There is a surprisingly frequent need to compute the quantity $O = 1 + G$ where both $O$ and $G$ are operators. These operators can be algebraic, often in the form of a matrix, or a derivative operator, and can be either finite- or infinite-dimensional. One typical example consists of constructing the local representation of a Lie group operator 'near' the identity - a construction that shows up most everywhere in quantum mechanics and in general relativity.

The interesting next step is to determine what is the inverse operator $O^{-1}$ in terms of $G$ such that

\[ O^{-1} O = 1 \; , \]

which translates to

\[ (1+G)^{-1} (1+G) = 1 \; .\]

To proceed, we can `decorated' the right hand side with a infinite string of suggestive zeros

\[ (1+G)^{-1} (1 + G) = 1 + \underbrace{(G - G)}\_{\text{=0}} + \underbrace{(G^2 - G^2)}\_{\text{=0}} + \cdots \; .\]

Next we indulge in some inspired rearrangement of terms to get

\[ (1+G)^{-1} (1 + G) = \left[ (1 + G) - (G + G^2) + (G^2 + G^3) - (G^3 + G^4) \cdots \right] \; . \]

We can now factor $(1+G)$ out to the right to get

\[ (1+G)^{-1} (1+G) = \left[ 1 - G + G^2 - G^3 + \cdots \right] (1 + G) \; .\]

Comparing the two sides we conclude that the quantity in the brackets must then be a series representation of the inverse

\[ (1+G)^{-1} = 1 - G + G^2 - G^3 + \cdots \; .\]

If the construction is correct, then the expansion must also serve as a right inverse. That is to say,

\[ (1 + G) (1 + G)^{-1} = 1 \; .\]

Substituting the form determined above yields

\[ (1 + G) (1 - G + G^2 - G^3 + \cdots) = 1 + G - G - G^2 + G^2 + G^3 - G^3 - G^4 + \cdots \; . \]

It is interesting to note that this operator series is formally similar the Taylor series expansion for the geometric series:

\[ \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \; .\]

Both series have limited domains of applicability. For the scalar function just expanded, it is well-known that the condition on $x$ for the expansion to be within the radius of convergence is $|x| < 1$, this latter the condition being the domain of applicability.

For the operator version, the corresponding condition isn't nearly as clear and that will be the subject of next month's blog.