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Reduction method for dimensionally regulated one-loop N-point Feynman integrals

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Abstract

We present a systematic method for reducing an arbitrary one−loop $N-$point massless Feynman integral with generic 4−dimensional momenta to a set comprised of eight fundamental scalar integrals: six box integrals in $D = 6$, a triangle integral in $D = 4$, and a general two−point integral in D space time dimensions. All the divergences present in the original integral are contained in the general two−point integral and associated coefficients. The problem of vanishing of the kinematic determinants has been solved in an elegant and transparent manner. Being derived with no restrictions regarding the external momenta, the method is completely general and applicable for arbitrary kinematics. In particular, it applies to the integrals in which the set of external momenta contains subsets comprised of two or more collinear momenta, which are unavoidable when calculating one−loop contributions to the hard−scattering amplitude for exclusive hadronic processes at large momentum transfer in PQCD. Further, a tensor decomposition scheme for $N-$point rank $P(\leq N)$ tensor integrals is formulated. Through the tensor decomposition and the scalar reduction presented, the computation of the massless one−loop integrals with an arbitrary number of external lines can be mastered. The iterative structure makes it easy to implement the formalism in an algebraic computer program. The conceptual problems related to the construction of multi−parton one−loop amplitudes in massless field theories are thus solved.

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1 Introduction

Scattering processes have played a crucial role in establishing the fundamental interactions of nature. They represent the most important source of information on short-distance physics. With increasing energy, multiparticle events are becoming more and more dominant. Thus, in testing various aspects of QCD, the high-energy scattering processes, both exclusive and inclusive, in which the total number of particles (partons) in the initial and final states is $N \geq 5$, have recently become increasingly important.

Owing to the well-known fact that the LO predictions in perturbative QCD do not have much predictive power, the inclusion of higher-order corrections is essential for many reasons. In general, higher-order corrections have a stabilizing effect, reducing the dependence of the LO predictions on the renormalization and factorization scales and the renormalization scheme. Therefore, to achieve a complete confrontation between theoretical predictions and experimental data, it is very important to know the size of radiative corrections to the LO predictions.

Obtaining radiative corrections requires the evaluation of one-loop integrals arising from the Feynman diagram approach. With the increasing complexity of the process under consideration, the calculation of radiative corrections becomes more and more tedious. Therefore, it is extremely useful to have an algorithmic procedure for these calculations, which is computerizable and leads to results which can be easily and safely evaluated numerically.

The case of Feynman integrals with massless internal lines is of special interest, because one often deals with either really massless particles (gluons) or particles whose masses can be neglected in high−energy processes (quarks). Owing to the fact that these integrals contain IR divergences (both soft and collinear), they need to be evaluated in an arbitrary number of space-time dimensions. As it is well known, in calculating Feynman diagrams mainly three difficulties arise: tensor decomposition of integrals, reduction of scalar integrals to several basic scalar integrals and the evaluation of a set of basic scalar integrals.

Considerable progress has recently been made in developing efficient approaches for calculating one−loop Feynman integrals with a large number ($N \geq 5$) of external lines [1, 2, 3, 4, 5, 6, 7, 8, 9]. Various approaches have been proposed for reducing the dimensionally regulated ($N \geq 5$)−point tensor integrals to a linear combination of $N-$ and lower−point scalar integrals multiplied by tensor structures made from the metric tensor $g^{\mu\nu}$ and external momenta [1, 2, 5, 6, 9]. It has also been shown that the general ($N > 5$)−point scalar one−loop integral can recursively be represented as a linear combination of ($N-1$)−point integrals provided the external momenta are kept in four dimensions [3, 4, 5, 6, 7]. Consequently, all scalar integrals occurring in the computation of an arbitrary one−loop ($N \geq 5$)−point integral with massless internal lines can be reduced to a sum over a set of basic scalar box ($N = 4$) integrals with rational coefficients depending on the external momenta and the dimensionality of space−time. The results for this set of integrals have been obtained in [10, 11] for arbitrary values of the relevant kinematic variables and are presented in a simple and compact form. This is not only useful on aesthetical grounds but it allows for fast and numerically stable computer programs.

As far as the calculation of one-loop N-point massless integrals is concerned, the most complete and systematic method is presented in [6]. It does not, however, apply to all cases of practical interest. Namely, being obtained for the non-exceptional external momenta it cannot be applied to the integrals in which the set of external momenta contains subsets comprised of two or three collinear on-shell momenta. The integrals of this type arise when performing the leading-twist NLO analysis of hadronic exclusive processes at large−momentum transfer in PQCD.

With no restrictions regarding the external kinematics, in this paper we formulate an efficient, systematic and completely general method for reducing an arbitrary one−loop $N-$point massless integral to a set of basic integrals.

The paper is organized as follows. Section 2 is devoted to introducing notation and to some preliminary considerations. In Sec. 3, a tensor reduction method for $N-$point tensor integrals is presented. It represents a massless version of the method originally obtained in Ref. [2] for

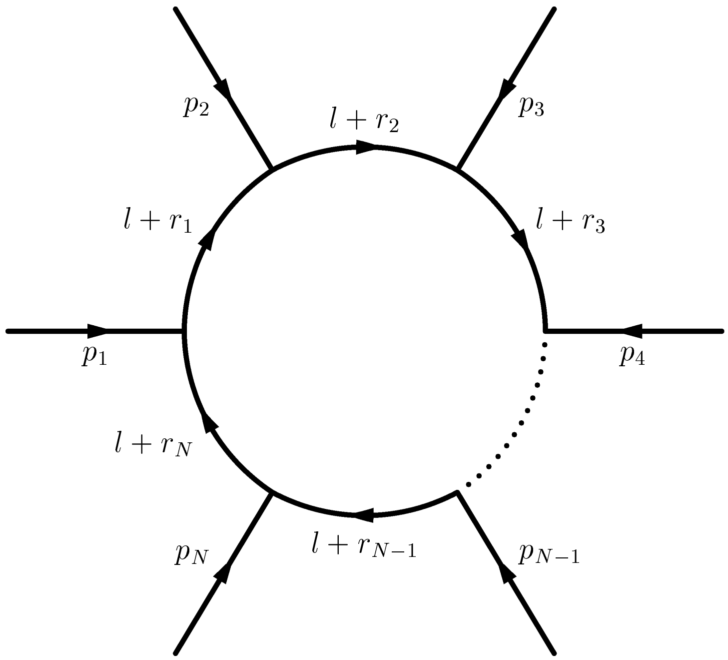

Figure 1: *One-loop N-point diagram*

massive integrals. In Sec. 4 we present a procedure to reduce one−loop $N-$point massless scalar integrals with generic 4−dimensional external momenta to a fundamental set comprised of eight integrals. Being derived with no restrictions to the external momenta, the method is completely general and applicable for arbitrary kinematics. Section 5 contains some considerations regarding the fundamental set of integrals. Section 6 is devoted to some concluding remarks. In the Appendix we give explicit expressions for the relevant basic massless box integrals in $D = 6$ space−time dimensions. These integrals constitute a subset of the fundamental set of scalar integrals.

2 Definitions and general properties

In order to obtain one−loop radiative coorections to physical processes in massless gauge theory, the integrals of the following type are required:

$$I^N_{\mu_1\cdots\mu_P}(D;\{p_i\}) \equiv (\mu^2)^{2-D/2}\int\frac{\mathrm{d}^D l}{(2\pi)^D}\frac{l_{\mu_1}\cdots l_{\mu_P}}{A_1A_2\cdots A_N} \quad . \tag{1}$$

This is a rank P tensor one−loop $N-$point Feynman integral with massless internal lines in $D-$dimensional space−time, where p_i, $(i = 1, 2, ..., N)$ are the external momenta, l is the loop momentum, and μ is the usual dimensional regularization scale.

The Feynman diagram with N external lines, which corresponds to the above integral, is shown in Fig. 1. For the momentum assignments as shown, i.e. with all external momenta taken to be incoming, the massless propagators have the form

$$A_i \equiv (l + r_i)^2 + \mathrm{i}\epsilon \qquad i = 1, \cdots, N \ , \tag{2}$$

where the momenta r_i are given by $r_i = p_i + r_{i-1}$ for i from 1 to N, and $r_0 = r_N$. The quantity $\mathrm{i}\epsilon(\epsilon > 0)$ represents an infinitesimal imaginary part, it ensures causality and after the integration determines the correct sign of the imaginary part of the logarithms and dilogarithms. It is customary to choose the loop momentum in a such a way that one of the momenta r_i vanishes. However, for general considerations, such a choice is not convenient, since by doing so, one loses the useful symmetry of the integral with respect to the indices $1, \cdots, N$.

The corresponding scalar integral is

$$I_0^N(D; \{p_i\}) \equiv (\mu^2)^{2-D/2} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{A_1 A_2 \cdots A_N} \ . \tag{3}$$

If $P + D - 2N \geq 0$, the integral (1) is UV divergent. In addition to UV divergence, the integral can contain IR divergence. There are two types of IR divergence: collinear and soft. A Feynman diagram with massless particles contains a soft singularity if it contains an internal gluon line attached to two external quark lines which are on mass−shell. On the other hand, a diagram contains a collinear singularity if it contains an internal gluon line attached to an external quark line which is on mass −shell. Therefore, a diagram containing a soft singularity at the same time contains two collinear singularities, i.e. soft and collinear singularities overlap.

When evaluating Feynman diagrams, one ought to regularize all divergences. Making use of the dimensional regularization method, one can simultaneously regularize UV and IR divergences, which makes the dimensional regularization method optimal for the case of massless field theories.

The tensor integral (1) is, as it is seen, invariant under the permutations of the propagators A_i, and is symmetric with respect to the Lorentz indices μ_i. Lorentz covariance allows the decomposition of the tensor integral (1) in the form of a linear decomposition consisting of the momenta p_i and the metric tensor $g_{\mu\nu}$. In the next section we propose a decomposition of the Feynman integral of the type (1) and show that the coefficients of the decomposition can be identified with the scalar Feynman integrals.

3 Decomposition of tensor integrals

Various approaches have been proposed for reducing the dimensionally regulated $(N \geq 5)$−point tensor integrals to a linear combination of $N-$ and lower−point scalar integrals multiplied by tensor structures made from the metric tensor $g^{\mu\nu}$ and external momenta. In this section we derive a tensor reduction formula for massless one−loop integrals of the form given by (1). We do this in parallel with the derivation given in [2, 5, 12], but corresponding to the case of massive integrals.

For the purpose of the following discussion, let us consider the tensor integral

$$I^N_{\mu_1 \cdots \mu_P}(D; \{\nu_i\}) \equiv (\mu^2)^{2-D/2} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l_{\mu_1} \cdots l_{\mu_P}}{A_1^{\nu_1} A_2^{\nu_2} \cdots A_N^{\nu_N}} \ , \tag{4}$$

and the corresponding scalar integral

$$I_0^N(D; \{\nu_i\}) \equiv (\mu^2)^{2-D/2} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{A_1^{\nu_1} A_2^{\nu_2} \cdots A_N^{\nu_N}} \ . \tag{5}$$

The above integrals represent generalizations of the integrals (1) and (3), in that they contain arbitrary powers $\nu_i \in \mathbf{N}$ of the propagators in the integrand, where $\{\nu_i\}$ is the shorthand notation for $(\nu_1, \cdots, \nu_N)$. Also, for notational simplicity, the external momenta are omitted from the argument of the integral. Now, making use of the integral representation of the Euler $\Gamma-$function, any propagator in (4) can be written in the form

$$\frac{1}{[(l + r_i)^2 + \mathrm{i}\epsilon]^{\nu_i}} = \frac{1}{\mathrm{i}^{\nu_i} \Gamma(\nu_i)} \int_0^\infty \mathrm{d}x \, x^{\nu_i - 1} e^{\mathrm{i}x\left[(l+r_i)^2 + \mathrm{i}\epsilon\right]}, \qquad \mathrm{Re}\, \nu_i > 0 \ . \tag{6}$$

By substituting (6) into (4), it follows that

$$I^N_{\mu_1\cdots\mu_P}(D;\{\nu_i\}) = \frac{(\mu^2)^{2-D/2}}{\mathrm{i}^{\Sigma_i\nu_i}\prod_i\Gamma(\nu_i)}\int\frac{\mathrm{d}^Dl}{(2\pi)^D}\,l_{\mu_1}\cdots l_{\mu_P}\int_0^\infty\Big(\prod\nolimits_i \mathrm{d}x_i x_i^{\nu_i-1}\Big)$$
$$\times\exp\Big[\,\mathrm{i}\,\Big(\,l^2\sum\nolimits_i x_i + 2l\sum\nolimits_i x_i r_i + \sum\nolimits_i x_i r_i^2 + \mathrm{i}\epsilon\sum\nolimits_i x_i\,\Big)\,\Big]\ , \tag{7}$$

while the index i in $\sum_i$ and $\prod_i$ takes on values from 1 to N. Representing the momenta l_{μ_j} in the numerator in terms of the derivative of the exponential function with respect to $(\sum_i x_i r_i)^{\mu_j}$, translating the integration variable l

$$l \longrightarrow l - \frac{\sum_i x_i r_i}{\sum_i x_i}$$

and performing the $D-$dimensional Gaussian integration with the help of the formula

$$\int\frac{\mathrm{d}^Dl}{(2\pi)^D}\,\exp\big(\,\mathrm{i}\,A\,l^2\,\big) = \frac{\mathrm{i}}{(4\pi\mathrm{i}A)^{D/2}}\ ,$$

it can be shown that the expression (7) reduces to

$$I^N_{\mu_1\cdots\mu_P}(D;\{\nu_i\}) = \frac{\mathrm{i}}{(4\pi)^2}(4\pi\mu^2)^{2-D/2}\frac{\mathrm{i}^{-\Sigma_i\nu_i-D/2}}{(2\mathrm{i})^P\prod_i\Gamma(\nu_i)}\int_0^\infty\Big(\prod\nolimits_i \mathrm{d}x_i x_i^{\nu_i-1}\Big)\,(\sum\nolimits_i x_i)^{-D/2}$$
$$\times\Big[\prod\nolimits_{j=1}^P\frac{\partial}{\partial\,(\sum_i x_i r_i)^{\mu_j}}\Big]\exp\Big[\,\mathrm{i}\,\Big(-\frac{(\sum_i x_i r_i)^2}{\sum_i x_i} + \sum\nolimits_i x_i r_i^2 + \mathrm{i}\epsilon\sum\nolimits_i x_i\Big)\Big]. \tag{8}$$

Now, introducing into (8) the following representation of "unity"

$$\int_0^\infty \mathrm{d}\lambda\,\delta\,(x_1+x_2+\ldots+x_N-\lambda) = 1\ ,$$

and passing to a new set of integration variables

$$x_i = \lambda y_i\ ,\quad (i=1,2,...,N),$$

we find that

$$I^N_{\mu_1\cdots\mu_P}(D;\{\nu_i\}) = \frac{\mathrm{i}}{(4\pi)^2}(4\pi\mu^2)^{2-D/2}\frac{\mathrm{i}^{-\Sigma_i\nu_i-D/2}}{(2\mathrm{i})^P\prod_i\Gamma(\nu_i)}\int_0^1\Big(\prod\nolimits_i \mathrm{d}y_i y_i^{\nu_i-1}\Big)\,\delta\,\Big(\sum\nolimits_{i=1}^N y_i - 1\Big)$$
$$\times\int_0^\infty \mathrm{d}\lambda\lambda^{(\Sigma_i\nu_i-D/2)-1}\Big[\prod\nolimits_{j=1}^P\frac{\partial}{\partial\,(\lambda\sum_i y_i r_i)^{\mu_j}}\Big]\exp\Big[\,\mathrm{i}\,\lambda\,\Big(-\Big(\sum\nolimits_i y_i r_i\Big)^2 + \sum\nolimits_i y_i r_i^2 + \mathrm{i}\epsilon\Big)\Big] \tag{9}$$

Next, performing the derivatives indicated in the above expression using the identity

$$\Big[\prod\nolimits_{j=1}^P\frac{\partial}{\partial\,(\lambda\sum_i y_i r_i)^{\mu_j}}\Big]\exp\Big[\mathrm{i}\lambda\Big(-\Big(\sum\nolimits_i y_i r_i\Big)^2 + \sum\nolimits_i y_i r_i^2 + \mathrm{i}\epsilon\Big)\Big]$$
$$= \sum_{\substack{k,j_1,\cdots,j_N\ge 0\\ 2k+\Sigma j_i=P}}\big\{[g]^k[r_1]^{j_1}\cdots[r_N]^{j_N}\big\}_{\mu_1\cdots\mu_P}\Big(\prod\nolimits_i y_i^{j_i}\Big)(-2\mathrm{i})^{P-k}\lambda^{-k}$$
$$\times\exp\Big[\mathrm{i}\lambda\Big(-\Big(\sum\nolimits_i y_i r_i\Big)^2 + \sum\nolimits_i y_i r_i^2 + \mathrm{i}\epsilon\Big)\Big], \tag{10}$$

we arrive at

$$I^N_{\mu_1\cdots\mu_P}(D;\{\nu_i\}) = \frac{\mathrm{i}}{(4\pi)^2}(4\pi\mu^2)^{2-D/2}\sum_{\substack{k,j_1,\cdots,j_N\ge 0\\ 2k+\Sigma j_i=P}}\big\{[g]^k[r_1]^{j_1}\cdots[r_N]^{j_N}\big\}_{\mu_1\cdots\mu_P}$$

$$\times\frac{(-1)^{P-k}\mathrm{i}^{-\Sigma_i\nu_i-D/2}}{(2\mathrm{i})^k\prod_i\Gamma(\nu_i)}\int_0^1\left(\prod\nolimits_i \mathrm{d}y_i y_i^{\nu_i+j_i-1}\right)\delta\left(\sum\nolimits_{i=1}^N y_i-1\right)$$
$$\times\int_0^\infty \mathrm{d}\lambda\lambda^{(\Sigma_i\nu_i-D/2-k)-1}\exp\left[\mathrm{i}\,\lambda\left(-\left(\sum\nolimits_i y_i r_i\right)^2+\sum\nolimits_i y_i r_i^2+\mathrm{i}\epsilon\right)\right], \qquad (11)$$

where $\{[g]^k[r_1]^{j_1}\cdots[r_N]^{j_N}\}_{\mu_1\cdots\mu_P}$ represents a symmetric (with respect to $\mu_1\cdots\mu_P$) combination of tensors, each term of which is composed of k metric tensors and j_i external momenta r_i. Thus, for example,

$$\{gr_1\}_{\mu_1\mu_2\mu_3}=g_{\mu_1\mu_2}r_{1\mu_3}+g_{\mu_1\mu_3}r_{1\mu_2}+g_{\mu_2\mu_3}r_{1\mu_1}.$$

Next, employing the representation (6), we can perform the λ integration in the region $D+2k-2\Sigma_i\nu_i<0$. The result is

$$I^N_{\mu_1\cdots\mu_P}(D;\{\nu_i\})=\frac{\mathrm{i}}{(4\pi)^2}(4\pi\mu^2)^{2-D/2}\sum_{\substack{k,j_1,\cdots,j_N\geq 0\\ 2k+\Sigma j_i=P}}\left\{[g]^k[r_1]^{j_1}\cdots[r_N]^{j_N}\right\}_{\mu_1\cdots\mu_P}$$
$$\times\frac{\Gamma\left(\sum_i\nu_i-D/2-k\right)}{2^k\left[\prod_i\Gamma(\nu_i)\right]}(-1)^{\Sigma_i\nu_i+P-k}\int_0^1\left(\prod\nolimits_i \mathrm{d}y_i y_i^{\nu_i+j_i-1}\right)$$
$$\times\delta\left(\sum\nolimits_{i=1}^N y_i-1\right)\left[\left(\sum\nolimits_i y_i r_i\right)^2-\sum\nolimits_i y_i r_i^2-\mathrm{i}\epsilon\right]^{k+D/2-\Sigma_i\nu_i}. \qquad (12)$$

It is worth pointing out that, in this section, we have not considered iϵ infinitesimal[3], so that the results obtained are valid for arbitrary values of $\epsilon>0$ allowing us to perform analytical continuation in the parameter D.

This implies that Eq. (12) is the representation of the tensor integral $I^N_{\mu_1\cdots\mu_P}(D;\{\nu_i\})$, given in (4), which is valid for arbitrary values of N, P, r_i and $\nu_i(>0)$, for the values of D for which the remaining integral is finite, and the $\Gamma-$function does not diverge.

As for the integral representation of the corresponding scalar integral (5), it can be obtained in the same way. The result is of the form

$$I^N_0(D;\{\nu_i\})=\frac{\mathrm{i}}{(4\pi)^2}(4\pi\mu^2)^{2-D/2}\frac{\Gamma\left(\sum_{i=1}^N\nu_i-D/2\right)}{\prod_{i=1}^N\Gamma(\nu_i)}(-1)^{\Sigma_{i=1}^N\nu_i}$$
$$\times\int_0^1\left(\prod_{i=1}^N \mathrm{d}y_i y_i^{\nu_i-1}\right)\delta\left(\sum_{i=1}^N y_i-1\right)\left[-\sum_{\substack{i,j=1\\ i<j}}^N y_i y_j\left(r_i-r_j\right)^2-\mathrm{i}\epsilon\right]^{D/2-\Sigma_{i=1}^N\nu_i}. \qquad (13)$$

In arriving at the above result, we have made use of the identity

$$\left(\sum\nolimits_i y_i r_i\right)^2-\sum\nolimits_i y_i r_i^2=-\sum_{\substack{i,j=1\\ i<j}}^N y_i y_j\left(r_i-r_j\right)^2, \qquad (14)$$

which is valid owing to the presence of the $\delta-$function in (13). Now, on the basis of (13) and (14), (12) can be written in the form

$$I^N_{\mu_1\cdots\mu_P}(D;\{\nu_i\})=\sum_{\substack{k,j_1,\cdots,j_N\geq 0\\ 2k+\Sigma j_i=P}}\left\{[g]^k[r_1]^{j_1}\cdots[r_N]^{j_N}\right\}_{\mu_1\cdots\mu_P}$$
$$\times\frac{(4\pi\mu^2)^{P-k}}{(-2)^k}\left[\prod_{i=1}^N\frac{\Gamma(\nu_i+j_i)}{\Gamma(\nu_i)}\right]I^N_0(D+2(P-k);\{\nu_i+j_i\}). \qquad (15)$$

[3]We make use of this fact in the next section where we derive the recursion relations which connect scalar integrals in a different number of space−time dimensions.

This is the desired decomposition of the dimensionally regulated $N-$point rank P tensor integral with massless internal lines given in (4). The general results (13) and (15) represent massless versions of the results that have originally been derived in [2] for the case of massive Feynman integrals. Based on (15), any dimensionally regulated $N-$point tensor integral can be expressed as a linear combination of $N-$point scalar integrals multiplied by tensor structures made from the metric tensor $g^{\mu\nu}$ and external momenta. Therefore, with the decomposition (15), the problem of calculating the tensor integrals has been reduced to the calculation of the general scalar integrals.

As is well known, the direct evaluation of the general scalar integral (5) (i.e. (13)) represents a non-trivial problem. However, with the help of the recursion relations, the problem can be significantly simplified in the sense that the calculation of the original scalar integral can be reduced to the calculation of a certain number of simpler fundamental (basic) intgerals.

4 Recursion relations for scalar integrals

Owing to the translational invariance, the dimensionally regulated integrals satisfy the following identity [5, 12, 13]:

$$0 \equiv \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\partial}{\partial l^\mu} \left(\frac{z_0 l^\mu + \sum_{i=1}^N z_i r_i^\mu}{A_1^{\nu_1} \cdots A_N^{\nu_N}} \right) , \tag{16}$$

where z_i ($i = 0 \cdots N$) are arbitary constants, while A_i are the propagators given by (2). After the differentiation, the identity (16) takes the form

$$0 \equiv \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{\prod_{k=1}^N A_k^{\nu_k}} \left[z_0 D - 2 \sum_{j=1}^N \frac{\nu_j}{A_j} \left(z_0 \; l^2 + z_0 \; l \cdot r_j + \sum_{i=1}^N z_i \; l \cdot r_i + \sum_{i=1}^N z_i \; r_i \cdot r_j \right) \right]. \tag{17}$$

Inserting the identity

$$l \cdot r_k \equiv \frac{1}{2} \left(A_k - l^2 - r_k^2 - \mathrm{i}\epsilon \right)$$

into the expression (17), one obtains

$$\begin{aligned} 0 \quad &\equiv \quad \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{\prod_{k=1}^N A_k^{\nu_k}} \Bigg[z_0 D - \sum_{j=1}^N \frac{\nu_j}{A_j} \Bigg((z_0 - \sum_{i=1}^N z_i)(l^2 - r_j^2) - \sum_{i=1}^N z_i (r_i - r_j)^2 \\ &\qquad + \sum_{i=1}^N z_i \; A_i + z_0 \; A_j - \mathrm{i}\epsilon (z_0 + \sum_{i=1}^N z_i) \Bigg) \Bigg]. \end{aligned} \tag{18}$$

If the constant z_0 is chosen to be $z_0 = \sum_{i=1}^N z_i$, (which we assume in the following), the relation (18) reduces to

$$0 \equiv \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{\prod_{k=1}^N A_k^{\nu_k}} \sum_{j=1}^N \left\{ D \; z_j + \frac{\nu_j}{A_j} \sum_{i=1}^N z_i \left[(r_i - r_j)^2 + 2\mathrm{i}\epsilon - A_i - A_j \right] \right\}. \tag{19}$$

Next, taking into account the scalar integral (5), the expression (19) can be written in the form

$$\begin{aligned} &\sum_{j=1}^N \left(\sum_{i=1}^N \left[(r_j - r_i)^2 + 2\mathrm{i}\epsilon \right] z_i \right) \nu_j I_0^N(D; \{\nu_k + \delta_{kj}\}) \\ &\quad = \sum_{i,j=1}^N z_i \nu_j I_0^N(D; \{\nu_k + \delta_{kj} - \delta_{ki}\}) - (D - \sum_{j=1}^N \nu_j) z_0 I_0^N(D; \{\nu_k\}), \end{aligned} \tag{20}$$

where δ_{ij} is the Kronecker delta symbol. In arriving at (20), it has been understood that

$$I_0^N(D;\nu_1,\cdots,\nu_{l-1},0,\nu_{l+1},\cdots,\nu_N)\equiv I_0^{N-1}(D;\nu_1,\cdots,\nu_{l-1},\nu_{l+1},\cdots,\nu_N). \tag{21}$$

The relation (20) represents the starting point for the derivation of the recursion relations for scalar integrals.

Before proceeding to derive the recursion relation, let us first derive a useful identity. To this end, we first take a partial derivative of (5) with respect to iϵ and find

$$\frac{\partial}{\partial(\mathrm{i}\epsilon)}I_0^N(D;\{\nu_k\})=-\sum\nolimits_{j=1}^N\nu_jI_0^N(D;\{\nu_k+\delta_{kj}\}). \tag{22}$$

In the next step, by partial differentiation of (13), which is an integral representation of (5), with respect to iϵ we obtain

$$\begin{aligned}\frac{\partial}{\partial(\mathrm{i}\epsilon)}I_0^N(D;\{\nu_k\})&=\frac{\mathrm{i}}{(4\pi)^2}(4\pi\mu^2)^{2-D/2}\frac{\Gamma\left(\sum_i\nu_i-D/2+1\right)}{\prod_i\Gamma(\nu_i)}(-1)^{\Sigma_i\nu_i}\\&\times\int_0^1\left(\prod\nolimits_i\mathrm{d}y_iy_i^{\nu_i-1}\right)\delta\left(\sum\nolimits_iy_i-1\right)\left[-\sum\nolimits_{i<j}y_iy_j\left(r_i-r_j\right)^2-\mathrm{i}\epsilon\right]^{D/2-\Sigma_i\nu_i-1}\\&\equiv(4\pi\mu^2)^{-1}I_0^N(D-2;\{\nu_k\}).\end{aligned} \tag{23}$$

Now, by equating the right−hand sides of Eqs. (22) and (23), we arrive at the following relation (identity):

$$-\sum\nolimits_{j=1}^N\nu_jI_0^N(D;\{\nu_k+\delta_{kj}\})=(4\pi\mu^2)^{-1}I_0^N(D-2;\{\nu_k\}). \tag{24}$$

From this point on we consider the parameter iϵ to be infinitesimal, which is the case in all practical calculations.

We have obtained the fundamental set of recursion relations by choosing the arbitrary constants z_i so as to satisfy the following system of linear equations:

$$\sum\nolimits_{i=1}^N(r_i-r_j)^2z_i=C,\quad j=1,\ldots,N, \tag{25}$$

where C is an arbitrary constant. Introducing the notation

$$r_{ij}=(r_i-r_j)^2, \tag{26}$$

the system (25) may be written in matrix notation as

$$\begin{pmatrix}0&r_{12}&\cdots&r_{1N}\\r_{12}&0&\cdots&r_{2N}\\\vdots&\vdots&\ddots&\vdots\\r_{1N}&r_{2N}&\cdots&0\end{pmatrix}\begin{pmatrix}z_1\\z_2\\\vdots\\z_N\end{pmatrix}=\begin{pmatrix}C\\C\\\vdots\\C\end{pmatrix}, \tag{27}$$

or more compactly as

$$R\cdot z=C\ , \tag{28}$$

where $R=(r_{ij})$ is the coeffcient matrix of the system. If (25) is taken into account, the relation (20) reduces to

$$C\sum_{j=1}^N\nu_jI_0^N(D;\{\nu_k+\delta_{kj}\})=\sum_{i=1}^Nz_i\sum_{j=1}^N\nu_jI_0^N(D;\{\nu_k-\delta_{ki}+\delta_{kj}\})-(D-\sum_{j=1}^N\nu_j)z_0I_0^N(D;\{\nu_k\}), \tag{29}$$

where the infinitesimal part proportional to iϵ has been omitted. By adding and subtracting δ_{ji} from the factor ν_j in the first term on the right-hand side in (29), one can write

$$
\begin{aligned}
C \sum_{j=1}^{N} \nu_j I_0^N(D;\{\nu_k+\delta_{kj}\}) &= \sum_{i=1}^{N} z_i \sum_{j=1}^{N} (\nu_j-\delta_{ji}) I_0^N(D;\{\nu_k-\delta_{ki}+\delta_{kj}\}) \\
&\quad -(D-1-\sum_{j=1}^{N}\nu_j) z_0 I_0^N(D;\{\nu_k\}).
\end{aligned} \tag{30}
$$

Taking into account the expression (24) on the left-hand side, and in the first term on the right-hand side in (30), we arrive at the recursion relation

$$
\begin{aligned}
C\, I_0^N(D-2;\{\nu_k\}) &= \sum_{i=1}^{N} z_i I_0^N(D-2;\{\nu_k-\delta_{ki}\}) \\
&\quad +(4\pi\mu^2)(D-1-\sum_{j=1}^{N}\nu_j) z_0 I_0^N(D;\{\nu_k\}),
\end{aligned} \tag{31}
$$

where z_i are given by the solution of the system (25). This is a generalized form of the recursion relation which connects the scalar integrals in a different number of dimensions [3, 4, 5, 6]. The use of the relation (31) in practical calculations depends on the form of the solution of the system of equations (25). For general considerations, it is advantageous to write the system (25) in the following way:

$$
\begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & r_{12} & \cdots & r_{1N} \\
1 & r_{12} & 0 & \cdots & r_{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & r_{1N} & r_{2N} & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
-C \\ z_1 \\ z_2 \\ \vdots \\ z_N
\end{pmatrix}
=
\begin{pmatrix}
z_0 \\ 0 \\ 0 \\ \vdots \\ 0
\end{pmatrix}. \tag{32}
$$

In writing (32), we have taken into account the fact that $z_0 = \sum_{i=1}^{N} z_i$. In this way, the only free parameter is z_0 and by choosing it in a convenient way, one can always find the solution of the above system and, consequently, be able to use the recursion relations (31).

In the following we frequently refer to two determinants, for which we introduce the notations: for the determinant of the system (27) we introduce $\det(R_N)$, while for the determinant of the system (32) we use $\det(S_N)$. Depending on whether the kinematic determinants $\det(R_N)$ and $\det(S_N)$ are equal to zero or not, we distinguish four different types of recursion relations following from (31). Before proceeding to consider various cases, note that in the case when $\det(S) \neq 0$, it holds

$$
C = -z_0 \frac{\det(R_N)}{\det(S_N)}. \tag{33}
$$

Let us now discuss all possible cases separately.

4.1 Case I: $\det(S_N) \neq 0$, $\det(R_N) \neq 0$

The most convenient choice in this case is $z_0 = 1$. It follows from (33) that $C \neq 0$, so that the recursion relation (31) can be written in the following form:

$$
\begin{aligned}
I_0^N(D;\{\nu_k\}) &= \frac{1}{4\pi\mu^2(D-1-\sum_{j=1}^{N}\nu_j)} \Big[C\, I_0^N(D-2;\{\nu_k\}) \\
&\quad - \sum_{i=1}^{N} z_i I_0^N(D-2;\{\nu_k-\delta_{ki}\}) \Big].
\end{aligned} \tag{34}
$$

This recursion relation, as it is seen, connects the scalar integral in D dimensions with the scalar integrals in $D-2$ dimensions and can be used to reduce the dimensionality of the scalar integral.

Since $\det(R_N) \neq 0$, some more recursion relations can be directly derived from (20). By directly choosing the constants z_i in (20) in a such a way that $z_i = \delta_{ik}$, for $k = 1, \cdots, N$, we arrive at a system of N equations which is always valid:

$$\sum_{j=1}^{N} (r_k - r_j)^2 \nu_j I_0^N(D; \{\nu_i + \delta_{ij}\}) = \sum_{j=1}^{N} \nu_j I_0^N(D; \{\nu_i + \delta_{ij} - \delta_{ik}\}) - (D - \sum_{j=1}^{N} \nu_j) I_0^N(D; \{\nu_i\}), \qquad k = 1, \cdots, N. \tag{35}$$

In the system (35) we have again disregarded the non-essential infinitesimal term proportional to iϵ. The matrix of the system (35) is the same as the matrix of the system (25), whose determinant is different from zero, so that the system (35) can be solved with respect to $I_0^N(D; \{\nu_i + \delta_{ij}\})$, $j = 1, \cdots, N$. The solutions represent the recursion relations which can be used to reduce the powers of the propagators in the scalar integrals. Making use of these relations and the relation (34), each scalar integral $I_0^N(D; \{\nu_i\})$ belonging to the type for which $\det(S_N) \neq 0$, $\det(R_N) \neq 0$ can be represented as a linear combination of integrals $I_0^N(D'; \{1\})$ and integrals with the number of propagators which is less than N. For the dimension D', one usually chooses $4 + 2\varepsilon$, where ε is the infinitesimal parameter regulating the divergences. Even in the case when one starts with $D < D'$, one can make use of the recursion (34) to change from the dimension D to the dimension D'.

In addition to the two sets of recursion relations presented above, by combining them one can obtain an additional and very useful set of recursion relations. This set at the same time reduces D and ν_i in all terms. By adding and subtracting the expression $\delta_{jk} I_0^N(D; \{\nu_i + \delta_{ij} - \delta_{ik})$ in the first term on the right-hand side of the system (35), one finds

$$\sum_{j=1}^{N} (r_k - r_j)^2 \nu_j I_0^N(D; \{\nu_i + \delta_{ij}\}) = \sum_{j=1}^{N} (\nu_j - \delta_{jk}) I_0^N(D; \{\nu_i + \delta_{ij} - \delta_{ik}\}) - (D - 1 - \sum_{j=1}^{N} \nu_j) I_0^N(D; \{\nu_i\}), \qquad k = 1, \cdots, N. \tag{36}$$

Now, on the right-hand side we make use of the relation (24)

$$\sum_{j=1}^{N} (r_k - r_j)^2 \nu_j I_0^N(D; \{\nu_i + \delta_{ij}\}) = -(4\pi\mu^2)^{-1} I_0^N(D-2; \{\nu_i - \delta_{ik}\}) - (D - 1 - \sum_{j=1}^{N} \nu_j) I_0^N(D; \{\nu_i\}), \qquad k = 1, \cdots, N. \tag{37}$$

The solution of this system of equations can in principle be used for reducing the dimension of the integral and the propagator powers. However, a much more useful set of the recursion relations is obtained by combining (37) and (34). Expressing the second term on the right−hand side of (37) with the help of (34), leads to

$$\sum_{j=1}^{N} (r_k - r_j)^2 \nu_j I_0^N(D; \{\nu_i + \delta_{ij}\}) = (4\pi\mu^2)^{-1} \left[\sum_{j=1}^{N} (z_j - \delta_{jk}) I_0^N(D-2; \{\nu_i - \delta_{ij}\}) - C I_0^N(D-2; \{\nu_i\}) \right], \qquad k = 1, \cdots, N, \tag{38}$$

where z_i and C represent solutions of the system (32) for $z_0 = 1$. Solutions of the system (38) represent the recursion relations which, at the same time, reduce (make smaller) the dimension and the powers of the propagators in all terms (which is very important). As such, they are especialy convenient for making a rapid reduction of the scalar integrals which appear in the tensor decomposition of high-rank tensor integrals.

4.2 Case II: $\det(S_N) \neq 0$, $\det(R_N) = 0$

The most convenient choice in this case is $z_0 = 1$. Unlike in the preceding case, it follows from (33) that $C = 0$, so that the recursion relation (31) can be written as

$$I_0^N(D; \{\nu_k\}) = \frac{1}{4\pi\mu^2(D - 1 - \sum_{j=1}^N \nu_j)} \left[-\sum\nolimits_{i=1}^N z_i I_0^N(D-2; \{\nu_k - \delta_{ki}\}) \right]. \tag{39}$$

It follows from (39) that it is possible to represent each integral of this type as a linear combination of scalar integrals with the number of propagators being less than N.

4.3 Case III: $\det(S_N) = 0$, $\det(R_N) \neq 0$

This possibility arises only if the first row of the matrix of the system (32) is a linear combination of the remaining rows. Then, the system (32) has a solution only for the choice $z_0 = 0$. With this choice, the remaining system of equations reduces to the system (25), where the constant C can be chosen at will. After the parameter C is chosen, the constants z_i are uniquely determined. Thus the recursion relation (31) with the choice $C = 1$ leads to

$$I_0^N(D; \{\nu_k\}) = \sum\nolimits_{i=1}^N z_i I_0^N(D; \{\nu_k - \delta_{ki}\}). \tag{40}$$

Consequently, as in the preceding case, the scalar integrals of the type considered can be represented as a linear combination of scalar integrals with a smaller number of propagators.

4.4 Case IV: $\det(S_N) = 0$, $\det(R_N) = 0$

Unlike in the preceding cases, in this case two different recursion relations arise. To derive them, we proceed by subtracting the last, $(N+1)-$th, equation of the system (32) from the second, third,... and $N-$th equation, respectively. As a result, we arrive at the following system of equations:

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & -r_{1N} & r_{12} - r_{2N} & \cdots & r_{1N} \\ 0 & r_{12} - r_{1N} & -r_{2N} & \cdots & r_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & r_{1,N-1} - r_{1N} & r_{2,N-1} - r_{2N} & \cdots & r_{N-1,N} \\ 1 & r_{1N} & r_{2N} & \cdots & 0 \end{pmatrix} \begin{pmatrix} -C \\ z_1 \\ z_2 \\ \vdots \\ z_{N-1} \\ z_N \end{pmatrix} = \begin{pmatrix} z_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \tag{41}$$

As it is seen, the first N equations of the above system form a system of equations in which the constant C does not appear, and which can be used to determine the constants z_i, $i = 1, \cdots N$. The fact that $\det(S_N) = 0$ implies that the determinant of this system vanishes. Therefore, for the system in question to be consistent (for the solution to exist), the choice $z_0 = 0$ has to be made. Consequently, the solution of the system, z_i $(i = 1, 2, ...N)$, will contain at least one free parameter. Inserting this solution into the last, $(N+1)-$th, equation of the system (41), we obtain

$$\sum\nolimits_{i=1}^N r_{iN} z_i = C \; . \tag{42}$$

Now, by arbitrarily choosing the parameter C, one of the free parameters on the left-hand side can be fixed.

Sometimes, (for instance, when there are collinear external lines) the left-hand side of Eq. (42) vanishes explicitly, although the solution for z_i contains free parameters. In this case the choice $C = 0$ has to be made.

Therefore, corresponding to the case when $\det(S_N)$=$\det(R_N) = 0$, one of the following two recursion relations holds:

$$I_0^N(D; \{\nu_k\}) = \sum_{i=1}^{N} z_i I_0^N(D; \{\nu_k - \delta_{ki}\}), \tag{43}$$

obtained from (31) by setting $z_0 = 0$ and $C = 1$, or

$$0 = \sum_{i=1}^{N} z_i I_0^N(D; \{\nu_k - \delta_{ki}\}), \tag{44}$$

obtained from (31) by setting $z_0 = 0$ and $C = 0$.

In the case (43), it is clear that the integral with N external lines can be represented in terms of the integrals with $N-1$ external lines. What happens, however, in the case (44)? With no loss of generality, we can take that $z_1 \neq 0$. The relation (44) can then be written in the form

$$z_1 I_0^N(D; \{\nu_k\}) = -\sum_{i=2}^{N} z_i I_0^N(D; \{\nu_k + \delta_{k1} - \delta_{ki}\}). \tag{45}$$

We can see that, in this case too, the integral with N external lines can be represented in terms of the integrals with $N-1$ external lines. In this reduction, $\sum_{i=1}^{N} \nu_i$ remains conserved.

Based on the above considerations, it is clear that in all the above cases with the exception of that when $\det(S_N) \neq 0$, $\det(R_N) \neq 0$, the integrals with N external lines can be represented in terms of the integrals with smaller number of external lines. Consequently, then, there exists a fundemantal set of integrals in terms of which all integrals can be represented as a linear combination.

We now turn to determine the fundamental set of integrals. To this end, let us first evaluate the determinant of the system (32), $\det(S_N)$, and determine the conditions under which this kinematic determinant vanishes.

By subtracting the last column from the second, third, ... and $N-$th column, respectively, and then the last row from the second, third, ... and $N-$th row, respectively, we find that the determinant $\det(S_N)$ is given by the following expression:

$$\det(S_N) = -\det\left[-2(r_i - r_N)(r_j - r_N)\right], \qquad i, j = 1, \ldots, N-1. \tag{46}$$

Denote by n the dimension of the vector space spanned by the vectors $r_i - r_N$, $(i = 1, \ldots, N-1)$. Owing to the linear dependence of these vectors, the determinant vanishes when $N > n + 1$. As in practice, we deal with the $4-$dimensional external kinematics, the maximum value for n equals 4. An immediate consequence of this is that all integrals with $N > 5$ can be reduced to the integrals with $N \leq 5$.

In view of what has been said above, all one-loop integrals are expressible in terms of the integrals $I_0^3(4+2\varepsilon; \{1\})$, $I_0^4(4+2\varepsilon; \{1\})$, $I_0^5(4+2\varepsilon; \{1\})$, belonging to Case I, and the general two$-$point integrals $I_0^2(D'; \nu_1, \nu_2)$, which are simple enough to be evaluated analytically.

Next, by substituting $D = 6 + 2\epsilon_{IR}$, $N = 5$ and $\nu_i = 1$ into the recursion relation (31), one finds

$$C\, I_0^5(4+2\varepsilon; \{1\}) = \sum_{i=1}^{5} z_i I_0^5(4+2\varepsilon; \{\delta_{kk} - \delta_{ki}\}) + (4\pi\mu^2)(2\varepsilon)\ z_0\ I_0^5(6+2\varepsilon; \{1\}). \tag{47}$$

Owing to the fact that the integral $I_0^5(6+2\varepsilon; \{1\})$ is IR finite[4], the relation (47) implies that the $N = 5$ scalar integral, $I_0^5(4+2\varepsilon; \{1\})$, can be expressed as a linear combination of the $N = 4$ scalar integrals, $I_0^4(4+2\varepsilon; \{1\})$, plus a term linear in ε. In massless scalar theories, the term linear in ε

can simply be omitted, with a consequence that the $N = 5$ integrals can be reduced to the $N = 4$ integrals. On the other hand, when calculating in renormalizable gauge theories (like QCD), the situation is not so simple, owing to the fact that the rank $P(\leq N)$ tensor integrals are required.

In the process of the tensor decomposition and then reduction of scalar integrals all way down to the fundamental set of integrals, there appears a term of the form $(1/\varepsilon)I_0^5(4 + 2\varepsilon; \{1\})$, which implies that one would need to know an analytical expression for the integral $I_0^5(4 + 2\varepsilon; \{1\})$, to order ε. Going back to the expression (47), we notice that all such terms can be written as a linear combination of the box (4−point) integrals in $4 + 2\varepsilon$ dimensions and 5−point integrals in $6+2\varepsilon$ dimensions. Therefore, at this point, the problem has been reduced to calculating the integral $I_0^5(6 + 2\varepsilon; \{1\})$, which is IR finite, and need to be calculated to order $\mathcal{O}(\varepsilon^0)$. It is an empirical fact [4, 5, 6, 14] that in final expressions for physical quantities all terms containing the integral $I_0^5(6 + 2\varepsilon; \{1\})$ always combine so that this integral ends up being multiplied by the coefficients $\mathcal{O}(\varepsilon)$, and as such, can be omitted in one−loop calculations. A few theoretical proofs of this fact can be found in literature [4, 6], but, to the best of our knowledge, the proof for the case of exceptional kinematics is still missing. That being the case, in concrete calculations, (to be sure and to have all the steps of the calculation under control), it is absolutely necessary to keep track of all the terms containing the integral $I_0^5(6+2\varepsilon; \{1\})$, add them up and check whether the factor multiplying it is of order $\mathcal{O}(\varepsilon)$. Even though the experience gained in numerous calculations shows that this is so, a situation in which the integral $I_0^5(6 + 2\varepsilon; \{1\})$ would appear in the final result for a physical quantity accompained by a factor $\mathcal{O}(1)$ would not, from practical point of view, present any problem. Namely, being IR finite, although extremely complicated to be evaluated analyticaly, the integral $I_0^5(6 + 2\varepsilon; \{1\})$ can always, if necessary, be evaluated numerically.

Based on the above considerations we may conclude that all one−loop integrals occurring when evaluating physical processes in massless field theories can be expressed in terms of the integrals

$$I_0^2(D'; \nu_1, \nu_2), \quad I_0^3(4 + 2\varepsilon; 1, 1, 1), \quad I_0^4(4 + 2\varepsilon; 1, 1, 1, 1).$$

These integrals, therefore, constitute a minimal set of fundamental integrals.

It is very important to point out that the scalar reduction formalism presented in this section has been formulated without any restrictions regarding the external momenta. As such, it is completely general and can be applied to arbitrary kinematics. This is in contrast to the procedure given in [6], which is obtained for non-exceptional kinematics, a consequence of which is that it cannot be used to deal with the one-loop N-point massless integrals in which the set of external momenta contains subsets comprised of two or more collinear momenta. As is well known, this type of integrals is encountered when performing the leading-twist NLO PQCD analysis of the exclusive hadronic processes at large momentum transfer. However, the formalism presented above is capable of dealing with this kind of integrals too.

5 On the fundamental set of integrals

In view of the above discussion, we conclude that the set of fundamental integrals is comprised of integrals with two, three and four external lines. Integrals with two external lines can be calculated analytically in arbitrary number of dimensions and with arbitrary powers of the propagators. They do not constitute a problem. As far as the integrals with three and four external lines are concerned, depending on how many kinematic variables vanish, we distinguish several different cases. We now show that in the case $N = 3$ we have only one fundamental integral, while in the case corresponding to $N = 4$ there are six integrals. For this purpose, we make use of the vanishing of the kinematic determinants $\det(R_N)$ and $\det(S_N)$.

5.1 The general scalar integral for $N = 2$

According to (5), the general massless scalar two−point integral in D space−time dimensions is of the form

$$I_0^2(D;\nu_1,\nu_2) \equiv (\mu^2)^{2-D/2} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{A_1^{\nu_1} A_2^{\nu_2}} \, . \tag{48}$$

The closed form expression for the above integral, valid for arbitrary $D = n + 2\varepsilon$, and arbitrary propagator powers ν_1 and ν_2, is given by

$$\begin{aligned} I_0^2(n+2\varepsilon;\nu_1,\nu_2) &= (4\pi\mu^2)^{2-n/2}(-1)^{\nu_1+\nu_2}\left(-p^2 - \mathrm{i}\epsilon\right)^{n/2-\nu_1-\nu_2} \\ &\times \frac{\Gamma\left(\nu_1+\nu_2-n/2-\varepsilon\right)}{\Gamma(-\varepsilon)} \frac{\Gamma\left(n/2-\nu_1+\varepsilon\right)}{\Gamma(1+\varepsilon)} \frac{\Gamma\left(n/2-\nu_2+\varepsilon\right)}{\Gamma(1+\varepsilon)} \\ &\times \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(2+2\varepsilon)}{\Gamma(n-\nu_1-\nu_2+2\varepsilon)} I_0^2(4+2\varepsilon,1,1)\, , \end{aligned} \tag{49}$$

where

$$I_0^2(4+2\varepsilon;1,1) = \frac{\mathrm{i}}{(4\pi)^2}\left(-\frac{p^2+\mathrm{i}\epsilon}{4\pi\mu^2}\right)^{\varepsilon} \frac{\Gamma\left(-\varepsilon\right)\Gamma^2\left(1+\varepsilon\right)}{\Gamma\left(2+2\varepsilon\right)}\, . \tag{50}$$

It is easily seen that in the formalism of the dimensional regularization the above integral vanishes for $p^2 = 0$.

5.2 The scalar integrals for $N = 3$

The massless scalar one−loop triangle integral in $D = 4 + 2\varepsilon$ dimensions is given by

$$I_0^3(4+2\varepsilon,\{1\}) = (\mu^2)^{-\varepsilon} \int \frac{\mathrm{d}^{4+2\varepsilon} l}{(2\pi)^{4+2\varepsilon}} \frac{1}{A_1 A_2 A_3}\, . \tag{51}$$

Making use of the representation (13), and introducing the external masses $p_i^2 = m_i^2$ $(i = 1, 2, 3)$, the integral (51) can be written in the form

$$\begin{aligned} I_0^3(4+2\varepsilon,\{1\}) &= \frac{-\mathrm{i}}{(4\pi)^2}\frac{\Gamma(1-\varepsilon)}{(4\pi\mu^2)^{\varepsilon}} \int_0^1 \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ \delta(x_1+x_2+x_3-1) \\ &\times \left(-x_1x_2\ m_2^2 - x_2x_3\ m_3^2 - x_3x_1\ m_1^2 - \mathrm{i}\epsilon\right)^{\varepsilon-1}\, . \end{aligned} \tag{52}$$

It is evident that the above integral is invariant under permutations of external massess m_i^2. Depending on the number of the external massless lines, and using the above mentioned symmetry, there are three relevant special cases of the above integral. We denote them by

$$I_3^{1m} \equiv I_0^3(4+2\varepsilon,\{1\};0,0,m_3^2), \tag{53}$$
$$I_3^{2m} \equiv I_0^3(4+2\varepsilon,\{1\};0,m_2^2,m_3^2), \tag{54}$$
$$I_3^{3m} \equiv I_0^3(4,\{1\};m_1^2,m_2^2,m_3^2), \tag{55}$$

The integrals I_3^{1m} and I_3^{2m} are IR divergent and need to be evaluated with $\varepsilon > 0$, while the integral I_3^{3m} is finite and can be calculated with $\varepsilon = 0$.

Now, it is easily found that the determinants of the systems of equations (27) and (32) are, for $N = 3$, given by

$$\det(R_3) = 2m_1^2m_2^2m_3^2, \tag{56}$$
$$\det(S_3) = (m_1^2)^2 + (m_2^2)^2 + (m_3^2)^2 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2, \tag{57}$$

As is seen from (56), if at least one of the external lines is on mass-shell, the determinant $\det(R_3)$ vanishes. Consequently, using the recursion relations (Case II or IV) the integrals I_3^{1m} and I_3^{2m} can be reduced to the integrals with two external lines. Therefore, we conclude that among the scalar integrals with three external lines the integral I_3^{3m} is the only fundamental one.

The result for this integral is well known [11, 12, 15]. In [11] it is expressed in terms of the dimensionless quantities of the form

$$x_{1,2} = \frac{1}{2}\left[1 - \frac{m_1^2}{m_2^2} + \frac{m_3^2}{m_2^2} \pm \sqrt{\left(1 - \frac{m_1^2}{m_2^2} - \frac{m_3^2}{m_2^2}\right)^2 - 4\frac{m_1^2}{m_2^2}\frac{m_3^2}{m_2^2}}\right] \tag{58}$$

and, being proportional to $1/(x_1 - x_2)$, appears to have a pole at $x_1 = x_2$. It appears that the final expression [11] is not well defined when $x_1 = x_2$.

On the basis of Eqs.(57) and (58), one finds that

$$x_1 - x_2 = \frac{1}{m_2^2}\sqrt{\det(S_3)}.$$

This equation implies that when $x_1 - x_2 = 0$, instead of examining the limit of the general expression in [11], one can utilize the reduction relations (40) (corresponding to $\det(R_3) \neq 0$ and $\det(S_3) = 0$) to reduce the IR finite integral with three external lines, I_3^{3m}, to the integrals with two external lines.

5.3 The scalar integrals for $N = 4$

The massless scalar one−loop box integral in $D = 4 + 2\varepsilon$ space−time dimensions is given by

$$I_0^4(4 + 2\varepsilon, \{1\}) = (\mu^2)^{-\varepsilon} \int \frac{\mathrm{d}^{4+2\varepsilon} l}{(2\pi)^{4+2\varepsilon}} \frac{1}{A_1 A_2 A_3 A_4}\ ,. \tag{59}$$

Making use of (13), introducing the external "masses" $p_i^2 = m_1^2$ $(i = 1, 2, 3, 4)$, and the Mandelstam variables $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$, the integral (59) becomes

$$\begin{aligned} I_0^4(4 + 2\varepsilon, \{1\}) = {} & \frac{\mathrm{i}}{(4\pi)^2} \frac{\Gamma(2 - \varepsilon)}{(4\pi\mu^2)^\varepsilon} \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4\ \delta(x_1 + x_2 + x_3 + x_4 - 1) \\ & \times \left(-x_1 x_3\ t - x_2 x_4\ s - x_1 x_2\ m_2^2\ - x_2 x_3\ m_3^2 - x_3 x_4\ m_4^2 - x_1 x_4\ m_1^2 - \mathrm{i}\epsilon\right)^{\varepsilon - 2} . \end{aligned} \tag{60}$$

Introducing the following set of ordered pairs

$$(s, t),\ (m_1^2, m_3^2),\ (m_2^2, m_4^2), \tag{61}$$

one can easily see that the integral (60) is invariant under the permutations of ordered pairs, as well as under the simultaneous exchange of places of elements in any two pairs.

The determinants of the coefficient matrices of the systems of equations (27) and (32), corresponding to the above integral, are

$$\begin{aligned} \det(R_4) \quad = \quad & s^2 t^2 + (m_1^2 m_3^2)^2 + (m_2^2 m_4^2)^2 \\ & - 2st m_1^2 m_3^2 - 2st m_2^2 m_4^2 - 2m_1^2 m_2^2 m_3^2 m_4^2 . \end{aligned} \tag{62}$$

$$\begin{aligned} \det(S_4) \quad = \quad & 2\,[\,st\,(m_1^2 + m_2^2 + m_3^2 + m_4^2 - s - t) \\ & + m_2^2 m_4^2 (s + t + m_1^2 + m_3^2 - m_2^2 - m_4^2) \\ & + m_1^2 m_3^2 (s + t - m_1^2 - m_3^2 + m_2^2 + m_4^2) \\ & - s(m_1^2 m_2^2 + m_3^2 m_4^2) - t(m_1^2 m_4^2 + m_2^2 m_3^2)\,] . \end{aligned} \tag{63}$$

By looking at the expression for $\det(R_4)$ given in (62) it follows that all box integrals I_0^4 that are characterized by the fact that in each of the ordered pairs in (61) at least one kinematic variable vanishes, are reducible. Therefore, for a box integral to be irreducible, it is necessary that both kinematic variables in at least one of the ordered pairs should be different from zero. Owing to the symmetries valid for the box integrals it is always possible to choose that pair to be (s,t).

Taking into account symmetries, and the number of external massless lines, there are six potentially irreducible special cases of the integral (60). Adopting the notation of Ref. [4], we denote them by

$$\begin{aligned}
I_4^{4m} &\equiv I_0^4(4,\{1\};s,t,m_1^2,m_2^2,m_3^2,m_4^2), &(64)\\
I_4^{3m} &\equiv I_0^4(4+2\varepsilon,\{1\};s,t,0,m_2^2,m_3^2,m_4^2), &(65)\\
I_4^{2mh} &\equiv I_0^4(4+2\varepsilon,\{1\};s,t,0,0,m_3^2,m_4^2), &(66)\\
I_4^{2me} &\equiv I_0^4(4+2\varepsilon,\{1\};s,t,0,m_2^2,0,m_4^2), &(67)\\
I_4^{1m} &\equiv I_0^4(4+2\varepsilon,\{1\};s,t,0,0,0,m_4^2), &(68)\\
I_4^{0m} &\equiv I_0^4(4+2\varepsilon,\{1\};s,t,0,0,0,0,), &(69)
\end{aligned}$$

with all kinematic variables appearing above being different from zero. The results for these integrals are well known [4, 10, 11, 16].

The integrals (65-69) are IR divergent, and as such need to be evaluated with $\varepsilon > 0$, while the integral (64) is finite and can be calculated in $D = 4$. The results for these integrals, obtained in [10, 11] for arbitrary values of the relevant kinematic variables, and presented in a simple and compact form, have the following structure:

$$\begin{aligned}
I_4^K(s,t;m_i^2) &= \frac{i}{(4\pi)^2}\,\frac{\Gamma(1-\varepsilon)\Gamma^2(1+\varepsilon)}{\Gamma(1+2\varepsilon)}\,\frac{1}{\sqrt{\det(R_4^K)}}\\
&\quad\times\left[\frac{G^K(s,t;\varepsilon;m_i^2)}{\varepsilon^2}+H^K(s,t;m_i^2)\right]+\mathcal{O}(\varepsilon),\\
&\quad K\in\{0m,1m,2me,2mh,3m,4m\}\,. \qquad (70)
\end{aligned}$$

The IR divergences (both soft and collinear) of the integrals are contained in the first term within the square brackets, while the second term is finite. The function $G^K(s,t;\varepsilon;m_i^2)$ is represented by a sum of powerlike terms, it depends on ε and is finite in the $\varepsilon \to 0$ limit. As for the function $H^K(s,t;m_i^2)$, it is given in terms of dilogarithm functions and constants. In the above, $\det(R_4^K)$ is the determinant corresponding to the integral I_4^K given in (64-69).

For the purpose of numerical integration, it is very useful to have the exact limit of the integral I_4^K when $\det(R_4^K) \to 0$. This limit can be determined in an elegant manner by noticing that for $\det(R_4^K) = 0$ the reduction relations corresponding to Cases II and IV apply, making it possible to represent the box integral I_4^K as a linear combination of the triangle integrals. This result can be made use of to combine box and triangle integrals (or pieces of these integrals) with the aim of obtaining numerical stability of the integrand [8].

The integrals (64-69) are irreducible only if the corresponding kinematic determinant $\det(R_4^K)$ does not vanish.

With the help of the tensor decomposition and the scalar reduction procedures, any dimensionally regulated one−loop $N-$point Feynman integral can be represented as a linear combination of the integrals:

$$\begin{aligned}
&I_0^2(D';\nu_1,\nu_2),\\
&I_3^{3m},\\
&I_4^{4m},\ I_4^{3m},\ I_4^{2mh},\ I_4^{2me},\ I_4^{1m},\ I_4^{0m}, \qquad (71)
\end{aligned}$$

multiplied by tensor structures made from the external momenta and the metric tensor. The integrals in (71) constitute a fundamental set of integrals. An alternative and more convenient set of fundamental integrals is obtained by noticing that all the relevant box integrals are finite in $D = 6$. On the basis of Eq. (31), all IR divergent box integrals can be expressed as linear combinations of triangle integrals in $D = 4 + 2\varepsilon$ dimension and a box integral in $D = 6 + 2\varepsilon$ dimension. Next, using the same equation, all triangle integrals can be decomposed into a finite triangle integral and two−point integrals. In the final expression thus obtained all divergences, IR as well as UV, are contained in the general two-point integrals and associated coefficients. Therefore, an alternative fundamental set of integrals is comprised of

$$\begin{aligned} &I_0^2(D';\nu_1,\nu_2), \\ &I_3^{3m}, \\ &I_4^{4m},\ J_4^{3m},\ J_4^{2mh},\ J_4^{2me},\ J_4^{1m},\ J_4^{0m}, \end{aligned} \tag{72}$$

where J_4^K denotes box integrals in $D = 6$ dimensions, explicit expressions for which are given in the Appendix. A characteristic feature of this fundamental set of integrals, which makes it particularly interesting, is that the integral I_0^2 is the only divergent one, while the rest of integrals are finite.

6 Conclusion

In this work we have considered one−loop scalar and tensor Feynman integrals with an arbitrary number of external lines which are relevant for construction of multi−parton one−loop amplitudes in massless field theories.

We have presented a tensor decomposition procedure for the dimensionally−regulated N−point rank $P(\leq N)$ massless integrals. The general results, expressed by formulas (13) and (15), represent a massless version of the results that have originally been obtained in Ref. [2] for the case of the massive Feynman integrals. Based on (15), any dimensionally regulated N−point tensor integral can be expressed as a linear combination of N−point scalar integrals multiplied by tensor structure made from the metric tensor $g^{\mu\nu}$ and external momenta. Therefore, with the decomposition (15), the problem of calculating the one−loop massless tensor integrals has been reduced to the calculation of the general scalar intgerals of the type (5).

Main result of this paper is a scalar reduction method by which an arbitrary $(N \geq 5)$−point scalar one−loop integral can be reqursively represented as a linear combination of scalar $(N-1)$−point integrals provided the external momenta are kept in four dimensions. Consequently, all scalar integrals occuring in the computation of an arbitrary one−loop $(N \geq 5)$−point integral with massless internal lines can be reduced to a sum over a set of six basic scalar box $(N = 4)$ integrals with rational coefficients depending on the external momenta and the dimensionality of space−time. The problem of vanishing of the kinematic determinants, which is a reflection of very complex singularity structure of these integrals, has been solved in an elegant and transparent manner. Namely, the approach has been taken according to which instead of solving the general system of linear equations given in (25), and then finding the limit of the obtained solution corresponding to a given singular kinematic situation, we first obtain and then solve the system of equations appropriate to the situation being considered.

In contrast to the procedure of Ref. [6], which has been obtained for the nonexceptional kinematics, our method has been derived without any restrictions regarding the external momenta. As such, it is completely general and, unlike the method of Ref. [6], applicable to arbitrary kinematics. In particular, it applies to the integrals in which the set of external momenta contains subsets comprised of two or more collinear momenta. This kind of integrals are encountered when performing leading−twist next−to−leading−order perturbative QCD analysis of the hadronic exclusive processes at large−momentum−transfer. Trough the tensor decomposition and scalar reduction presented, any massless one-loop Feynman integral with generic 4-dimensional momenta can be expressed as a linear combination of a fundamental set of scalar integrals: six box integrals in $D = 6$, a triangle integral in $D = 4$, and a general two−point integral. All the divergences present in the original integral are contained in the general two-point integral and associated coefficients.

In conclusion, the computation of IR divergent one−loop integrals for arbitrary number of external lines can be mastered with the reduction formulas presented above. The iterative structure makes it easy to implement the formalism in algebraic computer program. With this work all the conceptual problems concerning the construction of multi−parton one−loop amplitudes are thus solved.

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Appendix

In addition to the explicit calculation, the irreducible box integrals in $D = 6$ dimensions can be obtained using the existing analytical expressions for the irreducible box integrals in $D = 4 + 2\varepsilon$ dimensions and the reduction formula (31). To this end, we substitute $D = 6 + 2\varepsilon$, $N = 4$, $\nu_i = 1$ and $C = 1$ into the relation (31) and find

$$I_0^4(6+2\varepsilon;\{1\}) = \frac{1}{4\pi\mu^2(2\varepsilon+1)\ z_0}\left(I_0^4(4+2\varepsilon;\{1\}) - \sum_{i=1}^{4} z_i I_0^4(4+2\varepsilon;\{\delta_{kk}-\delta_{ki}\})\right)\ . \qquad (73)$$

Note that the IR divergences in $D = 4+2\varepsilon$ box integrals are exactly cancelled by the divergences of the triangle integrals.

The expressions for the relevant basic massless scalar box integrals in $D = 6$ space−time dimensions are listed below:

<u>The three−mass scalar box integral</u>

$$\begin{aligned}
I_4^{3m}(D=6;s,t;m_2^2,m_3^2,m_4^2) &= \frac{\mathrm{i}}{(4\pi)^2}\,\frac{1}{4\pi\mu^2}\\
&\times h^{3m}\Bigg\{\frac{1}{2}\ln\left(\frac{s+\mathrm{i}\epsilon}{m_3^2+\mathrm{i}\epsilon}\right)\ln\left(\frac{s+\mathrm{i}\epsilon}{m_4^2+\mathrm{i}\epsilon}\right) + \frac{1}{2}\ln\left(\frac{t+\mathrm{i}\epsilon}{m_2^2+\mathrm{i}\epsilon}\right)\ln\left(\frac{t+\mathrm{i}\epsilon}{m_3^2+\mathrm{i}\epsilon}\right)\\
&+ \mathrm{Li}_2\left(1-\frac{m_2^2+\mathrm{i}\epsilon}{t+\mathrm{i}\epsilon}\right) + \mathrm{Li}_2\left(1-\frac{m_4^2+\mathrm{i}\epsilon}{s+\mathrm{i}\epsilon}\right)\\
&+ \mathrm{Li}_2\Big[1-(s+\mathrm{i}\epsilon)\,f^{3m}\Big] + \mathrm{Li}_2\Big[1-(t+\mathrm{i}\epsilon)\,f^{3m}\Big]\\
&- \mathrm{Li}_2\Big[1-(m_2^2+\mathrm{i}\epsilon)\,f^{3m}\Big] - \mathrm{Li}_2\Big[1-(m_4^2+\mathrm{i}\epsilon)\,f^{3m}\Big]\\
&- \frac{1}{2}\left(t-m_2^2-m_3^2+2m_2^2m_3^2\frac{t-m_4^2}{st-m_2^2m_4^2}\right)\mathcal{I}_3(m_2^2,m_3^2,t)\\
&- \frac{1}{2}\left(s-m_3^2-m_4^2+2m_3^2m_4^2\frac{s-m_2^2}{st-m_2^2m_4^2}\right)\mathcal{I}_3(m_3^2,m_4^2,s)\Bigg\}. \qquad (74)
\end{aligned}$$

<u>The adjacent ("hard") two−mass scalar box integral</u>

$$I_4^{2mh}(D=6;s,t;m_3^2,m_4^2) = \frac{\mathrm{i}}{(4\pi)^2}\,\frac{1}{4\pi\mu^2}$$

$$\times\, h^{2mh}\Bigg\{ \frac{1}{2}\ln\left(\frac{s+\mathrm{i}\epsilon}{m_3^2+\mathrm{i}\epsilon}\right)\ln\left(\frac{s+\mathrm{i}\epsilon}{m_4^2+\mathrm{i}\epsilon}\right) + \mathrm{Li}_2\left(1-\frac{m_4^2+\mathrm{i}\epsilon}{s+\mathrm{i}\epsilon}\right)$$
$$-\,\mathrm{Li}_2\left(1-\frac{m_3^2+\mathrm{i}\epsilon}{t+\mathrm{i}\epsilon}\right) + \mathrm{Li}_2\Big[1-(s+\mathrm{i}\epsilon)\, f^{2mh}\Big]$$
$$+\,\mathrm{Li}_2\Big[1-(t+\mathrm{i}\epsilon)\, f^{2mh}\Big] - \mathrm{Li}_2\Big[1-(m_4^2+\mathrm{i}\epsilon)\, f^{2mh}\Big]$$
$$-\,\frac{1}{2}\left(s-m_3^2-m_4^2+2\frac{m_3^2m_4^2}{t}\right)\mathcal{I}_3(m_3^2,m_4^2,s)\Bigg\}. \tag{75}$$

<u>The opposite (”easy”) two−mass scalar box integral</u>

$$I_4^{2me}(D=6;s,t;m_2^2,m_4^2) = \frac{\mathrm{i}}{(4\pi)^2}\,\frac{1}{4\pi\mu^2}$$
$$\times\, h^{2me}\Bigg\{ \mathrm{Li}_2\Big[1-(s+\mathrm{i}\epsilon)\, f^{2me}\Big] + \mathrm{Li}_2\Big[1-(t+\mathrm{i}\epsilon)\, f^{2me}\Big]$$
$$-\,\mathrm{Li}_2\Big[1-(m_2^2+\mathrm{i}\epsilon)\, f^{2me}\Big] - \mathrm{Li}_2\Big[1-(m_4^2+\mathrm{i}\epsilon)\, f^{2me}\Big]\Bigg\}. \tag{76}$$

<u>The one−mass scalar box integral</u>

$$I_4^{1m}(D=6;s,t;m_4^2) = \frac{\mathrm{i}}{(4\pi)^2}\,\frac{1}{4\pi\mu^2}$$
$$\times\, h^{1m}\Bigg\{ \mathrm{Li}_2\Big[1-(s+\mathrm{i}\epsilon)\, f^{1m}\Big] + \mathrm{Li}_2\Big[1-(t+\mathrm{i}\epsilon)\, f^{1m}\Big]$$
$$-\,\mathrm{Li}_2\Big[1-(m_4^2+\mathrm{i}\epsilon)\, f^{1m}\Big] - \frac{\pi^2}{6}\Bigg\}, \tag{77}$$

<u>The zero−mass (massless) scalar box integral</u>

$$I_4^{0m}(D=6;s,t) = \frac{\mathrm{i}}{(4\pi)^2}\,\frac{1}{4\pi\mu^2}$$
$$\times\, h^{0m}\Bigg\{ \mathrm{Li}_2\Big[1-(s+\mathrm{i}\epsilon)\, f^{0m}\Big] + \mathrm{Li}_2\Big[1-(t+\mathrm{i}\epsilon)\, f^{0m}\Big] - \frac{\pi^2}{3}\Bigg\}, \tag{78}$$

where

$$h^K = \left(-2\frac{\sqrt{\det(R_4^K)}}{\det(S_4^K)}\right), \tag{79}$$

and

$$\frac{\mathrm{i}}{(4\pi)^2}\mathcal{I}_3(a,b,c) = I_3^{3m}(D=4;a,b,c)\ . \tag{80}$$

The functions appearing above are given by

$$f^{3m} = f^{2me} \quad = \quad \frac{s+t-m_2^2-m_4^2}{st-m_2^2m_4^2}\ ,$$

$$
\begin{aligned}
f^{2mh} = f^{1m} &= \frac{s+t-m_4^2}{st}\,, \\
f^{0m} &= \frac{s+t}{st}\,, \\
h^{3m} &= \left(s+t-m_2^2-m_3^2-m_4^2+m_3^2\frac{m_2^2t+m_4^2s-2m_2^2m_4^2}{st-m_2^2m_4^2}\right)^{-1}, \\
h^{2mh} &= \left(s+t-m_3^2-m_4^2+\frac{m_3^2m_4^2}{t}\right)^{-1}, \\
h^{2me} &= \left(s+t-m_2^2-m_4^2\right)^{-1}, \\
h^{1m} &= \left(s+t-m_4^2\right)^{-1}, \\
h^{0m} &= (s+t)^{-1}\,.
\end{aligned}
$$

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