{g}_A$ of Eq. (5.124) [see, e.g., Eq.

[34] Strictly speaking we should say $m_N - \stackrel{\circ}{m}_N = \mathcal{O}[M^2/\stackrel{\circ}{m}_N, M^3/(4\pi F_0)^2]$, where the second result originates from the loop contribution.

(50) of Ref. [Fea+ 97]]. Using the parameters of Eqs. (5.68) and (5.171) one finds that the counter term and the pion loop generate contributions to the physical nucleon mass of 0.0733 and -0.0163 in units of $\overset{\circ}{m}_N$, respectively.

The wave function renormalization constant Z_N is obtained from Eq. (5.190) as

$$Z_N = \frac{1}{1-\Sigma'(0,0)} \approx 1+\Sigma'(0,0) = 1 - \frac{m_N - \overset{\circ}{m}_N}{\overset{\circ}{m}_N} + \Sigma'_{\rm loop}(0,0). \qquad (5.194)$$

To the order we are considering, we have from Eqs. (5.193) and (5.184), respectively,

$$\begin{aligned}
\frac{m_N - \overset{\circ}{m}_N}{\overset{\circ}{m}_N} &= -\frac{4a_3 M^2}{\overset{\circ}{m}{}_N^2}, \\
\Sigma'_{\rm loop}(0,0) &= -\frac{9\,\overset{\circ}{g}{}_A^2\, M^2}{4(4\pi F_0)^2}\left[R + \ln\left(\frac{M^2}{\mu^2}\right) + \frac{2}{3}\right].
\end{aligned}$$

Finally, expressing all quantities in terms of physical quantities, the wave function renormalization constant Z_N reads

$$Z_N = 1 + \frac{4a_3 M_\pi^2}{m_N^2} - \frac{9g_A^2 M_\pi^2}{4(4\pi F_\pi)^2}\left[R + \ln\left(\frac{M_\pi^2}{\mu^2}\right) + \frac{2}{3}\right]. \qquad (5.195)$$

As in the pion case (see Table D.2 of Appendix D.2) Z_N contains the infinite constant R entering through dimensional regularization, i.e., Z_N is not a finite quantity. However, this is not a problem, because the wave function renormalization constant is not a physical observable. Moreover, as we have seen explicitly for the pion case, and as discussed in Ref. [Fea+ 97] for the heavy-baryon Lagrangian, Z_N will also depend on the specific parameterization of the Lagrangian.

In Ref. [EM 97] it was shown that the wave function renormalization "constant" Z_N is in fact a non-trivial differential operator and should, in momentum space, depend on the momentum of the initial or final nucleon. Here we argue that the findings of Ref. [EM 97] and the method used above do not seem to be in conflict with each other. To that end, we first note that Ref. [EM 97] made use of the external spinor $u_+(\vec{p}) = P_{v+}u(\vec{p})$. Using relativistic spinors normalized as in Eq. (5.131) this corresponds to a normalization of the heavy baryon spinors to $\bar{u}_+(\vec{p})u_+(\vec{p}) = (p\cdot v + m_N)$. To facilitate the comparison, let us consider the special choice $v^\mu = (1,0,0,0)$. In the framework of the reduction formula of Eq. (5.144), we work with a factor $Nu_v^\alpha/\sqrt{Z_N}$ for an external nucleon in the initial state, where Ecker and Mojžiš would have $u_+(\vec{p})/\sqrt{Z_N^{\rm EM}}$. It is now straightforward to show that the normalization factor N exactly produces the additional term which, in the approach of Ref. [EM 97], results from the additional term in the wave function renormalization. This explains why the two approaches, at least up to order $\mathcal{O}(p^3)$, generate the same result. For further discussion on this topic, the reader is referred to [Fea+ 97, EM 97, Ste+ 98, KM 99].

5.6 The Method of Infrared Regularization

In the discussion of the one-loop corrections to the nucleon self energy and pion-nucleon scattering of Sec. 5.4, we saw that the relativistic framework for baryons did not naturally provide a simple power counting scheme as for mesons. One major difference in comparison with the mesonic sector is related to the fact that the nucleon remains massive in the chiral limit which also introduces another mass scale into the problem. Thus, because of the zeroth component one can no longer argue that a derivative acting on the baryon field results in a small four-momentum. This problem is avoided in the heavy-baryon approach discussed in Sec. 5.5, where, through a field redefinition, the mass dependence has been shifted into an (infinite) string of vertices which are suppressed by powers of $1/m$. Since the derivatives in the heavy-baryon Lagrangian produce small residual four-momenta in the low-energy regime, a power counting scheme analogous to the mesonic sector can be formulated [see Eqs. (5.176) and (5.179)]. A vast majority of applications of chiral perturbation theory in the baryonic sector were performed in the framework of the heavy-baryon approach. However, it was realized some time ago that the heavy-baryon approach, under certain circumstances, may generate Green functions which do not satisfy the analytic properties resulting from a (fully) relativistic field theory [Ber+ 96b].

Clearly, it would be desirable to have a method which combines the advantages of the relativistic and the heavy-baryon approaches and, at the same time, avoids their shortcomings—absence of a power counting scheme on the one hand and failure of convergence on the other hand. Such approaches have been proposed and developed by various authors [Tan 96, ET 98, BL 99, Geg+ 99, Lut 00, Bec 02, LK 02] and here we will briefly outline the ideas of the so-called infrared regularization [BL 99]. Our presentation will closely follow Refs. [BL 99, Bec 02] to which we refer the reader for technical details. Some recent applications of the new approach deal with the electromagnetic form factors of the nucleon [KM 01a] and the baryon octet [KM 01b], πN scattering [BL 01], axial-vector current matrix elements [Zhu+ 01], and the generalized Gerasimov-Drell-Hearn sum rule [Ber+ 02b].

In order to understand the problems of the heavy-baryon approach regarding the analytic behavior of invariant functions let us start with a simple example [Bec 02]. To that end we consider the s channel of pion-nucleon scattering (see Sec. 5.3.2). The invariant amplitudes $B^\pm$ of Table 5.1 develop poles for $\nu = \pm\nu_B$ (the upper and lower signs correspond to $s = m_N^2$ and $u = m_N^2$, respectively). For example, the singularity due to the nucleon pole in the s channel is understood in terms of the relativistic propagator

$$\frac{1}{(p+q)^2 - m_N^2} = \frac{1}{2p\cdot q + M_\pi^2}, \tag{5.196}$$

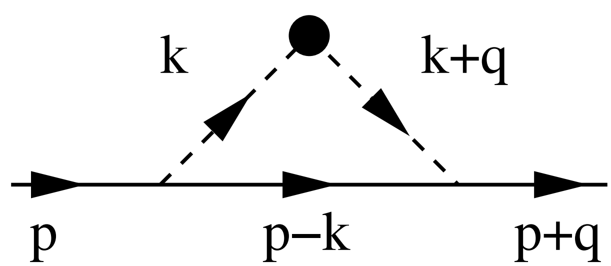
which, of course, has a pole at $2p\cdot q = -M_\pi^2$ or, equivalently, $s = m_N^2$. (Analogously, a second pole results from the u channel at $u = m_N^2$.) We also note that the propagator of Eq. (5.196) counts as $\mathcal{O}(p^{-1})$, because it is part of a tree-level diagram so that the four-momentum q is assumed to be small, i.e., of $\mathcal{O}(p)$. Although both poles are not in the physical region of pion-nucleon scattering, analyticity of the invariant amplitudes requires these poles to be present in the amplitudes. Let us compare the situation with a heavy-baryon type of expansion, where, for simplicity, we choose as the four-velocity $p^\mu = m_N v^\mu$,

$$\frac{1}{2p\cdot q + M_\pi^2} = \frac{1}{2m_N}\frac{1}{v\cdot q + \frac{M_\pi^2}{2m_N}} = \frac{1}{2m_N}\frac{1}{v\cdot q}\left(1 - \frac{M_\pi^2}{2m_N v\cdot q} + \cdots\right). \tag{5.197}$$

Clearly, to any finite order the heavy-baryon expansion produces poles at $v\cdot q = 0$ instead of a simple pole at $v\cdot q = -M_\pi^2/(2m_N)$ and will thus not generate the (nucleon) pole structures of the functions $B^\pm$.

As a second example, we consider the so-called triangle diagram of Fig. 5.8 which will serve to illustrate the different analytic properties of invariant functions obtained from loop diagrams in the relativistic and heavy-baryon approaches. A diagram of this type appears in many calculations such as the scalar or electromagnetic form factors of the nucleon, where • represents an external scalar or electromagnetic field, or πN or Compton scattering, where • stands for two pion or electromagnetic fields. In all of these cases a four-momentum q is transferred to the nucleon and the analytic properties of the Feynman diagram as a function of $t \equiv q^2$ are determined by the pole structure of the propagators.

Figure 5.8: Triangle diagram. The symbol • denotes an interaction which transfers the momentum q to the virtual pion.



Thus we need to discuss some properties of the integral

$$\gamma(t) \equiv i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M_\pi^2 + i0^+} \frac{1}{(k+q)^2 - M_\pi^2 + i0^+} \frac{1}{(p-k)^2 - m_N^2 + i0^+}, \tag{5.198}$$

where $t = q^2$. We assume the initial and final nucleons to be on the mass shell, $p^2 = m_N^2 = (p+q)^2$ which implies $2p \cdot q = -t$. Counting powers we see that the integral converges. The function $\gamma(t)$ is analytic in t except for a cut along the positive real axis starting at $t = 4M_\pi^2$ which expresses the fact that two on-shell pions can be produced for $t \geq 4M_\pi^2$. In the following discussion of the analytic properties of Eq. (5.198) we will concentrate on the imaginary part of $\gamma(t)$ which we will derive applying the Cutkosky (or cutting) rules [Cut 60, LeB 91, PS 95]. The rules, as summarized in [PS 95], read: In order to obtain $2i\mathrm{Im}\{\gamma(t)\}$ first cut through the diagram in all possible ways such that the cut propagators can simultaneously be put on shell. Next, for each cut one need to replace each cut propagator $1/(p^2 - m^2 + i0^+)$ by $-2\pi i\delta(p^2 - m^2)$. Finally, sum the contributions of all possible cuts. In the present case, the two terms where the nucleon propagator is simultaneously cut with either the first or the second pion propagator do not contribute.

The result from cutting the two pion propagators reads

$$2i\mathrm{Im}\{\gamma(t)\} = (-2\pi i)^2 i \int \frac{d^4k}{(2\pi)^4} \frac{\delta(k^2 - M_\pi^2)\delta((k+q)^2 - M_\pi^2)}{(p-k)^2 - m_N^2 + i0^+}. \tag{5.199}$$

In order to evaluate Eq. (5.199), we choose a frame where $q^\mu = (q_0, \vec{0})$ with $q_0 = \sqrt{t} > 0$, and $p^\mu = (-q_0/2, \vec{p})$. Using

$$\delta(k^2 - M_\pi^2)\delta((k+q)^2 - M_\pi^2) = \delta(k^2 - M_\pi^2)\frac{1}{2\sqrt{t}}\delta\left(k_0 + \frac{\sqrt{t}}{2}\right)$$

we find, as an intermediate result,

$$\mathrm{Im}\{\gamma(t)\} = -\frac{1}{16\pi^2\sqrt{t}} \int d^3k \delta\left(\vec{k}^2 + M_\pi^2 - \frac{t}{4}\right) \frac{1}{-\frac{t}{2} + 2\vec{p}\cdot\vec{k} + M_\pi^2 + i0^+}. \tag{5.200}$$

For $t < 4M_\pi^2$, the delta function in Eq. (5.200) always vanishes, showing that the cut starts, as anticipated, at $t = 4M_\pi^2$. Applying the mass-shell condition $p^2 = m_N^2$, we write

$$\begin{aligned} \vec{p} &= i\frac{\sqrt{4m_N^2 - t}}{2}\hat{e}_z \quad \text{for} \quad 4M_\pi^2 \le t \le 4m_N^2, \\ \vec{p} &= \frac{\sqrt{t - 4m_N^2}}{2}\hat{e}_z \quad \text{for} \quad 4m_N^2 \le t. \end{aligned}$$

Performing the integration using spherical coordinates, the result for the first case reads

$$\begin{aligned} \mathrm{Im}\{\gamma(t)\} &= \frac{\sqrt{t - 4M_\pi^2}}{16\pi\sqrt{t}} \int_{-1}^{1} dz \frac{1}{t - 2M_\pi^2 - i\sqrt{4m_N^2 - t}\sqrt{t - 4M_\pi^2}\, z - i0^+} \\ &= \frac{i}{16\pi\sqrt{t}\sqrt{4m_N^2 - t}} \ln\left(\frac{1 - iy}{1 + iy}\right) \\ &= \frac{1}{8\pi\sqrt{t(4m_N^2 - t)}} \arctan(y), \end{aligned} \tag{5.201}$$

where

$$y = \frac{\sqrt{(t - 4M_\pi^2)(4m_N^2 - t)}}{t - 2M_\pi^2}, \quad 4M_\pi^2 \le t \le 4m_N^2.$$

The second case, $t > 4m_N^2$, is obtained analogously by the replacement $i\sqrt{4m_N^2 - t} \to \sqrt{t - 4m_N^2}$:

$$\mathrm{Im}\{\gamma(t)\} = \frac{1}{16\pi\sqrt{t(t-4m_N^2)}} \ln\left(\frac{t - 2M_\pi^2 + \sqrt{t-4m_N^2}\sqrt{t-4M_\pi^2}}{t - 2M_\pi^2 - \sqrt{t-4m_N^2}\sqrt{t-4M_\pi^2}}\right). \tag{5.202}$$

Equations (5.201) and (5.202) agree with the results given in Eq. (B.43) of Ref. [Gas+ 88]. In the low-energy region $t \ll m_N^2$, and Eq. (5.201) becomes

$$\mathrm{Im}\{\gamma(t)\} \approx \frac{1}{16\pi m_N\sqrt{t}} \arctan(x), \quad x = \frac{2m_N\sqrt{t-4M_\pi^2}}{t-2M_\pi^2}. \tag{5.203}$$

Taking the factors resulting from the vertices and the relevant tensor structures of the loop integral into account, the contribution of Fig. 5.8 to the imaginary part of the scalar form factor of the nucleon reads [Gas+ 88, BL 99]

$$\mathrm{Im}\{\sigma(t)\} = \frac{3g_A^2 M_\pi^2 m_N}{4F_\pi^2}(t - 2M_\pi^2)\mathrm{Im}\{\gamma(t)\},$$

where $\sigma(t)$ is defined in terms of the u- and d-quark scalar densities $\bar{u}u$ and $\bar{d}d$ as

$$\langle N(p')|m[\bar{u}(0)u(0) + \bar{d}(0)d(0)]|N(p)\rangle = \bar{u}(p')u(p)\sigma(t), \tag{5.204}$$

where $m = m_u = m_d$ and $t = q^2 = (p' - p)^2$.

We will now investigate two limiting procedures. First, we consider a fixed value $t > 4M_\pi^2$ and let $m_N \to \infty$. In that case $x \gg 1$, and one would use the expansion

$$\arctan(x) = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \cdots, \quad x > 1.$$

Keeping only the leading order term, we find

$$\mathrm{Im}\{\sigma(t)\} = \frac{3g_A^2 M_\pi^2}{128F_\pi^2}\frac{t - 2M_\pi^2}{\sqrt{t}}, \tag{5.205}$$

which corresponds exactly to the result of HBChPT at $\mathcal{O}(p^3)$ [Ber+ 92b]. This result corresponds to the standard chiral expansion which treats the quantity x of Eq. (5.203) as $\mathcal{O}(p^{-1})$, because $m_N = \mathcal{O}(p^0)$ and $t, M_\pi^2 = \mathcal{O}(p^2)$.

However, for a fixed m_N we may also consider a small enough t close to the threshold value $t_{\rm thr} = 4M_\pi^2$ so that $x < 1$. In that case the expansion of the arctan reads

$$\arctan(x) = x - \frac{x^3}{3} + \cdots$$

yielding

$$\mathrm{Im}\{\sigma(t)\} \approx \frac{3g_A^2 M_\pi^2 m_N}{32\pi F_\pi^2} \frac{\sqrt{t - 4M_\pi^2}}{\sqrt{t}}, \tag{5.206}$$

where we have neglected higher powers of x. The critical value of t corresponding to $x = 1$ is given by

$$t_{\rm cr} = 4M_\pi^2 \left[1 + \frac{\mu^2}{4} + O(\mu^4)\right],$$

where $\mu = M_\pi/m_N$. Clearly the behavior of Eq. (5.206) is very different from the chiral expansion of Eq. (5.205) and, similar to the discussion of Eq. (5.197), a finite sum of terms in HBChPT cannot reproduce such a threshold behavior [BL 99]. The rapid variation of the imaginary part can be understood in terms of the analytic properties of the arctan which, as a function of the complex variable z, is analytic in the entire complex plane save for cuts along the positive and negative imaginary axis starting at $\pm i$. These branch points corresponding to $x = \pm i$ are obtained for $t = 4M_\pi^2(1 - \mu^2/4)$ which is just below the physical threshold $t_{\rm thr} = 4M_\pi^2$. For that reason an expansion around $x = 0$ corresponding to $t = 4M_\pi^2$ has a small radius of convergence.

Clearly, the heavy-baryon approach does not produce the correct analytic structure as generated by the relativistic loop diagram. Moreover the low-energy behavior of Eq. (5.203) cannot be accounted for in the standard chiral analysis because the argument x is of order $\mathcal{O}(p^{-1})$. What is needed is a method which produces both the relevant analytic structure and a consistent power counting.

Here we will illustrate the method of Ref. [BL 99] by means of the nucleon self energy diagram of Fig. 5.3. For a—at this stage—qualitative discussion of its properties we focus on the scalar loop integral

$$H(p^2, n) = -i \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - M_\pi^2 + i0^+} \frac{1}{k^2 - 2p\cdot k + (p^2 - m_N^2) + i0^+}, \tag{5.207}$$

where, as usual, the right-hand side is thought of as a Feynman integral which has to be analytically continued as a function of the space-time dimension n. Counting powers, we see that, for $n = 4$, the integrand behaves for large values of the integration variable k as k^3/k^4, producing a logarithmic ultraviolet divergence while, on the other hand, the integral converges for $n < 4$. Let us now consider the limit $M_\pi^2 \to 0$. In this case, for both $p^2 = m_N^2$ and $p^2 \neq m_N^2$, the integral is infrared regular for $n = 4$ because, for small momenta, the integrand behaves as k^3/k^3 and k^3/k^2, respectively. For $n = 3$ the integral is infrared regular for $p^2 \neq m_N^2$ but singular for $p^2 = m_N^2$. For any smaller value of n it is infrared singular for arbitrary p^2. The infrared singularity as $M_\pi^2 \to 0$ originates in the region, where the integration variable k is small, i.e., of the order $\mathcal{O}(p)$. Counting powers of momenta, we (naively) expect this part to be of order $\mathcal{O}(p^{n-3})$. On the other hand, for loop momenta of the order of and larger than the nucleon mass we expect power counting to fail, because the momentum of the nucleon propagating in loop integral is not constrained to be small in contrast to the case of tree-level diagrams [see Eq. (5.196)].

In order to explain these qualitative statements let us discuss the integral in more detail. We first introduce the Feynman parameterization[35]

$$H(p^2, n) = -i \int \frac{d^n k}{(2\pi)^n} \int_0^1 dz \frac{1}{[az + b(1-z)]^2}, \tag{5.208}$$

with $a = k^2 - 2k \cdot p + p^2 - m_N^2 + i0^+$ and $b = k^2 - M_\pi^2 + i0^+$, perform the shift $k \to k + pz$, and obtain

$$H(p^2, n) = -i \int_0^1 dz \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - A(z) + i0^+]^2},$$

where

$$A(z) = z^2 p^2 - z(p^2 - m_N^2 + M_\pi^2) + M_\pi^2.$$

We then apply Eq. (B.14) of Appendix B,

$$H(p^2, n) = \frac{\Gamma\left(2 - \frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}} \int_0^1 dz [A(z) - i0^+]^{\frac{n}{2}-2}. \tag{5.209}$$

[35] In order to make it easier for the interested reader to follow Ref. [BL 99] we have used the notation there, omitting a factor μ^{n-4} and choosing the opposite overall sign in comparison with previous sections.

The relevant properties can nicely be displayed at the threshold $p_{\rm thr}^2 = (m_N + M_\pi)^2$, where $A(z) = [z(m_N + M_\pi) - M_\pi]^2$ is particularly simple. The small imaginary part can be dropped in this case, because $A(z)$ is never negative. Splitting the integration interval into $[0, z_0]$ and $[z_0, 1]$ with $z_0 = M_\pi/(m_N + M_\pi)$, we have, for $n > 3$,

$$\begin{aligned}
\int_0^1 dz[A(z)]^{\frac{n}{2}-2} &= \int_0^{z_0} dz[M_\pi - z(m_N + M_\pi)]^{n-4} \\
&\quad + \int_{z_0}^1 dz[z(m_N + M_\pi) - M_\pi]^{n-4} \\
&= \frac{1}{(n-3)(m_N + M_\pi)}(M_\pi^{n-3} + m_N^{n-3}),
\end{aligned}$$

yielding, through analytic continuation, for arbitrary n

$$H((m_N + M_\pi)^2, n) = \frac{\Gamma\left(2 - \frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)} \left(\frac{M_\pi^{n-3}}{m_N + M_\pi} + \frac{m_N^{n-3}}{m_N + M_\pi} \right). \quad (5.210)$$

The first term, proportional to M_π^{n-3}, is defined as the so-called infrared singular part I which, as $M_\pi \to 0$, behaves as in the qualitative discussion above. Since $M_\pi \to 0$ implies $p_{\rm thr}^2 \to m_N^2$ this term is singular for $n \leq 3$. The second term, proportional to m_N^{n-3}, is defined as the infrared regular part R and can be thought of as originating from an integration region where k is of order m_N so that the tree-level counting rules no longer apply [see Eq. (5.196)]. Note that for non-integer n the infrared singular part contains non-integer powers of M_π, while an expansion of the regular part always contains non-negative integer powers of M_π only.

Let us now turn to a *formal* definition of the infrared singular and regular parts [BL 99] which makes use of the Feynman parameterization of Eq. (5.209). Introducing the dimensionless variables

$$\alpha = \frac{M_\pi}{m_N}, \quad \Omega = \frac{p^2 - m_N^2 - M_\pi^2}{2 m_N M_\pi}, \quad (5.211)$$

counting as $\mathcal{O}(p)$ and $\mathcal{O}(p^0)$ $[p^2 - m_N^2 = \mathcal{O}(p)]$, respectively, we rewrite $A(z)$ as

$$A(z) = m_N^2[z^2 - 2\alpha\Omega z(1-z) + \alpha^2(1-z)^2] \equiv m_N^2 C(z),$$

so that H is now given by

$$H(p^2, n) = \kappa(n) \int_0^1 dz[C(z) - i0^+]^{\frac{n}{2}-2}, \quad (5.212)$$

where

$$\kappa(n) = \frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}} m_N^{n-4}. \tag{5.213}$$

The infrared singularity originates from small values of z, where the function $C(z)$ goes to zero as $M_\pi \to 0$. In order to isolate the divergent part one scales the integration variable $z \equiv \alpha x$ so that the upper limit $z = 1$ in Eq. (5.212) corresponds to $x = 1/\alpha \to \infty$ as $M_\pi \to 0$. An integral I having the same infrared singularity as H is then defined which is identical to H except that the upper limit is replaced by ∞:

$$I \equiv \kappa(n) \int_0^\infty dz[C(z) - i0^+]^{\frac{n}{2}-2} = \kappa(n)\alpha^{n-3} \int_0^\infty [D(x) - i0^+]^{\frac{n}{2}-2}, \tag{5.214}$$

where

$$D(x) = 1 - 2\Omega x + x^2 + 2\alpha x(\Omega x - 1) + \alpha^2 x^2.$$

(The pion mass M_π is not sent to zero.) Accordingly, the regular part of H is defined as

$$R \equiv -\kappa(n) \int_1^\infty dz[C(z) - i0^+]^{\frac{n}{2}-2}, \tag{5.215}$$

so that

$$H = I + R. \tag{5.216}$$

Let us verify that the definitions of Eqs. (5.214) and (5.215) indeed reproduce the behavior of Eq. (5.210). To that end we make use of $\Omega_{\text{thr}} = 1$, yielding

$$I_{\text{thr}} = \kappa(n)\alpha^{n-3} \int_0^\infty dx \left\{[(1+\alpha)x - 1]^2 - i0^+\right\}^{\frac{n}{2}-2}, \tag{5.217}$$

which converges for $n < 3$. In order to continue the integral to $n > 3$, we write [BL 99]

$$\begin{aligned}
&\left\{[(1+\alpha)x - 1]^2 - i0^+\right\}^{\frac{n}{2}-2} = \\
&\quad = \frac{(1+\alpha)x - 1}{(1+\alpha)(n-4)} \frac{d}{dx} \left\{[(1+\alpha)x - 1]^2 - i0^+\right\}^{\frac{n}{2}-2},
\end{aligned}$$

and make use of a partial integration

$$\begin{aligned}
&\int_0^\infty dx \left\{[(1+\alpha)x - 1]^2 - i0^+\right\}^{\frac{n}{2}-2} = \\
&\quad \left[\frac{(1+\alpha)x - 1}{(1+\alpha)(n-4)} \left\{[(1+\alpha)x - 1]^2 - i0^+\right\}^{\frac{n}{2}-2}\right]_0^\infty \\
&\quad - \frac{1}{n-4} \int_0^\infty dx \left\{[(1+\alpha)x - 1]^2 - i0^+\right\}^{\frac{n}{2}-2}.
\end{aligned}$$

For $n < 3$, the first expression vanishes at the upper limit and, at the lower limit, yields $1/[(1+\alpha)(n-4)]$. Bringing the second expression to the left-hand side, we may then continue the integral analytically as

$$\int_0^\infty dx \left\{[(1+\alpha)x - 1]^2 - i0^+\right\}^{\frac{n}{2}-2} = \frac{1}{(n-3)(1+\alpha)}, \tag{5.218}$$

so that we obtain for $I_{\rm thr}$

$$I_{\rm thr} = \kappa(n)\alpha^{n-3}\frac{1}{(n-3)(1+\alpha)} = \frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)}\frac{M_\pi^{n-3}}{m_N + M_\pi}, \tag{5.219}$$

which agrees with the infrared singular part I of Eq. (5.210).

The threshold value of the regular part of Eq. (5.215) is obtained by analytic continuation from $n < 3$ to $n > 3$:

$$\begin{aligned} R_{\rm thr} &= -\frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}}\int_1^\infty [z(m_N+M_\pi) - M_\pi]^{n-4} \\ &= -\frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}}\frac{1}{(n-3)(m_N+M_\pi)}(\infty^{n-3} - m_N^{n-3}) \\ &\overset{n \leq 3}{=} \frac{\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)}\frac{m_N^{n-3}}{m_N+M_\pi}, \end{aligned} \tag{5.220}$$

which is indeed the regular part R of Eq. (5.210).

What distinguishes I from R is that, for non-integer values of n, the chiral expansion of I gives rise to non-integer powers of $\mathcal{O}(p)$, whereas the regular part R may be expanded in an ordinary Taylor series. For the threshold integral, this can nicely be seen by expanding $I_{\rm thr}$ and $R_{\rm thr}$ in the pion mass counting as $\mathcal{O}(p)$. On the other hand, it is the regular part which does not satisfy the counting rules valid at tree level. The basic idea of the infrared regularization consists of replacing the general integral H of Eq. (5.207) by its infrared singular part I, defined in Eq. (5.214), and dropping the regular part R, defined in Eq. (5.215). In the low-energy region H and I have the same analytic properties whereas the contribution of R, which is of the type of an infinite series in the momenta, can be included by adjusting the coefficients of the most general effective Lagrangian.

As discussed in detail in Ref. [BL 99], the method can be generalized to an arbitrary one-loop graph. Using techniques similar to those of Appendices C.1.2 and C.2.2, it is first argued that tensor integrals involving an expression of the type $k^{\mu_1} \cdots k^{\mu_2}$ in the numerator may always be reduced to scalar loop integrals of the form

$$-i \int \frac{d^n k}{(2\pi)^n} \frac{1}{a_1 \cdots a_m} \frac{1}{b_1 \cdots b_n},$$

where $a_i = (q_i+k)^2 - M_\pi^2 + i0^+$ and $b_i = (p_i-k)^2 - m_N^2 + i0^+$ are inverse meson and nucleon propagators, respectively. Here, the q_i refer to four-momenta of $\mathcal{O}(p)$ and the p_i are four-momenta which are not far off the nucleon mass shell, i.e., $p_i^2 = m_N^2 + \mathcal{O}(p)$. Using the Feynman parameterization, all pion propagators and all nucleon propagators are separately combined, and the result is written in such a way that it is obtained by applying $(m-1)$ and $(n-1)$ partial derivatives with respect to M_π^2 and m_N^2, respectively, to a master formula. A simple illustration is given by

$$\frac{1}{a_1 a_2} = \int_0^1 dz \frac{1}{[a_1 z + a_2(1-z)]^2} = \frac{\partial}{\partial M_\pi^2} \int_0^1 dz \frac{1}{a_1 z + a_2(1-z)},$$

where $a_i = (q_i + k)^2 - M_\pi^2 + i0^+$. Of course, the expressions become more complicated for larger numbers of propagators. The relevant property of the above procedure is that the result of combining the meson propagators is of the type $1/A$ with $A = (k+q)^2 - M_\pi^2 + i0^+$, where q is a linear combination of the m momenta q_i, with an analogous expression $1/B$ for the nucleon propagators. Finally, the expression

$$-i \int \frac{d^n k}{(2\pi)^n} \frac{1}{AB}$$

may then be treated in complete analogy to H of Eq. (5.207), i.e., the denominators are combined as in Eq. (5.208), and the infrared singular and regular pieces are identified by writing $\int_0^1 dz \cdots = \int_0^\infty dz \cdots - \int_1^\infty dz \cdots$.

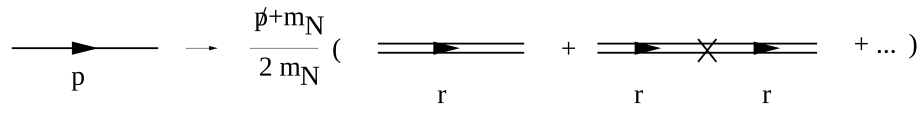
A crucial question is whether the infrared regularization respects the constraints of chiral symmetry as expressed through the chiral Ward identities. The argument given in Ref. [BL 99] that this is indeed the case is as follows. The total nucleon-to-nucleon transition amplitude of Eq. (5.1) is chirally symmetric, i.e., invariant under a simultaneous local transformation of the quark fields and the external fields (see Appendix A for an illustration). In terms of the effective theory, the contribution from all the tree-level diagrams is chirally symmetric so that the loop contribution must also be chirally symmetric. Since we work in dimensional regularization this statement holds for an arbitrary n. However, as we have seen in the example of Eq. (5.210), the separation into infrared singular and regular parts amounts to distinguishing between contributions of non-integer and non-negative integer powers in the momentum expansion. Since these powers do not mix for arbitrary n, the infrared singular and regular parts must be separately chirally symmetric. Finally, the regular part can be expanded in powers of either momenta or quark masses, and thus may as well be absorbed in the (modified) tree-level contribution.

Let us finally establish the connection between the infrared singular part I and the corresponding result in HBChPT. To that end, we first consider the relativistic propagator by expressing the (off-shell) four-momentum as $p = m_N v + r$,

$$\begin{aligned}
\frac{i}{\not p - m_N + i0^+} &= i\frac{\not p + m_N}{p^2 - m_N^2 + i0^+} = i\frac{\not p + m_N}{2m_N v\cdot r + r^2 + i0^+} \\
&= i\frac{\not p + m_N}{2m_N v\cdot r + i0^+}\,\frac{1}{1+\frac{r^2}{2m_N v\cdot r + i0^+}} \\
&\mapsto \frac{\not p + m_N}{2m_N}\frac{i}{v\cdot r + i0^+}\left[1 + \frac{ir^2}{2m_N}\frac{i}{v\cdot r + i0^+} + \left(\frac{ir^2}{2m_N}\frac{i}{v\cdot r + i0^+}\right)^2 + \cdots\right].
\end{aligned} \tag{5.221}$$

In the last step, we have assumed that r is small enough to allow for an expansion in terms of a geometric series. The result of Eq. (5.221) is displayed in Fig. 5.9 and may be interpreted as an infinite series in terms of the heavy-baryon propagator $i/(v\cdot r + i0^+)$ and the self-energy insertion $-i\Sigma = ir^2/2m_N$ which has the form of a non-relativistic kinetic energy. (Note that the expression still involves the operator $(\not p + m_N)/2m_N$.) Let us apply Eq. (5.221) to

Figure 5.9: Expansion of the relativistic propagator (single line) in terms of heavy-baryon propagators (double line) and self-energy insertions (cross).



the loop integral H of Eq. (5.207) by first expanding the integrand and then performing the summation. This corresponds to the prescription proposed in Refs. [Tan 96, ET 98] for identifying the low-energy or, in the nomenclature of Ref. [Tan 96], soft contribution to a Feynman graph. In the case at hand we obtain

$$H(p^2, n) \rightarrow \sum_{j=0}^{\infty} I_j, \tag{5.222}$$

where

$$I_j = \frac{-i}{2m_N} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - M_\pi^2 + i0^+} \frac{1}{v \cdot (r-k) + i0^+} \left[\frac{-(r-k)^2}{2m_N v \cdot (r-k) + i0^+} \right]^j, \tag{5.223}$$

which is somewhat easier to handle if we perform the shift $k \rightarrow k + r$ and then the substitution $k \rightarrow -k$,

$$I_j = i \frac{(-)^{j+1}}{(2m_N)^{j+1}} \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k-r)^2 - M_\pi^2 + i0^+} \frac{(k^2)^j}{(v \cdot k + i0^+)^{j+1}}. \tag{5.224}$$

As above, we explicitly discuss the threshold $p_{\rm thr}^2 = (m_N + M_\pi)^2$ by inserting $r = M_\pi v$ into Eq. (5.224). Defining $X = k^2 - 2M_\pi v \cdot k + i0^+$ and $Y = v \cdot k + i0^+$, we have

$$I_{j,\rm thr} = i \frac{(-)^{j+1}}{(2m_N)^{j+1}} \int \frac{d^n k}{(2\pi)^n} \frac{(X + 2M_\pi Y)^j}{XY^{j+1}}. \tag{5.225}$$

The different $I_{j,\rm thr}$ are related by a simple recursion relation

$$I_{j+1,\rm thr} = -\frac{M_\pi}{m_N} I_{j,\rm thr}, \quad j \geq 0, \tag{5.226}$$

implying

$$I_{j,\rm thr} = (-)^j \left(\frac{M_\pi}{m_N} \right)^j I_{0,\rm thr}, \quad j \geq 0. \tag{5.227}$$

Equation (5.226) is easily verified:

$$\begin{aligned}
I_{j+1,\rm thr} &= i \frac{(-)^{j+2}}{(2m_N)^{j+2}} \int \frac{d^n k}{(2\pi)^n} \frac{(X + 2M_\pi Y)^j}{XY^{j+1}} \frac{X + 2M_\pi Y}{Y} \\
&= i \frac{(-)^{j+2}}{(2m_N)^{j+2}} \int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^j}{(v \cdot k + i0^+)^{j+2}} \\
&\quad - \frac{M_\pi}{m_N} i \frac{(-1)^{j+1}}{(2m_N)^{j+1}} \int \frac{d^n k}{(2\pi)^n} \frac{(X + 2M_\pi Y)^j}{XY^{j+1}} \\
&= -\frac{M_\pi}{m_N} I_{j,\rm thr},
\end{aligned}$$

where we made use of the fact that the first term in the second line vanishes in dimensional regularization [see Eq. (C.37)]. We then obtain for the series, evaluated at threshold,

$$\sum_{j=0}^{\infty} I_{j,\mathrm{thr}} = I_{0,\mathrm{thr}} \sum_{j=0}^{\infty} (-)^j \left(\frac{M_\pi}{m_N}\right)^j = \frac{m_N}{m_N + M_\pi} I_{0,\mathrm{thr}}.$$

What remains to be determined is the threshold integral

$$I_{0,\mathrm{thr}} = \frac{-i}{2m_N} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - 2M_\pi v \cdot k + i0^+} \frac{1}{v \cdot k + i0^+}.$$

Performing a shift $k \to k + M_\pi v$, combining the denominators as in Eq. (C.17), performing another shift $k \to k - yv$, and making use of Eq. (B.14), one finds

$$I_{0,\mathrm{thr}} = \frac{1}{m_N} \frac{\Gamma\left(2 - \frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}} \int_0^\infty dy \left\{(y - M_\pi)^2 - i0^+\right\}^{\frac{n}{2}-2}.$$

Finally, performing a substitution $y = M_\pi x$ and using the analytic continuation of Eq. (5.218) with $\alpha = 0$, we obtain

$$I_{0,\mathrm{thr}} = \frac{\Gamma\left(2 - \frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)} \frac{M_\pi^{n-3}}{m_N}. \tag{5.228}$$

Inserting Eq. (5.228) into the series, the final result reads

$$\sum_{j=0}^{\infty} I_{j,\mathrm{thr}} = \frac{\Gamma\left(2 - \frac{n}{2}\right)}{(4\pi)^{\frac{n}{2}}(n-3)} \frac{M_\pi^{n-3}}{m_N + M_\pi}, \tag{5.229}$$

which is the same as I_{thr} of Eq. (5.219). This example shows that the infrared regularized amplitude is related to an *infinite* sum of heavy-baryon amplitudes with self-energy insertions in the heavy-baryon propagator, as depicted in Fig. 5.9. The advantage of the relativistic approach is obvious, because for a general one-loop amplitude it may be very difficult, if not impossible, to obtain a closed expression for the sum of all insertions. To conclude this section, the method of infrared regularization provides a fully relativistic framework producing amplitudes having the relevant analytic properties and satisfying the chiral power-counting rules. At the moment, it is not yet clear, whether it can be generalized beyond the one-loop level.

Chapter 6

Summary and Concluding Remarks

As we have discussed in great detail, the chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ symmetry of QCD in the limit of vanishing u-, d-, and s-quark masses (Sec. 2.3.4), together with the assumption of its spontaneous breakdown to $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$ in the ground state (Sec. 4.1), is one of the keys to understanding the phenomenology of the strong interactions in the low-energy regime. The importance of chiral symmetry was realized long before the formulation of QCD and led to a host of predictions within the current-algebra and PCAC approaches of the 1960's [AD 68]. Some of the consequences of an explicit symmetry breaking, yielding non-analytic terms in the perturbation, were worked out in the early 1970's, but the development came to a halt [Pag 75] because it was not clear how to systematically organize a perturbative expansion. From the present point of view, the explicit symmetry breaking is due to the finite u-, d-, and s-quark masses, leading to divergences of the symmetry currents (Sec. 2.3.6).

In 1979 Weinberg [Wei 79] laid the foundations for further progress with his observation that the constraints due to (chiral) symmetry may perturbatively be analyzed in terms of the most general effective field theory. A very important ingredient was the formulation of a consistent power-counting scheme (Secs. 4.4 and 5.5.8) which allowed for a systematic perturbative analysis in contrast to various commonly used *ad hoc* phenomenological approaches to the strong interactions at low energies. In particular, the inclusion of loop diagrams allowed for a perturbative restoration of unitarity which would be violated if only tree-level diagrams were used. Subsequently

Gasser and Leutwyler [GL 84, GL 85a] combined the ideas of Weinberg with other modern techniques of quantum field theory to analyze the Ward identities of QCD Green functions in terms of a local invariance of the generating functional under the chiral group (Sec. 2.4 and App. A). These papers were the starting point of what is nowadays called chiral perturbation theory.

The mesonic sector has generated a host of successful applications, some of which have reached two-loop accuracy. Here, we have concentrated on a few elementary observables and processes, namely: masses of the Goldstone bosons (Secs. 4.3 and 4.9.1), weak and electromagnetic π decays (Secs. 4.6.1 and 4.8), $\pi\pi$ scattering (Secs. 4.6.2 and 4.10.2), and electromagnetic form factors (Sec. 4.9.2). Moreover, we have discussed in quite some detail how to construct the mesonic effective Lagrangian (Secs. 4.2, 4.7, and 4.10.1).

At first sight, it might appear that the large number of low-energy parameters at $\mathcal{O}(p^6)$ would make any quantitative prediction at the two-loop level impossible. However, there are several reasons why this is not the case. To start with, there exist observables which do not depend on *any* new parameters at $\mathcal{O}(p^6)$, i.e., which can be predicted in terms of the $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$ low-energy constants only. An example is given by the correction to Sirlin's theorem discussed in Ref. [PS 97]. Clearly, such cases provide a natural testing ground for the convergence of the approach. Secondly, only a limited set of low-energy parameters contribute to any given process. It follows from the nature of the Ward identities that different physical processes are interrelated due to the underlying symmetries so that coefficients which have been fixed using one reaction can be used to *predict* another observable. In view of the ordinary implementation of symmetries, such as in the Wigner-Eckart theorem, this is not a surprise, because it is well-known that symmetries imply relations among S-matrix elements. However, the Ward identities provide *additional* constraints among Green functions of a different type and allow one to also include an explicit symmetry breaking (Sec. 2.4). It is this second case which can systematically be studied in the framework of ChPT and which provides interesting new insights into our understanding of both spontaneous and explicit symmetry breaking within QCD. Finally, different methods exist which allow one to estimate the value of the parameters and thus, in combination with the ChPT result, test our physical picture of the strong interactions.

In this work we have only considered elementary processes at an introductory level, not the extensions to and combinations with other methods. We omitted, for example, the weak interactions of kaons which are mediated by the exchange of W bosons *between* the quark currents [Eck 95, Raf 95, Pic 95]. We also did not discuss the breaking of isospin symmetry which requires the inclusion of the electromagnetic interaction in terms of dynamical (virtual) photons [Ure 95, NR 96, AM 02].

Chiral symmetry also dictates the interaction of the Goldstone bosons with other hadrons (Secs. 5.1 and 5.2). By studying the axial-vector current matrix element (Sec. 5.3.1) and πN scattering (Sec. 5.3.2) we verified that a tree-level calculation using the lowest-order Lagrangian reproduces the Goldberger-Treiman relation and the Weinberg-Tomozawa result for the s-wave scattering lengths, respectively. As we have seen, the first systematic study in the pion-nucleon sector [Gas+ 88] raised the question of a consistent power counting (Sec. 5.4). This problem was subsequently overcome in the framework of the heavy-baryon approach [JM 91] (Sec. 5.5) and most of the numerous applications in this sector have been performed in HBChPT [Ber+ 95b].

In the baryonic sector the chiral orders increase in units of one, because of the additional possibility of forming Lorentz invariants by contracting (covariant) derivatives with gamma matrices (Sec. 5.2). As a result, in the SU(2)×SU(2) baryonic sector at the one-loop level, up to and including $\mathcal{O}(p^4)$, one has in total $2 + 7 + 23 + 118 = 150$ [Fet+ 01] low-energy constants as opposed to the $2 + 7 = 9$ free parameters of the corresponding mesonic sector [GL 84]. Nevertheless, numerous results have been obtained in the baryonic sector because, at the same time, a large amount of very precise experimental data are available due to the existence of a stable proton target. (Neutron data can also be extracted, e.g, from experiments on the deuteron.) The availability of new high-precision data in combination with the techniques of chiral perturbation theory have led to a considerable improvement of our understanding of the strong interactions at low energies, in particular since systematic corrections to the old current-algebra predictions could be worked out and (successfully) tested.

We have not discussed the approach of the so-called small scale expansion to include the $\Delta(1232)$ resonance as an explicit degree of freedom [Hem+ 97c]. Clearly this is an important issue in the baryonic sector because the first nucleon excitation is such a prominent feature of the low-energy spectrum as seen, e.g., in the total pion-nucleon scattering or the total photo absorption cross sections.

Most recently, the method of infrared regularization [BL 99] has opened the possibility of reconciling the relativistic approach with a consistent chiral power counting scheme (Sec. 5.6). One may expect that this method will have a large impact insofar as many of the results obtained within the heavy-baryon framework will have to be checked with respect to relativistic corrections. The question regarding the radius of convergence in the baryonic sector remains a big challenge because, ultimately, a calculation at the two-loop level $\mathcal{O}(p^5)$ is, in general, required to quantitatively assess higher-order corrections [MB 99]. In comparison to two-loop calculations in the SU(2)$\times$SU(2) mesonic sector such an investigation in the (relativistic) nucleon sector is even more complicated for two reasons. First, due to the spin of the nucleon, the structure of vertices is richer than for spin-0 particles. Second, the nucleon mass introduces another mass in the propagators making the evaluation of the two-loop integrals more difficult than for a single mass.

Finally, we would like to mention that a description of the nucleon-nucleon interaction within the framework of effective field theory has made tremendous progress and that a rigorous treatment of nuclei within field theory is no longer out of reach [Wei 91, Ord+ 96, Kai+ 97, Kol 99, Epe+ 00, Bea+ 02, Fin+ 02].

In conclusion, chiral perturbation theory has added a new and unprecedented level of systematics to the description of strong-interaction processes at low energies and continues to be a very fruitful and rich field with promising perspectives. If this introductory review encourages students and newcomers to chiral perturbation theory to participate in this field of research, it has served its purpose.

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I dedicate this work to my family. I hope it was worth it!

Appendix A

Green Functions and Ward Identities

In this appendix we will show how to derive Ward identities for Green functions in the framework of canonical quantization on the one hand, and quantization via the Feynman path integral on the other hand, by means of an explicit example. In order to keep the discussion transparent, we will concentrate on a simple scalar field theory with a global O(2) or U(1) invariance. To that end, let us consider the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\Phi_1\partial^\mu\Phi_1+\partial_\mu\Phi_2\partial^\mu\Phi_2)-\frac{m^2}{2}(\Phi_1^2+\Phi_2^2)-\frac{\lambda}{4}(\Phi_1^2+\Phi_2^2)^2\\ &= \partial_\mu\Phi^\dagger\partial^\mu\Phi-m^2\Phi^\dagger\Phi-\lambda(\Phi^\dagger\Phi)^2,\end{aligned}\tag{A.1}$$

where

$$\Phi(x)=\frac{1}{\sqrt{2}}[\Phi_1(x)+i\Phi_2(x)],\quad \Phi^\dagger(x)=\frac{1}{\sqrt{2}}[\Phi_1(x)-i\Phi_2(x)],$$

with real scalar fields Φ_1 and Φ_2. Furthermore, we assume $m^2>0$ and $\lambda>0$, so there is no spontaneous symmetry breaking and the energy is bounded from below. Equation (A.1) is invariant under the global (or rigid) transformations

$$\Phi_1'=\Phi_1-\epsilon\Phi_2,\quad \Phi_2'=\Phi_2+\epsilon\Phi_1,\tag{A.2}$$

or, equivalently,

$$\Phi'=(1+i\epsilon)\Phi,\quad \Phi'^\dagger=(1-i\epsilon)\Phi^\dagger,\tag{A.3}$$

where ϵ is an infinitesimal real parameter. Applying the method of Gell-Mann and Lévy [GL 60], we obtain for a *local* parameter $\epsilon(x)$,

$$\delta\mathcal{L} = \partial_\mu\epsilon(x)(i\partial^\mu\Phi^\dagger\Phi - i\Phi^\dagger\partial^\mu\Phi), \tag{A.4}$$

from which, via Eqs. (2.42) and (2.43), we derive for the current corresponding to the global symmetry,

$$\begin{aligned} J^\mu &= \frac{\partial\delta\mathcal{L}}{\partial\partial_\mu\epsilon} = (i\partial^\mu\Phi^\dagger\Phi - i\Phi^\dagger\partial^\mu\Phi), && \text{(A.5)} \\ \partial_\mu J^\mu &= \frac{\partial\delta\mathcal{L}}{\partial\epsilon} = 0. && \text{(A.6)} \end{aligned}$$

Recall that the identification of Eq. (2.43) as the divergence of the current is only true for fields satisfying the Euler-Lagrange equations of motion.

We now extend the analysis to a *quantum* field theory. In the framework of canonical quantization, we first define conjugate momenta,

$$\Pi_i(x) = \frac{\partial\mathcal{L}}{\partial\partial_0\Phi_i}, \quad \Pi(x) = \frac{\partial\mathcal{L}}{\partial\partial_0\Phi}, \quad \Pi^\dagger(x) = \frac{\partial\mathcal{L}}{\partial\partial_0\Phi^\dagger}, \tag{A.7}$$

and interpret the fields and their conjugate momenta as operators which, in the Heisenberg picture, are subject to the equal-time commutation relations

$$[\Phi_i(\vec{x},t), \Pi_j(\vec{y},t)] = i\delta_{ij}\delta^3(\vec{x}-\vec{y}), \tag{A.8}$$

and

$$[\Phi(\vec{x},t), \Pi(\vec{y},t)] = [\Phi^\dagger(\vec{x},t), \Pi^\dagger(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}). \tag{A.9}$$

The remaining equal-time commutation relations, involving fields or momenta only, vanish. For the quantized theory, the current operator then reads

$$J^\mu(x) =: (i\partial^\mu\Phi^\dagger\Phi - i\Phi^\dagger\partial^\mu\Phi) :, \tag{A.10}$$

where : : denotes normal or Wick ordering, i.e., annihilation operators appear to the right of creation operators. For a conserved current, the charge operator, i.e., the space integral of the charge density, is time independent and serves as the generator of infinitesimal transformations of the Hilbert space states,

$$Q = \int d^3x J^0(\vec{x},t). \tag{A.11}$$

Applying Eq. (A.9), it is straightforward to calculate the equal-time commutation relations[1]

$$
\begin{aligned}
[J^0(\vec{x},t),\Phi(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y})\Phi(\vec{x},t),\\
[J^0(\vec{x},t),\Pi(\vec{y},t)] &= -\delta^3(\vec{x}-\vec{y})\Pi(\vec{x},t),\\
[J^0(\vec{x},t),\Phi^\dagger(\vec{y},t)] &= -\delta^3(\vec{x}-\vec{y})\Phi^\dagger(\vec{x},t),\\
[J^0(\vec{x},t),\Pi^\dagger(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y})\Pi^\dagger(\vec{x},t).
\end{aligned}
\tag{A.12}
$$

In particular, performing the space integrals in Eqs. (A.12), one obtains

$$
\begin{aligned}
[Q,\Phi(x)] &= \Phi(x),\\
[Q,\Pi(x)] &= -\Pi(x),\\
[Q,\Phi^\dagger(x)] &= -\Phi^\dagger(x),\\
[Q,\Pi^\dagger(x)] &= \Pi^\dagger(x).
\end{aligned}
\tag{A.13}
$$

In order to illustrate the implications of Eqs. (A.13), let us take an eigenstate $|\alpha\rangle$ of Q with eigenvalue q_α and consider, for example, the action of $\Phi(x)$ on that state,

$$Q\left(\Phi(x)|\alpha\rangle\right) = \left([Q,\Phi(x)]+\Phi(x)Q\right)|\alpha\rangle = (1+q_\alpha)\left(\Phi(x)|\alpha\rangle\right).$$

We conclude that the operators $\Phi(x)$ and $\Pi^\dagger(x)$ [$\Phi^\dagger(x)$ and $\Pi(x)$] increase (decrease) the Noether charge of a system by one unit.

We are now in the position to discuss the consequences of the U(1) symmetry of Eq. (A.1) for the Green functions of the theory. To that end, let us consider as our prototype the Green function

$$G^\mu(x,y,z) = \langle 0|T[\Phi(x)J^\mu(y)\Phi^\dagger(z)]|0\rangle, \tag{A.14}$$

which describes the transition amplitude for the creation of a quantum of Noether charge +1 at x, propagation to y, interaction at y via the current operator, propagation to z with annihilation at z. First of all we observe that under the global infinitesimal transformations of Eq. (A.3), $J^\mu(x)\mapsto J'^\mu(x)=J^\mu(x)$, or in other words $[Q,J^\mu(x)]=0$. We thus obtain

$$
\begin{aligned}
G^\mu(x,y,z)\mapsto G'^\mu(x,y,z) &= \langle 0|T[(1+i\epsilon)\Phi(x)J'^\mu(y)(1-i\epsilon)\Phi^\dagger(z)]|0\rangle\\
&= \langle 0|T[\Phi(x)J^\mu(y)\Phi^\dagger(z)]|0\rangle\\
&= G^\mu(x,y,z),
\end{aligned}
\tag{A.15}
$$

[1] The transition to normal ordering involves an (infinite) constant which does not contribute to the commutator.

the Green function remaining invariant under the U(1) transformation. (In general, the transformation behavior of a Green function depends on the irreducible representations under which the fields transform. In particular, for more complicated groups such as SU(N), standard tensor methods of group theory may be applied to reduce the product representations into irreducible components [BT 84, O'Ra 86, Jon 90]. We also note that for U(1), the symmetry current is charge neutral, i.e. invariant, which for more complicated groups, in general, is not the case.)

Moreover, since $J^\mu(x)$ is the Noether current of the underlying U(1) there are further restrictions on the Green function beyond its transformation behavior under the group. In order to see this, we consider the divergence of Eq. (A.14) and apply the equal-time commutation relations of Eqs. (A.12) to obtain (see Sec. 2.4.1)

$$\partial^y_\mu G^\mu(x,y,z) \quad = \quad [\delta^4(x-y)-\delta^4(z-y)]\langle 0|T[\Phi(x)\Phi^\dagger(z)]|0\rangle, \quad \text{(A.16)}$$

where we made use of $\partial_\mu J^\mu = 0$. Equation (A.16) is the analogue of the Ward identity of QED [see Eq. (2.85)] [War 50, Fra 55, Tak 57]. In other words, the underlying symmetry not only determines the transformation behavior of Green functions under the group, but also relates n-point Green functions containing a symmetry current to $(n-1)$-point Green functions [see Eq. (2.89)]. In principle, calculations similar to those leading to Eqs. (A.15) and (A.16), can be performed for any Green function of the theory. However, we will now show that the symmetry constraints can be compactly summarized in terms of an invariance property of a generating functional.

The generating functional is defined as the vacuum-to-vacuum transition amplitude in the presence of external fields,

$$\begin{aligned}
&W[j,j^*,j_\mu] = \langle 0,+\infty|0,-\infty\rangle_{j,j^*,j_\mu}\\
&\quad = \exp(iZ[j,j^*,j_\mu])\\
&\quad = \langle 0|T\left(\exp\left\{i\int d^4x[j(x)\Phi^\dagger(x)+j^*(x)\Phi(x)+j_\mu(x)J^\mu(x)]\right\}\right)|0\rangle,
\end{aligned} \quad \text{(A.17)}$$

where Φ and $\Phi^\dagger$ are the field operators and $J^\mu(x)$ is the Noether current. Note that the field operators and the conjugate momenta are subject to the equal-time commutation relations and, in addition, must satisfy the Heisenberg equations of motion. Via this second condition and implicitly through the ground state, the generating functional depends on the dynamics of the system which is determined by the Lagrangian of Eq. (A.1). The Green functions of the theory involving Φ, $\Phi^\dagger$, and J^μ are obtained through functional derivatives of Eq. (A.17). For example, the Green function of Eq. (A.14) is given by

$$G^\mu(x,y,z) = (-i)^3 \left. \frac{\delta^3 W[j,j^*,j_\mu]}{\delta j^*(x)\delta j_\mu(y)\delta j(z)}\right|_{j=0,j^*=0,j_\mu=0}. \tag{A.18}$$

In order to discuss the constraints imposed on the generating functional via the underlying symmetry of the theory, let us consider its path integral representation [Zin 89, Das 93],[2]

$$W[j,j^*,j_\mu] \;=\; \int [d\Phi_1][d\Phi_2] e^{iS[\Phi,\Phi^*,j,j^*,j_\mu]}, \tag{A.19}$$

where

$$S[\Phi,\Phi^*,j,j^*,j_\mu] = S[\Phi,\Phi^*] + \int d^4x[\Phi(x)j^*(x) + \Phi^*(x)j(x) + J^\mu(x)j_\mu(x)] \tag{A.20}$$

denotes the action corresponding to the Lagrangian of Eq. (A.1) in combination with a coupling to the external sources. Let us now consider a *local* infinitesimal transformation of the fields [see Eqs. (A.3)] together with a *simultaneous* transformation of the external sources,

$$j'(x) = [1+i\epsilon(x)]j(x), \quad j'^*(x) = [1-i\epsilon(x)]j^*(x), \quad j'_\mu(x) = j_\mu(x) - \partial_\mu\epsilon(x). \tag{A.21}$$

The action of Eq. (A.20) remains invariant under such a transformation,

$$S[\Phi',\Phi'^*,j',j'^*,j'_\mu] = S[\Phi,\Phi^*,j,j^*,j_\mu]. \tag{A.22}$$

We stress that the transformation of the external current j_μ is necessary to cancel a term resulting from the kinetic term in the Lagrangian. We can now verify the invariance of the generating functional as follows,

$$W[j,j^*,j_\mu] \;=\; \int [d\Phi_1][d\Phi_2] e^{iS[\Phi,\Phi^*,j,j^*,j_\mu]}$$

[2]Up to an irrelevant constant the measure $[d\Phi_1][d\Phi_2]$ is equivalent to $[d\Phi][d\Phi^*]$, with Φ and Φ^* considered as independent variables of integration.

$$
\begin{aligned}
&= \int [d\Phi_1][d\Phi_2] e^{iS[\Phi',\Phi'^*,j',j'^*,j'_\mu]} \\
&= \int [d\Phi'_1][d\Phi'_2] \left| \left(\frac{\partial \Phi_i}{\partial \Phi'_j} \right) \right| e^{iS[\Phi',\Phi'^*,j',j'^*,j'_\mu]} \\
&= \int [d\Phi_1][d\Phi_2] e^{iS[\Phi,\Phi^*,j',j'^*,j'_\mu]} \\
&= W[j', j'^*, j'_\mu]. \qquad \text{(A.23)}
\end{aligned}
$$

We made use of the fact that the Jacobi determinant is one and renamed the integration variables. In other words, given the *global* U(1) symmetry of the Lagrangian, Eq. (A.1), the generating functional is invariant under the *local* transformations of Eq. (A.21). It is this observation which, for the more general case of the chiral group SU(N)×SU(N), was used by Gasser and Leutwyler as the starting point of chiral perturbation theory.

We still have to discuss, how this invariance allows us to collect the Ward identities in a compact formula. We start from Eq. (A.23),

$$
\begin{aligned}
0 &= \int [d\Phi_1][d\Phi_2] \left(e^{iS[\Phi,\Phi^*,j',j'^*,j'_\mu]} - e^{iS[\Phi,\Phi^*,j,j^*,j_\mu]} \right) \\
&= \int [d\Phi_1][d\Phi_2] \int d^4x \left\{ \epsilon[\Phi j^* - \Phi^* j] - i J^\mu \partial_\mu \epsilon \right\} e^{iS[\Phi,\Phi^*,j,j^*,j_\mu]}.
\end{aligned}
$$

Observe that

$$
\Phi(x) e^{iS[\Phi,\Phi^*,j,j^*,j_\mu]} = -i \frac{\delta}{\delta j^*(x)} e^{iS[\Phi,\Phi^*,j,j^*,j_\mu]},
$$

and similarly for the other terms, resulting in

$$
\begin{aligned}
0 = &\int [d\Phi_1][d\Phi_2] \int d^4x \left\{ \epsilon(x) \left[-ij^*(x) \frac{\delta}{\delta j^*(x)} + ij(x) \frac{\delta}{\delta j(x)} \right] \right. \\
&\left. -\partial_\mu \epsilon(x) \frac{\delta}{\delta j_\mu(x)} \right\} e^{iS[\Phi,\Phi^*,j,j^*,j_\mu]}.
\end{aligned}
$$

Finally we interchange the order of integration, make use of partial integration, and apply the divergence theorem:

$$
0 = \int d^4x \epsilon(x) \left[ij(x) \frac{\delta}{\delta j(x)} - ij^*(x) \frac{\delta}{\delta j^*(x)} + \partial^x_\mu \frac{\delta}{\delta j_\mu(x)} \right] W[j, j^*, j_\mu]. \quad \text{(A.24)}
$$

Since Eq. (A.24) must hold for any $\epsilon(x)$ we obtain as the master equation for deriving Ward identities,

$$\left[j(x)\frac{\delta}{\delta j(x)} - j^*(x)\frac{\delta}{\delta j^*(x)} - i\partial^x_\mu \frac{\delta}{\delta j_\mu(x)}\right] W[j, j^*, j_\mu] = 0. \tag{A.25}$$

We note that Eqs. (A.23) and (A.25) are equivalent.

As a final illustration let us re-derive the Ward identity of Eq. (A.16) using Eq. (A.25). For that purpose we start from Eq. (A.18),

$$\partial^y_\mu G^\mu(x,y,z) = (-i)^3 \partial^y_\mu \left. \frac{\delta^3 W}{\delta j^*(x)\delta j_\mu(y)\delta j(z)}, \right|_{j=0,j^*=0,j_\mu=0},$$

apply Eq. (A.25),

$$= (-i)^2 \left\{ \frac{\delta^2}{\delta j^*(x)\delta j(z)} \left[j^*(y)\frac{\delta}{\delta j^*(y)} - j(y)\frac{\delta}{\delta j(y)} \right] W \right\}_{j=0,j^*=0,j_\mu=0},$$

make use of $\delta j^*(y)/\delta j^*(x) = \delta^4(y-x)$ and $\delta j(y)/\delta j(z) = \delta^4(y-z)$ for the functional derivatives,

$$= (-i)^2 \left\{ \delta^4(x-y)\frac{\delta^2 W}{\delta j^*(y)\delta j(z)} - \delta^4(z-y)\frac{\delta^2 W}{\delta j^*(x)\delta j(y)} \right\}_{j=0,j^*=0,j_\mu=0},$$

and, finally, use the definition of Eq. (A.17),

$$\partial^y_\mu G^\mu(x,y,z) = [\delta^4(x-y) - \delta^4(z-y)]\langle 0|T[\Phi(x)\Phi^\dagger(z)]|0\rangle$$

which is the same as Eq. (A.16). In principle, any Ward identity can be obtained by taking appropriate higher functional derivatives of W and then using Eq. (A.25).

Appendix B

Dimensional Regularization: Basics

For the sake of completeness we provide a simple illustration of the method of dimensional regularization. For a detailed account the interested reader is referred to Refs. [HV 72, Lei 75, HV 79, CL 84, Col 84, Vel 94].

Let us consider the integral

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i0^+} \tag{B.1}$$

which appears in the calculation of the masses of the Goldstone bosons [see Eq. (4.144)]. We introduce

$$a \equiv \sqrt{\vec{k}^2 + M^2} > 0$$

so that

$$k^2 - M^2 + i0^+ \quad = \quad [k_0 + (a - i0^+)][k_0 - (a - i0^+)],$$

and define

$$f(k_0) = \frac{1}{[k_0 + (a - i0^+)][k_0 - (a - i0^+)]}.$$

In order to determine $\int_{-\infty}^{\infty} dk_0 f(k_0)$ as part of the calculation of I, we consider f in the complex k_0 plane and make use of Cauchy's theorem

$$\oint_C dz f(z) = 0 \tag{B.2}$$

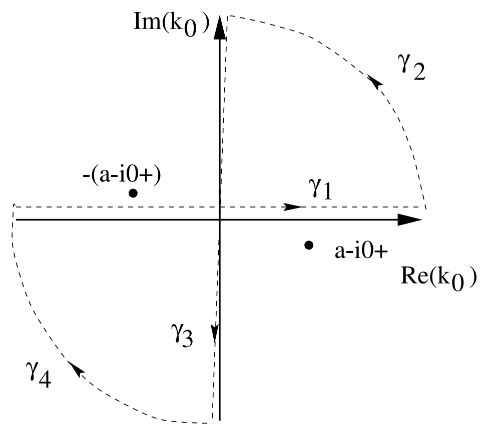

Figure B.1: Path of integration in the complex k_0 plane.

for functions which are differentiable in every point inside the closed contour C. We choose the contour as shown in Fig. B.1,

$$0 = \sum_{i=1}^{4} \int_{\gamma_i} dz f(z),$$

and make use of

$$\int_{\gamma} f(z) dz = \int_a^b f[\gamma(t)] \gamma'(t) dt$$

to obtain for the individual integrals

$$\begin{aligned}
\int_{\gamma_1} f(z) dz &= \int_{-\infty}^{\infty} f(t) dt, \\
\int_{\gamma_2} f(z) dz &= \lim_{R\to\infty} \int_0^{\frac{\pi}{2}} f(Re^{it}) iRe^{it} dt = 0, \text{ since } \lim_{R\to\infty} \underbrace{Rf(Re^{it})}_{\sim \frac{1}{R}} = 0, \\
\int_{\gamma_3} f(z) dz &= \int_{\infty}^{-\infty} f(it) i dt, \\
\int_{\gamma_4} f(z) dz &= \lim_{R\to\infty} \int_{\frac{3}{2}\pi}^{\pi} f(Re^{it}) iRe^{it} dt = 0.
\end{aligned}$$

In combination with Eq. (B.2) we obtain the so-called Wick rotation

$$\int_{-\infty}^{\infty} f(t)dt = -i\int_{\infty}^{-\infty} dt f(it) = i\int_{-\infty}^{\infty} dt f(it). \tag{B.3}$$

As an intermediate result the integral of Eq. (B.1) reads

$$I = \frac{1}{(2\pi)^4} i\int_{-\infty}^{\infty} dk_0 \int d^3k \frac{i}{(ik_0)^2 - \vec{k}^2 - M^2 + i0^+} = \int \frac{d^4l}{(2\pi)^4}\frac{1}{l^2+M^2-i0^+},$$

where $l^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2$ denotes a Euclidian scalar product. In this *special* case, the integrand does not have a pole and we can thus omit the $-i0^+$ which gave the positions of the poles in the original integral consistent with the boundary conditions. The degree of divergence can be estimated by simply counting the powers of momenta [Vel 94]. If the integral behaves asymptotically as $\int d^4l/l^2$, $\int d^4l/l^3$, $\int d^4l/l^4$ the integral is said to diverge quadratically, linearly, and logarithmically, respectively. Thus, our example I diverges quadratically. Various methods have been devised to regularize divergent integrals. We will make use of *dimensional* regularization, because it preserves algebraic relations between Green functions (Ward identities) if the underlying symmetries do not depend on the number of dimensions of space-time.

In dimensional regularization, we generalize the integral from 4 to n dimensions and introduce polar coordinates

$$\begin{aligned}
l_1 &= l\cos(\theta_1),\\
l_2 &= l\sin(\theta_1)\cos(\theta_2),\\
l_3 &= l\sin(\theta_1)\sin(\theta_2)\cos(\theta_3),\\
&\vdots\\
l_{n-1} &= l\sin(\theta_1)\sin(\theta_2)\cdots\cos(\theta_{n-1}),\\
l_n &= l\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-1}),
\end{aligned} \tag{B.4}$$

where $0 \le l$, $\theta_i \in [0,\pi], i = 1,\cdots,n-2$, $\theta_{n-1} \in [0,2\pi]$. A general integral is then symbolically of the form

$$\int d^nl\cdots = \int_0^{\infty} l^{n-1}dl \int_0^{2\pi} d\theta_{n-1}\int_0^{\pi} d\theta_{n-2}\sin(\theta_{n-2})\cdots\int_0^{\pi} d\theta_1 \sin^{n-2}(\theta_1)\cdots. \tag{B.5}$$

If the integrand does not depend on the angles, the angular integration can explicitly be carried out. To that end one makes use of

$$\int_0^\pi \sin^m(\theta)d\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

which can be shown by induction. We then obtain for the angular integration

$$\begin{aligned}\int_0^{2\pi} d\theta_{n-1}\cdots\int_0^\pi d\theta_1 \sin^{n-2}(\theta_1) &= 2\pi\underbrace{\frac{\sqrt{\pi}\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}\frac{\sqrt{\pi}\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\cdots\frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}}_{(n-2)\text{ factors}}\\ &= 2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \end{aligned}\tag{B.6}$$

We define the integral for n dimensions (n integer) as

$$I_n(M^2,\mu^2) = \mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{i}{k^2-M^2+i0^+},\tag{B.7}$$

where for convenience we have introduced the renormalization scale μ so that the integral has the same dimension for arbitrary n. (The integral of Eq. (B.7) is convergent only for $n=1$.) After the Wick rotation of Eq. (B.3) and the angular integration of Eq. (B.6) the integral formally reads

$$I_n(M^2,\mu^2) = \mu^{4-n}2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{(2\pi)^n}\int_0^\infty dl\frac{l^{n-1}}{l^2+M^2}.$$

For later use, we investigate the (more general) integral

$$\int_0^\infty\frac{l^{n-1}dl}{(l^2+M^2)^\alpha} = \frac{1}{(M^2)^\alpha}\int_0^\infty\frac{l^{n-1}dl}{(\frac{l^2}{M^2}+1)^\alpha} = \frac{1}{2}(M^2)^{\frac{n}{2}-\alpha}\int_0^\infty\frac{t^{\frac{n}{2}-1}dt}{(t+1)^\alpha},\tag{B.8}$$

where we made use of the substitution $t\equiv l^2/M^2$. We then make use of the Beta function

$$B(x,y) = \int_0^\infty\frac{t^{x-1}dt}{(1+t)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\tag{B.9}$$

where the *integral* converges for $x>0$, $y>0$ and diverges if $x\le 0$ or $y\le 0$. For non-positive values of x or y we make use of the analytic continuation in terms of the Gamma function to define the Beta function and thus the integral of Eq. (B.8).[1] Putting $x = n/2$, $x + y = \alpha$ and $y = \alpha - n/2$ our (intermediate) integral reads

$$\int_0^\infty \frac{l^{n-1}dl}{(l^2 + M^2)^\alpha} = \frac{1}{2}(M^2)^{\frac{n}{2}-\alpha}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} \tag{B.10}$$

which, for $\alpha = 1$, yields for our original integral

$$\begin{aligned} I_n(M^2, \mu^2) &= \mu^{4-n} \underbrace{2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}}_{\text{angular integration}} \frac{1}{(2\pi)^n}\frac{1}{2}(M^2)^{\frac{n}{2}-1}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(1 - \frac{n}{2}\right)}{\underbrace{\Gamma(1)}_{1}} \\ &= \frac{\mu^{4-n}}{(4\pi)^{\frac{n}{2}}}(M^2)^{\frac{n}{2}-1}\Gamma\left(1 - \frac{n}{2}\right). \end{aligned} \tag{B.11}$$

Since $\Gamma(z)$ is an analytic function in the complex plane except for poles of first order in $0, -1, -2, \cdots$, and $a^z = \exp[\ln(a)z]$, $a \in R^+$ is an analytic function in C, the right-hand side of Eq. (B.11) can be thought of as a function of a *complex* variable n which is analytic in C except for poles of first order for $n = 2, 4, 6, \cdots$. Making use of

$$\mu^{4-n} = (\mu^2)^{2-\frac{n}{2}}, \quad (M^2)^{\frac{n}{2}-1} = M^2(M^2)^{\frac{n}{2}-2}, \quad (4\pi)^{\frac{n}{2}} = (4\pi)^2(4\pi)^{\frac{n}{2}-2},$$

we define (for complex n)

$$I(M^2, \mu^2, n) = \frac{M^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{2-\frac{n}{2}}\Gamma\left(1 - \frac{n}{2}\right).$$

Of course, for $n \to 4$ the Gamma function has a pole and we want to investigate how this pole is approached. The property $\Gamma(z + 1) = z\Gamma(z)$ allows one to rewrite

$$\Gamma\left(1 - \frac{n}{2}\right) = \frac{\Gamma\left(1 - \frac{n}{2} + 1\right)}{1 - \frac{n}{2}} = \frac{\Gamma\left(2 - \frac{n}{2} + 1\right)}{\left(1 - \frac{n}{2}\right)\left(2 - \frac{n}{2}\right)} = \frac{\Gamma\left(1 + \frac{\epsilon}{2}\right)}{(-1)\left(1 - \frac{\epsilon}{2}\right)\frac{\epsilon}{2}},$$

[1] Recall that $\Gamma(z)$ is single valued and analytic over the entire complex plane, save for the points $z = -n$, $n = 0, 1, 2, \cdots$, where it possesses simple poles with residue $(-1)^n/n!$ [AS 72].

where we defined $\epsilon \equiv 4-n$. Making use use of $a^x = \exp[\ln(a)x] = 1 + \ln(a)x + O(x^2)$ we expand the integral for small ϵ

$$\begin{aligned}
I(M^2,\mu^2,n) &= \frac{M^2}{16\pi^2}\left[1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{M^2}\right)+O(\epsilon^2)\right] \\
&\quad \times\left(-\frac{2}{\epsilon}\right)\left[1+\frac{\epsilon}{2}+O(\epsilon^2)\right]\left[\underbrace{\Gamma(1)}_{1}+\frac{\epsilon}{2}\Gamma'(1)+O(\epsilon^2)\right] \\
&= \frac{M^2}{16\pi^2}\left[-\frac{2}{\epsilon}\quad\underbrace{-\Gamma'(1)}_{\gamma_E=0.5772\cdots}\quad -1-\ln(4\pi)+\ln\left(\frac{M^2}{\mu^2}\right)+O(\epsilon)\right],
\end{aligned}$$

where γ_E is Euler's constant. We finally obtain

$$I(M^2,\mu^2,n) = \frac{M^2}{16\pi^2}\left[R+\ln\left(\frac{M^2}{\mu^2}\right)\right]+O(n-4), \tag{B.12}$$

where

$$R = \frac{2}{n-4}-[\ln(4\pi)+\Gamma'(1)+1]. \tag{B.13}$$

Using the same techniques one can easily derive a very useful expression for the more general integral

$$\begin{aligned}
&\int \frac{d^nk}{(2\pi)^n}\frac{(k^2)^p}{(k^2-M^2+i0^+)^q} = \\
&\quad i(-)^{p-q}\frac{1}{(4\pi)^{\frac{n}{2}}}(M^2)^{p+\frac{n}{2}-q}\frac{\Gamma\left(p+\frac{n}{2}\right)\Gamma\left(q-p-\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(q)}.
\end{aligned} \tag{B.14}$$

We first assume $M^2>0$, $p=0,1,\cdots$, $q=1,2,\cdots$, and $p<q$. The last condition is used in the Wick rotation to guarantee that the quarter circles at infinity do not contribute to the integral. The transition to the Euclidian metric produces the factor $i(-)^{p-q}$. The angular integral in n dimensions is then performed as in Eq. (B.6). The remaining radial integration is done using Eq. (B.8) with the substitution $n-1\to 2p+n-1$ and $\alpha\to q$. The analytic continuation of the right-hand side of Eq. (B.14) is used to also define expressions with (integer) $q\leq p$ in dimensional regularization.

In the context of combining propagators by using Feynman's trick one encounters integrals of the type of Eq. (B.14) with M^2 replaced by $A-i0^+$, where A is a real number. In this context it is important to consistently deal with the boundary condition $-i0^+$ [Vel 94]. For example, let us consider a term of the type $\ln(A - i0^+)$. To that end one expresses a complex number z in its polar form $z = |z|\exp(i\varphi)$, where the argument φ of z is uniquely determined if, in addition, we demand $-\pi \leq \varphi < \pi$. For $A > 0$ one simply has $\ln(A - i0^+) = \ln(A)$. For $A < 0$ the infinitesimal imaginary part indicates that $-|A|$ is reached in the third quadrant from below the real axis so that we have to use the $-\pi$. We then make use of $\ln(ab) = \ln(a) + \ln(b)$ and obtain

$$\ln(A - i0^+) = \ln(|A|) + \ln(e^{-i\pi}) = \ln(|A|) - i\pi, \quad A < 0.$$

Both cases can be summarized in a single expression

$$\ln(A - i0^+) = \ln(|A|) - i\pi\Theta(-A) \quad \text{for } A \in R. \tag{B.15}$$

The preceding discussion is of importance for consistently determining imaginary parts of loop integrals.

Let us conclude with the general observation that (ultraviolet) divergences of one-loop integrals in dimensional regularization always show up as single poles in $\epsilon = 4 - n$.

Appendix C

Loop Integrals

In Appendix B we discussed the basic ideas of the method of dimensional regularization. Here we outline the calculation of more complicated one-loop integrals of mesonic as well as heavy-baryon chiral perturbation theory. We restrict ourselves to the cases needed to reproduce the examples discussed in the main text and refer the interested reader to Refs. [HV 72, Lei 75, HV 79, CL 84, Col 84, Vel 94] for more details.

C.1 One-Loop Integrals of the Mesonic Sector

In the mesonic sector we will use the following definition and nomenclature for the scalar loop integrals (i.e., no Lorentz indices) extended to n dimensions:

$$\begin{aligned} &I_{\pi\cdots\pi}(q_1,\cdots,q_m) \equiv \\ &\quad i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{1}{(k+q_1)^2-M_\pi^2+i0^+}\cdots\frac{1}{(k+q_m)^2-M_\pi^2+i0^+}, \end{aligned} \tag{C.1}$$

where we omit an explicit reference to the scale μ and the "number of dimensions" n.[1] In the SU(3)×SU(3) case one also needs loop integrals with different masses such as, e.g.,

$$I_{\pi K}(q_1,q_2)=i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{1}{(k+q_1)^2-M_\pi^2+i0^+}\frac{1}{(k+q_2)^2-M_K^2+i0^+}.$$

[1] If $m\geq 3$, the integral converges for $n=4$.

C.1.1 I_π

We define

$$I_\pi(q) \equiv i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k+q)^2 - M_\pi^2 + i0^+}. \tag{C.2}$$

Using a shift $k \to k - q$ (in the regularized integral) we obtain

$$I_\pi(q) = I_\pi(0).$$

However, this is just the basic integral I we discussed in detail in App. B:

$$I_\pi(0) = \frac{M_\pi^2}{16\pi^2} \left[R + \ln\left(\frac{M_\pi^2}{\mu^2}\right)\right] + O(n-4), \tag{C.3}$$

where

$$R = \frac{2}{n-4} - [\ln(4\pi) + \Gamma'(1) + 1]. \tag{C.4}$$

Later on we will also use the common notation $A_0(M_\pi^2)$ for the integral $I_\pi(0)$.

C.1.2 $I_{\pi\pi}$

In the calculation of the one-loop contribution of Fig. 4.12 to the electromagnetic form factor of the pion, Eq. (4.163), we encounter an integral of the type

$$I_{\pi\pi}(q_1, q_2) \equiv i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k+q_1)^2 - M_\pi^2 + i0^+} \frac{1}{(k+q_2)^2 - M_\pi^2 + i0^+}. \tag{C.5}$$

Using a shift $k \to k - q_2$ we obtain

$$I_{\pi\pi}(q_1, q_2) = I_{\pi\pi}(q_1 - q_2, 0).$$

It is thus sufficient to consider $I_{\pi\pi}(q, 0)$. To that end, we first combine the denominators using Feynman's trick:

$$\frac{1}{ab} = \int_0^1 dz \frac{1}{[az + b(1-z)]^2}, \tag{C.6}$$

with $a = (k+q)^2 - M_\pi^2 + i0^+$ and $b = k^2 - M_\pi^2 + i0^+$ to obtain

$$I_{\pi\pi}(q, 0) = i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \int_0^1 dz \frac{1}{[k^2 + 2k \cdot qz - (M_\pi^2 - zq^2) + i0^+]^2},$$

and perform the shift $k \to k - zq$, resulting in

$$I_{\pi\pi}(q,0) = \int_0^1 dz \left(i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{1}{\{k^2 - [M_\pi^2 + z(z-1)q^2] + i0^+\}^2} \right).$$

Performing the Wick rotation, applying the results of Eqs. (B.6) and (B.10), expanding the result in $(n-4)$, and finally performing the z integration, we obtain

$$I_{\pi\pi}(q,0) = \frac{1}{16\pi^2}\left[R + \ln\left(\frac{M_\pi^2}{\mu^2}\right) + 1 + J^{(0)}\left(\frac{q^2}{M_\pi^2}\right) + O(n-4)\right], \qquad \text{(C.7)}$$

where [Unk+ 00]

$$\begin{aligned} J^{(0)}(x) &= \int_0^1 dz \ln[1 + x(z^2 - z) - i0^+] \\ &= \begin{cases} -2 - \sigma \ln\left(\frac{\sigma-1}{\sigma+1}\right), & x < 0, \\ -2 + 2\sqrt{\frac{4}{x} - 1}\,\mathrm{arccot}\left(\sqrt{\frac{4}{x} - 1}\right), & 0 \leq x < 4, \\ -2 - \sigma \ln\left(\frac{1-\sigma}{1+\sigma}\right) - i\pi\sigma, & 4 < x, \end{cases} \end{aligned}$$

with

$$\sigma(x) = \sqrt{1 - \frac{4}{x}}, \quad x \notin [0,4].$$

Note that Eq. (C.7) represents a case where the $i0^+$ boundary condition has to be treated consistently, as discussed at the end of App. B. For later use we introduce the notation

$$B_0(q^2, M_\pi^2) = I_{\pi\pi}(q,0),$$

where the subscript 0 refers to the scalar character of the integral.

Next we want to determine the tensor integrals appearing in Eq. (4.163) by reducing them to already known integrals. The general idea consists of parameterizing the tensor structure in terms of the metric tensor and products of external four-vectors and multiplying the results by invariant functions of Lorentz scalars. We first consider

$$i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{k^\mu}{(k+q)^2 - M_\pi^2 + i0^+} \frac{1}{k^2 - M_\pi^2 + i0^+}, \qquad \text{(C.8)}$$

which must have the form

$$q^\mu B_1(q^2, M_\pi^2), \qquad \text{(C.9)}$$

where the subscript 1 refers to one four-vector k in the numerator of the integral. We contract Eq. (C.9) with q_μ and make use of $q \cdot k = [(k+q)^2 - M_\pi^2 - (k^2 - M_\pi^2) - q^2]/2$ to obtain

$$\begin{aligned}
q^2 B_1(q^2, M_\pi^2) &= \frac{1}{2} i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - M_\pi^2 + i0^+} \\
&\quad -\frac{1}{2} i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k+q)^2 - M_\pi^2 + i0^+} \\
&\quad -\frac{1}{2} q^2 i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k+q)^2 - M_\pi^2 + i0^+} \frac{1}{k^2 - M_\pi^2 + i0^+} \\
&= -\frac{1}{2} q^2 B_0(q^2, M_\pi^2),
\end{aligned}$$

where we used the argument in Appendix C.1.1 to show that the first two integrals cancel. We have thus reduced the determination of Eq. (C.8) to an already known integral:

$$B_1(q^2, M_\pi^2) = -\frac{1}{2} B_0(q^2, M_\pi^2). \tag{C.10}$$

Finally, we also need

$$\begin{aligned}
&q^\mu q^\nu B_{20}(q^2, M_\pi^2) + g^{\mu\nu} q^2 B_{21}(q^2, M_\pi^2) = \\
&\quad i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{k^\mu k^\nu}{(k+q)^2 - M_\pi^2 + i0^+} \frac{1}{k^2 - M_\pi^2 + i0^+},
\end{aligned} \tag{C.11}$$

where the first subscript 2 refers to two four-vectors k in the numerator of the integral and the second subscripts 0 and 1 refer to the number of metric tensors in the parameterization, respectively.

Contracting with $g_{\mu\nu}$ and making use of $g_{\mu\nu} g^{\mu\nu} = n$ in n dimensions we obtain

$$q^2 B_{20} + n q^2 B_{21} = A_0 + M_\pi^2 B_0. \tag{C.12}$$

Similarly, contracting with q_μ we obtain

$$q^2 B_{20} + q^2 B_{21} = \frac{1}{2} A_0 + \frac{q^2}{4} B_0. \tag{C.13}$$

By subtracting Eq. (C.13) from (C.12) we find

$$q^2 B_{21} = \frac{1}{n-1} \left[\frac{1}{2} A_0 + (M_\pi^2 - \frac{q^2}{4}) B_0 \right]$$

$$= \frac{1}{32\pi^2}\left\{\left(M_\pi^2-\frac{q^2}{6}\right)\left[R+\ln\left(\frac{M_\pi^2}{\mu^2}\right)\right]\right.$$
$$\left.+\frac{2}{3}J^{(0)}\left(\frac{q^2}{M_\pi^2}\right)\left(M_\pi^2-\frac{q^2}{4}\right)-\frac{q^2}{18}\right\}+O(n-4), \quad \text{(C.14)}$$

where we made use of

$$\frac{1}{n-1}=\frac{1}{3+(n-4)}=\frac{1}{3}\left(1-\frac{n-4}{3}+\cdots\right)$$

and Eqs. (C.3) and (C.7). From Eq. (C.13) we obtain

$$\begin{aligned} B_{20} &= \frac{1}{q^2(n-1)}\left(\frac{n-2}{2}A_0+n\frac{q^2}{4}B_0-M_\pi^2 B_0\right)\\ &= \frac{1}{3q^2}\left[A_0+\frac{6M_\pi^2-q^2}{96\pi^2}+(q^2-M_\pi^2)B_0\right]+O(n-4)\\ &= \frac{1}{48\pi^2}\left[R+\ln\left(\frac{M_\pi^2}{\mu^2}\right)+\frac{5}{6}+\left(1-\frac{M_\pi^2}{q^2}\right)J^{(0)}\left(\frac{q^2}{M_\pi^2}\right)\right]+O(n-4). \end{aligned} \quad \text{(C.15)}$$

In working out Eqs. (C.14) or (C.15) it must be remembered that R contains a term $2/(n-4)$ which, when multiplied by $(n-4)$, gives a term of order $(n-4)^0$. This must be done carefully to obtain, e.g., the $-q^2/18$ term in Eq. (C.14) and the $5/6$ term in Eq. (C.15).

C.2 One-Loop Integrals of the Heavy-Baryon Sector

C.2.1 Basic Loop Integral

The structure of the one-loop integrals in the heavy-baryon approach is slightly different from the integrals of the mesonic sector discussed in Sec. C.1. Here we will outline the calculation of a basic loop integral which serves as a starting point for more complicated calculations.

Consider an integral of the type

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{v\cdot k+\alpha+i0^+}\frac{1}{k^2-A+i0^+}, \quad \text{(C.16)}$$

where α and A are arbitrary real numbers and v is the four-velocity of the heavy-baryon approach. Counting powers of the momenta, the integral is linearly divergent. An integral of this type appears in the calculation of the nucleon self energy in Sec. 5.5.9. We combine the denominators using the following Feynman trick:

$$\frac{1}{ab} = 2\int_0^\infty dy \frac{1}{(2ya+b)^2}. \tag{C.17}$$

Below, this choice will allow us most easily to combine the y integration with the "radial" integration of the loop momentum after the Wick rotation. Inserting $a = v\cdot k + \alpha + i0^+$ and $b = k^2 - A + i0^+$, we obtain

$$2\int \frac{d^4k}{(2\pi)^4}\int_0^\infty dy \frac{1}{(k^2 + 2yv\cdot k + 2y\alpha - A + i0^+)^2}. \tag{C.18}$$

We now generalize to n dimensions:

$$2\mu^{4-n}\int_0^\infty dy \int \frac{d^nk}{(2\pi)^n}\frac{1}{(k^2 + 2yv\cdot k + 2y\alpha - A + i0^+)^2} \tag{C.19}$$

and perform a shift of integration variables $k \to k - yv$ so that there remain no terms linear in k in the denominator:

$$2\mu^{4-n}\int_0^\infty dy \int \frac{d^nk}{(2\pi)^n}\frac{1}{(k^2 - y^2 - A + 2y\alpha + i0^+)^2}, \tag{C.20}$$

where we made use of $v^2 = 1$. Finally, we shift the integration variable $y \to y + \alpha$ in order to eliminate terms linear in y in the denominator:

$$2\mu^{4-n}\int_{-\alpha}^\infty dy \int \frac{d^nk}{(2\pi)^n}\frac{1}{(k^2 - y^2 - A + \alpha^2 + i0^+)^2}. \tag{C.21}$$

The y integration is split into $[-\alpha, 0]$ and $[0, \infty[$. Making use of a Wick rotation and Eqs. (B.6), (B.10), and (B.15) we obtain for the first integral

$$\begin{aligned} & 2\mu^{4-n}\int_{-\alpha}^0 dy \int \frac{d^nk}{(2\pi)^n}\frac{1}{(k^2 - y^2 - A + \alpha^2 + i0^+)^2} \\ = & -\frac{i}{8\pi^2}\int_{-\alpha}^0 dy \left[R + 1 + \ln\left(\frac{|A + y^2 - \alpha^2|}{\mu^2}\right) - i\pi\Theta(\alpha^2 - A - y^2)\right] \\ & +O(n-4), \end{aligned} \tag{C.22}$$

where

$$R = \frac{2}{n-4} - [\ln(4\pi) + \Gamma'(1) + 1].$$

Performing a Wick rotation and using Eq. (B.6), the second integral reads

$$\begin{aligned} & 2\mu^{4-n} \int_0^\infty dy \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - y^2 - A + \alpha^2 + i0^+)^2} \\ = & \frac{4i\mu^{4-n}}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int_0^\infty dy \int_0^\infty dx \frac{x^{n-1}}{(x^2 + y^2 + A - \alpha^2 - i0^+)^2}. \end{aligned} \tag{C.23}$$

Using polar coordinates $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$ together with

$$\int_0^{\frac{\pi}{2}} \sin^{2\alpha+1}(\varphi)\cos^{2\beta+1}(\varphi)d\varphi = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+2)},$$

we rewrite the second integral as ($\Gamma(1/2) = \sqrt{\pi}$)

$$\frac{2i\mu^{4-n}\sqrt{\pi}}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)} \int_0^\infty dr \frac{r^n}{(r^2 + A - \alpha^2 - i0^+)^2}.$$

Finally, applying again Eq. (B.10) for the radial integral we obtain for the second integral (in four dimensions)

$$2\int_0^\infty dy \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - y^2 - A + \alpha^2 + i0^+)^2} = \frac{-i}{8\pi}\sqrt{A - \alpha^2 - i0^+}, \tag{C.24}$$

where we made use of $\Gamma(2) = 1$ and $\Gamma(-1/2) = -2\sqrt{\pi}$.

The remaining y integration of the first integral is elementary and we obtain as the final expression

$$\begin{aligned} & \mu^{4-n} \int \frac{d^n k}{(2\pi)^4} \frac{1}{v\cdot k + \alpha + i0^+} \frac{1}{k^2 - A + i0^+} \\ = & \frac{-i}{8\pi^2}\Bigg(\alpha\left[R + \ln\left(\frac{|A|}{\mu^2}\right) - 1\right] \\ & + \left\{\begin{array}{ll} 2\sqrt{\alpha^2 - A}\,\mathrm{arccosh}\left(\frac{\alpha}{\sqrt{A}}\right) & \\ -2\pi i\sqrt{\alpha^2 - A}, & A > 0 \wedge \alpha > \sqrt{A}, \\ 2\sqrt{A - \alpha^2}\,\arccos\left(-\frac{\alpha}{\sqrt{A}}\right), & A > \alpha^2, \\ -2\sqrt{\alpha^2 - A}\,\mathrm{arccosh}\left(-\frac{\alpha}{\sqrt{A}}\right), & A > 0 \wedge \alpha < -\sqrt{A}, \\ 2\sqrt{\alpha^2 - A}\,\mathrm{arcsinh}\left(\frac{\alpha}{\sqrt{-A}}\right) & \\ -i\pi\left(\alpha + \sqrt{\alpha^2 - A}\right), & A < 0, \end{array}\right\} \end{aligned}$$

$$+O(n-4)\Bigg), \tag{C.25}$$

with $R = \frac{2}{n-4} - [\ln(4\pi) + \Gamma'(1) + 1]$.

C.2.2 $J_{\pi N}$

In analogy to the mesonic case we define

$$J_{\pi N}(q;\omega) \equiv i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k+q)^2 - M_\pi^2 + i0^+} \frac{1}{v\cdot k + \omega + i0^+}. \tag{C.26}$$

Using a shift $k \to k - q$ we obtain

$$J_{\pi N}(q;\omega) = J_{\pi N}(0;\omega - v\cdot q).$$

It is thus sufficient to consider $J_{\pi N}(0;\omega)$ which, using the result of Eq. (C.25), is given by[2]

$$\begin{aligned}
&J_{\pi N}(0;\omega) \\
&= \frac{\omega}{8\pi^2}\left[R + \ln\left(\frac{M_\pi^2}{\mu^2}\right) - 1\right] \\
&\quad + \frac{1}{8\pi^2}\left\{\begin{array}{ll}
2\sqrt{\omega^2 - M_\pi^2}\,\mathrm{arccosh}\left(\frac{\omega}{M_\pi}\right) - 2\pi i\sqrt{\omega^2 - M_\pi^2}, & \omega > M_\pi, \\
2\sqrt{M_\pi^2 - \omega^2}\;\arccos\left(-\frac{\omega}{M_\pi}\right), & \omega^2 < M_\pi^2, \\
-2\sqrt{\omega^2 - M_\pi^2}\,\mathrm{arccosh}\left(-\frac{\omega}{M_\pi}\right), & \omega < -M_\pi,
\end{array}\right\} \\
&\quad + O(n-4).
\end{aligned} \tag{C.27}$$

In the calculation of the nucleon self energy we also need tensor integrals which, as in the mesonic case, one may reduce to already known integrals. Let us introduce the notation

$$C_0(\omega, M_\pi^2) = J_{\pi N}(0;\omega),$$

where the subscript 0 refers to the scalar character of the integral. Once again, the general idea in the determination of tensor integrals consists of

[2] Our $J_{\pi N}(0;\omega)$ corresponds to $-\mu^{4-n} J_0(\omega)$ of Ref. [Ber+ 95b].

parameterizing the tensor structure in terms of the metric tensor and products of the four-velocity v. We first consider

$$i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{k^\mu}{k^2-M_\pi^2+i0^+}\frac{1}{v\cdot k+\omega+i0^+}, \tag{C.28}$$

which must have the form

$$v^\mu C_1(\omega,M_\pi^2), \tag{C.29}$$

where the subscript 1 refers to one four-vector k in the numerator of the integral. We contract Eq. (C.29) with v_μ, make use of $v^2=1$, and add and subtract ω in the numerator of the integral, obtaining

$$\begin{aligned}C_1(\omega,M_\pi^2) &= i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{1}{k^2-M_\pi^2+i0^+}\\ &\quad -\omega i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{1}{k^2-M_\pi^2+i0^+}\frac{1}{v\cdot k+\omega+i0^+}\\ &= I_\pi(0)-\omega J_{\pi N}(0;\omega),\end{aligned} \tag{C.30}$$

where $I_\pi(0)$ is defined in Eq. (C.3). We have thus reduced the determination of Eq. (C.28) to the already known integrals I_π and $J_{\pi N}$. As our final example, let us discuss

$$i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{k^\mu k^\nu}{k^2-M_\pi^2+i0^+}\frac{1}{v\cdot k+\omega+i0^+},$$

which must be of the form

$$v^\mu v^\nu C_{20}(\omega,M_\pi^2)+g^{\mu\nu}C_{21}(\omega,M_\pi^2), \tag{C.31}$$

where the first subscript 2 refers to two four-vectors k in the numerator of the integral and the second subscripts 0 and 1 refer to the number of metric tensors in the parameterization, respectively. Contracting with v_μ and adding and subtracting ω in the numerator, we obtain

$$v^\nu C_{20}+v^\nu C_{21}=-\omega v^\nu C_1=-\omega v^\nu[I_\pi(0)-\omega J_{\pi N}(0;\omega)], \tag{C.32}$$

where we made use of Eq. (C.30) and

$$i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{k^\nu}{k^2-M_\pi^2+i0^+}=0.$$

Finally, contracting with $g_{\mu\nu}$ and making use of $g_{\mu\nu}g^{\mu\nu}=n$ in n dimensions we obtain

$$C_{20}+nC_{21}=M_\pi^2 C_0, \tag{C.33}$$

where we made use of

$$i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{1}{v\cdot k+\omega+i0^+}=0 \tag{C.34}$$

in dimensional regularization. In order to verify Eq. (C.34), one writes

$$\begin{aligned}
i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{1}{v\cdot k+\omega+i0^+} &= i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{v\cdot k}{(v\cdot k)^2-(\omega+i0^+)^2}\\
&\quad -\omega i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{1}{(v\cdot k)^2-(\omega+i0^+)^2}.
\end{aligned}$$

The first term vanishes, because the integrand is odd in k. For the evaluation of the second term we choose $v^\mu=(1,0,\cdots,0)$. Applying the residue theorem, we obtain for the integral

$$\int_{-\infty}^{\infty}dk_0\frac{1}{k_0^2-(\omega+i0^+)^2}=\frac{\pi i}{\omega},\quad \omega\neq 0,$$

so that we may define for arbitrary ω

$$\begin{aligned}
i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{1}{v\cdot k+\omega+i0^+} &= -i\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\frac{\omega}{k_0^2-(\omega+i0^+)^2}\\
&= \frac{\pi}{(2\pi)^n}\mu^{4-n}\int d^{n-1}k.
\end{aligned}$$

However, the last term vanishes in dimensional regularization.

From Eqs. (C.32) and (C.33) we obtain

$$\begin{aligned}
C_{21} &= \frac{1}{n-1}\left[(M_\pi^2-\omega^2)J_{\pi N}(0;\omega)+\omega I_\pi(0)\right]\\
&= \frac{1}{3}\left[(M_\pi^2-\omega^2)J_{\pi N}(0;\omega)+\omega I_\pi(0)\right]-\frac{\omega}{12\pi^2}\left(\frac{M_\pi^2}{2}-\frac{\omega^2}{3}\right),
\end{aligned} \tag{C.35}$$

and

$$\begin{aligned}
C_{20} &= M_\pi^2 J_{\pi N}(0,\omega)-C_{21}\\
&= \frac{1}{3}(2M_\pi^2+\omega^2)J_{\pi N}(0;\omega)-\frac{1}{3}\omega I_\pi(0)+\frac{\omega}{12\pi^2}\left(\frac{M_\pi^2}{2}-\frac{\omega^2}{3}\right),
\end{aligned} \tag{C.36}$$

where, as in the mesonic case, it is important to identify the finite terms resulting from the product of $O(n-4)$ terms and R.

Finally, when discussing the relation between the infrared regularization method of Ref. [BL 99] and the heavy-baryon approach in Sec. 5.6, we made use of the fact that an integral of the type

$$\int d^n k \frac{(k^2)^p}{(v\cdot k + i0^+)^q} \tag{C.37}$$

vanishes in dimensional regularization. This can be seen by substituting $k = \lambda k'$ and relabeling $k' = k$

$$= \lambda^{n+2p-q} \int d^n k \frac{(k^2)^p}{(v\cdot k + i0^+)^q}.$$

Since $\lambda > 0$ is arbitrary and, for fixed p and q, the result is to hold for arbitrary n, Eq. (C.37) is set to zero in dimensional regularization. We emphasize that the vanishing of Eq. (C.37) has the character of a prescription [IZ 80]. The integral does not depend on any scale and its analytic continuation is ill defined in the sense that there is no dimension n where it is meaningful. It is ultraviolet divergent for $n+2p-q \geq 0$ and infrared divergent for $n+2p-q \leq 0$.

Appendix D

Different Forms of $\mathcal{L}_4$ in SU(2) × SU(2)

D.1 GL Versus GSS

The purpose of this appendix is to explicitly relate the two commonly used $\mathcal{O}(p^4)$ SU(2)×SU(2) Lagrangians of Refs. [GL 84] (GL) and [Gas+ 88] (GSS).

In their pioneering work on mesonic SU(2) × SU(2) chiral perturbation theory [GL 84], Gasser and Leutwyler used a notation adopted from the O(4) nonlinear σ model, because the two Lie groups SU(2) × SU(2) and O(4) are locally isomorphic, i.e., their Lie algebras are isomorphic. The effective Lagrangian was written in terms of invariant scalar products of real four-vectors in contrast to the nowadays standard trace form. The dynamical pion degrees of freedom were expressed in terms of a four-component real vector field of unit length with components $U^A(x)$, $A = 0, 1, 2, 3$. The connection to the SU(2) matrix $U(x)$ of Sec. 4.2.2 can be expressed as

$$\begin{aligned} U(x) &= U^0(x) + i\vec{\tau}\cdot\vec{U}(x), \\ U^0(x) &= \frac{1}{2}\mathrm{Tr}[U(x)], \\ U^i(x) &= -\frac{i}{2}\mathrm{Tr}[\tau_i U(x)], \quad i = 1, 2, 3, \end{aligned} \tag{D.1}$$

with

$$\sum_{A=0}^{3}[U^A(x)]^2 = 1,$$

so that U is unitary. The lowest-order Lagrangian in the trace notation is given by the SU(2) $\times$ SU(2) version of Eq. (4.70), and the transcription of the $\mathcal{O}(p^4)$ SU(2) $\times$ SU(2) Lagrangian [see Eq. (5.5) of Ref. [GL 84]] reads[1]

$$\begin{aligned}
\mathcal{L}_4^{\rm GL} &= \frac{l_1}{4}\left\{\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\right\}^2 + \frac{l_2}{4}\mathrm{Tr}[D_\mu U(D_\nu U)^\dagger]\mathrm{Tr}[D^\mu U(D^\nu U)^\dagger] \\
&\quad + \frac{l_3}{16}\left[\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)\right]^2 + \frac{l_4}{4}\mathrm{Tr}[D_\mu U(D^\mu \chi)^\dagger + D_\mu \chi(D^\mu U)^\dagger] \\
&\quad + l_5\left[\mathrm{Tr}(f_{\mu\nu}^R U f_L^{\mu\nu} U^\dagger) - \frac{1}{2}\mathrm{Tr}(f_{\mu\nu}^L f_L^{\mu\nu} + f_{\mu\nu}^R f_R^{\mu\nu})\right] \\
&\quad + i\frac{l_6}{2}\mathrm{Tr}[f_{\mu\nu}^R D^\mu U(D^\nu U)^\dagger + f_{\mu\nu}^L (D^\mu U)^\dagger D^\nu U] \\
&\quad - \frac{l_7}{16}\left[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)\right]^2 \\
&\quad + \frac{h_1+h_3}{4}\mathrm{Tr}(\chi\chi^\dagger) + \frac{h_1-h_3}{16}\left\{\left[\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)\right]^2\right. \\
&\quad \left. + \left[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)\right]^2 - 2\mathrm{Tr}(\chi U^\dagger \chi U^\dagger + U\chi^\dagger U\chi^\dagger)\right\} \\
&\quad - 2h_2 \mathrm{Tr}(f_{\mu\nu}^L f_L^{\mu\nu} + f_{\mu\nu}^R f_R^{\mu\nu}). \qquad \text{(D.2)}
\end{aligned}$$

When comparing with the SU(3)$\times$SU(3) version of Eq. (4.104) we first note that Eq. (D.2) contains fewer independent terms which results from the application of the trace relations, as discussed in Sec. 4.10.1. The bare and the renormalized low-energy constants l_i and l_i^r are related by

$$l_i = l_i^r + \gamma_i \frac{R}{32\pi^2}, \qquad \text{(D.3)}$$

where $R = 2/(n-4) - [\ln(4\pi) + \Gamma'(1) + 1]$ and

$$\gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2, \quad \gamma_5 = -\frac{1}{6}, \quad \gamma_6 = -\frac{1}{3}, \quad \gamma_7 = 0.$$

In the SU(2)$\times$SU(2) sector one often uses the scale-independent parameters $\bar{l}_i$ which are defined by

$$l_i^r = \frac{\gamma_i}{32\pi^2}\left[\bar{l}_i + \ln\left(\frac{M^2}{\mu^2}\right)\right], \quad i = 1, \cdots, 6, \qquad \text{(D.4)}$$

[1]Note that Gasser and Leutwyler called the $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$ Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

where $M^2 = B_0(m_u + m_d)$. Since $\ln(1) = 0$, the $\bar{l}_i$ are proportional to the renormalized coupling constant at the scale $\mu = M$. Table D.1 contains numerical values for the scale-independent low-energy coupling constants $\bar{l}_i$ as obtained in Ref. [GL 84] together with more recent determinations.

Table D.1: Scale-independent low-energy coupling constants $\bar{l}_i$.

	Value	Obtained from	γ_i
$\bar{l}_1$	-2.3 ± 3.7 [GL 84] -1.7 ± 1.0 [Bij+ 94] -1.5 [Bij+ 97] -1.8 [Col+ 01b] -0.4 ± 0.6 [Col+ 01b]	$\pi\pi$ D-wave scattering lengths $\mathcal{O}(p^4)$ $\pi\pi$ and K_{l4} $\pi\pi$ D-wave scattering lengths $\mathcal{O}(p^6)$ $\pi\pi$ scattering $\mathcal{O}(p^4)$ + Roy equations $\pi\pi$ scattering $\mathcal{O}(p^6)$ + Roy equations	$\frac{1}{3}$
$\bar{l}_2$	6.0 ± 1.3 [GL 84] 6.1 ± 0.5 [Bij+ 94] 4.5 [Bij+ 97] 5.4 [Col+ 01b] 4.3 ± 0.1 [Col+ 01b]	$\pi\pi$ D-wave scattering lengths $\mathcal{O}(p^4)$ $\pi\pi$ and K_{l4} $\pi\pi$ D-wave scattering lengths $\mathcal{O}(p^6)$ $\pi\pi$ scattering $\mathcal{O}(p^4)$ + Roy equations $\pi\pi$ scattering $\mathcal{O}(p^6)$ + Roy equations	$\frac{2}{3}$
$\bar{l}_3$	2.9 ± 2.4 [GL 84] $\lvert\bar{l}_3\rvert \leq 16$ [Col+ 01a]	SU(3) mass formulae K_{l4} decay	$-\frac{1}{2}$
$\bar{l}_4$	4.3 ± 0.9 [GL 84] 4.4 ± 0.3 [Bij+ 98] 4.4 ± 0.2 [Col+ 01b]	F_K/F_π scalar form factor $\mathcal{O}(p^6)$ $\pi\pi$ scattering $\mathcal{O}(p^6)$ + Roy equations	2
$\bar{l}_5$	13.9 ± 1.3 [GL 84] 13.0 ± 0.9 [Bij+ 98]	$\pi \to e\nu\gamma$ $\mathcal{O}(p^4)$ $\pi \to e\nu\gamma$ $\mathcal{O}(p^6)$ [BT 97]	$-\frac{1}{6}$
$\bar{l}_6$	16.5 ± 1.1 [GL 84] $16.0 \pm 0.5 \pm 0.7$ [Bij+ 98]	$\langle r^2\rangle_\pi$ $\mathcal{O}(p^4)$ vector form factor $\mathcal{O}(p^6)$	$-\frac{1}{3}$
l_7	$O(5 \times 10^{-3})$ [GL 84]	π^0-η mixing	0

Secondly, the expression proportional to $(h_1 - h_3)$ can be rewritten so that the U's completely drop out, i.e., it contains only external fields. The trick is to use

$$
\begin{aligned}
&2\mathrm{Tr}(\chi U^\dagger \chi U^\dagger + U\chi^\dagger U \chi^\dagger) = \\
&\quad [\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)]^2 + [\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)]^2 \\
&\quad + [\mathrm{Tr}(\tau_i\chi)]^2 + [\mathrm{Tr}(\tau_i\chi^\dagger)]^2 - [\mathrm{Tr}(\chi)]^2 - [\mathrm{Tr}(\chi^\dagger)]^2.
\end{aligned}
\tag{D.5}
$$

The terms proportional to the l_i agree with Eq. (4.2) of Ref. [Bel+ 94] but the l_4 term is not yet in the form of the SU(3)×SU(3) version of Eq. (4.104).

By means of a total-derivative argument in combination with a field transformation as discussed in Sec. 4.7 we will transform $\mathcal{L}_4^{\rm GL}$ of Eq. (D.2) into another form which is often used in the literature. To that end let us make use of

$$\begin{aligned}&\mathrm{Tr}[D_\mu\chi(D^\mu U)^\dagger + D^\mu U(D_\mu\chi)^\dagger] =\\ &\qquad \partial_\mu \mathrm{Tr}[\chi(D^\mu U)^\dagger + D^\mu U\chi^\dagger] - \mathrm{Tr}[\chi(D^2U)^\dagger + D^2U\chi^\dagger],\end{aligned}$$

rewrite the D^2U and $(D^2U)^\dagger$ terms by using

$$\begin{aligned} 2D^2UU^\dagger &= D^2UU^\dagger - U(D^2U)^\dagger - 2D_\mu U(D^\mu U)^\dagger,\\ 2U(D^2U)^\dagger &= U(D^2U)^\dagger - D^2UU^\dagger - 2D_\mu U(D^\mu U)^\dagger,\end{aligned}$$

to obtain

$$\begin{aligned}&\mathrm{Tr}[D_\mu\chi(D^\mu U)^\dagger + D^\mu U(D_\mu\chi)^\dagger] =\\ &\qquad \text{tot. der.} + \mathrm{Tr}[D_\mu U(D^\mu U)^\dagger(\chi U^\dagger + U\chi^\dagger)]\\ &\qquad -\frac{1}{2}\mathrm{Tr}\{(\chi U^\dagger - U\chi^\dagger)[U(D^2U)^\dagger - D^2UU^\dagger]\},\end{aligned}\tag{D.6}$$

where "tot. der." refers to a total-derivative term which has no dynamical significance. We make use of a trace relation for arbitrary 2×2 matrices A_i [see Eqs. (4.192) and (4.194)],

$$\begin{aligned}&\mathrm{Tr}(A_1A_2A_3 + A_1A_3A_2) - \mathrm{Tr}(A_1)\mathrm{Tr}(A_2A_3) - \mathrm{Tr}(A_2)\mathrm{Tr}(A_3A_1)\\ &\qquad -\mathrm{Tr}(A_3)\mathrm{Tr}(A_1A_2) + \mathrm{Tr}(A_1)\mathrm{Tr}(A_2)\mathrm{Tr}(A_3) = 0,\end{aligned}\tag{D.7}$$

and $\mathrm{Tr}(D_\mu UU^\dagger) = 0$ [see Eq. (4.67)] to rewrite the first term of (D.6) as the product of two trace terms,

$$\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger(\chi U^\dagger + U\chi^\dagger)] = \frac{1}{2}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger).$$

By adding and subtracting appropriate χ terms to generate an expression proportional to the lowest-order equation of motion which, for SU(2)×SU(2), reads [see Eq. (4.77)]

$$\mathcal{O}^{(2)}_{\rm EOM}(U) = D^2UU^\dagger - U(D^2U)^\dagger - \chi U^\dagger + U\chi^\dagger + \frac{1}{2}\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger) = 0,\tag{D.8}$$

we re-express the last term of Eq. (D.6) as

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\{(\chi U^\dagger - U\chi^\dagger)[U(D^2U)^\dagger - D^2UU^\dagger]\} = \\
&\quad +\frac{1}{2}\mathrm{Tr}[(\chi U^\dagger - U\chi^\dagger)\mathcal{O}^{(2)}_{\mathrm{EOM}}(U)] \\
&\quad +\frac{1}{2}\mathrm{Tr}[(\chi U^\dagger - U\chi^\dagger)(\chi U^\dagger - U\chi^\dagger)] - \frac{1}{4}[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)]^2. \qquad \text{(D.9)}
\end{aligned}
$$

The l_4 term can thus be written as

$$
\begin{aligned}
&\mathrm{Tr}[D_\mu\chi(D^\mu U)^\dagger + D^\mu U(D_\mu\chi)^\dagger] = \\
&\quad \text{tot. der.} + \frac{1}{2}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) + \frac{1}{2}\mathrm{Tr}(\chi U^\dagger\chi U^\dagger + U\chi^\dagger U\chi^\dagger) \\
&\quad -\mathrm{Tr}(\chi\chi^\dagger) - \frac{1}{4}[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)]^2 + \frac{1}{2}\mathrm{Tr}[(\chi U^\dagger - U\chi^\dagger)\mathcal{O}^{(2)}_{\mathrm{EOM}}], \qquad \text{(D.10)}
\end{aligned}
$$

which, except for the total derivative and the equation-of-motion term, is the same as Eq. (5.9) of Gasser, Sainio, and Švarc (GSS) [Gas+ 88].

The difference between the Lagrangians of [GL 84] and [Gas+ 88] then reads

$$
\mathcal{L}_4^{\mathrm{GL}} - \mathcal{L}_4^{\mathrm{GSS}} = \frac{l_4}{4}\left\{\text{tot. der.} + \frac{1}{2}\mathrm{Tr}\left[(\chi U^\dagger - U\chi^\dagger)\mathcal{O}^{(2)}_{\mathrm{EOM}}\right]\right\}, \qquad \text{(D.11)}
$$

which agrees with Eq. (26) of Ecker and Mojžiš [EM 96] once their expressions are rewritten in the above notation. Let us also specify the field transformation required to connect the two Lagrangians. For that purpose we rewrite Eq. (D.11) in accord with Eq. (2.11) of [SF 95],

$$
\mathcal{L}_4^{\mathrm{GSS}}(U) = \mathcal{L}_4^{\mathrm{GL}}(U) + \text{tot. der.} - \frac{l_4}{8}\mathrm{Tr}[(\chi U^\dagger - U\chi^\dagger)\mathcal{O}^{(2)}_{\mathrm{EOM}}]. \qquad \text{(D.12)}
$$

According to Eqs. (4.118) and (4.119) we need to insert $\alpha_1 = 0$ and $\alpha_2 = -l_4/(2F_0^2)$ in Eq. (4.116) in order to relate the two Lagrangians.

Finally, making use of Eqs. (D.10) and (D.5) and dropping the total derivative as well as the equation-of-motion term let us explicitly write out the GSS Lagrangian:

$$
\mathcal{L}_4^{\mathrm{GSS}} = \frac{l_1}{4}\left\{\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\right\}^2 + \frac{l_2}{4}\mathrm{Tr}[D_\mu U(D_\nu U)^\dagger]\mathrm{Tr}[D^\mu U(D^\nu U)^\dagger]
$$

$$
\begin{aligned}
&+\frac{l_3+l_4}{16}\left[\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)\right]^2 + \frac{l_4}{8}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)\\
&+l_5\mathrm{Tr}(f^R_{\mu\nu}Uf^{\mu\nu}_L U^\dagger) + i\frac{l_6}{2}\mathrm{Tr}[f^R_{\mu\nu}D^\mu U(D^\nu U)^\dagger + f^L_{\mu\nu}(D^\mu U)^\dagger D^\nu U]\\
&-\frac{l_7}{16}\left[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)\right]^2 + \frac{h_1+h_3-l_4}{4}\mathrm{Tr}(\chi\chi^\dagger)\\
&+\frac{h_1-h_3-l_4}{16}\left\{\left[\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)\right]^2 + \left[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)\right]^2\right.\\
&\left.-2\mathrm{Tr}(\chi U^\dagger\chi U^\dagger + U\chi^\dagger U\chi^\dagger)\right\} - \frac{4h_2+l_5}{2}\mathrm{Tr}(f^L_{\mu\nu}f^{\mu\nu}_L + f^R_{\mu\nu}f^{\mu\nu}_R). \quad \text{(D.13)}
\end{aligned}
$$

D.2 Different Parameterizations

In App. D.1 we saw that two versions of the $\mathcal{O}(p^4)$ SU(2) $\times$ SU(2) mesonic Lagrangian, Eqs. (D.2) and (D.13), are used in the literature. Since they are related by a field transformation, they must yield the same results for physical observables [Chi 61, Kam+ 61]. Furthermore, in SU(2)$\times$SU(2) two different parameterizations of the SU(2) matrix $U(x)$ [see Eqs. (4.87) and (4.88)] are popular,

$$
U(x) = \exp\left[i\frac{\vec{\tau}\cdot\vec{\phi}(x)}{F_0}\right], \quad \text{(D.14)}
$$

$$
U(x) = \frac{1}{F_0}\left[\sigma(x) + i\vec{\tau}\cdot\vec{\pi}(x)\right], \quad \sigma(x) = \sqrt{F_0^2 - \vec{\pi}^{\,2}(x)}, \quad \text{(D.15)}
$$

where the pion fields of the two parameterizations are non-linearly related [see Eq. (4.89)].

In this appendix we collect the pion wave function renormalization constants entering a calculation at $\mathcal{O}(p^4)$ depending on which Lagrangian and parameterization is used. The actual calculation parallels that of Sec. 4.9.1 and will not be repeated here. The self energies up to $\mathcal{O}(p^4)$ can be written as

$$
\Sigma(p^2) = A + Bp^2. \quad \text{(D.16)}
$$

The renormalized mass and the wave function renormalization constant are, respectively to $\mathcal{O}(p^4)$ and $\mathcal{O}(p^2)$, given by

$$
M^2_{\pi,4} = M^2_{\pi,2}(1+B) + A, \quad \text{(D.17)}
$$

$$
Z = 1 + B, \quad \text{(D.18)}
$$

where $M_{\pi,2}^2 = 2B_0 m$ is the prediction at $\mathcal{O}(p^2)$. The different values for A, B, and Z are given in Table D.2. Note that the result for the pion mass is, as expected, independent of the Lagrangian and parameterization used:

$$M_{\pi,4}^2 = M^2\left(1+\frac{2}{3}\frac{I}{F_0^2}\right) - \frac{1}{6}\frac{M^2}{F_0^2}I + 2l_3\frac{M^4}{F_0^2} = M^2 - \frac{\bar{l}_3}{32\pi^2 F_0^2}M^4, \quad \text{(D.19)}$$

where $M^2 = 2B_0 m$ and

$$l_3 = -\frac{1}{64\pi^2}\left[\bar{l}_3 + \ln\left(\frac{M^2}{\mu^2}\right) + R\right]$$

[see Eqs. (D.3) and (D.4)]. On the other hand, the constants A, B, and Z are auxiliary mathematical quantities and thus depend on both Lagrangian and parameterization.

Table D.2: Self-energy coefficients and wave function renormalization constants for the Lagrangians of Eqs. (D.2) (GL) and (D.13) (GSS) and the field parameterizations of Eqs. (D.14) and (D.15). I denotes the dimensionally regularized integral of Eq. (B.12), $I = I(M^2,\mu^2,n) = \frac{M^2}{16\pi^2}\left[R + \ln\left(\frac{M^2}{\mu^2}\right)\right] + O(n-4)$, $R = \frac{2}{n-4} - [\ln(4\pi) + \Gamma'(1) + 1]$, $M^2 = 2B_0 m$.

	A	B	Z
GL, Eq. (D.14)	$-\frac{1}{6}\frac{M^2}{F_0^2}I + 2l_3\frac{M^4}{F_0^2}$	$\frac{2}{3}\frac{I}{F_0^2}$	$1+\frac{2}{3}\frac{I}{F_0^2}$
GL, Eq. (D.15)	$\frac{3}{2}\frac{M^2}{F_0^2}I + 2l_3\frac{M^4}{F_0^2}$	$-\frac{I}{F_0^2}$	$1-\frac{I}{F_0^2}$
GSS, Eq. (D.14)	$-\frac{1}{6}\frac{M^2}{F_0^2}I + 2(l_3+l_4)\frac{M^4}{F_0^2}$	$\frac{2}{3}\frac{I}{F_0^2} - 2l_4\frac{M^2}{F_0^2}$	$1+\frac{2}{3}\frac{I}{F_0^2} - 2l_4\frac{M^2}{F_0^2}$
GSS, Eq. (D.15)	$\frac{3}{2}\frac{M^2}{F_0^2}I + 2(l_3+l_4)\frac{M^4}{F_0^2}$	$-\frac{I}{F_0^2} - 2l_4\frac{M^2}{F_0^2}$	$1-\frac{I}{F_0^2} - 2l_4\frac{M^2}{F_0^2}$

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