Proof of NO4 | CS | 11/16/07

- We wish to show $\mathrm{grad}(\phi) = \sum_i \frac{1}{h_i} \frac{\partial}{\partial q^i} \phi \hat{e}_i$ using $\nabla\phi = \lim_{L\to 0} \frac{1}{L} \int_C \phi\, d\hat{r}$

F10

Consider an arbitrary curve. Then an arbitrary infinitesimal displacement along the curve is given by:

$$d\vec{r} = h_1 dq^1 \hat{e}_1 + h_2 dq^2 \hat{e}_2 + h_3 dq^3 \hat{e}_3$$

$$\left[= \frac{\partial \vec{r}}{\partial q^i} dq^i \right]$$

Now consider the component of $\nabla\phi$ along $\hat{e}_i$

$$\mathrm{grad}(\phi)\cdot\hat{e}_i = \lim_{L\to 0} \frac{1}{L} \int_C \phi \, dq_i \qquad \text{but } L = h_i dq_i$$

$$= \lim_{L\to 0} \frac{1}{h_i dq_i} \left[\phi(q_i + 1/2) - \phi(q_i - 1/2) \right]$$

$$= \lim_{L\to 0} \frac{1}{h_i dq_i} \frac{\partial \phi}{\partial q_i} dq_i = \frac{1}{h_i} \frac{\partial \phi}{\partial q_i}$$

so:

$$\mathrm{grad}(\phi) = \frac{1}{h_1} \frac{\partial \phi}{\partial q^1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial q^2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial q^3} \hat{e}_3$$

or

$$\mathrm{grad}(\phi) = \sum_i \vec{e}_i \frac{1}{h_i} \left(\frac{\partial}{\partial q^i} \phi \right)$$