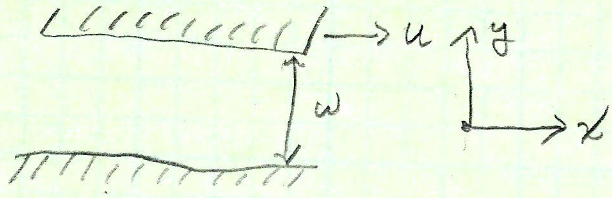
July 4TH 2020



$\mu_{water} = 0.9\ mPa \cdot s$ @ $25^{\circ}C$

$\nu_{water} = 0.8926\ \frac{mm^2}{s}$

$\vec{\nabla} \cdot \vec{v} = 0$ $\qquad \rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p + \rho \vec{g} + \mu \nabla^2 \vec{v}$

assume that for $t<0$, the fluid is at rest $\Rightarrow \vec{\nabla} P = 0$

assume that body forces are ignorable $\rho \vec{g} = 0$

at $t=0$ the top plate starts in motion with velocity $\vec{u} = u\hat{x}$

assume that the flow is only in the x direction and is a function of t & y only.

$$\vec{v} = \tilde{v}_x(t,y)\hat{x}$$

$$\frac{D\vec{v}}{Dt} = \partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v}$$

$$\vec{v} \cdot \vec{\nabla} = \tilde{v}_x \partial_x + v_y \partial_y + v_z \partial_z = v_x \partial_x$$

but since $\tilde{v}_x \neq \tilde{v}_x(x) \Rightarrow (\vec{v} \cdot \vec{\nabla})\vec{v} = 0$

$$\Rightarrow \boxed{\rho \partial_t \tilde{v}_x = \mu \partial_y^2 v_x}$$

BC's: $\begin{cases} v_x(t,0) = 0 \\ v_x(t,w) = u \end{cases}$

IC's: $v_x(0,y) = 0$

Now look at the steady solution : $\frac{u}{w} y$

$$v_x(t,y) = \frac{u}{w} y + \tilde{v}_x(t,y)$$

BC's on $\tilde{v}_x(t,y)$ $\begin{cases} \tilde{v}_x(t,0) = 0 \\ \tilde{v}_x(t,w) = 0 \end{cases}$

IC's : $\tilde{v}_x(0,y) = -u/w\, y$ $\leftarrow$ Key point

$$\tilde{v}_x(t,y) = f(t)\, g(y)$$

$$\frac{1}{\nu}\frac{1}{f}\partial_t f = -s^2 = \frac{1}{g}\partial_y^2 g$$

Start w/ spatial equation

$$g'' + s^2 g = 0 \qquad g(y) = A\sin(sy) + B\cos(sy)$$

$g(0) = 0 \Rightarrow B = 0$

$g(w) = 0 \Rightarrow A\sin(sw) = 0 \qquad sw = n\pi$

$$\boxed{g(y) = A\sin\left(\frac{n\pi}{w}y\right)} \qquad s = \frac{n\pi}{w} \quad n = 1, 2, 3, \ldots$$

$$\dot{f} = -\nu s f = -\nu\left(\frac{n\pi}{w}\right)^2 f$$

$$f(t) = \exp\left[-\left(\frac{n^2\pi^2}{w^2}\right)\nu t\right]$$

$$\tilde{v}_x(t,y) = \sum_{n=1}^{\infty} A_n \exp\left[-\left(\frac{n^2\pi^2}{w^2}\right)\nu t\right]\sin\left(\frac{n\pi}{w}y\right)$$

$$\tilde{V}_x(0,y) = \sum_n A_n \sin\left(\frac{n\pi}{w}y\right) = -\frac{u}{w}y$$

$$\sum_n \int_0^w \sin\left(\frac{m\pi}{w}y\right) A_n \sin\left(\frac{n\pi}{w}y\right)dy = \int_0^w -\frac{u}{w}y\sin\left(\frac{m\pi}{w}y\right)dy$$

since $\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$$\int_0^w \sin\left(\frac{m\pi}{w}y\right)\sin\left(\frac{n\pi}{w}y\right)dy = \int_0^w \frac{1}{2}\left[\cos\left(\frac{m-n}{w}\pi y\right) - \cos\left(\frac{m+n}{w}\pi y\right)\right]dy$$

- the second is simple as $m+n > 0 \Rightarrow$ it is zero since it integrates to $\sin((m+n)\pi) - \sin(0) = 0$.
- The first is a little more subtle, if $m \neq n$ then the above argument applies. But if $m = n$, then

$$\int_0^w \frac{1}{2}dy = \frac{w}{2}$$

$$\sum_n \int_0^{\infty} A_n \sin\left(\frac{m\pi}{w}y\right)\sin\left(\frac{n\pi}{w}y\right)dy = \sum_n A_n \delta_{mn}\frac{w}{2} = A_m\frac{w}{2}$$

$$A_m = -\frac{2}{w}\int_0^w u\frac{y}{w}\sin\left(\frac{n\pi y}{w}\right)dy$$

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$$A_m = -\frac{2}{w}\int_0^w u\,\frac{y}{w}\sin\left(\frac{n\pi y}{w}\right)dy$$

$$q = \frac{n\pi y}{w} \qquad q(y=0)=0 \qquad q(y=w)=n\pi \qquad dq = \frac{n\pi}{w}dy$$

$$A_m = -\frac{2}{w}\,\frac{uw}{n\pi}\int_0^{n\pi}\frac{q}{n\pi}\sin(q)\,dq$$

$$= \frac{-2u}{(n\pi)^2}\int_0^{n\pi} q\sin(q)\,dq$$

$$= \frac{-2u}{(n\pi)^2}\left[-y\cos y + \sin y\right]\Big|_0^{n\pi}$$

$$= \frac{2u}{n\pi}(\cos n\pi)$$

$$= \frac{2u(-1)^n}{n\pi}$$

- Now suppose, the upper-boundary oscillates with velocity $U\cos(\omega t)\,\hat{x}$

$$V_x = f(y)e^{i\omega t} \Rightarrow \underbrace{v_x = Ue^{-ky}\cos(ky-\omega t)}_{\text{can't work}} \Big\}\ \text{? Real part}$$

$$k = \left(\frac{\omega}{2\nu}\right)^{1/2}$$

$$\partial_t v_x = \nu\partial_y^2 v_x \qquad v_x = V_x(t,y)$$

$$\partial_t v_x = i\omega e^{i\omega t}f(y) \qquad \nu\partial^2_y v_x = \nu f'' e^{i\omega t}$$

$$f'' - i\omega/\nu\, f = 0 \qquad f = e^{\lambda y}$$

$$\left(\lambda^2 - i\frac{\omega}{\nu}\right)=0 \qquad \lambda = \pm\left(i\frac{\omega}{\nu}\right)^{1/2} = \pm(\lambda_r + i\lambda_i)$$

$$f(y) = c_1 e^{-\lambda y} + c_2 e^{\lambda y} \qquad \text{set } c_1 = c_2$$

$$V_x = \left\{e^{-(\lambda_R + i\lambda_I)y} + e^{(\lambda_R + i\lambda_I)y}\right\}e^{i\omega t}$$

$$= e^{-\lambda_R y}\left[e^{i(\omega t - \lambda_I y)} + e\right.$$

finite for $y \to \infty$

$$\lambda = \pm e^{i\pi/4}\sqrt{\frac{\omega}{\nu}} = \pm\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]\sqrt{\frac{\omega}{\nu}}$$

$$= \pm\left[1+i\right]\sqrt{\frac{\omega}{2\nu}} \equiv \pm\left[1+i\right]k$$

$c_2 = 0$

$$V_x = Ue^{-[1+i]ky}e^{i\omega t} = Ue^{-ky}e^{i(\omega t - ky)}$$

Re

$c_1 = U$

Viscometer

cylindrical coords

$\partial_\theta \hat{e}_r = \hat{e}_\theta \quad \partial_\theta \hat{e}_\theta = -\hat{e}_r$

$\vec{\nabla} = \frac{1}{r}\partial_r(r\ ) + \frac{1}{r}\partial_\theta + \partial_z$

$\nabla^2 = \frac{1}{r}\partial_r\left(r\frac{1}{r}\partial_r(r)\right) + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2$

$(\vec{v}\cdot\vec{\nabla}) = v_r\partial_r + \frac{v_\theta}{r}\partial_\theta + v_z\partial_z = v_\theta\partial_\theta$

$(\vec{v}\cdot\vec{\nabla})(v_\theta\hat{e}_\theta) = \left(\frac{v_\theta}{r}\partial_\theta v_\theta\right)\hat{e}_\theta - \frac{v_\theta^2}{r}\hat{e}_r$

$\nabla^2\vec{v} = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)(v_\theta\hat{e}_\theta)$

$= \left(\partial_r^2 v_\theta + \frac{1}{r}\partial_r v_\theta + \frac{1}{r^2}\partial_\theta^2 v_\theta\right)\hat{e}_\theta$

$- \frac{v_\theta}{r^2}\hat{e}_\theta$

$\rho\frac{D\vec{v}}{Dt} = \mu\nabla^2\vec{v} +$

$v_\theta(r_1) = Ar_1 + \frac{B}{r_1}$

$\frac{v_\theta(r_1)}{r_1} = \Omega_1$

$A + \frac{B}{r_1^2} = \Omega_1$

$\frac{v_\theta(r_2)}{r_2} = A + \frac{B}{r_2^2} = \Omega_2$

$\Omega_2 - \Omega_1 = B\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)$ ✓

$v_\theta = Ar + B/r$

$s = r\theta$

$\boxed{v = r\,\Omega}$

$\Omega = \frac{v}{r}$

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$$\boxed{\begin{aligned}\Omega_1 &= v_\theta(r_1)\,r_1 = Ar_1^2 + B\\ \Omega_2 &= v_\theta(r_2)\,r_2 = Ar_2^2 + B\end{aligned}}$$

$B = \Omega_1 - Ar_1^2 \qquad \Omega_2 = Ar_2^2 + \Omega_1 - Ar_1^2$

$\Omega_2 - \Omega_1 = A(r_2^2 - r_1^2) \qquad A = \frac{\Omega_2 - \Omega_1}{(r_2^2 - r_1^2)}$

$B = \Omega_1 \cdot (r_1^2 - r_2^2) - ($

$r^2\partial_r^2 v_\theta + r\partial_r v_\theta - v_\theta = 0$

$r^2 \qquad rA - Ar$ ✓

$\partial_r\frac{1}{r} = -\frac{1}{r^2} \qquad \partial_r\left(-\frac{1}{r^2}\right) = \frac{2}{r^3}$

$r^2\cdot\frac{2}{r^3} - r\cdot\frac{1}{r^2} - \frac{1}{r}$

$\frac{2}{r} - \frac{1}{r} - \frac{1}{r} = 0$