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# Graph Theory-Exercise Set 2

1. Which graphs have a diameter equal to 1?

Proof: All complete graphs have a diameter equal to 1 because every pair of vertices are adjacent.

2. If every degree vertex of a simple graph is greater than one, the graph $G$ contains a cycle.

Proof: Suppose that the graph doesn't contain a cycle. Consider one path $P = (v_0, v_1, ..., v_k)$ from $G$ with the maximum length. We know that the vertex $v_k$ is incident with at least one edge $e$ that hasn't been used from the path. If this edge $e$ is also incident with one vertex from path $P$, then $G$ has a cycle (which is a contradiction). If $e$ is incident with one vertex that doesn't belong to path $P$, then $P$ isn't the path with the maximum length, which again is a contradiction. So $G$ has a cycle.

3. Find all $n$ such that $K_n$ is Eulerian.

Proof: To be $K_n$ Eulerian, $n$ must be odd because if it isn't, every vertex degree is odd and can't be Eulerian. If n is odd, then every vertex degree is even and so contains an Eulerian circuit and $K_n$ is Eulerian.

4. Let $G$ be a simple graph. Prove that:
$i$) If $G$ is Eulerian, then $L(G)$ is Eulerian.
$ii$) If $L(G)$ is Eulerian, then we can't conclude that $G$ is Eulerian.

Proof: $i$) We know that each vertex degree of $G$ is even number. From the definition of line graph we can deduce that for edge $e(v, u) \in E(G)$ that becomes a vertex in $L(G)$, it has degree $deg(v) - 1 + deg(u) - 1$ that it is also an even number. So $L(G)$ is Eulerian.
$ii$) If $L(G)$ is Eulerian this means that each vertex degree of $L(G)$ is even number. This vertex will become an edge $e(v, u)$ on $G$ with $deg(v) + deg(u) = even$. In this case, we can't deduce whether $deg(v)$ and $deg(u)$ are even or not. This completes the proof.

6. Find a Hamiltonian graph that its closure is not complete.

Proof:

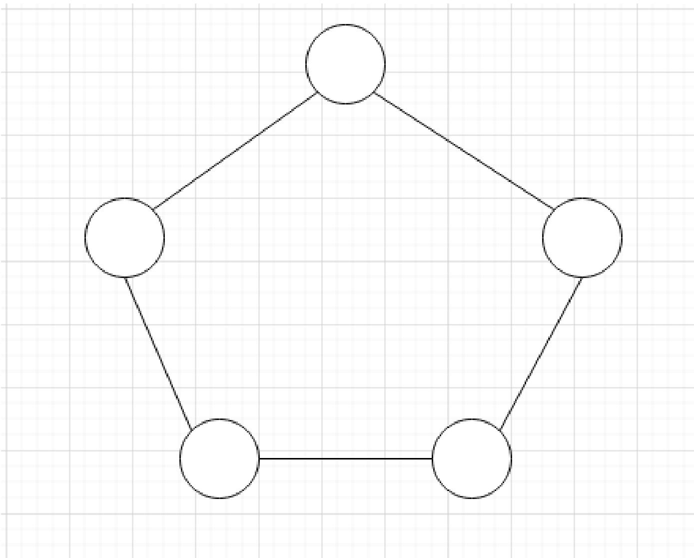

Figure 1: Hamiltonian graph that its closure is a complete graph

7. If simple graph $G$ is Eulerian, then $L(G)$ is Hamiltonian.

Proof: Because $G$ is Eulerian this means that exists an Eulerian circuit. Let the edges of this circuit be $(e_1, e_2, ..., e_n)$. Of course $e_1$ and $e_n$ has a common vertex $v_n \in V(G)$. We can easily construct a Hamiltonian cycle on $L(G)$ by replacing two incident edges with the common vertex. This means that the cycle $(v_{12}, v_{23}, ..., v_{n-1n})$ is a Hamiltonian cycle.

9. Give an $O(|V| + |E|)$ algorithm that takes as input a graph $G = (V, E)$ and one edge $e$ and checks if this edge is bridge.

Proof: Create a graph without this edge and run DFS algorithm counting the vertices that are visited (let $x$). If $x = |V|$ then e is not bridge, else is bridge.