1 Introduction

In this note we go through the various approaches for formally solving

$$\frac{d}{dt}|S\rangle = \mathbb{A}|S\rangle + |u\rangle \ ,$$

where $|S\rangle$ is the state of the system, $A(t)$ called the process matrix, and $|u(t)\rangle$ is an inhomogeneous forcing function. This equation is the most general way of writing a linear nonhomogeneous coupled set of differential equations. Despite its innocent form, this equation is general enough to encompass most linear equations (Maxwell's, Schrodingers, wave, heat, and selected Newtonian equations). The steps taken in this note will be to first solve the homogeneous equation formally in 'time' space and show how the state transition matrix (STM) serves as a kernel or Green's function for solving the inhomogeneous equation. We next explore the formal ways for constructing the STM and closely related matrix, called the evolution operator. The evolution operator has the advantage of allowing the formal solution to transformed into the 'frequency' space via the Fourier transform.

2 Homogeneous Equation

Begin by solving the homogeneous equation by setting the $|u\rangle$ term to zero. Now multiply both sides by dt resulting in

$$d|S\rangle = \mathbb{A}|S\rangle\, dt\,,$$

which integrates formally to

$$|S\rangle = |S_0\rangle + \int_{t_0}^{t} dt'\mathbb{A}(t')\,|S(t')\rangle \ .$$

Since the equation is linear, there must be an operator that connnects the state of the system at a latter time in terms of the initial conditions

$$|S\rangle = \Phi(t,t_0)\,|S_0\rangle \ .$$

This operator is known as the state transition matrix (STM), denoted by $\Phi(t,t_0)$, and can be determined by an interative procedure using the integral equation. Before starting the iteration define the operator

$$\mathbb{B}\left[|X\rangle\right] = |X_0\rangle + \int_{t0}^{t} dt'\mathbb{A}(t')\,|X(t')\rangle \ ,$$

where $|X_0\rangle$ is the argument evaluated at the initial conditions. The iteration to find Φ is affected by successive applications of $\mathbb{B}$ to produce the n^{th} approximation $|S\rangle^{(n)}$ to $|S\rangle$. Generalizing the obvious pattern to $|S\rangle$ gives Φ as an infinite nested set of integrals. The interation starts with $|S\rangle^{(0)} = |S_0\rangle$ and the first iteration gives

$$
\begin{aligned}
|S\rangle^{(1)} &= \mathbb{B}\left[|S\rangle^{(0)}\right] \\
&= |S_0\rangle + \int_{t0}^{t} dt' \mathbb{A}(t') \, |S_0\rangle \; .
\end{aligned}
$$

Pluggin this result into the next iteration yields

$$
\begin{aligned}
|S\rangle^{(2)} &= \mathbb{B}\left[|S\rangle^{(1)}\right] \\
&= |S_0\rangle + \int_{t_0}^{t} dt' \mathbb{A}(t') \left[|S_0\rangle + \int_{t_0}^{t'} dt'' \mathbb{A}(t'') \, |S_0\rangle \right] \\
&= \left[\mathbb{I} + \int_{t_0}^{t} dt' \mathbb{A}(t') + \int_{t_0}^{t} \int_{t_0}^{t'} dt' dt'' \mathbb{A}(t') \mathbb{A}(t'') \right] |S_0\rangle \; .
\end{aligned}
$$

Abstracting to $n \to \infty$ yields

$$
|S\rangle = \left[\mathbb{I} + \int_{t_0}^{t} dt' \mathbb{A}(t') + \int_{t_0}^{t} \int_{t_0}^{t'} dt' dt'' \mathbb{A}(t') \mathbb{A}(t'') + \ldots \right] |S_0\rangle
$$

from which the STM can be read off as

$$
\Phi(t, t_0) = \mathbb{I} + \int_{t_0}^{t} dt' \mathbb{A}(t') + \int_{t_0}^{t} \int_{t_0}^{t'} dt' dt'' \mathbb{A}(t') \mathbb{A}(t'') + \ldots .
$$

Finally, we can derive the equation of motion for the STM by first noting that

$$
\frac{d}{dt} |S\rangle = \frac{d}{dt} \Phi(t, t_0) \, |S_0\rangle
$$

and that

$$
\frac{d}{dt} |S\rangle = A(t) \, |S\rangle = A(t) \Phi(t, t_0) \, |S\rangle \; .
$$

Equatating the two right-hand sides yields

$$
\frac{d}{dt} \Phi(t, t_0) = \mathbb{A}(t) \Phi(t, t_0) \, .
$$

3 Solving the Inhomogeneous Solution

The STM offers a solution to the inhomogeneous equation by acting as a kernel that propagates the influence of the forcing function $|u(t)\rangle$ as follows

$$
|S\rangle = |S\rangle_h + \int_{t_0}^{t} \Phi(t, t') \, |u(t')\rangle \, dt' \, ,
$$

where $|S\rangle_h$ is a solution to the homogeneous equation. To see that the proposed solution satifies the inhomogeneous solution start by taking the time derivative of each side

$$\begin{aligned}
\frac{d}{dt}|S\rangle &= \frac{d}{dt}|S\rangle_h + \frac{\partial}{\partial t}\int_{t_0}^{t} \Phi(t,t')\,|u(t')\rangle\,dt' \\
&= \mathbb{A}(t)\,|S\rangle_h + \int_{t_0}^{t} \frac{\partial \Phi(t,t')}{\partial t}\,|u(t')\rangle\,dt' + \Phi(t,t)\,|u(t)\rangle \\
&= \mathbb{A}(t)\,|S\rangle_h + \int_{t_0}^{t} \mathbb{A}(t)\Phi(t,t')\,|u(t')\rangle\,dt' + |u(t)\rangle \\
&= \mathbb{A}(t)\left[|S\rangle_h + \int_{t_0}^{t} \Phi(t,t')\,|u(t')\rangle\,dt'\right] + |u(t)\rangle \\
&= \mathbb{A}(t)\,|S\rangle + |u(t)\rangle\ ,
\end{aligned}$$

where the Leibnitz rule for differentiating under an integral is used in line two, the evolution equation for Φ is used in line three, and a regrouping of terms in used in line four.

4 Time Evolution Operator

Generally the iterated integrals used in defining the STM are difficult to implement and give a slow convergence to the final solution as $n \to \infty$. The convergence of the process can be increased by using a technique that involves the separation of the using the

$$\mathbb{A} = \mathbb{H} + \mathbb{V}(t)$$

where $\mathbb{H}$ is time-invariant. Now define a new state $|T\rangle$ by

$$|S\rangle = e^{\mathbb{H}(t-t_0)}\,|T\rangle\ .$$

Defining the free propagator $\mathcal{P}(t-t_0) = e^{\mathbb{H}(t-t_0)}$ yields a time evolution for $|T\rangle$ which depends only on $\mathbb{V}$. To determine this relationship first derive the evolution equation for the free propagator. Since the free propagator is the matrix exponential of $\mathbb{H}$ it obeys the particularly simple evolution equation

$$\frac{d}{dt}\mathcal{P}(t-t_0) = \mathbb{H}\,\mathcal{P}(t-t_0)\,.$$

$$\frac{d}{dt}|S\rangle = \frac{d}{dt}\Pi(t-t_0)\,|T\rangle = \mathbb{H}\,\Pi(t-t_0)\,|T\rangle + \Pi(t-t_0)\frac{d}{dt}|T\rangle$$

$$\mathbb{H}\,\Pi(t-t_0)\,|T\rangle + \Phi(t-t_0)\frac{d}{dt}|T\rangle = (\mathbb{H}+\mathbb{V}(t))\Pi(t-t_0)\,|T\rangle + |U(t)\rangle$$

$$\frac{d}{dt}|T\rangle = \Pi(t_0-t)\mathbb{V}(t)\,\Pi(t-t_0)\,|T\rangle + \Pi(t_0-t)\,|U(t)\rangle$$

$$\mathcal{A}(t) = \Pi(t_0 - t) \mathbb{V}(t) \, \Pi(t - t_0)$$

$$|\mathcal{U}(t)\rangle = \Pi(t_0 - t) \, |U(t)\rangle$$

$$\frac{d}{dt} \, |T\rangle = \mathcal{A}(t) \, |T\rangle + |\mathcal{U}(t)\rangle$$

$$|T\rangle = \mathbb{U}(t, t_0) \, |T_0\rangle$$

$$\mathbb{U}(t, t_0) = \mathbb{I} + \int_{t_0}^{t} dt' \mathcal{A}(t') + \int_{t_0}^{t} \int_{t_0}^{t'} dt' dt'' \mathcal{A}(t') \mathcal{A}(t'') + ...$$

Suppose that there exists a matrix Q such that

$$Q \sum_{n=0}^{\infty} A^n = 1$$

then by definition Q^{-1} would be equal to $\sum_{n=0}^{\infty} A^n$. It turns out that such a matrix can be trivially constructed. Assume $Q = 1 - A$ then:

$$\begin{aligned}
(1 - A) \sum_{n=0}^{\infty} A^n &= (1 - A) \left(1 + A + A^2 + A^3 + ...\right) && (1) \\
&= (1 - A) + (1 - A)A + (1 - A)A^2 + (1 - A)A^3 + ... && (2) \\
&= 1 - A + A - A^2 + A^2 - A^3 + A^3 - A^4 + ... && (3) \\
&= 1 \, . && (4)
\end{aligned}$$

So therefore $(1 - A)^{-1} = \sum_{n=0}^{\infty} A^n$