How a infinitesimal volume evolves during a flow CJ

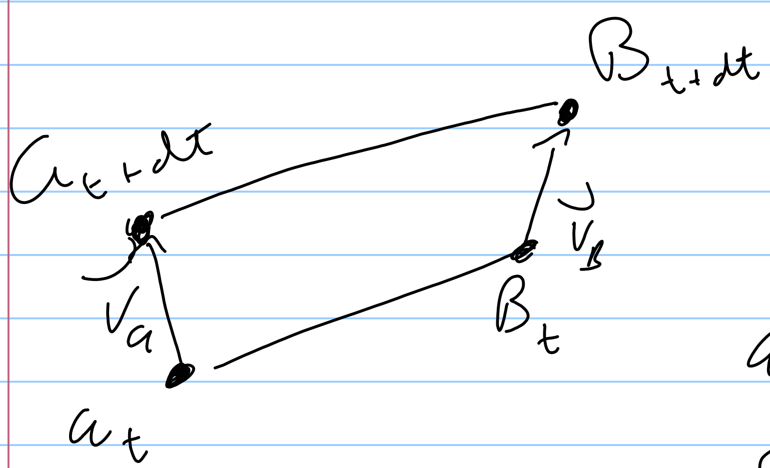
Note Title 11/25/2011

- Start by deriving a general formula for how a line segment evolves under a fluid flow.

- Consider the points A & B located at (x, y, z) & $(x+\delta x, y+\delta y, z+\delta z)$ respectively.

- Assume that a velocity field exists at each point, causing the points to move during elapsed time dt.

- The geometry looks like



$$A_t = (x, y, z)$$

$$B_t = (x+\delta x, y+\delta y, z+\delta z)$$

$$A_{t+dt} = A_t + \vec{V}_a\, dt$$

$$B_{t+dt} = A_t + \vec{V}_B\, dt$$

- The change in the distance between the two points in dt is obtained by subtracting the initial distance at time t from the distance at time $t+dt$

$$\delta l = l_{t+dt} - l_t$$

- The distance at time t is

$$l_t = |A_t - B_t| = \sqrt{(x+\delta x - x)^2 + (y+\delta y - y)^2 + (z+\delta z - z)^2}$$
$$= \sqrt{\delta x^2 + \delta y^2 + \delta z^2}$$

- While the distance at time $t+dt$ is

$$l_{t+dt} = |A_{t+dt} - B_{t+dt}| = \sqrt{(\delta x + V_{Bx}dt - V_{Ax}dt)^2 + (\delta y + V_{By}dt - V_{Ay}dt)^2 + (\delta z + V_{Bz}dt - V_{Az}dt)^2}$$

assume the velocity field is continuous

$$\vec{V}_a = \vec{v}(x, y, z) \qquad \vec{V}_B = \vec{v}(x+\delta x, y+\delta y, z+\delta z)$$
$$\simeq \vec{v}(x, y, z) + \frac{\partial \vec{v}}{\partial x}\delta x + \frac{\partial \vec{v}}{\partial y}\delta y + \frac{\partial \vec{v}}{\partial z}\delta z$$

$$\vec{V}_b - \vec{V}_a \simeq \frac{\partial \vec{V}}{\partial x}\delta x + \frac{\partial \vec{v}}{\partial y}\delta y + \frac{\partial \vec{v}}{\partial z}\delta z + \text{H.O.T.}$$

- Or in index notation

$$\left(\vec{V}_b - \vec{V}_a\right)_i \simeq \frac{\partial V_i}{\partial q_j}\delta q_j$$

- Using this result, l_{t+dt} becomes

$$l_{t+dt} = \sqrt{\Big(\delta x + dt\underbrace{\frac{\partial v_x}{\partial q_j}\delta q_j}_{=\vec{\nabla}_q v_x}\Big)^2 + \Big(\delta y + dt\frac{\partial v_y}{\partial q_j}\delta q_j\Big)^2 + \Big(\delta z + dt\frac{\partial v_z}{\partial q_j}\delta q_j\Big)^2}$$

- The difference δl is now

$$\delta l = l_{t+dt} - l_t$$

$$= \sqrt{(\delta x + dt\,\vec{\nabla}_q v_x)^2 + (\delta y + dt\,\vec{\nabla}_q v_y)^2 + (\delta z + dt\,\vec{\nabla}_q v_z)^2} - \sqrt{\delta x^2 + \delta y^2 + \delta z^2}$$

- This result is as simple as it gets without additional assumptions

- Now assume $\delta y = \delta z = 0$ (i.e. $\overline{AB}$ along x-axis). Then

$$\delta l = \sqrt{\Big(\delta x + dt\frac{\partial v_x}{\partial x}\delta x\Big)^2 + \Big(dt\frac{\partial v_y}{\partial x}\delta x\Big)^2 + \Big(dt\frac{\partial v_z}{\partial x}\delta x\Big)^2} - \delta x$$

$$= \delta x\left[\sqrt{\Big(1 + dt\frac{\partial v_x}{\partial x}\Big)^2 + \Big(\frac{\partial v_y}{\partial x}dt\Big)^2 + \Big(\frac{\partial v_z}{\partial x}dt\Big)^2} - 1\right]$$

- Keep terms only up to 1^{st} order in dt.

$$\delta l = \delta x \left[\sqrt{1 + 2dt \frac{\partial v_x}{\partial x}} - 1 \right]$$

$$= \delta x \left[1 + dt \frac{\partial v_x}{\partial x} - 1 \right]$$

$$\delta l = \delta x \, dt \frac{\partial v_x}{\partial x} \Rightarrow \frac{d}{dt} \delta x = \frac{\partial v_x}{\partial x} \delta x$$

Likewise $\quad \frac{d}{dt} \delta y = \frac{\partial v_y}{\partial y} \delta y$

$$\frac{d}{dt} \delta z = \frac{\partial v_z}{\partial z} \delta z$$

- So if we have an infinitisimal cube of sides $\delta x, \delta y, \& \delta z$ then its volume $\delta \mathcal{V}$ is given by

$$\delta \mathcal{V} = \delta x \, \delta y \, \delta z$$

and the time rate of change of this volume is

$$\frac{d}{dt} \delta \mathcal{V} = \frac{d}{dt}(\delta x) \delta y \delta z + \delta x \frac{d}{dt}(\delta y) \delta z + \delta x \delta y \frac{d}{dt}(\delta z)$$

$$= \frac{\partial v_x}{\partial x} \delta x \delta y \delta z + \delta x \frac{\partial v_y}{\partial y} \delta y \delta z + \delta x \delta y \frac{\partial v_z}{\partial z} \delta z$$

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \delta x \delta y \delta z$$

$$\boxed{\frac{d}{dt} \delta \mathcal{V} = \operatorname{div}(\vec{v}) \delta \mathcal{V}}$$