Fun with spinor indices

based on S-35

invariant symbol for raising and lowering spinor indices:

$$\psi^a(x) \equiv \varepsilon^{ab}\psi_b(x)$$

$$\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = \varepsilon_{21} = \varepsilon_{\dot{2}\dot{1}} = +1 \,, \qquad \varepsilon^{21} = \varepsilon^{\dot{2}\dot{1}} = \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -1$$

$$\varepsilon^{ab} = -\varepsilon_{ab} = i\sigma_2$$

another invariant symbol:

$$\sigma^\mu_{a\dot{a}} = (I, \vec{\sigma})$$

$$A_{a\dot{a}}(x) = \sigma^\mu_{a\dot{a}} A_\mu(x)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Simple identities:

$$\sigma^\mu_{a\dot{a}}\sigma_{\mu b\dot{b}} = -2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}$$

proportionality constants from direct calculation

$$\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} = -2g^{\mu\nu}$$

What can we learn about the generator matrices $(S_{\rm L}^{\mu\nu})_a{}^b$ from invariant symbols?

- from $\varepsilon_{ab} = L(\Lambda)_a{}^c L(\Lambda)_b{}^d \varepsilon_{cd}$:

  for an infinitesimal transformation we had:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$$

$$L_a{}^b(1+\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm L}^{\mu\nu})_a{}^b$$

  and we find:

$$\begin{aligned}\varepsilon_{ab} &= \varepsilon_{ab} + \tfrac{i}{2}\delta\omega_{\mu\nu}\Big[(S_{\rm L}^{\mu\nu})_a{}^c\varepsilon_{cb} + (S_{\rm L}^{\mu\nu})_b{}^d\varepsilon_{ad}\Big] + O(\delta\omega^2) \\ &= \varepsilon_{ab} + \tfrac{i}{2}\delta\omega_{\mu\nu}\Big[-(S_{\rm L}^{\mu\nu})_{ab} + (S_{\rm L}^{\mu\nu})_{ba}\Big] + O(\delta\omega^2)\ .\end{aligned}$$

$$\Longrightarrow \quad (S_{\rm L}^{\mu\nu})_{ab} = (S_{\rm L}^{\mu\nu})_{ba}$$

  similarly:

$$(S_{\rm R}^{\mu\nu})_{\dot a\dot b} = (S_{\rm R}^{\mu\nu})_{\dot b\dot a}$$

from $\sigma^\rho_{a\dot a} = \Lambda^\rho{}_\tau L(\Lambda)_a{}^b R(\Lambda)_{\dot a}{}^{\dot b} \sigma^\tau_{b\dot b}$:

for infinitesimal transformations we had:

$$(S^{\mu\nu}_{\rm V})^\rho{}_\tau \equiv \tfrac{1}{i}(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau)$$

$$\begin{aligned} \Lambda^\rho{}_\tau &= \delta^\rho{}_\tau + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm V})^\rho{}_\tau \ , \\ L_a{}^b(1{+}\delta\omega) &= \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm L})_a{}^b \ , \\ R_{\dot a}{}^{\dot b}(1{+}\delta\omega) &= \delta_{\dot a}{}^{\dot b} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b} \ , \end{aligned}$$

isolating linear terms in $\delta\omega_{\mu\nu}$ we have:

$$(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau)\sigma^\tau_{a\dot a} + i(S^{\mu\nu}_{\rm L})_a{}^b\sigma^\rho_{b\dot a} + i(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b}\sigma^\rho_{a\dot b} = 0$$

multiplying by $\sigma_{\rho c\dot c}$ we have:

$$\sigma^\mu_{c\dot c}\sigma^\nu_{a\dot a} - \sigma^\nu_{c\dot c}\sigma^\mu_{a\dot a} + i(S^{\mu\nu}_{\rm L})_a{}^b\sigma^\rho_{b\dot a}\sigma_{\rho c\dot c} + i(S^{\mu\nu}_{\rm R})_{\dot a}{}^{\dot b}\sigma^\rho_{a\dot b}\sigma_{\rho c\dot c} = 0$$

$$\sigma^\mu_{a\dot a}\sigma_{\mu b\dot b} = -2\varepsilon_{ab}\varepsilon_{\dot a\dot b}$$

$$\sigma^\mu_{c\dot c}\sigma^\nu_{a\dot a} - \sigma^\nu_{c\dot c}\sigma^\mu_{a\dot a} + 2i(S^{\mu\nu}_{\rm L})_{ac}\varepsilon_{\dot a\dot c} + 2i(S^{\mu\nu}_{\rm R})_{\dot a\dot c}\varepsilon_{ac} = 0$$

$$\sigma^\mu_{c\dot{c}}\sigma^\nu_{a\dot{a}} - \sigma^\nu_{c\dot{c}}\sigma^\mu_{a\dot{a}} + 2i(S^{\mu\nu}_{\rm L})_{ac}\varepsilon_{\dot{a}\dot{c}} + 2i(S^{\mu\nu}_{\rm R})_{\dot{a}\dot{c}}\varepsilon_{ac} = 0$$

multiplying by $\varepsilon^{\dot{a}\dot{c}}$ we get:

$\varepsilon^{\dot{a}\dot{c}}\varepsilon_{\dot{a}\dot{c}} = -2$

$\varepsilon^{\dot{a}\dot{c}}(S^{\mu\nu}_{\rm R})_{\dot{a}\dot{c}} = 0$

$$(S^{\mu\nu}_{\rm L})_{ac} = \tfrac{i}{4}\varepsilon^{\dot{a}\dot{c}}(\sigma^\mu_{a\dot{a}}\sigma^\nu_{c\dot{c}} - \sigma^\nu_{a\dot{a}}\sigma^\mu_{c\dot{c}})$$

similarly, multiplying by ε^{ac} we get:

$$(S^{\mu\nu}_{\rm R})_{\dot{a}\dot{c}} = \tfrac{i}{4}\varepsilon^{ac}(\sigma^\mu_{a\dot{a}}\sigma^\nu_{c\dot{c}} - \sigma^\nu_{a\dot{a}}\sigma^\mu_{c\dot{c}})$$

let's define:

$$\bar{\sigma}^{\mu\dot{a}a} \equiv \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^\mu_{b\dot{b}}$$

$$\sigma^\mu_{a\dot{a}} = (I, \vec{\sigma})$$

$$\bar{\sigma}^{\mu\dot{a}a} = (I, -\vec{\sigma})$$

we find:

$$(S^{\mu\nu}_{\rm L})_a{}^b = +\tfrac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_a{}^b$$

$\dot{c}$ $\dot{c}$

$$(S^{\mu\nu}_{\rm R})^{\dot{a}}{}_{\dot{b}} = -\tfrac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}{}_{\dot{b}}$$

c c

consistent with our previous choice! (homework, S-35.2)

Convention:

missing pair of contracted indices is understood to be written as:

$$\boxed{{}^{c}{}_{c}} \qquad\qquad \boxed{{}_{\dot{c}}{}^{\dot{c}}}$$

thus, for left-handed Weyl fields we have:

$$\chi\psi = \chi^a\psi_a \quad \text{and} \quad \chi^\dagger\psi^\dagger = \chi^\dagger_{\dot{a}}\psi^{\dagger\dot{a}}$$

spin 1/2 particles are fermions that anticommute:

the spin-statistics theorem (later)

$$\chi_a(x)\psi_b(y) = -\psi_b(y)\chi_a(x)$$

and we find:

$$\underline{\chi\psi} = \chi^a\psi_a = -\psi_a\chi^a = \psi^a\chi_a = \underline{\psi\chi}$$

$${}_a{}^a = -{}^a{}_a$$

$$\chi\psi = \chi^a\psi_a \quad \text{and} \quad \chi^\dagger\psi^\dagger = \chi^\dagger_{\dot a}\psi^{\dagger\dot a}$$

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$${}_a{}^a = -{}^a{}_a$$

for hermitian conjugate we find:

$$(\chi\psi)^\dagger = (\chi^a\psi_a)^\dagger = (\psi_a)^\dagger(\chi^a)^\dagger = \psi^\dagger_{\dot a}\chi^{\dagger\dot a} = \psi^\dagger\chi^\dagger$$

as expected if we ignored indices

and similarly:

$$\underline{\psi^\dagger\chi^\dagger} = \underline{\chi^\dagger\psi^\dagger}$$

we will write a right-handed field always with a dagger!

Let's look at something more complicated:

$$\psi^\dagger \bar{\sigma}^\mu \chi = \psi^\dagger_{\dot{a}} \bar{\sigma}^{\mu\dot{a}c} \chi_c$$

it behaves like a vector field under Lorentz transformations:

$$U(\Lambda)^{-1}[\psi^\dagger \bar{\sigma}^\mu \chi] U(\Lambda) = \Lambda^\mu{}_\nu [\psi^\dagger \bar{\sigma}^\nu \chi]$$

evaluated at $\Lambda^{-1}x$

the hermitian conjugate is:

$$\begin{aligned} [\psi^\dagger \bar{\sigma}^\mu \chi]^\dagger &= [\psi^\dagger_{\dot{a}} \bar{\sigma}^{\mu\dot{a}c} \chi_c]^\dagger \\ &= \chi^\dagger_{\dot{c}} (\bar{\sigma}^{\mu a\dot{c}})^* \psi_a \\ &= \chi^\dagger_{\dot{c}} \bar{\sigma}^{\mu\dot{c}a} \psi_a \\ &= \chi^\dagger \bar{\sigma}^\mu \psi \, . \end{aligned}$$

$\bar{\sigma}^\mu = (I, -\vec{\sigma})$ is hermitian

Lagrangians for spinor fields

based on S-36

we want to find a suitable lagrangian for left- and right-handed spinor fields.

it should be:

- Lorentz invariant and hermitian
- quadratic in ψ_a and $\psi^\dagger_{\dot{a}}$

equations of motion will be linear with plane wave solutions
(suitable for describing free particles)

terms with no derivative:

$$\psi\psi = \psi^a\psi_a = \varepsilon^{ab}\psi_b\psi_a$$

+ h.c.

terms with derivatives:

$$\cancel{\partial^\mu\psi\partial_\mu\psi}$$

would lead to a hamiltonian unbounded from below to get a bounded hamiltonian the kinetic term has to contain both ψ_a and $\psi^\dagger_{\dot a}$, a candidate is:

$$i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi$$

is hermitian up to a total divergence

$$\begin{aligned}(i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi)^\dagger &= (i\psi^\dagger_{\dot a}\,\bar\sigma^{\mu\dot a c}\partial_\mu\psi_c)^\dagger \\ &= -i\partial_\mu\psi^\dagger_{\dot c}\,(\bar\sigma^{\mu a\dot c})^*\psi_a \\ &= -i\partial_\mu\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\psi_a \\ &= i\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\partial_\mu\psi_a - i\partial_\mu(\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\psi_a). \\ &= i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - i\partial_\mu(\psi^\dagger\bar\sigma^\mu\psi)\ .\end{aligned}$$

$\bar\sigma^{\mu\dot a a} = (I, -\vec\sigma)$
are hermitian

does not contribute to the action

Our complete lagrangian is:

$$\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \tfrac{1}{2}m\psi\psi - \tfrac{1}{2}m^*\psi^\dagger\psi^\dagger$$

the phase of m can be absorbed into the definition of fields

$$m = |m|e^{i\alpha} \qquad \psi = e^{-i\alpha/2}\,\tilde{\psi}$$

and so without loss of generality we can take m to be real and positive.

Equation of motion:

$$0 = -\frac{\delta S}{\delta\psi^\dagger} = -i\bar{\sigma}^\mu\partial_\mu\psi + m\psi^\dagger$$

$$0 = -i\bar{\sigma}^{\mu\dot{a}c}\partial_\mu\psi_c + m\psi^{\dagger\dot{a}}$$

Taking hermitian conjugate:

$\bar{\sigma}^{\mu\dot{a}a} = (I, -\vec{\sigma})$

are hermitian

$\bar{\sigma}^{\mu\dot{a}a} \equiv \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^\mu_{b\dot{b}}$

$$\begin{aligned} 0 &= +i(\bar{\sigma}^{\mu a\dot{c}})^*\,\partial_\mu\psi^\dagger_{\dot{c}} + m\psi^a \\ &= +i\bar{\sigma}^{\mu\dot{c}a}\partial_\mu\psi^\dagger_{\dot{c}} + m\psi^a \\ &= -i\sigma^\mu_{a\dot{c}}\,\partial_\mu\psi^{\dagger\dot{c}} + m\psi_a\ . \end{aligned}$$

We can combine the two equations:

$$0 = -i\bar{\sigma}^{\mu\dot{a}c}\partial_\mu \psi_c + m\psi^{\dagger\dot{a}}$$

$$0 = -i\sigma^{\mu}_{a\dot{c}}\partial_\mu \psi^{\dagger\dot{c}} + m\psi_a$$

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^{\mu}_{a\dot{c}}\partial_\mu \\ -i\bar{\sigma}^{\mu\dot{a}c}\partial_\mu & m\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \psi_c \\ \psi^{\dagger\dot{c}} \end{pmatrix} = 0$$

which we can write using 4x4 gamma matrices:

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^{\mu}_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix}$$

and defining four-component Majorana field:

$$\Psi \equiv \begin{pmatrix} \psi_c \\ \psi^{\dagger\dot{c}} \end{pmatrix}$$

as:

$$(-i\gamma^\mu\partial_\mu + m)\Psi = 0$$

Dirac equation

using the sigma-matrix relations:

$$\sigma^{\mu}_{a\dot{a}} = (I, \vec{\sigma})$$

$$\bar{\sigma}^{\mu\dot{a}a} = (I, -\vec{\sigma})$$

$$(\sigma^{\mu}\bar{\sigma}^{\nu} + \sigma^{\nu}\bar{\sigma}^{\mu})_a{}^c = -2g^{\mu\nu}\delta_a{}^c$$

$$(\bar{\sigma}^{\mu}\sigma^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{a}}{}_{\dot{c}} = -2g^{\mu\nu}\delta^{\dot{a}}{}_{\dot{c}}$$

we see that

$$\gamma^{\mu} \equiv \begin{pmatrix} 0 & \sigma^{\mu}_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix}$$

$$\{\gamma^{\mu}, \gamma^{\nu}\} = -2g^{\mu\nu}$$

and we know that that we needed 4 such matrices;

recall:

$$i\hbar\frac{\partial}{\partial t}\psi_a(x) = \left(-i\hbar c(\alpha^j)_{ab}\partial_j + mc^2(\beta)_{ab}\right)\psi_b(x)$$

$$\{\alpha^j, \alpha^k\}_{ab} = 2\delta^{jk}\delta_{ab}\ , \quad \{\alpha^j, \beta\}_{ab} = 0\ , \quad (\beta^2)_{ab} = \delta_{ab}$$

$$\beta = \gamma^0$$

$$\alpha^k = \gamma^0\gamma^k$$

$$(-i\gamma^{\mu}\partial_{\mu} + m)\Psi = 0$$

consider a theory of two left-handed spinor fields:

$$\mathcal{L} = i\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i - \tfrac{1}{2} m\psi_i\psi_i - \tfrac{1}{2} m\psi_i^\dagger\psi_i^\dagger$$

i = 1,2

the lagrangian is invariant under the SO(2) transformation:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

it can be written in the form that is manifestly U(1) symmetric:

$$\chi = \tfrac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$

$$\xi = \tfrac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$$

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

$$\chi \to e^{-i\alpha}\chi$$

$$\xi \to e^{+i\alpha}\xi$$

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

Equations of motion for this theory:

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^\mu_{a\dot{c}}\,\partial_\mu \\ -i\bar{\sigma}^{\mu\dot{a}c}\,\partial_\mu & m\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix} = 0$$

we can define a four-component Dirac field: $\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}$

$$(-i\gamma^\mu\partial_\mu + m)\Psi = 0$$

Dirac equation

we want to write the lagrangian in terms of the Dirac field:

$$\Psi^\dagger = (\chi^\dagger_{\dot{a}}\,,\ \xi^a)$$

$$\beta \equiv \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}$$

Let's define:

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\xi^a,\ \chi^\dagger_{\dot{a}})$$

numerically $\beta = \gamma^0$

but different spinor index structure

Then we find:

$$\overline{\Psi} \equiv \Psi^\dagger \beta = (\xi^a,\, \chi^\dagger_{\dot a})$$

$$\overline{\Psi}\Psi = \xi^a \chi_a + \chi^\dagger_{\dot a} \xi^{\dagger \dot a}$$

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger \dot c} \end{pmatrix}$$

$$\overline{\Psi}\gamma^\mu \partial_\mu \Psi = \xi^a \sigma^\mu_{a\dot c}\, \partial_\mu \xi^{\dagger \dot c} + \chi^\dagger_{\dot a}\, \bar\sigma^{\mu \dot a c}\, \partial_\mu \chi_c$$

$$A\partial B = -(\partial A)B + \partial(AB)$$

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot c} \\ \bar\sigma^{\mu \dot a c} & 0 \end{pmatrix}$$

$$\xi^a \sigma^\mu_{a\dot c}\, \partial_\mu \xi^{\dagger \dot c} = -(\partial_\mu \xi^a)\sigma^\mu_{a\dot c}\, \xi^{\dagger \dot c} + \partial_\mu(\xi^a \sigma^\mu_{a\dot c}\, \xi^{\dagger \dot c})$$

$$-(\partial_\mu \xi^a)\sigma^\mu_{a\dot c}\, \xi^{\dagger \dot c} = +\xi^{\dagger \dot c} \sigma^\mu_{a\dot c}\, \partial_\mu \xi^a = +\xi^\dagger_{\dot c}\, \bar\sigma^{\mu \dot c a} \partial_\mu \xi_a$$

$$\bar\sigma^{\mu \dot a a} \equiv \varepsilon^{ab} \varepsilon^{\dot a \dot b} \sigma^\mu_{b\dot b}$$

Thus we have:

$$\overline{\Psi}\gamma^\mu \partial_\mu \Psi = \chi^\dagger \bar\sigma^\mu \partial_\mu \chi + \xi^\dagger \bar\sigma^\mu \partial_\mu \xi + \partial_\mu(\xi \sigma^\mu \xi^\dagger)$$

Thus the lagrangian can be written as: $\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m\chi\xi - m\xi^\dagger\chi^\dagger$

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi$$

The U(1) symmetry is obvious:

$$\Psi \to e^{-i\alpha}\,\Psi$$

$$\overline{\Psi} \to e^{+i\alpha}\,\overline{\Psi}$$

The Nether current associated with this symmetry is: $j^\mu(x) \equiv \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}\,\delta\varphi_a(x)$

$$j^\mu = \overline{\Psi}\gamma^\mu\Psi = \chi^\dagger\bar{\sigma}^\mu\chi - \xi^\dagger\bar{\sigma}^\mu\xi$$

later we will see that this is the electromagnetic current

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

There is an additional discrete symmetry that exchanges the two fields, charge conjugation:

$$C^{-1}\chi_a(x)C = \xi_a(x)$$
$$C^{-1}\xi_a(x)C = \chi_a(x)$$

unitary charge conjugation operator

$$C^{-1}\mathcal{L}(x)C = \mathcal{L}(x)$$

we want to express it in terms of the Dirac field:

Let's define the charge conjugation matrix:

$$\mathcal{C} \equiv \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon^{\dot{a}\dot{c}} \end{pmatrix}$$

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}$$

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\xi^a,\, \chi^\dagger_{\dot{a}})$$

$$\overline{\Psi}^{\mathrm{T}} = \begin{pmatrix} \xi^a \\ \chi^\dagger_{\dot{a}} \end{pmatrix}$$

then

$$\Psi^{\mathrm{C}} \equiv \mathcal{C}\overline{\Psi}^{\mathrm{T}} = \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot{a}} \end{pmatrix}$$

and we have:

$$C^{-1}\Psi(x)C = \Psi^{\mathrm{C}}(x)$$

The charge conjugation matrix has following properties:

$$\mathcal{C} \equiv \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon^{\dot{a}\dot{c}} \end{pmatrix}$$

$$\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{\dagger} = \mathcal{C}^{-1} = -\mathcal{C}$$

it can also be written as:

$$\mathcal{C} = \begin{pmatrix} -\varepsilon^{ac} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{c}} \end{pmatrix}$$

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu}_{e\dot{a}} \\ \bar{\sigma}^{\mu\dot{e}a} & 0 \end{pmatrix}$$

and then we find a useful identity:

$$\mathcal{C}^{-1}\gamma^{\mu}\mathcal{C} = \begin{pmatrix} \varepsilon^{ab} & 0 \\ 0 & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} 0 & \sigma^{\mu}_{b\dot{c}} \\ \bar{\sigma}^{\mu\dot{b}c} & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{ce} & 0 \\ 0 & \varepsilon^{\dot{c}\dot{e}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \varepsilon^{ab}\sigma^{\mu}_{b\dot{c}}\varepsilon^{\dot{c}\dot{e}} \\ \varepsilon_{\dot{a}\dot{b}}\bar{\sigma}^{\mu\dot{b}c}\varepsilon_{ce} & 0 \end{pmatrix}$$

transposed form of

$\bar{\sigma}^{\mu\dot{a}a} \equiv \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{b\dot{b}}$

$$= \begin{pmatrix} 0 & -\bar{\sigma}^{\mu a\dot{e}} \\ -\sigma^{\mu}_{\dot{a}e} & 0 \end{pmatrix}.$$

$$\boxed{\mathcal{C}^{-1}\gamma^{\mu}\mathcal{C} = -(\gamma^{\mu})^{\mathrm{T}}}$$

Majorana field is its own conjugate:

$$\Psi \equiv \begin{pmatrix} \psi_c \\ \psi^{\dagger \dot{c}} \end{pmatrix}$$

$$\Psi^{\mathrm{C}} = \Psi$$

similar to a real scalar field

$$\varphi^\dagger = \varphi$$

Following the same procedure with:

$$\chi \to \psi$$
$$\xi \to \psi$$

we get:

$$\mathcal{L} = \tfrac{i}{2}\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\overline{\Psi}\Psi$$

does not incorporate the Majorana condition

$$\Psi = \mathcal{C}\overline{\Psi}^{\mathrm{T}}$$
$$\overline{\Psi} = \Psi^{\mathrm{T}}\mathcal{C}$$

incorporating the Majorana condition, we get:

$$\mathcal{L} = \tfrac{i}{2}\Psi^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}m\Psi^{\mathrm{T}}\mathcal{C}\Psi$$

lagrangian for a Majorana field

If we want to go back from 4-component Dirac or Majorana fields to the two-component Weyl fields, it is useful to define a projection matrix:

$$\gamma_5 \equiv \begin{pmatrix} -\delta_a{}^c & 0 \\ 0 & +\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}$$

just a name

We can define left and right projection matrices:

$$P_L \equiv \tfrac{1}{2}(1-\gamma_5) = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_R \equiv \tfrac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}$$

And for a Dirac field we find:

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger \dot{c}} \end{pmatrix}$$

$$P_L \Psi = \begin{pmatrix} \chi_c \\ 0 \end{pmatrix}$$

$$P_R \Psi = \begin{pmatrix} 0 \\ \xi^{\dagger \dot{c}} \end{pmatrix}$$

The gamma-5 matrix can be also written as:

$$\begin{aligned}\gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= -\tfrac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\end{aligned}$$

$$\varepsilon_{0123} = -1$$

Finally, let’s take a look at the Lorentz transformation of a Dirac or Majorana field:

$$U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x)$$

$$D(1{+}\delta\omega) = 1 + \tfrac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu}$$

$$\tfrac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} +(S_{\rm L}^{\mu\nu})_a{}^c & 0 \\ 0 & -(S_{\rm R}^{\mu\nu})^{\dot a}{}_{\dot c}\end{pmatrix} \equiv S^{\mu\nu}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{e\dot a} \\ \bar\sigma^{\mu\dot e a} & 0\end{pmatrix}$$

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L(\Lambda)_a{}^c\,\psi_c(\Lambda^{-1}x)$$

$$U(\Lambda)^{-1}\psi^\dagger_{\dot a}(x)U(\Lambda) = R(\Lambda)_{\dot a}{}^{\dot c}\,\psi^\dagger_{\dot c}(\Lambda^{-1}x)$$

$$L(1{+}\delta\omega)_a{}^c = \delta_a{}^c + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm L}^{\mu\nu})_a{}^c$$

$$R(1{+}\delta\omega)_{\dot a}{}^{\dot c} = \delta_{\dot a}{}^{\dot c} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm R}^{\mu\nu})_{\dot a}{}^{\dot c}$$

$$(S_{\rm L}^{\mu\nu})_a{}^c = +\tfrac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_a{}^c$$

$$(S_{\rm R}^{\mu\nu})^{\dot a}{}_{\dot c} = -\tfrac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot a}{}_{\dot c}$$

$$\Psi \equiv \begin{pmatrix}\chi_c \\ \xi^{\dagger\dot c}\end{pmatrix}$$

compensates for ${}^{\dot c}{}_{\dot c} = -\ {}_{\dot c}{}^{\dot c}$

Canonical quantization of spinor fields I

based on S-37

Consider the lagrangian for a left-handed Weyl field:

$$\mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \tfrac{1}{2} m(\psi\psi + \psi^\dagger \psi^\dagger)$$

the conjugate momentum to the left-handed field is:

$$\pi^a(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a(x))}$$

$$= i\psi^\dagger_{\dot{a}}(x) \bar{\sigma}^{0\dot{a}a}$$

and the hamiltonian is simply given as:

$$\begin{aligned}
\mathcal{H} &= \pi^a \partial_0 \psi_a - \mathcal{L} \\
&= i\psi^\dagger_{\dot{a}} \bar{\sigma}^{0\dot{a}a} \dot{\psi}_a - \mathcal{L} \\
&= -i\psi^\dagger \bar{\sigma}^i \partial_i \psi + \tfrac{1}{2} m(\psi\psi + \psi^\dagger \psi^\dagger)
\end{aligned}$$

the appropriate canonical anticommutation relations are:

$$\{\psi_a(\mathbf{x},t),\psi_c(\mathbf{y},t)\} = 0\,,$$

$$\{\psi_a(\mathbf{x},t),\pi^c(\mathbf{y},t)\} = i\delta_a{}^c\,\delta^3(\mathbf{x}-\mathbf{y})$$

$$\pi^a(x) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_a(x))}$$

$$= i\psi^\dagger_{\dot a}(x)\bar\sigma^{0\dot a a}$$

or

$$\{\psi_a(\mathbf{x},t),\psi^\dagger_{\dot c}(\mathbf{y},t)\}\bar\sigma^{0\dot c c} = \delta_a{}^c\,\delta^3(\mathbf{x}-\mathbf{y})$$

using $\bar\sigma^0 = \sigma^0 = I$ we get

$$\{\psi_a(\mathbf{x},t),\psi^\dagger_{\dot c}(\mathbf{y},t)\} = \sigma^0_{a\dot c}\,\delta^3(\mathbf{x}-\mathbf{y})$$

or, equivalently,

$$\{\psi^a(\mathbf{x},t),\psi^{\dagger\dot c}(\mathbf{y},t)\} = \bar\sigma^{0\dot c a}\,\delta^3(\mathbf{x}-\mathbf{y})$$

For a four-component Dirac field we found:

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m(\chi\xi + \xi^\dagger\chi^\dagger)$$
$$= i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi\ .$$

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}$$

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\xi^a,\ \chi^\dagger_{\dot{a}})$$

and the corresponding canonical anticommutation relations are:

$$\{\psi_a(\mathbf{x},t), \psi^\dagger_{\dot{c}}(\mathbf{y},t)\} = \sigma^0_{a\dot{c}}\,\delta^3(\mathbf{x}-\mathbf{y})$$

$$\{\psi_a(\mathbf{x},t), \psi^\dagger_{\dot{c}}(\mathbf{y},t)\} = \sigma^0_{a\dot{c}}\,\delta^3(\mathbf{x}-\mathbf{y})$$

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0\ ,$$

$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})$$

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix}$$

can be also derived directly from $\partial\mathcal{L}/\partial(\partial_0\Psi) = i\overline{\Psi}\gamma^0$, ...

For a four-component Majorana field we found:

$$\Psi \equiv \begin{pmatrix} \psi_c \\ \psi^{\dagger \dot{c}} \end{pmatrix}$$

$$\begin{aligned} \mathcal{L} &= i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \tfrac{1}{2} m(\psi\psi + \psi^\dagger\psi^\dagger) \\ &= \tfrac{i}{2} \overline{\Psi} \gamma^\mu \partial_\mu \Psi - \tfrac{1}{2} m \overline{\Psi} \Psi \\ &= \tfrac{i}{2} \Psi^{\mathrm{T}} \mathcal{C} \gamma^\mu \partial_\mu \Psi - \tfrac{1}{2} m \Psi^{\mathrm{T}} \mathcal{C} \Psi \; . \end{aligned}$$

$$\overline{\Psi} \equiv \Psi^\dagger \beta = (\psi^a ,\, \psi^\dagger_{\dot{a}})$$

$$\overline{\Psi} = \Psi^{\mathrm{T}} \mathcal{C}$$

$$\mathcal{C} \equiv \begin{pmatrix} -\varepsilon^{ac} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{c}} \end{pmatrix}$$

and the corresponding canonical anticommutation relations are:

$$\{\Psi_\alpha(\mathbf{x}, t), \Psi_\beta(\mathbf{y}, t)\} = (\mathcal{C}\gamma^0)_{\alpha\beta}\, \delta^3(\mathbf{x} - \mathbf{y}) \; ,$$

$$\{\Psi_\alpha(\mathbf{x}, t), \overline{\Psi}_\beta(\mathbf{y}, t)\} = (\gamma^0)_{\alpha\beta}\, \delta^3(\mathbf{x} - \mathbf{y}) \; ,$$

Now we want to find solutions to the Dirac equation:

$$(-i\partial\!\!\!/ + m)\Psi = 0$$

where we used the Feynman slash: $a\!\!\!/ \equiv a_\mu \gamma^\mu$

$$\begin{aligned} a\!\!\!/ a\!\!\!/ &= a_\mu a_\nu \gamma^\mu \gamma^\nu \\ &= a_\mu a_\nu \left(\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \tfrac{1}{2}[\gamma^\mu, \gamma^\nu] \right) \\ &= a_\mu a_\nu \left(-g^{\mu\nu} + \tfrac{1}{2}[\gamma^\mu, \gamma^\nu] \right) \\ &= -a_\mu a_\nu g^{\mu\nu} + 0 \\ &= -a^2 \,. \end{aligned}$$

then we find:

$$\begin{aligned} 0 &= (i\partial\!\!\!/ + m)(-i\partial\!\!\!/ + m)\Psi \\ &= (\partial\!\!\!/ \partial\!\!\!/ + m^2)\Psi \\ &= (-\partial^2 + m^2)\Psi \,. \end{aligned}$$

the Dirac (or Majorana) field satisfies
the Klein-Gordon equation and so
the Dirac equation has plane-wave solutions!

Consider a solution of the form:

$$\Psi(x) = u(\mathbf{p})e^{ipx} + v(\mathbf{p})e^{-ipx}$$

$$p^0 = \omega \equiv (\mathbf{p}^2 + m^2)^{1/2}$$

four-component constant spinors

$$(-i\partial\!\!\!/ + m)\Psi = 0$$

plugging it into the Dirac equation gives:

$$(p\!\!\!/ + m)u(\mathbf{p})e^{ipx} + (-p\!\!\!/ + m)v(\mathbf{p})e^{-ipx} = 0$$

that requires:

$$(p\!\!\!/ + m)u(\mathbf{p}) = 0$$
$$(-p\!\!\!/ + m)v(\mathbf{p}) = 0$$

each eq. has two solutions (later)

The general solution of the Dirac equation is:

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx} \right]$$

$$\widetilde{dp} \equiv \frac{d^3p}{(2\pi)^3 2\omega}$$

Spinor technology

based on S-38

The four-component spinors obey equations:

$$(\not{p} + m)u_s(\mathbf{p}) = 0$$
$$(-\not{p} + m)v_s(\mathbf{p}) = 0$$

s = + or -

In the rest frame, $\mathbf{p} = \mathbf{0}$ we can choose:

for $m \neq 0$

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},$$

$$\not{p} = -m\gamma^0$$
$$\gamma^0 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

$$v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}.$$

convenient normalization and phase

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},$$

$$v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}.$$

this choice corresponds to eigenvectors of the spin matrix:

$$S_z = \tfrac{i}{4}[\gamma^1, \gamma^2] = \tfrac{i}{2}\gamma^1\gamma^2 = \begin{pmatrix}\frac{1}{2}\sigma_3 & 0\\ 0 & \frac{1}{2}\sigma_3\end{pmatrix}$$

$$S_z u_\pm(\mathbf{0}) = \pm\tfrac{1}{2}u_\pm(\mathbf{0})$$

$$S_z v_\pm(\mathbf{0}) = \mp\tfrac{1}{2}v_\pm(\mathbf{0})$$

$$S^{\mu\nu} \equiv \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]$$

this choice results in (we will see it later):

$$[J_z, b^\dagger_\pm(\mathbf{0})] = \pm\tfrac{1}{2}b^\dagger_\pm(\mathbf{0})$$

$$[J_z, d^\dagger_\pm(\mathbf{0})] = \pm\tfrac{1}{2}d^\dagger_\pm(\mathbf{0})$$

creates a particle with spin up (+) or down (-) along the z axis

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},$$

$$v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}.$$

let us also compute the barred spinors:

$$\overline{u}_s(\mathbf{p}) \equiv u_s^\dagger(\mathbf{p})\beta$$
$$\overline{v}_s(\mathbf{p}) \equiv v_s^\dagger(\mathbf{p})\beta$$

$$\beta = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

$$\beta^{\mathrm{T}} = \beta^\dagger = \beta^{-1} = \beta$$

we get:

$$\overline{u}_+(\mathbf{0}) = \sqrt{m}\,(1,\,0,\,1,\,0)\,,$$
$$\overline{u}_-(\mathbf{0}) = \sqrt{m}\,(0,\,1,\,0,\,1)\,,$$
$$\overline{v}_+(\mathbf{0}) = \sqrt{m}\,(0,\,-1,\,0,\,1)\,,$$
$$\overline{v}_-(\mathbf{0}) = \sqrt{m}\,(1,\,0,\,-1,\,0)\,.$$

We can find spinors at arbitrary 3-momentum by applying the matrix that corresponds to the boost:

$$U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x)$$

$$D(\Lambda) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})$$

$$K^j = \tfrac{i}{4}[\gamma^j,\gamma^0] = \tfrac{i}{2}\gamma^j\gamma^0$$

$$S^{\mu\nu} \equiv \tfrac{i}{4}[\gamma^\mu,\gamma^\nu]$$

$$\eta \equiv \sinh^{-1}(|\mathbf{p}|/m)$$

we find:

$$u_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0})$$

$$v_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})v_s(\mathbf{0})$$

and similarly:

$$\overline{u}_s(\mathbf{p}) = \overline{u}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})$$

$$\overline{v}_s(\mathbf{p}) = \overline{v}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})$$

$$\overline{K^j} = K^j$$

$$\overline{A} \equiv \beta A^\dagger\beta$$

For any combination of gamma matrices we define:

$$\overline{A} \equiv \beta A^\dagger \beta$$

It is straightforward to show:

$$\begin{aligned} \overline{\gamma^\mu} &= \gamma^\mu \ , \\ \overline{S^{\mu\nu}} &= S^{\mu\nu} \ , \\ \overline{i\gamma_5} &= i\gamma_5 \ , \\ \overline{\gamma^\mu\gamma_5} &= \gamma^\mu\gamma_5 \ , \\ \overline{i\gamma_5 S^{\mu\nu}} &= i\gamma_5 S^{\mu\nu} \ . \end{aligned}$$

$$S^{\mu\nu} \equiv \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$\overline{K^j} = K^j$$

homework

For barred spinors we get:

$$(\not p + m)u(\mathbf{p}) = 0$$
$$(-\not p + m)v(\mathbf{p}) = 0$$

$$\overline{u}_s(\mathbf{p})(\not p + m) = 0$$
$$\overline{v}_s(\mathbf{p})(-\not p + m) = 0$$

$$\overline{u}_s(\mathbf{p}) \equiv u_s^\dagger(\mathbf{p})\beta$$
$$\overline{v}_s(\mathbf{p}) \equiv v_s^\dagger(\mathbf{p})\beta$$

It is straightforward to derive explicit formulas for spinors, but will will not need them; all we will need are products of spinors of the form:

$$\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) = \overline{u}_{s'}(\mathbf{0})u_s(\mathbf{0})$$

$$u_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0})$$
$$v_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})v_s(\mathbf{0})$$

$$\overline{u}_s(\mathbf{p}) = \overline{u}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})$$
$$\overline{v}_s(\mathbf{p}) = \overline{v}_s(\mathbf{0})\exp(-i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})$$

which do not depend on p!

we find:

$$\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) = +2m\,\delta_{s's}\ ,$$
$$\overline{v}_{s'}(\mathbf{p})v_s(\mathbf{p}) = -2m\,\delta_{s's}\ ,$$
$$\overline{u}_{s'}(\mathbf{p})v_s(\mathbf{p}) = 0\ ,$$
$$\overline{v}_{s'}(\mathbf{p})u_s(\mathbf{p}) = 0\ .$$

Useful identities (Gordon identities):

$$2m\,\overline{u}_{s'}(\mathbf{p}')\gamma^\mu u_s(\mathbf{p}) = \overline{u}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]u_s(\mathbf{p})$$
$$-2m\,\overline{v}_{s'}(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) = \overline{v}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]v_s(\mathbf{p})$$

Proof:

$$\gamma^\mu \not{p} = \tfrac{1}{2}\{\gamma^\mu, \not{p}\} + \tfrac{1}{2}[\gamma^\mu, \not{p}\,] = -p^\mu - 2iS^{\mu\nu}p_\nu$$
$$\not{p}'\gamma^\mu = \tfrac{1}{2}\{\gamma^\mu, \not{p}'\} - \tfrac{1}{2}[\gamma^\mu, \not{p}'] = -p'^\mu + 2iS^{\mu\nu}p'_\nu$$

add the two equations, and sandwich them between spinors, and use:

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$$
$$S^{\mu\nu} \equiv \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$(\not{p} + m)u(\mathbf{p}) = 0 \qquad \overline{u}_s(\mathbf{p})(\not{p} + m) = 0$$
$$(-\not{p} + m)v(\mathbf{p}) = 0 \qquad \overline{v}_s(\mathbf{p})(-\not{p} + m) = 0$$

An important special case $p' = p$:

$$\overline{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) = 2p^\mu \delta_{s's}$$
$$\overline{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) = 2p^\mu \delta_{s's}$$

One can also show:

$$\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0$$
$$\overline{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) = 0$$

homework

Gordon identities with gamma-5:

$$\overline{u}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]\gamma_5 u_s(\mathbf{p}) = 0$$
$$\overline{v}_{s'}(\mathbf{p}')\Big[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu\Big]\gamma_5 v_s(\mathbf{p}) = 0$$

homework

We will find very useful the spin sums of the form:

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p})$$

can be directly calculated but we will find the correct for by the following argument: the sum over spin removes all the memory of the spin-quantization axis, and the result can depend only on the momentum four-vector and gamma matrices with all indices contracted.

In the rest frame, $\not{p} = -m\gamma^0$, we have:

$$\sum_{s=\pm} u_s(\mathbf{0})\overline{u}_s(\mathbf{0}) = m\gamma^0 + m$$

$$\sum_{s=\pm} v_s(\mathbf{0})\overline{v}_s(\mathbf{0}) = m\gamma^0 - m$$

Thus we conclude:

$$\boxed{\begin{aligned} \sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) &= -\not{p} + m \\ \sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) &= -\not{p} - m \end{aligned}}$$

if instead of the spin sum we need just a specific spin product, e.g.

$$u_+(\mathbf{p})\overline{u}_+(\mathbf{p})$$

we can get it using appropriate spin projection matrices:

in the rest frame we have

$$\tfrac{1}{2}(1+2sS_z)u_{s'}(\mathbf{0}) = \delta_{ss'}\, u_{s'}(\mathbf{0})$$
$$\tfrac{1}{2}(1-2sS_z)v_{s'}(\mathbf{0}) = \delta_{ss'}\, v_{s'}(\mathbf{0})$$

$$S_z u_\pm(\mathbf{0}) = \pm\tfrac{1}{2}u_\pm(\mathbf{0})$$
$$S_z v_\pm(\mathbf{0}) = \mp\tfrac{1}{2}v_\pm(\mathbf{0})$$

the spin matrix $S_z = \frac{i}{2}\gamma^1\gamma^2$ can be written as:

$$S_z = -\tfrac{1}{2}\gamma_5\gamma^3\gamma^0$$

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

in the rest frame we can write γ^0 as $-\not{p}/m$ and γ^3 as $\not{z}$ and so we have:

$$S_z = \frac{1}{2m}\gamma_5\not{z}\not{p}$$

$$z^\mu = (0, \hat{\mathbf{z}})$$
$$z^2 = 1$$
$$z\cdot p = 0$$

we can now boost it to any frame simply by replacing z and p with their values in that frame

frame independent

Boosting to a different frame we get:

$$\tfrac{1}{2}(1+2sS_z)u_{s'}(\mathbf{0}) = \delta_{ss'}\,u_{s'}(\mathbf{0})$$
$$\tfrac{1}{2}(1-2sS_z)v_{s'}(\mathbf{0}) = \delta_{ss'}\,v_{s'}(\mathbf{0})$$

$$u_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0})$$
$$v_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})v_s(\mathbf{0})$$

$$S_z = \frac{1}{2m}\gamma_5 \not{z}\not{p}$$

$$(\not{p}+m)u(\mathbf{p}) = 0$$
$$(-\not{p}+m)v(\mathbf{p}) = 0$$

$$\tfrac{1}{2}(1-s\gamma_5\not{z})u_{s'}(\mathbf{p}) = \delta_{ss'}\,u_{s'}(\mathbf{p})$$
$$\tfrac{1}{2}(1-s\gamma_5\not{z})v_{s'}(\mathbf{p}) = \delta_{ss'}\,v_{s'}(\mathbf{p})$$

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = -\not{p}+m$$
$$\sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = -\not{p}-m$$

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \tfrac{1}{2}(1-s\gamma_5\not{z})(-\not{p}+m)$$
$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = \tfrac{1}{2}(1-s\gamma_5\not{z})(-\not{p}-m)$$

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p} + m)$$
$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p} - m)$$

Let's look at the situation with 3-momentum in the z-direction:

The component of the spin in the direction of the 3-momentum is called the helicity (a fermion with helicity +1/2 is called right-handed, a fermion with helicity -1/2 is called left-handed.

rapidity

$$\tfrac{1}{m}p^\mu = (\cosh\eta, 0, 0, \sinh\eta)$$
$$z^\mu = (\sinh\eta, 0, 0, \cosh\eta)$$

$$z^2 = 1$$
$$z \cdot p = 0$$

In the limit of large rapidity

$$z^\mu = \tfrac{1}{m}p^\mu + O(e^{-\eta})$$

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p} + m)$$
$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = \tfrac{1}{2}(1 - s\gamma_5 \not{z})(-\not{p} - m)$$

In the limit of large rapidity

$$z^\mu = \frac{1}{m}p^\mu + O(e^{-\eta})$$

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) \to \tfrac{1}{2}(1 + s\gamma_5)(-\not{p})$$
$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) \to \tfrac{1}{2}(1 - s\gamma_5)(-\not{p})$$

dropped m, small relative to p

In the extreme relativistic limit the right-handed fermion (helicity +1/2) (described by spinors u+ for b-type particle and v- for d-type particle) is projected onto the lower two components only (part of the Dirac field that corresponds to the right-handed Weyl field). Similarly left-handed fermions are projected onto upper two components (left-handed Weyl field.

Formulas relevant for massless particles can be obtained from considering the extreme relativistic limit of a massive particle; in particular the following formulas are valid when setting $m = 0$:

$$(\not{p} + m)u_s(\mathbf{p}) = 0$$
$$(-\not{p} + m)v_s(\mathbf{p}) = 0$$

$$\overline{u}_s(\mathbf{p})(\not{p} + m) = 0$$
$$\overline{v}_s(\mathbf{p})(-\not{p} + m) = 0$$

$$\overline{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) = +2m\,\delta_{s's} \;,$$
$$\overline{v}_{s'}(\mathbf{p})v_s(\mathbf{p}) = -2m\,\delta_{s's} \;,$$
$$\overline{u}_{s'}(\mathbf{p})v_s(\mathbf{p}) = 0 \;,$$
$$\overline{v}_{s'}(\mathbf{p})u_s(\mathbf{p}) = 0 \;.$$

$$\overline{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) = 2p^\mu \delta_{s's}$$
$$\overline{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) = 2p^\mu \delta_{s's}$$

$$\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0$$
$$\overline{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) = 0$$

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = -\not{p} + m$$
$$\sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = -\not{p} - m$$

$$u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) \to \tfrac{1}{2}(1 + s\gamma_5)(-\not{p})$$
$$v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) \to \tfrac{1}{2}(1 - s\gamma_5)(-\not{p})$$

becomes exact

Canonical quantization of spinor fields II

based on S-39

Lagrangian for a Dirac field:

$$\mathcal{L} = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi$$

canonical anticommutation relations:

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0\,,$$
$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})$$

The general solution to the Dirac equation:

$$(-i\partial\!\!\!/ + m)\Psi = 0$$

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right]$$

four-component spinors

creation and annihilation operators

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right]$$

We want to find formulas for creation and annihilation operator:

$$\int d^3x \, e^{-ipx} \Psi(x) = \sum_{s'=\pm} \left[\tfrac{1}{2\omega} b_{s'}(\mathbf{p}) u_{s'}(\mathbf{p}) + \tfrac{1}{2\omega} e^{2i\omega t} d_{s'}^\dagger(-\mathbf{p}) v_{s'}(-\mathbf{p}) \right]$$

multiply by $\overline{u}_s(\mathbf{p})\gamma^0$ on the left:

$$\overline{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) = 2p^\mu \delta_{s's}$$

$$\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0$$

$$b_s(\mathbf{p}) = \int d^3x \, e^{-ipx} \, \overline{u}_s(\mathbf{p})\gamma^0 \Psi(x)$$

for the hermitian conjugate we get:

$$\left[\overline{u}_s(\mathbf{p})\gamma^0\Psi(x)\right]^\dagger = \overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})$$

$$b_s^\dagger(\mathbf{p}) = \int d^3x \, e^{ipx} \, \overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})$$

b’s are time independent!

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right]$$

similarly for d:

$$\int d^3x \; e^{ipx} \Psi(x) = \sum_{s'=\pm} \left[\tfrac{1}{2\omega} e^{-2i\omega t} b_{s'}(-\mathbf{p}) u_{s'}(-\mathbf{p}) + \tfrac{1}{2\omega} d_{s'}^\dagger(\mathbf{p}) v_{s'}(\mathbf{p}) \right]$$

multiply by $\overline{v}_s(\mathbf{p})\gamma^0$ on the left:

$$\overline{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) = 0$$

$$\overline{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) = 2p^\mu \delta_{s's}$$

$$d_s^\dagger(\mathbf{p}) = \int d^3x \; e^{ipx} \, \overline{v}_s(\mathbf{p})\gamma^0 \Psi(x)$$

for the hermitian conjugate we get:

$$d_s(\mathbf{p}) = \int d^3x \; e^{-ipx} \, \overline{\Psi}(x)\gamma^0 v_s(\mathbf{p})$$

$$b_s(\mathbf{p}) = \int d^3x \, e^{-ipx} \, \overline{u}_s(\mathbf{p})\gamma^0\Psi(x) \qquad d_s(\mathbf{p}) = \int d^3x \, e^{-ipx} \, \overline{\Psi}(x)\gamma^0 v_s(\mathbf{p})$$

$$d_s^\dagger(\mathbf{p}) = \int d^3x \, e^{ipx} \, \overline{v}_s(\mathbf{p})\gamma^0\Psi(x) \qquad b_s^\dagger(\mathbf{p}) = \int d^3x \, e^{ipx} \, \overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})$$

we can easily work out the anticommutation relations for b and d operators:

$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0 \, ,$$
$$\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta} \, \delta^3(\mathbf{x}-\mathbf{y})$$

$$\{b_s(\mathbf{p}), b_{s'}(\mathbf{p}')\} = 0 \qquad \{b_s^\dagger(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} = 0$$
$$\{d_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} = 0 \qquad \{d_s^\dagger(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} = 0$$
$$\{b_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} = 0 \qquad \{b_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\} = 0$$

$$b_s(\mathbf{p}) = \int d^3x \; e^{-ipx} \, \overline{u}_s(\mathbf{p})\gamma^0\Psi(x)$$

$$b_s^\dagger(\mathbf{p}) = \int d^3x \; e^{ipx} \, \overline{\Psi}(x)\gamma^0 u_s(\mathbf{p})$$

we can easily work out the anticommutation relations for b and d operators:

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$$\begin{aligned}
\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} &= \int d^3x \, d^3y \, e^{-ipx+ip'y} \, \overline{u}_s(\mathbf{p})\gamma^0\{\Psi(x), \overline{\Psi}(y)\}\gamma^0 u_{s'}(\mathbf{p}') \\
&= \int d^3x \, e^{-i(p-p')x} \, \overline{u}_s(\mathbf{p})\gamma^0\gamma^0\gamma^0 u_{s'}(\mathbf{p}') \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}') \, \overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) \\
&= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}') \, 2\omega\delta_{ss'} \; .
\end{aligned}$$

$$(\gamma^0)^2 = 1$$

$$\overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) = 2\omega\delta_{ss'}$$

$$d_s(\mathbf{p}) = \int d^3x \; e^{-ipx}\, \overline{\Psi}(x)\gamma^0 v_s(\mathbf{p})$$

$$d_s^\dagger(\mathbf{p}) = \int d^3x \; e^{ipx}\, \overline{v}_s(\mathbf{p})\gamma^0 \Psi(x)$$

similarly:

$$\begin{aligned}
\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} &= 0\,, \\
\{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} &= (\gamma^0)_{\alpha\beta}\, \delta^3(\mathbf{x}-\mathbf{y})
\end{aligned}$$

$$\begin{aligned}
\{d_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= \int d^3x\, d^3y\, e^{ipx - ip'y}\, \overline{v}_s(\mathbf{p})\gamma^0 \{\Psi(x), \overline{\Psi}(y)\}\gamma^0 v_{s'}(\mathbf{p}') \\
&= \int d^3x\, e^{i(p-p')x}\, \overline{v}_s(\mathbf{p})\gamma^0\gamma^0\gamma^0 v_{s'}(\mathbf{p}') \\
&= (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{p}')\, \overline{v}_s(\mathbf{p})\gamma^0 v_{s'}(\mathbf{p}) \\
&= (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{p}')\, 2\omega \delta_{ss'}\ .
\end{aligned}$$

$$b_s(\mathbf{p}) = \int d^3x\, e^{-ipx}\, \overline{u}_s(\mathbf{p})\gamma^0\Psi(x) \qquad\qquad d_s(\mathbf{p}) = \int d^3x\, e^{-ipx}\, \overline{\Psi}(x)\gamma^0 v_s(\mathbf{p})$$

and finally:

$$\begin{aligned} \{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} &= 0\,, \\ \{\Psi_\alpha(\mathbf{x},t), \overline{\Psi}_\beta(\mathbf{y},t)\} &= (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y}) \end{aligned}$$

$$\begin{aligned} \{b_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= \int d^3x\, d^3y\, e^{-ipx-ip'y}\, \overline{u}_s(\mathbf{p})\gamma^0\{\Psi(x), \overline{\Psi}(y)\}\gamma^0 v_{s'}(\mathbf{p}') \\ &= \int d^3x\, e^{-i(p+p')x}\, \overline{u}_s(\mathbf{p})\gamma^0\gamma^0\gamma^0 v_{s'}(\mathbf{p}') \\ &= (2\pi)^3\delta^3(\mathbf{p}+\mathbf{p}')\, \overline{u}_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p}) \\ &= 0\,. \end{aligned}$$

$$\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0$$

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right]$$

We want to calculate the hamiltonian in terms of the b and d operators; in the four-component notation we would find:

$$H = \int d^3x \, \overline{\Psi}(-i\gamma^i \partial_i + m)\Psi$$

let's start with:

$$\begin{aligned}
(-i\gamma^i \partial_i + m)\Psi &= \sum_{s=\pm} \int \widetilde{dp} \left(-i\gamma^i \partial_i + m\right)\Big(b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} \\
&\qquad\qquad + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \Big) \\
&= \sum_{s=\pm} \int \widetilde{dp} \Big[b_s(\mathbf{p})(+\gamma^i p_i + m) u_s(\mathbf{p}) e^{ipx} \\
&\qquad\qquad + d_s^\dagger(\mathbf{p})(-\gamma^i p_i + m) v_s(\mathbf{p}) e^{-ipx} \Big] \\
&= \sum_{s=\pm} \int \widetilde{dp} \Big[b_s(\mathbf{p})(\gamma^0 \omega) u_s(\mathbf{p}) e^{ipx} \\
&\qquad\qquad + d_s^\dagger(\mathbf{p})(-\gamma^0 \omega) v_s(\mathbf{p}) e^{-ipx} \Big].
\end{aligned}$$

$$(\not p + m) u_s(\mathbf{p}) = 0$$
$$(-\not p + m) v_s(\mathbf{p}) = 0$$

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right]$$

$$H = \int d^3x \, \overline{\Psi}(-i\gamma^i \partial_i + m)\Psi$$

$$(-i\gamma^i \partial_i + m)\Psi = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p})(\gamma^0 \omega) u_s(\mathbf{p}) e^{ipx} \right. \\ \left. + d_s^\dagger(\mathbf{p})(-\gamma^0 \omega) v_s(\mathbf{p}) e^{-ipx} \right]$$

thus we have:

$$\begin{aligned} H = \sum_{s,s'} \int \widetilde{dp}\, \widetilde{dp}'\, d^3x & \left(b_{s'}^\dagger(\mathbf{p}') \overline{u}_{s'}(\mathbf{p}') e^{-ip'x} + d_{s'}(\mathbf{p}') \overline{v}_{s'}(\mathbf{p}') e^{ip'x} \right) \\ & \times \omega \left(b_s(\mathbf{p}) \gamma^0 u_s(\mathbf{p}) e^{ipx} - d_s^\dagger(\mathbf{p}) \gamma^0 v_s(\mathbf{p}) e^{-ipx} \right) \end{aligned}$$

$$\begin{aligned} = \sum_{s,s'} \int \widetilde{dp}\, \widetilde{dp}'\, d^3x\, \omega \Big[& b_{s'}^\dagger(\mathbf{p}') b_s(\mathbf{p})\; \overline{u}_{s'}(\mathbf{p}') \gamma^0 u_s(\mathbf{p})\; e^{-i(p'-p)x} \\ & - b_{s'}^\dagger(\mathbf{p}') d_s^\dagger(\mathbf{p})\; \overline{u}_{s'}(\mathbf{p}') \gamma^0 v_s(\mathbf{p})\; e^{-i(p'+p)x} \\ & + d_{s'}(\mathbf{p}') b_s(\mathbf{p})\; \overline{v}_{s'}(\mathbf{p}') \gamma^0 u_s(\mathbf{p})\; e^{+i(p'+p)x} \\ & - d_{s'}(\mathbf{p}') d_s^\dagger(\mathbf{p})\; \overline{v}_{s'}(\mathbf{p}') \gamma^0 v_s(\mathbf{p})\; e^{+i(p'-p)x} \Big] \end{aligned}$$

$$
\begin{aligned}
H &= \int d^3x\, \overline{\Psi}(-i\gamma^i\partial_i + m)\Psi \\
&= \sum_{s,s'} \int \widetilde{dp}\, \widetilde{dp}'\, d^3x\, \omega \Big[\, b^\dagger_{s'}(\mathbf{p}')b_s(\mathbf{p})\, \overline{u}_{s'}(\mathbf{p}')\gamma^0 u_s(\mathbf{p})\, e^{-i(p'-p)x} \\
&\qquad\qquad - b^\dagger_{s'}(\mathbf{p}')d^\dagger_s(\mathbf{p})\, \overline{u}_{s'}(\mathbf{p}')\gamma^0 v_s(\mathbf{p})\, e^{-i(p'+p)x} \\
&\qquad\qquad + d_{s'}(\mathbf{p}')b_s(\mathbf{p})\, \overline{v}_{s'}(\mathbf{p}')\gamma^0 u_s(\mathbf{p})\, e^{+i(p'+p)x} \\
&\qquad\qquad - d_{s'}(\mathbf{p}')d^\dagger_s(\mathbf{p})\, \overline{v}_{s'}(\mathbf{p}')\gamma^0 v_s(\mathbf{p})\, e^{+i(p'-p)x} \Big] \\
&= \sum_{s,s'} \int \widetilde{dp}\;\; \tfrac{1}{2}\Big[\, b^\dagger_{s'}(\mathbf{p})b_s(\mathbf{p})\, \overline{u}_{s'}(\mathbf{p})\gamma^0 u_s(\mathbf{p}) \\
&\qquad - b^\dagger_{s'}(-\mathbf{p})d^\dagger_s(\mathbf{p})\, \overline{u}_{s'}(-\mathbf{p})\gamma^0 v_s(\mathbf{p})\, e^{+2i\omega t} \\
&\qquad + d_{s'}(-\mathbf{p})b_s(\mathbf{p})\, \overline{v}_{s'}(-\mathbf{p})\gamma^0 u_s(\mathbf{p})\, e^{-2i\omega t} \\
&\qquad - d_{s'}(\mathbf{p})d^\dagger_s(\mathbf{p})\, \overline{v}_{s'}(\mathbf{p})\gamma^0 v_s(\mathbf{p}) \Big] \\
&= \sum_{s} \int \widetilde{dp}\; \omega \Big[b^\dagger_s(\mathbf{p})b_s(\mathbf{p}) - d_s(\mathbf{p})d^\dagger_s(\mathbf{p}) \Big].
\end{aligned}
$$

$$
\begin{aligned}
\overline{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) &= 2\omega\delta_{ss'} \\
\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) &= 0 \\
\overline{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) &= 0
\end{aligned}
$$

$$H = \sum_s \int \widetilde{dp}\ \omega \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - d_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) \right]$$

$$\{ d_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}') \} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')\, 2\omega \delta_{ss'}$$

finally, we find:

$$H = \sum_{s=\pm} \int \widetilde{dp}\ \omega \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) + d_s^\dagger(\mathbf{p}) d_s(\mathbf{p}) \right] - 4\mathcal{E}_0 V$$

$$V = (2\pi)^3 \delta^3(\mathbf{0}) = \int d^3x$$

$$\mathcal{E}_0 = \tfrac{1}{2}(2\pi)^{-3} \int d^3k\ \omega$$

four times the zero-point
energy of a scalar field
and opposite sign!

we will assume that the zero-point energy is cancelled by a constant term