• Start with two traveling waves with different frequencies and wave numbers

$$\varphi_1 = a\cos(\omega_1 t - k_1 x)$$

$$\varphi_2 = a\cos(\omega_2 t - k_2 x)$$

where generally $\omega_i = \omega_i(k_i)$ with the relationship can be linear or non-linear as becomes obvious below.

• Now superpose the two waves

$$\varphi = \varphi_1 + \varphi_2$$

$$= a\cos(\omega_1 t - k_1 x) + a\cos(\omega_2 t - k_2 x)$$

• To make the expression wieldy; define the average frequency & wave number and their average deviations

$$\omega = \frac{\omega_1+\omega_2}{2} \qquad \delta\omega = \frac{\omega_1-\omega_2}{2}$$

$$k = \frac{k_1+k_2}{2} \qquad \delta k = \frac{k_1-k_2}{2}$$

solve: $\omega_1 = \omega + \delta\omega \qquad \omega_2 = \omega - \delta\omega$
$k_1 = k + \delta k \qquad k_2 = k - \delta k$

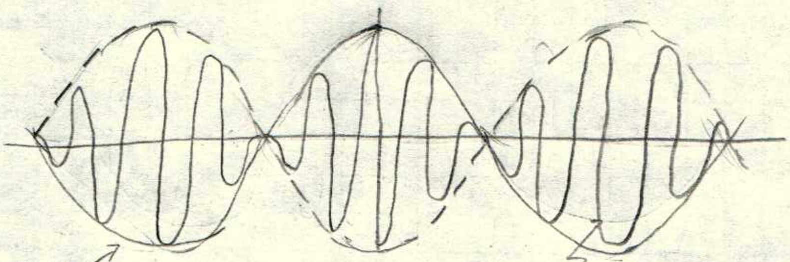
rewrite: $$\varphi = a\cos[(\omega+\delta\omega)t - (k+\delta k)x] + a\cos[(\omega-\delta\omega)t - (k-\delta k)x]$$

$$= a\cos[(\omega t - kx) + (\delta\omega t - \delta k x)] + a\cos[(\omega t - kx) - (\delta\omega t - \delta k x)]$$

expand: $$\varphi = a\{\cos(\omega t - kx)\cos(\delta\omega t - \delta k x) - \sin(\omega t - kx)\sin(\delta\omega t - \delta k x) + \cos(\omega t - kx)\cos[-(\delta\omega t - \delta k x)] - \sin(\omega t - kx)\sin[-(\delta\omega t - \delta k x)]\}$$

simplify: $$\boxed{\varphi = 2a\cos(\omega t - kx)\cos(\delta\omega t - \delta k x)}$$

The shape of φ is given roughly as:



envelope

actual shape of the wave group

- The actual shape of wave group has the frequency of $\omega = (\omega_1 + \omega_2)/2$ and a wave number $k = (k_1 + k_2)/2$.
- The envelope, in contrast, moves with a frequency of $\delta\omega = (\omega_1 - \omega_2)/2$ and a wave number $\delta k = (k_1 - k_2)/2$.
- The phase velocity of each of the constituent waves is

$$v_{p_1} = \frac{\omega_1}{k_1} \quad \text{d} \quad v_{p_2} = \frac{\omega_2}{k_2}$$

which is obtained by setting the phase of each wave as constant and solving for x/t

$$\text{phase}_i = \omega_i t - k_i x \; : \; \text{set phase}_i = 0 \;\Rightarrow\; \frac{x}{t} = \frac{\omega_i}{k_i}$$

- The group velocity (in this case) is obtained by doing an analogous operation with the envelope

$$\text{group-phase} = \delta\omega t - \delta k x \quad \text{set group-phase} = 0 \Rightarrow \frac{x}{t} = \frac{\delta\omega}{\delta k}$$

- Two cases to be considered:
  - a) phase velocities are equal
  - b) phase velocities are unequal

- <u>Phase velocities equal</u>

$$\omega_1 / k_1 = c \qquad \omega_2 / k_2 = c \qquad \Rightarrow \quad \boxed{\omega = kc}$$

- The frequency is linear in k & the group travels along with the individual components; everything moves together

- Phase velocities unequal

$$\omega_1/k_1 \neq \omega_2/k_2 \Rightarrow \omega = \Omega(k)$$ where $\Omega(k)$ is a non-linear function

- In this case ω/k is not constant and the waves are said to satisfy a dispersion relation

- The group velocity can be written as:

$$v_g = \frac{\delta\omega}{\delta k} \Rightarrow \frac{d\omega}{dk} = \frac{d}{dk}(k \underbrace{v_p}_{\text{phase velocity}}) = v_p + k\frac{dv_p}{dk}$$

or $\boxed{V_g = v_p - \lambda \frac{dv_p}{d\lambda}}$

if: $\frac{dv_p}{d\lambda} > 0$ $\quad V_g < v_p$ normal dispersion

if: $\frac{dv_p}{d\lambda} < 0$ $\quad V_g > v_p$ anomalous dispersion.

[Winter] • These relations and concepts hold for arbitrary wave forms.

$$\psi(x,t) = \int_{-\infty}^{\infty} A(k) \underbrace{e^{i[kx - \omega(k)t]}}_{f(kx-\omega(k)t) \Leftarrow \text{elementary wave}} dt$$

- The phase velocity of an elementary wave is given by the equation between x & t for a constant phase

$$kx - \omega(k)t = \text{const}$$

Solve for x: $kx = \omega(k)t + \text{const}$

$$x = \frac{\omega(k)}{k}t + \frac{\text{const}}{k}$$

$$\boxed{x = \frac{\omega(k)}{k}t + \varphi_0}$$

$$\boxed{v_p = \frac{\omega(k)}{k}}$$

- However the group velocity is the velocity of the hump formed by the constructive interference between all the waves.

- The hump is identified as coming from the phase value that changes little (essentially zero) for a small change in k:

$$\frac{d}{dk}\left[kx - \omega(k)t\right]\Big|_{k_0} = x - \frac{d\omega(k)}{dk}\Big|_{k_0} t = 0$$

$$x = \frac{d\omega(k)}{dk}\Big|_{k_0} t$$

$$\Rightarrow \boxed{v_{group} = \frac{d\omega(k)}{dk}}$$

- In quantum mechanics, we require the group velocity to be the usual classical velocity

$$\frac{d\omega(k)}{dk} = \frac{p}{m}$$

- From the de Broglie relation $p = \hbar k$

$$\frac{d\omega(k)}{dk} = \frac{\hbar k}{m} \Rightarrow \boxed{\omega(k) = \frac{\hbar k^2}{2m} + \alpha}$$

- From the Einstein relation $E = \hbar\omega$

$$\Rightarrow \boxed{E = \hbar\,\omega(k) = \frac{\hbar^2 k^2}{2m} + \alpha}$$

Q* • Can this constant of integration be exploited?