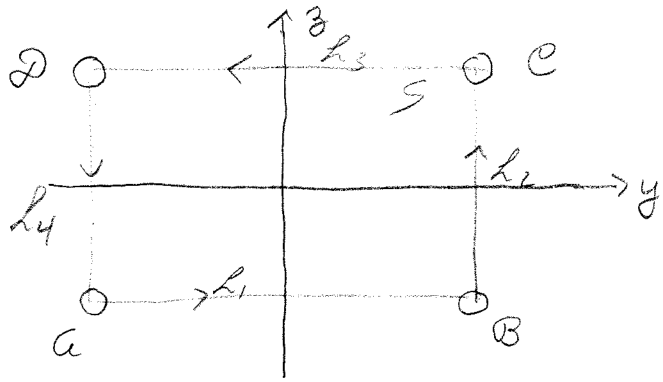

$$\mathcal{A} = (t, -t, -1)$$
$$\mathcal{B} = (t, t, -1)$$
$$\mathcal{C} = (t, t, 1)$$
$$\mathcal{D} = (t, -t, 1)$$

$$S \doteq \{ \vec{R}(u,v) = \mathcal{A} + u(\mathcal{B}-\mathcal{A}) + v(\mathcal{D}-\mathcal{A}) \mid u,v \in [0,1] \}$$

$$\vec{R}(u,v) = (t,\ (2u-1)t,\ 2v-1) \qquad u,v \in [0,1]$$

$$\frac{\partial \vec{R}}{\partial u} = (0, 2t, 0) \qquad \frac{\partial \vec{R}}{\partial v} = (0,0,2) \qquad d\vec{S} = \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\, du\, dv$$
$$= 4t\hat{\imath}\, du\, dv$$

$$\text{Area}(S) = \int_0^1 du \int_0^1 dv\, |d\vec{S}| = \int_0^1 du \int_0^1 dv\, 4t = 4t \quad \checkmark$$

Calculate the flux at some instant of time from the vector field $\vec{F} = (xy^2, yz^2, zx^2)$ through S.

$$\Phi[\vec{F}, S] = \int_S \vec{F}(\vec{R}(u,v)) \cdot d\vec{S} = \int_0^1 du \int_0^1 dv\, (xy^2)(u,v)\, 4t$$
$$= \int_0^1 du \int_0^1 dv\, t \cdot (2u-1)^2 t^2\, 4t$$
$$= \int_0^1 du\, 4t^4 (2u-1)^2$$
$$= 4t^4 \int_0^1 (2u-1)^2\, du$$

$$\int_0^1 (2u-1)^2 du = \frac{1}{2}\int_{-1}^{1} \lambda^2 d\lambda = \frac{1}{6}\lambda^3\Big|_{-1}^{1} = \frac{2}{6} = \frac{1}{3}$$
$$\lambda = 2u-1 \quad \lambda(u=0) = -1 \quad \lambda(u=1) = 1$$
$$d\lambda = 2du$$

$$\boxed{\Phi[\vec{F}, S] = \frac{4t^4}{3} \qquad \frac{d\Phi[\vec{F}, S]}{dt} = \frac{16t^3}{3}}$$

To use the Flux transport theorem, need to evaluate $\text{div}(\vec{F})$ and $\frac{\partial \vec{F}}{\partial t}$ on S and $\vec{F} \times \vec{v}$ along the boundary

$$\text{div}(\vec{F}) = x^2 + y^2 + z^2 \qquad \text{div}(\vec{F})(\vec{Q}(u,v)) = t^2 + (2u-1)^2 t^2 + (2v-1)^2$$

$$\vec{v} = \frac{\partial \vec{Q}(u,v)}{\partial t} = (1, 2u-1, 0) \qquad \frac{\partial \vec{F}(\vec{Q}(u,v))}{\partial t} = 0$$

$$Q_0 = \int_S \left[\text{div}(\vec{F})(\vec{Q}(u,v))\,\vec{v} + \frac{\partial \vec{F}(\vec{Q}(u,v))}{\partial t}\right] \cdot d\vec{S}$$

$$= \int_0^1 du \int_0^1 dv \; (t^2 + (2u-1)^2 t^2 + (2v-1)^2)\, 4t$$

$$= 4t \int_0^1 du \int_0^1 dv \; (t^2 + (2u-1)^2 t^2 + (2v-1)^2)$$

$$= 4t\left[t^2 \int_0^1 du \int_0^1 dv + t^2 \int_0^1 du\,(2u-1)^2 \int_0^1 dv + \int_0^1 du \int_0^1 (2v-1)^2 dv\right]$$

$$= 4t\left[t^2 + \frac{t^2}{3} + \frac{1}{3}\right]$$

$$= 4t\left[\frac{4t^2}{3} + \frac{1}{3}\right]$$

$$\boxed{Q_0 = \frac{16t^3}{3} + \frac{4t}{3}}$$

$$\vec{F} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ xy^2 & yz^2 & zx^2 \\ 1 & 2u-1 & 0 \end{vmatrix} = -zx^2(2u-1)\,\hat{\imath} + zx^2\,\hat{\jmath} + [xy^2(2u-1) - yz^2]\,\hat{k}$$

$$= -(2v-1)\cdot t^2(2u-1)\,\hat{\imath} + (2v-1)t^2\,\hat{\jmath} +$$

$$+ \left[t(2u-1)^3 t^2 - (2u-1)t(2v-1)^2\right]\hat{k}$$

$$\vec{L}_1(u) = \mathcal{A} + u(\mathcal{B} - \mathcal{A}) = (t, -t, -1) + u(0, 2t, 0) = (t, (2u-1)t, -1)$$
$$\vec{L}_2(v) = \mathcal{B} + v(\mathcal{C} - \mathcal{B}) = (t, t, -1) + v(0, 0, 2) = (t, t, 2v-1)$$
$$\vec{L}_3(u) = \mathcal{C} + (u)(\mathcal{D} - \mathcal{C}) = (t, t, 1) + (u)(0, -2t, 0) = (t, (1-2u)t, 1)$$
$$\vec{L}_4(v) = \mathcal{D} + v(\mathcal{A} - \mathcal{D}) = (t, -t, 1) + v(0, 0, -2) = (t, -t, 1-2v)$$

$$Q_1 = \int_{L_1} (\vec{F} \times \vec{v}) \cdot d\vec{L}_1 = \int_0^1 (2v-1)t^2\, 2t \; du \Big|_{v=0} = \int_0^1 du\,(-2t^3) = -2t^3$$

$$Q_2 = \int_{L_2} (\vec{F} \times \vec{v}) \cdot d\vec{L}_2 = \int_0^1 2\left[t(2u-1)^3 t^2 - (2u-1)t(2v-1)^2\right] dv \Big|_{u=1} =$$

$$= 2t^3 \int_0^1 dv - 2t \int_0^1 (2v-1)^2 = 2t^3 - \frac{2}{3}t = \frac{2(3t^3 - t)}{3}$$

$$Q_3 = \int_{L_3} (\vec{F} \times \vec{v}) \cdot d\vec{L}_3 = \int_0^1 (-2t)\, du\, (2v-1)t^2 \Big|_{v=1} = -2t^3 \int_0^1 du = -2t^3$$

$$Q_4 = \int_{L_4} (\vec{F} \times \vec{v}) \cdot d\vec{L}_4 = \int_0^1 (-2)\, dv \left[t(2u-1)3t^2 - (2v-1)t(2v-1)^2 \right] \Big|_{u=0}$$

$$= (-2) \int_0^1 dv \left[t^3(-1)3 - (-1)t(2v-1)^2 \right]$$

$$= (2)t^3 \int_0^1 dv + (-2)t \int_0^1 dv (2v-1)^2$$

$$= 2t^3 - \tfrac{2}{3} t$$

$$Q_c = Q_1 + Q_2 + Q_3 + Q_4$$

$$= -2t^3 + 2t^3 - \tfrac{2}{3}t - 2t^3 + 2t^3 - \tfrac{2}{3}t$$

$$\boxed{Q_c = -\tfrac{4}{3}t}$$

$$Q_{tot} = Q_0 + Q_c$$

$$= \frac{16t^3}{3} + \frac{4t}{3} - \frac{4t}{3}$$

$$\boxed{Q_{tot} = \frac{16t^3}{3}}$$

Alternatively Q_c can be re-written as

$$Q_c = \int_S d\vec{S} \cdot \vec{\nabla} \times (\vec{F} \times \vec{v})$$

But here $\vec{v}$ must be written in the image space (x, y, z) not in the starting space (u, v). In other words, we need:

$$\vec{v} = \vec{v}(x, y, z)$$ } : Eulerian view (push-forward) rather than Lagrangian (pull-back)

To obtain this, we push $\vec{v}(u, v; t)$ to $\vec{v}(x, y, z; t)$ by using $\vec{R}(u, v; t)$ to solve and invert as follows

$\vec{R}(u, v; t) \Rightarrow \quad x = t, \quad y = (2u-1)t \quad z = (2v-1)$

Then $\vec{v} = (1, 2u-1, 0) \Rightarrow \vec{v} = \left(\frac{x}{t}, \frac{y}{t}, 0\right)$

Thus $$\vec{F} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xy^2 & yz^2 & zx^2 \\ \frac{x}{t} & \frac{y}{t} & 0 \end{vmatrix} = -\frac{zx^2 y}{t} \hat{i} + \frac{zx^3}{t} \hat{j} + \frac{(xy^3 - xyz^2)}{t} \hat{k}$$

and $\vec{\nabla}\times(\vec{F}\times\vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{-3x^2y}{t} & \frac{3x^3}{t} & \frac{(xy^3 - xyz^2)}{t} \end{vmatrix}$

$$= \left(\frac{3xy^2 - xz^2}{t} - \frac{x^3}{t}\right)\hat{i} - \left[\frac{(y^3 - yz^2)}{t} + \frac{x^2y}{t}\right]\hat{j}$$

$$+ \left(\frac{3zx^2}{t} + \frac{3x^2}{t}\right)\hat{k} \quad \checkmark$$

$$\vec{\nabla}\times(\vec{F}\times\vec{v})\cdot d\vec{S} = 4\,du\,dv\,(3xy^2 - xz^2 - x^3)$$

Finally substitute in $x = t \quad y = (2u-1)t \quad z = (2v-1)$

$$\vec{\nabla}\times(\vec{F}\times\vec{v})\cdot d\vec{S} = 4\,du\,dv\,(3t\,(2u-1)^2t^2 - t\,(2v-1)^2 - t^3)$$

$$\int_S \vec{\nabla}\times(\vec{F}\times\vec{v})\cdot d\vec{S} = \int_0^1\int_0^1 du\,dv\; 4t(3t(2u-1)^2t^2 - t(2v-1)^2 - t^3)$$

$$= 12t^3\int_0^1\int_0^1 du\,dv\,(2u-1)^2 - 4t\int_0^1\int_0^1 du\,dv\,(2v-1)^2$$

$$- 4t^3\int_0^1\int_0^1 du\,dv$$

$$= 4t^3 - \frac{4}{3}t - 4t^3$$

$$\boxed{\int_S \vec{\nabla}(\vec{F}\times\vec{v})\cdot d\vec{S} = -\frac{4}{3}t = Q_c}$$

$$\Rightarrow \boxed{\int_S \vec{\nabla}\times(\vec{F}\times\vec{v})\cdot d\vec{S} = \int_{\partial S}(\vec{F}\times\vec{v})\cdot d\vec{L}}$$