$$\dot{\vec{r}} \cdot \left(\ddot{\vec{r}} + \frac{\mu \vec{r}}{r^3} = 0 \right)$$

expand : $\dot{\vec{r}} \cdot \ddot{\vec{r}} + \dfrac{\mu \vec{r} \cdot \dot{\vec{r}}}{r^3} = 0$

1st term : $\dfrac{d}{dt}\left(\dot{\vec{r}} \cdot \dot{\vec{r}} \right) = \ddot{\vec{r}} \cdot \dot{\vec{r}} + \dot{\vec{r}} \cdot \ddot{\vec{r}} = 2 \dot{\vec{r}} \cdot \ddot{\vec{r}}$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} \dot{\vec{r}} \cdot \dot{\vec{r}} \right) = \dot{\vec{r}} \cdot \ddot{\vec{r}}$$

2nd term : $\dfrac{d}{dt}\left(\dfrac{1}{r} \right) = \dfrac{d}{dt} \left(x^2 + y^2 + z^2 \right)^{-1/2}$

$$= -\frac{1}{2} \left(x^2 + y^2 + z^2 \right)^{-3/2} \frac{d}{dt} \left(x^2 + y^2 + z^2 \right)$$

$$= -\frac{1}{2r^3} \left(2x\dot{x} + 2y\dot{y} + 2z\dot{z} \right)$$

$$= -\frac{\vec{r} \cdot \dot{\vec{r}}}{r^3}$$

combine : $\dfrac{d}{dt}\left(\dfrac{1}{2} \dot{\vec{r}} \cdot \dot{\vec{r}} \right) - \mu \dfrac{d}{dt}\left(\dfrac{1}{r} \right) = 0$

$$\frac{d}{dt}\left(\frac{1}{2} \dot{\vec{r}} \cdot \dot{\vec{r}} - \frac{\mu}{r} \right) = 0$$

$$\Rightarrow \boxed{\frac{1}{2} \dot{\vec{r}} \cdot \dot{\vec{r}} - \frac{\mu}{r} = E \quad \text{conserved quantity}}$$