Last month's post discussed Freidberg's heuristic derivation of the Boltzmann equation, which focused on the flow of particles into and out a particular volume in phase space. This month we'll look at how, with the proper perspective, we can derive the conventional equations of fluid mechanics. The starting point for this derivation is the Boltzmann equation, which, when written as last time as,

\[ \frac{\partial f}{\partial t} + {\vec v} \cdot \nabla f + {\vec a} \cdot \nabla\_v f + f \, \nabla\_v \cdot {\vec a} = S \; , \]

describes how the phase space density $f({\vec r},{\vec v},t)$ changes as a function of independent variations in spatial location, ${\vec r}$, velocity, ${\vec v}$, and time, $t$.

The sinks and sources of phase space density, which are represented by $S$, are due to effects such as ionization, recombination, charge exchange, radioactive decay, and so on.

The term $f \, \nabla\_v \cdot {\vec a}$ is analogous to the $\rho \nabla \cdot {\vec v}$ from the Reynolds transport theorem in conventional fluid mechanics and, as a result, we'll refer to this terms as the Reynolds term. It results from carefully considering flux into the velocity portion of the region of phase space. The only reason a $f \, \nabla \cdot {\vec v}$ term is missing from the flows into the spatial portion of phase space is that ${\vec r}$ and ${\vec v}$ are independent degrees-of-freedom.

The terms involving the acceleration are divided into two pieces: those resulting from long-range forces ${\vec a}\_{L}$ and those resulting from forces that act over short range ${\vec a}\_{S}$. The long-range forces are externally applied and/or result from the collective nature of the system. These forces either have no velocity dependence or their dependence enters through the Lorentz force law. In both cases, the Reynolds term for ${\vec a}\_L$ is zero. The short-term forces are assumed to be elastic so that collisions mediated by these forces, typically imagined as Coulomb in nature, conserve particle number, momentum, and energy. The Reynolds term for the short range accelerations is usually rewritten in terms of the Boltzmann collision operator as

\[ - f \, \nabla\_v \cdot {\vec a}\_S \equiv \left. \frac{\delta f}{\delta t} \right|\_c \;. \]

Boltzmann's brilliance is most apparent in his analysis of this collisional term. Sadly, we don't have the space to delve into this deeply. We have to content ourselves with the fact that, due to conservation/symmetry considerations, the collisional term will behave in a very convenient way when certain averaging operations, defined below, are performed. In order to recover the equations of hydrodynamics we need to make only a few minor assumptions. Since we are seeking conventional fluid flow, we can set the sources and sinks to zero and assume that the external force is solely due to gravity so that ${\vec a}\_L = {\vec g}/m$.

Following [David Weinberg's lecture](https://www.astronomy.ohio-state.edu/weinberg.21/A825/notes2.pdf), we define an averaging process for a physical quantity $Q$ as

\[ \langle Q \rangle ({\vec r},t) = \frac{ \int d^3v \, Q \, f({\vec r},{\vec v},t) }{\int d^3v \, f({\vec r},{\vec v},t) } \; . \]

Defining $n({\vec r},t) = \int d^3v \, f({\vec r},{\vec v},t)$, we can express this average as

\[ n \langle Q \rangle = \int d^3v \, Q \, f({\vec r},{\vec v},t) \; .\]

Next, we integrate each term in the Boltzmann equation over the entire velocity space. We find that we need to deal with the three derivatives of the phase space density, $\partial\_t$, ${\vec v} \cdot \nabla$, and ${\vec a} \cdot \nabla\_v$. Each on of these is done via the product rule.

For the time derivative

\[ \int d^3v \, Q \, \partial\_t f = \int d^3v \, \partial\_t ( Q f ) - \int d^3v f \, \partial Q \; . \]

The time derivative can be brought outside the first integral and then we can use the average relation to arrive at

\[ \int d^3v \, Q \, \partial\_t f = \partial\_t (n \langle Q \rangle ) - n \langle \partial\_t Q \rangle \; .\]

For the spatial derivative

\[ \int d^3v \, Q \, {\vec v} \cdot \nabla f = \int d^3v \, {\vec v} \cdot \nabla ( Q f ) - \int d^3v \, f {\vec v} \cdot \nabla Q \; , \]

where we used $\partial v\_i / \partial r\_j = 0$, since the velocity and position are independent variables.

The spatial derivative can be brought outside the first integral and then we can use the average relation to arrive at

\[ \int d^3v \, Q {\vec v} \cdot \nabla f = \nabla \cdot (n \langle {\vec v} Q \rangle ) - n \langle {\vec v} \cdot \nabla Q \rangle \; . \]

For those terms involving the velocity derivative

\[ \int d^3v \, Q \, {\vec g} \cdot \nabla\_v f = \int d^3v \nabla\_v \cdot ( Q {\vec g} f ) - \int d^3v f {\vec g} \cdot \nabla\_v Q \; .\]

The first integral simplifies using the divergence theorem

\[ \int d^3v \, \nabla\_v \cdot ( Q {\vec g} f ) = \left. \left( Q {\vec g} \cdot {\hat n} f \right) \right|\_{boundary} = 0 \; , \]

where the last equality follows from the fact that a physically reasonable $f$ must go to zero at infinite speed. Finally, using the average relation, yields

\[ \int d^3v \, Q \, {\vec g} \cdot \nabla\_v f = - n {\vec g} \cdot \langle \nabla\_v Q \rangle \; .\]

Putting it all together gives what we'll call the reduced Boltzmann equation

\[ \partial\_t (n \langle Q \rangle ) - n \langle \partial\_t Q \rangle + \nabla \cdot (n \langle {\vec v} Q \rangle ) - n \langle {\vec v} \cdot \nabla Q \rangle - n {\vec g} \cdot \langle \nabla\_v Q \rangle = \int d^3v \, Q \left. \frac{\delta f}{\delta t} \right|\_c \; . \]

The final steps involve setting $Q$ to be $m$, $m {\vec v}$, and $1/2 m v^2$ to get the classical fluid equations for mass continuity, Cauchy momentum, and energy. We’ll save those steps for next month’s installment.