- We wish to prove $\int \phi \, d\vec{l} = \int (\hat{n} \times \nabla \phi) \, dS$

Start with Stokes theorem applied to $\vec{F} = \phi \vec{A}$ where $\vec{A}$ is a constant vector

$$\int \vec{F} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, dS$$

and expand both sides using the definition of $\vec{F}$

$$\int \phi \vec{A} \cdot d\vec{l} = \int \vec{\nabla} \times (\phi \vec{A}) \cdot \hat{n} \, dS$$

$$= \int (\vec{\nabla} \phi \times \vec{A}) \cdot \hat{n} \, dS$$

$$= \int (\hat{n} \times \nabla \phi) \cdot A \, dS$$

Re-arranging yields

$$\vec{A} \cdot \left(\int \phi \, d\vec{l} - \int \hat{n} \times \nabla \phi \, dS \right) = 0$$

and since $\vec{A}$ is arbitrary

$$\int \phi \, d\vec{l} = \int (\hat{n} \times \nabla \phi) \, dS$$