QFT.fields.nb

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both from Schwartz and Peskin

Schwarz scalar

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To connect special relativity to the simple harmonic oscillator we note that the simplest possible Lorentz-invariant equation of motion that a field can satisfy is $\Box\phi = 0$. That is

$$\Box\phi == \left(\partial t^2 - \vec{\nabla}^2\right)\phi = 0 \tag{0.1}$$

分离变量法求解，The classical solutions are plane waves.For example, one solution is (0.2)

$$\varphi(x) = a[p, t]\,\mathrm{Exp}\left[I * \vec{p}\cdot\vec{x}\right] \tag{0.3}$$

where,

$$\left(\partial t^2 + \vec{p}\cdot\vec{p}\right)a[p, t] = 0 \tag{0.4}$$

This is exactly the equation of motion of a harmonic oscillator. 正好是谐振子方程

A general solution is

$$\phi[x, t] = \int \frac{d^3 p}{(2\pi)^3}\left(a[p] * \mathrm{Exp}[-I * p\, x] + a[p]^* * \mathrm{Exp}[I * p\, x]\right)$$

with $a[p]$ and $a[p]^*$ now just number, $\mathrm{p}\mu = (\omega\mathrm{p}, \vec{p})$, with $\omega\mathrm{p} \equiv |\,\vec{p}\,|$, $p\,x$ contains an implicit 4-vector contraction

$$p\,x = \mathrm{p}\mu\,\mathrm{x}\mu = \omega\mathrm{p}\,\mathrm{x0} - \vec{p}\cdot\vec{x} \tag{0.5}$$

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We then define quantum fields as integrals over creation and annihilation operators for each momentum:

我们定义一个量子场是对产生湮灭算符的积分

$$\phi0[\vec{x}] = \int \frac{d^3 p}{(2\pi)^3}\frac{1}{\sqrt{2\,\omega\mathrm{p}}}\left(a[\vec{p}] * \mathrm{Exp}\left[I * \vec{p}\cdot\vec{x}\right] + a^\dagger[\vec{p}] * \mathrm{Exp}\left[-I * \vec{p}\cdot\vec{x}\right]\right)$$

where the subscript 0 indicates this is a free field. The factor of $\frac{1}{\sqrt{2\,\omega\mathrm{p}}}$ is included for later convenience.

The quantum equation, Eq. (2.75), should be taken as the definition of a field operator $\phi0[\vec{x}]$ constructed from the creation and annihilation operators $a[p]$ and $a[p]^\dagger$.

它应该被当成是场算符的定义

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看看$\phi0$ 能做些什么，把它作用到真空上，投影出一个动量分量

$$\begin{aligned}\left\langle\vec{p}\right|\phi0[\vec{x}]\left|0\right\rangle &= \left\langle 0\right|\sqrt{2\,\omega\mathrm{p}}\,a[p]\int \frac{d^3 k}{(2\pi)^3}\frac{1}{\sqrt{2\,\omega\mathrm{k}}}\left(a[\vec{k}] * \mathrm{Exp}\left[I * \vec{k}\cdot\vec{x}\right] + a^\dagger[\vec{k}] * \mathrm{Exp}\left[-I * \vec{k}\cdot\vec{x}\right]\right)\left|0\right\rangle \\ &= \int \frac{d^3 k}{(2\pi)^3}\sqrt{\frac{\omega\mathrm{p}}{\omega\mathrm{k}}}\left(\mathrm{Exp}\left[I * \vec{k}\cdot\vec{x}\right]\left\langle 0\right|a[\vec{p}].a[\vec{k}]\left|0\right\rangle + a^\dagger[\vec{k}] * \mathrm{Exp}\left[-I * \vec{k}\cdot\vec{x}\right]\left\langle 0\right|a[\vec{p}].a^\dagger[\vec{k}]\left|0\right\rangle\right)\end{aligned} \tag{0.6}$$

$= \mathrm{Exp}\left[-I * \vec{p} \cdot \vec{x}\right]$

This is the same thing as the projection of a position state on a momentum state in one-particle quantum mechanics

与单粒子量子力学中的位置本征态投影到动量本征态上的结果一样。

$$\langle \vec{p} \mid \vec{x} \rangle = \mathrm{Exp}\left[-I * \vec{p}\, \vec{x}\right] \tag{0.7}$$

$$\phi 0[\vec{x}]\, \left|0\right\rangle = \left|\vec{x}\right\rangle, \tag{0.8}$$

that is $\phi 0[\vec{x}]$ 在 $\vec{x}$ 处创造一个粒子。

$$\left\langle\vec{x}\right| = \left\langle 0\right| \phi 0[\vec{x}], \text{ as well} \tag{0.9}$$

顺便，Fock 空间中有许多态$\left|\psi\right\rangle$都满足$\langle \vec{p} \mid \vec{x} \rangle = \mathrm{Exp}\left[-I * \vec{p}\, \vec{x}\right]$。

$\left\langle p\right|$ 只跟单粒子态有non – zero 矩阵元，所以加上两粒子态或者0粒子态，比如 $\phi 0^2[\vec{x}]\, \left|0\right\rangle$,

对于$\langle \vec{p} \mid \vec{x} \rangle$没有影响，即

$$\left|\psi\right\rangle = \left(\phi 0[\vec{x}] + \phi 0[\vec{x}]^2\right) \left|0\right\rangle \tag{0.10}$$

也满足$\left\langle\vec{p}\right| \left|\psi\right\rangle = \mathrm{Exp}\left[-I\, \vec{p}\, \vec{x}\right]$

2.3.2 Time dependence

在量子场论中，通常我们工作 Heisenberg picture 下，所有的时间依赖在算符中，比如ϕ或者a[p]。

对于自由场，每个动量$\vec{p}$的产生湮灭算符就是简简单单谐振子的那些算符。

这些算符满足简单的时间演化关系 Eq.2.55,$a[\boldsymbol{p}, t] = \mathrm{Exp}[-I * \omega[\boldsymbol{p}] * t] * a[\boldsymbol{p}]$,

和它的共轭$a[p, t]^\dagger = \mathrm{Exp}[I * \omega[\boldsymbol{p}] * t] * a[\boldsymbol{p}]^\dagger$.

其中$a[\boldsymbol{p}]$和$a[\boldsymbol{p}]^\dagger$（没有时间参数）是时间无关的。

我们可以定义一个量子标量场为，

$$\phi_0\left[\vec{x}, t\right] = \int \frac{d^3 p}{(2\pi)^3} * \frac{1}{\sqrt{2\,\omega_p}} \left(a[p]\, \mathrm{Exp}[-I\, p\, x] + a[p]^\dagger\, \mathrm{Exp}[I\, p\, x]\right) \tag{0.11}$$

其中p$\mu \equiv \left(\omega_p, \vec{p}\right)$，且$\omega_p = \mathrm{Abs}[\vec{p}]$. 下标0用来表示它是自由场.

****

需要说清楚的是，上面方程中没有物理内容，它只是个定义。

物理内容在$a[p]$和$a[p]^\dagger$的代数关系，和哈密顿H0里面。

尽管如此，我们会看到$a[p]$和$a[p]^\dagger$的这种组织形式在量子场论中很重要。

例如，或许你注意到积分仅对三动量进行，但是相位组合成了一个明显的洛伦兹不变形式。

这个场自动满足$\square\, \phi[x] = 0$.

如果标量场有质量m， 我们仍可以这么写，除了色散关系改成$\omega[p] \equiv \sqrt{\boldsymbol{p}^2 + m^2}$.

那么，场将会满足经典运动方程$\left(\square + m^2\right) \phi[x] = 0$.

** ** ** ** ** ** ** ** ** ** ** ***

现在让我们检查，自由哈密顿和我们期望的场的时间演化是一致的。

将自由场和哈密顿对易我们发现，

$$[\mathrm{H0},\ \phi 0[\boldsymbol{x}, t]] = \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} * \frac{1}{\sqrt{2\,\omega_k}} \left[\omega\mathrm{p}\left(a[p]^\dagger . a[p] + \frac{1}{2}\right), a[k]\, \mathrm{Exp}[-I\, k\, x] + a[k]^\dagger\, \mathrm{Exp}[I\, k\, x]\right] \tag{0.12}$$

$$= \int \frac{d^3 p}{(2\pi)^3} * \frac{1}{\sqrt{2\,\omega_k}} \left(-\omega[p] * a[p] * \mathrm{Exp}[-I\, p\, x] + \omega_p * a[p]^\dagger * \mathrm{Exp}[I\, p\, x]\right) \tag{0.13}$$

$$= -I * \partial t . \phi 0[\boldsymbol{x}, t] \tag{0.14}$$

正好是期望的结果。

对于任意哈密顿而言，量子场满足 Heisenberg 运动方程

$$-I*\partial t.\phi[x]=[H,\phi] \tag{0.15}$$

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在自由理论中，$H=\mathrm{H0}$, 海森堡方程和场的展开式是一致的。

在相互作用理论中，哈密顿1跟自由哈密顿不同，但是海森堡方程仍然满足，但我们几乎没有能求解的。

为了研究相互作用理论，经常把场的展开式写成和自由场一样的形式，这还挺有用的，

$$\phi[\boldsymbol{x},t]=\int\frac{d^3p}{(2\pi)^3}*\frac{1}{\sqrt{2\,\omega_p}}\left(a[\boldsymbol{p},t]\,\mathrm{Exp}[-I\,p\,x]+a[\boldsymbol{p},t]^\dagger\,\mathrm{Exp}[I\,p\,x]\right) \tag{0.16}$$

在任意确定的时间，完全的相互作用产生和湮灭算符$a[\boldsymbol{p},t]$和$a[\boldsymbol{p},t]^\dagger$满足相同的代数，相比于自由理论。

--由于时间平移不变性，Fock 空间在任何时间总是相同的。

因此，我们可以在任意一给定时间，定义相互作用的湮灭（产生）算符$a[\boldsymbol{p},t]$等于自由理论中的算符，

所以$a[\boldsymbol{p},\mathrm{t0}]=a[\boldsymbol{p}]$

** ** ** ** ** ** ** ** ** ** ** **

然而，相互作用理论中，创造特定动量状态$|p\rangle$的算符在时间演化过程中会彼此混合。

一般来说，我们无法确切求解相互作用理论的动力学。

取而代之的是，我们将H展开$H=\mathrm{H0}+\mathrm{Hint}$，

利用含时的微扰理论和Hint，计算振幅，就像量子力学里学过的那样。

在Chapter 7,我们利用这种方法推导费曼规则。

peskin scalar field

2.3 The Klein-Gordon Field as Harmonic Oscillators

p19 peskin 的标量场求解过程

以实标量场为例，the real Klein-Gordon field.

拿 oscillator 作为类比

单粒子或多粒子离散系统的对易关系是

$$[\mathrm{qi},\mathrm{pj}]=I*\delta_{\mathrm{ij}}; \tag{0.17}$$

$$[\mathrm{qi},\mathrm{qj}]=[\mathrm{pi},\mathrm{pj}]=0 \tag{0.18}$$

推广到连续系统，用Dirac delta代替Kronecker delta，

$$[\phi[\boldsymbol{x}],\pi[\boldsymbol{y}]]=I*\delta^3[\boldsymbol{x}-\boldsymbol{y}]; \tag{0.19}$$

$$[\phi[\boldsymbol{x}],\phi[\boldsymbol{y}]]=[\pi[x],\pi[y]]=0 \tag{0.20}$$

现在，暂时工作在 Schrodinger 表象中，ϕ和π是不依赖时间的。

等我们切换到 Heisenberg表象时，这些等时对易关系仍然成立，假如算符在同一时间被考虑。

哈密顿，现在是ϕ和π的函数，也变成了一个算符。

下面来研究哈密顿的本征谱。

不知道咋办，就先来个傅里叶变换。

$$\phi[\boldsymbol{x},t]=\int\frac{d^3p}{(2\pi)^3}\,\mathrm{Exp}[I\,\boldsymbol{p}.\boldsymbol{x}]\,\phi[\boldsymbol{p},t] \tag{0.21}$$

with $\phi^*[p]=\phi[-p]$ so that $\phi[\mathbf{x}]$ is real.

Klein-Gordon equation becomes

$$\left(\frac{\partial^2}{\partial t^2}+\left(|\,\boldsymbol{p}\,|^2+m^2\right)\right)\phi[\boldsymbol{p},t]=0 \tag{0.22}$$

这跟一个简单谐振子的方程是一模一样的，其频率是

$$\omega[\boldsymbol{p}] = \sqrt{\text{Abs}[p]^2 + m^2} \tag{0.23}$$

谐振子的本征谱我们已经知道如何得到。首先把哈密顿写成

$$H[\text{SHO}] = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 \phi^2 \tag{0.24}$$

然后把ϕ和p组成阶梯算符：

$$\phi = \frac{1}{\sqrt{2\,\omega}} (a + a^\dagger);\; p = -I * \sqrt{\frac{\omega}{2}}\, (a - a^\dagger) \tag{0.25}$$

正则对易关系$[\phi, p] = I$等价于

$$[a, a^\dagger] = 1 \tag{0.26}$$

现在哈密顿可以写成

$$H[\text{SHO}] = \omega\left(a^\dagger a + \frac{1}{2}\right) \tag{0.27}$$

$|0\rangle$ 满足 a $|0\rangle = 0$ 是 H 的本征态，本征值是 $\frac{1}{2}\omega$，zero – point energy

另外，利用

$$[\text{H`SHO}, a^\dagger] = \omega\, a^\dagger,\; [\text{H`SHO}, a] = -\omega\, a \tag{0.28}$$

可以验证

$$|n\rangle \equiv (a^\dagger)^n\, |0\rangle \tag{0.29}$$

是H`SHO本征值为$\left(n + \frac{1}{2}\right)\omega$的本征态。这些态可以遍历本征谱。

我们可以用类似的手段找出Klein-Gordon 哈密顿的本征谱。

现在场的每一个傅里叶模式被看成是一个独立的谐振子，拥有自己的a and $a^\dagger$。

类比地，我们可以写出，

$$\phi[\boldsymbol{x}] = \int \frac{d^3 p}{(2\,\pi)^3} * \frac{1}{\sqrt{2\,\omega[\boldsymbol{p}]}} (a[\boldsymbol{p}]\,\text{Exp}[I\, \boldsymbol{p}.\boldsymbol{x}] + a[\boldsymbol{p}]^\dagger\,\text{Exp}[-I\, \boldsymbol{p}.\boldsymbol{x}]) \tag{0.30}$$

$$\pi[x] = \int \frac{d^3 p}{(2\,\pi)^3} * (-I) * \sqrt{\frac{\omega[\boldsymbol{p}]}{2}}\, (a[\boldsymbol{p}]\,\text{Exp}[I\, \boldsymbol{p}.\boldsymbol{x}] - a[\boldsymbol{p}]^\dagger\,\text{Exp}[-I\, \boldsymbol{p}.\boldsymbol{x}]) \tag{0.31}$$

反向的表达式也容易得到，但很少用到。

也可以这么写，利用积分的对称性？？

$$\phi[\boldsymbol{x}] = \int \frac{d^3 p}{(2\,\pi)^3} * \frac{1}{\sqrt{2\,\omega[\boldsymbol{p}]}} (a[\boldsymbol{p}] + a[-\boldsymbol{p}]^\dagger)\,\text{Exp}[I\, \boldsymbol{p}.\boldsymbol{x}] \tag{0.32}$$

$$\pi[x] = \int \frac{d^3 p}{(2\,\pi)^3} * (-I) * \sqrt{\frac{\omega[\boldsymbol{p}]}{2}}\, (a[\boldsymbol{p}] - a[-\boldsymbol{p}]^\dagger)\,\text{Exp}[I\, \boldsymbol{p}.\boldsymbol{x}] \tag{0.33}$$

对易关系变成，

$$[a[\boldsymbol{p}], a[\mathbf{p\prime}]^\dagger] = (2\,\pi)^3\, \delta^3[\boldsymbol{p} - \mathbf{p\prime}] \tag{0.34}$$

利用它你可以验证ϕ和π的对易关系，

$$[\phi[\boldsymbol{x}], \pi[\mathbf{x\prime}]] = \int \frac{d^3 p\, d^3 \text{p}\prime}{(2\,\pi)^6} * \frac{-I}{2} * \sqrt{\frac{\omega\text{p}\prime}{\omega\text{p}}}\, ([a[-\boldsymbol{p}]^\dagger, a[\mathbf{p\prime}]] - [a[\boldsymbol{p}], a[-\mathbf{p\prime}]^\dagger])\,\text{Exp}[I\,(\boldsymbol{p}.\boldsymbol{x} + \mathbf{p\prime}.\mathbf{x\prime})] \tag{0.35}$$

$$= I * \delta^3[\mathbf{x} - \mathbf{x}\prime] \tag{0.36}$$

****

类似地，可以推导出

$$H = \int d^3 x \mathcal{H} = \int d^3 x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right) \tag{0.37}$$

$$H = \int \frac{d^3 p\, d^3 \mathrm{p}\prime}{(2\pi)^6} * \mathrm{Exp}[I(\boldsymbol{p} + \boldsymbol{p}\prime).\boldsymbol{x}]\Bigg(-\frac{\sqrt{\omega\mathrm{p} * \omega\mathrm{p}\prime}}{4}(a[\boldsymbol{p}] - a[-\boldsymbol{p}]^\dagger)(a[\mathbf{p}\prime] - a[-\mathbf{p}\prime]^\dagger) + \frac{-\boldsymbol{p}.\mathbf{p}\prime + m^2}{4}(a[\boldsymbol{p}] + a[-\boldsymbol{p}]^\dagger)(a[\mathbf{p}\prime] + a[-\mathbf{p}\prime]^\dagger)\Bigg) \tag{0.38}$$

$$= \int \frac{d^3 p}{(2\pi)^3}\omega[\boldsymbol{p}]\left(a[\boldsymbol{p}]^\dagger.a[\boldsymbol{p}] + \frac{1}{2}[a[\boldsymbol{p}], a[\boldsymbol{p}]^\dagger]\right) \tag{0.39}$$

第二项正比于$\delta[0]$, 一个无限大的c-数。

它就是零电能的求和。$\omega[\boldsymbol{p}]/2$

****

$\nabla\cdot(\phi\nabla\phi)$ 是一个边界项，在场论中，认为等于0. (0.40)

$$\nabla\cdot(\phi\nabla\phi) == \nabla\phi.\nabla\phi + \phi\nabla^2\phi == 0, \text{ so,} \tag{0.41}$$

$$(\nabla\phi)^2 == -\phi\nabla^2\phi \tag{0.42}$$

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利用海森堡方程，可以得到场$\phi[\mathbf{x}]$的时间依赖

$$I * \frac{\partial}{\partial t} O = [O, H] \tag{0.43}$$

$$\phi[x] = \phi[\boldsymbol{x}, t] = \mathrm{Exp}[I\,H\,t].\phi[\boldsymbol{x}].\mathrm{Exp}[-I\,H\,t] \tag{0.44}$$

对于自由场，海森堡方程与Klein-Gordon方程不矛盾，且可以化简为更简单的依赖关系。

** ***

$$\mathrm{Exp}[I * H * t].a[\vec{p}].\mathrm{Exp}[-I * H * t] = a[\vec{p}] * \mathrm{Exp}[-I * \mathrm{Ep} * t] \tag{0.45}$$

$$\mathrm{Exp}[I * H * t].a[\vec{p}]^\dagger.\mathrm{Exp}[-I * H * t] = a[\vec{p}]^\dagger * \mathrm{Exp}[I * \mathrm{Ep} * t] \tag{0.46}$$

$$\mathrm{Exp}\left[-I * \vec{P} * \vec{x}\right].a[\vec{p}].\mathrm{Exp}\left[I * \vec{P} * \vec{x}\right] = a[\vec{p}] * \mathrm{Exp}\left[I * \vec{p} * \vec{x}\right] \tag{0.47}$$

$$\mathrm{Exp}\left[-I * \vec{P} * \vec{x}\right].a[\vec{p}]^\dagger.\mathrm{Exp}\left[I * \vec{P} * \vec{x}\right] = a[\vec{p}]^\dagger * \mathrm{Exp}\left[-I * \vec{p} * \vec{x}\right] \tag{0.48}$$

****

$$\phi[x]\left|0\right\rangle == \int \frac{d^3 p}{(2\pi)^3}\frac{1}{2\,\mathrm{Ep}}\mathrm{Exp}\left[-I * \vec{p} * \vec{x}\right]\left|p\right\rangle \tag{0.49}$$

$$\phi\left[\vec{x}, t\right] = \int \frac{d^3 p}{(2\pi)^3}\frac{1}{\sqrt{2\,\mathrm{Ep}}}\left(a[\vec{p}] * \mathrm{Exp}[-I * p\cdot x] + a^\dagger[\vec{p}] * \mathrm{Exp}[I * p\cdot x]\right), \ \mathrm{p0} == \mathrm{Ep} \tag{0.50}$$

$$\pi\left[\vec{x}, t\right] = \frac{\partial}{\partial t}\phi\left[\vec{x}, t\right] \tag{0.51}$$

$$\phi[x] = \mathrm{Exp}[I * P\cdot x].\phi[0].\mathrm{Exp}[-I * p\cdot x], \ \mathrm{P}\mu = \left(H, \vec{P}\right) \tag{0.52}$$

****

$$[\phi[\vec{x}],\ \pi[\vec{y}]] = I*\delta3[\vec{x}-\vec{y}] \tag{0.53}$$

$$[\phi[\vec{x}],\ \phi[\vec{y}]] = [\pi[\vec{x}],\ \pi[\vec{y}]] = 0 \tag{0.54}$$

$$[a[\vec{p}],\ a[\vec{p'}]^{\dagger}] = (2\pi)^3\,\delta3[\vec{p}-\vec{p'}] \tag{0.55}$$

$$\backslash\backslash[a,\ a^{\dagger}] = 1,\ 1 \leftrightarrow (2\pi)^3\,\delta3[\vec{p}-\vec{p'}]\ \backslash\backslash$$

$$\backslash\backslash \int \frac{d^3 p}{(2\pi)^3}\,\mathrm{Exp}[-I*\vec{p}*\vec{x}] == (2\pi)^3\,\delta3[\vec{x}]\ \backslash\backslash$$

****

$$H = \int \frac{d^3 p}{(2\pi)^3}\,\omega[\vec{p}]\left(a[\vec{p}]^{\dagger}.a[\vec{p}] + \frac{1}{2}[a[\vec{p}],\ a[\vec{p}]^{\dagger}]\right) \tag{0.56}$$

$$P = -\int d^3 x\,\pi[\vec{x}].\nabla\phi[x] = \int \frac{d^3 p}{(2\pi)^3}\,\vec{p}*a[\vec{p}]^{\dagger}.a[\vec{p}] \tag{0.57}$$

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we naturally choose to normalize the vacuum state so that

$$\langle 0 \mid 0\rangle = 1 \tag{0.58}$$

the one-particle state $|\vec{\mathrm{p}}\rangle \propto a[\vec{p}]^{\dagger}|0\rangle$ will also appear quite often.

the simplest normalization

$$\langle\vec{p}\mid\vec{q}\rangle\ =\ (2\pi)^3\,\delta3[\vec{p}-\vec{q}] \tag{0.59}$$

is not Lorentz invariant, under a 3-direction boost,

$$\mathrm{p3}\prime = \gamma*(\mathrm{p3}+\beta*E),\ \ \mathrm{E}\prime = \gamma*(E+\beta*\mathrm{p3}) \tag{0.60}$$

using the delta function identity,

$$\delta[f[x]-f[\mathrm{x0}]] = \frac{1}{|\mathrm{f}\prime[\mathrm{x0}]|}\,\delta[x-\mathrm{x0}] \tag{0.61}$$

$$\delta3[\vec{p}-\vec{q}] = \delta3[\vec{p'}-\vec{q'}]\cdot\frac{d\,\mathrm{p3}\prime}{d\,\mathrm{p3}} = \delta3[\vec{p'}-\vec{q'}]*\gamma*\left(1+\beta*\frac{dE}{d\,\mathrm{p3}}\right) \tag{0.62}$$

$$= \delta3[\vec{p'}-\vec{q'}]*\frac{\gamma}{E}*(E+\beta*\mathrm{p3}) \tag{0.63}$$

$$= \delta3[\vec{p'}-\vec{q'}]*\frac{\mathrm{E}\prime}{E} \tag{0.64}$$

The volumes are not invariant under boosts; V in rest into V/γ in a boosted frame

but Ep*$\delta3[\vec{p}-\vec{q}]$ is Lorentz invariant. We therefore define

$$|\vec{p}\rangle = \sqrt{2\,\mathrm{Ep}}\ a[\vec{p}]^{\dagger}\,|0\rangle \tag{0.65}$$

so that (0.66)

$$\langle\vec{p}\mid\vec{q}\rangle = 2\,\mathrm{Ep}*(2\pi)^3*\delta3[\vec{p}-\vec{q}] \tag{0.67}$$

the factor of 2 is unnecessary, but is convenient because of the factor of 2 in Eq.(2.25)

那里的2来自于正则量子化的时候谐振子的坐标表换

with this normalization we must divide by 2Ep in other places.

For example, the completeness relation for the one-particle states is

$$(\mathbb{1})_{1-\text{particle}} = \int \frac{d^3 p}{(2\pi)^3} \left|\vec{p}\right\rangle * \frac{1}{2\,\text{Ep}} * \left\langle\vec{p}\right| \tag{0.68}$$

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\,\text{Ep}} = \int \frac{d^4 p}{(2\pi)^4} (2\pi)\,\delta\left[p^2 - m^2\right]$$

Klein-Gorden propagator

$$\langle 0|[\phi[x], \phi[y]]\,|0\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\,\text{Ep}} (\text{Exp}[-I * p\cdot(x-y)] - \text{Exp}[I * p\cdot(x-y)]) \tag{0.69}$$

$$\underset{\text{x0>y0}}{=} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d\,\text{p0}}{2\pi I} * \frac{-1}{p^2 - m^2} \text{Exp}[-I * p\cdot(x-y)] \tag{0.70}$$

for x0>y0, we close the contour below, picking up both poles to obtain previous line

for x0<y0, we close the contour above, giving zero.

this is called retarded Green’s function

$$\text{DR}[x-y] \equiv \theta[\text{x0} - \text{y0}]\,\langle 0|[\phi[x], \phi[y]]\,|0\rangle \tag{0.71}$$

$$\left(\partial^2 + m^2\right).\text{DR}[x-y] == -I * \delta 4[x-y] \tag{0.72}$$

the DR[x-y] is a Green’s function of the Klein-Gordon Operator.

****

the p0-integral can be evaluated according to four different contours, a extremely useful one, called the Feynman prescription.

$$\text{DF}[x-y] \equiv \int \frac{d^4 p}{(2\pi)^4} * \frac{I}{p^2 - m^2 + I*\epsilon} \text{Exp}[-I * p\cdot(x-y)] \tag{0.73}$$

the poles are at

$$\text{p0} = \pm(\text{Ep} - I*\epsilon),\ \text{2 and 4 quadrant} \tag{0.74}$$

for x0>y0, we close the contour below,
for x0<y0, we close the contour above,

$$\text{DF}[x-y] = \theta[\text{x0} - \text{y0}]\,\langle 0|\,\phi[x].\phi[y]\,|0\rangle + \theta[\text{y0} - \text{x0}]\,\langle 0|\,\phi[y].\phi[x]\,|0\rangle \tag{0.75}$$

$$\equiv \langle 0|\,T[\phi[x].\phi[y]]\,|0\rangle \tag{0.76}$$

时间分量大的放在左边，与数轴顺序相反

$$\left(\partial^2 + m^2\right).\text{DF}[x-y]$$

$$= \left(\partial^2 + m^2\right).\left(\theta[\text{x0} - \text{y0}] * \left(\langle 0|\,\phi[x].\phi[y]\,|0\rangle - \langle 0|\,\phi[y].\phi[x]\,|0\rangle\right) + (\theta[\text{y0} - \text{x0}] + \theta[\text{x0} - \text{y0}]) * \langle 0|\,\phi[y].\phi[x]\,|0\rangle\right)$$

$$-I * \delta 4[x-y] + \left(\partial^2 + m^2\right).\langle 0|\,\phi[y].\phi[x]\,|0\rangle$$

$$= -I * \delta 4[x-y]$$

DF 对于x，y是对称的。

Complex scalar field

Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4 x \left(\partial\mu\,\phi^*\,\partial\mu\,\phi^* - m^2\,\phi^*\,\phi\right) \tag{0.77}$$

****

$$\phi[\vec{x}, t] = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\,\text{Ep}}} \left(a[\vec{p}] * \text{Exp}[-I * p\cdot x] + b^\dagger[\vec{p}] * \text{Exp}[I * p\cdot x]\right),\ \text{p0} == \text{Ep} \tag{0.78}$$

$$\phi[\vec{x}, t]^{\dagger} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\,\mathrm{Ep}}} \left(b[\vec{p}] * \mathrm{Exp}[-I * p \cdot x] + a^{\dagger}[\vec{p}] * \mathrm{Exp}[I * p \cdot x]\right),\ \mathrm{p0} == \mathrm{Ep} \tag{0.79}$$

****

$$\langle 0 | \, T[\phi(x).\phi(y)^{\dagger}] \, | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} * \frac{I}{p^2 - m^2 + I * \epsilon} \mathrm{Exp}[-I * p \cdot (x - y)] \tag{0.80}$$

if x0>y0

$$\langle 0 | \, T[\phi(x).\phi(y)^{\dagger}] \, | 0 \rangle = \langle 0 | \, \phi(x).\phi(y)^{\dagger} \, | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\,\mathrm{Ep}}} \int \frac{d^3 \mathrm{p}\prime}{(2\pi)^3} \frac{1}{\sqrt{2\,\mathrm{Ep}\prime}} \tag{0.81}$$

$$\langle 0 | \left(a[\vec{p}] * \mathrm{Exp}[-I * p \cdot x]\right).\left(a^{\dagger}[\vec{p}] * \mathrm{Exp}[I * \mathrm{p}\prime \cdot y]\right) | 0 \rangle \tag{0.82}$$

粒子 y to x

if x0<y0

$$\langle 0 | \, T[\phi(x).\phi(y)^{\dagger}] \, | 0 \rangle = \langle 0 | \, \phi(y)^{\dagger}.\phi(x) \, | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\,\mathrm{Ep}}} \int \frac{d^3 \mathrm{p}\prime}{(2\pi)^3} \frac{1}{\sqrt{2\,\mathrm{Ep}\prime}} \tag{0.83}$$

$$\langle 0 | \left(b[\vec{p}] * \mathrm{Exp}[-I * \mathrm{p}\prime \cdot y]\right).\left(b^{\dagger}[\vec{p}] * \mathrm{Exp}[I * p \cdot x]\right) | 0 \rangle \tag{0.84}$$

反粒子运行 x to y

两部分加起来，构成费曼传播子

exercise

It is easiest to analyze this theory by considering ϕ[x] and ϕ[x]*

rather than the real and imaginary parts of ϕ[x] as the basic dynamical variables.

1. Show that the Hamiltonian is

$$H = \int d^3 x \left(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi\right) \tag{0.85}$$

2. Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m

Rewrite the conserved charge

$$Q = \int d^3 x * \frac{I}{2} (\phi^* \pi^* - \pi \phi) \tag{0.86}$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

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7.1 Field-Strength Renormalization

在本章中我们将审查两点关联函数的解析结构，

$$\langle \Omega | \, T[\phi[x] * \phi[y]] \, | \Omega \rangle,\ \text{or},\ \langle \Omega | \, T\left[\psi[x].\overline{\psi}[y]\right] | \Omega \rangle \tag{0.87}$$

在自由场论中，两点函数$\langle \Omega \,|\, T[\phi[x] * \phi[y]] \,|\, \Omega \rangle$有一个简单的解释，它是粒子从y点传播到x点的振幅。

这种诠释在相互作用理论中能在多大程度上成立呢？

这个分析只跟相对论和量子理论的general principles 有关。不依赖于相互作用的细节或者微扰展开。

以标量场为例。

****

插入状态的完备集。

由于场的（总）动量算符 P 和哈密顿 H 对易，所以可以选取 P 的本征态做为标记。

利用Lorentz 不变性，所有状态可以写成 0 动量本征态的 boost，

The eigenvalues of the 4-momentum operator $P\mu = \{H,\ P\}$ organize themselves into hyperboloids, as shown in Fig.7.1

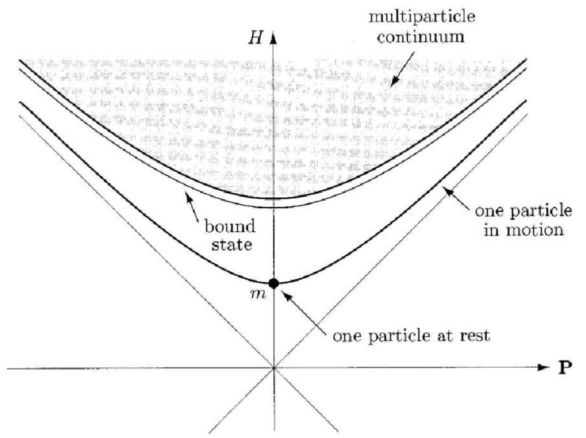

Figure 7.1. The eigenvalues of the 4-momentum operator $P^\mu = (H, \mathbf{P})$ occupy a set of hyperboloids in energy-momentum space. For a typical theory the states consist of one or more particles of mass m. Thus there is a hyperboloid of one-particle states and a continuum of hyperboloids of two-particle states, three-particle states, and so on. There may also be one or more bound-state hyperboloids below the threshold for creation of two free particles.

****

Recall form Chapter 2 单粒子态的完备性关系为

$$(\mathbf{1})_{1-\text{particle}} = \int \frac{d^3 p}{(2\pi)^3} * \frac{1}{2\,\mathrm{Ep}} \left|\boldsymbol{p}\right\rangle \left\langle\boldsymbol{p}\right| \tag{0.88}$$

可以类似地写出整个 Hilbert space 的完备性关系。

令 $\left|\lambda\mathbf{p}\right\rangle$ 是 $\left|\lambda\mathbf{0}\right\rangle$ 的 boost，其动量为$\boldsymbol{p}$，并假设态 $|\lambda\mathrm{p}\rangle$ 如同 $|\,p\rangle$ 那样，经过了适当地相对论性归一化。

$$\text{let, } \mathrm{Ep}[\lambda] \equiv \sqrt{|\,\boldsymbol{P}\,|^2 + \mathrm{m}\lambda^2}\ , \tag{0.89}$$

where $\mathrm{m}\lambda$ is the “mass” of the state $\left|\lambda\mathbf{p}\right\rangle$, that is, the energy of the state $\left|\lambda\mathbf{0}\right\rangle$.

则欲得到的完备性关系为，

$$\mathbf{1} = \left|\Omega\right\rangle\left\langle\Omega\right| + \sum_\lambda \int \frac{d^3 p}{(2\pi)^3} * \frac{1}{2\,\mathrm{Ep}[\lambda]} \left|\boldsymbol{\lambda}\mathbf{p}\right\rangle\left\langle\boldsymbol{\lambda}\mathbf{p}\right| \tag{0.90}$$

sum runs over all zero-momentum states $\left|\lambda\mathbf{0}\right\rangle$.

把完备性关系插入到两点关联函数中，真空项目由于洛伦兹不变性为0，

假设x0 > y0 , 两点关联函数变成，

$$\left\langle\Omega\right| T[\phi[x] * \phi[y]] \left|\Omega\right\rangle = \sum_\lambda \int \frac{d^3 p}{(2\pi)^3} * \frac{1}{2\,\mathrm{Ep}[\lambda]} \left\langle\Omega\right| \phi[x] \left|\lambda\boldsymbol{p}\right\rangle \left\langle\lambda\boldsymbol{p}\right| \phi[y] \left|\Omega\right\rangle \tag{0.91}$$

利用Lorentz 不变性，

$$\begin{aligned} &\left\langle\Omega\right| \phi[x] \left|\lambda\boldsymbol{p}\right\rangle = \left\langle\Omega\right| \mathrm{Exp}[I\,P.x].\phi[0].\mathrm{Exp}[-I\,P.x] \left|\lambda\boldsymbol{p}\right\rangle \\ &= \left\langle\Omega\right| \phi[0] \left|\lambda\boldsymbol{p}\right\rangle \mathrm{Exp}[-I * p.x],\ [\mathrm{p0} = \mathrm{Ep}] \end{aligned} \tag{0.92}$$

此处的p由算符变成动量

$$= \left\langle\Omega\right| \phi[0] \left|\lambda\mathbf{0}\right\rangle \mathrm{Exp}[-I * p.x],\ [\mathrm{p0} = \mathrm{Ep}] \tag{0.93}$$

implements a Lorentz boost from **P to 0,** and use $U\,\phi[0]\,U^{-1} = \phi[0]$. For a field with spin, we would need to keep track of its nontrivial transformation.

引入一个对 p0 的积分，我们把两点函数变成

$$\left\langle\Omega\right| T[\phi[x] * \phi[y]] \left|\Omega\right\rangle = \sum_\lambda \int \frac{d^4 p}{(2\pi)^4} * \frac{1}{p^2 - \mathrm{m}\lambda^2 + I\,\epsilon}\ \mathrm{Exp}[-I\,p.(x-y)]\ \mathrm{Abs}\left[\left\langle\Omega\right| \phi[0] \left|\lambda 0\right\rangle\right]^2 \tag{0.94}$$

注意到相当于是费曼传播子DF$[x-y]$ 中的m换成了mλ

类似的关系对于y0 > x0 也成立。两种情形可以被总结为一个通用的表示(Kallen-Lehmann spectral representation)

$$\langle\Omega|\, T[\phi[x]*\phi[y]]\,|\Omega\rangle = \int_0^\infty \frac{d M^2}{2\pi}\,\rho[M^2]*\mathrm{DF}[x-y;\, M^2] \tag{0.95}$$

$$= \int_0^\infty \frac{d M^2}{2\pi}\,\rho[M^2]*\int \frac{d^4 p}{(2\pi)^4}*\frac{I}{p^2-\mathrm{m}\lambda^2+I\,\epsilon}\,\mathrm{Exp}[-I\,p.(x-y)] \tag{0.96}$$

where $\rho[M^2]$ is a positive spectral density function

$$\rho[M^2] = \sum_\lambda (2\pi)\,\delta[M^2-\mathrm{m}\lambda^2]\,\mathrm{Abs}[\langle\Omega|\,\phi[0]\,|\lambda 0\rangle]^2 \tag{0.97}$$

典型的谱密度$\rho[M^2]$ 见下图

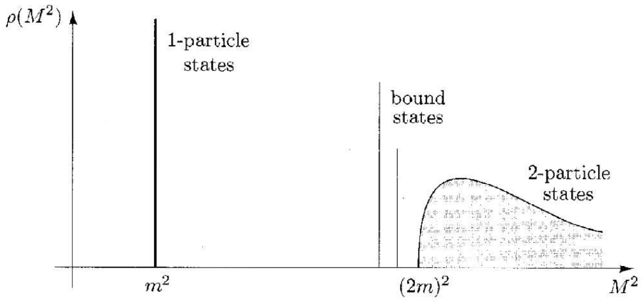

Figure 7.2. The spectral function $\rho(M^2)$ for a typical interacting field theory. The one-particle states contribute a delta function at m^2 (the square of the particle's mass). Multiparticle states have a continuous spectrum beginning at $(2m)^2$. There may also be bound states.

单粒子态贡献一个孤立的 delta 函数，

$$\rho[M^2] = 2\pi\,\delta[M^2-m^2]*Z + \left(\text{nothing else until } M^2 \gtrsim (2\,m)^2\right) \tag{0.98}$$

Z 是公式中由平方矩阵元给出的一个数字。我们把Z叫做场强重整化常数。

量m是单粒子态的质量，是粒子静止状态的能量本征值。

这个量一般与出现在拉氏量中的mass parameter不同。

为了区分，把拉氏量中的叫做 m0，把这里的叫做 physical 质量 of the ϕ boson.

only the physical mass m is observable.

****

自由场的各个动量模式互不干扰，

相互作用场的各个动量模式可以通过拉氏量中的相互作用项进行动量能量转移，

也就是产生湮灭算符会发生混合。

所以只要能量足够，单粒子态可以变成多粒子态，但是此过程应该仍要遵守能动量守恒，总概率守恒。

因为算符的演化是幺正的，哈密顿和总动量算符是时空平移不变的。

****

单粒子态必须是on-shell的，自由场的产生湮灭算符的单个模式必须是on-shell的，由相对论决定。

但是态是可以线性叠加的，场算符是可以线性叠加的，场一般来说，不必是on-shell 的

海森堡运动方程的解

$$a \to \mathrm{Exp}[I\,H*t].a.\mathrm{Exp}[-I\,H*t] \tag{0.99}$$

$$= a[\vec{p}\,]*\mathrm{Exp}[-I*\mathrm{Ep}*t]\text{，对于自由场} \tag{0.100}$$

$$= \mathrm{XXXX}\text{，对于相互作用场，一般无法求解，} \tag{0.101}$$

无法保持结果为，算符简单乘上一个时间因子，不同动量模式的算符会互相混合，

也就是只要能量够高，就可以产生其他动量的粒子，

既关联函数的传播过程中，可以产生多粒子中间态。

但是强度最高的应该还是p^2 pole 的单粒子态。

****

上面的谱分解产生了如下的傅里叶空间两点函数形式，

$$\backslash * \langle\Omega|\, T[\phi[x]*\phi[y]]\, |\Omega\rangle = \int_0^\infty \frac{d M^2}{2\pi}\, \rho[M^2]*\mathrm{DF}[x-y;\, M^2]*\backslash$$

$$\langle\Omega|\, T[\phi[x]*\phi[0]]\, |\Omega\rangle = \int_0^\infty \frac{d M^2}{2\pi}\, \rho[M^2]*\mathrm{DF}[x;\, M^2]$$

$$\int d^4 x\, \mathrm{Exp}[I*p.x]\, \langle\Omega|\, T[\phi[x]*\phi[0]]\, |\Omega\rangle = \int d^4 x\, \mathrm{Exp}[I\, p.x] \int_0^\infty \frac{d M^2}{2\pi}\, \rho[M^2]*\mathrm{DF}[x;\, M^2] = \tag{0.102}$$

$$= \int d^4 x\, \mathrm{Exp}[I\, p.x] \left(\int_0^\infty \frac{d M^2}{2\pi} \left(2\pi\,\delta[M^2-m^2]*Z + \left(\text{nothing else until } M^2 \gtrsim (2\,m)^2\right)\right)* \right.$$

$$\left. \int \frac{d^4 \mathrm{p}\prime}{(2\pi)^4} * \frac{I}{\mathrm{p}\prime^2 - M^2 + I\,\epsilon}\, \mathrm{Exp}[-I\, \mathrm{p}\prime.(x)] \right)$$

first term,

$$= \int d^4 x\, \mathrm{Exp}[I\, p.x] * \int \frac{d^4 \mathrm{p}\prime}{(2\pi)^4} * \frac{I*Z}{\mathrm{p}\prime^2 - m^2 + I\,\epsilon}\, \mathrm{Exp}[-I\, \mathrm{p}\prime.(x)]$$

$$= (2\pi)^4 \int \frac{d^4 \mathrm{p}\prime}{(2\pi)^4} * \frac{I*Z}{\mathrm{p}\prime^2 - m^2 + I\,\epsilon} * \delta[p-\mathrm{p}\prime]$$

$$= \frac{I*Z}{p^2 - m^2 + I\,\epsilon} \tag{0.103}$$

similarly, second term,

$$\int_{\sim 4\,m^2}^{\infty} \frac{d M^2}{2\pi}\, \rho[M^2] * \frac{I}{p^2 - M^2 + I\,\epsilon} \tag{0.104}$$

****

The analytic structure of this function 在 p^2 复平面上，显示在下图

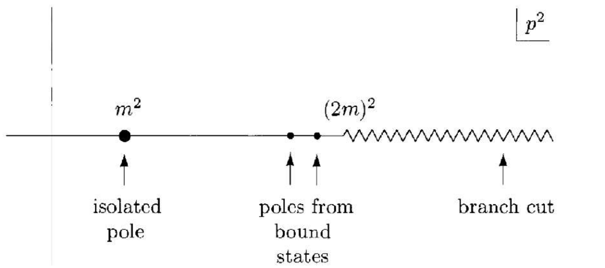

Figure 7.3. Analytic structure in the complex p^2-plane of the Fourier transform of the two-point function for a typical theory. The one-particle states contribute an isolated pole at the square of the particle mass. States of two or more free particles give a branch cut, while bound states give additional poles.

第一项贡献一个单极点在 $p^2 = m^2$, 第二项贡献一个支割线，从$p^2 = (2\,m)^2$开始。

（这里的意思应该是两粒子散射态之间可以有势能，所以是连续的）

如果有两粒子束缚态的话，会出现额外的pole，在支割线起点的能量以下。

****

在section 2.4， 我们得到对于自由标量场的两点关联函数：

$$\int d^4 x \,\mathrm{Exp}[I\, p.x] \left\langle 0 \right| T[\phi[x] * \phi[0]] \left| 0 \right\rangle = \frac{I}{p^2 - m^2 + I\,\epsilon} \tag{0.105}$$

解释成，for x0 > 0, 粒子从0到x点的振幅。

****

对于一个general 的相互作用场，两点函数的形式是类似的。

振幅是一系列标量传播子的求和，传播由算符$\phi[0]$ 创造的各种状态。

自由和相互作用情形有两个不同。

first，Eq.7.9 包含场强重整化因子 $Z = \mathrm{Abs}[\langle \lambda 0 \mid \phi[0] \mid \Omega \rangle]^2$, $\phi[0]$ 从真空创造一个指定状态λ0的概率。

在式 7.10 中，这个因子默认为1，$\left\langle p \right| \phi[0] \left| 0 \right\rangle = 1$ 在自由理论中。

second，式7.9 包含多粒子中间态的贡献，其质量谱是连续的。

在自由理论中，$\phi[0]$只能从真空创造单粒子。

考虑了这两点不同，7.9是7.10的直接推广。

****

单粒子态和多粒子态对于 式7.9的贡献可以区分，通过它们解析结构奇点的强度。

单粒子中间态的贡献是通过 p^2的pole，而多粒子是通过 weaker 的支割线奇点。

推广到多点函数，这一点很重要，可以用来推导LSZ公式。

****

推广到高自旋的场是很直接的。主要的复杂来自于洛伦兹boosts 的时候，会有一些 nontrivial 的变换。

总的来说，会需要几个不变函数来表示多粒子态。

但这个复杂性不影响主要结果，就是 p^2的pole 只来自于单粒子态。

例如 Dirac 场的两点函数，结构如下，

$$\int d^4 x \,\mathrm{Exp}[I\, p.x] \left\langle \Omega \right| T\left[\psi[x] * \overline{\psi}[0]\right] \left| \Omega \right\rangle = \frac{I * \mathrm{Z2} * \sum_s u^s[p].\overline{u}^s[p]}{p^2 - m^2 + I\,\epsilon} + \cdots \tag{0.106}$$

$$= \frac{I * \mathrm{Z2} * (\gamma.p + m)}{p^2 - m^2 + I\,\epsilon} + \cdots, \tag{0.107}$$

其中省略的项给出多粒子支割线。

如同标量场那里，Z2 是量子场创造或者湮灭一个H单粒子本征态的概率

$$\left\langle \Omega \right| \psi[0] \left| p, s \right\rangle = \sqrt{\mathrm{Z2}}\; u^s[p] \tag{0.108}$$

对于反粒子，把 u 替换成v。在 p^2 pole 处，两点函数跟自由场一样，除了质量换成物理质量，再呈上尺度变换因子Z2

Dirac Equation

$$\mathcal{L} = \overline{\psi}.(I\, \partial\!\!\!/ - m).\psi \tag{0.109}$$

$$\left(I\, \partial\!\!\!/ - m\right)\psi = 0 \;\&\; - I\, \partial_\mu \overline{\psi}\, \gamma^\mu - m\, \overline{\psi} = 0 \rightarrow I\, \partial_\mu \overline{\psi}\, \gamma^\mu + m\, \overline{\psi} = 0 \tag{0.110}$$

$$\psi = \int \frac{d^3 p}{(2\,\pi)^3} * \frac{1}{\sqrt{2\, E_p}} \sum_s \left(a_p^s\, u^s[p]\, \mathrm{Exp}[-I\, p\cdot x] + b_p^{s\dagger}\, v^s[p]\, \mathrm{Exp}[I\, p\cdot x]\right) \tag{0.111}$$

$$\overline{\psi} = \int \frac{d^3 p}{(2\,\pi)^3} * \frac{1}{\sqrt{2\, E_p}} \sum_s \left(b_p^s\, \overline{v}^s[p]\, \mathrm{Exp}[-I\, p\cdot x] + a_p^{s\dagger}\, \overline{u}^s[p]\, \mathrm{Exp}[I\, p\cdot x]\right) \tag{0.112}$$