Last month’s posting derived the the reduced Boltzmann equation

\[ \partial\_t (n \langle Q \rangle ) - n \langle \partial\_t Q \rangle + \nabla \cdot (n \langle {\vec v} Q \rangle ) - n \langle {\vec v} \cdot \nabla Q \rangle - n {\vec g} \cdot \langle \nabla\_v Q \rangle = \int d^3v \, Q \left. \frac{\delta f}{\delta t} \right|\_c \; \]

by integrating the Boltzmann equation over all velocity space. In this month’s column, we’ll assign the physical quantity $Q$ various roles ($m$, $m {\vec v}$, and $1/2 m v^2$) to get the classical fluid equations for mass continuity, Cauchy momentum, and energy.

Setting $Q = m$ and recognizing that $\rho({\vec r},t) = m n({\vec r},t)$, $\langle m \rangle = m$ and $\langle {\vec v} \rangle = {\vec u}({\vec r},t)$ yields

\[ \partial\_t \rho + \nabla \cdot (\rho \, {\vec u} ) = 0 \; , \]

which is the equation of continuity. It is important to recognize two things. First, the spatial dependence in this equation is carried entirely by the phase space density and that it is the averaging process that imparts that dependence to the first moment of the velocity ${\vec u}$. Second, the average of the collisional term on the right-hand side is zero due to the fact that the number and type of each particle is preserved.

Next we set $Q = m {\vec v}$ and employ the same set of rules as before to get

\[ \partial\_t (\rho \, {\vec U}) + \nabla \cdot (\rho \langle {\vec v}\,{\vec v} \rangle ) - \rho {\vec g} = 0 \; .\]

Next, we interpret the term $\langle {\vec v} {\vec v} \rangle$ by first defining

\[ {\vec v} = \delta {\vec v} - {\vec U} \; \]

to get that

\[ \langle {\vec v} {\vec v} \rangle = \langle ( \delta {\vec v} - {\vec U})(\delta {\vec v} - {\vec U}) \rangle = \langle \delta {\vec v} \delta {\vec v} \rangle - 2 {\vec U} \langle \delta {\vec v} \rangle + {\vec U }{\vec U} \; . \]

If we assume that the deviations from the bulk velocity average to zero (which is a far more subtle point than Weinberg conveys in his write-up) then

\[ \langle {\vec v} {\vec v} \rangle = \langle \delta {\vec v} \delta {\vec v} \rangle + {\vec U }{\vec U} \; . \]

The first term, which is the variance in the velocity distribution, captures the deviation in the marginal distribution left after the integral over all velocities from the bulk fluid speed. Since it is the second moment of the velocity distribution, it can be expressed in terms of the second central moment and the variance

\[ \langle {\vec v} {\vec v} \rangle = \sigma\_v^2 + \langle {\vec v} \rangle ^2 = \sigma\_v^2 + {\vec U}{\vec U} \; . \]

The physical meaning of $\sigma\_v$ is that it is the deviation of the local velocity value at a point in phase space (i.e., a specific value of ${\vec r}, {\vec v}, t$) from the bulk velocity, which suggests the definition

\[ {\vec v} = {\vec w} - {\vec U} \; . \]

If we assume that the variations in phase space density are largely uncorrelated in its velocity dependence, then the expectation value of ${\vec w}$ should be zero leaving us with

\[ \langle {\vec v} {\vec v} \rangle = \sigma\_v^2 + \langle {\vec v} \rangle ^2 = \sigma\_v^2 + {\vec U}{\vec U} \; . \]

This same assumption suggests that the tensor $\rho \langle {\vec w} {\vec w} \rangle$ can be decomposed into a diagonal piece and a symmetric traceless piece as

\[\rho \langle {\vec w} {\vec w} \rangle = P \stackrel{\leftrightarrow}{1} - \stackrel{\leftrightarrow}{\pi} \; ,\]

where $P$ is the scalar pressure ($Tr$ is the trace)

\[ P = \frac{1}{3} Tr \left( \langle {\vec w} {\vec w} \rangle \right) \; \]

and $\pi$ is the viscous stress tensor

\[ \pi\_{ij} = P \delta\_{ij} - \langle w\_i w\_j \rangle \; .\]

The form of the reduce Boltzmann equation is now

\[ \partial\_t ( \rho {\vec U} ) + \nabla \cdot (\rho {\vec U} {\vec U} ) = - \nabla \cdot( P \stackrel{\leftrightarrow}{1}) + \rho {\vec g} + \nabla \cdot \stackrel{\leftrightarrow}{\pi} \; , \]

which is the usual (up to variations of notation) form of the momentum equation in fluid mechanics (see e.g. https://underthehood.blogwyrm.com/?p=1250). Note that we assumed due to conservation/symmetry considerations that $\int d^3v \, m {\vec v} \, (\delta f/\delta t |\_c )= 0$.

Setting $Q = (1/2) m v^2$ yields

\[ \partial\_t (1/2 \rho \langle v^2 \rangle ) + \nabla \cdot (1/2 \rho \langle {\vec v} v^2 \rangle ) - 1/2 \rho {\vec g} \cdot \langle \nabla\_v v^2 \rangle = 0 \; . \]