1 Introduction to Case Study 0

We want to examine the solutions to the equation

$$\frac{d^2}{dt^2}z + p\frac{d}{dt}z + qz = te^{2t} \tag{1}$$

with $p = -4$ and $q = 4$, by comparing and contrasting a variety of methods. First we will solve the homogeneous equation equation through 2 separate methods. The first method will be through the standard second-order differential equation methods. The second method is based on the state space formalism.

2 Standard Second Order Method

In this section, we tackle Eq.(1) by first calculating the characteristic equation for the homogeneous equation. Ordinarily the characteristic equation yields two separate roots yielding two independent solutions. However, in this case, the roots are repeated and the reduction of order method must be used. Once two independent solutions are obtained the particular solution is constructed by using the Variation of Parameters method.

2.1 Charateristic Equation

To determine the characteristic equation assume that the solution has the form

$$z = \exp(rt) \ ,$$

Substituting this form into Eq. (1) yields

$$r^2 - 4r + 4 = 0 \ ,$$

which is solved immediately as $(r-2)^2 = 0$. The first independent solution

$$y_1 = e^{2t}$$

is then arrived at. However, this is as far as it goes due to the double root.

2.2 Reduction of Order

To obtain the second independent solution we turn to the *reduction of order* technique. Start by defining

$$y(t) = y_1(t)v(t).$$

Taking the first and second derivatives yields

$$\dot{y} = y_1\dot{v} + \dot{y}_1 v$$

and

$$\ddot{y} = \ddot{y}_1 v + 2\dot{y}_1 \dot{v} + y_1 \ddot{v},$$

respectively. Substituting these forms for $\dot{y}$ and $\ddot{y}$ into the differential equation $\ddot{y} + p\dot{y} + qy = 0$ yields

$$(\ddot{y}_1 + p\dot{y}_1 + qy_1)\, v + (2\dot{y}_1 + py_1)\, \dot{v} + y_1 \ddot{v} = 0.$$

The first term cancels since y_1 is a solution of the differential equation. This leaves the following equation for $w = \dot{v}$

$$y_1 \dot{w} + (2\dot{y}_1 + py_1) w = 0$$

which can be rearranged as

$$\frac{\dot{w}}{w} = -\frac{(2\dot{y}_1 + py_1)\dot{v}}{y_1}$$

which, since it is first-order, can be immediately integrated to

$$\begin{aligned} w &= c \exp\left[-\int dt \left(\frac{2\dot{y}_1}{y_1} + p\right)\right] \\ &= c \exp\left[-2\int \frac{\dot{y}_1'}{y_1} dt\right] \exp\left[-\int dtp\right] \\ &= c\frac{1}{{y_1}^2} \exp\left[-\int dtp\right] \end{aligned}$$

Now in the case of constant coeffcients, $p \neq p(t)$ and the integral can be performed trivially and w becomes

$$w = \frac{c}{{y_1}^2} e^{-pt}.$$

Since in this particular case we have equal roots $y_1 = e^{-pt/2}$ and w can be written as

$$w = \frac{e^{-pt}}{\left(e^{-pt/2}\right)^2} = 1$$

where the arbitrary constant of integration has been set to 1 since we are interested only in the functional form. This equation can be integrated immediately to give

$$v = \int dt\, w = \int dt = t$$

so that the new independent solution is

$$y_2 = vy_1 = ty_1 = te^{2t}.$$

2.3 Variation of Parameters

To respond to the inhomogeneous term te^{2t} start by assuming a solution

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where $u_1(t)$ and $u_2(t)$ are the now time varying 'constants' of the homogeneous solution. Taking the first derivative gives

$$\dot{y} = u_1\dot{y}_1 + u_2\dot{y}_2 + (\dot{u}_1 y_1 + \dot{u}_2 y_2) \; .$$

Taking the second derivative gives

$$\ddot{y} = \dot{u}_1\dot{y}_1 + u_1\ddot{y}_1 + \dot{u}_2\dot{y}_2 + u_2\ddot{y}_2 + \frac{d}{dt}(\dot{u}_1 y_1 + \dot{u}_2 y_2) \; .$$

Substituting these terms into the differential equation $\ddot{y} + p\dot{y} + qy = f$ and rearranging yields

$$\begin{aligned} u_1(\ddot{y}_1 + p\dot{y}_1 + qy_1) \; + u_2(\ddot{y}_2 + p\dot{y}_2 + qy_2) & \qquad (2) \\ +\dot{u}_1\dot{y}_1 + \dot{u}_2\dot{y}_2 + \left(\frac{d}{dt} + p\right)(\dot{u}_1 y_1 + \dot{u}_2 y_2) \quad = \quad f \, . & \qquad (3) \end{aligned}$$

The first two terms equal to zero since y_1 and y_2 are solutions to the homogeneous equation $\ddot{y} + p\dot{y} + qy = 0$. The equation now becomes

$$\dot{u}_1\dot{y}_1 + \dot{u}_2\dot{y}_2 + \left(\frac{d}{dt} + p\right)(\dot{u}_1 y_1 + \dot{u}_2 y_2) = f \, ,$$

which can be simplfied by assuming that the term in parentheses is zero. Doing so yields the set of equations

$$\begin{aligned} \dot{u}_1 y_1 + \dot{u}_2 y_2 &= 0 & (4) \\ \dot{u}_1\dot{y}_1 + \dot{u}_2\dot{y}_2 &= f \, . & (5) \end{aligned}$$

These last two can be separated easily in two steps. In the first, multiply the first equation by $\dot{y}_2$ and second by y_2 and then subtract the second from the first. In the second, multiply the first equation by $\dot{y}_1$ and the second by y_1 and subtract the first from the second. The following equations result:

$$\dot{u}_1(y_1\dot{y}_2 - \dot{y}_1 y_2) = -f y_2$$

and

$$\dot{u}_2(y_1\dot{y}_2 - \dot{y}_1 y_2) = f y_1 \, .$$

Defining the Wronskian as $W(y_1, y_2) = y_1\dot{y}_2 - \dot{y}_1 y_2$ gives the final equations for u_1

$$\dot{u}_1 = \frac{-f\,y_2}{W(y_1, y_2)}$$

and u_2

$$\dot{u}_2 = \frac{-f\, y_1}{W(y_1, y_2)}\,.$$

These equations can be integrated directly to get the form for u_1 and u_2 to give

$$u_1(t) = -\int_{t_0}^{t} \frac{f(\tau)\, y_2(\tau)}{W(y_1, y_2)(\tau)} d\tau$$

and

$$u_2(t) = \int_{t_0}^{t} \frac{f(\tau)\, y_1(\tau)}{W(y_1, y_2)(\tau)} d\tau\,.$$

Substituting these forms for $u_1(t)$ and $u_2(t)$ into the expression $y_p(t) = u_1(t)y_1(t)+ u_2(t)y_2(t)$ yields

$$y_p(t) = \int_{t_0}^{t} \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{W(y_1, y_2)(\tau)} f(\tau) d\tau\,.$$

Defining the one-sided Green's function

$$g_1(t, \tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{W(y_1, y_2)(\tau)}$$

gives the usual form

$$y_p(t) = \int_{t_0}^{t} g_1(t, \tau) f(\tau) d\tau\,.$$

In the current case of $f = t \exp 2t$, these integrations yield ($W = \exp 4t$), $u_1 = -t^3/3$ and $u_2 = t^2/2$. So the particular solution is

$$y_p = \frac{t^3 e^{2t}}{6}\,,$$

to which can be added any linear multiples of the homogeneous solutions $y_1 = \exp 2t$ and $y_2 = t \exp 2t$.
All things considered, the state space methods that

3 State Space Methods

Since the

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The eigenvalues of $\mathbb{A}$ are degenerate with the value of 2 occuring twice. The implication of this degeneracy is that there will only be one eigenvector that cvan be obtained with the usual method.

$$\begin{bmatrix} \lambda & -1 \\ 4 & \lambda - 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

yields $b = 2a$ with a arbitrary. To obtain another, independent eigenvalue first take a vector that is not a multiple of the eigenvector determined above. Determine the action of the operator $\mathbb{A} - \lambda\mathbb{I}$ on this vector. It will always be a multiple of the previous eigenvector. Scaling the new vector so that exactly yields the eigenvector results in a valid independent vector for diagonalization. For concreteness, consider $\langle e_2'| = [1, 0]$.

$$(\mathbb{A} - \lambda\mathbb{I})\,|e_2'\rangle = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = |e_1\rangle$$

So a proper second vector is $\langle e_2| = [-1/2, 0]$ with the resulting

$$\mathbb{S} = \begin{bmatrix} 1 & -1/2 \\ 2 & 0 \end{bmatrix}$$

and corresponding inverse

$$\mathbb{S}^{-1} = \begin{bmatrix} 0 & 1/2 \\ -2 & 1 \end{bmatrix}$$

$$\mathbb{S}^{-1}\mathbb{A}\mathbb{S} = \mathbb{M}$$

where $\mathbb{M}$ takes the form

$$\mathbb{M} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \equiv 2\mathbb{I} + \mathbb{V}$$

$$e^{\mathbb{M}t} = e^{2\mathbb{I}t} e^{\mathbb{V}t}$$

$$\Phi = \mathbb{S}\mathbb{M}\mathbb{S}^{-1} = \begin{bmatrix} (1-2t)e^{2t} & te^{2t} \\ -4te^{2t} & (1+2t)e^{2t} \end{bmatrix}$$

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbb{S} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbb{S}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\mathbb{M} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \equiv \mathbb{I} + \mathbb{V}$$

$$e^{\mathbb{M}t} = e^{\mathbb{I}t} e^{\mathbb{V}t}$$

$$e^{\mathbb{M}t} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

$$\Phi = \mathbb{S}\mathbb{M}\mathbb{S}^{-1} = \begin{bmatrix} (1-t)e^{2t} & te^{2t} \\ -te^{2t} & (1+t)e^{2t} \end{bmatrix}$$