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Reduction formalism for dimensionally regulated one-loop N-point integrals

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Abstract

We consider one-loop scalar and tensor integrals with an arbitrary number of external legs relevant for multi-parton processes in massless theories. We present a procedure to reduce N-point scalar functions with generic 4-dimensional external momenta to box integrals in $(4-2\epsilon)$ dimensions. We derive a formula valid for arbitrary N and give an explicit expression for $N=6$.

Further a tensor reduction method for N-point tensor integrals is presented. We prove that generically higher dimensional integrals contribute only to order ϵ for $N \geq 5$. The tensor reduction can be solved iteratively such that any tensor integral is expressible in terms of scalar integrals. Explicit formulas are given up to $N=6$.

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1. Introduction

At future colliders multi-particle/jet final states will become more and more dominant. A precise theoretical description of the QCD reactions is desirable in order to have a better control on the backgrounds for various search experiments. Especially at hadron colliders lots of multi-jet data will be collected which have to be confronted with theoretical predictions. This motivates the accurate calculation of multi-parton reactions.

For tree level calculations the construction of matrix elements with large numbers of particles in the final state is well established [1–3]. However, tree level results are very unstable with respect to variations of the renormalization and factorization scales. As

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$2 \to N$ parton cross sections behave as α_S^N, at least next-to-leading order precision is necessary to stabilize the predictions. The knowledge of the respective one-loop amplitudes is thus mandatory. For $2 \to 3$ parton scattering NLO contributions have been calculated in Refs. [4–7], for $e^+e^- \to 4\, jets$ see [8–11].

The main technical difficulties for constructing amplitudes consist in the treatment of the occurring N-point scalar and tensor integrals. Because of infrared (IR) divergencies the standard methods of [12–14] are not directly applicable. The authors of [5,6] used a formalism working in Feynman parameter space. The generation of parameter integrals with nontrivial numerators connected to tensor integrals was done by differentiation techniques. The formalism produced formally higher dimensional integrals which in the end canceled out. This cancellation had to be shown by explicit calculation. In this paper we present a proof that this cancellation mechanism is true for arbitrary N. Furthermore, a scalar reduction formula was derived in Ref. [5,6] which relates scalar integrals in different space-time dimensions to each other. For 4-dimensional external momenta the formula could not be shown to hold true in general. The problem stemmed from the presence of singular matrices. We rederive the formula and treat the necessary inversions of these matrices by using so-called pseudo-inverse matrices. To keep the external momenta in 4 dimensions is advantageous because it allows one to use helicity techniques [15].

A formula relating tensor integrals to scalar integrals with shifted space-time dimensions has been derived in Ref. [16]. This work was extended in Ref. [17,18], where scalar integrals were used as a generating functional for tensor integrals. The reduction of the higher dimensional scalar integrals works in basically the same way as in Ref. [5,6] with the same problems in the case of 4-dimensional kinematics. The method of [17,18] has been designed for massive integrals and the applicability to the massless case has not been worked out. The authors state that a regulator mass is needed for infrared divergencies in the case of 4-dimensional kinematics.

A generalization of the Passarino–Veltman techniques dealing with the problem of vanishing Gram determinants has been developed in Ref. [19,20]. The authors show explicitly how to reduce box tensor integrals to scalar integrals. Another approach which uses helicity methods for the reduction of tensor integrals to scalar integrals has been discussed in Refs. [21–23].

The complete reduction to scalar integrals is also possible in our tensor reduction scheme. Our formalism is in a sense the generalization of the Passarino–Veltman methods used for the calculation of electroweak radiative corrections [24] to the massless case.

The paper is organized as follows. In Section 2 we rederive a reduction formula for scalar integrals. We concentrate on the case of 4-dimensional kinematics from the start and show that any N-point scalar integral with $N \geq 6$ is a linear combination of pentagon integrals which in turn are combinations of box integrals plus terms of $\mathcal{O}(\epsilon)$. We give an explicit formula for the 6-point function with all external legs on-shell. In Section 3 we derive our tensor reduction formalism which combines Passarino–Veltman-like methods with Feynman parameter space techniques. We prove that for generic 4-dimensional kinematics all higher dimensional N-point functions drop out for arbitrary $N \geq 5$ and generalize the reduction methods of [5,6,25] to arbitrary N. We construct a hierarchy of tensor formulas up to $N = 6$ and rank $\leq N$ which can be solved by iteration. The explicit expressions and the derivation of some basic formulas are given in the appendix.

2. Reduction formula for massless scalar integrals

In this section we will first derive a scalar reduction formula valid for an arbitrary number of external legs. Then we will use these formulas to derive explicit representations for scalar integrals up to $N=6$.

2.1. Derivation

Consider a one-loop, scalar N-point function with massless propagators for $N \geq 4$. If all legs are off-shell the integral is finite and can be treated in four dimensions [14]. If at least one external leg is massless it is infrared divergent and needs a regulator. In the framework of dimensional regularization, 4-dimensional methods are not applicable anymore. We work in $n = 4 - 2\epsilon$ dimensions in the following with the external momenta kept in four dimensions.

With the momentum flows as indicated in Fig. 1, we define the propagator momenta as $q_l = k - r_l$ with $r_l = p_l + r_{l-1}$ for l from 1 to N and $r_0 = r_N$. Momentum conservation allows us to choose $r_N = 0$.

The corresponding analytic expression in momentum and Feynman parameter space is

$$I_N^n(R) = \int d\kappa \, \frac{1}{\prod_{l=1}^{N} q_l^2} = (-1)^N \Gamma(N - n/2) \int_0^\infty d^N z \, \frac{\delta\left(1 - \sum_{l=1}^{N} z_l\right)}{(z \cdot S \cdot z)^{N-n/2}} \,. \qquad (1)$$

Herein $R = (r_1, \ldots, r_N)$, $d\kappa = d^n k / (i\pi^{n/2})$. The kinematic information is contained in the matrix S which is related to the Gram matrix G by

$$S_{kl} = -\tfrac{1}{2}(r_l - r_k)^2 = \tfrac{1}{2}(G_{kl} - v_l - v_k)\,,$$
$$G_{kl} = 2\, r_k \cdot r_l\,, \qquad v_k = G_{kk}/2\,, \quad k,l = 1, \ldots, N. \qquad (2)$$

Although it is well known [5,6,17,18,25] that the N-point integral can be split into a finite, $(6 - 2\epsilon)$-dimensional integral and a part with less external legs containing the infrared poles, we want to present our derivation which later allows us to deal with the problem of vanishing Gram determinants in a transparent manner. As an ansatz we write (1) as a sum of (one-propagator) reduced diagrams and a remainder,

$$I_N^n = I_{\mathrm{div}} + I_{\mathrm{fin}} = \int d\kappa \, \frac{\sum_{l=1}^{N} b_l \, q_l^2}{\prod_{l=1}^{N} q_l^2} + \int d\kappa \, \frac{\left[1 - \sum_{l=1}^{N} b_l \, q_l^2\right]}{\prod_{l=1}^{N} q_l^2} \,. \qquad (3)$$

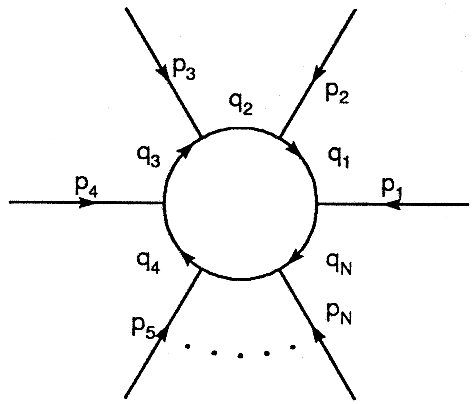

Fig. 1. N-point graph.

We want to show in the following that one can find coefficients b_l such that $I_{\rm fin}$ contains no IR poles. In standard Feynman parametrization one gets with $D=\sum_{l=1}^N z_l q_l^2$,

$$I_{\rm fin}=\Gamma(N)\int_0^\infty d^Nz\,\delta\Bigl(1-\sum\nolimits_{i=1}^N z_i\Bigr)\int d\kappa\,\frac{\Bigl[1-\sum\nolimits_{l=1}^N b_l q_l^2\Bigr]}{D^N}\,, \tag{4}$$

and after a shift $k=\tilde{k}+\sum_{l=1}^N z_l\,r_l$,

$$I_{\rm fin}=\Gamma(N)\int_0^\infty d^Nz\,\delta\Bigl(1-\sum\nolimits_{l=1}^N z_l\Bigr)\int d\tilde{\kappa}\,\frac{\Bigl[1-\sum\nolimits_{l=1}^N b_l\tilde{q}_l^2\Bigr]}{(\tilde{k}^2-M^2)^N}\,, \tag{5}$$

with

$$M^2=z\cdot S\cdot z\,,\qquad \tilde{q}_j=\tilde{k}-\sum_{l=1}^N(\delta_{lj}-z_l)r_l\,. \tag{6}$$

Now the term in square brackets in Eq. (5) can be written as

$$\Bigl[1-\sum\nolimits_{i=1}^N b_i\,\tilde{q}_i^2\Bigr]=-(\tilde{k}^2+M^2)\sum_{j=1}^N b_j+\sum_{j=1}^N z_j\bigl(1+2(S\cdot b)_j\bigr)\,. \tag{7}$$

If now the equations

$$(S\cdot b)_j=-\tfrac{1}{2}\,,\quad j=1,\dots,N, \tag{8}$$

are fulfilled, the second term on the right-hand side of (7) vanishes and one finds

$$I_{\rm fin}=-\Gamma(N)\Bigl(\sum_{l=1}^N b_l\Bigr)\int_0^\infty d^Nz\,\delta\Bigl(1-\sum\nolimits_{l=1}^N z_l\Bigr)\int d\tilde{\kappa}\,\frac{\tilde{k}^2+M^2}{(\tilde{k}^2-M^2)^N}\,. \tag{9}$$

Finally, the loop momentum integration gives

$$I_{\text{fin}} = \left(\sum_{l=1}^{N} b_l \right) (-1)^{N+1} \Gamma\left(N - 1 - \frac{n}{2} \right) (N - n - 1) \int_0^\infty d^N z \, \frac{\delta\left(1 - \sum_{l=1}^{N} z_l\right)}{(M^2)^{N-(n+2)/2}}$$

$$= -\left(\sum_{l=1}^{N} b_l \right) (N - n - 1) \, I_N^{n+2} \,. \tag{10}$$

The $(6-2\epsilon)$-dimensional integral is IR finite, as can be seen by a power-counting argument in the corresponding momentum integral.

It remains to solve Eq. (8). In the case of non-exceptional 4-dimensional kinematics, $\text{rank}(S) = \min(N,6)$ holds. For $N \leq 6$ one has $b_l = -1/2 \sum_{k=1}^{N} S_{kl}^{-1}$. In the case $N = 5$, I_{fin} contains a factor $(N-1-n)$ which is $\mathscr{O}(\epsilon)$. As is well known, pentagon integrals are just a sum of box integrals up to a remainder which drops out in phenomenological applications. In the case $N > 6$ and 4-dimensional kinematics, Eq. (8) does not have a unique solution. To clarify this point it is useful to rewrite Eq. (8). Using momentum conservation, $r_N = 0$, Eq. (8) leads to the following equations:

$$\sum_{l=1}^{N-1} G_{kl} b_l = v_k B_N \,, \qquad \sum_{l=1}^{N-1} v_l b_l = 1 \,, \qquad B_N = \sum_{l=1}^{N} b_l \,. \tag{11}$$

Herein G is now the Gram-matrix of the vectors $r_{l=1\ldots N-1}$. In the case of four dimensional kinematics it is at most of rank 4 for all $N \geq 5$. Generically any four vectors out of $r_1, \ldots, r_{N-1}$ are linearly independent. Such a configuration is called non-exceptional in the following. Choosing four linearly independent vectors $E^{\mu}_{l=1,\ldots,4}$ as a basis of the physical Minkowski space, one can define a coefficient matrix R and a Gram matrix $\tilde{G}$ made out of these basis vectors,

$$r_j^{\mu} = \sum_{l=1}^{4} R_{lj} E_l^{\mu} \,, \qquad \tilde{G}_{jk} = 2 \, E_j \cdot E_k \,. \tag{12}$$

The Gram matrix G is expressible as $G = R^T \tilde{G} R$.

Based on (12) one can construct the most general solution for the case $\det(G) = 0$ by means of pseudo-inverse matrices. This concept will also be useful for the tensor integrals.

A pseudo-inverse H to G is defined by the conditions $HGH = H$ and $GHG = G$. Given a pseudo-inverse matrix H to G, the following statement can be proven. A solution to the linear equation $G \cdot x = y$ exists if and only if $y = GH \cdot y$. Then the most general solution can be written as $x = H \cdot y + (1_{N-1} - HG) \cdot u$, where the last term – with $u \in R^{N-1}$ arbitrary – is parametrizing the solutions of the homogeneous equation. As G is symmetric, its pseudo-inverse is uniquely defined [27,28]. The pseudo-inverse to G is given by

$$H = R^T (RR^T)^{-1} \tilde{G}^{-1} (RR^T)^{-1} R \,, \tag{13}$$

where R is defined in (12). For $N = 5$ and non-exceptional external momenta the vectors $r_1, \ldots, r_4$ are the natural basis. Then $R = 1_4$, $\tilde{G} = G$, $H = G^{-1}$. Note that the concept of the pseudo-inverse always allows the inversion of linear equations containing the Gram matrix, even if the external momenta are exceptional[2].

If the Gram matrix is not of maximal rank, the condition $v = GH \cdot v$ for the existence of a solution of the inhomogeneous equation in (11) is never fulfilled. This can be seen by diagonalizing the symmetric matrix GH by an orthogonal transformation O, $GH = O^T DO$, where D is a diagonal matrix with elements $(1,1,1,1,0,\ldots,0)$. Now it is evident that in general $O \cdot v \neq DO \cdot v$. Thus a solution to Eq. (11) exists only for the case $B_N = 0$. The solution of the homogeneous equation spans a $(N-5)$-dimensional space which is just the kernel of the Gram matrix. It can be parametrized by $(N-5)$ vectors $U^{(1,\ldots,N-5)}$. Defining

$$K = 1_{N-1} - HG = 1_{N-1} - R^T \left(RR^T \right)^{-1} R\,, \tag{14}$$

one has $K \cdot v \in \ker(G)$. Now one can choose $U^{(N-5)} = K \cdot v / (v \cdot K \cdot v)$ parallel to v and the others orthogonal, $v \cdot U^{(k=1,\ldots,N-6)} = 0$. A general vector in $\ker(G)$ is then parametrized by $U = \sum_{k=1}^{N-6} \beta_k U^{(k)} + \alpha U^{(N-5)}$ and the general solution is given by

$$b_i = \frac{(K \cdot v)_i + \sum\limits_{k=1}^{N-6} \beta_k U_i^{(k)}}{v \cdot K \cdot v}\,, \quad i = 1,\ldots,N-1\,,$$

$$B_N = \sum_{k=1}^{N} b_k = 0\,, \tag{15}$$

where $\alpha = 1$ is imposed by $v \cdot b = 1$. Thus for $N = 6$ the solution to (11) is unique and therefore equal to the one defined by the inverse of the matrix S. Eq. (15) proves constructively that for all $N \geq 6$, I_N^n can be expressed in terms of $(N-1)$-point functions without the higher dimensional remainder term. This is a consequence of the linear dependence of propagators if the external momenta are 4-dimensional. For the special case $N = 7$ this has already been demonstrated in Ref. [5,6]. Obviously one can choose the β's to eliminate $(N-6)$ b's from the set $\{b_1,\ldots,b_{N-1}\}$. Doing so one observes that I_N^n can be expressed by only 6 $(N-1)$-point graphs for arbitrary $N \geq 6$. Here we make contact to a result in Refs. [14,26], where a similar relation has been derived for the IR finite case in integer dimensions by using 4-dimensional Schouten identities.

For $N \leq 6$ one has $\det(S) \neq 0$ and the following relation (see also [5,6]) holds:

$$\sum_{l=1}^{N} b_l = -\tfrac{1}{2} \sum_{l,k=1}^{N} S_{lk}^{-1} = -\frac{\det(G)}{2^N \det(S)}\,, \tag{16}$$

which shows that the vanishing of the finite remainder terms (10) is related to the vanishing of the Gram determinant.

[2] Clearly one could also solve Eq. (8) by means of the pseudo-inverse to S. With O_S defining the orthogonal transformation which diagonalizes S and the non-zero eigenvalues $\lambda_1,\ldots,\lambda_6$ of S, the pseudo-inverse to S is given by $O_S^T \cdot D_S^{\mathrm{inv}} \cdot O_S$, where $D_S^{\mathrm{inv}} = \mathrm{diag}(1/\lambda_1,\ldots,1/\lambda_6,0,\ldots,0)$. Its computation is algebraically more involved than the computation of H though.

In summary, with the definition of reduced graphs,

$$I^n_{N-p,l_1,\dots,l_p} = \int d\kappa \, \frac{\prod_{m=1}^{p} q^2_{l_m}}{\prod_{l=1}^{N} q^2_l} \,,$$

$$I^n_{N-p,N-l_1,\dots,N-l_p} = \int d\kappa \, \frac{\prod_{m=1}^{p} \left(q^2_N - q^2_{l_m} \right)}{\prod_{l=1}^{N} q^2_l} \,, \tag{17}$$

the scalar reduction formula for a regular matrix S is

$$I^n_N = -\tfrac{1}{2} \sum_{k,l=1}^{N} S^{-1}_{kl} \, I^n_{N-1,l} + (N-n-1) \, \frac{\det(G)}{2^N \det(S)} \, I^{n+2}_N \,, \qquad \det(S) \neq 0 \,. \tag{18}$$

In the case of a singular matrix S, as outlined above, we can always achieve a representation without higher-dimensional integrals,

$$I^n_N = -\frac{1}{v \cdot K \cdot v} \sum_{l=1}^{N-1} \left((K \cdot v)_l + \sum_{k=1}^{N-6} \beta_k U^{(k)}_l \right) I^n_{N-1,N-l} \,, \qquad \det(S) = 0 \,. \tag{19}$$

Remember that the $(N-6)$ β's are free parameters and that the $U^{(k=1,\dots,N-6)}$ are lying in the kernel of G and are orthogonal to v. The case $N=6$ is special in the sense that it is of the form (18) with vanishing higher-dimensional term. By applying the above formulas iteratively, any N-point function can be reduced to linear combinations of box integrals plus irrelevant $\mathcal{O}(\epsilon)$ terms in a constructive way.

2.2. Application

We will now discuss the formulas for the cases up to $N=6$ in more detail. In these cases one has $\det(S) \neq 0$ and relation (16) holds. Note that the vanishing of the Gram determinant for $N<6$ typically occurs at a border of the respective phase space. In the case $N=3$ (4,5) and $\det(G)=0$ the scalar integrals are just sums of 2-point, (3-point,4-point) functions, respectively, an observation made some time ago by Stuart [19,20] and reflected by Eq. (18). This fact can be used as a guideline to define nontrivial groupings of $(N-1)$- and N-point functions which is helpful for numerical purposes [25]. The other dangerous determinant is $\det(S)$ which vanishes if $N \geq 7$. We note that in this case our formulas are mathematically well-defined and that for the generation of explicit expressions one just has to apply formula (19). Due to the freedom to choose the parameters β, the representation of the N-point function in terms of reduced integrals is not unique as it is the case for $N \leq 6$. This is a reflection of the fact that the respective reduced integrals are not linearly independent.

As the reduction formulas were derived under the assumption $N \geq 4$, we discuss now the special cases $N=1,2,3$ first. We will represent the kinematic information in terms of the matrix S. The relation to the Gram matrix is defined in Eq. (2).

Cases N = 1,2

As massless tadpole integrals are zero in dimensional regularization, $I^n_1 = 0$. The case $N=2$ is trivial in the sense that reduced graphs are tadpoles. The formula simply gives a relation between n- and $(n+2)$-dimensional two-point functions. If p_1 is the external momentum with $p_1^2 = m_1^2$, one obtains

$$I_2^n = 2\frac{(n-1)}{m_1^2} I_2^{n+2} . \tag{20}$$

For vanishing Gram determinant, i.e. $m_1^2 = 0$, $I_2^n = 0$ in dimensional regularization.

Case N = 3

The matrix S generally looks like $(p_k^2 = m_k^2)$

$$S = -\tfrac{1}{2}\begin{pmatrix} 0 & m_2^2 & m_1^2 \\ m_2^2 & 0 & m_3^2 \\ m_1^2 & m_3^2 & 0 \end{pmatrix} . \tag{21}$$

Note that in the cases of one or two external lines on-shell, S is not of maximal rank. If for example $m_1^2 = m_2^2 = 0$, or $m_1^2 = 0, m_2^2 \neq m_3^2$, there exists no solution to (8) and no reduction is possible. If all legs are off-shell one finds a solution which relates an $(n+2)$-dimensional off-shell triangle to an n-dimensional one,

$$I_3^n = -\tfrac{1}{2}\sum_{l,k=1}^{3} S_{lk}^{-1} I_{2,l}^n + \frac{\det(G)}{8\det(S)}(2-n)\, I_3^{n+2} . \tag{22}$$

We will see later that the $(n+2)$-dimensional off-shell triangle appears in the tensor reduction of rank ≥ 3 tensor 4-point functions.

As any N-point integral in $(4-2\epsilon)$ dimensions can be reduced to triangles plus a finite remainder, we see that they are the atoms of any scalar reduction formula. Moreover, at the one-loop level they define an IR counterterm structure to any N-point function[3].

The explicit expressions for the three types of triangle graphs may be found for example in Refs. [25,29].

Case N = 4

With $(p_1+p_2)^2 = s$, $(p_2+p_3)^2 = t$, one has

$$S = -\tfrac{1}{2}\begin{pmatrix} 0 & m_2^2 & t & m_1^2 \\ m_2^2 & 0 & m_3^2 & s \\ t & m_3^2 & 0 & m_4^2 \\ m_1^2 & s & m_4^2 & 0 \end{pmatrix} . \tag{23}$$

S is of maximal rank for all on-shell cases as long as s and t are nonzero. The reduction

[3] This is actually to be expected from looking at the reduced diagrams (obtained by shrinking all finite propagators, if the loop momentum becomes soft/collinear) of the one-loop N-point function corresponding to solutions of the Landau equations. The respective reduced diagrams are just the reduced diagrams of triangle graphs.

formula (18) applies and relates a 4-point function in n dimensions with triangles and $(n+2)$-dimensional 4-point functions,

$$I_4^n = -\tfrac{1}{2}\sum_{l,k=1}^{4} S^{-1}_{lk} I^n_{3,l} + \frac{\det(G)}{16\det(S)}(3-n)\, I_4^{n+2}\,. \tag{24}$$

For $n = 4-2\epsilon$ the triangles carry all the infrared poles whereas the $(6-2\epsilon)$-dimensional part is a finite remainder. It can be calculated directly by setting $\epsilon = 0$. Explicit expressions for the box integrals can be found in Refs. [5,6,25].

Case N = 5

With $(p_j + p_{j+1})^2 = s_{j,j+1}$ (j mod 5) one has

$$S = -\tfrac{1}{2}\begin{pmatrix} 0 & m_2^2 & s_{23} & s_{51} & m_1^2 \\ m_2^2 & 0 & m_3^2 & s_{34} & s_{12} \\ s_{23} & m_3^2 & 0 & m_4^2 & s_{45} \\ s_{51} & s_{34} & m_4^2 & 0 & m_5^2 \\ m_1^2 & s_{12} & s_{45} & m_5^2 & 0 \end{pmatrix}. \tag{25}$$

For $N = 5$ the higher dimensional integrals come with a prefactor $(4-n)$ and thus drop out in phenomenological applications. One only has to know box integrals,

$$I_5^n = -\tfrac{1}{2}\sum_{l,k=1}^{5} S^{-1}_{lk} I^n_{4,l} + \mathcal{O}(\epsilon)\,. \tag{26}$$

Explicit expressions for the 5-point function with zero or one massive legs may be found in Refs. [5,6,25]. Other cases are easily constructed by using known representations of off-shell box-integrals.

Case N = 6

Now the Gram matrix is not invertible anymore for 4-dimensional kinematics whereas an inverse for S still exists. With $(p_j + p_{j+1})^2 = s_{j,j+1}$, $(p_j + p_{j+1} + p_{j+2})^2 = s_{j,j+1,j+2}$ (j mod 6), one has

$$S = -\tfrac{1}{2}\begin{pmatrix} 0 & m_2^2 & s_{23} & s_{234} & s_{61} & m_1^2 \\ m_2^2 & 0 & m_3^2 & s_{34} & s_{345} & s_{12} \\ s_{23} & m_3^2 & 0 & m_4^2 & s_{45} & s_{123} \\ s_{234} & s_{34} & m_4^2 & 0 & m_5^2 & s_{56} \\ s_{61} & s_{345} & s_{45} & m_5^2 & 0 & m_6^2 \\ m_1^2 & s_{12} & s_{123} & s_{56} & m_6^2 & 0 \end{pmatrix}. \tag{27}$$

Before going on, it is instructive to clarify the dependences between the Mandelstam variables which define the matrix S. Since they are directly related to the cuts of the N-point graph, the following counting holds for $N \geq 4$, where M is the number of off-shell legs, C the number of cuts, D the number of independent Lorentz invariants built out of the vectors $r_1, \ldots, r_N$, and $Z = C - D$ the number of constraints:

$$C = N(N-3)/2 + M, \qquad D = 3N - 10 + M, \qquad Z = (N-4)(N-5)/2. \tag{28}$$

This makes manifest that for $N \geq 6$ one encounters subtleties due to the appearing constraints. Here, the (nonlinear) constraint $\det(G) = 0$ shows up in the fact that $\sum_{l=1}^{N} b_l = -\frac{1}{2}\sum_{k,l=1}^{N} S_{kl}^{-1} = 0$. The reduction formula for $N = 6$ reads

$$I_6^n = -\frac{1}{2} \sum_{l,k=1}^{6} S_{lk}^{-1} I_{5,l}^n. \tag{29}$$

If all six external legs are on-shell, the occurring pentagon integrals $I_{5,l}^n$ have one external leg off-shell. The explicit expression for the on-shell 6-point function is given in Appendix A.

3. Reduction of tensor-integrals

In this section we show that any rank L, N-point tensor integral can, by recursion and scalar reduction, be expressed in terms of scalar box, triangle and two-point integrals. We prove that for $N \geq 5$ and non-exceptional external momenta higher dimensional integrals drop out.

The rank L, N-point tensor integral is given by

$$I_N^{\mu_1 \cdots \mu_L} = \int d\kappa \, \frac{k^{\mu_1} \ldots k^{\mu_L}}{\prod_{l=1}^{N} q_l^2}. \tag{30}$$

We omit the superscript n indicating the dimension in the tensor integrals to simplify the notation. If the dimension is different from $(4 - 2\epsilon)$, it will always be written out explicitly. After introducing Feynman parameters and making a shift of the loop momentum as in the last section, the odd powers of the loop momentum in the numerator can be dropped. Using the standard integral

$$\int d\kappa \, \frac{k^{\mu_1} \ldots k^{\mu_{2m}}}{(k^2 - M^2)^N} = (-1)^N \left[g_{(m)}^{\cdot\cdot}\right]^{\{\mu_1 \cdots \mu_{2m}\}} \left(-\frac{1}{2}\right)^m \frac{\Gamma(N - (n+2m)/2)}{\Gamma(N)} (M^2)^{-N+(n+2m)/2}, \tag{31}$$

one finds the following formula:

$$I_N^{\mu_1 \cdots \mu_L} = \sum_{m=0}^{[L/2]} \left(-\frac{1}{2}\right)^m \sum_{j_1, \ldots, j_{L-2m}=1}^{N-1} \left[g_{(m)}^{\cdot\cdot} r_{j_1}^{\cdot} \ldots r_{j_{L-2m}}^{\cdot}\right]^{\{\mu_1 \cdots \mu_L\}} I_N^{n+2m}(j_1, \ldots, j_{L-2m}). \tag{32}$$

$[L/2]$ is the nearest integer less or equal to $L/2$ and $\left[g^{\cdot\cdot}_{(m)} r^{\cdot}_{j_1} \dots r^{\cdot}_{j_{L-2m}} \right]^{\{\mu_1 \dots \mu_L\}}$ stands for the sum over all different combinations of L Lorentz indices distributed to m metric tensors and $(L-2m)$ r-vectors. These are $\binom{L}{2m} \prod_{k=1}^{m}(2k-1)$ terms. A dot appearing as an index at objects inside a square bracket stands for one index out of the set specified in curly brackets at the outside of the square bracket. $X^{(m)}$ or $X_{(m)}$ denotes the product of m terms of X with adequate indices[4]. For example, a tensor object like $\left[X^{\cdot\cdot}_{(2)} Y^{\cdot} \right]^{\{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5\}}$ with X a symmetric tensor of rank 2 and Y a vector means

$$\left[X^{\cdot\cdot}_{(2)} Y^{\cdot} \right]^{\{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5\}} = \left(X^{\mu_1 \mu_2} X^{\mu_3 \mu_4} + X^{\mu_1 \mu_3} X^{\mu_2 \mu_4} + X^{\mu_1 \mu_4} X^{\mu_2 \mu_3} \right) Y^{\mu_5}$$

$$+ 5 \,\text{permutations}.$$

$I_N^D(j_1, \dots, j_M)$ are scalar integrals with nontrivial numerators in D dimensions, defined by

$$I_N^D(j_1, \dots, j_M) = (-1)^N \Gamma(N - D/2) \int_0^\infty d^N z \, \delta\left(1 - \sum_{l=1}^{N} z_l\right) \frac{z_{j_1} \dots z_{j_M}}{(z \cdot S \cdot z)^{N-D/2}} \,. \tag{33}$$

Recursion relations for this kind of integrals were given for $M \leq 4$ in Ref. [25]. We derive such relations for general M in Appendix B.

From now on we use standard summation conventions for the indices. Eq. (32) implies that tensor integrals in momentum space are linear combinations of scalar integrals in different dimensions. We note that formula (32) is equivalent to a more general formula given in Ref. [16] for the case of integer powers of the propagators.

We want to prove now that for non-exceptional momenta and $N \geq 5$ the higher dimensional scalar integrals drop out. To this end we solve Eq. (32) for $I_N^n(j_1, \dots, j_L)$ by contracting it with $2^L r_{l_1}^{\mu_1} \dots r_{l_L}^{\mu_L}$. For the inversion of the Gram matrices we use its (pseudo) inverse as defined above. With $2 r_l \cdot k = q_N^2 - q_l^2 + v_l$ one finds

$$\begin{aligned} I_N^n(j_1, \dots, j_L) = & \sum_{k=0}^{L} \left[w^{(L-k)}_{\cdot} H_{\cdot l_1} \dots H_{\cdot l_k} \right]_{\{j_1, \dots, j_L\}} I^n_{N-k, N-l_1, \dots, N-l_k} \\ & - \sum_{m=1}^{[L/2]} (-1)^m \left[H^{(m)}_{\cdot\cdot} I_N^{n+2m}((L-2m) \text{ indices}) \right]_{\{j_1, \dots, j_L\}}, \end{aligned} \tag{34}$$

where $w = H \cdot v$ and H is defined in Eq. (13). On the right-hand side of Eq. (34), differences of reduced integrals appear as defined in Eq. (17). In principle one has to add to (34) also the solutions stemming from the homogeneous equation which are present if G is not of maximal rank. But as they contain the matrix $K = 1_{N-1} - HG$, they vanish after contraction with the r's, i.e. $r^\mu \cdot (1_{N-1} - HG) = 0$. If G is invertible,

[4] The ‘‘power’’ (m) may appear as a lower index for convenience of notation if it would interfere with the dots standing for upper indices. Hence we write $X^{\cdot\cdot}_{(m)}$ instead of $(X^{\cdot\cdot})^{(m)}$.

$H = G^{-1}$. The bracket of the kth term of the first sum stands for $\binom{L}{K}$ terms whereas the bracket of the mth term of the second sum stands for $\binom{L}{2m}\prod_{k=1}^{m}(2k-1)$ terms. Inserting (34) into (32), we find that the tensor integral decays into a part containing $(4-2\epsilon)$-dimensional objects, $K_N^{\mu_1\cdots\mu_L}$, and a part built out of higher-dimensional objects, $J_N^{\mu_1\cdots\mu_L}$,

$$I_N^{\mu_1\cdots\mu_L} = K_N^{\mu_1\cdots\mu_L} + J_N^{\mu_1\cdots\mu_L}\,, \tag{35}$$

$$K_N^{\mu_1\cdots\mu_L} = \sum_{k=0}^{L}\left[\mathscr{W}^{\cdot}_{(L-k)}\mathscr{K}^{\cdot}_{l_1}\cdots\mathscr{K}^{\cdot}_{l_k}\right]^{\{\mu_1,\dots,\mu_L\}} I^n_{N-k,N-l_1,\dots,N-l_k}\,, \tag{36}$$

$$\begin{aligned} J_N^{\mu_1\cdots\mu_L} = &\sum_{m=1}^{[L/2]}(-1)^m \\ &\times\sum_{j_1,\dots,j_{L-2m}=1}^{N-1}\left[\left((g/2)^{\cdot\cdot}_{(m)} - \mathscr{H}^{\cdot\cdot}_{(m)}\right) r^{\cdot}_{j_1}\cdots r^{\cdot}_{j_{L-2m}}\right]^{\{\mu_1\dots\mu_L\}} \\ &\times I_N^{n+2m}(j_1,\dots,j_{L-2m})\,. \end{aligned} \tag{37}$$

The Lorentz indices are carried by the objects

$$\begin{aligned} &\mathscr{H}^{\mu\nu} = r^\mu\cdot H\cdot r^\nu\,,\\ &\mathscr{K}_l^\mu = (r^\mu\cdot H)_l\,,\\ &\mathscr{W}^\mu = r^\mu\cdot w = \mathscr{K}^\mu\cdot v\,. \end{aligned} \tag{38}$$

We recall that $w = H\cdot v$ and $v_l = G_{ll}/2$.

If the r's span 4-dimensional Minkowski space, which is generically the case if $N \geq 5$, $\mathscr{H}^{\mu\nu}$ is just proportional to the metric tensor $g_4^{\mu\nu}$ in 4 dimensions,

$$\mathscr{H}^{\mu\nu} = r^\mu\cdot H\cdot r^\nu = \sum_{i,j=1}^{N-1} r_i^\mu r_j^\nu H_{ij} = \sum_{l,k=1}^{4} E_l^\mu E_k^\nu \tilde{G}^{-1}_{lk} = \tfrac{1}{2} g_4^{\mu\nu}\,. \tag{39}$$

Thus, the coefficients of the higher dimensional integrals in (37) are proportional to $g^{(m)} - g_4^{(m)}$ which is of order ϵ. As by power counting the higher dimensional integrals are finite objects, it follows that the whole contribution is $\mathscr{O}(\epsilon)$, if the external momenta are 4-dimensional. This proves the cancellation of higher dimensional integrals in tensor reductions for arbitrary $N \geq 5$ and non-exceptional external momenta, and we obtain

$$I_N^{\mu_1\cdots\mu_L} = \sum_{k=0}^{L}\left[\mathscr{W}^{\cdot}_{(L-k)}\mathscr{K}^{\cdot}_{l_1}\cdots\mathscr{K}^{\cdot}_{l_k}\right]^{\{\mu_1,\dots,\mu_L\}} I^n_{N-k,N-l_1,\dots,N-l_k} + \mathscr{O}(\epsilon)\,,\quad N\geq 5\,. \tag{40}$$

For $N < 5$, Eq. (39) is not valid since the external momenta cannot span Minkowski space anymore. Thus one has to calculate the terms $J_N^{\mu_1\cdots\mu_L}$ for $N < 5$. The explicit expressions are given in Appendix D.

Now we want to rewrite Eq. (35) as a recursion formula for arbitrary tensor integrals. To this effect we express the contracted tensor integrals $I^n_{N-k,N-l_1,\dots,N-l_k}$ $(k>0)$ as

$(N-1)$-point tensor integrals which are maximally of rank $(L-1)$. By using $(k-1)$ times the relation $q_N^2 - q_l^2 = 2\, r_l \cdot k - v_l$, one gets

$$I^n_{N-k,N-l_1,\dots,N-l_k} = \sum_{j=0}^{k-1} (-1)^{k-j-1} \frac{2^j}{j!} \left[v_{\cdot}^{(k-j-1)}\, r_{\cdot\nu_1} \dots r_{\cdot\nu_j} \right]_{\{l_1 \dots l_{k-1}\}} I^{\nu_1 \dots \nu_j}_{N-1,N-l_k}\,,$$

$$k \geq 1\,. \tag{41}$$

Insertion of expression (41) into Eq. (36) leads to the recursion formula,

$$\begin{aligned} K_N^{\mu_1 \cdots \mu_L} = {} & \frac{1}{L} \left[\mathscr{W}^{\cdot} K_N^{\{L-1\,\mathrm{dots}\}} \right]^{\{\mu_1 \dots \mu_L\}} \\ & + \frac{2^{(L-1)}}{L!} \left[\mathscr{K}_l^{\cdot} \mathscr{H}^{\cdot}_{\nu_1} \dots \mathscr{H}^{\cdot}_{\nu_{L-1}} \right]^{\{\mu_1 \dots \mu_L\}} I^{\nu_1 \dots \nu_{L-1}}_{N-1,N-l}\,. \end{aligned} \tag{42}$$

The derivation of (42) is given in Appendix C.

Since for $N \geq 5$ the terms $J_N^{\mu_1 \cdots \mu_L}$ contribute only to order ϵ, $K_N^{\mu_1 \cdots \mu_L}$ in Eq. (42) can be replaced by $I_N^{\mu_1 \cdots \mu_L}$ to obtain the recursion formula for $N \geq 5$. For $N < 5$, one cannot drop the $J_N^{\mu_1 \cdots \mu_L}$ terms, such that the general recursion formula reads

$$\begin{aligned} I_N^{\mu_1 \cdots \mu_L} = J_N^{\mu_1 \cdots \mu_L} + {} & \frac{1}{L} \left[\mathscr{W}^{\cdot} \left(I_N - J_N \right)^{\{L-1\,\mathrm{dots}\}} \right]^{\{\mu_1 \dots \mu_L\}} \\ & + \frac{2^{(L-1)}}{L!} \left[\mathscr{K}_l^{\cdot} \mathscr{H}^{\cdot}_{\nu_1} \dots \mathscr{H}^{\cdot}_{\nu_{L-1}} \right]^{\{\mu_1 \dots \mu_L\}} I^{\nu_1 \dots \nu_{L-1}}_{N-1,N-l}\,. \end{aligned} \tag{43}$$

As a consequence, higher dimensional integrals $I_{N'}^{n+2m}$ $(m = 1,2)$ will appear in the reduction of tensor N-point integrals only with $N' \leq 4$. All the higher dimensional integrals can be remapped to n-dimensional integrals with the scalar reduction formulas (20), (22) and (24). In the case of exceptional kinematics and $N \geq 5$, higher dimensional $(N \geq 5)$-point functions can be present.

As some of the reduced integrals on the right-hand side of (43) do not contain a trivial propagator, they are not in the standard form for applying the reduction formula again. Therefore one has to perform a shift $k \to k + r_l$ in the tensor integral, leading to

$$\begin{aligned} I^{\mu_1 \dots \mu_P}_{N-1,N-l}(R) &= I^{\mu_1 \dots \mu_P}_{N-1}\left(\hat{R}_{[N]}\right) - I^{\mu_1 \dots \mu_P}_{N-1}\left(\hat{R}_{[l]}\right) \\ &= \sum_{k=0}^{P} \left[r_l^{\cdot(k)} I_{N-1}^{\{P-k\,\mathrm{dots}\}}\left(R_{[l]}\right) \right]^{\{\mu_1 \dots \mu_P\}} - I^{\mu_1 \dots \mu_P}_{N-1}\left(\hat{R}_{[l]}\right). \end{aligned} \tag{44}$$

The argument vectors are

$$\begin{aligned} R &= (r_1, \dots, r_N)\,, \\ \hat{R}_{[k]} &= (r_1, \dots, \hat{r}_k, \dots, r_N)\,, \\ R_{[l]} &= (r_{l+1} - r_l, r_{l+2} - r_l, \dots, r_{N-2+l} - r_l, 0)\,, \end{aligned} \tag{45}$$

where $\hat{r}$ means that the respective vector has to be left out. The vector indices are understood to be taken cyclically symmetric with periodicity N, i.e. $r_{N+l} = r_l$ for

$l \in \{1, \ldots, N\}$. As we assume $r_N = 0$, the integrals in the last line of (44) have again at least one trivial propagator and are suited for a further reduction step.

A full tensor reduction of a rank L tensor integral is obtained by employing Eq. (42) for $N \geq 5$ resp. Eq. (43) for $N < 5$ to do the first reduction step. Then the integrals $I_{N-1,N-l}^{\nu_1 \cdots \nu_{L-1}}$ have to be shifted by using Eq. (44) to end up with expressions of rank $\leq L - 1$ to which the same reduction procedure can be applied again. Explicit tensor reduction formulas for $N = 2, \ldots, 6$ are given in Appendix D.

For $N \geq 5$ one can express by recursion any rank L, N-point tensor integral in terms of $(4 - 2\epsilon)$-dimensional scalar N-point and tensor box integrals, which in turn can be reduced further down to scalar box, triangle and two-point integrals, taking also into account the scalar reduction formulas given in the previous section. This defines an algorithm which can be easily programmed.

4. Conclusion

We have presented reduction formulas for massless N-point scalar and tensor functions.

We have shown in a constructive manner how scalar N-point functions can be represented as linear combinations of $(4 - 2\epsilon)$-dimensional scalar $(N - 1)$-point functions for arbitrary $N \geq 5$. In particular, we pointed out how to treat the linear equations which determine the reduction coefficients, b_l, for $N \geq 6$ in terms of the pseudo-inverse of the singular Gram matrix. In this way all mathematical operations are valid for 4-dimensional external kinematics throughout the whole reduction procedure. We applied the reduction formalism to scalar integrals up to $N = 6$ explicitly and gave an expression for the on-shell 6-point function.

For N-point tensor integrals we have formulated a reduction scheme for arbitrary N. We have proven that higher-dimensional integrals always vanish for $N \geq 5$ in the case of non-exceptional kinematics. We derived a recursion formula for the remaining n-dimensional part. For the derivation we used methods à la Passarino–Veltman, such as contracting tensor integrals with external vectors and inverting Gram matrices. In the case of a singular Gram matrix the inversion has to be done with its pseudo-inverse. By iteration any tensor integral can be expressed as a linear combination of scalar integrals. Higher dimensional scalar integrals appear only in the reduction of $N \leq 4$ tensor integrals. These higher dimensional integrals are expressible in terms of $(4 - 2\epsilon)$-dimensional integrals as has been shown explicitly in the discussion of the scalar reduction formulas. To make the general formalism more user-friendly we gave explicit expressions up to $N = 6$ in an appendix.

We also derived a reduction formula in Feynman parameter space in an appendix. The formula generalizes results in the literature avoiding a projective transformation. As we gave all the formulas to translate momentum space expressions into Feynman parameter space expressions, the equivalence between the two approaches is manifest. In applications this will allow us to employ methods similar to the ones in Ref. [25] to get numerically stable expressions, a problem we did not address in this paper.

We conclude that the computation of IR divergent one-loop integrals for arbitrary numbers of legs can be mastered with the reduction formulas presented here. The iterative structure makes it easy to implement the formalism in algebraic computer programs. The conceptual problems for the construction of multi-parton one-loop amplitudes are thus solved.

The generalization of our method to include also massive particles is postponed to a future publication.

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Appendix A. The scalar 6-point function

Here we present an explicit formula for the scalar hexagon function with all external legs on-shell. The reduction formula (29) allows us to write it as a sum of six pentagon integrals with one external leg off-shell.

$$I_6^n(s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{234}, s_{345})$$
$$= -\tfrac{1}{2} \sum_{k,l=1}^{6} S_{kl}^{-1} I_{5,1mass}(s_{l+2,l+3}, s_{l+3,l+4}, s_{l+4,l+5}, s_{l+5,l,l+1}, s_{l,l+1,l+2}; s_{l,l+1}) . \tag{A.1}$$

The function $I_{5,1mass}$ has been given in Ref. [5,6]. We rederived the formula and agree with it for Euclidean kinematics (up to a trivial typo[5]).

The other ingredient, the matrix S, is given explicitly by Eq. (27) after setting all $m_j^2 = 0$. The indices in (A.1) have to be taken modulo 6, and due to momentum conservation one has $s_{456} = s_{123}$, $s_{612} = s_{345}$, $s_{561} = s_{234}$. The nine Lorentz invariants are not independent since they fulfill the constraint $\det(G) = 0$. We find

$$I_6^n = \sum_{k=1}^{6} \left[\frac{r_\Gamma}{\epsilon^2} (A_k + B_k) + C_k + D_k \right],$$

$$A_1 = \frac{(-s_{12})^{-\epsilon}}{s_{61} s_{12} s_{23} s_{234}},$$

$$B_1 = \frac{(s_{12} - s_{123})^2 \left[(-s_{12})^{-\epsilon} - (-s_{123})^{-\epsilon} \right]}{s_{12} s_{23} s_{34} s_{123} E_1}$$
$$+ \frac{(s_{12} - s_{345})^2 \left[(-s_{12})^{-\epsilon} - (-s_{345})^{-\epsilon} \right]}{s_{56} s_{61} s_{12} s_{345} E_1},$$

[5] The last term in the curly bracket of Eq. (5.8) should read $[1 - (m_5^2 s_{23} / s_{45} / s_{51})^{-\epsilon}]$.

$$
\begin{aligned}
C_1 = {} & b_1\left(\frac{1}{s_{34}s_{45}s_{56}} + \frac{s_{45}-s_{345}}{s_{34}s_{45}E_1} + \frac{s_{45}-s_{123}}{s_{45}s_{56}E_1}\right)\mathrm{Li}_2\left(1-\frac{s_{123}s_{345}}{s_{12}s_{45}}\right) \\
& - \frac{b_1}{s_{34}s_{45}s_{56}}\frac{\pi^2}{3} + \frac{(s_{123}-s_{12})^2}{s_{12}s_{23}s_{34}s_{123}E_1}\left[\mathrm{Li}_2\left(1-\frac{s_{12}}{s_{123}}\right) - \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right)\right] \\
& + \frac{(s_{345}-s_{12})^2}{s_{56}s_{61}s_{12}s_{345}E_1}\left[\mathrm{Li}_2\left(1-\frac{s_{12}}{s_{345}}\right) - \mathrm{Li}_2\left(1-\frac{s_{345}}{s_{12}}\right)\right], \\
D_1 = {} & -\frac{1}{2}\left[b_4\left(\frac{1}{s_{61}s_{12}s_{23}} + \frac{s_{12}-s_{123}}{s_{12}s_{23}E_1} - \frac{s_{12}-s_{345}}{s_{61}s_{12}E_1}\right)\right. \\
& \left. + b_3\left(\frac{1}{s_{56}s_{61}s_{12}} + \frac{s_{61}-s_{234}}{s_{56}s_{61}E_3} - \frac{s_{61}-s_{345}}{s_{61}s_{12}E_3}\right)\right] \\
& \times\log^2\left(\frac{s_{12}}{s_{61}}\right) - \frac{1}{2}\left[b_3\left(\frac{1}{s_{56}s_{61}s_{12}} - \frac{1}{s_{56}s_{61}s_{345}} - \frac{1}{s_{61}s_{12}s_{234}}\right)\right. \\
& \left. + b_1\left(\frac{s_{12}-s_{345}}{s_{56}s_{345}E_1} - \frac{s_{12}-s_{123}}{s_{34}s_{123}E_1}\right) + b_5\left(\frac{s_{56}-s_{234}}{s_{12}s_{234}E_2} - \frac{s_{56}-s_{123}}{s_{34}s_{123}E_2}\right)\right] \\
& \times\log^2\left(\frac{s_{12}}{s_{56}}\right) - \frac{1}{2}\left[b_2\left(\frac{1}{s_{45}s_{56}s_{61}} + \frac{s_{56}-s_{123}}{s_{45}s_{56}E_2} + \frac{s_{56}-s_{234}}{s_{56}s_{61}E_2}\right)\right. \\
& \left. + b_5\left(\frac{1}{s_{12}s_{23}s_{34}} + \frac{s_{23}-s_{123}}{s_{12}s_{23}E_2} + \frac{s_{23}-s_{234}}{s_{23}s_{34}E_2}\right)\right] \\
& \times\log^2\left(\frac{s_{123}}{s_{234}}\right) - \frac{1}{2}\left[b_1\left(\frac{s_{12}-s_{123}}{s_{34}s_{123}E_1}\right) - b_4\left(\frac{1}{s_{61}s_{12}s_{23}} + \frac{s_{12}-s_{345}}{s_{61}s_{12}E_1}\right)\right. \\
& \left. + b_5\left(\frac{s_{56}-s_{123}}{s_{34}s_{123}E_2} + \frac{s_{23}-s_{123}}{s_{12}s_{23}E_2}\right)\right]\log^2\left(\frac{s_{12}}{s_{123}}\right) \\
& - \frac{1}{2}\left[b_3\left(\frac{1}{s_{61}s_{12}s_{234}} - \frac{1}{s_{56}s_{61}s_{12}} - \frac{s_{61}-s_{234}}{s_{56}s_{61}E_3} - \frac{s_{34}-s_{234}}{s_{12}s_{234}E_3}\right)\right. \\
& \left. + b_5\left(\frac{1}{s_{12}s_{23}s_{234}} - \frac{1}{s_{12}s_{23}s_{34}} - \frac{s_{23}-s_{234}}{s_{23}s_{34}E_2} - \frac{s_{56}-s_{234}}{s_{12}s_{234}E_2}\right)\right]\log^2\left(\frac{s_{12}}{s_{234}}\right) \\
& - \frac{1}{2}\left[b_1\left(\frac{s_{12}-s_{345}}{s_{56}s_{345}E_1}\right) - b_4\left(\frac{1}{s_{61}s_{12}s_{23}} + \frac{s_{12}-s_{123}}{s_{12}s_{23}E_1}\right)\right. \\
& \left. + b_3\left(\frac{s_{34}-s_{345}}{s_{56}s_{345}E_3} + \frac{s_{61}-s_{345}}{s_{61}s_{12}E_3}\right)\right]\log^2\left(\frac{s_{12}}{s_{345}}\right),
\end{aligned}
\tag{A.2}
$$

with the abbreviations

$$E_1 = E_4 = s_{123}\,s_{345} - s_{12}\,s_{45}\,,$$

$$E_2 = E_5 = s_{234}\,s_{123} - s_{23}\,s_{56}\,,$$

$$E_3 = E_6 = s_{345}\,s_{234} - s_{34}\,s_{61}\,,$$

and

$$r_\Gamma = \frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,.$$

The functions A_k, B_k, C_k, D_k for $k>1$ are obtained by cyclic permutations of the indices. The other necessary ingredients are

$$\begin{aligned} b_1 = \big(&-s_{345}^2\,s_{56}^2\,s_{23} + s_{12}\,s_{234}\,s_{45}^2\,s_{34} - s_{56}\,s_{23}\,s_{345}\,s_{45}\,s_{34} - s_{45}\,s_{234}\,s_{345}\,s_{123}\,s_{34} \\ &+ s_{34}^2\,s_{45}\,s_{61}\,s_{123} - s_{34}\,s_{123}\,s_{12}\,s_{234}\,s_{45} - s_{34}\,s_{123}\,s_{56}\,s_{23}\,s_{345} + 2\,s_{34}\,s_{45}\,s_{23}\,s_{12}\,s_{56} \\ &+ s_{34}\,s_{123}^2\,s_{345}\,s_{234} - s_{34}^2\,s_{61}\,s_{123}^2 - s_{34}\,s_{345}\,s_{56}\,s_{123}\,s_{61} + s_{345}^2\,s_{56}\,s_{123}\,s_{234} \\ &- s_{45}\,s_{234}\,s_{123}\,s_{56}\,s_{345} + s_{45}^2\,s_{234}\,s_{12}\,s_{56} + s_{56}^2\,s_{23}\,s_{345}\,s_{45} - s_{56}\,s_{34}\,s_{45}\,s_{61}\,s_{123} \\ &+ 2\,s_{34}\,s_{12}\,s_{56}\,s_{61}\,s_{45} - s_{12}\,s_{56}\,s_{45}\,s_{234}\,s_{345} - s_{12}^2\,s_{234}\,s_{45}^2 - s_{123}^2\,s_{234}\,s_{345}^2 \\ &+ s_{123}\,s_{56}\,s_{23}\,s_{345}^2 + 2\,s_{45}\,s_{345}\,s_{234}\,s_{123}\,s_{12} - 2\,s_{34}\,s_{45}^2\,s_{12}\,s_{56} + 2\,s_{45}\,s_{34}\,s_{123}\,s_{56}\,s_{345} \\ &- s_{45}\,s_{34}\,s_{61}\,s_{123}\,s_{12} - s_{45}\,s_{56}\,s_{23}\,s_{345}\,s_{12} + s_{34}\,s_{123}^2\,s_{61}\,s_{345}\big)/F\,, \end{aligned} \tag{A.3}$$

where the other b_k's are again obtained by cyclic permutation. Finally,

$$\begin{aligned} F = &-s_{345}^2\,s_{56}^2\,s_{23}^2 + 4\,s_{23}\,s_{34}\,s_{12}\,s_{56}\,s_{61}\,s_{45} - 2\,s_{23}\,s_{34}\,s_{345}\,s_{56}\,s_{123}\,s_{61} \\ &- 2\,s_{23}\,s_{12}\,s_{56}\,s_{45}\,s_{234}\,s_{345} + 2\,s_{23}\,s_{345}^2\,s_{56}\,s_{123}\,s_{234} - s_{123}^2\,s_{61}^2\,s_{34}^2 \\ &- 2\,s_{34}\,s_{12}\,s_{123}\,s_{234}\,s_{61}\,s_{45} + 2\,s_{34}\,s_{123}^2\,s_{61}\,s_{345}\,s_{234} - s_{45}^2\,s_{234}^2\,s_{12}^2 \\ &- s_{345}^2\,s_{123}^2\,s_{234}^2 + 2\,s_{12}\,s_{123}\,s_{234}^2\,s_{45}\,s_{345} = 64\det(S)\,. \end{aligned} \tag{A.4}$$

Note that in the ϵ-dependent part of the formula no b's appear. We did not succeed in finding a more compact form for the part which does not depend on ϵ without spoiling this nice feature.

The expression for I_6^n in the form as given above is strictly only correct in the Euclidean region where all Mandelstam variables are negative. For most of the terms, the analytic continuation to positive values is defined by simply using the replacement $s \to s + i\delta$, where s stands here for any of the s_{ij}, s_{ijk}. No cut will be hit by the logarithms, the dilogarithms with a single ratio of Mandelstam variables and the exponentials $(-s-i\delta)^{-\epsilon}$. Concerning the dilogarithms of a product of ratios, more care has to be taken. To avoid the crossing of a cut one has to make the replacement

$$Li_2\left(1-\frac{s_1 s_2}{s_3 s_4}\right) \to Li_2\left(1-\frac{s_1+i\delta}{s_3+i\delta}\frac{s_2+i\delta}{s_4+i\delta}\right)+\eta\left(\frac{s_1+i\delta}{s_3+i\delta},\frac{s_2+i\delta}{s_4+i\delta}\right) \times\log\left(1-\frac{s_1+i\delta}{s_3+i\delta}\frac{s_2+i\delta}{s_4+i\delta}\right), \tag{A.5}$$

$$\eta(x,y)=\log(xy)-\log(x)-\log(y). \tag{A.6}$$

Appendix B. Recursion relation for integrals with Feynman parameters in the numerator

Here we want to comment on the reduction of Feynman parameter integrals with nontrivial numerators as defined in (33). In Ref. [25] recursion relations for integrals with up to 4 Feynman parameters are given. The derivation is based on the approach of [5,6] which uses a projective transformation [30]. We present an independent derivation and generalize the formulas to arbitrary N and numbers of Feynman parameters in the numerator.

Consider the following identity ($j=1,\ldots,N-1$):

$$\int_{-\infty}^{\infty} d^N z \frac{\partial}{\partial z_j}\left(\prod_{l=1}^{N}\theta(z_l)\,\delta\left(1-\sum_{k=1}^{N} z_k\right) z_{l_1}\ldots z_{l_p}(z\cdot S\cdot z)^{-N+n/2+1}\right)=0. \tag{B.1}$$

Setting $r_N=0$ and using $\sum_{k=1}^{N} z_k=1$, one obtains $z\cdot S\cdot z=\sum_{k,l=1}^{N-1} z_l G_{lk} z_k/2-\sum_{k=1}^{N-1} v_k z_k$. The first step consists in eliminating the δ-function by integrating out z_N. If now the derivative acts on θ-functions, terms with δ-function insertions are produced. The reduced integrals $I^n_{N-1,j}$ (defined in Eq. (B.2) below), obtained by pinching the jth propagator line in an N-point graph, correspond to these δ-function insertions. If the derivative acts on the Feynman parameters, the monomial in the numerator is reduced by one degree, and if it acts on the $z\cdot S\cdot z$ term it formally decreases the dimension by two and increases the numerator by $(G\cdot z)_j-v_j$. After these operations one reintroduces the z_N-integration with a delta-function insertion. Using the following generalized definition for pinched scalar integrals with nontrivial numerators:

$$I^n_{N-1,j}(l_1,\ldots,l_p)=(-1)^{N-1}\Gamma(N-1-n/2)\int_0^{\infty} d^N z\,\delta\left(1-\sum_{l=1}^{N} z_l\right) \times\frac{z_{l_1}\ldots z_{l_p}\delta(z_j)}{(z\cdot S\cdot z)^{N-1-n/2}}, \tag{B.2}$$

we find

$$\sum_{l_0=1}^{N-1} G_{jl_0} I^n_N(l_0,\ldots,l_p)=\sum_{k=1}^{p}\delta_{jl_k} I^{n+2}_N\left(l_1,\ldots,\hat{l}_k,\ldots l_p\right)+I^n_{N-1,N}(l_1,\ldots,l_p) -I^n_{N-1,j}(l_1,\ldots,l_p)+v_j I^n_N(l_1,\ldots,l_p). \tag{B.3}$$

Herein $\hat{l}_k$ means that the respective index does not appear.

Now we want to sketch the proof for a second equation,

$$I^n_{N-1,N}(l_1,\ldots,l_p) = (N-n-p-1)\, I^{n+2}_N(l_1,\ldots,l_p) + \sum_{l_0=1}^{N-1} v_{l_0} I^n_N(l_0,\ldots,l_p)\,. \tag{B.4}$$

The proof is done by induction to p. In order to show that the induction start ($p=0$) holds true, one directly calculates

$$I^n_{N-1,N}(1) = \int d\kappa\, \frac{k^2}{\prod\limits_{l=1}^{N} q_l^2} = (N-n-1)\, I^{n+2}_N(1) + \sum_{l_0=1}^{N-1} v_{l_0} I^n_N(l_0)\,. \tag{B.5}$$

For the induction step, one assumes that (B.4) is fulfilled for a given p. Viewing the v_j as independent variables, one differentiates the formula with respect to $v_{l_{p+1}}$ and finds the formula for $p+1$ and $n-2$. As the dimension is arbitrary we can replace $n-2$ by n in the expression. This proves the validity of Eq. (B.4) for all p.

Combining (B.3) and (B.4) one finds ($j=1,\ldots,N$)

$$\begin{aligned} 2\sum_{l_0=1}^{N} S_{jl_0} I^n_N(l_0,\ldots,l_p) = {}& \sum_{k=1}^{p} \delta_{jl_k} I^{n+2}_N(l_1,\ldots,\hat{l}_k,\ldots,l_p) \\ & + (N-n-p-1)\, I^{n+2}_N(l_1,\ldots,l_p) - I^n_{N-1,j}(l_1,\ldots,l_p)\,. \end{aligned} \tag{B.6}$$

In the case $N \le 6$, S is invertible and the final reduction formula reads

$$\begin{aligned} I^n_N(l_0,\ldots,l_p) = {}& \tfrac{1}{2}\sum_{k=1}^{p} S^{-1}_{l_0 l_k} I^{n+2}_N(l_1,\ldots,\hat{l}_k,\ldots,l_p) \\ & + \tfrac{1}{2}\sum_{j=1}^{N} S^{-1}_{jl_0} (N-n-p-1)\, I^{n+2}_N(l_1,\ldots,l_p) \\ & - \tfrac{1}{2}\sum_{j=1}^{N} S^{-1}_{jl_0} I^n_{N-1,j}(l_1,\ldots,l_p)\,. \end{aligned} \tag{B.7}$$

This is the generalization of the formulas given in Ref. [25] for the case of four Feynman parameters in the numerator ($p=3$). (In their conventions $S^{-1}_{ik} = \eta_{ik}\,\alpha_i\,\alpha_k/N_n$ and $\sum_i S^{-1}_{ik} = \gamma_k/N_n$, which is identical to our S^{-1} up to a trivial relabeling of indices). It means that all Feynman parameter integrals with numerators can be reduced to ordinary scalar integrals by iteration.

In the case $N > 6$, S is not invertible. The linear dependence of the row (resp. column) vectors of S makes it difficult to generalize the Feynman parameter space based techniques to arbitrary N [5,6,17,18]. In our approach one can use Eq. (B.6) directly in this case, where the inversion of the Gram matrix should be done with its pseudo-inverse as explained in the main text. As already noted earlier, it is also possible to work with the pseudo-inverse to S. In any way, as we have proven, it is the (pseudo) inverse to the Gram matrix which induces the cancellation of the higher dimensional terms by virtue of Eq. (39).

Finally, with the formulas given above and in the main text, it is possible to translate expressions in Feynman parameter space into expressions in momentum space and vice versa for arbitrary N.

Appendix C. Derivation of the recursion formula for tensor reduction

In this appendix we give the derivation of the recursion relation (42),

$$K_N^{\mu_1\cdots\mu_L} = \frac{1}{L}\left[\mathscr{W}^{\cdot} K_N^{\{L-1\,\mathrm{dots}\}}\right]^{\{\mu_1\cdots\mu_L\}} + \frac{2^{(L-1)}}{L!}\left[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\cdots\mathscr{H}_{\nu_{L-1}}^{\cdot}\right]^{\{\mu_1\cdots\mu_L\}} I_{N-1,N-l}^{\nu_1\cdots\nu_{L-1}}. \tag{C.1}$$

To Eq. (36),

$$K_N^{\mu_1\cdots\mu_L} = \mathscr{W}^{\mu_1}\cdots\mathscr{W}^{\mu_L} I_N^n + \sum_{k=1}^{L}\left[\mathscr{W}_{(L-k)}^{\cdot}\mathscr{K}_{l_1}^{\cdot}\cdots\mathscr{K}_{l_k}^{\cdot}\right]^{\{\mu_1\cdots\mu_L\}} I_{N-k,N-l_1,\ldots,N-l_k}^n,$$

we apply Eq. (41),

$$I_{N-k,N-l_1,\ldots,N-l_k}^n = \sum_{j=0}^{k-1}(-1)^{k-j-1}\frac{2^j}{j!}\left[v_{\cdot}^{(k-j-1)}\, r_{\cdot\nu_1}\cdots r_{\cdot\nu_j}\right]_{\{l_1\cdots l_{k-1}\}} I_{N-1,N-l_k}^{\nu_1\cdots\nu_j},$$

and get, by taking into account combinatorial factors from contracting the tensor brackets

$$K_N^{\mu_1\cdots\mu_L} = \mathscr{W}^{\mu_1}\cdots\mathscr{W}^{\mu_L} I_N^n + \sum_{k=1}^{L}\frac{1}{k}\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{L-j-1}{k-j-1}\frac{2^j}{j!} \times\left[\mathscr{W}_{(L-j-1)}^{\cdot}\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\cdots\mathscr{H}_{\nu_j}^{\cdot}\right]^{\{\mu_1\cdots\mu_L\}} I_{N-1,N-l}^{\nu_1\cdots\nu_j}. \tag{C.2}$$

Using the fact that

$$\sum_{k=1}^{L}\sum_{j=0}^{k-1} a_{kj} = \sum_{k=1}^{L}\sum_{j=0}^{L-1}\theta(j\le k-1)\,a_{kj} = a_{L,L-1} + \sum_{j=0}^{L-2}\sum_{k=j+1}^{L} a_{kj}, \tag{C.3}$$

we can separate terms with and without $\mathscr{W}^{\cdot}$-vectors,

$$K_N^{\mu_1\cdots\mu_L} = \mathscr{W}^{\mu_1}\ldots\mathscr{W}^{\mu_L} I_N^n + \sum_{k=1}^{L}\sum_{j=0}^{L-2}\theta(j\le k-1)\frac{2^j}{j!}\frac{(-1)^{k-j-1}}{k}\binom{L-j-1}{k-j-1}$$

$$\cdot\left[\mathscr{W}^{\cdot}_{(L-j-1)}\mathscr{H}^{\cdot}_{\nu_1}\ldots\mathscr{H}^{\cdot}_{\nu_j}\mathscr{K}^{\cdot}_{l}\right]^{\{\mu_1\ldots\mu_L\}} I_{N-1,N-l}^{\nu_1\ldots\nu_j}$$

$$+\frac{2^{L-1}}{L!}\left[\mathscr{H}^{\cdot}_{\nu_1}\ldots\mathscr{H}^{\cdot}_{\nu_{L-1}}\mathscr{K}^{\cdot}_{l}\right]^{\{\mu_1\ldots\mu_L\}} I_{N-1,N-l}^{\nu_1\ldots\nu_{L-1}}. \tag{C.4}$$

Now one has to rearrange the terms containing $\mathscr{W}$-vectors.

First we write $1/k$ in (C.4) as $1/k = (L-k)/(Lk) + 1/L$. The $1/L$ part vanishes since the sum over k just represents $(1-1)^{L-j-1} = 0$. The $(L-k)/(Lk)$ part makes the upper bound of the k-sum to be $L-1$. From the tensor bracket a $\mathscr{W}^{\cdot}$ can be factored in a symmetric way,

$$\left[\mathscr{W}^{\cdot}_{(L-j-1)}\mathscr{H}^{\cdot}_{\nu_1}\ldots\mathscr{H}^{\cdot}_{\nu_j}\mathscr{K}^{\cdot}_{l}\right]^{\{\mu_1\ldots\mu_L\}}$$

$$=\frac{\mathscr{W}^{\mu_1}}{L-j-1}\left[\mathscr{W}^{\cdot}_{(L-j-2)}\mathscr{H}^{\cdot}_{\nu_1}\ldots\mathscr{H}^{\cdot}_{\nu_j}\mathscr{K}^{\cdot}_{l}\right]^{\{\mu_2\ldots\mu_L\}}$$

$$+\cdots\quad+\frac{\mathscr{W}^{\mu_L}}{L-j-1}\left[\mathscr{W}^{\cdot}_{(L-j-2)}\mathscr{H}^{\cdot}_{\nu_1}\ldots\mathscr{H}^{\cdot}_{\nu_j}\mathscr{K}^{\cdot}_{l}\right]^{\{\mu_1\ldots\mu_{L-1}\}}.$$

Hence Eq. (C.4) is equivalent to

$$K_N^{\mu_1\cdots\mu_L} - \frac{2^{L-1}}{L!}\left[\mathscr{H}^{\cdot}_{\nu_1}\ldots\mathscr{H}^{\cdot}_{\nu_{L-1}}\mathscr{K}^{\cdot}_{l}\right]^{\{\mu_1\ldots\mu_L\}} I_{N-1,N-l}^{\nu_1\ldots\nu_{L-1}}$$

$$=\frac{\mathscr{W}^{\mu_1}}{L}\Biggl(\mathscr{W}^{\mu_2}\ldots\mathscr{W}^{\mu_L} I_N^n + \sum_{k=1}^{L-1}\sum_{j=0}^{k-1}\frac{2^j}{j!}\frac{(-1)^{k-j-1}}{k}\binom{L-j-2}{k-j-1}$$

$$\cdot\left[\mathscr{W}^{\cdot}_{(L-j-2)}\mathscr{H}^{\cdot}_{\nu_1}\ldots\mathscr{H}^{\cdot}_{\nu_j}\mathscr{K}^{\cdot}_{l}\right]^{\{\mu_2\ldots\mu_L\}} I_{N-1,N-l}^{\nu_1\ldots\nu_j}\Biggr)$$

$$+(L-1)\ \text{permutations}. \tag{C.5}$$

Comparing with Eq. (C.2) we see that the right-hand side of Eq. (C.5) is just

$$\frac{\mathscr{W}^{\mu_1}}{L}K_N^{\mu_2\cdots\mu_L} + (L-1)\ \text{permutations} = \frac{1}{L}\left[\mathscr{W}^{\cdot} K_N^{\{L-1\,\text{dots}\}}\right]^{\{\mu_1\cdots\mu_L\}}. \tag{C.6}$$

Combining (C.5) and (C.6) we obtain the recursion relation (C.1).

Appendix D. Explicit reduction for $N = 2,3,4,5,6$

We now give explicit formulas for tensor integrals up to $N = 6$, and rank $L \leq N$. They have been derived by applying Eqs. (43), (44) and (37). The reduction stops because tadpole integrals of massless propagators are zero, $I_1, I_1^{\mu} = 0$. This shows that any tensor integral with an arbitrary number of legs can be expressed in terms of scalar integrals only. For non-exceptional kinematics higher dimensional integrals $I_{N'}^{n+2m}$ ($m = 1,2$) occur only with $N' \leq 4$ in the reduction of arbitrary N-point tensor integrals, as has been explained above. All the higher dimensional integrals can be remapped to n-dimensional integrals with the scalar reduction formulas (20), (22) and (24). Explicitly, the expressions for the higher dimensional integrals are given by

$$
\begin{aligned}
J_{N=2,3,4}^{\mu_1\mu_2} &= (-1)\left(g^{\mu_1\mu_2}/2 - \mathscr{H}^{\mu_1\mu_2}\right) I_N^{n+2}\,, \\
J_{N=3,4}^{\mu_1\mu_2\mu_3} &= (-1)\Big[\left(g/2 - \mathscr{H}\right)^{\cdot\cdot}\mathscr{W}^{\cdot}\Big]^{\{\mu_1\mu_2\mu_3\}} I_N^{n+2} \\
&\quad - \Big[\left(g/2 - \mathscr{H}\right)^{\cdot\cdot}\mathscr{K}_l^{\cdot}\Big]^{\{\mu_1\mu_2\mu_3\}} I_{N-1,N-l}^{n+2}\,, \\
J_4^{\mu_1\mu_2\mu_3\mu_4} &= \Big[\left(g/2 - \mathscr{H}\right)_{(2)}^{\cdot\cdot}\Big]^{\{\mu_1\mu_2\mu_3\mu_4\}} I_4^{n+4} \\
&\quad - \Big[\left(g/2 - \mathscr{H}\right)^{\cdot\cdot}\mathscr{W}_{(2)}^{\cdot}\Big]^{\{\mu_1\mu_2\mu_3\mu_4\}} I_4^{n+2} \\
&\quad - \tfrac{1}{2}\Big[\left(g/2 - \mathscr{H}\right)^{\cdot\cdot}\mathscr{W}^{\cdot}\mathscr{K}_l^{\cdot}\Big]^{\{\mu_1\mu_2\mu_3\mu_4\}} I_{3,4-l}^{n+2} \\
&\quad - \Big[\left(g/2 - \mathscr{H}\right)^{\cdot\cdot}\mathscr{H}_{\nu}^{\cdot}\mathscr{K}_l^{\cdot}\Big]^{\{\mu_1\mu_2\mu_3\mu_4\}} I_{3,4-l}^{\nu,n+2}\,.
\end{aligned}
\tag{D.1}
$$

Note that rank one, $(n+2)$-dimensional three-point functions appear. As the reduction rules do not depend on the space-time dimension all the formulas given above are also valid for higher dimensional tensor integrals. The tensor structure is carried by the well-defined 4-dimensional objects $\mathscr{W}^{\mu}, \mathscr{K}_l^{\mu}, \mathscr{H}^{\mu\nu}$ given in Eq. (38). Note that metric tensors occur only in the combination $(g/2 - \mathscr{H})$ as coefficients of the higher dimensional integrals I_N^{n+2m}.

For $N \geq 5$, the equal signs are only valid up to $\mathscr{O}(\epsilon)$ since the $J_N^{\mu_1\cdots\mu_L}$ terms have been dropped. The shift of the integrals $I_{N-1,N-l}^{\nu_1\cdots\nu_L}(R)$ has been done explicitly up to $N = 3$ in order to give examples for the algorithm defined in (44). How to proceed for $N > 3$ then should be obvious.

The reduction rules can be implemented easily in algebraic manipulation programs.

N = 2

Necessary for a non-vanishing result is that the external momentum is not light-like. The Gram matrix is trivial. One has $\mathscr{W}^{\mu}(r) = r^{\mu}/2$, $\mathscr{H}^{\mu_1\mu_2} = r^{\mu_1}r^{\mu_2}/(2r\cdot r)$,

$$
\begin{aligned}
I_2^{\mu_1}(r) &= \frac{r^{\mu_1}}{2}\, I_2^n(r)\,, \\
I_2^{\mu_1\mu_2}(r) &= \frac{1}{4(n-1)}\left(n r^{\mu_1}r^{\mu_2} - r^2 g^{\mu_1\mu_2}\right) I_2^n(r)\,.
\end{aligned}
\tag{D.2}
$$

$N = 3$

We use the short-hand notation for the arguments of the tensors and integrals defined in Eq. (45) in the following: $R = [r_1, r_2, 0]$, $R_{[1]} = [r_2 - r_1, 0]$, $\hat{R}_{[1]} = [r_2, 0]$, etc. The arguments for the tensors $\mathscr{W}^{\mu}, \mathscr{K}_l^{\mu}, \mathscr{H}^{\mu\nu}$ are not written explicitly, e.g. it is always understood that $\mathscr{W}^{\mu} = \mathscr{W}^{\mu}(R)$, etc.,

$$I_3^{\mu}(R) = \mathscr{W}^{\mu} I_3^n(R) + \sum_{l=1}^{2} \mathscr{K}_l^{\mu}\Big(I_2^n(R_{[l]}) - I_2^n\big(\hat{R}_{[l]}\big)\Big),$$

$$\begin{aligned} I_3^{\mu_1\mu_2}(R) = {} & \tfrac{1}{2}\big(\mathscr{W}^{\mu_1} I_3^{\mu_2}(R) + \mathscr{W}^{\mu_2} I_3^{\mu_1}(R)\big) + \sum_{l=1}^{2} \big(\mathscr{K}_l^{\mu_1}\mathscr{H}_{\nu}^{\mu_2} + \mathscr{K}_l^{\mu_2}\mathscr{H}_{\nu}^{\mu_1}\big) \\ & \times\Big(r_l^{\nu} I_2^n(R_{[l]}) + I_2^{\nu}(R_{[l]}) - I_2^{\nu}\big(\hat{R}_{[l]}\big)\Big) \\ & - \big(g^{\mu_1\mu_2}/2 - \mathscr{H}^{\mu_1\mu_2}\big) I_3^{n+2}(R), \end{aligned}$$

$$\begin{aligned} I_3^{\mu_1\mu_2\mu_3}(R) = {} & \tfrac{1}{3}\Big(\mathscr{W}^{\mu_1}\big\{I_3^{\mu_2\mu_3} - J_3^{\mu_2\mu_3}\big\} \\ & + \mathscr{W}^{\mu_2}\big\{I_3^{\mu_1\mu_3} - J_3^{\mu_1\mu_3}\big\} + \mathscr{W}^{\mu_3}\big\{I_3^{\mu_1\mu_2} - J_3^{\mu_1\mu_2}\big\}\Big) \\ & + \tfrac{2}{3}\sum_{l=1}^{2}\Big[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\mathscr{H}_{\nu_2}^{\cdot}\Big]^{\{\mu_1\mu_2\mu_3\}}\Big(r_l^{\nu_1} r_l^{\nu_2} I_2^n(R_{[l]}) \\ & + r_l^{\nu_1} I_2^{\nu_2}(R_{[l]}) + r_l^{\nu_2} I_2^{\nu_1}(R_{[l]}) + I_2^{\nu_1\nu_2}(R_{[l]}) - I_2^{\nu_1\nu_2}\big(\hat{R}_{[l]}\big)\Big) \\ & - \Big[\big(g/2 - \mathscr{H}\big)^{\cdot\cdot}\mathscr{W}^{\cdot}\Big]^{\{\mu_1\mu_2\mu_3\}} I_3^{n+2}(R) \\ & - \Big[\big(g/2 - \mathscr{H}\big)^{\cdot\cdot}\mathscr{K}_l^{\cdot}\Big]^{\{\mu_1\mu_2\mu_3\}} I_{2,3-l}^{n+2}(R). \end{aligned} \tag{D.3}$$

$N = 4$

$R = [r_1, r_2, r_3, 0]$. The integrals $I_{3,4-l}^{\nu_1\cdots\nu_L}$ are given in unshifted form. The integrals $J_4^{\mu_1\cdots\mu_L}$ are given in Eq. (D.1),

$$I_4^{\mu_1}(R) = \mathscr{W}^{\mu_1} I_4^n(R) + \sum_{l=1}^{3} \mathscr{K}_l^{\mu_1} I_{3,4-l}^n(R),$$

$$\begin{aligned} I_4^{\mu_1\mu_2}(R) = {} & \tfrac{1}{2}\big(\mathscr{W}^{\mu_1} I_4^{\mu_2}(R) + \mathscr{W}^{\mu_2} I_4^{\mu_1}(R)\big) + \sum_{l=1}^{3}\Big[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu}^{\cdot}\Big]^{\{\mu_1\mu_2\}} I_{3,4-l}^{\nu}(R) \\ & + J_4^{\mu_1\mu_2}, \end{aligned}$$

$$\begin{aligned} I_4^{\mu_1\mu_2\mu_3}(R) = {} & \tfrac{1}{3}\Big(\mathscr{W}^{\mu_1}\big\{I_4^{\mu_2\mu_3} - J_4^{\mu_2\mu_3}\big\} + \mathscr{W}^{\mu_2}\big\{I_4^{\mu_1\mu_3} - J_4^{\mu_1\mu_3}\big\} \\ & + \mathscr{W}^{\mu_3}\big\{I_4^{\mu_1\mu_2} - J_4^{\mu_1\mu_2}\big\}\Big) + \tfrac{2}{3}\sum_{l=1}^{3}\Big[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\mathscr{H}_{\nu_2}^{\cdot}\Big]^{\{\mu_1\mu_2\mu_3\}} I_{3,4-l}^{\nu_1\nu_2}(R) \\ & + J_4^{\mu_1\mu_2\mu_3}, \end{aligned}$$

$$I_4^{\mu_1\mu_2\mu_3\mu_4}(R) = \tfrac{1}{4}\Big(\mathscr{W}^{\mu_1}\left\{I_4^{\mu_2\mu_3\mu_4} - J_4^{\mu_2\mu_3\mu_4}\right\} + \mathscr{W}^{\mu_2}\left\{I_4^{\mu_1\mu_3\mu_4} - J_4^{\mu_1\mu_3\mu_4}\right\}$$
$$+\mathscr{W}^{\mu_3}\left\{I_4^{\mu_1\mu_2\mu_4} - J_4^{\mu_1\mu_2\mu_4}\right\} + \mathscr{W}^{\mu_4}\left\{I_4^{\mu_1\mu_2\mu_3} - J_4^{\mu_1\mu_2\mu_3}\right\}\Big)$$
$$+\tfrac{1}{3}\sum_{l=1}^{3}\left[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\mathscr{H}_{\nu_2}^{\cdot}\mathscr{H}_{\nu_3}^{\cdot}\right]^{\{\mu_1\mu_2\mu_3\mu_4\}} I_{3,4-l}^{\nu_1\nu_2\nu_3}(R)$$
$$+ J_4^{\mu_1\mu_2\mu_3\mu_4}. \tag{D.4}$$

$N = 5$

As discussed above, no higher dimensional integrals appear in the reduction of tensor N-point integrals for $N > 4$, as long as the external momenta are non-exceptional. They occur only implicitly by reduction of 2,3,4-point tensor integrals. Now, $R = [r_1,r_2,r_3,r_4,0]$. The formulas are valid up to $\mathscr{O}(\epsilon)$ since $J_5^{\{\mu\}}$ terms have been dropped,

$$I_5^{\mu_1}(R) = \mathscr{W}^{\mu_1}\, I_5^n(R) + \sum_{l=1}^{4}\mathscr{K}_l^{\mu_1}\, I_{4,5-l}^n(R),$$

$$I_5^{\mu_1\mu_2}(R) = \tfrac{1}{2}\Big(\mathscr{W}^{\mu_1}\, I_5^{\mu_2}(R) + \mathscr{W}^{\mu_2}\, I_5^{\mu_1}(R)\Big) + \sum_{l=1}^{4}\left[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu}^{\cdot}\right]^{\{\mu_1\mu_2\}} I_{4,5-l}^{\nu}(R),$$

$$I_5^{\mu_1\mu_2\mu_3}(R) = \tfrac{1}{3}\Big(\mathscr{W}^{\mu_1}\, I_5^{\mu_2\mu_3} + \mathscr{W}^{\mu_2}\, I_5^{\mu_1\mu_3} + \mathscr{W}^{\mu_3}\, I_5^{\mu_1\mu_2}\Big)$$
$$+\tfrac{2}{3}\sum_{l=1}^{4}\left[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\mathscr{H}_{\nu_2}^{\cdot}\right]^{\{\mu_1\mu_2\mu_3\}} I_{4,5-l}^{\nu_1\nu_2}(R),$$

$$I_5^{\mu_1\mu_2\mu_3\mu_4}(R) = \tfrac{1}{4}\Big(\mathscr{W}^{\mu_1}\, I_5^{\mu_2\mu_3\mu_4} + \mathscr{W}^{\mu_2}\, I_5^{\mu_1\mu_3\mu_4} + \mathscr{W}^{\mu_3}\, I_5^{\mu_1\mu_2\mu_4} + \mathscr{W}^{\mu_4}\, I_5^{\mu_1\mu_2\mu_3}\Big)$$
$$+\tfrac{1}{3}\sum_{l=1}^{4}\left[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\mathscr{H}_{\nu_2}^{\cdot}\mathscr{H}_{\nu_3}^{\cdot}\right]^{\{\mu_1\mu_2\mu_3\mu_4\}} I_{4,5-l}^{\nu_1\nu_2\nu_3}(R),$$

$$I_5^{\mu_1\mu_2\mu_3\mu_4\mu_5}(R) = \tfrac{1}{5}\Big(\mathscr{W}^{\mu_1}\, I_5^{\mu_2\mu_3\mu_4\mu_5} + \ldots + \mathscr{W}^{\mu_5}\, I_5^{\mu_1\mu_2\mu_3\mu_4}\Big)$$
$$+\tfrac{2}{15}\sum_{l=1}^{4}\left[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\mathscr{H}_{\nu_2}^{\cdot}\mathscr{H}_{\nu_3}^{\cdot}\mathscr{H}_{\nu_4}^{\cdot}\right]^{\{\mu_1\mu_2\mu_3\mu_4\mu_5\}} I_{4,5-l}^{\nu_1\nu_2\nu_3\nu_4}(R). \tag{D.5}$$

$N = 6$

For non-exceptional kinematics any set of four vectors out of $\{r_1,r_2,r_3,r_4,r_5\}$ spans 4-dimensional Minkowski space. We will express the tensor 6-point functions in terms of scalar 6-point functions and pentagon integrals. The latter then can be reduced further by using (29) and the reduction formulas for $N=5$ with the corresponding new argument vectors,

$$I_6^{\mu_1}(R) = \mathscr{W}^{\mu_1}\, I_6^n(R) + \sum_{l=1}^{5} \mathscr{K}_l^{\mu_1}\, I_{5,6-l}^n(R)\,,$$

$$I_6^{\mu_1\mu_2}(R) = \tfrac{1}{2}\left(\mathscr{W}^{\mu_1}\, I_6^{\mu_2}(R) + \mathscr{W}^{\mu_2}\, I_6^{\mu_1}(R)\right) + \sum_{l=1}^{5} \left[\mathscr{K}_l^{\cdot}\mathscr{H}_\nu^{\cdot}\right]^{\{\mu_1\mu_2\}} I_{5,6-l}^{\nu}(R)\,,$$

$$\begin{aligned} I_6^{\mu_1\mu_2\mu_3}(R) = {} & \tfrac{1}{3}\left(\mathscr{W}^{\mu_1}\, I_6^{\mu_2\mu_3} + \mathscr{W}^{\mu_2}\, I_6^{\mu_1\mu_3} + \mathscr{W}^{\mu_3}\, I_6^{\mu_1\mu_2}\right) \\ & + \tfrac{2}{3}\sum_{l=1}^{5}\left[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\mathscr{H}_{\nu_2}^{\cdot}\right]^{\{\mu_1\mu_2\mu_3\}} I_{5,6-l}^{\nu_1\nu_2}(R)\,, \end{aligned}$$

$$\begin{aligned} I_6^{\mu_1\mu_2\mu_3\mu_4}(R) = {} & \tfrac{1}{4}\left(\mathscr{W}^{\mu_1}\, I_6^{\mu_2\mu_3\mu_4} + \mathscr{W}^{\mu_2}\, I_6^{\mu_1\mu_3\mu_4} + \mathscr{W}^{\mu_3}\, I_6^{\mu_1\mu_2\mu_4} + \mathscr{W}^{\mu_4}\, I_6^{\mu_1\mu_2\mu_3}\right) \\ & + \tfrac{1}{3}\sum_{l=1}^{5}\left[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\mathscr{H}_{\nu_2}^{\cdot}\mathscr{H}_{\nu_3}^{\cdot}\right]^{\{\mu_1\mu_2\mu_3\mu_4\}} I_{5,6-l}^{\nu_1\nu_2\nu_3}(R)\,, \end{aligned}$$

$$\begin{aligned} I_6^{\mu_1\mu_2\mu_3\mu_4\mu_5}(R) = {} & \tfrac{1}{5}\left(\mathscr{W}^{\mu_1}\, I_6^{\mu_2\mu_3\mu_4\mu_5} + \ldots + \mathscr{W}^{\mu_5}\, I_6^{\mu_1\mu_2\mu_3\mu_4}\right) \\ & + \tfrac{2}{15}\sum_{l=1}^{5}\left[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\mathscr{H}_{\nu_2}^{\cdot}\mathscr{H}_{\nu_3}^{\cdot}\mathscr{H}_{\nu_4}^{\cdot}\right]^{\{\mu_1\mu_2\mu_3\mu_4\mu_5\}} I_{5,6-l}^{\nu_1\nu_2\nu_3\nu_4}(R)\,, \end{aligned}$$

$$\begin{aligned} I_6^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}(R) = {} & \tfrac{1}{6}\left(\mathscr{W}^{\mu_1}\, I_6^{\mu_2\mu_3\mu_4\mu_5\mu_6} + \ldots + \mathscr{W}^{\mu_6}\, I_6^{\mu_1\mu_2\mu_3\mu_4\mu_5}\right) \\ & + \tfrac{2}{45}\sum_{l=1}^{5}\left[\mathscr{K}_l^{\cdot}\mathscr{H}_{\nu_1}^{\cdot}\mathscr{H}_{\nu_2}^{\cdot}\mathscr{H}_{\nu_3}^{\cdot}\mathscr{H}_{\nu_4}^{\cdot}\mathscr{H}_{\nu_5}^{\cdot}\right]^{\{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\}} \\ & \times I_{5,6-l}^{\nu_1\nu_2\nu_3\nu_4\nu_5}(R)\,. \end{aligned} \tag{D.6}$$

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