Solving Bernoulli's equation

$$\frac{dy}{dt} + f(t)\,y = g(t)$$

- First look at the homogeneous equation

$$\frac{dy}{dt} + f(t)\,y = 0$$

$$\frac{dy}{y} = -f(t)\,dt$$

$$\int \frac{dy}{y} = -\int f(t)\,dt$$

$$\ln y = -\int_0^t f(t')\,dt' + \text{const}$$

$$y = y_0 \exp\left[-\int_0^t dt' f(t')\right]$$

- To solve the inhomogeneous equation; assume $y_0 = y_0(t)$

$$\frac{dy}{dt} = y_0 \frac{d}{dt} \exp\left[-\int_0^t dt' f(t')\right] + \frac{dy_0}{dt} \exp\left[-\int_0^t dt' f(t')\right]$$

$$= y_0 \exp\left[-\int_0^t dt' f(t')\right] \frac{d}{dt}\left[-\int_0^t dt' f(t')\right] + \frac{dy_0}{dt} \exp\left[-\int_0^t dt' f(t')\right]$$

$$= y_0 \exp[\quad](-f(t)) + \frac{dy_0}{dt} \exp[\;]$$

$$\Rightarrow \frac{dy}{dt} + f y = g \quad \Rightarrow -y_0 \exp[\;] f(t) + \frac{dy_0}{dt} \exp[\;] + y_0 \exp[\;] f(t) = g$$

$$\Rightarrow \frac{dy_0}{dt} \exp[\;] = g$$

$$dy_0 = g \exp\left[\int_0^t dt' f(t')\right] dt$$

$$y_0 = y_0^* + \int_0^t dt' g(t') \exp\left[\int_0^{t'} dt'' f(t'')\right]$$

defining $Q_\pm(t) = \exp\left[\pm\int_0^t dt' f(t')\right]$

$$y_0 = y_0^* + \int_0^t dt' g(t') Q_+(t')$$

& $$y(t) = \left(y_0^* + \int_0^t dt' g(t') Q_+(t')\right) Q_-(t)$$

$$\boxed{y(t) = y_0^* Q_-(t) + Q_-(t)\int_0^t dt' g(t') Q_+(t')}$$

- Battin problem 10-1.(1)

$$\frac{dy}{dt} - \alpha y = e^{\alpha t} \quad (\alpha \text{ constant})$$

$f(t) = -\alpha \qquad Q_{\pm} = \exp\left[\mp \int_0^t \alpha \, dt'\right] = \exp\left[\mp \alpha t\right]$

$g(t) = e^{\alpha t} \qquad y(t) = y_0^* Q_-(t) + Q_-(t) \int_0^t dt' \, g(t') \, Q_+(t')$

$$= y_0^* e^{\alpha t} + e^{\alpha t} \int_0^t dt' \, e^{\alpha t'} e^{-\alpha t'}$$

$$\boxed{y(t) = y_0^* e^{\alpha t} + e^{\alpha t} t}$$

check w/ sympy