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Graph Theory-Exercise Set 1

1. Prove that at least two vertices have the same degree in any simple graph with $n(vertices) > 2$.

Proof: Let $G(V, E)$ be a simple graph that for every pair of vertices v_1 and v_2 we have that $d(v1) \neq d(v2)$ where $d(v)$ is the degree of vertex v. Then, each vertex has a degree from the set $\{0, 1, 2, ..., n-1\}$. This means that there is a vertex with degree 0 and vertex with degree n - 1. But this is a contradiction.

2. Let H be subgraph of G. Which of the following statements is correct?
$i) d(G) \geqslant d(H)$
$ii) D(G) \geqslant D(H)$

Proof: For the $i)$ it is easy to find a counterexample. Let G with $d(G) = 1$ and there are only one vertex v_1 with this degree. We can delete this vertex and the new subgraph H will have $d(H) = 2$.

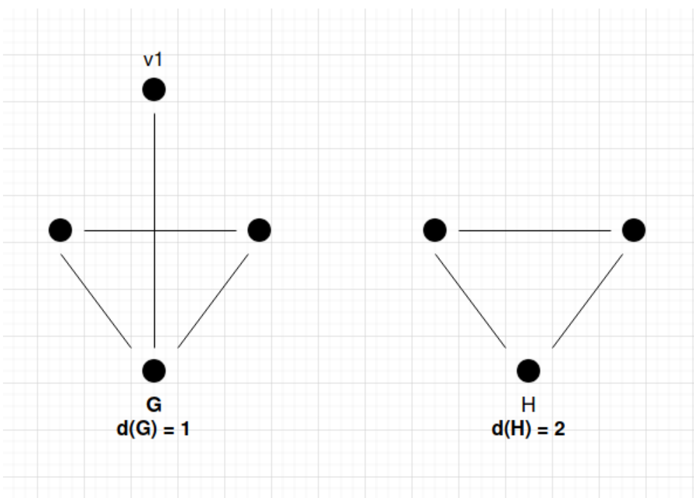

Figure 1: A counterexample

For the $ii)$ assume that exists $G(V, E)$ and its subgraph $H(\hat{V}, \hat{E})$ with $D(G) < D(H)$. This means that H contains an edge that G not. But this is a contradiction because $\hat{E} \subseteq E$.

3. Prove that doesn't exist a 2k-regular graph with bridge.

Proof: Assume that exists a 2k-regular graph with bridge. This means that each vertex has an even degree. If we delete the bridge edge, then we will have two components. Each of these has exactly one vertex with odd degree and with the rest having even degree. This means that the sum of degrees of all vertices for each component is odd number. This is a contradiction because we know that the sum of degrees is an even number.

4. Prove that for every finite graph G, exists $m > 0$ with $G^{m+1} = G^m$.

Proof: Let diameter(d) be the maximum distance of G between any pair of vertices. Let the pair of vertices (v_1, v_2) be that with the maximum distance. Then it is easy to see that G^d is a complete graph because if it isn't, the path from v_1 to v_2 is more than one, so d wasn't the maximum distance. A contradiction. Let $m := d$. Then G^{m+1} is the same complete graph. So $G^m = G^{m+1}$.

5. Prove that for every simple graph G(V,E) with $n = |V|$ and $d(G) \geqslant \frac{n-1}{2}$ is connected.

Proof: Suppose that G is not connected. Then G has at least two components. Each of these have at least one vertex with degree $\frac{n-1}{2}$. Now counting all the vertices of graph (including the two vertices) we see that $|V| = \frac{n-1}{2} + \frac{n-1}{2} + 1 + 1 = n + 1$. A contradiction.

6. Give an efficient algorithm that takes as input a degree sequence and construct a simple graph from that sequence. Run the algorithm with input $(5, 4, 3, 3, 2, 2, 1)$.

Proof: For the algorithm we have used the fact that obtained from Havel-Hakimi theorem, that from a degree sequence, a v_i vertex connects with the next d_i vertices.

Algorithm 1 Construct graph from a degree sequence

Input: A degree sequence $s = (d_1, d_2, ..., d_k)$ with $d_1 \geqslant d_2 ... \geqslant d_k, k \geqslant 1$
Output: A graph $G(V, E)$ from sequence s

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vertices[] := List that contains the vertices v_i for each s_i = d_i, 1 ⩽ i ⩽ k
e[] := empty                                  ▷ Contains pairs with edges
while all elements from s is non-zero do
    if s contains negative number then
        print "Invalid degree sequence"
        exit
    end if
    i := 1
    for j := 1 to d_i do
        e.push(create_edge(vertices[i], vertices[i + j]))
        s[i + j] = s[i + j] − 1
    end for
    s[i] = 0
    i := i + 1
    sort(s, vertices, i)    ▷ Sort s with decreasing order from i to k and simultaneously the
vertices list with the same order
end while
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The following figure shows what lists s and $vertices$ contain on each loop for the input $(5, 4, 3, 3, 2, 2, 1)$.

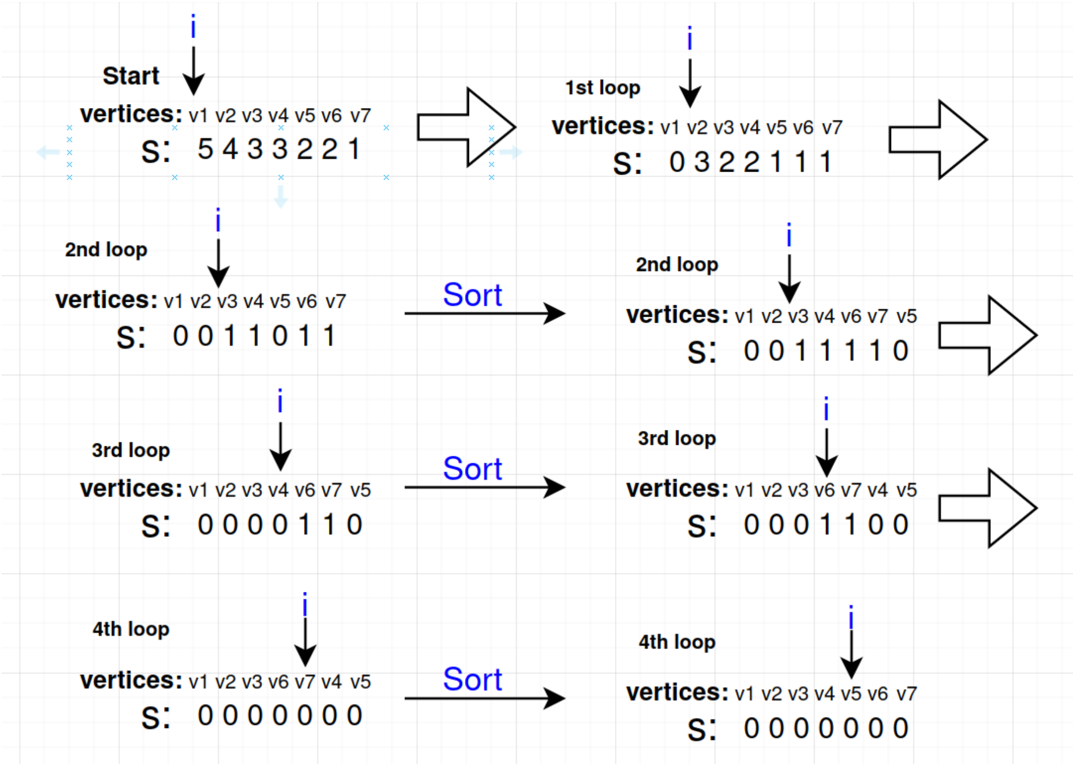

Figure 2: vertices and s for input (5,4,3,3,2,2,1)

7. Prove that for every graph G, this graph or its complement $\tilde{G}$ is connected.

Proof: Suppose that G is disconnected. Let two vertices $u, v \in V(G)$. Then these vertices will belong to the same component or not. If they belong to different components, then u and v will be adjacent in $\tilde{G}$, so there exists a path from u to v. On the other hand, if u and v belong to the same component, then exists a path on G from u to v that doesn't exist on $\tilde{G}$. But in this case, exists a vertex $w \in V(G)$ that belongs to a different component on G. So w will be adjacent to u and v on $\tilde{G}$ so we can construct a path (u, w, v). Therefore $\tilde{G}$ is connected.

8. Show that for every graph G exists path with length $d(G)$.

Proof: Let the path $(v_1, v_2, ..., v_d)$ with length equals the diameter d, of graph G (maximum length). Then all the neighbors of v_d lie on this path, because if they don't, then this path is not this with the maximum length (a contradiction). So, $d \geqslant d(v_d) \geqslant d(G)$. This completes the proof.