Euclidean Geometric Analysis

Basic Geometric Objects

point:	$\mathcal{P}, \mathcal{Q}, \ldots$
bounding points:	$\partial\,\mathcal{C}$
curve:	$\mathcal{C}$
bounding curve:	$\partial\,\mathcal{S}$
surface:	$\mathcal{S}$
bounding surface:	$\partial\,\mathcal{V}$
volume:	$\mathcal{V}$
scalar:	m, n, etc.
indices:	$i, j = 1, 2, \ldots, N$
coordinates:	x^i, $q^{i'}$ $\quad i,\, j' = 1{,}2{,}3$
transformations:	$x^i = x^i\left(q^{j'}\right)$
field:	$\phi = \phi\left(q^i\right)$
parametric curve:	$q^i\,(s)$
implicit surface:	$\phi_{\mathcal{S}}\left(q^i\right) - c = 0$
parametric surface:	$q^i\,(u, v)$
Kronecker delta:	δ_{ij}
permutation symb:	$[i, j, k]$
column array:	$\lvert\mathbf{q}\rangle$
row array:	$\langle\mathbf{q}\rvert$

Defining Vectors

A00:	$\mathbf{A} + \mathbf{B} \in \mathbb{V}$
A00a:	$m\mathbf{A} \equiv \mathbf{A}m \in \mathbb{V}$
A01:	$\mathbf{A} + \mathbf{B} \equiv \mathbf{B} + \mathbf{A}$
A02:	$\mathbf{A} + (\mathbf{B} + \mathbf{C}) \equiv (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
A03:	$1\mathbf{A} = \mathbf{A}$
A04:	$m\,(n\mathbf{A}) \equiv (mn)\,\mathbf{A}$
A05:	$(m + n)\,\mathbf{A} \equiv m\mathbf{A} + m\mathbf{A}$
A06:	$m\,(\mathbf{A} + \mathbf{B}) \equiv m\mathbf{A} + m\mathbf{B}$
A06a:	$\mathbf{A} + \mathbf{0} = \mathbf{A}$
A06b:	$\mathbf{A} + inv\mathbf{A} = \mathbf{0}$
A07:†	$\mathbf{A} \cdot \mathbf{B} \equiv \mathbf{B} \cdot \mathbf{A}$
A08:†	$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) \equiv \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$
A09:†	$m\,(\mathbf{A} \cdot \mathbf{B}) \equiv (m\mathbf{A}) \cdot \mathbf{B}$ $\equiv \mathbf{A} \cdot (m\mathbf{B}) \equiv (\mathbf{A} \cdot \mathbf{B})\, m$
A10:†	$\mathbf{e}_i \cdot \mathbf{e}_j \equiv g_{ij}$
A11:†	$\mathbf{A} \equiv A^i\mathbf{e}_i$ with $A^i \equiv \mathbf{A} \cdot \mathbf{e}_i$
A12:†	$\mathbf{A} \cdot \mathbf{A} \equiv \lvert\mathbf{A}\rvert^2$
A13:†	$\mathbf{A} \cdot \mathbf{B} \equiv \lvert\mathbf{A}\rvert\lvert\mathbf{B}\rvert \cos(\theta)$
A14:‡	$\mathbf{A} \times \mathbf{B} \equiv -\mathbf{B} \times \mathbf{A}$
A15:‡	$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) \equiv \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
A16:‡	$m\,(\mathbf{A} \times \mathbf{B}) \equiv (m\mathbf{A}) \times \mathbf{B}$ $\equiv \mathbf{A} \times (m\mathbf{B}) \equiv (\mathbf{A} \times \mathbf{B})\, m$
A17:‡	$\mathbf{e}_i \times \mathbf{e}_j \equiv [ijk]\mathbf{e}_k$
A18:‡	$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$
A19:† ‡	$\lvert\mathbf{A} \times \mathbf{B}\rvert \equiv \lvert\mathbf{A}\rvert\lvert\mathbf{B}\rvert \sin(\theta)$
A20:† ‡	$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$

Defining Vector Fields

F01:	vector field $\mathbf{F} \equiv \mathbf{F}\left(q^i\right)$
F02:	fundamental field $\mathbf{r} \equiv x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$
F03:†	covariant basis $\mathbf{e}_i \equiv \frac{\partial_{q^i}\mathbf{r}}{\lvert\partial_{q^i}\mathbf{r}\rvert}$
F04:	curve tangent $\mathbf{e}_{\mathcal{C}} \equiv \frac{d\mathbf{r}_{\mathcal{C}}}{ds} \equiv \hat{t}$
F05:	surface tangents $\mathbf{e}_u \equiv \frac{\partial\mathbf{r}_{\mathcal{S}}}{\partial u}$, $\mathbf{e}_v \equiv \frac{\partial\mathbf{r}_{\mathcal{S}}}{\partial v}$
F06:† ‡	surface normal (parametric): $\mathbf{n}_{\mathcal{S}} \equiv \frac{\mathbf{e}_u \times \mathbf{e}_v}{\lvert\mathbf{e}_u \times \mathbf{e}_v\rvert}$
F07:†	surface normal (implicit): $\mathbf{n}_{\mathcal{S}} \equiv \frac{\nabla\phi_{\mathcal{S}}}{\lvert\nabla\phi_{\mathcal{S}}\rvert}$
F08:†	$div(\mathbf{F}) \equiv \lim_{\mathcal{V}->0} \frac{\int_{\partial\mathcal{V}} \mathbf{F}\cdot\mathbf{n}d\mathcal{S}}{\mathcal{V}}$
F09:† ‡	$curl(\mathbf{F}) \cdot \mathbf{n} \equiv \lim_{\mathcal{S}->0} \frac{\int_{\partial\mathcal{S}} \mathbf{F}\cdot\hat{t}dl}{\mathcal{S}}$
F10:†	$grad(\phi) \equiv \lim_{\mathcal{C}->0} \frac{\int_{\partial\mathcal{C}} \phi\hat{t}dl}{\mathcal{C}}$
F11:†	$\int_{\mathcal{V}} div(\mathbf{F})d\mathcal{V} = \int_{\partial\mathcal{V}} \mathbf{F} \cdot \mathbf{n}d\mathcal{S}$
F12:†	$\int_{\mathcal{S}} curl(\mathbf{F}) \cdot \mathbf{n}d\mathcal{S} = \int_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{r}$

Orthogonal Coordinates† ‡

N01:	$h_i \equiv \lvert\partial\mathbf{r}/\partial q^i\rvert$
N02:	$\mathbf{e}_i \equiv \frac{1}{h_i}\partial\mathbf{r}/\partial q^i$ (no sum)
N03:	$\Omega \equiv h_1 h_2 h_3$
N04:	$grad(\phi) = \sum_i \mathbf{e}_i \frac{1}{h_i}\ \partial_{q^i}\phi$
N05:	$curl(\mathbf{F}) = \frac{1}{\Omega}\sum_{ijk} \mathbf{e}_i[ijk]h_i\partial_{q^j}\,(h_k F_k)$
N06:	$div(\mathbf{F}) = \sum_i \frac{1}{\Omega}\partial_{q^i}\left(\frac{\Omega F_i}{h_i}\right)$
N07:	$laplacian(\phi) = \frac{1}{\Omega}\sum_i \partial_{q^i}\left(\frac{\Omega}{{h_i}^2}\partial_{q^i}\phi\right)$

grad, *div*, and *curl* can all be represented by a single operator ∇, which gives functionally the relations $grad(\phi) = \nabla\phi$, $div(\mathbf{F}) = \nabla \cdot \mathbf{F}$, $curl(\mathbf{F}) = \nabla \times \mathbf{F}$, and $laplacian(\phi) = \nabla \cdot \nabla\phi = \nabla^2\phi$. In Cartesian components, ∇ takes on the form:

N08:	$\nabla = \mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y + \mathbf{e}_z\partial_z$

Derivative Theorems† ‡

D01:	$\nabla \cdot \nabla\phi = \nabla^2\phi$
D02:	$\nabla(\phi\psi) = (\nabla\phi)\psi + \phi(\nabla\psi)$
D03:	$\nabla \times \nabla\phi = 0$
D04:	$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F} \times (\nabla \times \mathbf{G})$ $+(\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{G} \times (\nabla \times \mathbf{F})$
D05:	$\frac{1}{2}\nabla^2\mathbf{F} = \mathbf{F} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{F}$
D06:	$\nabla \cdot (\nabla \times \mathbf{F}) = 0$
D07:	$\nabla \cdot (\phi\mathbf{F}) = \mathbf{F} \cdot \nabla\phi + \phi\nabla \cdot \mathbf{F}$
D08:	$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - (\nabla \times \mathbf{G}) \cdot \mathbf{F}$
D09:	$\nabla \times (\phi\mathbf{F}) = (\nabla\phi) \times \mathbf{F} + \phi\nabla \times \mathbf{F}$
D10:	$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$
D11:	$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G}$ $+(\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$
D12:	$\nabla \cdot \mathbf{r} = 3$
D13:	$\nabla \times \mathbf{r} = 0$
D14:	$(\mathbf{G} \cdot \nabla)\mathbf{r} = \mathbf{G}$
D15:	$\nabla^2\mathbf{r} = 0$
D16:	$\phi(r)$ or $\mathbf{F}(r)$: $\nabla = \frac{\mathbf{r}}{r}\frac{d}{dr}$
D17:	$\nabla(\mathbf{A} \cdot \mathbf{r}) = \mathbf{A}$
D18:	$\phi(\mathbf{A} \cdot \mathbf{r})$ or $\mathbf{F}(\mathbf{A} \cdot \mathbf{r})$: $\nabla = \mathbf{A}\frac{d}{d(\mathbf{A}\cdot\mathbf{r})}$

Integral Theorems† ‡ ∗

I01:	$\int_{\mathcal{V}} \nabla \times \mathbf{F}\, d\mathcal{V} = \int_{\partial\mathcal{V}} \hat{n} \times \mathbf{F}\, d\mathcal{S}$
I02:	$\int_{\mathcal{V}} \nabla\phi\, d\mathcal{V} = \int_{\partial\mathcal{V}} \phi\,\hat{n}\, d\mathcal{S}$
I03:	$\int_{\mathcal{V}}(\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi)d\mathcal{V}$ $= \int_{\partial\mathcal{V}} \phi\nabla\psi \cdot \hat{n}d\mathcal{S}$
I04:	$\int_{\mathcal{V}}(\phi\nabla^2\psi - \psi\nabla^2\phi)d\mathcal{V}$ $= \int_{\partial\mathcal{V}}(\phi\nabla\psi - \psi\nabla\phi) \cdot \hat{n}d\mathcal{S}$
I05:	$\int_{\mathcal{V}} d\mathcal{V}\nabla\phi \cdot \mathbf{F} = \int_{\partial\mathcal{V}} \phi\mathbf{F} \cdot \hat{n}\, d\mathcal{S} -$ $\int_{\partial\mathcal{V}} \phi\nabla \cdot \mathbf{F}d\mathcal{S}$
I06:	$\int_{\partial\mathcal{S}} \phi\,\hat{t}d\ell = \int_{\mathcal{S}}(\hat{n} \times \nabla\phi)\, d\mathcal{S}$
I07:	$\int_{\partial\mathcal{S}} d\ell\,\hat{t} \times \mathbf{F} = \int_{\mathcal{S}}(\hat{n} \times \nabla) \times \mathbf{F}d\mathcal{S}$
I08:	$\int_{\partial\mathcal{V}} d\mathcal{S}\,\hat{n}\circ = \int_{\mathcal{V}}\ d\mathcal{V}\nabla\circ$
I09:	$\int_{\partial\mathcal{S}} d\ell\,\hat{t}\circ = \int_{\mathcal{S}} d\,\mathcal{S}(\hat{n} \times \nabla)\circ$
I10:	$\frac{d\Phi}{dt} = \int_{\mathcal{S}}\left[\frac{\partial\mathbf{F}}{\partial t} + (\nabla \cdot \mathbf{F})\mathbf{v}\right] \cdot d\mathcal{S}$ $+ \int_{\partial\mathcal{S}} \mathbf{F} \times \mathbf{v} \cdot \hat{t}d\ell$
I11:	$\frac{d}{dt}\int_{\mathcal{V}} \rho d\mathcal{V} = \int_{\mathcal{V}} \frac{\partial\rho}{\partial t}d\mathcal{V} + \int_{\partial\mathcal{V}} \rho\mathbf{v} \cdot d\mathcal{S}$

† - inner product required ‡ - valid only in 3 dimensions ∗ - ∘ any product (I08 & I09) transport theorems I10 & I11: $\mathbf{v}$ velocity of the region

Jacobian
Let a n-dimensional space be spanned by a set of vectors $\{\mathbf{e}_i\}$. Then the components collected column-wise

$$\mathbf{J} = [\,\lvert\mathbf{e}_1\rangle,\, \lvert\mathbf{e}_2\rangle,\, \ldots,\, \lvert\mathbf{e}_n\rangle\,]$$

is know as the *Jacobian matrix*. Its inverse

$$\mathbf{J}^{-1} = \begin{bmatrix} \langle\mathbf{e}^1\rvert \\ \langle\mathbf{e}^2\rvert \\ \vdots \\ \langle\mathbf{e}^n\rvert \end{bmatrix}$$

gives the *dual vector space* with corresponding components collected row-wise and obeying the contraction rule

$$\mathbf{J}^{-1}\mathbf{J} = \begin{bmatrix} \langle\mathbf{e}^1\lvert\mathbf{e}_1\rangle & \langle\mathbf{e}^1\lvert\mathbf{e}_2\rangle & \ldots & \langle\mathbf{e}^1\lvert\mathbf{e}_n\rangle \\ \langle\mathbf{e}^2\lvert\mathbf{e}_1\rangle & \langle\mathbf{e}^2\lvert\mathbf{e}_2\rangle & \ldots & \langle\mathbf{e}^2\lvert\mathbf{e}_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\mathbf{e}^n\lvert\mathbf{e}_1\rangle & \langle\mathbf{e}^n\lvert\mathbf{e}_2\rangle & \ldots & \langle\mathbf{e}^n\lvert\mathbf{e}_n\rangle \end{bmatrix}$$

$$= \langle\mathbf{e}^i\lvert\mathbf{e}_j\rangle = \delta^i{}_j$$

and the completeness relation

$$\mathbf{J}\mathbf{J}^{-1} = \sum_i \lvert\mathbf{e}_i\rangle\langle\mathbf{e}^i\rvert = \mathrm{Id}_{n\times n}.$$

Finally $\det(\mathbf{J}) \equiv J$ and $\det(\mathbf{J}^{-1}) \equiv J^{-1}$.

Permutation Tensor

E01: $\epsilon_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k = J[ijk]$

E02: $\epsilon^{ijk} = (\mathbf{e}^i \times \mathbf{e}^j) \cdot \mathbf{e}^k = J^{-1}[ijk]$

E03: $\epsilon^{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta^i{}_p & \delta^i{}_q & \delta^i{}_p \\ \delta^j{}_p & \delta^j{}_q & \delta^j{}_r \\ \delta^k{}_p & \delta^k{}_q & \delta^k{}_r \end{vmatrix}$

E04: $\epsilon^{ijk}\epsilon_{pqk} = \delta^i{}_p\delta^j{}_q - \delta^i{}_q\delta^j{}_p$

E05: $\epsilon^{ijk}\epsilon_{pjk} = 2\delta^i{}_p$

E06: $\epsilon^{ijk}\epsilon_{ijk} = 6$

Transport Theorems

TT01: $\frac{d\Phi}{dt} = \int_{\partial\mathcal{V}_t}\left[\frac{\partial\mathbf{F}}{\partial t} + (\nabla \cdot \mathbf{F})\mathbf{v}\right] \cdot \mathbf{n}d\mathcal{S}$

TT02: $\frac{d}{dt}\int_{\mathcal{V}} \rho$

Afrken Arcana

Collected Electromagnetism Results

Interaction energy two dipoles:	$V = -\frac{\mu_1 \cdot \mu_2}{r^3} + \frac{3(\mu_1 \cdot \mathbf{r})(\mu_2 \cdot \mathbf{r})}{r^5}$
Induce magnetic field by a moving charge:	$\mathbf{B} = \frac{\mu_0}{4\pi} q_1 \frac{\mathbf{v}\times\mathbf{r}}{r^3}$
Field of an electric dipole:	$\psi(\mathbf{r}) = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3}$
curve:	$\mathcal{C}$
bounding curve:	$\partial\,\mathcal{S}$
surface:	$\mathcal{S}$
bounding surface:	$\partial\,\mathcal{V}$
volume:	$\mathcal{V}$
scalar:	m, n, etc.
indices:	$i, j = 1, 2, \ldots, N$
coordinates:	x^i, $q^{i'}$ $\quad i,\ j' = 1{,}2{,}3$
transformations:	$x^i = x^i\left(q^{j'}\right)$
field:	$\phi = \phi\left(q^i\right)$
parametric curve:	$q^i\left(s\right)$
implicit surface:	$\phi_{\mathcal{S}}\left(q^i\right) - c = 0$
parametric surface:	$q^i\left(u, v\right)$
Kronecker delta:	δ_{ij}
permutation symb:	$[i, j, k]$
column array:	$\vert\mathbf{q}\rangle$
row array:	$\langle\mathbf{q}\vert$

Vector Results

Arf00:	$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) = (AB)^2 - (\mathbf{A}\cdot\mathbf{B})^2$
Arf01:	$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$
Arf02:	$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$
Arf03:	$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{B}\times\mathbf{D})\mathbf{C} - (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\mathbf{D}$
Arf04:	$\mathbf{L} = -i(\mathbf{r}\times\nabla)$
Arf05:	$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$
Arf06:	$\{\sigma_i, \sigma_j\}\, 2\delta_{ij}\mathbb{I}$
A04:	$m\,(n\mathbf{A}) \equiv (mn)\,\mathbf{A}$
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A13:†	$\mathbf{A}\cdot\mathbf{B} \equiv \vert\mathbf{A}\vert\vert\mathbf{B}\vert\cos(\theta)$
A14:‡	$\mathbf{A}\times\mathbf{B} \equiv -\mathbf{B}\times\mathbf{A}$
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A19:†‡	$\vert\mathbf{A}\times\mathbf{B}\vert \equiv \vert\mathbf{A}\vert\vert\mathbf{B}\vert\sin(\theta)$
A20:†‡	$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$

$$F = ma$$

$$L[y] = \left(\frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)\right) y$$

$$L[y] = g(t) \quad y(t_0) = y_0 \quad \frac{dy}{dt}(t_0) = \dot{y}_0$$

Let $p(t)$ and $q(t)$ be continuous in the open interval (α, β) then there exists one and only one function $y(t)$ which satisfies $L[y] = 0$ and $y(t_0) = y_0$ and $\frac{dy}{dt}(t_0) = \dot{y}_0$.
Wronskian:

$$W[y_1, y_2](t) \equiv y_1\dot{y}_2 - \dot{y}_1 y_2$$

$$\dot{W} + p(t)W = 0 \Rightarrow W(t) = W_0 \exp\left(-\int_{t0}^{t} p(s)ds\right)$$

Constant Coefficients:

$$p(t) = b \quad q(t) = c$$

Reduction of order:

$$y_2 = v(t)y_1$$

$$v(t) = \int u(s)ds \quad u(t) = \frac{1}{{y_1}^2} exp\left(-\int p(s)ds\right)$$

distinct complex roots

$$\gamma_{12} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

repeated roots

$$y_1 = e^{-\frac{bt}{2}}$$

$$y_2 = ty_1$$

VOP:

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$\frac{d}{dt}u_1(t) = -\frac{g(t)y_2(t)}{W\left[y_1, y_2\right](t)}$$

$$\frac{d}{dt}u_2(t) = \frac{g(t)y_1(t)}{W\left[y_1, y_2\right](t)}$$

One-sided Green's functions

$$g(t, \tau) = \theta(t - \tau)g_1(t, \tau)$$

$$g_1(t, \tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{W[y_1, y_2](\tau)}$$

$$N[g(t, t\tau)] = \delta(t - \tau) \quad y(t_0) = 0 \; \dot{y}(t_0) = 0$$

$$g(\tau^-, \tau) = g(\tau^+, \tau)$$

$$\left.\frac{\partial g}{\partial t}\right|_{t=\tau^-}^{t=\tau^+} = 1$$

Operators

$$M = A_2(x)D^2 + A_1(x)D + A_0(x)$$

$$L = D[p(x)D] + q(x) \quad L[y] + \lambda r(x)y = 0$$

$$N = D^2 + a_1(x)D + a_0(x)$$

$$B_i[y_k] = a_{ijk\ell}D^j y_k(x_ell)$$

The conversion $M \to L$ can be affected by multiplying $\mu(x) = p(x)/A_2(x)$ with $p(x) = \exp\left(\int \frac{A_1}{A_2}\right)(x)$. Likewise, the conversion $M \to N$ by dividing by $A_2(x)$.

$$N[\theta(t-\tau)g_1(t,\tau)] = \left(\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0\right)[\theta(t-\tau)g_1(t-\tau)]$$

Using the short-hand $g_1 = g_1(t-\tau)$, $\theta = \theta(t-\tau)$, $D = \frac{d}{dt}$ the equation becomes

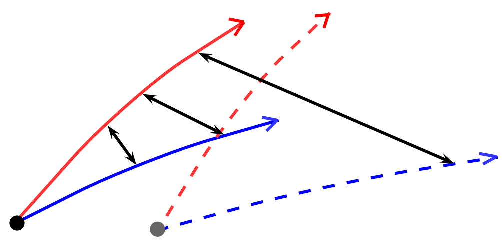
$$\begin{aligned}
N[\theta g_1] &= (D^2 + a_1 D + a_0)[\theta g_1] \\
&= D^2(\theta g_1) + a_1 D(\theta g_1) + a_0\theta g_1 \\
&= D(g_1 D\theta + \theta D g_1) + a_1(g_1 D\theta + \theta D g_1) + a_0\theta g_1 \\
&= Dg_1 D\theta + g_1 D^2\theta + D\theta D g_1 + \theta D^2 g_1 + a_1 g_1 D\theta + a_1\theta D g_1 + a_0\theta g_1
\end{aligned}$$

Now when $t \neq \tau$, θ is a constant and its derivative is zero. So it is only at the point when $t = \tau$ where care must be exercised. At this time, $g_1 = 0$, $D\theta = \delta$, $Dg_1 = 1$, $g_1 D\delta = -\delta g_1 = -\delta$. Plugging these results in yields

$$\begin{aligned}
N[\theta g_1] &= \delta - \delta + \delta + \theta(D^2 g_1 + a_1 g_1 + a_0 g_1) + a_1 g_1 \delta \\
&= \delta\theta N[g_1] + a_1 g_1 \delta\,.
\end{aligned}$$

Finally note that $N[g_1] = 0$ and that the $x * \delta(x) = 0$. Since $g_1(\tau,\tau) = 0$, then $g_1\delta = 0$. The final result is (restoring the arguments)

$$N[\theta(t-\tau)g_1(t,\tau)] = \delta(t-\tau)\,.$$



Integrals

Improper Integrals

An integral is *singular* or *improper* if one or both limits are infinte or the integrand becomes unbounded at one or more points in the interval. The plan of attack is to: 1) decide on integrability, 2) test for convergence, and 3) calculate the integral (if integrable and convergent). Case 1: Integral limits approach $\pm\infty$.

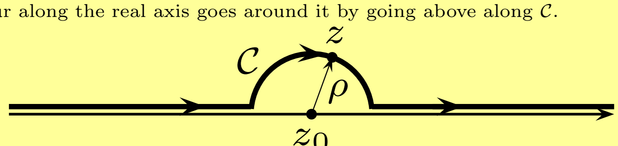
Complex Analysis

Basic Results

mapping:	$\phi : z \equiv (x+iy) \to w \equiv (u(x,y) + iv(x,y))$
Cauchy-Riemann:	$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial v} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \iff w$ analytic
contour integral:	$\int_{z_1}^{z_2} f(z)dz = \int_{(x_1,y_1)}^{(x_2,y_2)} [u(x,y) + iv(x,y)]\,[dx + idy]$ with the path from (x_1, y_1) to (x_2, y_2) specified
Cauchy's theorem:	$\oint_{\mathcal{C}} f(z)dz = 0 \iff f(z)$ is analytic throughout some simply connected region $\mathcal{R}$ bound by $\mathcal{C}$
Cauchy's formula:	$\oint_{\mathcal{C}} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$ where $f(z)$ is analytic
Cauchy's formula:	$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)dz}{(z-z_0)^n}$
residue primitve:	$\oint_{\mathcal{C}} (z-z_0)^n\, dz = \begin{cases} 2\pi i & n=-1 \quad \mathcal{C} \quad ccw \\ -2\pi i & n=-1 \quad \mathcal{C} \quad cw \\ 0 & n \neq -1 \end{cases}$
Laurent series:	$q(z) = \sum_{n=-\infty}^{n=\infty} a_n (z-z_0)^n$
residue:	$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^{m-1} q(z) \right|_{z=z_0}$

Pole on the Real Axis - Principal Value

When there is a pole on the real axis of an integral, the approach is to 'side-step' the singularity by either going around it above or below in the complex plane along a semi-circular contour whose radius ρ will be allowed to go to zero. The figure below shows the situation with the pole at z_0 and where the contour along the real axis goes around it by going above along $\mathcal{C}$.



Let $z - z_0 = \rho e^{i\theta}$ on $\mathcal{C}$ with $\theta \in [\pi, 0]$ and $dz = \rho i e^{i\theta} d\theta$. The residue primitive integral

$$I_n = \int_{\mathcal{C}} (z-z_0)^n dz$$

becomes

$$I_n = i\rho^{n+1} \int_{\pi}^{0} e^{i(n+1)\theta} d\theta = \begin{cases} i\int_{\pi}^{0} d\theta = -i\pi & n=-1 \\ \frac{i\rho^{n+1}}{n+1}\left(1 - e^{i(n+1)\pi}\right) & n \neq -1 \end{cases}.$$

In the limit as ρ goes to zero,

$$I_n = \begin{cases} -i\pi & n=-1 \\ 0 & n > -1 \\ 0 & n < -1 \text{ and odd} \\ \text{undefined} & n < -1 \text{ and even} \end{cases}.$$

These results bascially say that only the $n = -1$ term in the Laurent series contributes a well defined non-zero value to this 'side-stepping', known as the Cauchy principal value. In the case where $n > -1$ the integrand is finite and the length of the contour goes to zero, effectively killing the contribution along $\mathcal{C}$. In the case where $n < -1$ and n is odd, areas of equal magnitude and opposite sign around the pole cancel each other out as the arc length of $\mathcal{C}$ goes to zero. In the final case where $n < -1$ and even, no such cancellation can occur and no well-defined limit can be assigned.
If the lower contour is instead chosen, dropping below the pole into the lower half of the complex plane, then the range of θ is now from π to 2π meaning that the value of the integral for $n = -1$ changes to $i\pi$.
The complex analysis presented here justifies the real analysis definition of the principal value of an integral whose integrand has a singularity at $x = c \in [a, b]$ as

$$PV \int_a^b f(x)dx = \lim_{\epsilon \to 0} \left(\int_a^{c-\epsilon} f(x) + \int_{c+\epsilon}^{b} f(x) \right).$$

Residue Primitive

The point of this article is to establish the residue primitive result by explicitly evaluating the contour integral

$$I = \oint_{\mathcal{C}} (z-z_0)^n\, dz,$$

where $\mathcal{C}$ is a circular contour around z_0 in a counterclock-wise (ccw) direction. Assume on $\mathcal{C}$ that $z - z_0 = Re^{i\theta}$ where θ goes from 0 to 2π. With this parametrization $dz = Rie^{i\theta} d\theta$ and

$$I = iR^{n+1} \int_0^{2\pi} e^{i\theta(n+1)} d\theta.$$

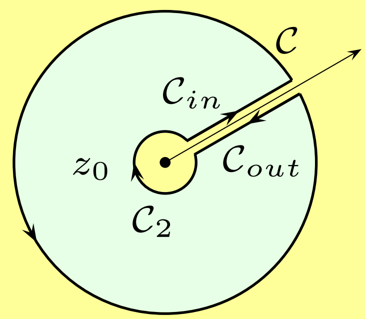
There are two cases to consider: $n \neq -1$ and $n = -1$. In the first case

$$\begin{aligned} I_{n\neq-1} &= iR^{n+1} \int_0^{2\pi} e^{i\theta(n+1)} d\theta \\ &= \frac{R^{n+1}}{n+1} e^{i\theta(n+1)} \Big|_0^{2\pi} \\ &= 0 \end{aligned}$$

and in the second

$$I_{n=-1} = i \int_0^{2\pi} d\theta = 2\pi i$$

To determine the value of the integral if the contour were traversed in the clockwise direction, the limits of the integral can be reversed so that it went from 2π to 0 but ir is more instructive to consider the closed contour $\mathcal{C} \to \mathcal{C}_{in} \to \mathcal{C}_{out} \to \mathcal{C}_2$.



From Cauchy's theorem $(z - z_0)^n$ is analytic in the region bound by this combined contour and the integrals over $\mathcal{C}_{in}$ and $\mathcal{C}_{out}$ cancel leaving $I_{\mathcal{C}} = -I_{\mathcal{C}_2}$.

Fourier Analysis

Consider a periodic function $f(x-2\ell) = f(x) = f(x+2\ell)$ defined on the interval $[-\ell, \ell]$. Because of the underlying periodicity of $f(x)$, both the derivative and the integral also inherit periodic properties.
In the case of the derivative, the observation is evident simply from the standard definition as follows:

$$\begin{aligned} \frac{df}{dx}(x) &= \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx} \\ &= \lim_{dx \to 0} \frac{f(x+2\ell+dx) - f(x+2\ell)}{dx} \\ &= \frac{df}{dx}(x+2\ell). \end{aligned}$$

In the case of the integral, a bit more thinking and manipulation is needed. To begin, consider the integral of $f(x)$ over an interval $[a, b]$. The value of this integral is invariant if the entire interval is shifted by 2ℓ. This is proven by direct substitution as

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^b f(x-2\ell)dx \qquad (1) \\ &= \int_{a+2\ell}^{b+2\ell} f(y)dy \\ &= \int_{a+2\ell}^{b+2\ell} f(x)dx. \end{aligned}$$

With this result, it is easy (although somewhat subtle) to show that the integral of $f(x)$ over the interval $[c-\ell, c+\ell]$ is equal to the integral over the interval $[-\ell, \ell]$. To do so, start by letting $a = c-\ell$ and $b = -\ell$ in (1), which yields the identity

$$\int_{c-\ell}^{-\ell} f(x)dx = \int_{c+\ell}^{\ell} f(x)dx, \qquad (2)$$

which is then used to manipulate the integral over $[c-\ell, c+\ell]$ as follows,

$$\begin{aligned} \int_{c-\ell}^{c+\ell} f(x)dx &= \int_{c-\ell}^{-\ell} f(x)dx + \int_{-\ell}^{c+\ell} f(x)dx \qquad (3) \\ &= \int_{c+\ell}^{\ell} f(x)dx + \int_{-\ell}^{c+\ell} f(x)dx \\ &= \int_{-\ell}^{c+\ell} f(x)dx + \int_{c+\ell}^{\ell} f(x)dx \\ &= \int_{-\ell}^{\ell} f(x)dx \end{aligned}$$

where (2) was used in going from the first to the second line.

$$\int_c^{c+2L} \cos\left(\frac{m\pi x}{L}\right) dx = 0$$

$$\int_c^{c+2L} \sin\left(\frac{m\pi x}{L}\right) dx = 0$$

$$\int_c^{c+2L} dx \, \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) = \begin{cases} 2L\,\delta_{mn} & m = 0 \\ L\,\delta_{mn} & m \neq 0 \end{cases}$$

$$\int_c^{c+2L} dx \, \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} 0\,\delta_{mn} & m = 0 \\ L\,\delta_{mn} & m \neq 0 \end{cases}$$

$$u_n \equiv \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, \ldots$$

$$v_n \equiv \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi x}{L}\right) \qquad n = 0, 1, 2, \ldots$$

$$f = \sum_{n=1}^{\infty} \{v_n(v_n \cdot f) + u_n(u_n \cdot f)\} + \frac{1}{2}(v_0 \cdot f)v_0$$

$$\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$a_n = \frac{1}{L} \int_c^{c+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \equiv \frac{1}{\sqrt{L}}(f \cdot v_n)$$

$$b_n = \frac{1}{L} \int_c^{c+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\tilde{f}(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$c_n = \frac{1}{2L} \int_c^{c+2L} f(x) e^{-in\pi x/L} dx$$

Fourier Transform

Definitions

Constants	a, b
General function	f, g
Schwartz function	ϕ, ψ
Tempered distribution	$\mathcal{T}, \mathcal{T}_f, \mathcal{S}$
Shift Operator	$(\tau_{\pm b} f)(x) \equiv f(x \mp b)$
Distributational pairing	$\langle \mathcal{T}_f, \phi \rangle \equiv \int_{-\infty}^{\infty} f(x)\phi(x)dx$
Distributional linearity	$\langle \mathcal{T}, a\phi + b\psi \rangle \equiv a\langle \mathcal{T}, \phi \rangle + b\langle \mathcal{T}, \psi \rangle$
Distributional derivative	$\langle \mathcal{T}', \phi \rangle \equiv -\langle \mathcal{T}, \phi' \rangle$
Distributional reversal	$\langle \mathcal{T}^-, \phi \rangle \equiv \langle \mathcal{T}, \phi^- \rangle$
Convolution	$(g * f)(t) \equiv \int_{-\infty}^{\infty} g(t-\tau)f(\tau)d\tau$
Dist. Fourier Transform	$\langle \mathcal{F}\mathcal{T}, \phi \rangle \equiv \langle \mathcal{T}, \mathcal{F}\phi \rangle$
Dist. Inverse FT	$\langle \mathcal{F}^{-1}\mathcal{T}, \phi \rangle \equiv \langle \mathcal{T}, \mathcal{F}^{-1}\phi \rangle$

Specific Distributions

delta	$\delta = \begin{cases} \delta(x) = 0 \; x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)dx = 1 \end{cases}$
signum	$sgn(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$
unit step	$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$
unit ramp	$R(x) = \begin{cases} 0 & x \leq 0 \\ x & x > 0 \end{cases}$
delta pairing	$\langle \delta, \phi \rangle = \phi(0)$
delta pairing	$\langle \delta_a, \phi \rangle = \phi(a)$
unit step deriv.	$H' = \delta$
signum deriv.	$sgn' = 2\delta$
delta deriv.	$\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0)$

Specific Fourier Transforms

delta	$\mathcal{F}\delta = 1$
1	$\mathcal{F}1 = \delta$
unit step	$\mathcal{F}\delta_{\pm a} = e^{\mp 2\pi i s a}$
unit ramp	$\mathcal{F}e^{\pm 2\pi i x a} = \delta_{\pm a}$
delta pairing	$\mathcal{F}\left[\frac{1}{2}\left(\delta_a + \delta_{-a}\right)\right] = \cos 2\pi s a$
delta pairing	$\mathcal{F}cos 2\pi a x = \frac{1}{2}\left[\delta_a + \delta_{-a}\right]$
unit step deriv.	$H' = \delta$
signum deriv.	$sgn' = 2\delta$
delta deriv.	$\mathcal{F}\left[\frac{1}{2i}\left(\delta_{-a} - \delta_a\right)\right] = \sin 2\pi s a$

Fourier Transform Notation

There are many different conventions and notations for the Fourier transform. The convention used here will be that the complex exponentials will have the 2π explicitly indicated and paired with the transform variable rather than the usual physics standard of angular frequency, ($e.g.$, $2\pi s$ instead of ω). This usage supresses the needs for $1/\sqrt{2\pi}$ or similar terms outside the integral.
For notation, two types are employed. The first is this simple pairing $f(t) \rightleftharpoons F(s)$ which tends to emphasize the two domains (time and frequency) but which clouds the duality. The second is the script notation $\mathcal{F}f(s)$ which allows for greater insight into the transform but which should be interpretted as follows:

1. $\mathcal{F}f$ means substitute f into a Fourier integral
2. the absence or presence of the inverse symbol $^{-1}$ indicates which sign to use ($-$ - absence, $+$ - presence)
3. the variable that follows (s) indicates the free variable in the integral (can be suppressed), the other being dummy.

Finally, there will be occasions when the reversal of a signal, defiend as $f(t) \to f(-t)$, will be examined. When pairing this $f(-t)$ with a Fourier integral in the script notation, a convenient way to suppress the dummy variable ($i.e.$ t) is with the notation f^-. It can also be applied to the transform itself - $\mathcal{F}f(-s) \equiv (\mathcal{F}f)^-$.

Basic Fourier Transform Results

Pairing Notation 1	$f(t) \rightleftharpoons F(s)$
Pairing Notation 2	$f(t) \rightleftharpoons \mathcal{F}f(s)$
FT01: Forward Transform	$F(s) \equiv \int_{-\infty}^{\infty} f(t)e^{-2\pi i s t}dt$
FT02: Inverse Transform	$f(t) \equiv \int_{-\infty}^{\infty} F(s)e^{2\pi i s t}ds$
FT03: Forward Reversal	$(\mathcal{F}\mathcal{T})^- = \mathcal{F}^{-1}\mathcal{T}$
FT04: Inverse Reversal	$(\mathcal{F}^{-1}\mathcal{T})^- = \mathcal{F}\mathcal{T}$
FT05: Reversed Forward	$\mathcal{F}\mathcal{T}^- = \mathcal{F}^{-1}\mathcal{T}$
FT06: Reversed Inverse	$\mathcal{F}^{-1}\mathcal{T}^- = \mathcal{F}\mathcal{T}$
FT07: Linearity	$\mathcal{F}(a\mathcal{T} + b\mathcal{S}) = a\mathcal{F}\mathcal{T} + b\mathcal{F}\mathcal{S}$
FT08: Shift & Pairing	$\langle \tau_{\pm b} f, \phi \rangle = \langle f, \tau_{\mp b}\phi \rangle$
FT09: Time Shift	$\mathcal{F}(\tau_{\pm b} f) = e^{\mp 2\pi i s b}\mathcal{F}f$
FT10: Frequency Shift	$\tau_{\pm b}\mathcal{F}f(s) \rightleftharpoons f(t)e^{\pm 2\pi i b t}$
FT11: Shift Theorem	$\mathcal{F}(\tau_{\pm b}\mathcal{T}) = e^{\mp 2\pi i b x}\mathcal{T}$
FT12: Scaling	$f(at) \rightleftharpoons \frac{1}{\lvert a \rvert}\mathcal{F}f\left(\frac{s}{a}\right)$
FT13: Deriv. Func.	$f' \rightleftharpoons (2\pi i s)F(s)$
FT14: Power Rule Func.	$(-2\pi i t)f(t) \rightleftharpoons F'(s)$
FT15: Deriv. Dist.	$\mathcal{T}' \rightleftharpoons 2\pi i x\mathcal{F}\mathcal{T}$
FT16: Power Rule Dist.	$-2\pi i t\mathcal{T} \rightleftharpoons (\mathcal{F}\mathcal{T})'$
FT17: Convolution Theorem	$f * g \rightleftharpoons F(s)G(s)$
FT18: Convolution Assoc.	$(k * g) * f = k * (g * f)$

Unit Step

Unit Step

definition:	$U(x-q) = \begin{cases} 0 & x < q \\ 1 & x > q \end{cases}$
integral property:	$\int_a^b dx\, U(x-q) = \int_a^q dx \quad a < q < b$
reversal:	$\int_a^b dx\, U(q-x) = \int_q^b dx \quad a < q < b$
ϵ representation:	$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ixt}}{x-i\epsilon} dx$
PV representation:	$\frac{1}{2} + PV\int_{-\infty}^{\infty} \frac{e^{ixt}}{x} dx$

$$\begin{aligned}
\int_a^b dx\, \theta(q-x) &= \int_a^b dx \left\{1 - \theta(x-q)\right\} \\
&= \int_a^b dx - \int_a^q dx \\
&= \int_a^q dx + \int_q^b dx - \int_a^q dx \\
&= \int_q^b dx
\end{aligned}$$

The Fourier transform (FT) of the unit step can be obtained by considering the Schwartz function approximation

$$U_\alpha(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{4}$$

in the limit as $\alpha \to \infty$. The FT of U_α is given by

$$\begin{aligned}
\mathcal{F}U_\alpha &= \int_{-\infty}^{\infty} U_\alpha(t) e^{-2\pi i s t} dt \qquad (5) \\
&= \int_0^{\infty} e^{-\alpha t} e^{-2\pi i s t} dt \\
&= \frac{-1}{\alpha + 2\pi i s} e^{-(\alpha + 2\pi i s)t} \Big|_0^{\infty} \\
&= \frac{1}{\alpha + 2\pi i s}.
\end{aligned}$$

The next step is to take the limit, but to make the limit meaningful, $\mathcal{F}U_\alpha$ must be separated into real and imaginary parts

$$\mathcal{F}U_\alpha = \frac{\alpha}{\alpha^2 + 4\pi^2 s^2} - \frac{2\pi s}{\alpha^2 + 4\pi^3 s^2} i. \tag{6}$$

The limit of the imaginary part is straightforward and yields $\frac{1}{2\pi i s}$ but the real part requires more care. First calculate the integral of the real part $s \in [-\infty, \infty]$

$$I = \int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2 + 4\pi^2 s^2} ds. \tag{7}$$

The value of I is obtained by making the subsitution $s = \frac{\alpha}{2\pi}\tan(\theta)$ to yield

$$I = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta}{1 + \tan^2(\theta)} d\theta = \frac{1}{2}.$$

Now note that the real part is zero as $s = \pm\infty$ and $1/\alpha$ at $s = 0$, which means that the real part is strongly-peaked at $s = 0$. The only function that integrates to a constant over $[-\infty, \infty]$, is strongly-peaked at the origin, where it tends to infinity, and is zero everywhere else is a function proportional to the δ function. Thus

$$\lim_{\alpha \to 0} \mathcal{F}U_\alpha = \mathcal{F}U = \frac{1}{2}\delta(s) + \frac{1}{2\pi i s}. \tag{8}$$

Differential Forms Version of Classical Vector Analysis

Basic Geometric Objects1

$$\vec{A} <=> \phi_A$$

$$\phi_A = A^x dx + A^y dy + A^z dz$$

$$\vec{\nabla} \times \vec{A} <=> *d\phi_A$$

$$div(\vec{A}) <=> *d * \phi_A$$

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{A} <=> (*d*)(*d\phi_A)$$

$$\vec{\nabla} \times \vec{\nabla} f = *d(df) = *d^2 f = 0$$

$$\vec{\nabla} \times (\vec{\nabla}\vec{A}) <=> (*d)(*d\phi_A) = *d * d\phi_A$$

$$(*d * d + d * d*) f \Leftrightarrow \nabla^2 f$$

$$\phi_{\mathbf{A}} \Leftrightarrow \left(A^i \delta_{ij}\right) dq^j \equiv A_j dq^j$$

$$*d\phi_{\mathbf{A}} \Leftrightarrow \nabla \times \mathbf{A}$$

$$*d * \phi_{\mathbf{A}} \Leftrightarrow div(\mathbf{A})$$

$$(*d*)(*d\phi_{\mathbf{A}}) \Leftrightarrow \nabla \cdot \nabla \times \mathbf{A}$$

$$\mathcal{L}_{\mathbf{A}} f = A^\alpha \partial_{q^\alpha} f$$

$$\mathcal{L}_{\mathbf{A}} \mathbf{B} = [\mathbf{A}, \mathbf{B}]$$

$$\mathcal{L}_{\mathbf{A}} \omega = (\omega_\sigma A^\sigma{}_{,\gamma} + A^\sigma \omega_{\gamma,\sigma}) dq^\gamma$$

$$\Delta_L = \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \ddot{q}^\alpha \frac{\partial}{\partial \dot{q}^\alpha}$$

$$\Theta_L = \frac{\partial L}{\partial \dot{q}^\alpha} dq^\alpha$$

$$\mathcal{L}_{\Delta_L} \Theta_L = dL$$

$$\Delta_H = \dot{\xi}^j \partial_j = \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \dot{p}^\alpha \frac{\partial}{\partial p^\alpha}$$

$$\Theta_H = p_\alpha dq^\alpha$$

$$i_{\Delta_H} \omega = dH$$

$$\mathcal{L}_\Delta f = \{f, H\} + \partial_t f$$

$$i_{\mathbf{X}} \alpha = \langle \alpha, \mathbf{X} \rangle$$

$$i_{\mathbf{X}} \omega = \omega\left(\bullet, \bullet, \ldots, \bullet, \mathbf{X}\right)$$

$$\mathcal{L}_{\mathbf{X}} [\mathbf{Y}, \mathbf{Z}] = [\mathcal{L}_{\mathbf{X}} \mathbf{Y}, \mathbf{Z}] + [\mathbf{Y}, \mathcal{L}_{\mathbf{X}} \mathbf{Z}]$$

$$\mathcal{L}_{\mathbf{X}} (\alpha \wedge \beta) = (\mathcal{L}_{\mathbf{X}} \alpha) \wedge \beta = \alpha \wedge (\mathcal{L}_{\mathbf{X}} \beta)$$

$$[\mathcal{L}_{\mathbf{X}}, \mathcal{L}_{\mathbf{Y}}] \alpha = \mathcal{L}_{[\mathbf{X}, \mathbf{Y}]} \alpha$$

$$[\mathcal{L}_{\mathbf{X}}, i_{\mathbf{Y}}] \alpha = [i_{\mathbf{Y}}, \mathcal{L}_{\mathbf{X}}] \alpha = i_{[\mathbf{X}, \mathbf{Y}]} \alpha = i_{(\mathcal{L}_{\mathbf{X}} \mathbf{Y})} \alpha$$

$$\mathcal{L}_{\mathbf{X}} \alpha = i_{\mathbf{X}} d\alpha + d i_{\mathbf{X}} \alpha$$

Classical Mechanics

The Euler Lagrange equations are invariant to a basic change in coordinates from q^i to y^j of the form

$$q^i = q^i(y^j). \tag{9}$$

To see this first note that the from the form of the transformation equation (9) we get

$$\dot{q}^i = \frac{\partial q^i}{\partial y^j}\dot{y}^j \tag{10}$$

where $\dot{f} = \frac{d}{dt}f$. Next note that the Lagrangian $\tilde{L}(y^j, \dot{y}^j; t)$ in the y^j coordinates is related to the Lagrangian $L(q^i, \dot{q}^i; t)$ in q^i coordinates by virtue of a substitution of (9) and (10) yielding

$$\tilde{L}(y^j, \dot{y}^j; t) = L(q^i(y^j), \dot{q}^i(y^j, \dot{y}^j); t). \tag{11}$$

The parts of the Euler-Lagrange equation in terms of the y^j coordinates in relation to the q^i coordinates are

$$\frac{\partial \tilde{L}}{\partial y^j} = \frac{\partial L}{\partial q^i}\frac{\partial q^i}{\partial y^j} + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial y^j} \tag{12}$$

and

$$\frac{\partial \tilde{L}}{\partial \dot{y}^j} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial \dot{y}^j}. \tag{13}$$

Now substituting (12) and (13) into the Euler-Lagrange equations yields

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{y}^j}\right) - \frac{\partial \tilde{L}}{\partial y^j} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\frac{\partial \dot{q}^i}{\partial \dot{y}^j} \qquad (14) \\ &\quad + \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\left(\frac{\partial \dot{q}^i}{\partial \dot{y}^j}\right) \\ &= -\frac{\partial L}{\partial q^i}\frac{\partial q^i}{\partial y^j} - \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial y^j}. \end{aligned}$$

But from (10) $\partial \dot{q}^i/\partial \dot{y}^j = \partial q^i/\partial y^j$ and thus the second and fourth terms in (14) cancel, leaving

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{y}^j}\right) - \frac{\partial \tilde{L}}{\partial y^j} = \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right]\frac{\partial q^i}{\partial y^j}. \tag{15}$$

Using the definition $\Xi_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}$, Eq. (15) takes on the more obvious form of

$$\Xi_{\tilde{j}} = \Xi_i \Lambda^i{}_{\tilde{j}} \tag{16}$$

which shows that the Euler-Lagrange equations transform like the components of a covariant vector.

The Euler-Lagrange equations can be cast into 1^{st}-order form by first making the identification

$$\frac{d}{dt}q^\alpha = \dot{q}^\alpha$$

and then by expanding

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) = \frac{\partial^2 L}{\partial q^\beta \partial \dot{q}^\alpha}\dot{q}^\beta + \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\beta}\ddot{q}^\beta + \frac{\partial^2 L}{\partial t \partial q^\alpha}$$

Substituting this form in the EL equations and solving for the $\ddot{q}^\alpha$ yields

$$\begin{aligned} \frac{d}{dt}\dot{q}^\alpha &= \ddot{q}^\alpha \\ &= \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\beta}\right)^{-1} \\ &\quad \times \left(\frac{\partial L}{\partial q^\beta} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\beta} - \frac{\partial^2 L}{\partial q^\beta \partial \dot{q}^\gamma}\dot{q}^\gamma\right) \end{aligned}$$

which requires that the Hessian, defined by

$$\left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\beta}\right)$$

be invertible.

Classical Electromagnetism

Electromagnetic theory is built on::

- Maxwell's equations - 4 linear PDEs in 5 vector and 1 scalar functions (16 variables)
  - $\vec{E}$ - electric field $[N/C]$ or $[V/m]$
  - $\vec{D}$ - electric displacement $[C/m^2]$
  - $\vec{B}$ - magnetic flux density $[Ns/Cm]$ or $[kg/Cs]$
  - $\vec{H}$ - magnetic intensity $[A/m]$
  - $\vec{J}$ - current density $[C/m^2/s]$
  - ρ - charge density $[C/m^3]$
- Lorentz Force Law $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$ relating the force to the charge, electric field, velocity, and magnetic flux density.
- Constitutive relations
  - $\vec{D} = \vec{D}(\vec{E})$; usually through $\vec{D} = \vec{E} + \vec{P}(\vec{E}) = \epsilon\vec{E}$ for linear polarization
  - $\vec{H} = \vec{H}(\vec{B})$; usually through $\vec{H} = \vec{B} + \vec{M}(\vec{B}) = \frac{1}{\mu}\vec{B}$ for linear magnetization
  - $\vec{J} = \vec{J}(\vec{E})$; usually through $\vec{J} = g\vec{E}$ for linear conductivity (Ohm's law)
  - Note that ϵ (permittivity) $[C^2/Nm]$, μ (permeability) $[N/A^2]$, & g $[A/V]$ can either be scalars or rank 2 tensors for isotropic or anisotropic linear materials, respectively
  - Nonlinear behavior occurs for all three cases

The Maxwell Equations exhibit the following properties:

- Charge conservation
- Well-pose initial-value problem
- Compact expression in terms of potentials
  - Vector & scalar potentials - connected to fields
  - Connecting potentials & sources - inhomogenous wave eqns.
  - Super potential - all in one
  - Gauge invariance
  - Lorentz gauge
  - Coulomb gauge
- Wave excitations
  - Magnetic Intensity
  - Electric Field
- Conservation of energy
  - Poynting Theorem
  - Energy & momentum flow
- Well-defined boundary conditions

Point Form	Integral Form	Label and Name
$\vec{\nabla} \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$	$\oint \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_c + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{S}$	ME1 - Ampere's Law
$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint \vec{E} \cdot d\vec{\ell} = \int_S \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{S}$	ME2 - Faraday's Law (S fixed)
$\vec{\nabla} \cdot \vec{D} = \rho$	$\oint_S \vec{D} \cdot d\vec{S} = \int_V \rho dV$	ME3 - Gauss' Law
$\vec{\nabla} \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{S} = 0$	ME4 - nonexistance of monopoles

$\vec{D} = \epsilon\vec{E}$
$\vec{B} = \mu\vec{H}$
$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$
$\epsilon = \epsilon_r\epsilon_0$
$\mu = \mu_r\mu_0$

simplifying assumptions: $EMA1 : \epsilon = const \& \mu = const$

$\vec{H}$	magnetic field strength
$\vec{E}$	electric field strength
$\vec{B}$	magnetic flux density
$\vec{D}$	electric flux density
$\vec{J}$	current density
ρ	charge density
$\vec{F}$	force
q	charge
$\vec{v}$	velocity
μ	
ϵ	

$\vec{E} = \frac{\partial \vec{A}}{\partial t} - \nabla\phi$
$\vec{B} = \nabla \times \vec{A}$

Magnetic Fields	*Electric Fields*
$B_{n1} = B_{n2}$	$\left(\vec{D}_1 - \vec{D}_2\right) \cdot \vec{a}_{n12} = -\rho_s$
$\left(\vec{H}_1 - \vec{H}_2\right) \times \vec{a}_{n12} = \vec{K}$	$E_{t1} = E_{t2}$
$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_{r2}}{\mu_{r1}}$ (current-free)	$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_{r2}}{\epsilon_{r1}}$ (charge-free)

Quantum Mechanics

Structure

QM1 System state is denoted by the ket $|S\rangle$
- a belongs to a vector space which obeys the 10 axioms
- b all physical information is contained in $|S\rangle$

QM2 The vector space containing $|S\rangle$
- a has a dual space $|S\rangle \longleftrightarrow \langle S| = |S\rangle^\dagger$
- b has a positive semi-definite inner product $\langle S|S\rangle \geq 0$
- c which is linear $\langle S|\Big(\alpha|S_1\rangle + \beta|S_2\rangle\Big) = \alpha\langle S|S_1\rangle + \beta\langle S|S_2\rangle$
- d $\langle S|T\rangle^* = \langle T|S\rangle$

QM3 Linear operators $\hat{A}$ exist in the space
- a $\hat{A}\,(\alpha|S\rangle + \beta|T\rangle) = \alpha\hat{A}|S\rangle + \beta\hat{A}|T\rangle$
- b Classical observables are Hermitian $\hat{A} = \hat{A}^\dagger$
- c Observables need not commute $\left[\hat{A}, \hat{B}\right] \neq 0$
- d Commutator $\leftrightarrow$ Poisson bracket $\{A, B\}_{PB} = (i\hbar)\left[\hat{A}, \hat{B}\right]$

QM5 Observables are associated with Hermitian operators $\langle\phi|\hat{O}|\psi\rangle = \langle\psi|\hat{O}^\dagger|\phi\rangle = \langle\psi|\hat{O}|\phi\rangle$)

QM6 Statistical average of an observable is $O = \langle\psi|\hat{O}|\psi\rangle$

QM7 Statisitical fluctuations are given by: $\sigma_O^2 = \langle\psi|\hat{O}^2|\psi\rangle - \langle\psi|\hat{O}|\psi\rangle^2$

QM8 Non-commuting operators $\hat{A}$, $\hat{B}$ have

Translation Operator

A translation operator should satisfy:

QM_TO1:	$\mathcal{T}(d\vec{x})|\vec{x}\rangle = |\vec{x} + d\vec{x}\rangle$
QM_TO2:	$\mathcal{T}^\dagger(d\vec{x})\mathcal{T}(d\vec{x}) = \mathcal{T}(d\vec{x})\mathcal{T}^\dagger(d\vec{x}) = 1$
QM_TO3:	$\mathcal{T}(d\vec{x})\mathcal{T}(d\vec{x}') = \mathcal{T}(d\vec{x} + d\vec{x}')$
QM_TO4:	$\mathcal{T}(-d\vec{x}) = \mathcal{T}^{-1}(d\vec{x})$
QM_TO5:	$\lim_{|d\vec{x}|\to 0}\mathcal{T}(d\vec{x}) = 1$

These conditions are met by $\mathcal{T}(d\vec{x}) = 1 - i\vec{K}\cdot d\vec{x}$, where $\vec{K} \doteq \{\hat{K}_i\}$ are Hermitian operators. The commutator is found to be $[\hat{X}_i, \hat{K}_j] = i\,\delta_{ij}$ and since classically $\{x_i, p_j\}_{PB} = \delta_{ij}$ then $\hat{K}_i = \hat{p}_i/\hbar$.

Representations

Position	$\hat{q}\,|q\rangle = q\,|q\rangle$ $\langle q'|\,q\rangle = \delta\left(q - q'\right)$ $1 = \int dq\,|q\rangle\,\langle q|$ $\psi(q) = \langle q|\,\Psi\rangle$
Momentum 1	$\hat{p}\,|p\rangle$ $\langle p'|\,p\rangle = \delta\left(p - p'\right)$ $1 = \int \frac{dp}{2\pi\hbar}\,|p\rangle\,\langle p\rangle$ $\psi(p) = \langle p|\,\Psi\rangle$
Momentum 2	$\hat{p}\,|p\rangle$ $\langle p'|\,p\rangle = 2\pi\hbar\delta\left(p - p'\right)$ $1 = \int \frac{dp}{2\pi\hbar}\,|p\rangle\,\langle p\rangle$ $\psi(p) = \langle p|\,\Psi\rangle$
Number	$N\,|n\rangle = n\,|n\rangle$ $a = b$

Fourier Transforms & Representations

QMFT1	$\langle p|x\rangle = \exp(-ip\cdot x)$

Basis Vectors & Operator Representations

QMB1:	Basis	$\{|a_i\rangle\}$
QMB2:	Closure	$\sum_i |a_i\rangle\langle a_i| = 1$
QMB3:	Normalization	$\langle a_i|a_j\rangle = \delta_{ij}$
QMB4:	Projection	$\Lambda_{a_i} = |a_i\rangle\langle a_i|$
QMB5:	Operator Rep. (General)	$A \doteq \sum_{ij} |a_i\rangle\langle a_i|A|a_j\rangle\langle a_j|$
QMB6:	Operator Rep. (Diagonal)	$A \doteq \sum_i a_i|a_i\rangle\langle a_i| = \sum_i a_i\Lambda_{a_i}$
QMB7:		

Quantum Mechanics

Numerical Values

speed of light	c	$2.998x10^8 m/s$
vacuum permittivity	$\epsilon_0 = 1/\mu_0 c^2$	$8.854x10^{-12} F/m$
Planck's constant	h	$6.626x10^{-34} Js$
hbar	$\hbar = h/2\pi$	$1.055x10^{-34} Js$
elementary charge	e	$1.602x10^{-19} C$
Boltzmann constant	k_B	$1.381x10^{-23} J/K$
Avogadro constant	N_A	$6.022x10^{23} /mol$
electron mass	m_e	$9.109x10^{-31} kg$
proton mass	m_p	$1.673x10^{-27} kg$
neutron mass	m_n	$1.675x10^{-27} kg$
atomic mass unit	$m_u = 10^{-3}/N_A$	$1.661x10^{-27} kg$
Bohr radius	$a_0 = 4\pi\epsilon_0\hbar^2/e^2 m_e$	$5.292x10^{-11} m$
Rydberg constant	$R_\infty = \hbar/4\pi c m_e {a_0}^2$	$1.097x10^7 /m$
fine structure const	$\alpha = e^2/4\pi\epsilon_0 c\hbar$	$7.297x10^{-3}$
Bohr magneton	$\mu_B = e\hbar/2m_e$	$9.274x10^{-24} J/T$
nuclear magneton	$\mu_N = e\hbar/2m_p$	$5.051x10^{-27} J/T$

States are represented by kets $|`label'\rangle$
$\forall$ kets there is a dual correspondence of bras $\langle`label'|$
Both kets and bras form a vector space and obey the 10 axioms
The space is equipped with an inner product $\langle b|a\rangle \in \mathbb{C}$ such that $\langle a|a\rangle \geq 0$ and $\langle b|a\rangle = \langle a|b\rangle^*$.
Completeness: $\sum_i |a_i\rangle\langle a_i| = 1$ where $\{|a_i\rangle\}$ is a basis.

Classical Mechanics

Work Energy

$$\frac{d\,(mT)}{dt} = \mathbf{F} \cdot \mathbf{p}$$

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v}$$

Cancellation of Dots

$$\frac{d}{dt}\left(\frac{\partial G}{\partial q^i}\right) = \frac{\partial \dot{G}}{\partial q^i}$$

$$\frac{d}{dt}\left(\frac{\partial G}{\partial \dot{q}^j}\right) = \frac{\partial \dot{G}}{\partial \dot{q}^j} - \frac{\partial G}{\partial q^j}$$

$$\frac{d}{dt}\left(\frac{\partial \dot{G}}{\partial \dot{q}^j}\right) = \frac{\partial \dot{G}}{\partial q^j} + \frac{d^2}{dt^2}\left(\frac{\partial G}{\partial \dot{q}^j}\right)$$

Generalized Force

$$Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^k}\right) - \frac{\partial T}{\partial q^k}$$

Hamiltonian

$$H(q,p,t) = \dot{q}^i p_i - L(q,\dot{q},t)$$

$$p_i = \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}^i}$$

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} = \frac{dH}{dt}$$

D'Alembert's Principle

$$\delta W - \mathbf{p} \cdot \delta \mathbf{r} = 0$$

Delta Functions

Delta Function Sequences

box	$\phi_B(x;n)$	$\begin{cases} 0 & \lvert x\rvert \geq 1/n \\ n/2 & \lvert x\rvert < 1/n \end{cases} \quad n = 1, 2, \ldots$
Lorentz	$\phi_L(x;n)$	$\frac{n}{\pi}\frac{1}{1+n^2x^2}$
Gaussian	$\phi_G(x;n)$	$\frac{n}{\sqrt{\pi}}e^{-n^2x^2}$
Sinc	$\phi_S(x;n)$	$\frac{1}{n\pi}\frac{\sin^2 nx}{x^2}$

Basic Properties

definition	$\delta(x) = \frac{d}{dx}H(x)$
normalization	$\int_{-\infty}^{\infty}\delta(x)dx = 1$
sifting	$\int_{-\infty}^{\infty}\delta(x)f(x)dx = f(0)$
derivative	$\int_{-\infty}^{\infty}\left(\frac{d^m}{dx^m}\delta(x)\right)f(x)dx = (-1)^m \left.\frac{d^m}{dx^m}f(x)\right\|_{x=0}$
scaling	$\delta(ax) = 1/\lvert a\rvert\delta(x)$
parity	$x\delta(x) = 0$

$$\frac{d}{dt}\mathbf{S}_\phi = \mathbf{F}_\phi\left(\mathbf{S}_\phi; t\right)$$

$$\frac{d}{dt}\mathbf{S}_\eta = \mathbf{F}_\eta\left(\mathbf{S}_\eta; t\right)$$

$$\frac{d}{dt}\mathbf{T}_\phi = \mathbf{F}_\phi\left(\mathbf{T}_\phi; t\right)$$

$$\frac{d}{dt}\mathbf{T}_\eta = \mathbf{F}_\eta\left(\mathbf{T}_\eta; t\right)$$

$$\theta(x-q) = \left\{ \begin{array}{ll} 0 & x < q \\ 1 & x > q \end{array} \right.$$

$$\int_a^b dx\, \theta(x-q) = \int_a^q dx$$

$$\begin{aligned} \int_a^b dx\, \theta(q-x) &= \int_a^b dx \left\{1 - \theta(x-q)\right\} \\ &= \int_a^b dx - \int_a^q dx \\ &= \int_a^q dx + \int_q^b dx - \int_a^q dx \\ &= \int_q^b dx \end{aligned}$$

Define an integrand

$$L(a,b)[f] = af(x)^2 + bf'(x)^2$$

$$I_v(a,b)[f] = \int_0^1 L(a,b)[f]dx = \int_0^1 \left(af(x)^2 + bf'(x)^2\right)dx$$

$I_p(1,1)[x]$	1.33333
$I_p(1,1)\left[x^2\right]$	1.53333
$I_p(1,1)\left[x^3\right]$	1.94286

subject to the boundary conditions $f(0)=0$ and $f(1)=1$. I_p has as its domain any well-behaved function that is continuous and differentiable (this may be too restrictive). Let the set of such functions be called $\mathcal{D}$. Then let $\mathcal{D}'$ be the subset of $\mathcal{D}$ that satisfies the boundary conditions. Examples of $\mathcal{D}$ are: $const, 0, \cos(x), x, x^2, (1-x^3), ln(x), \sin\left(\frac{\pi}{2}x\right), e^x$. Examples of $\mathcal{D}'$ are: $x, x^2, \sin\left(\frac{\pi}{2}\right)$.

The goal is to find $q \ni \mathcal{D}'$ extremizes $I_p(a,b)[f]$ by taking the variation δI_p and solve for f

$$\delta I_p = 2\int_0^1 dx\left(af\delta f + bf'\delta f'\right)$$

Harmonic Oscillators

The equation for a damped simple harmonic oscillator is

$$m\ddot{x} + \Gamma\dot{x} + kx = 0.$$

Assume a solution of the form $x = Ae^{\imath\omega t}$ and then substitute into the equation of motion. Factoring out $Ae^{i\omega t}$, yields the characteristic equation

$$-m\omega^2 + i\Gamma\omega + k = 0.$$

Before solving the charateristic equation, divide by m and then define the parameters ${\omega_0}^2 = k/m$ and $\gamma = \Gamma/2$. The equation then becomes

$$\omega^2 - i\gamma\omega + {\omega_0}^2 = 0$$

with corresponding roots

$$\omega = \frac{i\gamma}{2} \pm \sqrt{{\omega_0}^2 - \frac{\gamma^2}{4}} = \frac{i\gamma}{2} \pm \omega_D$$

Thus the final solution looks like

$$x = e^{-i\gamma/2}\left[x_0\cos(\omega_D t) + \frac{v_0}{\omega_D}\sin(\omega_D t)\right]$$