· We wish to prove Green's 1st Theorem

$$\int_{\mathcal{V}} (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, d\mathcal{V} = \int_{S} \phi \nabla \psi \cdot \hat{n} \, dS$$

Start by defining the vector $\vec{A} = \phi \nabla \psi$. Then $\text{div}(\vec{A})$ is given by

$$\text{div}(\vec{A}) = \vec{\nabla} \cdot (\phi \nabla \psi) = \vec{\nabla} \phi \cdot \vec{\nabla} \psi + \phi \nabla^2 \psi$$

So we can write

$$\int_{\mathcal{V}} (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, d\mathcal{V} = \int_{\mathcal{V}} \text{div}(\vec{A}) \, d\mathcal{V}$$

and using the divergence theorem

$$\int_{\mathcal{V}} \text{div}(\vec{A}) \, d\mathcal{V} = \int_{S} \vec{A} \cdot \hat{n} \, d\mathcal{V} = \int_{S} \phi \nabla \psi \cdot \hat{n} \, dS$$