A Simple Inventory System

Lawrence M. Leemis and Stephen K. Park, Discrete-Event Simulation: A First Course, Prentice Hall, 2006

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Discrete or Continuous Variables?

- Single Server Queue
  - both arrival and service times are continuous variables
- Simple Inventory System
  - Input variables (e.g., arrival and service times) are inherently discrete

A Simple Inventory System

- Distributes items from current inventory to customers
- Order items from supplier to replenish the inventory
- Customer demand is discrete
  - Customers do not want a portion of an item
- Balancing *holding cost*, *shortage cost*, and *ordering cost*
- Simple
  - One type of item

Different Inventory Policies

- Transaction Reporting Policy
  - Inventory review after each transaction
  - Significant labor may be required
  - Less likely to experience shortage
- Periodic Inventory Review
  - Inventory review is periodic
  - Items are ordered, if necessary, only at review times
  - Defined by two parameters (s, S)
    - s: minium inventory level
    - S: maximum inventory level
    - s and S are constnat in time, with $0 \leq s < S$
- Assume *periodic inventory review*
- Search for (s, S) that minimize cost

Cost and Assumption

- Inventory System Costs
  - Holding cost: for items in inventory
  - Shortage cost: for unmet demand
  - Ordering cost: sum of setup and item costs
    - Setup cost: fixed cost when order is placed
    - Item cost: per-item order cost
- Additional Assumptions
  - Back ordering is possible
  - No delivery lag
  - Initial inventory level is S
  - Terminal inventory level is S

Specification Model

- Time begins at $t = 0$
- Review times are $t = 0, 1, 2, \ldots$
  - i-th time interval begins at time $t = i - 1$ and ends at $t = i$
- l_{i-1}: inventory level at beginning of the i–th interval
- o_{i-1}: amount ordered at time $t = i - 1$ is an integer, $o_{i-1} \geq 0$
- d_i: demand quantity during the i–th interval, $d_i \geq 0$
- Inventory at end of interval can be negative
  - As a result of back-ordering

Inventory Level Consideration

- Inventory level is reviewed at $t = i - 1$
- If $l_{i-1} \geq s$, no order is placed; item If $l_{i-1} < s$, inventory is replenished to S

$$o_{i-1} = \begin{cases} 0 & l_{i-1} \geq s \\ S - l_{i-1} & l_{i-1} < s \end{cases} \tag{1}$$

- Items are delivered immediately
- At end of i-th interval, inventory diminished by d_i

$$l_i = l_{i-1} + o_{i-1} - d_i \tag{2}$$

Time Evolution of Inventory Level

Algorithm 1.3.1

$l_0 = S;$
$i = 0;$
while (more demand to process) $\{$
  $i++;$
  if $(l_{i-1} < s)$
    $o_{i-1} = S - l_{i-1};$
  else
    $o_{i-1} = 0;$
  $d_i = GetDemand();$
  $l_i = l_{i-1} + o_{i-1} - d_i;$
$\}$
$n = i;$
$o_n = S - l_n$
$l_n = S;$
return l_1, l_2, l_3, ...l_n and o_1, o_2, ...o_n;

Example 1.3.1: SIS with Sample Demands

- Let $(s, S) = (20, 60)$ and apply Algorithm 1.3.1 to process $n = 12$ time intervals of operation as follows.
  Note that $l_0 = S = 60$.

	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i												
inventory	l_i												

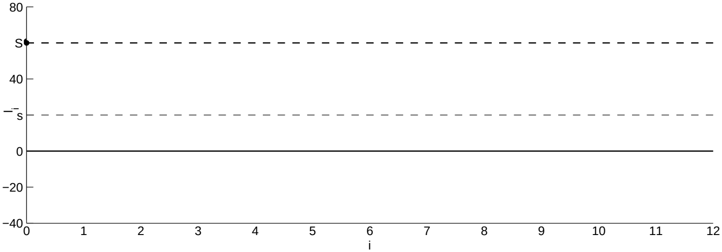

Figure : Inventory Levels

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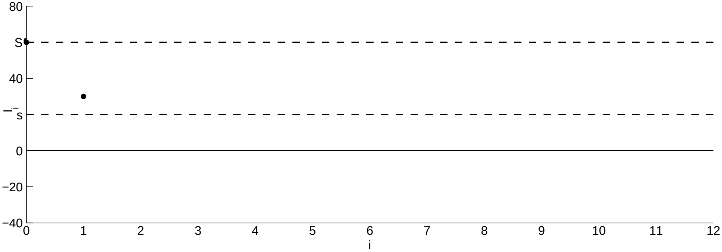
	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0											
inventory	l_i	30											



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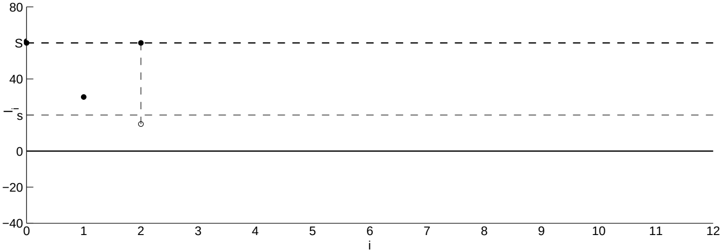
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0	45										
inventory	l_i	30	15										



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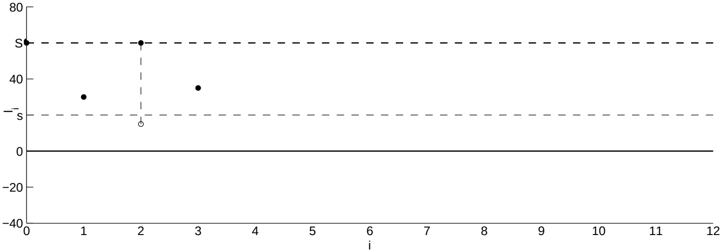
  Note that $l_0 = S = 60$.

	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0	45	0									
inventory	l_i	30	15	35									



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input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0	45	0	0								
inventory	l_i	30	15	35	20								

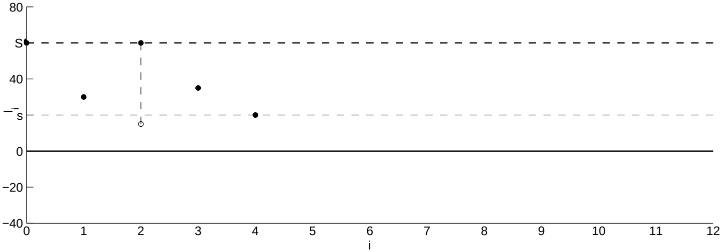

Figure : Inventory Levels

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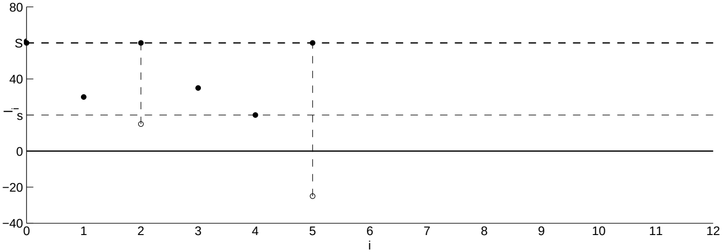
  Note that $l_0 = S = 60$.

	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0	45	0	0	85							
inventory	l_i	30	15	35	20	-25							



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- Let $(s, S) = (20, 60)$ and apply Algorithm 1.3.1 to process $n = 12$ time intervals of operation as follows.
  Note that $l_0 = S = 60$.

	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0	45	0	0	85	...						
inventory	l_i	30	15	35	20	-25	...						

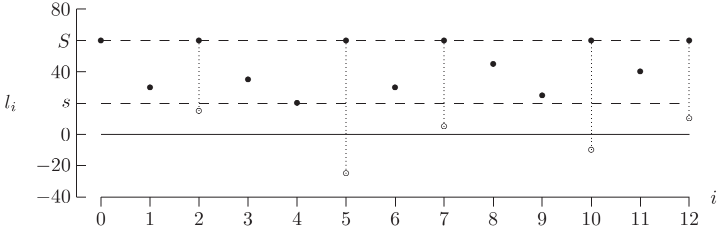


Figure : Inventory Levels

Output Statistics

- *Average demand* and *average order* per time interval

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i \tag{3}$$

$$\bar{o} = \frac{1}{n}\sum_{i=1}^{n} o_i \tag{4}$$

- For Example 1.3.1 Data $\bar{d} = \bar{o} = 305/12 \cong 25.42$ items per time interval

Exercise L3-1

- Let $(s, S) = (20, 60)$ and apply Algorithm 1.3.1 *by tracing the algorithm* to process $n = 12$ time intervals of operation as follows.

		1	2	3	4	5	6	7	8	9	10	11	12
	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i												
inventory	l_i												

- Calculate average demand. You must show the steps.
- Calculate average order. You must show the steps.

Flow Balance

- Average demand and order must be equal
- Ending inventory level is S
- Over the simulated period, all demand is satisfied
- Average "flow" of items in equals average "flow" of items out

- The inventory system is *flow balanced*

Constant Demand Rate Assumption

- Holding and shortage costs are proportional to time-averaged inventory levels
- Must know inventory level for all t
- Assume *the demand rate is constant between review times*
  - As a result, the continuous-time evolution of the inventory level is piecewise linear, as illustrated below.

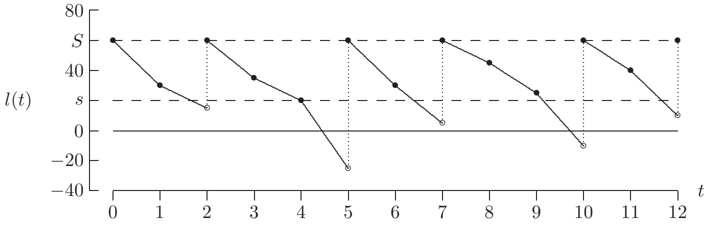


Figure : Piecewise-linear inventory levels

Inventory Level as a Function of Time

- Under condition that the demand rate (d_i) is constant between review times, then the inventory level at any time t in i-th interval is,

$$l(t) = l'_{i-1} - (t - i + 1)d_i \quad (5)$$

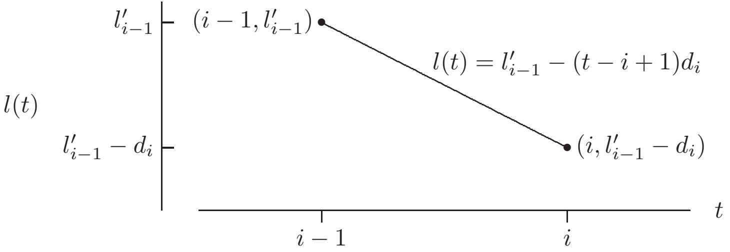


Figure : Linear inventory level in time interval i

- $l'_{i-1} = l_{i-1} + o_{i-1}$ represents inventory level after review.

Inventory Level is Not Linear!

- Inventory level at any time t is an *integer*
- $l(t)$ should be rounded to an integer value
- As a result, $l(t)$ is a stair-step, rather than linear, function
- However, it can be shown, it has no effect on the values of *Time Averaged Inventory Level* $\bar{l}_i^+$ and $\bar{l}_i^-$

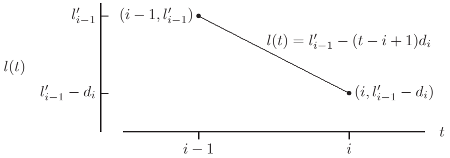


Figure : Linear inventory level in time interval i

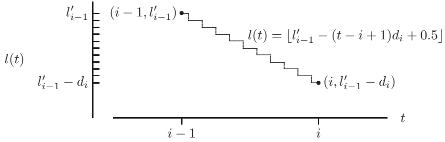


Figure : Linear inventory level in time interval i

Time Averaged Inventory Level at Time Interval i

- $l(t)$ is the basis for computing the time-averaged inventory level
  - Case 1: If $l(t)$ remains non-negative over i-th interval

$$\bar{l}_i^+ = \int_{i-1}^{i} l(t)dt \tag{6}$$

  - Case 2: If $l(t)$ becomes negative at some time τ

$$\bar{l}_i^+ = \int_{i-1}^{\tau} l(t)dt \tag{7}$$

$$\bar{l}_i^- = -\int_{\tau}^{i} l(t)dt \tag{8}$$

where $\bar{l}_i^+$ is the time-averaged *holding level* and $\bar{l}_i^-$ is the time-averaged *shortage level*

Case 1: No Back-Ordering

- No shortage during i-th time interval if and only if $d_i \leq l'_{i-1}$

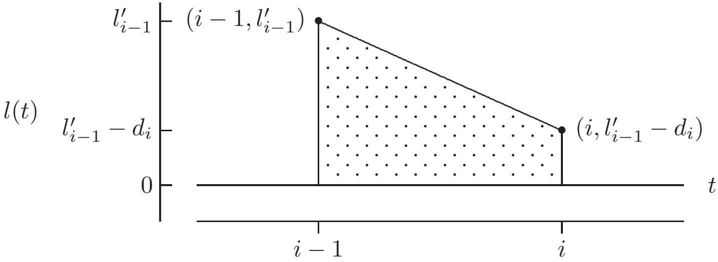


Figure : Inventory level in time interval i without back-ordering

- Time-averaged holding level: area of a trapezoid

$$\bar{l}_i^+ = \int_{i-1}^{i} l(t)dt = \frac{l'_{i-1} + (l'_{i-1} - d_i)}{2} = l'_{i-1} - \frac{1}{2}d_i \qquad (9)$$

Case 2: Back-Ordering

- Inventory $l(t)$ becomes negative if and only if $d_i > l'_{i-1}$, i.e., at $t = \tau = i - 1 + l'_{i-1}/d_i$

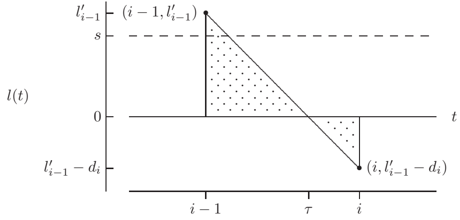


Figure : Inventory level in time interval i with back-ordering

- Time-averaged holding and shortage levels for i-th interval computed as the areas of triangles

$$\bar{l}_i^+ = \int_{i-1}^{\tau} l(t)dt = \dots \frac{(l'_{i-1})^2}{2d_i} \quad (10) \qquad \bar{l}_i^- = -\int_{\tau}^{i} l(t)dt = \dots = \frac{(d_i - l'_{i-1})^2}{2d_i} \quad (11)$$

$\bar{l}_i^+$ and $\bar{l}_i^-$

$$
\begin{aligned}
\bar{l}_i^+ &= \int_{i-1}^{\tau} l(t)dt = \frac{1}{2}[\tau - (i-1)]l'_{i-1} = \frac{(l'_{i-1})^2}{2d_i} \\
\bar{l}_i^- &= -\int_{\tau}^{i} l(t)dt = -\frac{1}{2}(i-\tau)(l'_{i-1} - d_i) \\
&= -\frac{1}{2}[i - (i - 1 + l'_{i-1}/d_i)](l'_{i-1} - d_i) \\
&= -\frac{1}{2}(1 - l_{i-1}'/d_i)(l'_{i-1} - d_i) \\
&= -\frac{1}{2}\frac{d_i - l_{i-1}'}{d_i}(l'_{i-1} - d_i) \\
&= -\frac{(d_i - l_{i-1}')(l'_{i-1} - d_i)}{2d_i} = \frac{(d_i - l'_{i-1})^2}{2d_i}
\end{aligned}
$$

Time-Averaged Inventory Level

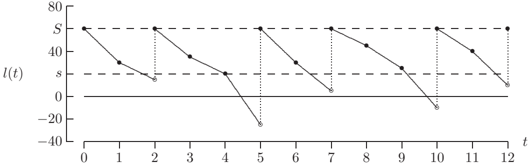


Figure : Piecewise-linear inventory levels

- *Time-averaged holding level* and *time-averaged shortage level*

$$\bar{l}^{+} = \frac{1}{n}\sum_{i=1}^{n}\bar{l}_i^{+} \qquad (12) \qquad\qquad \bar{l}^{-} = \frac{1}{n}\sum_{i=1}^{n}\bar{l}_i^{-} \qquad (13)$$

- Note that time-averaged shortage level is positive
- The time-averaged inventory level is

$$\bar{l} = \frac{1}{n}\int_0^n l(t)dt = \bar{l}^{+} - \bar{l}^{-} \qquad (14)$$

Exercise L3-2

- Let $(s, S) = (20, 60)$ and process $n = 3$ time intervals of operation as follows (see slides 23, 24, and 26 for steps)

	i	1	2	3
input	d_i	15	45	30

- Calculate time-averaged holding level
- Calculate time-averaged shortage level

Computational Model

- Program *sis1* is a trace-driven computational model of the SIS
- Computes the statistics
  - $\bar{d}$: Average demand
  - $\bar{o}$: Average order
  - $\bar{l}^{+}$: Time-averaged holding level
  - $\bar{l}^{-}$: Time-averaged shortage level

  and the order frequency $\bar{u}$

$$\bar{u} = \frac{\textit{number of orders}}{n}$$

- Consistency check: compute $\bar{d}$ and $\bar{o}$ separately, then compare. They should be equal.

Example 1.3.4: Executing sis1

- Trace file *sis1.dat* contains data from $n = 100$ time intervals
- Inventory-policy parameter values (i.e., mininum & maximum inventory levels) $(s, S) = (20, 80)$
- Program outputs:

$$\bar{o} = \bar{d} = 29.29 \quad \bar{u} = 0.39 \quad \bar{l}^{+} = 42.90 \quad \bar{l}^{-} = 0.25$$

Exercise L3-3: sis1

- Run either C/C++ or Java program against the trace with the following inventory levels,
  - $S = 80$ and $s = 2, 4, 6, \ldots 40$

submit the result.

Operating Cost

- A facility's cost of operation is determined by
  - c_{item}: unit cost of new item
  - c_{setup}: fixed cost for placing an order
  - c_{hold}: cost to hold one item for one time interval
  - c_{short}: cost of being short one time for one time interval

Example 1.3.5: Case Study

- An automobile dealership that uses weekly periodic inventory review
  - The facility is the showroom and surrounding areas holding cars
  - The items are cars for sell
  - The supplier is the car manufacturer
  - The customers are the people who purchase cars from the dealership
  - Assume SIS: sells one type of car

Example 1.3.5: Results of Case Study

- Limited to a maximum of $S = 80$ cars
- Limited to a minimum of $s = 20$ cars
- Inventory reviewed every Monday
  - if inventory falls below s, order cars sufficient to restore the inventory to S
- For now, ignore delivery lag
- Then costs:
  - Item cost is $c_{item} = \$8,000$ per item
  - Setup cost is $c_{setup} = \$1,000$ per order from manufacturer
  - Holding cost is $c_{hold} = \$25$ per week
  - Shortage cost is $c_{hold} = \$700$ per week

Per-Interval Average Operating Costs

- The average operating costs per time interval are
  - $c_{item}\bar{o}$: item cost
  - $c_{setup}\bar{u}$: setup cost
  - $c_{hold}\bar{l}^{+}$: holding cost
  - $c_{short}\bar{l}^{-}$: shortage cost
- The average *total* operating cost *per time interval* is their sum

$$\bar{C}_{total} = c_{item}\bar{o} + c_{setup}\bar{u} + c_{hold}\bar{l}^{+} + c_{short}\bar{l}^{-} \tag{15}$$

- *Total* cost of operation is the product of the average *total* operating cost *per time interval* and the number of intervals

$$C_{total} = n\bar{C}_{total} \tag{16}$$

Example 1.3.6: Operating Cost

- Use the statistics in Example 1.3.4

$$\bar{o} = \bar{d} = 29.29 \quad \bar{u} = 0.39 \quad \bar{l}^{+} = 42.90 \quad \bar{l}^{-} = 0.25$$

  and the constants in Example 1.3.5

$$c_{item} = \$8,000 \quad c_{setup} = \$1,000 \quad c_{hold} = \$25 \quad c_{hold} = \$700$$

- For the dealership
  - item cost: $\$8,000 \times 29.29 = \$234,320$ per week
  - setup cost: $\$1,000 \times 0.39 = \$390$ per week
  - holding cost: $\$25 \times 42.40 = \$1,060$ per week
  - shortage cost: $\$700 \times 0.25 = \$175$ per week

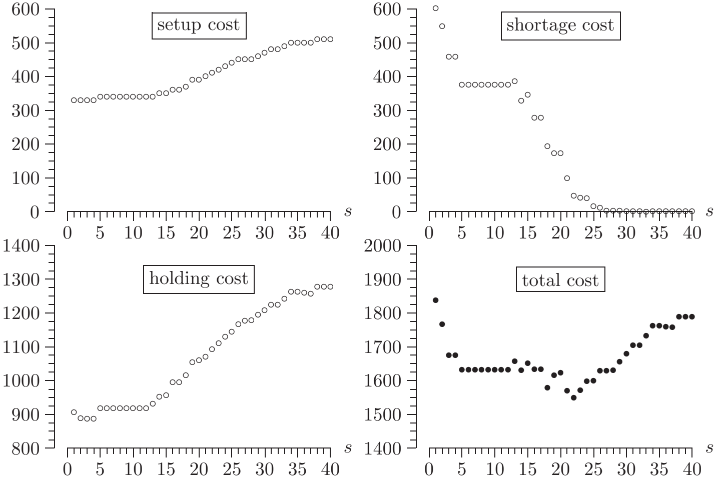
Cost Minimization

- By varying minimum inventory level s (and possibly maximum inventory level S), an *optimal policy* can be determined
- Optimal $\Longleftrightarrow$ minimum average total cost per time interval
- Note that $\bar{o} = \bar{d}$ and $\bar{d}$ depends only on the demands
- Hence, item cost per time interval $c_{item}\bar{o}$ is independent of (s, S)
- Average dependent cost per time interval is the sum of these three
  - $c_{setup}\bar{u}$: average setup cost per time interval
  - $c_{hold}\bar{l}^{+}$: average holding cost per time interval
  - $c_{short}\bar{l}^{-}$: average shortage cost per time interval

Experiment

- Let S be fixed, and let the demand sequence be fixed
- If s is systematically increased, we expect:
  - Average setup cost and holding cost per time interval will increase as s increases
  - Average shortage cost per time interval will decrease as s increases
  - Average dependent cost per time interval will have "U" shape, yielding an optimum (i.e., minimum cost $= \$1,550$ at $s = 22$)

Example 1.3.7: Optimal Periodic Inventory Review Policy



- Minimum cost $= \$1,550$ at $s = 22$

Exercise L3-4: sis1 and Optimizing Operating Cost

- Use the sample data (sis1.dat) and the parameters in the lecture nodes, verify the optimal inventory review policy by showing the results in slides 35 – 37, produce the data necessary to reproduce the figures in slide 38.
- Submit the result.

Summary

- SIS
- Cost model and case studies