These next two posts are a bit of a departure from the thermal physics theme that had been the central point for the last many months. They grew out of some discussions on classical field theory that arose in several venues with different people and it seemed important to capture what is a clean (and perhaps new) argument for the beginning student on the best way to transform differential equations into curvilinear coordinates.

The starting point is the recasting of the Euler equation for an ideal fluid (typically a gas)

\[ \rho \frac{D {\vec V}}{Dt} = -{\vec \nabla p} + {\vec f} \; , \]

where $\rho$ and $p$ are the mass density and pressure of the fluid, ${\vec V}$ is its velocity, $frac{D}{Dt}$ is the material derivative, and ${\vec f}$ is the body force per unit mass.

Typically, within basic discussions of fluid mechanics, Euler’s equation is expressed in Cartesian coordinates (assumed here, without loss of generality to the method, to cover a two dimensional space) where the velocity is given by

\[ {\vec V} = V\_x {\hat x} + V\_y {\hat y} \; ,\]

the material derivative takes on the simple form

\[ \frac{D}{Dt} = V\_x \partial\_x + V\_y \partial\_y + \partial\_t \; ,\]

and Euler’s equation, in component form is

\[ \rho \frac{D}{Dt} V\_x = -\partial\_x p + f\_x \; ,\]

and

\[ \rho \frac{D}{Dt} V\_y = -\partial\_y p + f\_y \; .\]

This relatively, simple form allows the student to focus on the Lagrangian nature of following a fluid flow but it typical hides a subtle complication when using curvilinear (or even rotating coordinates). For example, the corresponding version of Euler’s equations in cylindrical coordinates (see also Acheson’s Appendix A.6) uses

\[ \frac{D}{Dt} = \partial\_t + V\_r \partial\_r + \frac{V\_{\theta}}{r} \partial\_{\theta} \; \]

for the material derivative with the component equations being

\[ \rho \frac{D}{Dt} V\_r - \frac{{V\_{\theta}}^2}{r^2} = - \partial\_r p + f\_r \; \]

and

\[ \rho \frac{D}{Dt} V\_{\theta} + frac{V\_r V\_{\theta}}{r} = -\frac{1}{r} \partial\_{\theta} p + f\_{\theta} \; . \]

Suddenly there are new multiplicative terms (e.g. the $1/r$ multiplying the derivative with respect to the polar angle $\theta$) as well as additive terms on the left-hand side of the component equations (e.g. $-\frac{{V\_{\theta}}^2}{r^2}$) that weren’t there in the Cartesian version. The student is left to wonder about just why they are there.

Many books and lecture notes on the internet try to justify one or the other (but rarely both) with varying degrees of success. The aim of this note is to suggest a simple mantra: the multiplicative terms are strictly the result of minding units and the additive terms are strictly the result of the curvilinear basis vectors changing from point to point.

The strategy behind the mantra is that even if the students don’t fully connect all the dots the first few times, they will have an explanation that is rock solid and easy to remember to guide them in exploring on their own.

Let’s examine each of these claims in turn.

The first claim of the mantra is that the multiplication of the $\partial\_{\theta}$ term by $1/r$ is the result of minding units. Of the two claims of the mantra, this one is the more conceptually difficult even though it is the easier of the two claims to understand mathematically. The conceptual hurdle is rooted in the arguments used to define it in terms of the partial derivatives of a scalar field, $f(x,y,t)$, expressed in Cartesian coordinates

\[ df = \partial\_x f dx + \partial\_y f dy + \partial\_t f dt \; .\]

Dividing by $dt$ immediately gives the Cartesian form the material derivative

\[ \frac{Df}{Dt} = V\_x \partial\_x f + V\_y \partial\_y f + \partial\_t f \; .\]

The student then asks why doesn’t a similar relationship hold for curvilinear coordinates. For example, why isn’t the material derivative in cylindrical coordinates not based on the differential of $g(r,\theta,t)$

\[ dg = \partial\_r g dr + \partial\_\theta g d\theta + \partial\_t g dt \; ?\]

This is a point most often most clearly discussed within the realm of continuum mechanics or general relativity. Schutz, in his book *A First Course in General Relativity*, notes, in Section 5.5, that defining the gradient of $g$ essentially in terms of the differential given above is perfectly acceptable but that the price paid for using it is that the basis vectors that are not normalized, which he summarizes with the equation

\[ {\vec e}\_{\alpha} \cdot {\vec e}\_{\beta} = g\_{\alpha \beta} \neq \delta\_{\alpha \beta} \; .\]

While this is certainly true and quite clearly argued, the beginning student consulting Schutz (or some similar text) as a reference has to know either the definition of the metric or the difference between vectors and differential forms and the natural duality between them. In the first case, they need to know that the metric encodes all of the possible dot products between the basis vectors. In the second, they are confronted with notation that expresses the duality between basis forms and vectors in the coordinate version as

\[ \left<d\theta, \partial\_{\theta} \right> = 1 \; \]

and in the non-coordinate version as

\[ \left< {\tilde \omega}^{\hat \theta}, {\vec e}\_{\hat \theta} \right> = 1 \; .\]

These mathematical distinctions are quite beyond the beginning student who, by definition, is struggling with a host of other things.

A cleaner way of justifying the first point of the mantra is to perform a unit analysis on the differential $dg$. It doesn’t matter what units $g$ possesses but for the sake of this argument lets assume $g$ has units of temperature. The idea of a temperature field is familiar and the units are well known. We will denote the units of a physical quantity by square brackets so that in this case $[g] = T$.

The differential must also have units of temperature which means that the partial derivatives have mixed units. The partial derivative with respect to the radius $r$ has units of temperature per length

\[ \left[ \partial\_r g \right] = T/L \; \]

while the partial derivative with respect to the azimuth $\theta$ has units of temperature

\[ \left[ \partial\_{\theta} g \right] = T \; .\]

Dividing by $dt$ gives a material derivative of the form

\[ \frac{Dg}{Dt} = V\_r \partial\_r g + U\_{\theta} \partial\_{\theta} g + \partial\_t g \; .\]

The units on the radial velocity $V\_r \equiv dr/dt$ are length per unit time as we expect of a conventional derivative but the units on the azimuthal velocity $U\_{\theta} \equiv d\theta/dt$ are radians per unit time, which are quite different (hence the use of the letter $U$ in place of $V$). The next step is to challenge the student to think about how any lab would measure this angular velocity and to then argue that a much better way to link to experiments is to multiply $U\_{\theta}$ by the radius $r$.

Once this step is done, the remaining piece involves rewriting the differential as

\[ dg = dr \partial\_r g + (r d\theta) (\frac{1}{r} \partial\_{\theta} g) + dt \partial\_t g \; , \]

where we’ve multiplied the second term by unity in the form of $r/r$. Dividing by $dt$ immediately gives

\[ \frac{Dg}{Dt} = V\_r \partial\_r g + V\_{\theta} \frac{1}{r} \partial\_{\theta} g + \partial\_t g \; , \]

which is the accepted form of the material derivative.

Next month’s post will cover the second part of the mantra by showing that the additive terms result from how the basis vectors in curvilinear components change from point-to-point in space.