Last month’s post set the stage for one of the more interesting and important aspects of elastic motion – the propagation of elastic deformation waves in a material. The familiar travelling and standing waves seen on stringed instruments are the prototype example but the richness and complexity is increased when the allowed deformations take place in more dimensions. The complex relationship of the generalized Hooke’s law relating the strain to the stress tensor given by

\[ T\_{ij} = c\_{ijkl} \epsilon\_{kl} \; ,\]

leads to many more propagation modes compared to the one-dimensional textbook case. Since every solid material can support elastic waves, understanding these modes becomes important in the mechanical design of all manner of objects, such as bridges, support columns, cars, planes, skyscrapers, and so on. This analysis also serves as a cornerstone in seismology for understanding one of the most powerful and catastrophic events on the planet, the earthquake.

As discussed in the last post, there are 81 components in the stiffness tensor $$c\_{ijkl}$$ but there are never more than 21 independent components due to various symmetries of the physical relationships connecting the stiffness tensor to the stress and strain tensors. This is a substantial reduction but is still far too complicated for an initial foray into elastic waves so the first step is to look at the waves that arise in an isotropic medium.

To perform this analysis requires three steps. The first step simplifies the generalized Hooke’s law in terms of the elastic moduli discussed in detail in the previous post. The second step adapts Newton’s law for the elastic medium yielding the wave equation. The last step is the decomposition of the wave equation into its independent components.

## Step 1 – Simplifying Generalized Hooke’s Law

To simplify the generalized Hooke’s law for an isotropic material begin by focusing in on the strains that result along the $$x\_1$$-axis as a result of the application of various stresses. Applying a small tensile stress along the $$x\_1$$-axis gives

\[ E \epsilon\_{11} = T\_{11} \; , \]

where $$E$$ is Young’s modulus.

Application of the same small tensile stress along the $$x\_2$$- and $$x\_3$$-axes yields

\[ -\frac{E}{\nu} \epsilon\_{11} = T\_{22} \; \]

and

\[ -\frac{E}{\nu} \epsilon\_{11} = T\_{33} \; , \]

where $$\nu$$ is Poisson’s ratio. Note that no indices are needed on this Poisson’s ratio since the material is isotropic.

Putting these pieces together gives

\[ E \epsilon\_{11} = (1+\nu) T\_{11} - \nu \left(T\_{11} + T\_{22} + T\_{33} \right) = (1+\nu) T\_{11} + \nu Tr({\mathbf T}) \; ,\]

where $$Tr({\mathbf T})$$ is the trace of the stress tensor.

Arfken presents a rigorous argument for generalizing the above equation but it is obvious that it is consistent with the form

\[ E \, \epsilon\_{ij} = (1+\nu) T\_{ij} - \nu Tr({\mathbf T}) \delta\_{ij} \; . \]

Taking the trace of both sides and performing some straightforward algebra gives

\[ T\_{ij} = \left( \frac{E}{1+\nu} \right) \epsilon\_{ij} + \left(\frac{\nu}{1+\nu}\right) \left(\frac{E}{1-2\nu}\right) Tr({\mathbf \epsilon}) \delta\_{ij} \; . \]

It is traditional to recast this equation slightly by defining the Lame coefficients

\[ \lambda = \frac{\nu E}{ (1+\nu)(1-2\nu) } \; ,\]

and

\[ \mu = \frac{E}{2 (1+\nu)} \; .\]

The isotropic stress-strain relationship now takes on the compact form

\[T\_{ij} = 2 \mu \epsilon\_{ij} + \lambda Tr({\mathbf \epsilon}) \delta\_{ij} \; , \]

which is the desired relation.

## Step 2 – Adapting Newton’s Law

The next step is to adapt Newton’s law to a piece of the continuum. The standard way of doing this is to look at a small cube of the continuum whose sides are spanned by the components $$u\_i$$, similar defined as in the earlier post in which the strain tensor is first derived. The resulting equation is then (see <https://www.geophysik.uni-muenchen.de/~igel/Lectures/Sedi/sedi_weq.pdf>)

\[ \rho \partial\_t^2 u\_i = T\_{ij,j} \; .\]

Substituting in for $$T\_{ij}$$ with the relationship from step 1 yields

\[ \rho \partial\_t^2 u\_i = \partial\_j \left( 2 \mu \epsilon\_{ij} + \lambda Tr({\mathbf \epsilon}) \delta\_{ij} \right) \; .\]

By definition of the strain tensor

\[2 \epsilon\_{ij} = \partial\_i u\_j + \partial\_j u\_i \; \]

and

\[ Tr({\mathbf \epsilon}) = \partial\_k u\_k \; . \]

Substituting these definitions in yields

\[ \rho \partial\_t^2 u\_i = \lambda \partial\_i \partial\_k u\_k + \mu \partial\_i \partial\_j u\_j + \mu \partial^2\_j u\_i \; .\]

In vector terms

\[ \rho \partial\_t^2 {\vec u} = (\lambda + \mu) \nabla (\nabla \cdot \vec u) + \mu \nabla^2 \vec u \; . \]

## Step 3 – Decomposing the Wave Equation

As it stands the, wave equation mixes the various independent modes. This observation isn’t at all obvious but it will become apparent with the application of the vector identity

\[ \nabla^2 \vec u = \nabla(\nabla \cdot \vec u) - \nabla \times (\nabla \times \vec u) \; .\]

Substituting this relation into the wave equation and collecting terms gives a new form of the wave equation

\[ \rho \partial\_t^2 \vec u = (\lambda + 2 \mu) \nabla (\nabla \cdot \vec u) - \mu \nabla \times (\nabla \times \vec u) \; .\]

The importance of this form is that there are now two uncoupled modes: the ‘divergence’ mode and the ‘curl’ mode. To see this, take the divergence of both sides and exchange all partial derivatives with abandon. Since the divergence of a curl is identically zero the result is

\[ \rho \partial\_t^2 \nabla \cdot \vec u = (\lambda + 2 \mu) \nabla^2 (\nabla \cdot \vec u) \; .\]

From this equation, we can see that a compression/expansion wave propagates with a speed given by

\[ v\_{P} = \sqrt{ \frac{\lambda + 2 \mu}{\rho} } \; . \]

The ‘P’ label stands for [primary or pressure wave](https://en.wikipedia.org/wiki/P-wave).

Likewise, taking the curl gives

\[ \rho \partial\_t^2 \nabla \times \vec u = \mu \nabla^2 (\nabla \times \vec u) \; .\]

From this equation, we can see that a shear wave propagates with a speed given by

\[ v\_{S} = \sqrt{ \frac{\mu}{\rho} \; . \]

The ‘S’ label stands for [secondary or shear wave](https://en.wikipedia.org/wiki/S-wave).

As the links to Wikipedia show, this simple theory has been used to analyze earthquake propagation.

* Better variation of Arfken:
  + Deformation: - my improvement - measures the extent of deformation (Lie derivative?) with
  + Focus on two points:
    - Before deformation:
    - After deformation:
  + Measure the distance between the points
    - Before deformation:
    - After deformation:
* <http://web.mit.edu/16.20/homepage/2_Strain/Strain_files/module_2_with_solutions.pdf>
  + Find the length difference
    - Define the Green-Lagrange strain tensor:
  + Now lose the idea of by assuming that
    - Can usually drop the non-linear term

This synthesis results in the generalize Hooke’s law

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