Referee's report on "Block bootstrap consistency under weak assumptions", by Calhoun

This paper establishes conditions for bootstrap consistency under as weak as possible conditions on the weak memory of the stochastic process involved. This is a research agenda with merit and should be of interest to econometricians. I found it often hard to follow the arguments, and I have below a large number of requests for clarification. Most of my comments concern issues with the treatment of uniformity issues and the conditioning on blocks and block lengths. Regarding the conditioning on blocks and block lengths, it would be preferable if the author made it explicit where exactly the property of independence of the block lengths of the data is used, and how results conditioned on $\mathcal{M}_n$ and $\mathcal{N}_n$ translate to unconditional results. There also are uniformity issues in the proof. It appears to me that issues involving uniformity are not always treated correctly; see my detailed comments below.

While the author sets out to prove an FCLT for bootstrapped partial sums (see abstract), it is somewhat odd and counterintuitive that the proof only appears to use de Jong's CLT arguments. The suggestion seems to be that the bootstrapped FCLT holds under the same conditions as the CLT for the original data, instead of the FCLT for the original data. Some explanation of this aspect of the paper would be welcome.

Detailed comments

1. On page 3 and 4, the notation $\rightarrow^d$: does this signify weak convergence or pointwise convergence here? The notation suggests pointwise convergence, yet the paper mentions the FCLT. In Corollary 3, does the "converge in distribution" mean pointwise, or did you mean to indicate that an FCLT holds here?

2. Page 7: "for any t_0 and k such that the index of the following summation are well defined": this read a bit odd, perhaps you need to explain better what is meant here.

3. Lemma A.3: in the definition of l_n, p seems to be missing a subscript.

4. The definition of the "mixingale conditional on $\mathcal{M}_n$" is a tricky thing. It is clear from the followup how it is intended to be defined; yet, the mixingale numbers depend on $\mathcal{M}_n$ now; so that seems okay, and equalities such as (19) are then "almost sure" inequalities. The proof however proceeds then with Equation (22), at which point I get lost in the argument. You use McLeish's 1975 Theorem 1.6, which is a maximal inequality for partial sums, which suggests that perhaps the author intends to have a supremum inside the expectation - yet that does not seem to be the author's intention.

Then after applying the maximal inequality, an object that depends on $\mathcal{M}_n$ results, which is $O_p(l_n p_n)$ apparently, yet no argument is provided. More detail would be welcome here.

5. The argument then continues by a reference to de Jong (1997). Again, it is not clear how that argument can be uniform in τ and γ. Also, I cannot see how the conditioning on $\mathcal{M}_n$ is without apparent consequence here; this may be simple, but this reader would have been helped by more detail.

6. A similar comment holds for equations (25)-(27).

7. On page 11, bottom, the author states that "clearly Z_{nj}^{**} is an MDS conditional on $\mathcal{M}_n$ that has finite variance and is globally covariance stationary condition by Lemma 5." (sic). This statement I cannot interpret at all. Lemma 5 does not seem to have that information. I do not understand how conditional on $\mathcal{M}_n$ there can be a stationarity property, as block lengths as different conditional on $\mathcal{M}_n$. In short, you may consider rewriting this area of your paper and providing additional detail.

8. On page 11, bottom, the exact mathematical reasoning is not clear to me. You verify conditions conditional on $\mathcal{M}_n$. This means that where formerly some object and its limit were less than δ apart for $n \geq N_\delta$, now, we have this N_δ depend on $\mathcal{M}_n$. This then makes N_δ random as well as dependent on n. Therefore, at this point, this referee needs more detail on why your argument suffices.

9. Pages 11/12: "Since this limit does not depend on $\mathcal{M}_n$, it holds unconditionally as well": this statement needs elaboration. In particular, what does "this limit does not depend on $\mathcal{M}_n$" mean exactly and how is it used in the argument exactly?

10. Page 13, equation (36): the application of McLeish's Theorem 1.6 is again unclear. The notation suggests regular convergence to 0, while the initial object in the (36) is random.

11. p. 14, middle and below Equation (43): "$\{Z_{nj}(\tau, \mathcal{M}\}_j$": here $\mathcal{M}$ has lost its n subscript, why?

12. Page 14, Equation (41): it seems that Lemma A2 and independence is used here; perhaps that should be mentioned.

13. Page 15, above Equation 44: "can be made arbitrarily small by choosing large enough C": this appears to me to be an ambiguous statement, as this concerns a random

variable, so I suggest that you state the exact mathematical assertion here. C may depend on $\mathcal{M}_n$ here now.

14. p. 15, below the middle: "which converges to zero": the object is random, so you need to assert a mode of convergence.

15. p. 15, bottom: "To prove (45)" and onwards: here, I do not understand where the ζ_n function is coming from. You assert the existence of such a sequence of functions with the property $\zeta_n(x) = \zeta_n(x-1)$ for all x, but I do not understand why such a sequence of functions should exist. Then, you use Lemma A.3 and A.4 and "the fact that $\sum_{\tau=1}^{n} n^{-1}\zeta(x + \tau/n) \to 1$ uniformly in x by assumption". That last property I cannot see being assumed, and how this property plus A.3 and A.4 adds up to the assertion I do not know.