UF01 - Liouville's Theorem

Saturday, August 7, 2021 6:35 AM

Following Ryu Sec. 1.4

- Assume a dynamical system which evolves according to Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} \qquad H = H(q_i, p_i; t)$$

- Next form the density in phase space as the # of trajectories passing thru a unit-volume of phase space and let's denote that quantity as ρ_m

- Since we are working with Hamilton's theory then $\rho_m = \rho_m(q_i, p_i; t)$

- Now look at how the density changes along the flow

$$\frac{D\rho_m}{Dt} = \lim_{\delta t \to 0} \frac{\rho_m(q_i + \delta q_i, p_i + \delta p_i; t + \delta t) - \rho_m(q_i, p_i; t)}{\delta t}$$

$$\boxed{\frac{D\rho_m}{Dt} = \frac{\partial \rho_m}{\partial q_i}\dot{q}_i + \frac{\partial \rho_m}{\partial p_i}\dot{p}_i + \frac{\partial \rho_m}{\partial t}}$$

- Next, use the equation of continuity to get:

$$\frac{\partial \rho_m}{\partial t} + \vec{\nabla}_6 \cdot (\rho_m \vec{v}_6) = 0 \quad \text{or} \quad \frac{\partial \rho_m}{\partial t} + \frac{\partial}{\partial q_i}(\rho_m \dot{q}_i) + \frac{\partial}{\partial p_i}(\rho_m \dot{p}_i) = 0$$

- Eliminate $\frac{\partial \rho_m}{\partial t}$ in the equation above

$$\frac{D\rho_m}{Dt} = \frac{\partial \rho_m}{\partial q_i}\dot{q}_i + \frac{\partial \rho_m}{\partial p_i}\dot{p}_i - \frac{\partial}{\partial q_i}(\rho_m \dot{q}_i) - \frac{\partial}{\partial p_i}(\rho_m \dot{p}_i)$$

$$= \frac{\partial \rho_m}{\partial q_i}\dot{q}_i + \frac{\partial \rho_m}{\partial p_i}\dot{p}_i - \frac{\partial \rho_m}{\partial q_i}\dot{q}_i - \rho_m \frac{\partial \dot{q}_i}{\partial q_i} - \frac{\partial \rho_m}{\partial p_i}\dot{p}_i - \rho_m \frac{\partial \dot{p}_i}{\partial p_i}$$

$$= -\rho_m \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right)$$

- Finally use Hamilton's equations:

$$\frac{D\rho_m}{Dt} = -\rho_m \left[\frac{\partial}{\partial q_i} \left(\frac{\partial H}{\partial p_i} \right) + \frac{\partial}{\partial p_i} \left(-\frac{\partial H}{\partial q_i} \right) \right]$$

$$= -\rho_m \left[\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right] = 0$$

$$\Rightarrow \boxed{\frac{D\rho_m}{Dt} = 0}$$

→ total derivative
→ Hamilton's equations
→ Continuity with $\vec{v}_6$ in phase space