- A particular solution to an inhomogeneous 2^{nd}-order linear canonical ODE can be constructed by the variation-of-parameters method.

1) Assume the two linearly independent solutions to the homogeneous problem exist and are denoted as $y_1(x)$ & $y_2(x)$

2) Form the general linear combination with the constants in the homogeneous case now being thought of as functions

$$u(x) = v_1(x)\,y_1(x) + v_2(x)\,y_2(x)$$

3) The two functions $v_1(x)$ & $v_2(x)$ are subject to two conditions

a) <u>algebraic arbitrarily</u> - v_1 & v_2 can be adjusted arbitrarily as follows:

$$\tilde{u}(x) = [v_1(x) + g(x)]y_1(x) + [v_2(x) + h(x)]y_2(x)$$
$$= v_1(x)y_1(x) + v_2(x)y_2(x)$$
$$+ g(x)y_1(x) + h(x)y_2(x)$$

$\tilde{u}(x)$ can be made equal to $u(x)$ by setting the last two terms equal to zero and solving for $h(x)$ as

$$h(x) = \frac{-g(x)\,y_1(x)}{y_2(x)}$$

b) <u>differential</u> - using the algebraic freedom $v_1(x)$ and $v_2(x)$ can be adjusted such ~~that~~ $v_1(x)$ and $v_2(x)$ behave as if they were constants for the first derivative of u

$$u' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2'$$
$$= (v_1'y_1 + v_2'y_2) + v_1y_1' + v_2y_2'$$
$$= v_1y_1' + v_2y_2'$$

$$\boxed{\Rightarrow \quad v_1'y_1 + v_2'y_2 = 0 \quad \text{condition of osculation}}$$

4) Substituting in the particular solution $u(x)$ into the LHS of the ODE yields

$$
\begin{aligned}
u'' + pu' + Qu &= (v_1 y_1 + v_2 y_2)'' + p(v_1 y_1 + v_2 y_2)' + Qu \\
&= (v_1 y_1' + v_2 y_2')' + p(v_1 y_1' + v_2 y_2') + Qu \\
&= v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'' + p(v_1 y_1' + v_2 y_2') + Qu \\
&= v_1' y_1' + v_2' y_2' + v_1(y_1'' + py_1' + Qy_1) + v_2(y_2'' + py_2' + Qy_2) \\
&= v_1' y_1' + v_2' y_2'
\end{aligned}
$$

where two facts were used: a) y_1 & y_2 are solutions of the homogeneous ODE so that $y_1'' + py_1' + Qy_1 = 0$ and $y_2'' + py_2' + Qy_2 = 0$ and b) $v_1' y_1 + v_2' y_2 = 0$.

5) Combining the condition of osculation and the particular equation in item 4), yields two differential equations for v_1' & v_2'

$$
\begin{aligned}
v_1' y_1 + v_2' y_2 &= 0 \\
v_1' y_1' + v_2' y_2' &= R
\end{aligned}
\quad \Rightarrow \quad
\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}
\begin{bmatrix} v_1' \\ v_2' \end{bmatrix}
=
\begin{bmatrix} 0 \\ R \end{bmatrix}
$$

which is readily solved to yield

$$
\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{1}{W}
\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}
\begin{bmatrix} 0 \\ R \end{bmatrix} \qquad W - \text{Wronskian}
$$

$$
= \begin{bmatrix} \dfrac{-y_2}{WR} \\[2mm] \dfrac{y_1}{WR} \end{bmatrix}
$$

so that

$$
\boxed{
\begin{aligned}
v_1(x) &= -\int \frac{y_2(x')}{W(x')R(x')}\, dx' \\
v_2(x) &= \int \frac{y_1(x')}{W(x')R(x')}\, dx'
\end{aligned}
}
$$