- We wish to prove $\int d\hat{\ell} \times \vec{F} = \int (\hat{n} \times \vec{\nabla}) \times \vec{F} \, dS$

Define $\vec{G} = \vec{F} \times \vec{A}$ where $\vec{A}$ is a constant vector and apply Stokes theorem

$$\int d\hat{\ell} \cdot \vec{G} = \int (\vec{\nabla} \times \vec{G}) \cdot \hat{n} \, dS$$

Now the LHS can been rewritten

$$\int d\hat{\ell} \cdot \vec{G} = \int d\hat{\ell} \cdot (\vec{F} \times \vec{A}) = \int \vec{A} \cdot (d\hat{\ell} \times \vec{F})$$

and the RHS can also be rewritten by re-writing the integrand

$$\begin{aligned}
\hat{n} \cdot (\vec{\nabla} \times \vec{G}) &= n_i \varepsilon_{ijk} \partial_j G_k \\
&= n_i \varepsilon_{ijk} \partial_j \varepsilon_{k\ell m} F_\ell A_m \\
&= A_m \varepsilon_{mk\ell} (\varepsilon_{kij} n_i \partial_j) F_\ell \\
&= A_m \varepsilon_{mk\ell} \{\hat{n} \times \vec{\nabla}\}_k F_\ell \\
&= \vec{A} \cdot (\hat{n} \times \vec{\nabla}) \times \vec{F}
\end{aligned}$$

- Equating and re-arranging yields

$$\vec{A} \cdot \left[ \int d\hat{\ell} \times \vec{F} - \int (\hat{n} \times \vec{\nabla}) \times \vec{F} \, dS \right] = 0$$

and since $\vec{A}$ is arbitrary

$$\boxed{\int d\hat{\ell} \times \vec{F} = \int (\hat{n} \times \vec{\nabla}) \times \vec{F} \, dS}$$