A GENERALIZATION OF THE DIRAC EQUATIONS

BY EDWARD N. LORENZ

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

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If $x_1, \ldots, x_4$ are rectangular coördinates in a Euclidean four dimensional space, the Dirac equations can be written

$$\frac{1}{\lambda} \sum_{m=1}^{4} A_m \frac{\partial \psi\cdot}{\partial x_m} - \psi\cdot = 0, \tag{1}$$

where $\lambda = 2\pi i/h$, $\psi\cdot = (\psi_{ij}) = (\psi_i)$ is a square matrix with four identical columns, and $A_m = (a_{ij}^{(m)})$, $m = 1, \ldots, 4$ are any particular four-by-four matrices such that

$$A_m A_n + A_n A_m = 2\delta_{mn} I. \tag{2}$$

It is known that if $\{A_m\}$ and $\{A'_m\}$ are any two sets of matrices satisfying (2), then there exists a matrix, C, unique except for a scalar factor, such that $A'_m = CA_mC^{-1}$ for each m. In this work we shall find it convenient to choose

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

as fixed values for A_1, A_2, A_3, A_4, respectively.

We now introduce a fifth matrix $A_0 = (a_{ij}^{(0)}) = iI$, which is equal to $-i$ times the coefficient of $\psi\cdot$ in (1), and let

$$A^* = (a_{ij}^*) = \sum_{m=0}^{4} A_m y_m,$$

where $y_0, \ldots, y_4$ form a set of independent variables. Then

(I). $|A^*| = Q^2 = (y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2)^2$,

(II). The cofactor of a_{ij}^* in $|A^*|$ is equal to A_{ij}^*Q, for some linear function A_{ij}^* of $y_0, \ldots, y_4$,

for all sets $\{A_m\}$ satisfying (2).[1]

It is easily verified that if $A_{ij}^* = \Sigma A_{ij}^{(m)} y_m$, then $A_{ij}^{(m)}$ is the cofactor of $a_{ij}^{(m)}$ in $|A_m|$, and

$$\left.\begin{aligned} A_{ij}^{(m)} &= a_{ji}^{(m)} \text{ for } m = 1, \ldots, 4, \text{ since } Am^{-1} = A_m, \\ A_{ji}^{(0)} &= -a_{ji}^{(0)}, \text{ since } A_0^{-1} = -A_0. \end{aligned}\right\} \tag{4}$$

We see that (I) is essentially a weaker assertion than (II), since (II) implies that $| A^* | = cQ^2$ for some constant c. In this note we shall use (I) and (II) as a starting point, and examine the equations which arise.

We begin by observing two groups of transformations which leave (I) and (II) invariant and which leave (I) invariant when (II) does not apply. First, if (C, D) is a pair of matrices such that $| C |\cdot| D | = 1$, and B^* satisfies (I) and (II), then so does CB^*D. For $| CB^*D | = | B^* |$, and each third order minor of $| CB^*D |$ is a linear combination of third order minors of $| B^* |$. Evidently we do not throw away any solutions by requiring that $| C | = | D | = 1$. If we then identify the pairs

$$(C,\ D),\ (iC,\ -iD),\ (-C,\ -D),\ (-iC,\ iD),$$

and define multiplication as

$$(C_1,\ D_1)\cdot(C_2,\ D_2) = (C_1C_2,\ D_2D_1),$$

the group of transformations is isomorphic to the group of pairs, which has 30 degrees of freedom. We shall use *pair* to denote either a pair of matrices or a transformation of the group.

Secondly, take any five-by-five orthogonal matrix $G = (g_{mn})$, $m, n = 0, \ldots, 4$, and let $y_m' = \sum\limits_n g_{mn}y_n$. Let $B^* = \Sigma B_m y_m$ be any solution, and define $B'^* = \Sigma B_m' y_m$ by the identity $\Sigma B_m' y_m' = \Sigma B_m y_m$. Since G is orthogonal, $\Sigma y_m'^2 = \Sigma y_m^2$. Hence B'^* is a solution. We see that $B_m' = \sum\limits_n g_{mn}B_n$, so that the set $\{B_m\}$ undergoes an orthogonal transformation. The group of such transformations, which we shall call *rotations*, has 10 degrees of freedom.

We now state that if B^* satisfies (I) and (II), then there exists a unique pair (C, D) such that $B^* = CA^*D$. The proof is roughly as follows: Since $| B_0 | = 1$, (I, iB_0^{-1}) is a pair transforming B^* to a solution B'^* with $B_0' = A_0$. For any $m \neq 0$ we can find C so that $B_m'' = CB_m'C^{-1}$ is in diagonal or almost-diagonal form. Equating the coefficients of $y_0^3 y_m$, $\ldots$, y_m^4 in $| B''^* |$ to the values required by (I), and solving, we see that the diagonal elements of B_m'' are those of A_1. The assumption that B_m'' is not diagonal leads to a contradiction of (II). Hence $CB_m'C^{-1} = A_1$, and $B_m'^2 = I$, for $m = 1, \ldots, 4$. Letting $B_0'' = A_0$, $B_m'' = A_1$, we see by similar reasoning that $B_m'B_n' + B_n'B_m' = 0$ for $n \neq 0$ or m. Hence $B_1', \ldots, B_4'$ satisfy (2), and the existence of a pair follows. The uniqueness is readily verified.

We have observed that for any rotation G, $A_m' = \Sigma g_{mn}A_n$ is a solution. It follows that to each rotation G there corresponds a unique pair (C_G, D_G) such that

$$\sum_{n=0}^{4} g_{mn}A_n = C_G A_m D_G,\ m = 0, \ldots, 4.^2 \qquad (5)$$

If G leaves the axis of y_0 fixed, i.e., $g_{om} = g_{mo} = \delta_{mo}$, then $D_G = C_G^{-1}$.

We now remove condition (II) and show that (I) possesses more general solutions. In particular, we shall determine the most general matrix $R = (r_{ij}) = (r_is_j)$ of rank $\leqq 1$, with elements of an unknown form, which may be added to A^* without changing the determinant. Since any two rows of R are proportional,

$$|A^* + R| - |A^*| = \sum_{i,j=1}^{4} r_is_jA_{ij}^*Q = 0.$$

Using (4), we obtain the five equations

$$\sum_{i,j=1}^{4} r_is_ja_{ji}^{(m)} = 0,\ m = 0, \ldots, 4. \qquad (6)$$

The special values (3) of A_m show that equations (6) are consistent and determine s_j in terms of r_i to within a factor of proportionality.

$$s_1 = cr_2,\ s_2 = -cr_1,\ s_3 = cr_4,\ s_4 = -cr_3. \qquad (7)$$

When R is a matrix of constants, we can choose r_i and s_j so that $c = 1$.

It is worth noting that R is also the most general matrix of rank $\leqq 1$ such that

$$RA_mR = 0 \text{ for } m = 0, \ldots, 4, \qquad (8)$$

$$\text{i.e., } \sum_{j,k=1}^{4} r_is_ja_{jk}^{(m)}r_ks_p = 0,\quad k, p = 1, \ldots, 4.$$

For unless each $r_i = 0$ or each $s_p = 0$, in which case $R = 0$, this reduces to (6). Any matrix $R = (r_is_j)$ satisfying (6) will be called an *R-matrix*.

We wish solutions of (I) whose elements are linear homogeneous functions of the y_m. Taking any R-matrix R of constants and multiplying it by an arbitrary linear function $t^* = \sum_{m=0}^{4} t_my_m$ gives a solution $A^* + t^*R$. The set of such solutions has 8 degrees of freedom.

Using the two groups of transformations, we can obtain further solutions. First, every pair (C, D) gives solutions

$$C(A^* + t^*R)D. \qquad (9)$$

Secondly, every rotation G gives a solution B^* with

$$\begin{aligned} B_m &= \sum_n g_{mn}(A_n + t_nR) \\ &= C_G[A_m + (\sum_n g_{mn}t_n)(C_G^{-1}RD_G^{-1})]D_G. \end{aligned}$$

Since $C_G^{-1}RD_G^{-1}$ is of rank ≤ 1, it must be an R-matrix. Hence B^* is of the form (9). So the solutions (9) form a set closed under pairs and rotations. We shall say that under a rotation G, t_m transforms to $\sum_n g_{mn}t_n$, and R to $C_G^{-1}RD_G^{-1}$.

We can show that if $C(A^* + t^*R)D = C'(A^* + t'^*R')D'$, then $(C, D) = (C', D')$, and $t^*R = t'^*R'$. We first reduce the problem by a rotation to the case where $t_0 = t_1 = t_2 = 0$ and $t_0' = 0$. It is then tedious but not too difficult to verify the result. Thus the set of solutions (9) has 38 degrees of freedom.[3]

We have seen that any solution of (I) and (II) leads to the Dirac equations. Similarly, any solution B^* of (I) leads to a set of equations whose multiplier equation[4] is

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1,$$

being determined by (I). It is thus identical with the multiplier equation of (1). In particular, (9) yields the equations

$$\frac{1}{\lambda}\sum_{m=1}^{4}(A_m + t_mR)\frac{\partial\psi\cdot}{\partial x_m} - (I + t_0R)\psi\cdot = 0 \qquad (10)$$

When B^* is transformed by an operation leaving (I) invariant, the corresponding equations will be changed. First, a pair (C, D) leads to equations which are satisfied by $\psi'\cdot = D^{-1}\psi\cdot$ if $\psi\cdot$ satisfies the original equations. Hence a pair corresponds to a linear transformation of the dependent variables ψ_i. Secondly, a rotation G causes a transformation of the independent variables x_m, leaving fixed the quadric $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. This will be an orthogonal transformation of $x_1, \ldots, x_4$ if and only if G leaves the axis of y_0 fixed. In the following paragraphs we shall use *rotation* to refer only to those G mentioned above, whence $D_G = C_G^{-1}$.

Since equations (10) depend largely upon R, it is desirable to develop some more properties of R-matrices. Let $A_5 = A_1A_2A_3A_4$. Then for all G,

$$\begin{aligned} C_GA_5C_G^{-1} &= C_GA_1C_G^{-1} \ldots C_GA_4C_G^{-1} \\ &= \sum_{m_1, m_2, m_3, m_4} g_{1m_1}g_{2m_2}g_{3m_3}g_{4m_4}A_{m_1}A_{m_2}A_{m_3}A_{m_4}. \end{aligned}$$

Using (2) and the fact that G is orthogonal, we see that

$$C_GA_5C_G^{-1} = A_5 \text{ if } |G| = +1,$$
$$C_GA_5C_G^{-1} = -A_5 \text{ if } |G| = -1.$$

Now let $z_m = z_m(R) =$ trace of A_mA_5R, $m = 1, \ldots, 4$. Then if $R' = C_G^{-1}RC_G$,

$$
\begin{aligned}
z_m' = z_m(R') &= \text{trace of } A_m A_5 C_G^{-1} R C_G \\
&= \text{trace of } C_G A_m C_G^{-1} A_5 R \\
&= \text{trace of } \sum_n g_{mn} A_n A_5 R \\
&= \sum_n g_{mn} z_n,
\end{aligned}
$$

if $|G| = +1$. Thus $z(R) = (z_1, z_2, z_3, z_4)$ is actually a vector under rotations of determinant $+1$. If $|G| = -1$, $z(R)$ does not behave exactly like a vector, but instead $z_m' = -\sum_n g_{mn} z_n$. However, the set of all transformations on $z(R)$ is the same as the set of all rotations. Using the values (3) for A_m, and letting $c = 1$ in (7),

$$
\begin{aligned}
z_1 &= 2i(r_1 r_3 - r_2 r_4) \\
z_2 &= -i(r_1^2 - r_2^2 - r_3^2 + r_4^2) \\
z_3 &= -(r_1^2 + r_2^2 + r_3^2 + r_4^2) \\
z_4 &= 2i(r_1 r_2 + r_3 r_4).
\end{aligned}
$$

We see that $\sum_m z_m^2 = 0$. Hence there corresponds to each R-matrix a unique isotropic (or null) vector.

On the other hand, there is an infinity of R-matrices corresponding to each isotropic vector z. Consider first the case $z \neq 0$. Here there exists a rotation transforming z to $(0, -i, -1, 0)$. Solving,

$$r_1 = \cos\alpha, \quad r_2 = 0, \quad r_3 = 0, \quad r_4 = \sin\alpha, \text{ for some } \alpha, \tag{11}$$

and the set of R-matrices corresponding to z has one degree of freedom. Explicit calculation reveals that if G is a rotation in the y_1-y_4 plane through -2α, then G transforms R, as given by (11), to an R-matrix with $r_1 = 1$, $r_2 = r_3 = r_4 = 0$. Hence all R with $z(R) \neq 0$ are equivalent under rotations. If G leaves R fixed, it obviously preserves $z(R)$. Since the set of all G preserving $z(R)$ has three degrees of freedom, the set of G preserving R must have two degrees of freedom. It follows that no unique vector independent of $z(R)$ can be associated with R. For the set of rotations preserving two linearly independent vectors has but one degree of freedom.

Similar methods show that the set of all $R \neq 0$ with $z(R) = 0$ has two degrees of freedom, and that all these R are equivalent under rotations, a normal form being $r_1 = 1$, $r_2 = 0$, $r_3 = 0$, $r_4 = i$. The set of G preserving R has four degrees of freedom; hence no unique non-null vector can be associated with R.

If we take the first term of (1), and substitute it for $\psi\cdot$ in (1), we build a second order equation, which reduces by (2) to

$$(\square - I)\psi\cdot = 0, \tag{12}$$

where $\square$ may be treated as a scalar-matrix operator with diagonal elements

$$\frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2} + \frac{\partial^2}{\partial x_4{}^2}\right)$$

Every solution of the Dirac equations must satisfy (12). Conversely, the Dirac equations have been obtained and studied as the most general system of four equations of first order, all of whose solutions satisfy (12).[5] Evidently $\square$, and hence (12), are invariant under rotations.

If we use the same procedure with (10), we obtain a second order equation which will not reduce to a simple form. But if we continue and build a fourth order equation, it reduces by (2) and (8) to

$$[\square - I][\square - (I + t_0R)]\psi\cdot = 0. \tag{13}$$

Thus if $\psi\cdot$ satisfies (10), $[\square - (I + t_0R)]\psi\cdot$ satisfies (12). Since $\square$ is invariant under rotations, (13) is invariant under those rotations preserving R.

Thus the generalized equations (10) and the related equation (13), with $z(R) \neq 0$, might be suitable for a universe in which there is a preferred istropic direction, whereas the Dirac equations belong to a universe in which no direction is preferred. Such an isotropic direction may be identified with the possible path of a light ray or a photon.

Of particular interest is the case where $t_0 = 0$. Here the four dependent variables ψ_i are separated in (13), and each satisfies the same fourth order equation, which is invariant under all rotations. Of interest also may be the case where the vector (t_1, t_2, t_3, t_4) is a multiple of $z(R)$. For then any rotation which preserves R also preserves t^*, and equations (10) are themselves invariant under such rotations, in the sense that the Dirac equations are invariant. That is, under such rotations, the new solutions are linear combinations of the old ones. Finally, we mention that these two cases may be combined.

[1] Properties (I) and (II) were pointed out by Professor G. D. Birkhoff in lectures at Peiping (1934) as affording a possible starting point for a generalization of the Dirac equations.

[2] This fact is well known. The approach presented here merely affords an additional proof.

[3] At present I have been unable to determine whether (9) is the most general solution of (I).

[4] G. D. Birkhoff, "Quantum Mechanics and Asymptotic Series," *Bull. Amer. Math. Soc.*, **39**, 681–700 (1933).

[5] This approach was used by Birkhoff in the lectures at Peiping cited above.