- We wish to prove $\int \vec{\nabla}\phi \cdot \vec{F}\, dv = \int \phi \vec{F} \cdot \hat{n}\, ds - \int \phi \vec{\nabla} \cdot \vec{F}\, dv$

Let $\vec{Q} = \phi \vec{F}$. Then $\vec{\nabla} \cdot \vec{Q} = \phi \vec{\nabla} \cdot \vec{F} + \vec{\nabla}\phi \cdot \vec{F}$. But from the divergence theorem:

$$\int \vec{\nabla} \cdot \vec{Q}\, dv = \int \vec{Q} \cdot \hat{n}\, ds$$

Plugging in the expansion of $\vec{\nabla} \cdot \vec{Q}$ we get

$$\int (\phi \vec{\nabla} \cdot \vec{F} + \nabla\phi \cdot \vec{F})\, dv = \int \phi \vec{F} \cdot \hat{n}\, ds$$

or solving for the $\nabla\phi$ term

$$\boxed{\int \nabla\phi \cdot \vec{F}\, dv = \int \phi \vec{F} \cdot \hat{n}\, ds - \int \phi \vec{\nabla} \cdot \vec{F}\, dv}$$