

Discrete-Event Simulation

Lawrence M. Leemis and Stephen K. Park, Discrete-Event Simul A First Course, Prentice Hall, 2006

Hui Chen

Computer Science
Virginia State University
Petersburg, Virginia

February 1, 2016

Introduction

- ▶ Programs ssq1 and sis1 are *trace-driven* discrete-event simulations
 - ▶ Both rely on input data from an external source
- ▶ These realizations of naturally occurring stochastic processes are limited
- ▶ Cannot perform “what if” studies without modifying the data
- ▶ Solution
 - ▶ Convert the single server service node and the simple inventory system to utilize *randomly generated input*
 - ▶ Use a random-number generator to produce the randomly generated input
 - ▶ Discrete-event simulation programs using the randomly generated input does not depend on external trace data

Single Queue Service Node: Revisited

- ▶ Need two stochastic assumptions
 - ▶ arrival times
 - ▶ service times
- ▶ The assumptions governs how arrival and service times are randomly generated in discrete-event simulation programs

Example: Generating Service Times: Uniform Distribution

- ▶ Service time

- ▶ Range: between 1.0 and 2.0

- ▶ Distribution within the range?

Without further knowledge, we assume no time is more likely than any other

- ▶ To generate service times: use $u = \text{Uniform}(1.0, 2.0)$ random variate

Example: Generating Service Times: Uniform Distribution

Is it reasonable to assume that service times are uniformly distributed, e.g., service times are generated using $u = \text{Uniform}(1.0, 2.0)$ random variate?

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It depends.

Example: Generating Service Times: Uniform Distribution

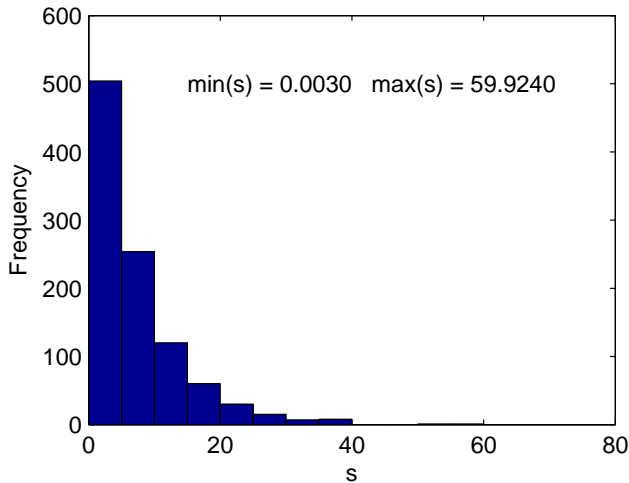
Is it reasonable to assume that service times are uniformly distributed, e.g., service times are generated using $u = \text{Uniform}(1.0, 2.0)$ random variate?

It depends.

In most applications, it is unrealistic to assume service times are uniformly distributed.

Service Time in *ssq1.dat* Trace Data

Is service times in *ssq1.dat* uniformly distributed?



Example: Generating Service Times: Exponential Distribution

- ▶ In general, it is unreasonable to assume that all possible values are equally likely.
- ▶ Frequently, small values are more likely than large values
- ▶ Need a non-linear transformation that maps $0 \rightarrow 1$ to $0 \rightarrow \infty$ since $0 < u = \text{Uniform}(0, 1) < 1$

Example: Generating Service Times: Exponential Distribution

- ▶ A common nonlinear transformation is

$$x = -\mu \ln(1 - u) \quad (1)$$

- ▶ The transformation is monotone increasing, one-to-one, and onto

$$0 < \mu < 1 \iff 0 > -u > -1 \quad (2)$$

$$\iff 0 + 1 > -u + 1 < -1 + 1 \quad (3)$$

$$\iff 1 > 1 - u > 0 \quad (4)$$

$$\iff \ln(1) > \ln(1 - u) > \ln(0) \quad (5)$$

$$\iff 0 > \ln(1 - u) > \infty \quad (6)$$

$$\iff 0 < -\ln(1 - u) < \infty \quad (7)$$

$$\iff 0 < -\mu \ln(1 - u) < \infty \quad (8)$$

$$\iff 0 < x < \infty \quad (9)$$

Example: Generating Service Times: Exponential Distribution

- ▶ The common nonlinear transformation $x = -\mu \ln(1 - u)$ is monotone increasing, one-to-one, and onto

$$0 < \mu < 1 \iff 0 < -\mu \ln(1 - u) < \infty \iff 0 < x < \infty \quad (10)$$

which generates $\text{Exponential}(\mu)$ random variate

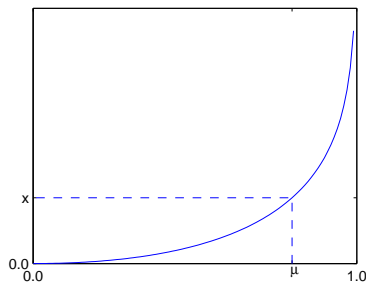


Figure : Exponential-variate-generation Geometry

Example: Generating Service Times: Exponential Distribution

- ▶ The common nonlinear transformation

$$x = -\mu \ln(1 - u) \quad (11)$$

generates $Exponential(\mu)$ random variate

- ▶ Note that $0 < u < 1$ and

$$\int_0^1 -\mu \ln(1 - u) du = -\mu \int_0^1 \ln(1 - u) du \quad (12)$$

$$= -\mu \int_0^1 -\ln(1 - u) d(1 - u) = \mu \int_0^1 \ln(1 - u) d(1 - u) \quad (13)$$

$$= \mu \{ \ln(1 - u)(1 - u) |_0^1 - \int_0^1 (1 - u) d\ln(1 - u) \} \quad (14)$$

$$= \mu \{ 0 - (1 - u) \frac{1}{1 - u} (1 - u) |_0^1 \} \quad (15)$$

$$= -\mu (1 - u) \frac{1}{1 - u} (1 - u) |_0^1 = -\mu (1 - u) |_0^1 \quad (16)$$

$$= \mu \quad (17)$$

i.e., the parameter μ specifies the sample mean

Generating $Exponential(\mu)$ Random Variate

Definition 3.1.1 ANSI C Function for $Exponential(\mu)$

```
double Exponential(double  $\mu$ )  
{  
    return -  $\mu$  * log(1.0 - Random());  
}
```

where $Random()$ generates $u = Uniform(0, 1)$ random variate and μ is the sample mean.

Example: Generating Service Times: Exponential Distribution

In the single-server service node simulation, we use $Exponential(\mu_s)$ to generate service times,

$$s_i = Exponential(\mu_s); \quad i = 1, 2, 3, \dots, n \quad (18)$$

where μ_s is the sample mean of service times.

Example: Generating Interarrival Times: Exponential Distribution

In the single-server service node simulation, we use $Exponential(\mu_a)$ to generate interarrival times,

$$a_i = a_{i-1} + Exponential(\mu_a); \quad i = 1, 2, 3, \dots, n \quad (19)$$

where μ_a is the sample mean of interarrival times.

Example: Recap

- ▶ Inter-arrival times
 - ▶ Generating $u = \text{Uniform}(a, b)$ random variate
 - ▶ Generating $u = \text{Exponential}(a)$ random variate
- ▶ Service times
 - ▶ Generating $u = \text{Uniform}(a, b)$ random variate
 - ▶ Generating $u = \text{Exponential}(a)$ random variate

Simulation Program *ssq2.m*

- ▶ Program *ssq2* is an extension of *ssq1*
 - ▶ Interarrival times are drawn from *Exponential*(2.0)
 - ▶ Service times are drawn from *Uniform*(1.0, 2.0)
- ▶ The program generates job-averaged and time-averaged statistics
 - ▶ \bar{r} : average interarrival time
 - ▶ \bar{w} : average wait
 - ▶ \bar{d} : average delay
 - ▶ \bar{s} : average service time
 - ▶ \bar{l} : average # in the node
 - ▶ \bar{q} : average # in the queue
 - ▶ \bar{x} : server utilization

Exercise L4-1

In this exercise, you are required to complete the following tasks,

- ▶ Compile and run the *ssq2* program.
- ▶ Make a copy of the *ssq2* program, revise it to meet the following,
 - ▶ Interarrival times are drawn from *Uniform*(0.0, 6.0)
 - ▶ Service times are drawn from *Exponential*(2.0)and then compile and run the program.
- ▶ Submit your work including both versions of the *ssq2* program and the results of both runs in Blackboard

Example 3.1.3: Theoretical Result from Analytic Model

- ▶ The theoretical averages for a single-server service node using *Exponential*(2.0) inter-arrivals and *Uniform*(1.0, 2.0) service times are (Gross and Harris, 1985),

\bar{r}	\bar{w}	\bar{d}	\bar{s}	\bar{l}	\bar{q}	\bar{x}
2.00	3.83	2.33	1.50	1.92	1.17	0.75

- ▶ Although the server is busy only 75% of the time, on average there are approximately two jobs in the service node
- ▶ A job can expect to spend more time in the queue than in service
- ▶ To achieve these averages, many jobs must pass through node

Example 3.1.3: Results from Simulation Program *ssq2*

- ▶ The accumulated average wait was printed every 20 jobs

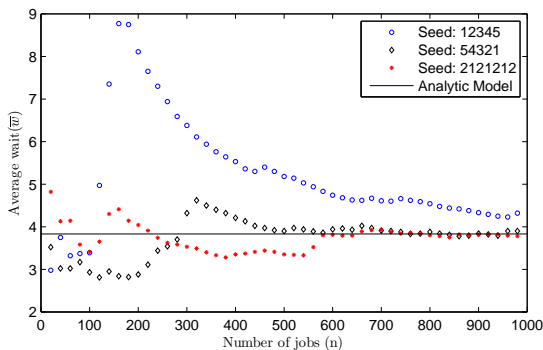


Figure : Average wait times

- ▶ The convergence of \bar{w} is slow, erratic, and dependent on the initial seed

Use of Program *ssq2*

- ▶ The program can be used to study the steady-state behavior
 - ▶ Will the statistics converge independent of the initial seed?
 - ▶ How many jobs does it take to achieve steady-state behavior?
- ▶ It can be used to study the transient behavior
 - ▶ Fix the number of jobs processed and replicate the program with the initial state fixed
 - ▶ Each replication uses a different initial rng seed

Exercise L4-2

You are required to reproduce the figure in slide 20. You may take steps below (using the C/C++ program as an example),

- ▶ Convert the main function *int main(void)* to function *void SimulateOnce(long seed, long last)*.
 - ▶ *seed*: seed of RNG
 - ▶ *last*: the number of jobs to process
 - ▶ Replace in the function the *printf* statements to

```
printf("%ld,%ld,%6.2f,%6.2f,%6.2f,%6.2f,%6.2f,%6.2f,%6.2f\n",  
    seed, index, sum.interarrival/index, sum.wait/index,  
    sum.delay/index, sum.service/index, sum.wait/departure,  
    sum.delay/departure, sum.service/departure);
```

- ▶ Add the *main* function in which you call *SimulateOnce* with *seed* and *last* in a loop with *last* as the loop variable to simulate with the number of jobs as 20, 40, ..., 1000.
- ▶ Run the program and graph the results
- ▶ Submit the program, the results, and the graph in Blackboard

Geometric Random Variables

- ▶ The *Geometric*(p) random variate is the discrete analog to a continuous *Exponential*(μ) random variate

Let $x = \text{Exponential}(\mu) = \mu \ln(1 - \mu)$, $y = \lfloor x \rfloor$, and $p = \Pr(y \neq 0)$

$$y = \lfloor x \rfloor \neq 0 \iff x \geq 1 \quad (20)$$

$$\iff \mu \ln(1 - \mu) \geq 1 \quad (21)$$

$$\iff \ln(1 - \mu) \leq -1/\mu \quad (22)$$

$$\iff 1 - \mu \leq e^{-1/\mu} \quad (23)$$

Since $1 - \mu$ is also *Uniform*(0.0, 1.0) and $p = \Pr(y \neq 0) = e^{-1/\mu}$

Finally, since $\mu = -1/\ln(p)$, $y = \lfloor \ln(1 - \mu)/\ln(p) \rfloor$

Generating $Geometric(p)$ Random Variates

Definition 3.1.2 ANSI C Function for $Geometric(p)$

```
long Geometric(double p)  use  $0.0 < p < 1.0$ 
{
    return (long)(log(1.0 - Random()) / log(p));
}
```

- ▶ $Random()$ generates $u = Uniform(0, 1)$ random variate.
- ▶ The mean of a $Geometric(p)$ random variate is $p/(1 - p)$
 - ▶ If p is close to zero then the mean will be close to zero
 - ▶ If p is close to one, then the mean will be large

Example 3.1.4: Composite Service Model

Now consider a composite service model

- ▶ Assume that jobs arrive at random with a steady-state arrival rate of 0.5 jobs per minute
- ▶ Assume that Job service times are composite with two components
 - ▶ The number of service tasks is $1 + \text{Geometric}(0.9)$
 - ▶ The time (in minutes) per task is $\text{Uniform}(0.1, 0.2)$

Example 3.1.4: Composite Service Model

ANSI C Function for the Composite Service Model

```
double GetService(void)
{
    long k;
    double sum = 0.0;
    long tasks = 1 + Geometric(0.9);
    for (k = 0; k < tasks; k++)
        sum += Uniform(0.1, 0.2);
    return (sum);
}
```

Example 3.1.4: Composite Service Model: Analytic Model

- ▶ The theoretical steady-state statistics for this model are

\bar{r}	\bar{w}	\bar{d}	\bar{s}	\bar{l}	\bar{q}	\bar{x}
2.00	5.77	4.27	1.50	2.89	2.14	0.75

- ▶ The arrival rate, service rate, and utilization are identical to Example 3.1.3 (See slide 19)
- ▶ The other four statistics are significantly larger
- ▶ Performance measures are sensitive to the choice of service time distribution

Simple Inventory System: Example 3.1.5

- ▶ Program *sis2* has randomly generated demands using an *Equilikely*(a, b) random variate
- ▶ Using random data, we can study transient and steady-state behaviors
- ▶ If $(a, b) = (10, 50)$ and $(s, S) = (20, 80)$, then the approximate steady-state statistics are

\bar{d}	\bar{o}	\bar{u}	\bar{l}^+	\bar{l}^-
30.00	30.00	0.39	42.86	0.26

Exercise L4-3

In this exercise, you are required to complete the following tasks,

- ▶ Compile and run the *sis2* program. Document the results.
- ▶ Make a copy of the *sis2* program, revise it to meet the following,
 - ▶ The demand is drawn from *Geometric*(0.967742)and then compile and run the program.
- ▶ Submit your work including both version of the *sis2* program and the results of both runs in Blackboard

Effects of Number of Time Intervals and Seed of RNG

- ▶ The average inventory level $\bar{l} = \bar{l}^+ - \bar{l}$ approaches steady state after several hundred time intervals

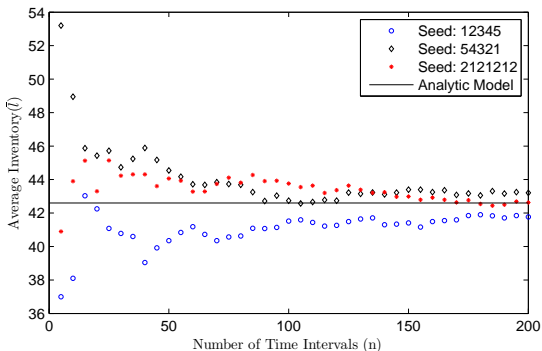


Figure : Number of Time Intervals (n)

- ▶ Convergence is slow, erratic, and dependent on the initial seed

Exercise L4-4

You are required to reproduce the figure in slide 30. You may take steps below (using the Java program as an example),

- ▶ Convert the main function *public static void main(String[] args)* to function *public static void SimulateOnce(long seed, long stop)*.
 - ▶ *seed*: seed of RNG; *stop*: the number of intervals to process
 - ▶ Replace in the function the *System.out.print* and *System.out.println* statements to

```
System.out.println(seed+","+ stop + "," + f.format(sum.demand/index)+","
  + MINIMUM + "," + MAXIMUM + "," + f.format(sum.order/index) + ", "
  + f.format(sum.setup/index) + "," + f.format(sum.holding/index) + ","
  + f.format(sum.shortage/index) + ", "
  + f.format(sum.holding/index - sum.shortage/index));
```

- ▶ Add the *public static void main(String[] args)* function in which you call *SimulateOnce* with *seed* and *stop* in a loop with *stop* as the loop variable to simulate with the number of jobs as 5, 10, 15, ..., 200.
- ▶ Run the program and graph the results
- ▶ Submit the program, the results, and the graph in Blackboard

Example 3.1.7: Optimal Inventory Policy

- If we fix S , we can find the optimal cost by varying s

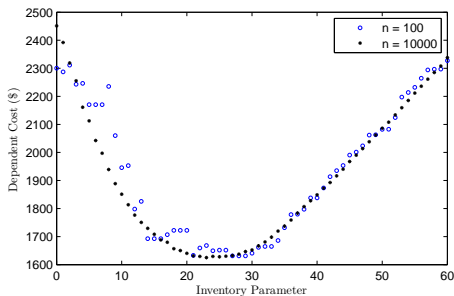


Figure : Dependent Cost for (s, S) Inventory System

where $c_{setup} = \$1,000$, $c_{hold} = 25$, $c_{short} = 700$,
 $\min(\text{DependentCost}) = \$1,624.86$, and $s = 24$.

- Recall that the dependent cost ignores the fixed cost of each item

Example 3.1.7: Discussion

- ▶ Using a fixed initial seed guarantees the exact same demand sequence
 - ▶ Any changes to the system are caused solely by the change of s
- ▶ A steady state study of this system is unreasonable
 - ▶ All parameters would have to remain fixed for many years
 - ▶ When $n = 100$ we simulate approximately 2 years
 - ▶ When $n = 10000$ we simulate approximately 192 years

Statistical Considerations

- ▶ Example 3.1.7 illustrates two consideration
 - ▶ Variance reduction
 - ▶ Robust estimation
- ▶ With Variance Reduction, we eliminate all sources of variance except one
 - ▶ Transient behavior will always have some inherent uncertainty
 - ▶ We kept the same initial seed and changed only s
- ▶ Robust Estimation occurs when a data point that is not sensitive to small changes in assumptions
 - ▶ Values of s close to 23 have essentially the same cost
 - ▶ Would the cost be more sensitive to changes in S or other assumed values?

Exercise L4-5

You are required to reproduce the figure in slide 32.

Hints (using the Java program as an example):

- ▶ Revise

```
public static void SimulateOnce(long seed, long stop)
    throws IOException { .....
```

to

```
public static void SimulateOnce(long seed, long stop, int slower)
    throws IOException { .....
```

where *slower* is s is (s, S) in the inventory system.

- ▶ In the main method/function, call the *SimulateOnce* method/function with $stop = 100$ and $stop = 10000$, respectively in two loops whose loop variable changes from $slower = 0$ to $slower = 60$ with increment 1.
- ▶ Let $c_{setup} = \$1,000$, $c_{hold} = 25$, and $c_{short} = 700$. Compute the dependent cost in an Excel workbook. Graph the cost versus s for the two $stop$ values.

$$C_{dependent} = c_{setup}\bar{u} + c_{hold}\bar{I}^+ + c_{short}\bar{I}^-$$

- ▶ Submit your work in Blackboard including both the program and the Excel workbook.

Summary

- ▶ Discrete-Event Simulations: random variate vs. trace
- ▶ Revisited SSQ
- ▶ Revisited SIS
- ▶ Variance reduction and robust estimation