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Generalizations of the Dirac equation in covariant and Hamiltonian form

R-K Loide[1], I Ots[2] and R Saar[3]

[1] Tallinn Technical University, Ehitajate tee 5, 19086 Tallinn, Estonia
[2] Institute of Physics, Riia 142, 51014 Tartu, Estonia
[3] Institute of Theoretical Physics, Tartu University, Tähe 4, 51010 Tartu, Estonia

E-mail: karl@edu.ttu.ee, ots@fi.tartu.ee and reinsaar@physic.ut.ee

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Abstract

The Dirac wave equation can be treated equally in covariant and Hamiltonian forms. Recently the equations in Hamiltonian form which are in some sense the generalizations of the Dirac Hamiltonian form to the arbitrary spin case have become popular. Here we give similar generalization in the covariant form for the field with n bispinor indices and investigate the physics behind these two generalizations. We show that both generalizations are related to the representations of the de Sitter group and give the multiplets with certain mass and spin. It appears that covariant and Hamiltonian forms are not physically equivalent, the latter one offers nonphysical solutions which should be eliminated using some sort of additional conditions.

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1. Introduction

The problem of the relativistic particle with arbitrary spin has had a long history, but in spite of the enormous amount of related papers there are still a lot of open questions. Due to the known difficulties with higher-spin interactions (Velo and Zwanziger 1969a, b) different single- and multi-spin theories have been proposed. In multi-spin case the acausality difficulties may be avoided, but there appear other problems, such as the indefiniteness of energy and charge, and the presence of unphysical states.

The most important wave equation in modern field theory is the Dirac spin-$1/2$ equation (Dirac 1928a, b). For that reason its various generalizations to the higher-spin case are also of great interest. There are two main ways to generalize. The first one starts from the Dirac equation in its covariant form and leads to set of relativistic wave equations related with the de Sitter group $SO(1,4)$. The general case of Bhabha equations (Bhabha 1945, Lubansky 1942a, b) related to the representations of the de Sitter group was recently treated in Loide *et al* (1997a, b), here we obtain a special case. As we shall demonstrate later, the Bargmann–Wigner equations (Bargmann and Wigner 1948) have similar de Sitter structure. The second one starts from the Dirac theory in its Hamiltonian form. In that case we obtain another interesting set of equations, recently proposed and investigated in the work of Moshinsky and del Sol Mesa (1996) and Moshinsky and Smirnov (1996). In this paper we will investigate and compare these two main sets of equations and we will demonstrate that they are not physically equivalent. The covariant form gives a set of Poincaré states with certain mass and spin, but the corresponding Hamiltonian form leads in general to different mass and spin content, and in the integer-spin case it has also nonphysical solutions having vanishing energy.

2. The Dirac equation and its generalization

Massive spin-$1/2$ is described with the help of the Dirac equation

$$(p_\mu \gamma^\mu - m)\psi = 0 \tag{2.1}$$

where γ^μ are the usual Dirac matrices, ψ is a four-component bispinor and m is a mass of a given particle.

The most natural generalization of equation (2.1) is obtained by introducing the field with n bispinor indices

$$\psi_{\alpha_1,\alpha_2\ldots\alpha_n} \tag{2.2}$$

which is a solution of the equation

$$(p_\mu \beta^\mu - m)\psi = 0 \tag{2.3}$$

where the β-matrices are

$$\beta^\mu = \frac{1}{n}\sum_{i=1}^{n} \gamma_i^\mu \tag{2.4}$$

and γ_i^μ is a direct product:

$$\gamma_i^\mu = I \otimes I \otimes \cdots \otimes \gamma^\mu \otimes \cdots \otimes I \tag{2.5}$$

with Dirac matrices in the ith place.

In order to give a more thorough analysis of equation (2.3) we first note that equation (2.3) and also the Dirac equation we started with, are Bhabha equations connected with the de Sitter group $SO(1,4)$. If we define operators

$$S^{\mu 5} = \frac{n}{2}\beta^\mu \tag{2.6}$$

we get the de Sitter algebra $SO(1,4)$, where the Lorentz generators $S^{\mu\nu}$ are generated in the following way:

$$S^{\mu\nu} = \frac{n^2}{4}[\beta^\mu, \beta^\nu] \equiv [S^{\mu 5} S^{\nu 5}]. \tag{2.7}$$

Equation (2.3) is therefore written as a Bhabha equation

$$\left(p_\mu S^{\mu 5} - \frac{nm}{2}\right)\psi = 0. \tag{2.8}$$

The most general form of Bhabha equation (2.8), corresponding to the arbitrary irreducible representations (n_1, n_2) of the de Sitter group $SO(1,5)$ was previously treated in Loide (1975), Loide *et al* (1997a, b). Equation (2.8) with a specific choice of matrices (2.4) and (2.6) is a special Bhabha equation, corresponding to a fixed set of representations (Loide *et al* 1997a, b). It gives a certain multiplet of states up to spin $s = n/2$.

In the Dirac case ($n = 1$) we have

$$S^{\mu 5} = \tfrac{1}{2}\gamma^\mu \tag{2.9}$$

which corresponds to a four-component representation $(1/2, 1/2)$ of $SO(1, 4)$. The general case, of course, leads to the direct product of representations

$$(1/2, 1/2) \otimes (1/2, 1/2) \otimes \cdots \otimes (1/2, 1/2) \tag{2.10}$$

which reduces to the direct sum of irreducible representations

$$(n/2, n/2), (n/2, n/2 - 1), \ldots, (n/2 - 1, n/2 - 1), \ldots, (1/2, 1/2) \text{ (or } (0, 0)) \tag{2.11}$$

with different multiplicity (Kõiv *et al* 1970).

Next we treat the lowest cases $n = 2$ and 3 more thoroughly.

In the $n = 2$ case the de Sitter generators $S^{\mu 5}$ are

$$S^{\mu 5} \equiv \beta^{\mu} = \tfrac{1}{2}(\gamma^{\mu} \otimes I + I \otimes \gamma^{\mu}). \tag{2.12}$$

The 16-component field $\psi_{\alpha\beta}$ corresponds to the representation

$$(1/2, 1/2) \otimes (1/2, 1/2) = (1, 1) \oplus (1, 0) \oplus (0, 0) \tag{2.13}$$

which is a direct sum of ten-, five- and one-component irreducible representations of $SO(1, 4)$.

Obtaining a separate equation for each irreducible representation gives us the possibility to analyse them separately. In the ten-component case $(1, 1)$ we get the well known Kemmer–Duffin spin-1 equation (Duffin 1938, Kemmer 1939) corresponding to the symmetrical field $\psi_{\alpha\beta} = \psi_{\beta\alpha}$. The antisymmetrical field $\psi_{\alpha\beta} = -\psi_{\beta\alpha}$ reduces to a five-component irreducible representation $(1, 0)$ and one-component scalar representation $(0, 0)$. Equation (2.6) corresponding to the irreducible representation $(1, 0)$ is a Kemmer–Duffin spin-0 equation. The scalar field gives due to the vanishing $S^{\mu 5}$ matrices no physical states.

In the $n = 3$ case the de Sitter generators $S^{\mu 5}$ are

$$S^{\mu 5} \equiv \tfrac{3}{2}\beta^{\mu} = \tfrac{1}{2}(\gamma^{\mu} \otimes I \otimes I + I \otimes \gamma^{\mu} \otimes I + I \otimes I \otimes \gamma^{\mu}). \tag{2.14}$$

The 64-component field $\psi_{\alpha\beta\gamma}$ corresponds to the following $SO(1, 4)$ representation:

$$(1/2, 1/2) \otimes (1/2, 1/2) \otimes (1/2, 1/2) = (3/2, 3/2) \oplus 2(3/2, 1/2) \oplus 3(1/2, 1/2) \tag{2.15}$$

which is therefore the direct sum of 20-, 16- and four-component irreducible representations with various multiplicities.

The mass and spin content of corresponding equations will be given later, here we only mention that the equation describes four spin-3/2 and eight spin-1/2 states in total.

3. Bargmann–Wigner equations

Bargmann and Wigner (1948) proposed an interesting single-spin theory, which is also a generalization of the Dirac spin-1/2 theory. Bargmann and Wigner considered the symmetrical field $\psi_{\alpha_1\alpha_2\ldots\alpha_n}$ demanding Dirac equation for each component α_i. Therefore, the Bargmann–Wigner $s = n/2$ equation for a given n is a set of Dirac equations

$$(p_{\mu}\gamma_i^{\mu} - m)\psi = 0 \qquad i = 1, 2, \ldots, n. \tag{3.1}$$

As we have already demonstrated in Loide *et al* (1997a) the given set of equations is closely related to the Bhabha equations. Adding and subtracting equations it is easy to rewrite equation (3.1) as an equation

$$\left(p_{\mu}\left(\frac{1}{n}\sum_{i=1}^{n}\gamma_i^{\mu}\right) - m\right)\psi = 0 \tag{3.2}$$

and $n - 1$ subsidiary conditions

$$p_{\mu}(\gamma_i^{\mu} - \gamma_{i+1}^{\mu})\psi = 0 \qquad i = 1, \ldots, n - 1. \tag{3.3}$$

By comparing equation (3.2) with equations (2.3) and (2.4) one can see that (3.2) is a Bhabha equation treated above. The symmetrical field $\psi_{\alpha_1\alpha_2\ldots\alpha_n}$ corresponds to the $SO(1,4)$ representation $(n/2, n/2)$. In general it contains $n+1$ spin-$n/2$ states and also lower spin states, but the subsidiary conditions (3.3) leave only one spin-$n/2$ state with mass m.

In the $n = 2$ case equation (3.2) is the well known Kemmer–Duffin spin-1 equation. In that case the conditions (3.3) are automatically satisfied.

4. Physical states

Physical states with a given mass and spin are unitary irreducible representations (m, s) of the Poincaré group. As it was demonstrated in Loide *et al* (1993) the role of relativistic wave equations is to define a proper Poincaré basis of physical states.

Next we briefly recall the role of wave equations and give the basis for analysis of mass and spin spectrum of equations (2.3). Without the loss of generality we may use the rest system $\vec{p} = 0$. In the rest system equation (2.3) reduces to

$$(\hat{p}_0\beta^0 - m)\psi = 0. \tag{4.1}$$

The latter is an eigenvalue problem of the β^0-matrix

$$\beta^0\psi = h\psi \equiv \frac{m}{\hat{p}_0}\psi. \tag{4.2}$$

From (4.2) it is obvious that to each real pair of nonzero eigenvalues $\pm h$ we get

$$\hat{p}_0 = \pm\frac{m}{|h|} \equiv \pm m_h \tag{4.3}$$

i.e. physical states with mass m_h.

The possible eigenvalue $h = 0$ is due to $m \neq 0$ eliminated and gives no physical states.

As we see, the primary role of relativistic wave equation (2.3) is to determine the masses of physical states. Since β^0 commutes with the generators S^{kl} of space rotations, in addition to (4.2) we may demand that ψ is at the same time the eigenstate with a given spin and spin projection

$$\vec{S}^2\psi = s(s+1)\psi \qquad S^3\psi = \sigma\psi \tag{4.4}$$

where $S^3 = \mathrm{i}S^{12}$ and $\vec{S}^2 = -(S^{23})^2 - (S^{31})^2 - (S^{12})^2$. The solution of equations (4.2) and (4.4) is a physical state

$$\psi_{m_h s\sigma} \tag{4.5}$$

with a given mass, spin and spin projection.

The above given procedure defines a proper Poincaré basis with correct transformation properties (Loide *et al* 1993).

In the case of our generalized Dirac equations the analysis is performed similarly. If we present equation (2.3) in the $SO(1,4)$-form (2.8), instead of (4.2) we have the eigenvalue problem of S^{05} matrix

$$S^{05}\psi = h\psi \tag{4.6}$$

which in our particular case gives masses

$$m_h = \frac{nm}{2|h|}. \tag{4.7}$$

Since the analysis of general Bhabha equations, there corresponding to the irreducible representation (n_1, n_2) is given elsewhere (Loide *et al* 1997a), here we shall treat only two special cases, $n = 2$ and 3, which demonstrate the main structure of equations (2.3).

$n = 2$. *Representation* $(1, 1)$. Eigenvalues of S^{05} are

$$h = 0, \pm 1. \tag{4.8}$$

We have a single spin-1 state with mass $m_1 = m$. It is interesting to note, that to $h = 0$, which is not a solution of equation (2.3), there correspond two spins: $s = 1$ and 0.

$n = 2$. *Representation* $(1, 0)$. Eigenvalues of S^{05} are the same as in the previous case, but they correspond to different spins. Now the solution with mass m corresponds to spin 0, eigenvalue $h = 0$ to spin 1.

$n = 3$. *Representation* $(3/2, 3/2)$. Eigenvalues of S^{05} are

$$h = \pm 3/2, \pm 1/2. \tag{4.9}$$

From (4.7) it follows that we have two mass states m and $3m$. Since $h = \pm 3/2$ correspond to spin $3/2$ and $h = \pm 1/2$ to spins $3/2$ and $1/2$, equation (2.6) has the following mass and spin content:

$$m_{3/2} = m \qquad s = 3/2 \qquad m_{1/2} = 3m \qquad s = 3/2, 1/2. \tag{4.10}$$

$n = 3$. *Representation* $(3/2, 1/2)$. Eigenvalues of S^{05} are the same as in the previous case, but the general spin content is different. Now $h = \pm 3/2$ correspond to spin $1/2$ and $h = \pm 1/2$ to spins $3/2$ and $1/2$, equation (2.6) has the following mass and spin content:

$$m_{3/2} = m \qquad s = 1/2 \qquad m_{1/2} = 3m \qquad s = 3/2, 1/2. \tag{4.11}$$

$n = 3$. *Representation* $(1/2, 1/2)$. Eigenvalues of S^{05} are $h = \pm 1/2$ and we have

$$m_{3/2} = 3m \qquad s = 1/2. \tag{4.12}$$

As we have mentioned above, the $n = 3$ case gives, in general, four spins $3/2$ with masses m or $3m$ and eight spins $1/2$ with masses m or $3m$. Since S^{05} may be decomposed into irreducible representations of $SO(1, 4)$, we get separate equations for each irreducible representation for which the mass and spin content is given by (4.10), (4.11) or (4.12).

5. Hamiltonian wave equations

The Dirac equation offers one more interesting generalization recently proposed in the works of Moshinsky and del Sol Mesa (1996) and Moshinsky and Smirnov (1996). These equations are in Hamiltonian form and therefore allow, as in ordinary quantum mechanics, the treatment of different potentials and therefore extend the results obtained for spin $1/2$ to an arbitrary spin case.

Dirac equation (2.1), if multiplied to $\gamma^0 \equiv \beta$, may be presented in the Hamiltonian form

$$p_0 \psi = (\vec{p} \cdot \vec{\alpha} + m\beta)\psi \tag{5.1}$$

where $\vec{\alpha} = \gamma^0 \vec{\gamma}$. Here

$$H = \vec{p} \cdot \vec{\alpha} + m\beta \tag{5.2}$$

is a Dirac Hamiltonian and allows the addition of some potential energy when treating different physical problems, such as the relativistic Coulomb problem, the Dirac oscillator and others. Equation (5.1) is, of course, a well known energy eigenvalue problem;

$$H\psi = E\psi. \tag{5.3}$$

In a similar way to section 2, we give the generalization of (5.1) to the arbitrary field $\psi_{\alpha_1\alpha_2\ldots\alpha_n}$ case

$$p_0\psi = (\vec{p}\cdot\vec{A} + mB)\psi \tag{5.4}$$

where

$$\vec{A} = \frac{1}{n}\sum_{i=1}^{n}\gamma_i^0\vec{\gamma}_i \qquad B = \frac{1}{n}\sum_{i=1}^{n}\gamma_i^0. \tag{5.5}$$

So we have reached the arbitrary-spin Hamiltonian equations, proposed by Moshinsky and del Sol Mesa (1996) and Moshinsky and Smirnov (1996) using a different approach.

Next we shall give a thorough analysis of equation (5.4) in the free-field case and discuss the equivalency with the covariant case discussed above. As we shall see, these equations are generally not equivalent.

In a similar way to as in the covariant case, we exploit the generators of the de Sitter group $SO(1,4)$. It is easy to verify that the Hamiltonian of equation (5.4) may be rewritten as

$$H = \frac{2}{n}(p_k S^{0k} + mS^{05}) \tag{5.6}$$

where S^{0k} are the generators of Lorentz boosts.

Exploiting the knowledge of $SO(1,4)$ representations it is possible to study the physics behind the equations written in the Hamiltonian form and to compare with the results obtained for the similar covariant case.

It should be noted that the Hamiltonian (5.6) was already given in connection with the generalized Foldy–Wouthuysen transformation in our earlier papers (Loide 1975, Loide *et al* 1997b), but the physics behind it was not analysed.

Without the loss of generality we also treat only the rest case $\vec{p} = 0$. Equation (5.4), using (5.6), gives

$$\frac{2m}{n}S^{05}\psi = \hat{p}_0\psi \tag{5.7}$$

which is also the eigenvalue problem of S^{05}

$$S^{05}\psi = h\psi \equiv \frac{n\hat{p}_0}{2m}\psi. \tag{5.8}$$

The results, compared with the covariant case ((4.6) and (4.7)), are in general different. From (5.8) it follows that the rest-system energy, i.e. the masses of the physical states are

$$\hat{p}_0 \equiv \pm m_h = \pm\frac{2m|h|}{n}. \tag{5.9}$$

As compared with the covariant case the most important difference here is that in the integer-spin case the eigenvalue $h = 0$ is not excluded and leads to unphysical solutions having zero mass and energy. In Moshinky and del Sol Mesa (1994) these solutions are called a cockroach nest. Independently of what we call them, the presence of unphysical solutions is not a positive point of a given theory and one must find some procedure (additional conditions) to exclude them. The presence of such solutions may lead to nonzero energies, if interactions are considered, and therefore to the generation of some ghost states.

Next we shall see that the mass content is also different. Only for the maximum possible eigenvalue $h = n/2$ do we get the same mass m. Unfortunately the masses of other states are different. If, for example, we take the eigenvalue $h = n/2 - 1$ which is next to the maximal one, from (5.9) we get

$$m_h = \frac{(n-2)m}{n} \tag{5.10}$$

then the corresponding covariant state has mass

$$m_h = \frac{nm}{(n-2)}. \tag{5.11}$$

Let us take some specific examples.

$n = 2$. *Representation* $(1, 1)$. For $h = \pm 1$ we get mass m and spin 1 as in the Kemmer–Duffin case. The eigenvalue $h = 0$, which is connected with spins 1 and 0, gives solutions with vanishing energy. As we have already mentioned above, these unphysical solutions are absent in the Kemmer–Duffin theory. Representation $(1, 0)$ gives similar results. Now $h = \pm 1$ gives mass m and spin 0 as in the Kemmer–Duffin spin-0 case: $h = 0$ which is connected with spin 1 leads to unphysical solutions.

$n = 3$. Independently on irreducible representations $(3/2, 3/2)$, $(3/2, 1/2)$ or $(1/2, 1/2)$ we have masses m and $m/3$, the latter corresponds to $h = \pm 1/2$. In the covariant case the representation $(3/2, 3/2)$ describes spin 3/2 with mass m and spins 3/2 and 1/2 with mass $3m$. In the Hamiltonian case the latter solutions have mass $m/3$.

Of course, one is ready to ask if the two generalizations, treated above, are both physically important? As we have seen there are similar states in both realizations, but also states which are not physically equivalent. As we have stated in section 4, the solutions of the covariant realization have direct physical meaning and transform as unitary irreducible representations of the Poincaré group (Loide *et al* 1993). Similarly it is possible to built the Poincaré basis for the Hamiltonian equations finding the corresponding covariant form. Of course, the covariant equation corresponding to the Hamiltonian equation is of higher order. Therefore, treating the covariant and Hamiltonian forms one must use different Poincaré bases. The main reason why the covariant and Hamiltonian forms are different, is that in the $n \neq 1$ case $(\beta^0)^2 \neq I$, and consequently, equation (5.4) does not follow from equation (2.3). As we have already mentioned the drawback of Hamiltonian realization is the presence of unphysical solutions in the integer spin case, which should be eliminated using some sort of additional conditions.

The Hamiltonian form (5.4) is useful when treating various interactions, since one may exploit potentials known from quantum mechanics and solve the corresponding eigenvalue problems to see the physics behind the spin-1/2 case. However, the analysis given above demonstrates that dealing with the results obtained for the higher-spin case one must be very careful to separate the results which correspond to the true physical states from the false, nonphysical solutions.

To clarify the point we exploit the results for a free particle with arbitrary spin in a magnetic field given in Moshinsky and Smirnov (1996). The potential used is as follows:

$$V(B) = \frac{e}{2}\vec{r} \times \vec{B} \tag{5.12}$$

where $\vec{B}$ is some external magnetic field.

In the spin-1 case, which corresponds to the $SO(1, 4)$ representation $(1, 1)$, the following equations for energies were given by Moshinsky and Smirnov (1996):

$$E^2(E^2 - 1 - 2\omega^2(2\mu + 1)) = 0 \tag{5.13}$$

$$E^6 - 2E^4[1 + 2\omega^2(2\mu + 1)] + E^2[1 + 2\omega^2(2\mu + 1)^2] - 4\omega^2 = 0 \tag{5.14}$$

where μ is a projection of total momentum and $\omega = (eB/2)^{1/2}$. In addition, m is chosen to be 1. Similarly, as in ordinary quantum mechanics, the solutions with different spin projection

σ separate. Solutions (5.13) correspond to $\sigma = 0$ and (5.14) to $\sigma = \pm 1$. In the limit $\omega \to 0$ from (5.13) we get

$$E^2 = 0 \qquad \text{and} \qquad E^2 = 1 \tag{5.15}$$

and from (5.14)

$$E^2 = 0 \qquad \text{and} \qquad E^2 = 1 \qquad E^2 = 1. \tag{5.16}$$

From (5.15) and (5.16) we see that we have normal solutions for spin 1, and four unphysical solutions $E = 0$ for spin 1 and spin 0. However, in the general magnetic field case (5.14) may lead to a situation where the solutions $E = 0$ may be lost and there may appear complex energy values, since (5.14), as the cubic equation in E^2 may have one to three real solutions depending on the choice of ω and μ.

To conclude this section we give some remarks on the group structure of the equations investigated above. We have stressed the connection with the de Sitter group, because the Lorentz generators and matrices β^μ form a closed algebra, which in our metric corresponds to $SO(1,5)$. Therefore the covariant form, we analysed here, gives a specific set of well known Bhabha equations. In addition, the Hamiltonian form was expressed and analysed with the help of $SO(1,5)$ generators. In Moshinsky and del Sol Mesa (1996), Moshinsky and Smirnov (1996) and Moshinsky *et al* (1998), equation (5.4) is treated from a viewpoint of the $SU(4)$ group, which is reduced to the direct product $SU(2)\otimes SU(2)$, associated with the ordinary and so-called sign spin. The $SU(4)$ structure is specific to the given realization of matrices (2.5) or (5.5) and is useful when one treats the interactions which connect different irreducible representations of the $SO(1,5)$ group. Interactions, treated so far do not interlock $SO(1,5)$ representations and in that case it is simpler to treat them separately. Moreover, using the $SO(1,5)$ structure, one may analyse the covariant and Hamiltonian equations corresponding to the arbitrary representation of the de Sitter group and not limit oneself with the direct products of Dirac bispinors. On the other hand, one may always use representations of larger groups, if necessary. In our previous analysis the $SO(1,5)$ structure was more transparent.

6. Conclusions

In this paper we have examined the two possible generalizations of the well known Dirac theory, the first one being the covariant generalization

$$(p_\mu \beta^\mu - m)\psi = 0$$

where the β-matrices were the following:

$$\beta^\mu = \frac{1}{n}\sum_{i=1}^{n} \gamma_i^\mu.$$

The second one was the Hamiltonian generalization

$$p_0\psi = (\vec{p} \times \vec{A} + mB)\psi$$

where

$$\vec{A} = \frac{1}{n}\sum_{i=1}^{n} \gamma_i^0 \vec{\gamma}_i \qquad B = \frac{1}{n}\sum_{i=1}^{n} \gamma_i^0.$$

Matrices $S^{\mu 5} = \frac{n}{2}\beta^\mu$ and $S^{\mu\nu}$ from the closed Lie algebra of the de Sitter group $SO(1,5)$ and the covariant and Hamiltonian equations may be rewritten as

$$\left(p_\mu S^{\mu 5} - \frac{nm}{2}\right)\psi = 0 \qquad \text{and} \qquad p^0\psi = \frac{2}{n}(p_k S^{0k} + mS^{05})\psi.$$

Therefore both realizations are most naturally analysed from the viewpoint of irreducible representations of $SO(1,4)$. Of course, the field ψ with n bispinor indices is specific and one may similarly treat other fields corresponding to arbitrary representations of $SO(1,4)$. As we have demonstrated above, the covariant and Hamiltonian realizations are different, corresponding to different Poincaré bases and leading to different physics when interactions are considered.

It should be mentioned that the de Sitter group must be considered as an important auxiliary group which provides an unified approach to some classes of relativistic field equations. In some sense the de Sitter group is the group most closely related with the Poincaré group. As it was demonstrated by Inonü and Wigner (1953), the contraction of the de Sitter group is the Poincaré group. In the theory of group deformations, on the other hand, it is demonstrated that the deformation of the Poincaré group is the de Sitter group (Levy-Nahas 1967).

In the specific case treated here, the field with n bispinor indices is also the representation of the larger group $SU(4)$ and the problem of an arbitrary spin relativistic equation can be treated through a subgroup of $SU(4)$ which is the unitary symplectic group $Sp(4)$ or, equivalently $O(5)$, as was performed by Moshinsky *et al* (1998).

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