7. 有质量矢量场的路径积分量子化

有质量的矢量场的拉氏量为（场强张量 $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$）

$$\mathcal{L}_0\left[A_\mu,\partial_\nu A_\mu\right]=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\left(m^2-i\epsilon\right)A_\mu A^\mu$$

作用量的泛函二次型形式（利用分部积分）

$$S[A]=\int d^4x\left[\mathcal{L}_0[A_\mu,\partial_\nu A_\mu]\right]=-\int d^4x\frac{1}{2}A^\mu\left(-g_{\mu\nu}\left(\partial^2+m^2-i\epsilon\right)+\partial_\mu\partial_\nu\right)A^\nu$$

矢量场的生成泛函

$$Z[J]=\int DA\exp\left\{i\int d^4x\ \left[\mathcal{L}_0[A_\mu,\partial_\nu A_\mu]+J_\mu(x)A^\mu(x)\right]\right\}$$

高斯型的路径积分

$$Z[J]=Z[0]\exp\left(-\frac{1}{2}\iint d^4x d^4y J_\mu(x)D_F^{\mu\nu}(x-y;m)J_\nu(y)\right)$$

其中 $D_F^{\mu\nu}(x-y;m)$ 满足

$$i\left(-g_{\mu\nu}\left(\partial^2+m^2-i\epsilon\right)+\partial_\mu\partial_\nu\right)D_F^{\nu\rho}(x-y;m)=\delta_\mu^\rho\delta^4(x-y)$$

Fourier 变换,

$$G_F^{\mu\nu}(x-y;m)=\int\frac{d^4p}{(2\pi)^4}e^{-ik\cdot(x-y)}\tilde{G}_F^{\mu\nu}(k;m)$$

前面方程在动量空间的表达

$$i\left(g_{\mu\nu}\left(k^2-m^2+i\epsilon\right)-k_\mu k_\nu\right)\tilde{G}_F^{\nu\rho}(k)=\delta_\mu^\rho$$

由于 k 和 $g_{\mu\nu}$ 只能构成两种洛伦兹二阶张量 $g_{\mu\nu}$和 $k_\mu k_\nu$，我们将 $\tilde{G}_F^{\mu\nu}(k;m)$ 参数化为

$$\tilde{G}_F^{\mu\nu}(k;m)=A\left(k^2;m\right)g^{\mu\nu}+B\left(k^2;m\right)k^\mu k^\nu$$

其中 $A(k^2;m)$ 和 $B(k^2;m)$ 是洛伦兹标量函数，带入上面方程得到，

$$A(k^2;m)=\frac{-i}{k^2-m^2+i\epsilon},\qquad B(k^2;m)=\frac{1}{m^2}\frac{i}{k^2-m^2+i\epsilon}$$

即

$$\widetilde{G}_F^{\mu\nu}(k;m)=\frac{-i}{k^2-m^2+i\epsilon}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{m^2}\right)$$

$$G_F^{\mu\nu}(x-y;m)=\int\frac{d^4p}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{-i}{k^2-m^2+i\epsilon}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{m^2}\right)$$

这正是正则量子化中矢量场传播子中的洛伦兹协变部分，非协变项不出现，

8. Grassmann数路径积分和Dirac场量子化

旋量场满足的反对易关系。尽管路径积分量子化中出现的场是经典场，但为了体现正确的的自旋-统计关系，我们也需要将经典的旋量场用反对易的数来描述，反对易的数有称作 Grassmann 数，它们满足特殊的代数关系，即 Grassmann 代数。经典费米场 $\psi(x)$ 在 x 的取值为 Grassmann 数。

8. 1 Grassmann 代数

对于一个 n –维 Grassmann 代数

1. 生成元： $\{\xi_i | i = 1, 2, \dots, n\}$，满足反对易关系（反对易数）

$$\xi_i \xi_j = -\xi_j \xi_i, \qquad \{\xi_i, \xi_j\} = \xi_i \xi_j + \xi_j \xi_i = 0, \qquad \xi_i^2 = 0$$

2. 数乘

- Grassmann 数和玻色型量相乘仍是 Grassmann 数，即数乘。

3. 函数 $f(\xi)$：

$$f(\xi)=f_0+f_1^i\xi_i+f_2^{ij}\xi_i\xi_j+\cdots+f_n^{i_1i_2\dots i_n}\xi_{i_1}\xi_{i_2}\dots\xi_{i_n}$$

其中 $f_n^{i_1i_2\dots i_n}$ 为复数。显然，上面展开只有 $n+1$ 项，指标 $i_1i_2\dots i_n$ 在1 到 n 取不同的值，不重复；$f_n^{i_1i_2\dots i_n}$ 对上指标全反对称。特别地，

$$n=1,\qquad \xi^2=0,\qquad f(\xi)=f_0+f_1\xi$$

$$n=2,\qquad \xi_2^2=\xi_1^2=0,$$

$$f(\xi_1,\xi_2)=f_0+f_1^1\xi_1+f_1^2\xi_2+f_2^{12}\xi_1\xi_2,\qquad f_2^{12}=-f_2^{21}$$

4. Grassmann 数的乘法

- 偶数个 Grassmann 数的乘积为复数（正常的数，Bosonic）

$$a=\xi_1\xi_2,\qquad a\xi_3=\xi_1\xi_2\xi_3=\xi_3\xi_1\xi_2=\xi_3a$$

- 奇数个 Grassmann 数的乘积仍为 Grassmann 数（Fermionic）

$$\eta=\xi_1\xi_2\xi_3,\qquad \eta\xi_4=\xi_1\xi_2\xi_3\xi_4=-\xi_4\xi_1\xi_2\xi_3=-\xi_4\eta$$

5. 复 Grassmann 数及其复共轭

θ_1 和 θ_2 都是实的 Grassmann 数，则复的 Grassmann 数为

$$\theta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2)$$

其复共轭 θ^* 定义为

$$\theta^* = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2)$$

θ 和 θ^* 是相互独立的 Grassmann 数

6. 两个复 Grassmann 数乘积的复共轭

定义两个 Grassmann 数 θ 和 η 的乘积的复共轭为

$$(\theta\eta)^* = \eta^*\theta^* = -\theta^*\eta^*$$

这和算符的厄米共轭的定义类似。

7.2 Grassmann 数积分

1. 积分微元：

Grassmann数的积分元 $d\xi_i$ 仍为 Grassmann 数：

$$\{\xi_i, \xi_j\} = \{d\xi_i, \xi_j\} = \{d\xi_i, d\xi_j\} = 0$$

$$\int d\xi = 0, \qquad \int d\xi \ \xi = 1$$

2. 线性积分：

积分应该是线性的，即

$$\int d\xi \ (af_1(\xi) + bf_2(\xi)) = a \int d\xi \ f_1(\xi) + b \int d\xi \ f_2(\xi)$$

① 一维积分：

$$\int d\xi \ (a + b\xi) = a \int d\xi + b \int d\xi \ \xi$$

② 平移不变性：$\xi \to \xi + \eta$，其中 η 是和 ξ 无关的。

$$\int d\xi \ (\xi + \eta) = \int d\xi \ \xi = 1, \quad \int d\xi \, f(\xi + \eta) = \int d\xi \ f(\xi)$$

3. Grassmann 数积分和 Grassmann 数微商等价 ：

$$\int d\xi\,(a+b\xi)=b \quad \Longleftrightarrow \quad \frac{d}{d\xi}(a+b\xi)=b$$

4. 多维积分

规定 $$\int d\xi_1 d\xi_2\cdots d\xi_n\,\xi_n\xi_{n-1}\cdots\xi_1=1$$

$$\int d\xi_1 d\xi_2\cdots d\xi_n\,F(\xi)=\frac{\partial}{\partial\xi_1}\frac{\partial}{\partial\xi_2}\cdots\frac{\partial}{\partial\xi_n}F(\xi)$$

积分顺序由里到外，特别是

$$\int d\xi_1 d\xi_2\,\xi_2\xi_1=-\int d\xi_1 d\xi_2\,\xi_1\xi_2=1$$

5. 高斯型积分（重要）

$$\int d\xi_1 d\xi_2\,e^{-\xi_1 A_{12}\xi_2}=\int d\xi_1 d\xi_2\,(1-A_{12}\xi_1\xi_2)=A_{12}$$

这里，注意在对指数函数做泰勒展开时高阶项由于 $\xi_2^2=\xi_1^2=0$，及 Grassmann 数的反对易性都没有贡献。

6. 积分变换（重要）

普通的一维积分，我们一般有 $$\int d\tilde{x}\, f(\tilde{x}) = \int dx \left(\frac{d\tilde{x}}{dx}\right) f(\tilde{x}(x))$$

普通的多维积分，$x'_i = f_i(x)$

$$\int dx'_1 dx'_2 \cdots dx'_n\, F(x') = \int dx_1 dx_2 \cdots dx_n \left[\det\left(\frac{\partial f_i}{\partial x_j}\right)\right] F(x'(x))$$

但是对于 Grassmann 数积分，令 $\tilde{\theta} = a + b\theta$, $f(\tilde{\theta}) = f_0 + f_1\tilde{\theta}$,

$$\int d\tilde{\theta}\ f(\tilde{\theta}) = f_1$$

所以

$$\int d\theta\, f(\tilde{\theta}) = \int d\theta\ f(a + b\theta) = bf_1$$

$$\int d\tilde{\theta}\ f(\tilde{\theta}) = \frac{1}{b}\int d\theta\, f(\tilde{\theta}) = \int d\theta \left(\frac{d\tilde{\theta}}{d\theta}\right)^{-1} f\left(\tilde{\theta}(\theta)\right)$$

这是和普通的复数型积分完全不同的。

Grassmann 数多维积分，$\xi'_i = A_{ij}\xi_j$，$F(\xi') = a\xi'_1\xi'_2\cdots\xi'_n$

$$\int d\xi'_n d\xi'_{n-1}\cdots d\xi'_1\, F(\xi') = \int d\xi'_n d\xi'_{n-1}\cdots d\xi'_1\, a\xi'_1\xi'_2\cdots\xi'_n = a$$

$$\xi'_1\xi'_2\cdots\xi'_n = A_{1i_1}A_{2i_2}\cdots A_{3i_n}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}$$

$\det A$

$$= \epsilon^{i_1 i_2\cdots i_n}A_{1i_1}A_{2i_2}\cdots A_{ni_n}\xi_1\xi_2\cdots\xi_n \quad = \det A\,\xi_1\xi_2\cdots\xi_n$$

$$\int d\xi_n d\xi_{n-1}\cdots d\xi_1\, F(A_{ij}\xi_j) = a\det A\int d\xi_n d\xi_{n-1}\cdots d\xi_1\,\xi_1\xi_2\cdots\xi_n$$
$$= a\det A$$

$$\boxed{d\xi_n d\xi_{n-1}\cdots d\xi_1 = \det A\, d\xi'_n d\xi'_{n-1}\cdots d\xi'_1}$$

高斯型 Grassmann 复数积分 $\boxed{\int d\xi^* d\xi\; e^{-b\xi^*\xi} = b}$

$$\xi = \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \qquad \xi^* = \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2), \quad \xi^*\xi = i\xi_1\xi_2$$

$$\begin{pmatrix}\xi^*\\ \xi\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & -i/\sqrt{2}\\ 1/\sqrt{2} & i/\sqrt{2}\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \equiv A\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \qquad \det A = i$$

$$\int d\xi^* d\xi\; e^{-b\xi^*\xi} = i^{-1}\int d\xi_1 d\xi_2\, e^{-ib\xi_1\xi_2} = i^{-1}\cdot ib = b$$

高维高斯型积分

$$\boxed{\int d\xi_1^* d\xi_2^* \cdots d\xi_n^* \, d\xi_n d\xi_{n-1} \cdots d\xi_1 e^{-\xi^* A \xi} = \det A}$$

令 $\tilde{\xi}_i = A_{ij}\xi_j$，利用 $d\xi_n d\xi_{n-1} \cdots d\xi_1 = \det A \, d\tilde{\xi}_n d\tilde{\xi}_{n-1} \cdots d\tilde{\xi}_1$

$$\int d\xi_1^* d\xi_2^* \cdots d\xi_n^* \, d\xi_n d\xi_{n-1} \cdots d\xi_1 e^{-\xi^* A \xi}$$

$$= \det A \int d\xi_1^* d\xi_2^* \cdots d\xi_n^* \, d\tilde{\xi}_n d\tilde{\xi}_{n-1} \cdots d\tilde{\xi}_1 e^{-\xi^* \tilde{\xi}} = \det A$$

对比 Boson 型的高斯型积分：

$$\int dx_1 dx_2 \cdots dx_n \, e^{-\frac{1}{2} x_i A_{ij} x_j} = \left(\frac{(2\pi)^n}{\det A} \right)^{\frac{1}{2}}$$

行列式在分母上。

8.3 Dirac场路径积分量子化

费米子体系的正则量子化中，费米子场（算符）的正则对易关系为

$$\{\psi_n, \Pi_m\} = i\delta_{mn}, \qquad \Pi_m = i\psi_m^+ \quad \Longrightarrow \quad \{\psi_n, \psi_m^+\} = \delta_{mn}$$

在场论的路径积分量子化在相干态表象更容易描述，而相干态表象要用场量的吸收部分（湮灭算符部分）来定义。如果我们用 ψ_n 代表费米场的吸收部分，

$$\psi^{(+)}(x) = \sum_{s=\pm 1} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} u_p^{(s)} a_p^{(s)} e^{-ip\cdot x}$$

令 $|0\rangle$ 代表裸真空态

$$\psi_n|0\rangle = 0, \qquad \langle 0|\psi_n^+ = 0$$

1. 算符 ψ_n 的本征态

引入态矢量

$$|\xi\rangle = \exp(-\xi\cdot\psi^+)\,|0\rangle, \qquad \langle\xi| = \langle 0|\exp(-\psi\cdot\xi^*)$$

$\xi\cdot\psi^+ = \xi_n\psi_n^+$ (重复指标求和约定)，其中 ξ_n 是复的 Grassmann 数。

可以证明，$|\xi\rangle$ 是 ψ_n 的本征态：

$$\psi_n|\xi\rangle = \xi_n|\xi\rangle, \qquad \langle\xi|\psi_n^+ = \xi_n^*$$

两个不同的本征态 $|\xi\rangle$ 和 $|\eta\rangle$ 不是正交的，其内积为 $\langle\xi|\eta\rangle = \mathrm{e}^{\xi^*\cdot\eta}$

可以证明，由 $|\xi\rangle$ 构成的Hilbert空间是完备的

$$\int\prod_n d\xi_n^* d\xi_n\,|\xi\rangle\,\exp(-\xi^*\cdot\xi)\,\langle\xi| = 1$$

证明：

$$\int\prod_n (d\xi_n^* d\xi_n)\,\langle\eta|\xi\rangle e^{-\xi^*\xi}\langle\xi|\eta'\rangle = \int\prod_n (d\xi_n^* d\xi_n)\,e^{\eta^*\cdot\xi}e^{-\xi^*\cdot\xi}e^{\xi^*\cdot\eta'}$$

$$= \int\prod_n (d\xi_n^* d\xi_n)\,e^{\eta^*\cdot\xi-\xi^*\cdot\xi+\xi^*\cdot\eta'} = e^{\eta^*\cdot\eta'}\int\prod_n (d\xi_n^* d\xi_n)\,e^{-(\xi-\eta)^*\cdot(\xi-\eta')}$$

$$= e^{\eta^*\cdot\eta'} \equiv \langle\eta|\eta'\rangle$$

证明： $\psi_n|\xi\rangle = \xi_n|\xi\rangle$

先做泰勒展开 $$\psi_n|\xi\rangle = \psi_n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (\xi \cdot \psi^+)^m |0\rangle$$

利用归纳法，当 $m = 1$ 时，利用 ψ_n 满足的反对易关系，有

$$\psi_n \xi \cdot \psi^+ = -\xi_k \psi_n \psi_k^+ = -\xi_n + (\xi \cdot \psi^+)\, \psi_n$$

当 $m = 2$ 时，

$$\psi_n (\xi \cdot \psi^+)^2 = (-\xi_n + (\xi \cdot \psi^+)\, \psi_n)(\xi \cdot \psi^+) = -2\xi_n (\xi \cdot \psi^+) + (\xi \cdot \psi^+)^2 \psi_n$$

假设 $m = k$ 时有

$$\psi_n (\xi \cdot \psi^+)^k = -k\xi_n \psi_n (\xi \cdot \psi^+)^{m-1} + (\xi \cdot \psi^+)^k \psi_n$$

则当 $m = k + 1$ 时

$$\psi_n (\xi \cdot \psi^+)^{k+1} = \psi_n (\xi \cdot \psi^+)^k (\xi \cdot \psi^+) = -m\, \xi_n (\xi \cdot \psi^+)^k + (\xi \cdot \psi^+)^k \psi_n (\xi \cdot \psi_n^+)$$

$$= -(m+1)\xi_n (\xi \cdot \psi^+)^{k+1} + (\xi \cdot \psi^+)^{k+1} \psi_n$$

所以有

$$\psi_n|\xi\rangle = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \psi_n (\xi \cdot \psi^+)^m |0\rangle = \sum_{m=0}^{\infty} \frac{(-1)^{m-1}}{(m-1)!} \xi_n\, (\xi \cdot \psi^+)^{m-1} |0\rangle$$

$$= \xi_n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \psi_n (\xi \cdot \psi^+)^m |0\rangle = \xi_n|\xi\rangle$$

证明：$\langle \xi | \eta \rangle = \exp(\xi^* \cdot \eta)$

$$\langle \xi | \eta \rangle = \langle 0 | \exp(-\psi \cdot \xi^*) \exp(-\eta \cdot \psi^+) | 0 \rangle = \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}}{m!\, n!} \langle 0 | (\psi \cdot \xi)^m (\eta \cdot \psi^+) | 0 \rangle$$

令 $A = \psi \cdot \xi^*,\ B = \eta \cdot \psi^+$，则 $[A, B] = \xi^* \cdot \eta$；当 $m \neq n$ 时，$\langle 0 | A^m B^n | 0 \rangle = 0$；

当 $m = n$ 时：

当 $m = n = 1$ 时，

$$\langle 0 | AB | 0 \rangle = \langle 0 | [A, B] | 0 \rangle + 2 \langle 0 | AB | 0 \rangle = \xi^* \cdot \eta$$

当 $m = n = 2$ 时，

$$A^2 B^2 = AABB = A[A, B]B + ABAB = A[A, B]B + AB[A, B] + ABBA$$

所以，

$$\langle 0 | A^2 B^2 | 0 \rangle = 2\xi^* \cdot \eta \langle 0 | AB | 0 \rangle = 2\, (\xi^* \cdot \eta)^2$$

假设当 $m = n = k$ 时有

$$\langle 0 | A^k B^k | 0 \rangle = k!\, (\xi^* \cdot \eta)^k$$

则当 $m = n = k + 1$ 时，

$$\begin{aligned} A^{k+1} B^{k+1} &= A(A^k B^k)B = A[A^k B^k, B] + AB\, (A^k B^k) \\ &= A[A^k, B]B^k + [A, B](A^k B^k) + BA(A^k B^k) \\ &= kA[A, B]A^{k-1} B^k + [A, B](A^k B^k) + BA(A^k B^k) \end{aligned}$$

于是，

$$\langle 0 | A^{k+1} B^{k+1} | 0 \rangle = (k+1)\xi^* \cdot \eta \langle 0 | A^k B^k | 0 \rangle = (k+1)!\, (\xi^* \cdot \eta)^{n+1}$$

所以有

$$\langle \xi | \eta \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} (\xi^* \cdot \eta)^n = \exp(\xi^* \cdot \eta)$$

2.　路径积分量子化

定义了这个完备集后，我们现在考虑无穷小时间间隔的跃迁矩阵元

$$\left\langle \xi^{(t+\Delta t)}, t+\Delta t \middle| \xi^{(t)}, t \right\rangle \equiv \left\langle \xi^{(t+\Delta t)}, t \middle| e^{-i\hat{H}(\psi^{+},\psi)\Delta t} \middle| \xi^{(t)}, t \right\rangle$$

$$= \exp\left(-iH(\xi^{(t+\Delta t),*}, \xi^{(t)}\right) \exp\left(\xi^{(t+\Delta t),*} \cdot \xi^{(t)}\right)$$

有限时间段的跃迁矩阵元可以通过将该时间段分成无穷小时间段，比如将时间段 $[t_i, t_f]$ 分成 N 段，

$$\left\langle \xi_f, t_f \middle| \xi_i, t_i \right\rangle = \int \prod_{k=1}^{N-1} (d\xi_k^* d\xi_k) \prod_{k=1}^{N-1} e^{-\xi_k^* \cdot \xi_k} \left\langle \xi_f, t_f \middle| e^{-i\hat{H}(\psi^{+},\psi)\Delta t} \middle| \xi_{N-1}, t_{N-1} \right\rangle$$

$$\times \left\langle \xi_{N-1}, t_{N-1} \middle| e^{-i\hat{H}(\psi^{+},\psi)\Delta t} \middle| \xi_{N-2}, t_{N-2} \right\rangle \cdots \left\langle \xi_1, t_1 \middle| e^{-i\hat{H}(\psi^{+},\psi)\Delta t} \middle| \xi_i, t_i \right\rangle$$

$$= \lim_{\Delta t \to 0} \int \prod_{k=1}^{N-1} (d\xi_k^* d\xi_k) \prod_{k=1}^{N} e^{-\xi_k^* \cdot (\xi_k - \xi_{k-1}) - i\Delta t H(\xi_{k+1}^*, \xi_k)} \, e^{\xi_f^* \cdot \xi_f}$$

$$= \lim_{\Delta t\to 0}\int \prod_{k=1}^{N-1}(d\xi_k^* d\xi_k)\exp\left(\frac{1}{2}\left(\xi_f^*\cdot\xi_f+\xi_i^*\cdot\xi_i\right)\right)$$

$$\times\exp\left[\sum_{k=1}^{N-1}\left(-\left(\xi_{k+1}^*\frac{\xi_{k+1}-\xi_k}{2}-\frac{\xi_{k+1}^*-\xi_k^*}{2}\xi_k\right)\right)\right]e^{-i\Delta t\sum_{k=0}^{N-1}H(\xi_{k+1}^*,\xi_k)}$$

$$= \exp\left(\frac{1}{2}\left(\xi_f^*\cdot\xi_f+\xi_i^*\cdot\xi_i\right)\right)$$

$$\times\int D\xi^* D\xi\exp\left[i\int_{t_i}^{t_f}dt\left(\frac{i}{2}\left(\xi^*(t)\cdot\dot{\xi}(t)-\dot{\xi}^*(t)\cdot\xi(t)\right)\right)-H(\xi^*,\xi)\right]$$

令 $\psi(x)$ 代表经典的Dirac旋量场，在每个时空点上的四个分量都是Grassmann数（空间坐标和旋量指标对应于 ξ_i 的下指标），其厄米共轭 $\psi^+(x)=\bar{\psi}(x)\gamma^0$ 对应上面表达式中的 ξ^* ，我们应该可以得到用 $\psi(x)$ 和 $\bar{\psi}(x)$ 表达的Dirac场的路径积分量子化形式。首先，考虑场在时空无穷远取值为0，利用关系

$$0=\int dt\ \partial^0\left(\bar{\psi}(x)\gamma^0\psi(x)\right)=\int dt\left(\bar{\psi}(x)\gamma^0\left(\overleftarrow{\partial^0}+\vec{\partial}^0\right)\psi(x)\right)$$

可以知道 $$\int dt\ \bar{\psi}(x)\gamma^0\overleftarrow{\partial}^0\psi(x) = -\int dt\ \bar{\psi}(x)\gamma^0\overrightarrow{\partial}^0\ \psi(x)$$

从而我们可以类比地写出

$$\lim_{\substack{t_i\to-\infty,\\ t_f\to+\infty}}\langle\psi_f,t_f|\psi_i,t_i\rangle = \lim_{\substack{t_i\to-\infty,\\ t_f\to+\infty}}\exp\left(\frac{1}{2}\int d^3\vec{x}\left(\psi_f^+(\vec{x})\cdot\psi_f(\vec{x})+\psi_i^+(\vec{x})\cdot\psi_i(\vec{x})\right)\right)$$
$$\times\int D\psi D\bar{\psi}\exp\left[i\int_{t_i}^{t_f}dt\int d^3\vec{x}\left(\bar{\psi}(x)i\gamma^0\partial^0\psi(x)-\mathcal{H}(\psi,\bar{\psi})\right)\right]$$

其中右边地第一个因子是一个常数。

哈密顿密度 $$\mathcal{H}(\psi,\bar{\psi}) = \bar{\psi}\left(-i\vec{\gamma}\cdot\vec{\nabla}+m\gamma^0\right)\psi$$

$$\bar{\psi}(x)i\gamma^0\partial^0\psi(x)-\mathcal{H}(\psi,\bar{\psi}) = \bar{\psi}(x)\left(i\gamma^\mu\partial_\mu-m+i\epsilon\right)\psi(x)\equiv\mathcal{L}_0(x)$$

正好给出了自由Dirac场的拉氏量（加入了 ϵ 项来体现正确的端条件。）

和玻色体系类似，费米体系地格林函数可以用路径积分表达为

$$\langle\Omega|T\hat{\mathcal{O}}_a(x_a)\hat{\mathcal{O}}_b(x_b)\cdots|\Omega\rangle = \mathcal{N}\int D\bar{\psi}D\psi\ e^{i\int d^4x\mathcal{L}_0(x)}\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\cdots$$

3. 格林函数生成泛函

$$Z_0[\eta,\bar{\eta}] = \mathcal{N}\int D\bar{\psi}D\psi\ \exp\left[i\int d^4x\left(\mathcal{L}_0(x)+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)\right)\right]$$

为了方便起见，我们可以选定归一化因子 $\mathcal{N}$ 使得 $Z_0[\eta,\bar{\eta}]_{\eta=\bar{\eta}=0}=1$。

利用Grassmann数的高斯型积分公式

$$\int d\xi^* d\xi\,\exp(-\xi^*\cdot A\cdot\xi+\eta^*\cdot\xi+\xi^*\cdot\eta) = \det A\ e^{\eta^* A^{-1}\eta}$$

类比：

$$A_{xy}\to -i\left(i\gamma^\mu\partial_\mu - m + i\epsilon\right)\delta^4(x-y),$$

$$\eta_x\to i\eta(x),\qquad \bar{\eta}_x\to i\bar{\eta}(x),$$

$$\left(A^{-1}\right)_{xy}\to S_F(x-y)$$

也就是说，

$$-i\left(i\gamma^\mu\partial_\mu - m + i\epsilon\right)S_F(x-y) = \delta^4(x-y)$$

对 $S_F(x-y)$ 做Fourier变换，

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)}\tilde{S}_F(p)$$

则在动量空间有

$$-i\left(\gamma^\mu p_\mu - m + i\epsilon\right)\tilde{S}_F(p) = 1 \quad \Rightarrow \quad \tilde{S}_F(p) = \frac{i}{\gamma^\mu p_\mu - m + i\epsilon}$$

最终得到

$$Z_0[\eta,\bar{\eta}] = \mathcal{N}\det A \exp\left[-\int d^4x\ \bar{\eta}(x)S_F(x-y)\eta(y)\right]$$

取 $\mathcal{N}\det A = 1$，则有

$$Z_0[\eta,\bar{\eta}] = \exp\left[-\int d^4x\ \bar{\eta}(x)S_F(x-y)\eta(y)\right]$$

定义对经典Dirac场的泛函微商采用左微商

$$\frac{\delta}{\delta\eta(x)}\int d^4y\,\bar{\psi}(x)\eta(y) = -\bar{\psi}(x), \qquad \frac{\delta}{\delta\bar{\eta}(x)}\int d^4y\,\bar{\eta}(x)\psi(y) = \psi(x)$$

则从生成泛函 $Z_0[\eta,\bar{\eta}]$ 出发，我们可以对其进行多次对外源的泛函微商，就可以得到格林函数，

$$\langle 0|T\psi(x_1)\cdots\bar{\psi}(x_2)\cdots\psi(x_l)\cdots|0\rangle$$

$$= \left[\frac{\delta}{i\delta\bar{\eta}(x_1)}\cdots\left(-\frac{\delta}{i\delta\eta(x_2)}\right)\cdots\frac{\delta}{i\delta\bar{\eta}(x_l)}\cdots Z_0[\eta,\bar{\eta}]\right]_{\eta=\bar{\eta}=0}$$

a. 两点函数

$$\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle = \frac{\delta}{i\delta\bar{\eta}(x_1)}\left(-\frac{\delta}{i\delta\eta(x_2)}\right)Z_0[\eta,\bar{\eta}]\Big|_{\eta=\bar{\eta}=0}$$

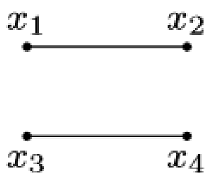
$$= \frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}\left[\exp\left(-\int d^4x\;\bar{\eta}(x)S_F(x-y)\eta(y)\right)\right]_{\eta=\bar{\eta}=0} = S_F(x_1-x_2)$$

b. 四点函数

$$\langle 0|T\psi(x_1)\bar{\psi}(x_2)T\psi(x_3)\bar{\psi}(x_4)|0\rangle = \left[\frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}\frac{\delta}{\delta\bar{\eta}(x_3)}\frac{\delta}{\delta\eta(x_4)}Z_0[\eta,\bar{\eta}]\right]_{\eta=\bar{\eta}=0}$$

$$= S_F(x_3-x_4)S_F(x_1-x_2) - S_F(x_1-x_4)S_F(x_3-x_2)$$

$$=$$



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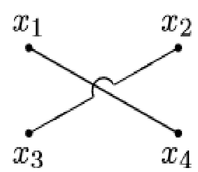


8.4 有相互作用时 Dirac 场路径积分量子化

我们考虑一个电磁场(有质量)、实标量场和费米场的理论，实标量场和费米场有Yukawa耦合，这个体系的拉氏量为

$$\mathcal{L}(x) = \mathcal{L}_0(\phi, A^\mu, \psi) - e\,\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) + g\bar{\psi}(x)\psi(x)\phi(x)$$

$$\mathcal{L}_0(\phi, A^\mu, \psi) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_A^2 A_\mu A^\mu + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

对电磁场 $A^\mu(x)$、$\phi(x)$、 $\bar{\psi}(x)$ 和 $\psi(x)$ 分别引入外源函数 $J_\mu(x)$、$J(x)$、$\eta(x)$ 和 $\bar{\eta}(x)$，生成泛函为

$$Z[J_\mu, J, \eta, \bar{\eta}] = \int DA D\phi D\bar{\psi} D\psi \exp\left[i\int d^4x \left(\mathcal{L}(x) + J_\mu A^\mu + J\phi + \bar{\eta}\psi + \bar{\psi}\eta\right)\right]$$

将相互作用部分分离出来：

$$Z[J_\mu, J, \eta, \bar{\eta}] = \int DA D\phi D\bar{\psi} D\psi \exp\left[i\int d^4x \left(\mathcal{L}_0(x) + J_\mu A^\mu + J\phi + \bar{\eta}\psi + \bar{\psi}\eta\right)\right]$$
$$\times \exp\left[-ie\int d^4x\ \bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\right]\exp\left[ig\int d^4x\ \bar{\psi}(x)\psi(x)\phi(x)\right]$$

$$
\begin{aligned}
&= \exp\left[-ie\int d^4x \frac{\delta}{\delta\eta(x)}\gamma^\mu \frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{i\delta J^\mu(x)}\right]\exp\left[ig\int d^4y\ \frac{\delta}{\delta\eta(y)}\frac{\delta}{\delta\bar{\eta}(y)}\frac{\delta}{i\delta J(y)}\right] \\
&\quad\times\int DAD\phi D\bar{\psi}D\psi \exp\left[i\int d^4x\left(\mathcal{L}_0(x)+J_\mu A^\mu+J\phi+\bar{\eta}\psi+\bar{\psi}\eta\right)\right] \\
&= \mathcal{N}\exp\left[-ie\int d^4x \frac{\delta}{\delta\eta(x)}\gamma^\mu \frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{i\delta J_\mu(x)}\right]\exp\left[ig\int d^4y\ \frac{\delta}{\delta\eta(y)}\frac{\delta}{\delta\bar{\eta}(y)}\frac{\delta}{i\delta J(y)}\right] \\
&\quad\times\exp\left[-\frac{1}{2}\iint d^4xd^4y\, J_\mu(x)D_F^{\mu\nu}(x-y)J_\nu(y)\right] \\
&\quad\times\exp\left[-\frac{1}{2}\iint d^4xd^4y\, J(x)D_F(x-y)J(y)\right] \\
&\quad\times\exp\left[-\int d^4x\ \bar{\eta}(x)S_F(x-y)\eta(y)\right]
\end{aligned}
$$

根据生成泛函的定义，各种格林函数都可以通过对上式中的外源相应地求泛函微商得到，如果对相互作用项再进行微扰展开，就会给出由耦合常数和传播子表示的微扰展开表达式。实际上，这种表达式和正则量子化中通过Wick收缩得到的形式是完全等价的。

1. $\psi - A - \psi$ 的顶角的 Feynman 规则

经典费米场取值是Grassmann数，我们约定按照就近求泛函微商的原则，即拉氏量中 $\bar{\psi}$ 出现在所有费米场最左侧，我们对其首先求泛函微商

$$i\Gamma_\mu^{(\bar{\psi}A\psi)}(x_1,x_2,x_3) = \frac{\delta^3(iS)}{\delta\psi(x_1)\delta\bar{\psi}(x_2)\delta A^\mu(x_3)}$$

$\boxed{-ie\gamma^\mu}$

$$= \frac{\delta}{\delta\psi(x_1)\delta\bar{\psi}(x_2)\delta A^\mu(x_3)}\left(-ie\int d^4x\ \bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x)\right)$$

$$= -ie\delta^4(x-x_3)\delta^4(x-x_2)\delta^4(x-x_1)\gamma_\mu$$

$$(2\pi)^4\delta^4(p_1+p_2+p_3)\ i\tilde{\Gamma}_\mu(p_1,p_2,p_3)$$

$$= \int d^4x_1 d^4x_2 d^4x_3\ e^{ip_1\cdot x_1}e^{ip_2\cdot x_2}e^{ip_3\cdot x_3}\, i\Gamma_\mu(x_1,x_2,x_3)$$

$$= -ie\gamma_\mu(2\pi)^4\ \delta^4(p_1+p_2+p_3)$$

2. $\psi - \phi - \psi$ 的顶角的 Feynman 规则

$\boxed{ig}$

$$i\tilde{\Gamma}^{\bar{\psi}\psi\phi}(p_1,p_2,p_3) = ig$$

9. 路径积分中的对称性

量子理论中的对称性体现在格林函数之间的关系上。

格林函数生成泛函：

$$Z[J_n] = \int D\phi_n \exp\left(i\int d^4x\ (\mathcal{L} + J_n\phi_n)\right) = \int D\phi_n\, exp\left(i\left(S[\phi_n] + \int d^4x J_n\phi_n\right)\right)$$

生成泛函 $Z[J_n]$ 只是外源函数 $J_n(x)$ 的泛函；场做变换，生成泛函不变！

$$\boxed{\phi \to \phi + \delta\phi, \qquad Z[J_n] \to Z[J_n]}$$

场的无穷小变换： $\phi'_n(x) = \phi_n(x) + \epsilon F_n[\phi_{n'}; x]$,

- 积分测度会变：$\phi'_{n'} = D\phi_n \det\left[\frac{\delta\phi'_{n'}}{\delta\phi_n}\right]$, $\frac{\delta\phi'_{n'}(y)}{\delta\phi_n(x)} = \delta_{nn'}\delta_{xy} + \epsilon\frac{\delta F_{n'}[\phi_{n''};y]}{\delta\phi_n(x)}$

$$\det\left[\frac{\delta\phi'_{n'}}{\delta\phi_n}\right] = \exp\left[\mathrm{Tr}\ln\left(1 + \epsilon\frac{\delta F_{n'}[\phi_{n''};y]}{\delta\phi_n(x)}\right)\right] \qquad \boxed{\det A = e^{\mathrm{Tr}\ln A}}$$

$$= 1 + \epsilon\,\mathrm{Tr}\frac{\delta F_{n'}[\phi_{n''};y]}{\delta\phi_n(x)} = 1 + \epsilon\int d^4x \frac{\delta F_n[\phi_{n'};x]}{\delta\phi_n(x)}$$

- 作用量会变：$\delta S[\phi] = \int d^4x \, \frac{\delta S[\phi]}{\delta \phi_n(x)} \delta \phi_n(x) = \epsilon \int d^4x \, \frac{\delta S[\phi]}{\delta \phi_n(x)} F_n[\phi_{n'}; x]$
- 外源项会变：$\int d^4x J_n \phi_n \to \int d^4x J_n \phi_n + \epsilon \int d^4x J_n F_n[\phi_{n'}; x]$

变量变换后的生成泛函

$$Z[J_n] = \int D\phi_n \, det\left[\frac{\delta \phi'_{n'}}{\delta \phi_n}\right] \exp\left(i \left(S[\phi] + \int d^4x J_n \phi_n\right)\right)$$

$$\times \exp\left(i \int d^4x \left[\frac{\delta S[\phi]}{\delta \phi_n} + J_n\right] F_n[\phi_{n'}; x] + \mathcal{O}(\epsilon^2)\right)$$

$$= \int D\phi_n \exp\left(i \left(S[\phi] + \int d^4x J_n \phi_n\right)\right)$$

$$\times \left[1 + \epsilon \int d^4x \left\{\frac{\delta F_n[\phi_{n'}; x]}{\delta \phi_n(x)} + i \left(\frac{\delta S[\phi]}{\delta \phi_n} + J_n\right) F_n[\phi_{n'}; x]\right\} + \mathcal{O}(\epsilon^2)\right]$$

$$\equiv \int D\phi_n \, exp\left(i \left(S[\phi] + \int d^4x J_n \phi_n\right)\right)$$

从而，我们得到泛函积分下的主方程（Schwinger-Dyson方程）

$$\int D\phi_n \exp\left(i\left(S[\phi]+\int d^4x J_n\phi_n\right)\right) \times \int d^4x \left\{\frac{\delta F_n[\phi_{n'};x]}{\delta\phi_n(x)}+i\left(\frac{\delta S[\phi]}{\delta\phi_n(x)}+J_n(x)\right)F_n[\phi_{n'};x]\right\}=0$$

量子理论中的对称性：场的关联函数（Green’s Function）可以通过泛函积分进行计算；量子理论的对称性体现在拉格朗日量的对称性（不变性）；最终体现在格林函数之间的关系上

9.1 运动方程

经典情形：物理体系运动轨迹满足作用量最小原理；
最小作用原理给出运动方程(Euler-Lagrange 方程)

$$0=\delta S=\int dx^4\left(\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi \quad \Longrightarrow \quad \frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}=0$$

量子理论：场量的平移变换给出格林函数之间相应的关系

$$\phi'_n(x) \to \phi_n(x) + \epsilon_n(x) \quad \Rightarrow \quad \frac{\delta F_n[\phi_{n'};x]}{\delta \phi_n(x)} = \frac{\delta \epsilon_n(x)}{\delta \phi_n(x)} = 0$$

（泛函积分测度不变）

由前面的主方程，有

$$I[J_n] = \int D\phi_{n'} \exp\left(i \left(S[\phi] + \int d^4x\, J_{n'} \phi_{n'} \right) \right) \left(\frac{\delta S[\phi]}{\delta \phi_n} + J_n \right) = 0$$

由

$$\left. \frac{\delta^k I[J_n]}{i\delta J_{n_1}(x_1) \cdots i\delta J_{n_k}(x_k)} \right|_{J=0} = 0$$

$$\boxed{\begin{aligned} \langle \dots \rangle_{J=0} &= \frac{1}{Z[0]} \int D\phi_{n'}\, e^{iS[\phi_{n'}]} \dots \\ &= \langle \Omega | T^* \dots | \Omega \rangle \end{aligned}}$$

当 $k = 0$ 时，

$$\left\langle \frac{\delta S[\phi]}{\delta \phi_n(x)} \right\rangle_{J=0} = 0$$

当 $k = 1$ 时，

$$\left\langle \frac{\delta S[\phi]}{\delta \phi_n(x)} \phi_{n_1}(x_1) \right\rangle_{J=0} = i\delta_{nn_1} \delta^4(x - x_1)$$

当 $k=2$ 时，$\left\langle \Omega \middle| T^* \frac{\delta S[\phi]}{\delta \phi_n(x)} \phi_{n_1}(x_1)\phi_{n_2}(x_2) \middle| \Omega \right\rangle$

$$= i\delta_{nn_1}\delta^4(x-x_1)\langle\Omega|\phi_{n_1}(x_1)|\Omega\rangle + i\delta_{nn_2}\delta^4(x-x_2)\langle\Omega|\phi_{n_2}(x_2)|\Omega\rangle$$

最一般地，有

$$\left\langle \Omega \middle| T^* \frac{\delta S[\phi]}{\delta \phi_n(x)} \phi_{n_1}(x_1)\phi_{n_2}(x_2)\cdots\phi_{n_k}(x_k)\right) \middle| \Omega \right\rangle$$

$$= \sum_{j=1}^{k} i\delta_{nn_j}\delta^4(x-x_j)\left\langle \Omega \middle| T^* \phi_{n_1}(x_1)\,\phi_{n_2}(x_2)\cdots\widehat{\phi}_{n_j}(x_j)\cdots\phi_{n_k}(x_k) \middle| \Omega \right\rangle$$

S-D 方程物理意义：相对于经典运动方程 $\frac{\delta S}{\delta \phi_n(x)} = 0$，量子理论中场的格林函数对运动方程的偏离体现在出现了一些接触项（有时又称作Schwinger项）。

T^* 算符：和编时乘积算符 T 有一些不同，

$$\langle\Omega|T^*\partial_\mu \ldots|\Omega\rangle = \partial_\mu\langle\Omega|T^* \ldots|\Omega\rangle$$

在正则量子化中，当考虑场的时空导数构成的格林函数时，由于时间导数算符和编时乘积算符 T是不对易的，所以当把时空导数算符编时移出乘积时，一般会多出一个接触项。

$$\langle\Omega|T^*\partial_\mu\mathcal{O}(x)\mathcal{O}'(y)\cdots|\Omega\rangle=\frac{1}{Z[0]}\int D\phi_{n'}\,e^{iS[\phi_{n'}]}\partial_\mu\mathcal{O}(x)\mathcal{O}'(y)\cdots$$

$$=\frac{1}{Z[0]}\partial_\mu\int D\phi_{n'}\,e^{iS[\phi_{n'}]}\partial_\mu\mathcal{O}(x)\mathcal{O}'(y)\cdots$$

$$=\partial_\mu\langle\Omega|T^*\partial_\mu\mathcal{O}(x)\mathcal{O}'(y)\cdots|\Omega\rangle$$

自由实标量场的情形，

$$\boxed{\int d^4x\,f(x)\partial^2 g(x)=\int d^4x\,g(x)\partial^2 f(x)}$$

$$\frac{\delta S}{\delta\phi(x)}=\frac{\delta}{\delta\phi(x)}\int d^4y\left[\frac{1}{2}\phi(y)\left(-\partial_y^2-m^2\right)\phi(y)\right]$$

$$=\int d^4y\left[\frac{1}{2}\delta^4(x-y)\left(-\partial_y^2-m^2\right)\phi(y)+\frac{1}{2}\phi(y)\left(-\partial_y^2-m^2\right)\delta^4(x-y)\right]$$

$$=\left(-\partial^2-m^2\right)\phi(x)$$

自由场满足运动方程 $\left(\partial^2+m^2\right)\phi(x)=0$， 所以有

$$\langle 0|T(-\partial^2-m^2)\phi(x)\phi(y)|0\rangle=0$$

按照 T^* 的定义，

$$\langle 0|T^*(-\partial^2-m^2)\phi(x)\phi(y)|0\rangle=\left(-\partial^2-m^2\right)\langle 0|T\phi(x)\phi(y)|0\rangle$$

$$=\left(-\partial^2-m^2\right)D_F(x-y)=i\delta^4(x-y)$$

T^* 算符和 T 算符关系的另一个例子

$$\partial_x^0\partial_y^0 T[\phi(x)\phi(y)] = \partial_x^0\partial_y^0[\theta(x^0-y^0)\phi(x)\phi(y)+\theta(y^0-x^0)\phi(y)\phi(x)]$$

$$= \partial_x^0[-\delta(x^0-y^0)\phi(x)\phi(y) + -\delta(x^0-y^0)\phi(y)\phi(x)$$

$$+\theta(x^0-y^0)\phi(x)\dot{\phi}(y)+\theta(y^0-x^0)\dot{\phi}(y)\phi(x)]$$

$$= \partial_x^0\left\{-\delta(x^0-y^0)[\phi(\vec{y},x^0),\phi(\vec{x},x^0)]\right\}$$

$$+\left(\delta(x^0-y^0)\phi(x)\dot{\phi}(y)-\delta(x^0-y^0)\dot{\phi}(y)\phi(x)\right)$$

$$+\left(\theta(x^0-y^0)\dot{\phi}(x)\dot{\phi}(y)+\theta(y^0-x^0)\dot{\phi}(y)\dot{\phi}(x)\right)$$

$$= i\delta^4(x-y) + T\left[\partial_x^0\phi(x)\partial_y^0\phi(y)\right]$$

在以上推导中，我们用到了等式对易关系

$$\left[\phi(\vec{y},x^0),\phi(\vec{x},x^0)\right]=0, \qquad \left[\phi(\vec{y},x^0),\dot{\phi}(\vec{x},x^0)\right]=i\delta^3(\vec{x}-\vec{y})$$

用 T^* 算符表示

$$\langle T^*\partial_x^0\phi(x)\partial_y^0(y)\rangle = \partial_x^0\partial_y^0\langle T\phi(x)\phi(y)\rangle = \left\langle 0\middle|T\partial_x^0\phi(x)\partial_y^0\phi(y)\middle|0\right\rangle + i\delta^4(x-y)$$

正则量子化中的编时乘积：

当考虑场的时空导数构成的格林函数时，由于时间导数算符和编时乘积算符不对易，当把时空导数算符移出编时乘积时，一般会多出一个接触项。也就说，编时乘积算符在两个算符等时的情况下没有很好的定义。

路径积分中的编时乘积：

路径积分是对场的所有的可能取值积分，积分测度和场的时空导数算符是对易的，可以移动到积分外面。所以在路径积分中的编时乘积是用T*算符来定义的。

S-D 方程也可以直接从格林函数的泛函积分定义得到：

$$
\begin{aligned}
G^{(k)}(x_1,x_2,\cdots,x_n) &= \langle\Omega|T\phi_{n_1}(x_1)\phi_{n_2}(x_2)\cdots\phi_{n_k}(x_k)|\Omega\rangle \\
&= \mathcal{N}\int D\phi\ e^{iS[\phi]}\phi_{n_1}(x_1)\phi_{n_2}(x_2)\cdots\phi_{n_k}(x_k)
\end{aligned}
$$

由于 $G^{(k)}$ 和 $\phi(x)$ 无关，在变换 $\phi(x)\to\phi(x)+\epsilon(x)$ 下， $\delta G^{(k)}=0$ ，即对任意的 $\phi_n(x)$，有 $\frac{\delta G^{(k)}}{\delta\phi_n(x)}\equiv 0$，则有（注意，积分测度不变）

$$
\begin{aligned}
&\left\langle\frac{\delta S[\phi]}{\delta\phi_n(x)}\phi_{n_1}(x_1)\phi_{n_2}(x_2)\cdots\phi_{n_k}(x_k)\right\rangle \\
&= (-i)\sum_{i=1}^{k}\delta_{nn_i}\delta(x-x_i)\left\langle\phi_{n_1}(x_1)\,\phi_{n_2}(x_2)\cdots\widehat{\phi}_{n_j}(x_j)\cdots\phi_{n_k}(x_k)\right\rangle
\end{aligned}
$$

9.2 内部对称变换

- 内部整体对称变换：$\phi_n(x) \to \phi'_n(x)$, $S[\phi_n] \to S[\phi'_n] = S[\phi_n]$
- Ward恒等式：如果泛函积分测度不变，相应的格林函数间满足的关系，
- 反常（anomaly）：如果泛函积分测度改变，则会带来反常。

1. $SU(N)$ 整体对称性

无穷小 $SU(N)$ 变换

$$\phi'_n(x) = \phi_n(x) + \epsilon\theta^a F_n^a[\phi_{n'};x] = \phi_n(x) + i\epsilon\theta^a T^a_{nn'}\phi_{n'}(x)$$

对称变换：
$$S[\phi_n] = S[\phi'_n] = S\left[\phi_n(x) + i\epsilon^a T^a_{nn'}\phi_{n'}(x)\right]$$

$$= S[\phi_n] + i\epsilon \int d^4x \frac{\delta S[\phi_n]}{\delta\phi_n(x)} F_n^a[\phi_{n'};x]$$

$$\int d^4x \frac{\delta S[\phi_n]}{\delta\phi_n(x)} F_n^a[\phi_{n'};x] = 0$$

积分测度不变：$SU(N)$ 群的生成元是无迹的（traceless）

$$\frac{\delta F_n^a[\phi_{n'};x]}{\delta\phi_n(x)} = i\,T^a_{nn}V_4 = 0$$

主方程（Master equation）

$$K[J] = \int D\phi_n \, e^{i\left(S[\phi]+\int d^4x J_n \phi_n\right)} \int d^4x \;\; J_n(x) F_n[\phi_{n'}; x] \equiv 0$$

由此可以得到格林函数满足的关系：

$n = 1$：

$$\frac{1}{Z}\frac{\delta K^a[J]}{i\delta J_{n'}(x_1)}\bigg|_{J=0} = -i\,\langle\Omega|TF_n^a[\phi_{n'}; x_1]|\Omega\rangle = 0$$

$n = 2$：

$$\frac{1}{Z}\frac{\delta^2 K^a[J\}}{i\delta J_{n_1}(x_1) i\delta J_{n_2}(x_2)}\bigg|_{J=0} = -i\langle\Omega|T\phi_{n_2}(x_2)F_{n_1}^a[\phi_{n'}; x_1]|\Omega\rangle$$

$$-i\langle\Omega|TF_{n_2}^a[\phi_{n'}; x_2]\phi_{n_1}(x_1)|\Omega\rangle$$

Ward-Takahashi 恒等式

$$\sum_{j=1}^{m}\Big\langle\Omega\Big|T\phi_{n_1}(x_1)\phi_{n_2}(x_2)\cdots F_{n_j}^a[\phi_{n'}; x_j]\cdots\phi_{n_m}(x_m)\Big|\Omega\Big\rangle = 0$$

W-T-I 恒等式也可以直接从格林函数的泛函积分定义得到：

$$
\begin{aligned}
G^{(k)}(x_1, x_2, \cdots, x_n) &= \langle \Omega | T\phi_{n_1}(x_1)\phi_{n_2}(x_2)\cdots\phi_{n_k}(x_k) | \Omega \rangle \\
&= \mathcal{N}\int D\phi \; e^{iS[\phi]}\phi_{n_1}(x_1)\phi_{n_2}(x_2)\cdots\phi_{n_k}(x_k)
\end{aligned}
$$

- 对称变换，积分测度不变；
- $G^{(k)}$ 和 $\phi(x)$ 无关，在变换 $\phi_n(x) \to \phi_n(x) + \epsilon\theta^a F_n^a[\phi_{n'}; x]$ 下， $\delta G^{(k)} = 0$

$$
\frac{\delta G^{(k)}}{\delta \phi_n(x)} F_n^a[\phi_{n'}; x] = 0
$$

$$
\sum_{i=1}^{k} \langle \phi_{n_1}(x_1)\, \phi_{n_2}(x_2) \cdots F_{n_i}^a[\phi_{n'}; x_i] \cdots \phi_{n_k}(x_k) \rangle = 0
$$

- W-T-I 在讨论非阿贝尔规范理论的 BRST 对称性时十分有用。

以复标量场（ $U(1)$ 对称性）为例来看Ward-Takahashi恒等式的应用

$$\delta\phi = i\theta\phi, \qquad \delta\phi^{+} = -i\theta\phi^{+} \Longrightarrow F_n[\phi] = i\phi, \qquad F_n[\phi^{+}] = -i\phi^{+}$$

两点函数满足关系

$$\langle\Omega|T\phi(x_1)\phi(x_2)|\Omega\rangle = \langle\Omega|T\phi^{+}(x_1)\phi^{+}(x_2)|\Omega\rangle = 0$$

$$\langle\Omega|T\phi(x_1)\phi^{+}(x_2)|\Omega\rangle - \langle\Omega|T\phi^{+}(x_1)\phi(x_2)|\Omega\rangle = 0$$

三点函数

$$\langle\Omega|T\phi(x_1)\phi(x_2)\phi^{+}(x_3)|\Omega\rangle = 0$$

它们是整体的荷守恒的反映。由于 ϕ 的量子数和 ϕ^{+} 的量子数符号相反，只有ϕ 和 ϕ^{+} 数目相同时，格林函数才不为零。

2. 诺特流(Noether's current)和相应的路径积分表述

- 诺特定理：一个连续的对称性对应一个守恒定律，存在一个守恒流。
- 经典场论中，对称变换是指运动方程在变换下不变，这是由作用量的不变性来保证的。
- 作用量是拉氏量的四维积分，拉氏量会有一个全散度项的不确定性，即 $\mathcal{L}(x)$ 和 $\mathcal{L}(x)+\theta^a\partial_\mu J^{a,\mu}(x)$ 具有相同的作用量。

考虑场的无穷小整体连续变换

$$\phi(x)\to\phi'(x)=\phi(x)+\theta^a F_n^a[\phi_{n'},x]$$

拉氏量的改变量

$$\Delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\phi_n}\theta^a F^a[\phi_{n'},x]+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\mu(\theta^a F^a[\phi_{n'},x])$$

$$=\theta^a\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}F^a[\phi_{n'},x]\right)+\theta^a\left[\frac{\partial\mathcal{L}}{\partial\phi_n}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\right]F^a[\phi_{n'},x]$$

经典运动方程（Euler-Lagrangian 方程）

$$\frac{\partial\mathcal{L}}{\partial\phi_n}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}=0$$

所以 $$\Delta\mathcal{L} \equiv \theta^a \partial_\mu J^{a,\mu} = \theta^a \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} F^a[\phi_{n'}, x] \right)$$

即存在守恒流：

$$\partial_\mu j^{a,\mu}(x) = 0, \qquad j^{a,\mu}(x) = J^{a,\mu}(x) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} F^a[\phi_{n'}, x] \right)$$

量子理论中的守恒流：

对场量做定域对称变换（并不是定域对称性）

$$\phi'_n(x) = \phi_n(x) + \theta^a(x) F_n^a[\phi_{n'}, x]$$

则 $$\delta S = \int d^4x \, \frac{\delta S[\phi]}{\delta \phi_n(x)} \theta^a(x) F_n^a[\phi_{n'}, x]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}(x)}{\partial \phi_n(x)} - \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial\left(\partial_\mu \phi_n(x)\right)} \right] \theta^a(x) F_n^a[\phi_{n'}, x \Big]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}(x)}{\partial \phi_n(x)} F_n^a[\phi_{n'}, x] + \frac{\partial \mathcal{L}(x)}{\partial\left(\partial_\mu \phi_n(x)\right)} \partial_\mu F_n^a[\phi_{n'}, x] \right] \theta^a(x)$$

$$- \int d^4x \; \partial_\mu \left(\frac{\partial \mathcal{L}(x)}{\partial\left(\partial_\mu \phi_n(x)\right)} F_n^a[\phi_{n'}, x] \right) \theta^a(x)$$

$$= \int d^4x \left\{\partial_\mu J^{a,\mu} - \partial_\mu\left(\frac{\partial\mathcal{L}(x)}{\partial\left(\partial_\mu\phi_n(x)\right)} F_n^a[\phi_{n'}, x]\right)\right\} \theta^a(x)$$

$$= \int d^4x\, \partial_\mu j^{a,\mu}(x)\theta^a(x)$$

显然，当 θ 和时空坐标无关时（即我们上述的变换是一个整体变换），可以将它提到积分外面，被积函数就是前面我们得到的守恒流的四散度，所以积分为零，将回到我们前一节的结论。但是当 θ 是时空的函数时，则情况完全不同。

考虑正常的对称变换，积分测度不变，则根据主方程有

$$\int D\phi\ e^{iS[\phi]+i\int d^4x J_n(x)\phi_n(x)} \int d^4x \left[\partial_\mu j^{a,\mu}(x) + J_n(x)F_n^a[\phi_{n'}; x]\right] \theta^a(x) = 0$$

由于 $\theta(x)$ 是任意的时空函数，上式要求在任意时空点都有：

$$B^a[J; x] = \int D\phi\ e^{iS[\phi]+i\int d^4x J_n(x)\phi_n(x)} \left[\partial_\mu j^{a,\mu}(x) + J_n(x)F_n^a[\phi_{n'}; x]\right] = 0$$

$B^a[J]$ 给出了又一个新的生成泛函，对外源 $J(x)$ 做泛函微商就可得到格林函数满足的一系列重要的恒等式，这类和守恒流有关的恒等式就是我们熟知的Ward恒等式。

Ward恒等式

$$\left\langle \Omega \middle| T^* \partial^{\mu}_{(x)} j^a_{\mu}(x) \phi_{n_1}(x_1) \phi_{n_2}(x_2) \cdots \phi_{n_m}(x_m) \middle| \Omega \right\rangle$$

$$= \partial^{\mu}_{(x)} \langle \Omega | T j^a_{\mu}(x) \phi_{n_1}(x_1) \phi_{n_2}(x_2) \cdots \phi_{n_m}(x_m) | \Omega \rangle$$

$$= i \sum_{j=1}^{m} \delta^4(x - x_j) \left\langle \Omega \middle| T \phi_{n_1}(x_1) \phi_{n_2}(x_2) \cdots F^a_{n_j}\left[\phi_{n'}, x_j\right] \cdots \phi_{n_m}(x_m) \middle| \Omega \right\rangle$$

Ward 恒等式告诉我们，经典场论中的守恒流关系 $\partial_{\mu} j^{a,\mu}(x) = 0$ 在算符层面上不再满足。在量子理论中，当流的四散度出现在格林函数中时，由于 T^* 算符定义的编时乘积的作用，当把时空导数提出矩阵元时，将会有一系列的Schwinger项出现，这是量子效应的体现。

需要强调的是，以上在路径积分中推导出来的Schwinger-Dyson 方程、Ward-Takahashi恒等式和Ward恒等式与微扰论无关，应该在微扰论的所有阶都成立。

3. QED中的Ward恒等式

量子电动力学的拉氏量为

$$\mathcal{L}_{QED} = -\frac{1}{4}F^2 + \bar{\psi}(i\gamma^\mu(\partial_\mu + ieA_\mu)\psi$$

该拉氏量在对称变换

$$\psi(x) \to (1 + ie\alpha(x))\psi, \qquad \bar{\psi}(x) \to \bar{\psi}(x)(1 - ie\alpha(x))$$

下是不变的（没有额外的四散度项），相应的守恒流为

$$j^\mu(x) = -\frac{\partial\mathcal{L}_{QED}(x)}{\partial\left(\partial_\mu\psi(x)\right)}\left(ie\psi(x)\right) = e(\bar{\psi}\gamma^\mu\psi)(x)$$

主方程

$$B[\eta, \bar{\eta}; x] = \int D\phi \; e^{i(S+\bar{\eta}\psi+\bar{\psi}\eta)}\left[\partial_\mu j^\mu(x) + ie\bar{\eta}(x)\psi(x) - ie\bar{\psi}(x)\eta(x)\right] = 0$$

$$\frac{\delta^2 B[\eta,\bar{\eta};x]}{\delta\bar{\eta}(x_1)\delta\eta(x_2)} = \frac{\delta}{\delta\bar{\eta}(x_1)}\int D\phi\ e^{i(S+\bar{\eta}\psi+\bar{\psi}\eta]}$$

$$\times\left\{-i\bar{\psi}(x_2)\left[\partial_\mu j^\mu(x) + ie\bar{\eta}(x)\psi(x) - ie\bar{\psi}(x)\eta(x)\right] + ie\bar{\psi}(x)\delta^4(x-x_2)\right\}$$

$$= \int D\phi\ e^{i(S+\bar{\eta}\psi+\bar{\psi}\eta]}\left\{\psi(x_1)\bar{\psi}(x_2)\left[\partial_\mu j^\mu(x) + \cdots\right]\right.$$

$$\left. - e\psi(x_1)\bar{\psi}(x)\delta^4(x-x_2) + e\psi(x)\bar{\psi}(x_2)\delta^4(x-x_1)\right\}$$

当取外源为零，并作适当归一化后，可以得到

$$\partial_\mu^{(x)}\langle\Omega|Tj^\mu(x)\psi(x_1)\bar{\psi}(x_2)|\Omega\rangle = -e\delta^4(x-x_1)\langle\Omega|\psi(x_1)\bar{\psi}(x_2)|\Omega\rangle$$
$$+e\delta^4(x-x_2)\langle\Omega|\psi(x_1)\bar{\psi}(x_2)|\Omega\rangle$$

用传播子和正规顶点表示就是

$$\partial_\mu^{(x)}\int d^4z\, d^4w\ G^{(2)}(x_1,z)e\Gamma^\mu(x,z,w)G^{(2)}(w,x_2)$$

$$= -e\left(\delta^4(x-x_1) - \delta^4(x-x_2)\right)G^{(2)}(x_1,x_2)$$

动量空间

$$LHS = e\int d^4x d^4x_1 d^4x_2\, e^{-ik\cdot x}e^{-ip\cdot x_2}e^{iq\cdot x_1}$$

$$\times\int d^4z d^4w \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} e^{-ip_1\cdot(x_1-z)}\widetilde{G}^{(2)}(p_1)\partial_\mu^{(x)}\Gamma^\mu(x,w,z)e^{-ip_2\cdot(w-x_2)}\widetilde{G}^{(2)}(p_2)$$

$$= e\int d^4x d^4z d^4w\, e^{-ik\cdot x}e^{-ip\cdot w}e^{iq\cdot z}\widetilde{G}^{(2)}(q)\partial_\mu^{(x)}\Gamma^\mu(x,w,z)\widetilde{G}^{(2)}(p)$$

$$= (2\pi)^4\delta^4(p+k-q)\left(ik_\mu\right)\widetilde{G}^{(2)}(p+k)e\tilde{\Gamma}^\mu(k,p,q)\widetilde{G}^{(2)}(p)$$

$$\boxed{(2\pi)^4\delta^4(k+p-q)\widetilde{\Gamma}^\mu(k+p,p) = \int d^4x d^4z d^4w\, e^{-ik\cdot x}e^{-ip\cdot w}e^{iq\cdot z}\Gamma^\mu(x,z,w)}$$

$$RHS = \int d^4x d^4x_1 d^4x_2\, e^{-ik\cdot x}e^{-ip\cdot x_2}e^{iq\cdot x_1}(-e)$$

$$\times\left(\delta(x-x_1)-\delta(x-x_2)\right)\int\frac{d^4p'}{(2\pi)^4}e^{-ip'\cdot(x_1-x_2)}\widetilde{G}^{(2)}(p')$$

$$= (-e)\int\frac{d^4p'}{(2\pi)^4}\widetilde{G}^{(2)}(p')\int d^4x_1 d^4x_2$$

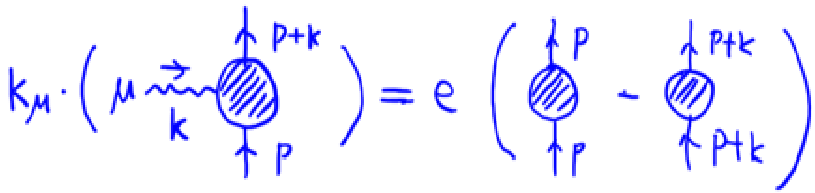
$$\times\left[e^{i(q-k-p')\cdot x_1}e^{i(p'-p)\cdot x_2} - e^{i(q-p')\cdot x_1}e^{i(p'-p-k)\cdot x_2}\right]$$

$$= (-e)\,(2\pi)^4(p+k-q)\left(\widetilde{G}^{(2)}(p)-\widetilde{G}^{(2)}(p+k)\right)$$

$$\tilde{G}^{(2)}(p+k)\left(ik_\mu\right)\tilde{\Gamma}^\mu(k,p,q)\tilde{G}^{(2)}(p) = -\left(\tilde{G}^{(2)}(p) - \tilde{G}^{(2)}(p+k)\right)$$

即

$$-ik_\mu\tilde{\Gamma}^\mu(p+k,p) = \tilde{G}^{(2)^{-1}}(p+k) - \tilde{G}^{(2)^{-1}}(p)$$



$i\Gamma^\mu(p+k,p)$

我们已经知道，费米子的全传播子可以表示为

$$\tilde{G}^{(2)}(p) = \frac{i}{\gamma^\mu p_\mu - m_0 - \Sigma(p) + i\epsilon} = \frac{iZ_2}{\gamma^\mu p_\mu - m + i\epsilon}$$

其中，$-i\Sigma(p)$ 电子的两点不可约顶点；如果 m 是电子的物理质量，则 Z_2 是电子场的在壳波函数重正化常数

$$Z_2 = 1 + \left.\frac{d\Sigma_2(p)}{d(\gamma^\mu p_\mu)}\right|_{p=m} = 1 + \delta Z_2,$$

带入 Ward 恒等式得到

$$k_\mu \tilde{\Gamma}^\mu(k+p,p) = Z_2^{-1}\gamma^\mu k_\mu$$

如果我们定义顶角重正化常数 Z_1:

$$\lim_{k\to 0}\tilde{\Gamma}^\mu(k+p,p) = Z_1^{-1}\gamma^\mu$$

则很显然有 $Z_1 = Z_2$，这是QED中的Ward恒等式的重要结论。这个关系可以在QED的单圈水平上得到验证。