Supplemental Appendix for “Block bootstrap consistency under weak assumptions”

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2015-12-31, commit 5032b34

Abstract

This appendix contains detailed proofs of several of the supporting results in the paper, “Block bootstrap consistency under weak assumptions.” As we have stated elsewhere these proofs are fundamentally identical to existing proofs in this literature, but our indexing changes the presentation slightly. We are not in any way trying to take credit for the insight that led to these results; they are presented only for readers' reference and convenience.

Lemma 6. *Suppose the conditions of Theorem 1 hold. For any positive δ, there exist positive and finite constants C, n_0, and ϵ such that, for all $n > n_0$, $m = 1, \dots, n$, $\tau = 0, \dots, m$, and $\ell = 1, \dots, m-1$,*

$$\mathrm{E}\Big|\sum_{i=0}^{\lfloor n/m\rfloor-1}\big[Z_n'(\tau+im, m-\ell)^2 - \mathrm{E}\big(Z_n'(\tau+im,m-\ell)^2\big)\big]\Big| \leq 2\delta + C\cdot\big(\tfrac{m}{n}\big)^{1/2}\big(\tfrac{m}{\ell^{1+\epsilon}}\big)^{1/2}. \quad (37)$$

Also, there exists a constant C and bounded function $D(x)$ with $D(x)\to 0$ as $x\to\infty$ such that, for large enough n,

$$\mathrm{E}\Big|\sum_{i=0}^{\lfloor n/m\rfloor-1}\mathrm{E}(Z_n'(\tau+im,m-\ell)^2 - \sigma^2\Big| \leq C\,D(\ell). \quad (38)$$

Results (37) and (38) are direct extensions of de Jong's (1997) Lemmas 5 and 4, respectively, replacing de Jong's implicit use of inequalities with explicit inequalities. We can assume that $\mu_{nt} = 0$ without loss of generality in these proofs.

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Proof of (37). Define $h(x,B) = x\,1\{|x| \le B\} + \operatorname{sgn}(x) B\,1\{|x| > B\}$. Since Lemma 7 ensures uniform integrability, we can choose B large enough that

$$\mathrm{E}\Big|\sum_{i=0}^{\lfloor n/m\rfloor-1}\big(Z_n'(\tau+im,m-\ell)^2 n/m - h(Z_n'(\tau+im,m-\ell),B\sqrt{m/n})^2\big)\Big| < \delta,$$

so it suffices to bound

$$\mathrm{E}\Big|\sum_{i=0}^{\lfloor n/m\rfloor-1}\big(h(Z_n'(\tau+im,m-\ell),B\sqrt{m/n})^2 - \mathrm{E}\,h(Z_n'(\tau+im,m-\ell),B\sqrt{m/n})^2\big)\Big|.$$

As in de Jong's proof, $h(Z_n'(\tau+im,m-\ell),B\sqrt{m/n})^2$ is L_2 NED with respect to

$$\mathcal{V}_{n,i-j}^{i+j} = \sigma(V_{n,\tau+(i-j-1)m+\ell+1},\dots,V_{n,\tau+(i+j)m}).$$

This can be seen through the sequence of inequalities (for $j > 0$):

$$\begin{aligned}
\|h(&Z_n'(\tau+im,m-\ell),B\sqrt{m/n})^2 - \mathrm{E}(h(Z_n'(\tau+im,m-\ell),B\sqrt{m/n})^2 \mid \mathcal{V}_{n,i-j}^{i+j})\|_2 \\
&\le 2B\sqrt{\tfrac{m}{n}}\|h(Z_n'(\tau+im,m-\ell),B\sqrt{m/n}) \\
&\quad - \mathrm{E}(h(Z_n'(\tau+im,m-\ell),B\sqrt{m/n}) \mid \mathcal{V}_{n,i-j}^{i+j})\|_2 \\
&\le 2B\tfrac{\sqrt{m}}{n}\sum_{t\in I_n(\tau+im,m-\ell)}\|X_{nt} - \mathrm{E}(X_{nt}\mid \mathcal{V}_{n,i-j}^{i+j})\|_2 \\
&\le 2B\tfrac{\sqrt{m}}{n}\sum_{t\in I_n(\tau+im,m-\ell)} d_{nt}v_{j\ell} \\
&= 2BD\tfrac{m^{3/2}}{n}(\ell j)^{-1/2-\epsilon}
\end{aligned}$$

for some $\epsilon > 0$ and $D \ge \max_t d_{nt}$. For $j = 0$, we have

$$\begin{aligned}
\|h(Z_n'(\tau+im,m-\ell),B\sqrt{m/n})^2 - \mathrm{E}(h(Z_n'(\tau+im,m-\ell),B\sqrt{m/n})^2 \mid \mathcal{V}_{n,i}^{i})\|_2& \\
\le 2B\sqrt{\tfrac{m}{n}}\|Z_n'(\tau+im,m-\ell)\|_2& \\
\le 2BD\tfrac{m}{n}&
\end{aligned}$$

with the last inequality holding by Lemma 7.

These NED inequalities further imply that $h(Z_n'(\tau+im,m-\ell))^2$ is an L_2-mixingale of size $-1/2$. Define

$$\mathcal{H}_{nk} = \sigma(V_{n,\tau+km},V_{n,\tau+(k-1)m},\dots)$$

For $j > 0$,

$$\begin{aligned}
&\|h(Z_n'(\tau+im, m-\ell, B\sqrt{m/n}))^2 - \mathrm{E}(h(Z_n'(\tau+im, m-\ell), B\sqrt{m/n})^2 \mid \mathcal{H}_{n,i-2j})\|_2 \\
&\qquad \le \|h(Z_n'(\tau+im, m-\ell), B\sqrt{m/n})^2 \\
&\qquad\quad - \mathrm{E}(h(Z_n'(\tau+im, m-\ell), B\sqrt{m/n})^2 \mid V_n(i+j, i-j))\|_2 \\
&\qquad\quad + \|\mathrm{E}(h(Z_n'(\tau+im, m-\ell), B\sqrt{m/n})^2 \mid V_n(i+j, i-j)) \\
&\qquad\quad - \mathrm{E}(h(Z_n'(\tau+im, m-\ell), B\sqrt{m/n})^2 \mid \mathcal{H}_{n,i-2j})\|_2 \\
&\qquad \le 2BD\tfrac{m^{3/2}}{n}(\ell j)^{-1/2-\epsilon} + 2BD\tfrac{m}{n}\psi(j\ell)
\end{aligned}$$

where $\psi(x) = x^{1/2-1/r}$ if V_{nt} is strong mixing and $\psi(x) = x^{1-1/r}$ if V_{nt} is uniform mixing. By assumption, $\psi(x) = O(x^{-1/2-\epsilon'})$ for some $\epsilon' > 0$. Assume without loss of generality that the ϵ we defined earlier satisfies this requirement as well, so $\psi(j\ell) \le D'(j\ell)^{-1/2-\epsilon}$ for some constant $D' > 1$.

For the rest of the mixingale inequalities, let $j \ge 0$ and then

$$\begin{aligned}
&\|h(Z_n'(\tau+im, m-\ell, B\sqrt{m/n}))^2 - \mathrm{E}(h(Z_n'(\tau+im, m-\ell), B\sqrt{m/n})^2 \mid \mathcal{H}_{n,i+2j})\|_2 \\
&\qquad \le \|h(Z_n'(\tau+im, m-\ell), B\sqrt{m/n})^2 \\
&\qquad\quad - \mathrm{E}(h(Z_n'(\tau+im, m-\ell), B\sqrt{m/n})^2 \mid V_n(i+j, i-j))\|_2 \\
&\qquad \le 2BD\tfrac{m^{3/2}}{n}(\ell j)^{-1/2-\epsilon},
\end{aligned}$$

completing the argument that $h(Z_n'(\tau+im, m-\ell, B\sqrt{m/n}))^2$ is an L_2-mixingale of size $-1/2$.

Now let $C = 4BDD'$. We can now apply de Jong's Lemma 2 (originally presented in McLeish, 1975a) to this mixingale, giving

$$\begin{aligned}
\Big\| \sum_{i=0}^{\lfloor n/m\rfloor-1} \big(h(Z_n'(\tau+im, m-\ell, B\sqrt{m/n}))^2 - \mathrm{E}\, h(Z_n'(\tau+im, m-\ell), B\sqrt{m/n})^2\big) \Big\|_2 \\
\le \Bigg(\sum_{i=0}^{\lfloor n/m\rfloor-1} \big(C\tfrac{m}{n}\tfrac{m^{1/2}}{\ell^{1/2+\epsilon}}\big)^2 \Bigg)^{1/2} \\
\le C\big(\tfrac{m}{n}\big)^{1/2}\tfrac{m^{1/2}}{\ell^{1/2+\epsilon}}
\end{aligned}$$

which gives the final result. □

Proof of (38). Observe that

$$
\begin{aligned}
&\Big|\sum_{j=0}^{\lfloor n/m\rfloor-1}\sum_{i=j+1}^{\lfloor n/m\rfloor-1}\mathrm{E}\,Z_n'(\tau+im,m-\ell)Z_n'(\tau+jm,m-\ell)\Big| \\
&\qquad\le \tfrac{1}{n}\sum_{j=0}^{\lfloor n/m\rfloor-1}\sum_{i=j+1}^{\lfloor n/m\rfloor-1}\sum_{t=jm+\ell+1}^{(j+1)m}\sum_{k=im+\ell+1-t}^{(i+1)m-t}1\{k\ge\ell\}\,\mathrm{E}|X_{nt}X_{n,t+k}| \\
&\qquad\le \tfrac{1}{n}\sum_{t=1}^{n}\sum_{k=0}^{n-t}1\{k\ge\ell\}\sum_{v=0}^{\infty}\big(\mathrm{E}\,\Delta_{1n}(t,v)\,\mathrm{E}\,\Delta_{2n}(t,k,v)\big)^{1/2} && (40)\\
&\qquad\quad + \tfrac{1}{n}\sum_{t=1}^{n}\sum_{k=0}^{n-t}1\{k\ge\ell\}\sum_{v=1}^{\infty}\big(\mathrm{E}\,\Delta_{3n}(t,v)\,\mathrm{E}\,\Delta_{4n}(t,k,v)\big)^{1/2} && (41)
\end{aligned}
$$

by de Jong's Lemma 3, with

$$
\begin{aligned}
\Delta_{1n}(t,v) &= (\mathrm{E}_{t-v}\,X_{nt})^2-(\mathrm{E}_{t-v-1}\,X_{nt})^2\\
\Delta_{2n}(t,k,v) &= (\mathrm{E}_{t-v}\,X_{n,t+k})^2-(\mathrm{E}_{t-v-1}\,X_{n,t+k})^2\\
\Delta_{3n}(t,v) &= (X_{nt}-\mathrm{E}_{t+v-1}\,X_{nt})^2-(X_{nt}-\mathrm{E}_{t+v}\,X_{nt})^2\\
\Delta_{4n}(t,k,v) &= (X_{n,t+k}-\mathrm{E}_{t+v-1}\,X_{n,t+k})^2-(X_{n,t+k}-\mathrm{E}_{t+v}\,X_{n,t+k})^2
\end{aligned}
$$

Both (40) and (41) can be bounded by very similar arguments, so we just present the first. Now

$$
\begin{aligned}
&\tfrac{1}{n}\sum_{t=1}^{n}\sum_{k=0}^{n-t}1\{k\ge\ell\}\sum_{v=0}^{\infty}\big(\mathrm{E}\,\Delta_{1n}(t,v)\,\mathrm{E}\,\Delta_{2n}(t,k,v)\big)^{1/2}\le\\
&\qquad\sum_{v=0}^{\infty}\Big\{\Big(\tfrac{1}{n}\sum_{t=1}^{n}\mathrm{E}\,\Delta_{1n}(t,v)\Big)^{1/2}\Big(\tfrac{1}{n}\sum_{t=1}^{n}\Big(\sum_{k=0}^{n-t}1\{k\ge\ell\}(\mathrm{E}\,\Delta_{2n}(t,k,v))^{1/2}\Big)^2\Big)^{1/2}\Big\}
\end{aligned}
$$

by the Cauchy-Schwarz inequality. By the same inequality, we have

$$
\begin{aligned}
&\tfrac{1}{n}\sum_{t=1}^{n}\Big(\sum_{k=0}^{n-t}1\{k\ge\ell\}(\mathrm{E}\Delta_{2n}(t,k,v))^{1/2}\Big)^2\\
&\qquad\qquad\le (C/n)\sum_{t=1}^{n}\sum_{k=\ell}^{n-t}k\log(k)^2\,\mathrm{E}\,\Delta_{2n}(t,k,v)\\
&\qquad\qquad\le (C/n)\sum_{t=1}^{n}c_{nt}^2\Big(\ell\log(\ell)^2\psi(\ell)^2+\sum_{k=\ell}^{\infty}\log(k)^2\psi(k)^2\Big)
\end{aligned}
$$

where C is any constant that bounds $\sum_{k=1}^{\infty} k^{-1}\log(k)^{-2}$ and ψ is defined as $\psi(k) \equiv v_k + 6\alpha_{k/2}^{1/p-1/r}$ (if the underlying mixing process is strong mixing with coefficients α_k) or $\psi(k) \equiv v_k + 6\phi_{k/2}^{1-1/r}$ (if the underlying process is uniform mixing with coefficients ϕ_k).[1] Notice that this last expression does not depend on v.

Similarly,

$$\begin{aligned}\sum_{v=0}^{\infty}\Big(\tfrac{1}{n}\sum_{t=1}^{n}\mathrm{E}\,\Delta_{1n}(t,v)\Big)^{1/2} &\leq \Big((C/n)\sum_{t=1}^{n}\Big(\mathrm{E}\,\Delta_{1n}(t,0)+\sum_{v=1}^{\infty}\log(v)^2\,\mathrm{E}\,\Delta_{1n}(t,v)\Big)\Big)^{1/2}\\ &\leq \Big((C/n)\sum_{t=1}^{n}c_{nt}^2\Big(\psi(0)^2+\sum_{v=1}^{\infty}\log(v)^2\psi(v)^2\Big)\Big)^{1/2}.\end{aligned}$$

So we can bound (40) with

$$\begin{aligned}\text{const}\times D(\ell) = \Big((C/n)\sum_{t=1}^{n}c_{nt}^2\Big)&\\ \times\Big(\ell\log(\ell)^2\psi(\ell)^2+\sum_{k=\ell}^{\infty}\log(k)^2\psi(k)^2\Big)^{1/2}&\Big(\psi(0)^2+\sum_{v=1}^{\infty}\log(v)^2\psi(v)^2\Big)^{1/2}\end{aligned}$$

and $D(\ell)\to 0$ as $\ell\to\infty$. A similar proof holds for (41), so (38) holds with

$$D(\ell) = \Big(\ell\log(\ell)^2\psi(\ell)^2+\sum_{k=\ell}^{\infty}\log(k)^2\psi(k)^2\Big)^{1/2}. \qquad \square$$

Lemma 7. *Under the conditions of Theorem 1,*

$$\begin{aligned}\lim_{C\to 0}\limsup_{n\to\infty}\sup_{\substack{\tau=0,\dots,n-1\\ m'=1,\dots,n}}\mathrm{E}\,\Big(\big(\max_{m=1,\dots,m'}Z_n'(\tau,m)^2 n/m'\big)&\\ \times 1\{\max_{m=1,\dots,m'}Z_n'(\tau,m)^2 n/m' > C\}\Big) = 0.&\qquad (39)\end{aligned}$$

Proof of Lemma 7. The argument follows McLeish (1975b, Lemma 6.5) and McLeish (1977, Lemma 3.5) almost exactly and is also presented as Theorem 16.13 in Davidson (1994). We present the proof here to show that it continues to hold under our indexing strategy.

Let ϵ be an arbitrary positive number. Without loss of generality, assume that X_{nt} has mean zero for all n and t. Define $w_n(\tau,m')^2 = \sum_{t\in I_n(\tau,m')}c_{nt}^2$ and separate X_{nt} into three components,

$$X_{nt} = U_{nt} + T_{nt} + Y_{nt}$$

[1] Note that these values of $\psi(k)$ are the mixingale indices corresponding to our NED array. See Davidson (1994, Theorem 17.15) for details.

with

$$
\begin{aligned}
U_{nt} &= X_{nt} - \mathrm{E}_{t+k} X_{nt} + \mathrm{E}_{t-k} X_{nt} \\
T_{nt} &= \mathrm{E}_{t+k} X_{nt} 1\{|X_{nt}| > C' c_{nt}\} - \mathrm{E}_{t-k} X_{nt} 1\{|X_{nt}| > C' c_{nt}\} \\
Y_{nt} &= \mathrm{E}_{t+k} X_{nt} 1\{|X_{nt}| \leq C' c_{nt}\} - \mathrm{E}_{t-k} X_{nt} 1\{|X_{nt}| \leq C' c_{nt}\}
\end{aligned}
$$

where k and C' are arbitrary constants that will be constrained later in the proof — this representation holds for any value of these constants. For convenience, define

$$
\begin{aligned}
x_n(\tau, m') &= \max_{m \in 1,\dots,m'} \Big(\sum_{t \in I_n(\tau,m)} X_{nt} \Big)^2 / w_n(\tau, m')^2 \\
u_n(\tau, m') &= \max_{m \in 1,\dots,m'} \Big(\sum_{t \in I_n(\tau,m)} U_{nt} \Big)^2 / w_n(\tau, m')^2 \\
y_n(\tau, m') &= \max_{m \in 1,\dots,m'} \Big(\sum_{t \in I_n(\tau,m)} Y_{nt} \Big)^2 / w_n(\tau, m')^2
\end{aligned}
$$

and

$$
z_n(\tau, m') = \max_{m \in 1,\dots,m'} \Big(\sum_{t \in I_n(\tau,m)} T_{nt} \Big)^2 / w_n(\tau, m')^2.
$$

Using the Cauchy-Schwarz inequality and basic algebra, we have for any τ and m' the inequality

$$
x_n(\tau, m') \leq 3\big(u_n(\tau, m') + y_n(\tau, m') + z_n(\tau, m')\big)
$$

which, along with a well known inequality (Theorem 9.29 in Davidson, 1994) gives the bounds

$$
\begin{aligned}
&\mathrm{E}(x_n(\tau, m') 1\{x_n(\tau, m') > C\}) \\
&\qquad \leq 6\big(\mathrm{E}(u_n(\tau, m') 1\{u_n(\tau, m') > C/6\}) + \mathrm{E}(y_n(\tau, m') 1\{y_n(\tau, m') > C/6\})\big) \\
&\qquad\quad + \mathrm{E}(z_n(\tau, m') 1\{z_n(\tau, m') > C/6\}) \\
&\qquad \leq 6\big(\mathrm{E}\, u_n(\tau, m') + \mathrm{E}(y_n(\tau, m') 1\{y_n(\tau, m') > C/6\}) + \mathrm{E}\, z_n(\tau, m')\big)
\end{aligned}
$$

for any positive C.

Now observe that $\| \mathrm{E}_{t-l} U_{nt} \|_2 \leq d_{nt} v_{\max(k,l)}$ and $\| U_{nt} - \mathrm{E}_{t+l} U_{nt} \|_2 \leq d_{nt} v_{\max(k,l)+1}$ for positive l, making U_{nt} an L_2-mixingale of size $-1/2$. Consequently, for any fixed τ and m' satisfying $\tau + m' \leq n$, we can apply Theorem 1.6 of McLeish (1975a) to get the bound

$$
\mathrm{E}\, u_n(\tau, m') \leq 8\Big((k+1)k^{-1-\delta} + \sum_{s=k+1}^{\infty} s^{-1-\delta}\Big)\Big(v_k^2\, k^{1+\delta} + 2 \sum_{s=k+1}^{\infty} v_s^2\, s^{\delta}\Big), \tag{42}
$$

where $\delta > 0$ satisfies $v_k = O(k^{-1/2-\delta})$. (This δ must exist because of our assumptions on the size of the NED array.) This bound is $O(k^{-\delta})$ as $k \to \infty$ uniformly in n, m', and τ. When $\tau + m' > n$, we have (from the Cauchy-Schwarz inequality)

$$\mathrm{E}\, u_n(\tau, m') \leq 4 \max\big(\mathrm{E}\, u_n(\tau, n-\tau), \mathrm{E}\, u_n(0, m'+\tau-n)\big),$$

and both terms individually satisfy (42). As a result, we can choose k large enough that

$$\mathrm{E}\, u_n(\tau, m') \leq \epsilon/2$$

for all n, m', and τ.

We can apply essentially the same argument to $z_n(\tau, m')$. For positive l, we have

$$\begin{aligned}
\|\mathrm{E}_{t-l} T_{nt}\|_2 &= \big(\mathrm{E}[(\mathrm{E}_{t-\min(l,k)} T_{nt})^2 - (\mathrm{E}_{t-k} T_{nt})^2]\big)^{1/2} \\
&\leq \big(\mathrm{E}(X_{nt}^2/c_{nt}^2\, 1\{|X_{nt}/c_{nt}| > C'\})\big)^{1/2}\, 1\{l < k\}
\end{aligned}$$

and

$$\begin{aligned}
\|T_{nt} - \mathrm{E}_{t+l} T_{nt}\|_2 &= \big(\mathrm{E}[(\mathrm{E}_{t+k} T_{nt})^2 - (\mathrm{E}_{t+\min(l,k)} T_{nt})^2]\big)^{1/2} \\
&\leq \big(\mathrm{E}(X_{nt}^2/c_{nt}^2\, 1\{|X_{nt}/c_{nt}| > C'\})\big)^{1/2}\, 1\{l < k\}
\end{aligned}$$

so T_{nt} is an L_2-mixingale as well. Applying the same steps for $z_n(\tau, m')$ as for $u_n(\tau, m')$ gives the upper bound

$$\begin{aligned}
\mathrm{E}\, z_n(\tau, m') &\leq 16(k+1) \max_{t \in I_n(\tau, m')} \mathrm{E}\,\big(X_{nt}^2/c_{nt}^2\, 1\{|X_{nt}/c_{nt}| > C'\}\big) \\
&\leq 16(k+1) \max_{t=1,\dots,n} \mathrm{E}\,\big(X_{nt}^2/c_{nt}^2\, 1\{|X_{nt}/c_{nt}| > C'\}\big)
\end{aligned}$$

for any τ, m', and n satisfying $\tau + m' \leq n$, and

$$\mathrm{E}\, z_n(\tau, m') \leq 4 \max\big(\mathrm{E}\, z_n(\tau, n-\tau), \mathrm{E}\, z_n(0, m'+\tau-n)\big)$$

when $\tau + m' > n$. Since X_{nt}^2/c_{nt}^2 is uniformly integrable, set C' large enough that this upper bound is less than $\epsilon/2$ for all n, m', and τ as well.

Finally, to bound $\mathrm{E}(y_n(\tau, m')\, 1\{y_n(\tau, m') > C/6\})$, use the inequality

$$\mathrm{E}(y_n(\tau, m')\, 1\{y_n(\tau, m') > C/6\}) \leq 6\, \mathrm{E}\, y_n(\tau, m')^2/C.$$

We can write $Y_{nt} = \sum_{l=1-k}^{k} \xi_n(t, l)$, where

$$\xi_n(t, l) = \mathrm{E}_{t+l} X_{nt}\, 1\{|X_{nt}| \leq C' c_{nt}\} - \mathrm{E}_{t+l-1} X_{nt}\, 1\{|X_{nt}| \leq C' c_{nt}\},$$

and $\{\xi_n(t,l), \mathcal{F}_{n,t+l}\}$ forms a martingale difference array for each l. Then, when $\tau + m' \le n$, we have

$$\begin{aligned}
\mathrm{E}\, y_n(\tau, m')^2 &= \mathrm{E} \max_{m=1,\dots,m'} \Big| \sum_{t \in I_n(\tau,m)} \sum_{l=1-k}^{k} \xi_n(t,l) \Big|^4 \Big/ w_n(\tau, m')^4 \\
&\le \sum_{l=1-k}^{k} \mathrm{E} \max_{m=1,\dots,m'} \Big| \sum_{t \in I_n(\tau,m)} \xi_n(t,l) \Big|^4 \Big/ w_n(\tau, m')^4 \\
&\le \frac{4^4 (2k+1)^3}{3^4 w_n(\tau,m')^4} \sum_{l=1-k}^{k} \mathrm{E} \Big| \sum_{t \in I_n(\tau,m')} \xi_n(t,l) \Big|^4
\end{aligned}$$

where the second inequality follows from a maximal inequality for MDSes (Davidson, 1994, Theorem 16.8). Since the $\xi_n(\tau, l)$ are all bounded by $2c_{nt}C'$, we can expand the expectation recursively to derive the bound

$$\mathrm{E} \Big| \sum_{t \in I_n(\tau,m')} \xi_n(t,l) \Big|^4 \le 11 (2C')^4 w_n(\tau, m')^4,$$

(For details, see Davidson, 1994, Equations 16.69–16.72) giving

$$\mathrm{E}\, y_n(\tau, m')^2 \le \frac{11 \times 4^6\, (2k+1)^4 C'^4}{3^4}.$$

When $\tau + m' > n$, the same bound holds, but with 4^7 replacing 4^6. Then, as $C \to \infty$, this quantity converges to zero, completing the proof of (39). $\square$

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