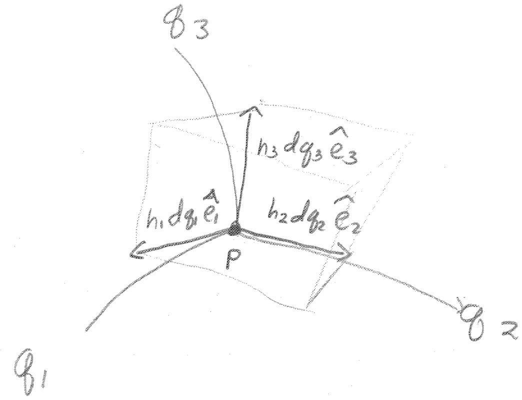
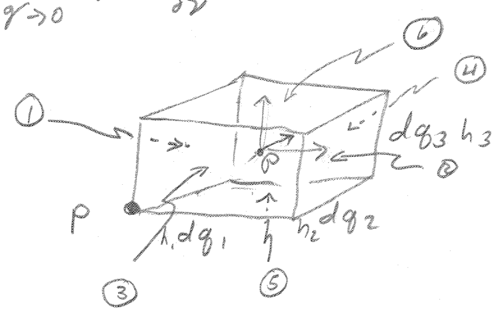
- We wish to show that $\operatorname{div}(\vec{F}) = \sum_i \frac{1}{\Omega}\partial_{q_i}\left(\frac{\Omega F_i}{h_i}\right)$ where $\Omega = h_1 h_2 h_3$, $h_i = |\vec{e}_i|$.

Consider an infinitesimal volume whose boundaries are the coordinate planes in curvilinear coordinates



$$\text{Volume @ } P = h_1 dq_1 \cdot h_2 dq_2 \cdot h_3 dq_3 = \Omega\, dq_1 dq_2 dq_3$$

$$\operatorname{div}(\vec{F}) = \lim_{\mathcal{V}\to 0} \frac{1}{\mathcal{V}} \int_{\partial \mathcal{V}} \vec{F}\cdot\hat{n}\, dS$$



$\vec{F}$ evaluated at the center of each face

Face ①: $\hat{n} = -\hat{e}_1$ $\quad dA = h_2 \cdot h_3 (q_1 - \frac{dq_1}{2}, q_2, q_3) dq_2 dq_3 \quad \vec{F}(q_1 - \frac{dq_1}{2}, q_2, q_3)$
Face ②: $\hat{n} = \hat{e}_1$ $\quad dA = h_2 \cdot h_3 (q_1 + \frac{dq_1}{2}, q_2, q_3) dq_2 dq_3 \quad \vec{F}(q_1 + \frac{dq_1}{2}, q_2, q_3)$
Face ③: $\hat{n} = -\hat{e}_2$ $\quad dA = h_1 \cdot h_3 (q_1, q_2 - \frac{dq_2}{2}, q_3) dq_1 dq_3 \quad \vec{F}(q_1, q_2 - \frac{dq_2}{2}, q_3)$
Face ④: $\hat{n} = \hat{e}_2$ $\quad dA = h_1 \cdot h_3 (q_1, q_2 + \frac{dq_2}{2}, q_3) dq_1 dq_3 \quad \vec{F}(q_1, q_2 + \frac{dq_2}{2}, q_3)$
Face ⑤: $\hat{n} = -\hat{e}_3$ $\quad dA = h_1 \cdot h_2 (q_1, q_2, q_3 - \frac{dq_3}{2}) dq_1 dq_2 \quad \vec{F}(q_1, q_2, q_3 - \frac{dq_3}{2})$
Face ⑥: $\hat{n} = \hat{e}_3$ $\quad dA = h_1 \cdot h_2 (q_1, q_2, q_3 + \frac{dq_3}{2}) dq_1 dq_2 \quad \vec{F}(q_1, q_2, q_3 + \frac{dq_3}{2})$

$$\begin{aligned}
\int_{\partial \mathcal{V}} \vec{F}\cdot\hat{n}\, dS = &\left(F_1 h_2 h_3 (q_1 + \tfrac{dq_1}{2}, q_2, q_3) - F_1 h_2 h_3 (q_1 - \tfrac{dq_1}{2}, q_2, q_3)\right) dq_2 dq_3 \\
&+ \left(F_2 h_1 h_3 (q_1, q_2 + \tfrac{dq_2}{2}, q_3) - F_2 h_1 h_3 (q_1, q_2 - \tfrac{dq_2}{2}, q_3)\right) dq_1 dq_3 \\
&+ \left(F_3 h_1 h_2 (q_1, q_2, q_3 + \tfrac{dq_3}{2}) - F_3 h_1 h_2 (q_1, q_2, q_3 - \tfrac{dq_3}{2})\right) dq_1 dq_2
\end{aligned}$$

$$= \left(\frac{\partial}{\partial q_1}(F_1 h_2 h_3) + \frac{\partial}{\partial q_2}(F_2 h_1 h_3) + \frac{\partial}{\partial q_3}(F_3 h_1 h_2)\right) dq_1 dq_2 dq_3$$

$$= \frac{\partial}{\partial q_1}\left(\frac{F_1 \Omega}{h_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{F_2 \Omega}{h_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{F_3 \Omega}{h_3}\right)$$

$$\Rightarrow \boxed{\operatorname{div}(\vec{F}) = \sum \frac{1}{\Omega}\frac{\partial}{\partial q_i}\left(\frac{F_i \Omega}{h_i}\right)}$$