# A Brief Note on First-order Gauss Markov Processes

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The aim of this note is to explain how to solve and then numerically model a First-order Gauss Markov process. The physical justification for the model or its application domain are not covered.

## 1 Analytic Solution

The basic aim is model the First-order Gauss Markov process as a stochastic differential equation

$$\dot{x} = -\frac{1}{\tau}x + w \quad , \tag{1}$$

where $x(t)$ is defined as the state and $w$ is a inhomogenous noise term. In the absence of the noise, the solution is given by a trivial time integration to yield

$$x_h(t) = x_0 e^{-(t-t_0)/\tau} \quad , \tag{2}$$

assuming that $x(t = t_0) = x_0$. The solution in Eq. (2) is known as the homogenous solution and it will help in integrating the inhomogenous term $w$. Ignore, for the moment, that $w$ (and, as a result, $x$) is a random variable. Treated as just an ordinary function, $w$ is a driving term that can be handled via the state transition matrix (essentially a one-sided Green's function for an initial-value problem). Recall that a state transition matrix (STM) is defined as the object linking the state at time $t_0$ to the state at time $t$ according to

$$x(t) = \Phi(t, t_0)x(t_0) \quad . \tag{3}$$

By the definition in Eq. (3), the STM is obtained as

$$\Phi(t, t_0) = \frac{\partial x(t)}{\partial x(t_0)} \quad . \tag{4}$$

Taking the partial derivative of Eq. (2) as required by Eq. (4) gives

$$\Phi(t, t_0) = e^{-(t-t_0)/\tau} \quad . \tag{5}$$

The solution of the inhomogenous equation is then given as

$$x(t) = x_h(t) + \int_{t_0}^{t} \Phi(t,t')w(t')dt' \quad . \tag{6}$$

To see how this is true, take the time derivative of Eq. (6) to get

$$\begin{aligned} \dot{x}(t) &= \dot{x}_h(t) + \frac{d}{dt}\left[\int_{t_0}^{t} \Phi(t,t')w(t')dt'\right] \\ &= \dot{x}_h(t) + \left[\Phi(t,t)w(t) + \int_{t_0}^{t} \frac{\partial\Phi(t,t')}{\partial t}w(t')dt'\right] \\ &= -\frac{1}{\tau}x_h(t) + \Phi(t,t)w(t) + \int_{t_0}^{t} \frac{\partial\Phi(t,t')}{\partial t}w(t')dt' \quad . \end{aligned} \tag{7}$$

Now suppose that the following two conditions are met

$$\Phi(t,t) = 1 \quad , \tag{8}$$

and

$$\frac{\partial\Phi(t,t')}{\partial t} = -\frac{1}{\tau}\Phi(t,t') \quad , \tag{9}$$

and that these are substituted into Eq. (7). Doing so yields

$$\begin{aligned} \dot{x}(t) &= -\frac{1}{\tau}x_h(t) + w(t) - \frac{1}{\tau}\int_{t_0}^{t} \Phi(t,t')w(t')dt' \\ &= -\frac{1}{\tau}\left[x_h(t) + \int_{t_0}^{t} \Phi(t,t')w(t')dt'\right] + w(t) \\ &= -\frac{1}{\tau}x(t) + w(t) \quad . \end{aligned} \tag{10}$$

It is worth noting that Eq. (8) is always true and Eq. (9) is a specific example of the general equation

$$\frac{\partial\Phi(t,t')}{\partial t} = A(t)\Phi(t,t') \quad , \tag{11}$$

where $A(t)$, known sometimes as the process matrix, is given, generally by

$$A(t) = \frac{\partial\mathbf{f}(\mathbf{x}(t))}{\partial\mathbf{x}(t)} \tag{12}$$

and $\mathbf{f}(\mathbf{x}(t))$ is given by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t);t) \quad . \tag{13}$$

Comparison of Eqs. (1) and (13) shows that

$$\mathbf{f}(\mathbf{x}(t);t) = -\frac{1}{\tau}x(t) \quad , \tag{14}$$

which is consistent with Eq. (9).

# 2 Constructing Solutions

As demonstrated in the previous section

$$x(t) = x_0 e^{-(t-t_0)/\tau} + \int_{t_0}^{t} e^{-(t-t')/\tau} w(t') dt' \tag{15}$$

is a complete analytic solution of Eq. (1). However, Eq. (15) is not very useful since $w(t)$ is a random variable. Instead of immediately calculating $x(t)$, calculate the solutions for the statistical moments about the origin (i.e. $E[x^n(t)]$, where $E[\cdot]$ denotes the expectation value of the argument).

In order to do this, we need to say something about the statistical distribution of the noise. We will assume that $w(t)$ is a Gaussian white-noise source with zero mean, a variance of $q$, and is uncorrelated. Mathematically these assertions amount to the condition $E[w(t)] \equiv \bar{w} = 0$ and the condition

$$\begin{aligned} E[(w(t)-\bar{w})(w(s)-\bar{w})] &= E[w(t)w(s) - \bar{w}w(t) - \bar{w}w(s) - \bar{w}^2] \\ &= E[w(t)w(s)] \\ &= q\delta(t-s) \quad . \end{aligned} \tag{16}$$

In addition, we assume that the state and the noise are independent giving

$$E[x(t)w(s)] = E[x(t)]E[w(s)] = 0 \quad . \tag{17}$$

Now we can compute the statistical moments of $x(t)$ about the origin.

## 2.1 First Moment

The first moment is given by

$$\begin{aligned} E[x(t)] &= E\left[x_0 e^{-(t-t_0)/\tau} + \int_{t_0}^{t} e^{-(t-t')/\tau} w(t') dt'\right] \\ &= E\left[x_0 e^{-(t-t_0)/\tau}\right] + E\left[\int_{t_0}^{t} e^{-(t-t')/\tau} w(t') dt'\right] \\ &= E\left[x_0\right] e^{-(t-t_0)/\tau} + \int_{t_0}^{t} e^{-(t-t')/\tau} E\left[w(t')\right] dt' \\ &= E\left[x_0\right] e^{-(t-t_0)/\tau} = E\left[x(t_0)\right] e^{-(t-t_0)/\tau}, \end{aligned} \tag{18}$$

where, in the last term, $x_0$ is explicately written as $x(t_0)$ for comparison with a later numerical result.

## 2.2 Second Moment

The second moment is given by

$$E[x(t)^2] = E\left[x_0^2 e^{-2(t-t_0)/\tau} + 2x_0 e^{-(t-t_0)/\tau} \int_{t_0}^{t} e^{-(t-t')/\tau} w(t') dt'\right.$$

$$
\begin{aligned}
& + \int_{t_0}^{t} e^{-(t-t')/\tau} w(t')dt' \int_{t_0}^{t} e^{-(t-t'')/\tau} w(t'')dt'' \Big] \\
= \; & E\left[x_0^2\right] e^{-2(t-t_0)/\tau} + \int_{t_0}^{t}\int_{t_0}^{t} e^{-(2t-t'-t'')/\tau} E\left[w(t')w(t'')\right] dt'dt'' \\
= \; & E\left[x_0^2\right] e^{-2(t-t_0)/\tau} + \int_{t_0}^{t}\int_{t_0}^{t} e^{-(2t-t'-t'')/\tau} q\delta(t'-t'')dt'dt'' \\
= \; & E\left[x_0^2\right] e^{-2(t-t_0)/\tau} + \int_{t_0}^{t} e^{-2(t-t')/\tau} q dt' \\
= \; & E\left[x_0^2\right] e^{-2(t-t_0)/\tau} + \frac{q\tau}{2} e^{-2(t-t')/\tau}\Big|_{t_0}^{t} \\
= \; & E\left[x_0^2\right] e^{-2(t-t_0)/\tau} + \frac{q\tau}{2}\left(1 - e^{-2(t-t_0)/\tau}\right),
\end{aligned} \tag{19}
$$

where $x_0$ is shorthand for $x(t_0)$.

### 2.3 Variance

$$
\begin{aligned}
E\left[x^2\right] - E\left[x\right]^2 \; = \; & E\left[x_0^2\right] e^{-2(t-t_0)/\tau} + \frac{q\tau}{2}\left(1 - e^{-2(t-t_0)/\tau}\right) \\
& - E\left[x_0\right]^2 e^{-2(t-t_0)/\tau} \\
= \; & \left(E\left[x_0^2\right] - E\left[x_0\right]^2\right) e^{-2(t-t_0)/\tau} \\
& + \frac{q\tau}{2}\left(1 - e^{-2(t-t_0)/\tau}\right)
\end{aligned} \tag{20}
$$

Eq. (20) can be re-written as

$$
\mathcal{P}(t) = \mathcal{P}_0 e^{-2(t-t_0)/\tau} + \frac{q\tau}{2}\left(1 - e^{-2(t-t_0)/\tau}\right) \quad , \tag{21}
$$

where $\mathcal{P}_0 = \left(E\left[x_0^2\right] - E\left[x_0\right]^2\right)$ is the initial covariance.

### 2.4 Generating Numerical Trials

A numerical realization of the first-order Gauss Markov process is not obtained by directly solving Eq. (1) but rather by exploting the Markov property that states that the 'system has no memory'. This means that the system's evolution only depends on the its current state. As a result we expect that there exists an equation that relates $x(t + \Delta t)$ to $x(t)$. As a candidate to fill this roll, consider the equation

$$
x(t + \Delta t) = x(t) e^{-\Delta t/\tau} + \eta(t)\sqrt{\frac{q\tau}{2}\left(1 - e^{-2\Delta t/\tau}\right)}, \tag{22}
$$

which is proposed as the correct numerical method for simulating Eq. (1). In Eq. (22) one imagines the elapsed time from $t_0$ to $t$ as being divided up into

$N$ increments, each of duration $\Delta t$ (i.e., $t = t_0 + N\Delta t$). At each time step, one constructs the 'new' value of $x$ from the 'old' value and from a zero-mean, unit-variance normal random number $\eta(t)$. To verify Eq. (22), first look at it's mean value, which is

$$\begin{aligned}
E\left[x(t)\right] &= E\left[x(t-\Delta t)e^{-\Delta t/\tau} + \eta(t)\sqrt{\frac{q\tau}{2}\left(1-e^{-2\Delta t/\tau}\right)}\ \right] \\
&= E\left[x(t-\Delta t)\right]e^{-\Delta t/\tau} \\
&= E\left[x(t-2\Delta t)e^{-\Delta t/\tau} + \eta(t-\Delta t)\sqrt{\frac{q\tau}{2}\left(1-e^{-2\Delta t/\tau}\right)}\ \right]e^{-\Delta t/\tau} \\
&= E\left[x(t-2\Delta t)\right]e^{-2\Delta t/\tau} \\
&= \ldots \\
&= E\left[x(t-N\Delta t)\right]e^{-N\Delta t/\tau} \\
&= E\left[x(t_0)\right]e^{-(t-t_0)/\tau} \\
&= E\left[x_0\right]e^{-(t-t_0)/\tau} \quad .
\end{aligned} \tag{23}$$

Comparison of Eq. (23) with Eq. (18) shows that two expectations are identical. Next look at its second moment about the origin. Before carrying out this computation, define $S = \sqrt{\frac{q\tau}{2}\left(1-e^{-2\Delta t/\tau}\right)}$ and recall that since $E[x(t)\eta(t)] = E[x(t)]E[\eta(t)] = 0$ all cross terms in the computation can be ignored. With these points in mind, the second moment about the origin becomes

$$\begin{aligned}
E\left[x^2(t)\right] &= E\left[\left(x(t-\Delta t)e^{-\Delta t/\tau} + \eta S\right)^2\right] \\
&= E\left[x^2(t-\Delta t)e^{-2\Delta t/\tau} + \eta^2 S^2\right] \\
&= E\left[x^2(t-\Delta t)\right]e^{-2\Delta t/\tau} + S^2 \\
&= E\left[\left(x(t-2\Delta t)e^{-\Delta t/\tau} + \eta S\right)^2\right]e^{-2\Delta t/\tau} + S^2 \\
&= E\left[x^2(t-2\Delta t)e^{-2\Delta t/\tau} + \eta^2 S^2\right]e^{-2\Delta t/\tau} + S^2 \\
&= E\left[x^2(t-2\Delta t)\right]e^{-4\Delta t/\tau} \\
&\quad +S^2\left(1+e^{-2\Delta t/\tau}\right) \\
&= E\left[\left(x(t-3\Delta t)e^{-\Delta t/\tau} + \eta S\right)^2\right]e^{-4\Delta t/\tau} + S^2\left(1+e^{-2\Delta t/\tau}\right) \\
&= E\left[x^2(t-3\Delta t)\right]e^{-6\Delta t/\tau} \\
&\qquad +S^2\left(1+e^{-2\Delta t/\tau}+e^{-4\Delta t/\tau}\right) \\
&= \ldots \\
&= E\left[x^2(t-N\Delta t)\right]e^{-2N\Delta t/\tau} \\
&\qquad +S^2\left(1+e^{-2\Delta t/\tau}+e^{-4\Delta t/\tau}+\ldots+e^{-2(N-1)\Delta t/\tau}\right)
\end{aligned}$$

$$
\begin{aligned}
&= E\left[x(t_0)^2\right] e^{-2(t-t_0)/\tau} \\
&\quad + S^2\left(1 + e^{-2\Delta t/\tau} + e^{-4\Delta t/\tau} + ... + e^{-2(N-1)\Delta t/\tau}\right) . \qquad (24)
\end{aligned}
$$

The last term simplifies as

$$
\begin{aligned}
S^2\left(1 + e^{-2\Delta t/\tau} + ... + e^{-2(N-1)\Delta t/\tau}\right) &= \frac{q\tau}{2}\left(1 - e^{-2N\Delta t/\tau}\right) \\
&= \frac{q\tau}{2}\left(1 - e^{-2(t-t_0)/\tau}\right) \quad , \qquad (25)
\end{aligned}
$$

and when substituted into Eq. (24) gives

$$
E\left[x^2(t)\right] = E\left[x_0^2\right] e^{-2(t-t_0)/\tau} + \frac{q\tau}{2}\left(1 - e^{-2(t-t_0)/\tau}\right) \quad . \qquad (26)
$$

Eq. (26) is identical to Eq. (19). Since the first and second moments resulting from Eq. (22) match those of the analytic solution it must give realizations of the random process that are also consistent and this completes the proof.