Chap. 3 Ludvigsen | Sec. 3.1 | 6/30/03

- ## Distance, Time, and Angle

Flat Spacetime — a region of spacetime where gravitat
tidal effects may be neglected.

The properties of flat spacetime can be explored using light rays (i.e. photons).

In general, if two observers communicate using light rays then the transmit frequency will differ fro the receive frequency.

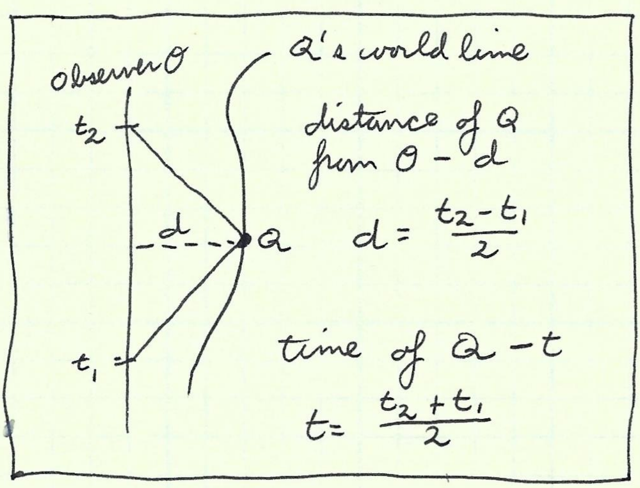
The equivalence class of observers, for which the frequencies are the same are said to be moving with zero relative speed and their world lines are termed parallel.

The equivalence class of observers with parallel world lines, defines a universal time function. This works sinc they always are the same distance apart and their time ca be synchronized as follows.

1) determine the distance between the observers using the process shown in the diagram below.

2) have observer 1 set his time to zero at which time he sends out his signal to observer 2. The signal should tell observer 2 that the time is zero (or any othe value)

3) when observer 2 receives the signal he sets his own clock to $0 + d$, where d is the 'light-distance' between the two observers.



With 3 observers, angles can be defined. Since these observers live on parallel worl lines, each appears as a stationary point of light to the others. The angle betw these points of light is defined as the angle between these even

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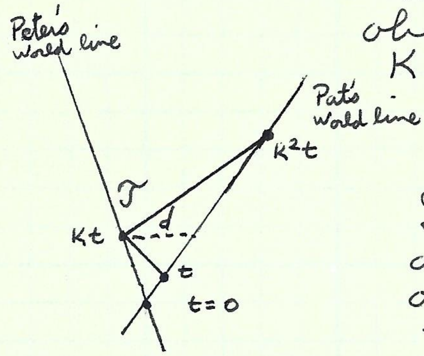
- <u>Speed and the Doppler Effect</u>

Bondi Factor — The number K that depends only on the relative motion between two observers and relates the time between subsequent signals from one observer and the time ~~received~~ between their arrival at another observer. The range of K is always $K \geq 1$



From Pat's perspective, the turn around point, T, on Peter's world line occurs at a distance d and at a time t given by:

$$d = \frac{K^2 t - t}{2} \qquad \tau = \frac{K^2 t + t}{2}$$

The velocity of Peter's world line with respect to Pat at the time τ is given by

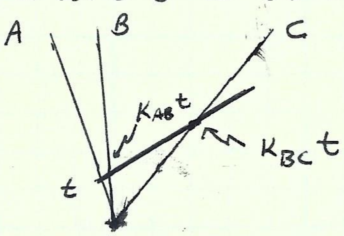
$$\boxed{V = \frac{d}{\tau} = \frac{K^2 t - t}{K^2 t + t} = \frac{K^2 - 1}{K^2 + 1}}$$

Solving for K yields

$$V(K^2+1) = K^2 - 1 \quad \Rightarrow \quad K^2(V-1) = -1-V$$

$$K^2 = \frac{1+V}{1-V} \quad \text{or} \quad \boxed{K = \sqrt{\frac{1+V}{1-V}}}$$

Suppose we have three observers A, B, & C and wish to know how to add the velocity of B wrt A with the velocity of C wrt B to obtain the velocity of C wrt A



$$K_{AC} = K_{AB} K_{BC} = \sqrt{\left(\frac{1+V_{AB}}{1-V_{AB}}\right)\left(\frac{1+V_{BC}}{1-V_{BC}}\right)}$$

but $K_{AC} = \sqrt{\dfrac{1+V_{AC}}{1-V_{AC}}}$

equating $\Rightarrow \quad \dfrac{1+V_{AC}}{1-V_{AC}} = \dfrac{(1+V_{AB})(1+V_{BC})}{(1-V_{AB})(1-V_{BC})}$

$$V_{AC} = \frac{K_{AC}^2 - 1}{K_{AC}^2 + 1} = \frac{\dfrac{(1+V_{AB})(1+V_{BC})}{(1-V_{AB})(1-V_{AC})} - 1}{\dfrac{(1+V_{AB})(1+V_{BC})}{(1-V_{AB})(1-V_{BC})} + 1}$$

$$= \frac{(1+V_{AB})(1+V_{BC}) - (1-V_{AB})(1-V_{BC})}{(1+V_{AB})(1+V_{BC}) + (1-V_{AB})(1-V_{BC})}$$

$$= \frac{1 + V_{AB} + V_{BC} + V_{AB}V_{BC} - 1 + V_{AB} + V_{BC} + V_{AB}V_{BC}}{1 + V_{AB} + V_{BC} + V_{AB}V_{BC} + 1 - V_{AB} + V_{BC} + V_{AB}V_{BC}}$$

$$= \frac{2(V_{AB} + V_{BC})}{2(1 + V_{AB}V_{BC})}$$

$$\boxed{V_{AC} = \frac{V_{AB} + V_{BC}}{1 + V_{AB}V_{BC}}}$$

Finally the formula for the Doppler shift is derived. If an observer transmits at frequency f_{tr} he sends out, by definition, n crests in time Δt. The receiver will obtain the n crests in time $K\Delta t$, thus

$$\boxed{f_{re} = \frac{n}{K\Delta t} = \frac{1}{K} f_{tr} = \sqrt{\frac{1-V}{1+V}}\, f_{tr}}$$

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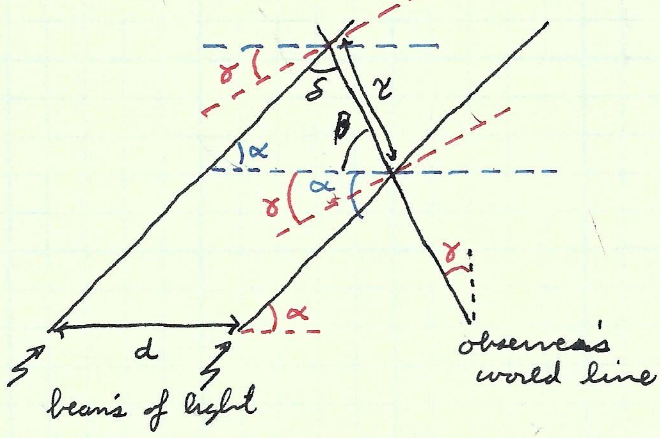
- Looking in depth at the Bondi factor. Consider two rays of light with a spatial separation of d with respect to an observer whose world line is perpendicular to the spatial slice. Then we wish to see how the time between the arrival of each beam of light onto an observer changes with the speed of the observer.



The triangle above yields the relation

$$\alpha + \beta + \delta = \pi$$

Since the red line is perpendicular to our observer

$$\beta + \gamma = \frac{\pi}{2} \quad \Rightarrow \quad \beta = \frac{\pi}{2} - \gamma$$

Combining : $\alpha + \delta + \frac{\pi}{2} - \gamma = \pi$

$$\delta = \frac{\pi}{2} + \gamma - \alpha$$

Law of Sines : $\frac{\tau}{\sin\alpha} = \frac{d}{\sin\delta} \Rightarrow \tau = d\,\frac{\sin\alpha}{\sin(\frac{\pi}{2}+\gamma-\alpha)}$

Now $\alpha = \frac{\pi}{4}$ since light rays are always at 45°

$$\tau = d\,\frac{1}{\sqrt{2}\sin(\gamma + \frac{\pi}{4})}$$

if $\gamma = 0 \quad \Rightarrow \quad \tau = d \cdot \frac{\sqrt{2}}{\sqrt{2}} = d$ which is corr

if $\gamma > 0 \quad \Rightarrow \quad \tau = \frac{d}{\sqrt{2}}\frac{1}{\sin(\frac{\pi}{4}+\gamma)} < d$ since $\sin(\frac{\pi}{4}+\gamma) > s$

if $\gamma < 0 \quad \Rightarrow \quad \tau = \frac{d}{\sqrt{2}}\frac{1}{\sin(\frac{\pi}{4}+\gamma)} > d$ since $\sin(\frac{\pi}{4}+\gamma) <$

so if the observer is moving in the same direction as the light, the Bondi factor is >1 if the observer is moving in the opposite direction, the Bondi factor is <1

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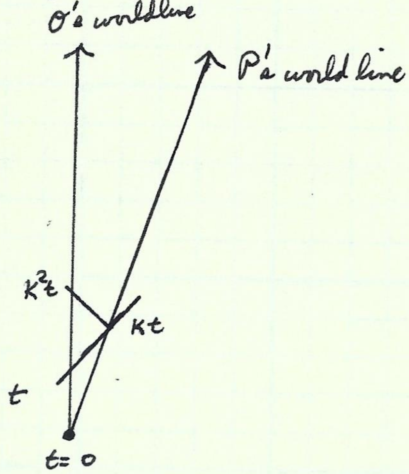
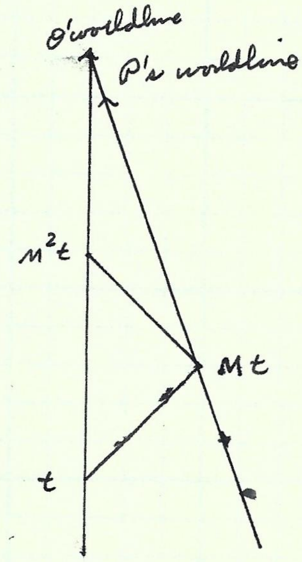
- Ludvigsen's presentation of the Bondi factor leaves much to be desired.

1) He uses the concept of speed, not velocity, but fail to include absolute value signs. So, what I would have found better is (in outline form)





$M = K$

receding

approaching

Insert $K>1$ (see Sec 3.5.1 for geometry) and $M<1$ (see Sec 3.5.1 for geometry). Then the parallel computations go as

Receding

$$x = \frac{K^2t - t}{2}$$

$$\tau = \frac{K^2t + t}{2}$$

$$v = \frac{|x|}{|\tau|} = \frac{K^2-1}{K^2+1} \quad \text{since } K>1$$

$$\Rightarrow v(K^2+1) = K^2-1$$

$$1+v = K^2(1-v)$$

$$K = \sqrt{\frac{1+v}{1-v}}$$

$$\mathcal{V}_R = \frac{x}{\tau} = \frac{K^2-1}{K^2+1}$$

$$\Rightarrow K = \sqrt{\frac{1+\mathcal{V}_R}{1-\mathcal{V}_R}}$$

$$\mathcal{V}_R = v$$

Approaching

$$x = \frac{M^2t - t}{2}$$

$$\tau = \frac{M^2t + t}{2}$$

$$v = \frac{|x|}{|\tau|} = \frac{1-M^2}{1+M^2} \quad \text{since } M<1$$

$$\Rightarrow v(1+M^2) = 1-M^2$$

$$M^2(1+v) = 1-v$$

$$M = \sqrt{\frac{1-v}{1+v}} = K^{-1}$$

$$\mathcal{V}_A = \frac{x}{\tau} = \frac{M^2-1}{M^2+1}$$

$$M = \sqrt{\frac{1+\mathcal{V}_A}{1-\mathcal{V}_A}}$$

$$\mathcal{V}_A = -v$$

- So the Doppler shift frequency formulae are also easily obtained

$$f_{re} = \frac{n}{K\Delta t} = \frac{1}{K}\frac{n}{\Delta t} \qquad\qquad f_{re} = \frac{n}{M\Delta t} = \frac{1}{M}\frac{n}{\Delta t}$$

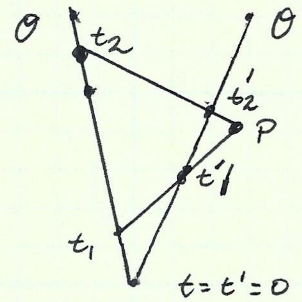
$$f_{re} = \underbrace{\sqrt{\frac{1-v}{1+v}}}_{<1} f_{em} \qquad\qquad f_{re} = \underbrace{\sqrt{\frac{1+v}{1-v}}}_{>1} f_{em}$$

red shift — <u>Receding</u>   blue shift — <u>Approaching</u>

- A nice expression of this is found on page 80 of 'Relativity and Common Sense' by H. Bondi, Dover 1980

— The relation between any two of our inertial observers is completely specified by the ratio of the interval of reception to the interval of transmission. If this ratio is unity ... then the two observers are at relative rest; if it is greater than one they are receding from each other; and if the ratio is less than one they are approaching each other. Note that the Principle of Relativity, by insisting on the equivalence of all inertial observers, makes it quite clear that the ratio must be the same whichever of a pair of inertial observers does the transmitting. It is through this rule that our work on light differs so sharply from the work on sound where, it will be remembered, the speed of transmitter and receiver relative to the air had also to be taken into account.

3.1 Consider the figure below. Observer $\mathcal{O}$ will say that event p occurred at distance $x=(t_2-t_1)/2$ and at time $t=(t_1+t_2)/2$ and Observer $\mathcal{O}'$ will say that p occurred at distance $x'=(t_2'-t_1')/2$ at time $t'=(t_2'+t_1')/2$. If $\mathcal{O}$ & $\mathcal{O}'$ have relative speed v, find the transformation between (t,x) and (t',x')



• Using the Bondi factor

$$\boxed{\begin{aligned} t_1' &= K t_1 \\ t_2 &= K t_2' \end{aligned}}$$

• Now isolate t_2 & t_1:

$$x+t = \frac{(t_2-t_1)}{2} + \frac{(t_2+t_1)}{2} = t_2$$

$$t-x = \frac{(t_2+t_1)}{2} - \frac{(t_2-t_1)}{2} = t_1$$

$$\Rightarrow \begin{aligned} t_1 &= \tfrac{1}{K} t_1' = \tfrac{1}{K}(t'-x') \\ t_2 &= K t_2' = K(t'+x') \end{aligned} \Rightarrow \begin{aligned} x &= \frac{K(t'+x') - \frac{1}{K}(t'-x')}{2} \\ t &= \frac{K(t'+x') + \frac{1}{K}(t'-x')}{2} \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} x &= \frac{K-\frac{1}{K}}{2} t' + \frac{K+\frac{1}{K}}{2} x' \\ t &= \frac{K+\frac{1}{K}}{2} t' + \frac{K-\frac{1}{K}}{2} x' \end{aligned}}$$

• Since $K=\sqrt{\frac{1+v}{1-v}}$ $\Rightarrow$

$$\left(K+\tfrac{1}{K}\right)^2 = K^2 + \frac{1}{K^2} + 2 = \frac{1+v}{1-v} + \frac{1-v}{1+v} + 2$$

$$= \frac{(1+v)^2 + (1-v)^2 + 2(1-v^2)}{(1-v^2)}$$

$$= \frac{1+2v+v^2+1-2v+v^2+2-2v^2}{(1-v^2)}$$

$$= \frac{4}{(1-v^2)}$$

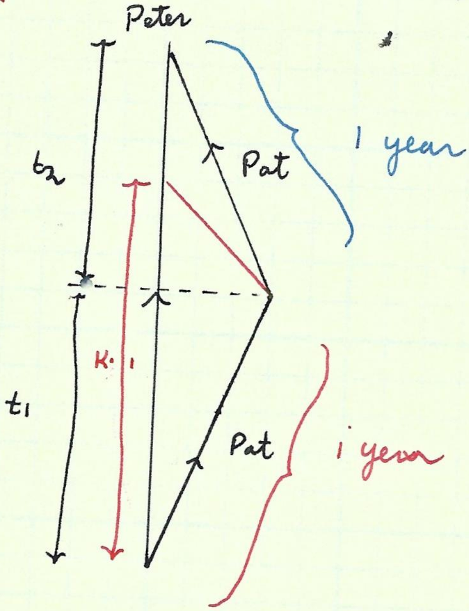
$$\left(K-\tfrac{1}{K}\right)^2 = K^2 + \frac{1}{K^2} - 2 = \frac{1+v}{1-v} + \frac{1-v}{1+v} - 2$$

$$= \frac{(1+v)^2 + (1-v)^2 - 2(1-v^2)}{(1-v^2)}$$

$$= \frac{1+2v+v^2+1-2v+v^2-2+2v^2}{(1-v^2)} = \frac{4v^2}{(1-v^2)}$$

$$\Rightarrow \boxed{\begin{aligned} x &= \frac{1}{\sqrt{1-v^2}}(vt'+x') \\ t &= \frac{1}{\sqrt{1-v^2}}(t'+vx') \end{aligned}} \quad \& \quad \boxed{\begin{aligned} x' &= \frac{1}{\sqrt{1-v^2}}(x-vt) \\ t' &= \frac{1}{\sqrt{1-v^2}}(t-vx) \end{aligned}}$$

3,3] Pat says goodbye to Peter, gets into her spaceship, a
travels off at half the speed of light. After one year she be
homesick and returns to her friend Peter, again at half the sp
of light, returning when she is two years ~~later~~ older. How m
older will Peter be on her return.



$V = 1/2$

$K = \sqrt{\frac{1+V}{1-V}} = \sqrt{\frac{3/2}{1/2}}$

t_1: receding $\Rightarrow$ $K t_1 = 1$ year

t_2: approaching $\Rightarrow$ $M t_2 = 1$ year

$$\Rightarrow \quad t_1 + t_2 = 1\,\text{year}\left(\frac{1}{K} + \frac{1}{M}\right)$$

$$= 1\,\text{year}\left(\frac{1}{K} + K\right)$$

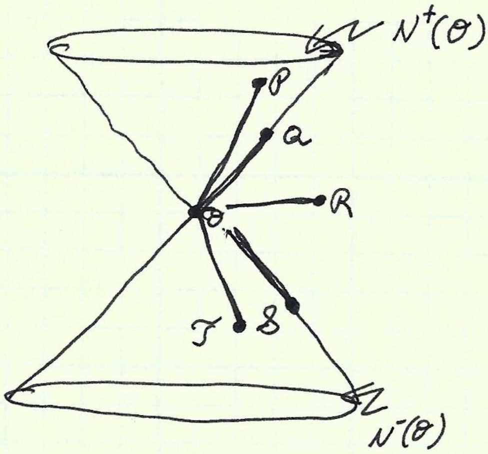
$$t_{tot} = t_1 + t_2 = 1\,\text{year}\left(\frac{1}{K} + K\right)$$

$$\boxed{t_{tot} = 1\,\text{year}\left(\frac{1}{\sqrt{3}} + \sqrt{3}\right)}$$

3.4] A train, 100 yards long, hurtles through a tunnel, 50 yards long. Show that if its speed is $\sqrt{3}/2$, then the driver (at the front) will leave the tunnel just as the guard (at the back) is entering it, according to a stationary observer in the tunnel.

• Spacetime Vectors

spacetime displacement - ordered pair of spacetime points



$\mathcal{O}P$ - future-pointing timelike

$\mathcal{O}Q$ - future-pointing null

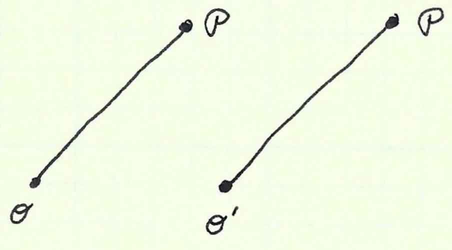
$\mathcal{O}R$ - spacelike

$\mathcal{O}S$ - past-pointing null

$\mathcal{O}T$ - past-pointing timelike

rules: $\mathcal{O}P = -P\mathcal{O}$; $\mathcal{O}P + PQ = \mathcal{O}Q$

All displacements can be partitioned into equivalence classes based on points connected by parallel lines with equal amounts of elapsed path parameters.



$\Rightarrow \quad \vec{v} = \{\mathcal{O}P\}$

spacetime vectors: equivalence class of all displacements equivalent to $\mathcal{O}P$.

rules: $\alpha \vec{v} = \{\alpha \mathcal{O}P\}$

$\vec{v} + \vec{w} = \{\mathcal{O}P + PQ\}$

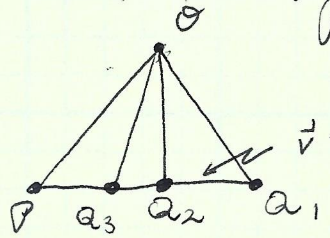
Mathematically, flat spacetime M is an affine space.

[Ref - Chap. 1 of Bamberg & Sternberg]

• <u>Spacetime Metric</u>

four-velocity vector — defined as the spacetime vector equivalent to the displacement OP provided OP is timelike and $t(P) - t(O) = 1$

three-dimensional subspace — Take some origin O and a four-velocity vector $\vec{v} = \{OP\}$, then the set of all vectors $\vec{x} = \{OQ\}$ where Q is synchronous with P (from the perspective of an observer with four-velocity $\vec{v}$) forms a three-dimensional subspace $\vec{v}^{\perp}$

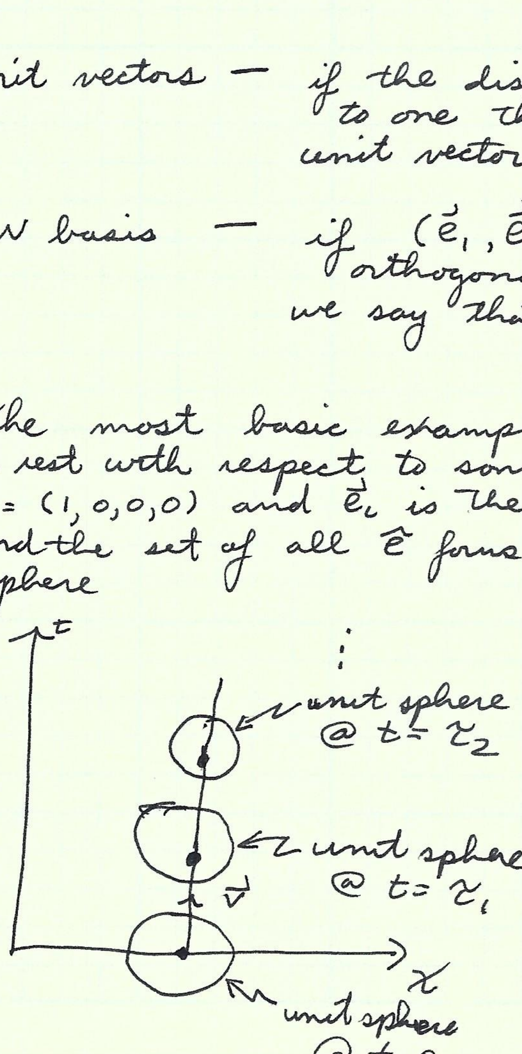


orthogonal vectors — if $\vec{e}_1 = OQ_1$ and $\vec{e}_2 = OQ_2$ are such that the angle between them is 90° then $\vec{e}_1$ and $\vec{e}_2$ are orthogonal

unit vectors — if the distance between O & Q_i is equal to one then $\vec{e}_i = OQ_i$ is termed a unit vector.

ON basis — if $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is a right-handed orthogonal basis of unit vectors in $\vec{v}^{\perp}$ we say that $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ where $\vec{v} = \vec{e}_0$

The most basic exampl of an ON basis is an observe at rest with respect to some coordinate system. Then $\vec{v} = (1,0,0,0)$ and $\vec{e}_i$ is the normal concept of a triad $\{\hat{i}, \hat{j},$ and the set of all $\hat{e}$ forms a two-sphere called the unit sphere



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Convention — Greek indices 0 to 3 $\{\vec{e}_\alpha\} = (\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}$
Latin indices 1 to 3 $\{\vec{e}_i\} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$

Expressing a vector — Any vector $\vec{w}$ can be expressed as
$\vec{w} = w^0\vec{e}_0 + w^1\vec{e}_1 + w^2\vec{e}_2 + w^3\vec{e}_3 = w^\alpha \vec{e}_\alpha$
where $w^\alpha = (w^0, w^1, w^2, w^3)$ are the compone

Projection — The projection of $\vec{w}$ into $\vec{v}^\perp$ is easily writte as $\overline{w} = w^i \vec{e}_i$

Spacetime Metric — There exists a unique quadratic form g, called the spacetime metric such that

i) $g(\vec{x}, \vec{x}) = 1, 0,$ or -1 for $\vec{x}$ - four-velocity vector, $\vec{x}$ - null vector, $\vec{x}$ - unit spacelike vector, respectively

ii) $g(\vec{v}, \vec{e}) = 0$ if $\vec{v}$ - four velocity and $\vec{e} \in \vec{v}^\perp$

iii) $g(\vec{e}_1, \vec{e}_2) = 0$ if $\vec{e}_1$ & $\vec{e}_2 \in \vec{v}^\perp$ and $\vec{e}_1$ & $\vec{e}_2$ orthogonal

Bilinear mapping — Q is a bilinear mapping iff $Q(\vec{a}, \vec{b}) = \pm Q(\vec{b},$
$Q(\alpha\vec{a}, \vec{b}) = \alpha Q(\vec{a}, \vec{b})$, $Q(\vec{a}, \alpha\vec{b}) = \alpha Q(\vec{a}, \vec{b})$
for the '+' sign Q is called a quadratic for for the '−' sign Q is called a two-form. In terms of a basis $\{\vec{e}_\alpha\}$, the components of Q are given by $Q_{\alpha\beta} = Q(\vec{e}_\alpha, \vec{e}_\beta)$

Linear transformation — define an object $\hat{Q}$ with one slot that is related to Q as follows $\hat{Q}(\vec{a}) = Q(\vec{a}, \;)$
Then $\hat{Q}(\vec{a}) : V \to V$ such that $\hat{Q}(\vec{a}) \cdot \vec{b} = Q(\vec{a}, \vec{b}$