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ADJUNCTION OF SEMIFUNCTORS: CATEGORICAL STRUCTURES IN NONEXTENSIONAL LAMBDA CALCULUS

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Abstract. Some connections between λ-calculus and category theory have been known. Among them, it has been shown by Lambek that cartesian closed categories (ccc for short) can be identified with extensional typed λ-calculus (cf. Lambek (1980), and Lambek and Scott (1986)). In this paper we introduce the notion of adjunction of semifunctors (for simplicity, we refer to this as 'semiadjunction') and, by the aid of this notion, we define the notion of semi cartesian closed category (semi-ccc for short). Some categorical or algebraic systems aimed to represent λ-calculus will turn out to be special cases of semi-ccc.

Another intersting connection between ccc and λ-calculus is Scott's embedding of λ-theory into a ccc (cf. Scott (1980)). (This will be referred to as *Scott embedding.*) We will show that any semiadjunction is embeddable in an adjunction (of functors) and Scott embedding is a special case.

1. Adjunction of semifunctors

In this section the notions of semifunctors and adjunction of them will be introduced and some basic facts will be shown.

1.1. Definition and notation

Definition 1.1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be categories. A *semifunctor* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a pair consisting of an *object function* from $\mathrm{Obj}(\boldsymbol{A})$ to $\mathrm{Obj}(\boldsymbol{B})$, where $\mathrm{Obj}(\boldsymbol{A})$ is the set of objects of $\boldsymbol{A}$, and of a *morphism function* $F_{X,Y}: \boldsymbol{A}(X, Y) \to \boldsymbol{B}(X, Y)$ *preserving compositions*, i.e., $F(f \circ g) = F(f) \circ F(g)$. Note that semifunctors need not preserve identity morphisms. Let G be a semifunctor from $\boldsymbol{A}$ to $\boldsymbol{B}$ and let F be a semifunctor from $\boldsymbol{B}$ to $\boldsymbol{A}$. A quadruple pair $(F, G, \{\alpha_{X,Y}\}_{X,Y}, \{\beta_{X,Y}\}_{X,Y})$ is an *adjunction of semifunctors* F and G (or *semiadjunction* of F and G) if and only if four squares in the following diagram are commutative:

$$\begin{tikzcd}
\boldsymbol{A}(F(Y), X) \arrow[r, shift left, "\alpha_{X,Y}"] \arrow[d, "\varphi"'] & \boldsymbol{B}(Y, G(X)) \arrow[l, shift left, "\beta_{X,Y}"] \arrow[d, "\psi"] \\
\boldsymbol{A}(F(Y'), X') \arrow[r, shift left, "\alpha_{X',Y'}"] & \boldsymbol{B}(Y', G(X')) \arrow[l, shift left, "\beta_{X',Y'}"]
\end{tikzcd}$$

where $f \in \boldsymbol{B}(Y', Y)$, $g \in \boldsymbol{A}(X, X')$, $\varphi = \boldsymbol{A}(F(f), g)$, and $\psi = \boldsymbol{B}(f, G(g))$. The commutativity conditions mean the following equations:

$$\psi \circ \alpha_{X,Y} = \alpha_{X',Y'} \circ \varphi, \qquad \varphi \circ \beta_{X,Y} = \beta_{X',Y'} \circ \psi,$$

$$\varphi = \beta_{X',Y'} \circ \psi \circ \alpha_{X,Y}, \qquad \psi = \alpha_{X',Y'} \circ \varphi \circ \beta_{X,Y}.$$

Note that the first and second equations mean the *naturality* of α and β. (For simplicity, we will denote $\{\alpha_{X,Y}\}_{X,Y}$ and $\{\beta_{X,Y}\}_{X,Y}$ by α and β, respectively.)

Let F and G be functors and let (F, G, α, β) be an adjunction of semifunctors F and G. Let f be id_X and g be id_Y. Then φ and ψ are identical functions, for $F(\mathrm{id}_X)$ and $G(\mathrm{id}_Y)$ are identical morphisms. So α and β are inverse functions of each other and (F, G, α, β) gives an adjunction of functors F and G in the usual sense. This justifies the terminology 'adjunction of semifunctors'. But this terminology is sometimes confusing, so we will often say semiadjunction instead of 'adjunction of semifunctors'.

The notions of semifunctor and semiadjunction are very similar to the notions of functors and adjunctions. So notions such as covariant or contravariant semifunctors, right or left semiadjoints, etc. are defined as the corresponding notions on functors and adjunctions. We will use such notions without any explicit definitions. (Consult [7] for the terminologies on functors and adjunctions.) The adjoint of a functor is unique up to isomorphism. Contrary to this, a semiadjoint of a semifunctor is *not* unique up to isomorphism. This means semiadjunction is *not extensional* in a sense.

1.2. Completion of semiadjunction

In this section we embed adjunctions of *semifunctors* into adjunction of *functors*. For this aim, we will use the Karoubi envelope.

Definition 1.2 (*Karoubi envelope*). Let $\boldsymbol{A}$ be a category. Then its Karoubi envelope $\tilde{\boldsymbol{A}}$ is the category defined as follows:

$$\mathrm{Obj}(\tilde{\boldsymbol{A}}) = \{f \mid f \circ f = f\}.$$

A morphism f such that $\mathrm{dom}(f) = \mathrm{codom}(f)$ and $f \circ f = f$ will be called *idemponent*, and the object $\mathrm{dom}(f)$ is denoted by ∂f. Let f and g be objects of $\tilde{\boldsymbol{A}}$. Then hom-sets are defined by

$$\tilde{\boldsymbol{A}}(X, Y) = \{h \in \boldsymbol{A}(\partial f, \partial g) \mid g \circ h \circ f = h\}.$$

The canonical embedding functor $\varepsilon_{\boldsymbol{A}} : \boldsymbol{A} \to \tilde{\boldsymbol{A}}$ is defined by

$$\varepsilon_{\boldsymbol{A}}(X) = \mathrm{id}_X \quad (X \in \mathrm{Obj}(\boldsymbol{A})),$$

$$\varepsilon_{\boldsymbol{A}}(f) = f \qquad (f \in \boldsymbol{A}(X, Y)).$$

This was first introduced by Karoubi [4] for an entirely different purpose. Scott [9] used the same idea (independently from Karoubi) to embed λ-theory into a ccc, and Lambek and Scott [9] pointed out that Scott's construction can be regarded as a Karoubi envelope.

Definition 1.3. For clarity, we will denote the morphism function and the object function of a functor G by G_m and G_o, respectively. Assume that $f \in \tilde{A}(X, Y)$. Set

$$\tilde{F}_o(X) = F_m(X) \quad (X \in \mathrm{Obj}(\tilde{A})),$$
$$\tilde{F}_m(f) = F_m(f) \quad (f \in \tilde{A}(X, Y)).$$

Obviously, $\tilde{F}$ is a semifunctor. For each object X of $\tilde{A}$, its identity morphism id_X is X itself. So

$$\tilde{F}_m(\mathrm{id}_X) = F_m(X) = \mathrm{id}_{\tilde{F}_o(X)}.$$

Hence, $\tilde{F}$ is a functor. This functor $\tilde{F}$ is called the *completion of* F.

Proposition 1.4. *A semifunctor is determined by its completion. Namely, if G is a functor from $\tilde{A}$ to $\tilde{B}$, then there exists at most one semifunctor F such that $\tilde{F} = G$.*

The proof is trivial by the definition of completion.

Definition 1.5. Let f be an idemponent of a category and let A be the object ∂f. An object X is called a *quotient* of A by f iff there are two morphisms e and m such that

$$X \underset{e}{\overset{m}{\rightleftarrows}} A$$

satisfying $m \circ e = f$, $e \circ m = \mathrm{id}_X$, and $e \circ f \circ m = \mathrm{id}_X$. The morphisms e and m are called the *retraction* and *coretraction* of the quotient, respectively. It is easy to check that two quotients of A by f are isomorphic. Each object X of $\tilde{A}$ is a quotient of $\varepsilon_A(A)$ by $\varepsilon_A(f)$, where f is an idemponent of A and $A = \partial f$. Thus every idemponent in a Karoubi envelope $\tilde{A}$ splits (cf. [1, 6]).

Proposition 1.6. *Let F be a semifunctor from A to B. Then the following hold:*

(1) *$\tilde{F} \circ \varepsilon_A(X)$ is a quotient of $\varepsilon_B \circ F(X)$ by $\varepsilon_B \circ F(\mathrm{id}_X)$.*

(2) *For any $f \in A(X, Y)$, the following diagram is commutative:*

$$\begin{tikzcd}
\varepsilon_B \circ F(X) \arrow[rr, "\varepsilon_B \circ F(f)"] \arrow[d, "e"'] & & \varepsilon_B \circ F(Y) \arrow[d, "e'"] \\
\tilde{F} \circ \varepsilon_A(X) \arrow[rr, "\tilde{F} \circ \varepsilon_A"] & & \tilde{F} \circ \varepsilon_A(Y)
\end{tikzcd}$$

where e and e' are the retractions of the quotients assured in (1).

(3) *$\tilde{F}$ is uniquely determined from F (up to isomorphism) by these two conditions.*

(4) *F is a functor iff $\varepsilon_B \circ F = G \circ \varepsilon_A$ holds.*

Remark 1.7. It is possible to characterize $\tilde{A}$ by a universal property as in [4, Section 6.10]. Thus, $\tilde{A}$ and $\tilde{F}$ determine a functor from the category of categories and *semifunctors* to the category of categories in which idemponents split and *functors*. See [1] for a detailed description of the essentially same functor.

Definition 1.8. Let F and G be semifunctors and let α be a natural transformation (of semifunctors) as follows:

$$\boldsymbol{A}(F(B), A) \xrightarrow{\alpha} \boldsymbol{B}(B, G(A)).$$

Set

$$\tilde{\alpha}(f) = \alpha(f),$$

for $f \in \tilde{\boldsymbol{A}}(\tilde{F}(Y), X)$. This $\tilde{\alpha}$ is called *the completion of* α.

By the naturality of α and the assumption $f \in \tilde{\boldsymbol{A}}(\tilde{F}(Y), X)$,

$$\tilde{G}(X) \circ \tilde{\alpha}(f) \circ Y = G(X) \circ \alpha(f) \circ Y = \alpha(X \circ f \circ F(Y)) = \alpha(f) = \tilde{\alpha}(f).$$

Hence, $\tilde{\alpha}(f)$ belongs to $\tilde{\boldsymbol{B}}(Y, \tilde{G}(X))$. So $\tilde{\alpha}$ is a natural transformation from $\tilde{\boldsymbol{A}}(\tilde{F}(Y), X)$ to $\tilde{\boldsymbol{B}}(Y, \tilde{G}(X))$.

Theorem 1.9 (generalized Scott embedding). *Let* (F, G, α, β) *be an adjunction of semifunctors. Then* $(\tilde{F}, \tilde{G}, \tilde{\alpha}, \tilde{\beta})$ *is an adjunction of functors. This adjunction is called the completion of* (F, G, α, β).

The proof is obvious from the definitions of $\tilde{F}$ and $\tilde{\alpha}$.

Proposition 1.10 (the inverse of Theorem 1.9). *Let* β *be a natural transformation from* $\tilde{\boldsymbol{A}}(\tilde{F}(X), Y)$ *to* $\tilde{\boldsymbol{B}}(X, \tilde{G}(Y))$. *Then there is a natural transformation* α *from* $\boldsymbol{A}(F(X), Y)$ *to* $\boldsymbol{B}(X, G(Y))$ *such that* $\tilde{\alpha} = \beta$.

The proof is trivial and therefore left to the reader.

Proposition 1.11. *The completion* $\tilde{\alpha}$ *is the unique natural transformation commuting the following diagram*:

$$\begin{tikzcd}
\tilde{\boldsymbol{A}}(\tilde{F}(\varepsilon_{\boldsymbol{B}}(B)), \varepsilon_{\boldsymbol{A}}(A)) \arrow[r, "\tilde{\alpha}"] & \tilde{\boldsymbol{B}}(\varepsilon_{\boldsymbol{B}}(B), \tilde{G}(\varepsilon_{\boldsymbol{A}}(A))) \\
\boldsymbol{A}(F(B), A) \arrow[u, "\varphi"] \arrow[r, "\alpha"] & \boldsymbol{B}(B, G(A)) \arrow[u, "\psi"']
\end{tikzcd}$$

where $\varphi = \tilde{\boldsymbol{A}}(m, \mathrm{id}) \circ \varepsilon_{\boldsymbol{A}}$, $\psi = \tilde{\boldsymbol{B}}(\mathrm{id}, e') \circ \varepsilon_{\boldsymbol{B}}$, m *is the coretraction of the quotient* $\tilde{F} \circ \varepsilon_{\boldsymbol{B}}(B)$ *of* $\varepsilon_{\boldsymbol{A}} \circ F(B)$ *and* e' *is the retraction of the quotient* $\tilde{G} \circ \varepsilon_{\boldsymbol{A}}(A)$ *of* $\varepsilon_{\boldsymbol{B}} \circ G(A)$.

The proof is easy and therefore left to the reader.

Remark 1.12. If the natural transformation α of Definition 1.8 satisfies

$$G(\mathrm{id}_A) \circ \alpha(f) = \alpha(f), \tag{1}$$

then not only the commutativity in Proposition 1.11 but also the following equation holds:

$$\tilde{\boldsymbol{B}}(\mathrm{id}, m') \circ \tilde{\alpha} \circ \varphi = \varepsilon_B \circ \alpha,$$

where φ is the same as in Proposition 1.11 and m' is the coretraction of the quotient of $\tilde{G} \circ \varepsilon_A(A)$ of $\varepsilon_B \circ G(A)$. This means that α is determined by $\tilde{\alpha}$. Equation (1) means that α is natural with respect to A (*not* with respect to A and B). Such an α will be said to be *normal.* Let α be a natural transformation as in Definition 1.8. Set

$$\alpha' = \alpha(f \circ F(\mathrm{id}_B)).$$

Then α' is a normal natural transformation, and its completion is identical to $\tilde{\alpha}$. Let (F, G, α, β) be a semiadjunction. Then it is easy to see that (F, G, α', β), (F, G, α, β'), and (F, G, α', β') are semiadjunctions and their completions are identical to $(F, G, \tilde{\alpha}, \tilde{\beta})$. In this sense we may assume that the two natural transformations in a semiadjunction are normal without loss of generality.

2. Semi cartesian closed category

2.1. Definition and notation

A ccc is a category $\boldsymbol{A}$ equipped with the following three adjunctions (cf. [7, Chapter IV. 6]):

$$0 \dashv 1_{(-)},$$

$$\Delta(-) \dashv - \times -,$$

$$- \times b \dashv (-)^b.$$

A semi cartesian closed cateogry is defined by replacing these adjunctions by semiadjunctions.

Definition 2.1. A *semi cartesian closed category* (*semi-ccc*) is a category equipped with the following three semiadjunctions:

$$1 \rightleftarrows \boldsymbol{A}(X, 1),$$

$$\boldsymbol{A}^2(\Delta(X), (Y, Z)) \rightleftarrows \boldsymbol{A}(X, Y \times Z),$$

$$\boldsymbol{A}(X \times Y, Z) \rightleftarrows \boldsymbol{A}(X, Z^Y).$$

Such adjunctions will be called a *semi-ccc structure on* $\boldsymbol{A}$. Note that there may be many different semi-ccc structures on a category. (Contrary to this, ccc structures on a category are unique up to isomorphism.) A *morphism* from a semi-ccc $\boldsymbol{A}$ to a semi-ccc $\boldsymbol{B}$ is a semifunctor from $\boldsymbol{A}$ to $\boldsymbol{B}$ which is a *map* of each of the above three semiadjunctions (see [7] for the definition of a map of adjunctions).

Remark 2.2. The second semiadjunction in the definition of semi-ccc is a *semiadjunction with a parameter* Y (cf. [7] for adjunction with a parameter). By a semiadjunction version of [7, Ch. IV.7, Theorem 3], a canonical semiadjunction with a parameter Y exists, if there is a semiadjunction for each Y.

2.2. Algebraic description of a semi-ccc

It is well known how to describe a ccc *algebraically* with the aid of pairing operators and evaluation morphisms. An algebraic description of a semi-ccc is given by the following theorem.

Theorem 2.3. *A category* $\boldsymbol{A}$ *is a semi-ccc if and only if there is an algebraic structure on* $\boldsymbol{A}$

$$(\langle *, *\rangle, * \times *, p, q, a_0, 1_{(*)}, *^*, \Lambda(*), \mathrm{ev}).$$

satisfying the following conditions:

(1) *For morphisms* $a: x \to y$ *and* $b: x \to z$, $\langle a, b\rangle$ *is a morphism from* x *to* $y \times z$ *such that*

$$\langle a, b\rangle \circ c = \langle a \circ c, b \circ c\rangle.$$

(2) *For objects* y *and* z, *there are morphisms*

$$p: y \times z \to y, \qquad q: y \times z \to z$$

such that

$$p \circ \langle a, b\rangle = a, \qquad q \circ \langle a, b\rangle = b.$$

(3) *For a morphism* $h: x \times y \to z$, $\Lambda(h)$ *is a morphism from* x *to* z^y.

(4) *For objects* y *and* z, *there is a morphism*

$$\mathrm{ev}: z^y \times y \to z$$

such that

$$\mathrm{ev} \circ \langle \Lambda(h) \circ u, v\rangle = h \circ \langle u, v\rangle, \qquad \Lambda(h) \circ u = \Lambda(h \circ \langle u \circ p, q\rangle),$$

where

$$u: a \to x, \qquad v: a \to y, \qquad p: a \times y \to a, \qquad q: a \times y \to y.$$

(5) a_0 *is an object of* $\boldsymbol{A}$. *For each object* a, 1_a *is a morphism from* a *to* a_0 *such that* $f \circ 1_{\mathrm{codom}(f)} = 1_{\mathrm{dom}(f)}$ *holds for any morphism* f.

(6) $\mathrm{ev} \circ \langle p, q\rangle = \mathrm{ev}$ *holds.*

Proof. Assume h is a morphism from $x \times y$ to z. Then $\Lambda(h)$ is the image of $h \circ \mathrm{id} \times \mathrm{id}$ by the natural transformation $A(x \times y, z) \to A(x, z^y)$. The definitions of the other parts of the structure and the details of the proof are left to the reader. (The proofs of [3, Theorems 6.5 and 6.6] will serve as a good reference.) □

Definition 2.4. A category A equipped with an algebraic structure such as the above theorem is called an *algebraic semi-ccc.*

Remark 2.5. Conditions (5) and (6) of the algebraic semi-ccc are superfluous in a sense. In fact, they are not necessary to prove the 'if' part of Theorem 2.3. If A has an object A_0 and satisfies (1)–(4), then set $1 = A_0$ and set $1_B = \Lambda(q)$, where $q : B \times A_0 \to A_0$. Then they satisfy condition (5). Let $S_0 = (\langle *, * \rangle, \ldots, \mathrm{ev})$ be an algebraic structure on A satisfying (1)–(5). Set $\mathrm{ev}_1 = \mathrm{ev} \circ \langle p, q \rangle$. Then $S_1 = (\langle *, * \rangle, \ldots, \mathrm{ev}_1)$ satisfies the conditions (1)–(6) and S_0 and S_1 give the same ccc. Furthermore, the same equations on λ-terms hold in S_0 and S_1 in the sense of the semantics of Section 3.

Theorem 2.6. *Let A be a semi-ccc. Then $\tilde{A}$ is a ccc. Namely, A is embedded into a ccc $\tilde{A}$ by ε_A. The ccc structure on $\tilde{A}$ will be called the completion of the semi-ccc structure on A.*

This theorem is easily proved as a direct consequence of Theorem 1.9.

Theorem 2.7. *Let A be a category whose Karoubi envelope $\tilde{A}$ is a ccc. Then there is a canonical semi-ccc structure on A such that the ccc $\tilde{A}$ is its completion.*

This theorem is easily proved as a direct consequence of Proposition 1.10.

2.3. *Examples of semi-ccc*

In this subsection we will examine some categorical or algebraic systems introduced to characterize type-free λ-calculus.

2.3.1. *CCM, weak cartesian closed monoid and C-domain*

Koymans [3], Lambek and Scott [6], and Yokouchi [11] introduced a sort of monoid which corresponds to $\lambda\beta$-calculus. Their definitions are different but are essentially the same.

Koymans's CCM is an algebraic semi-ccc with just one object which need not satisfy condition (5). But this condition is superfluous as was noted in Remark 2.5. His vision of Scott embedding ('if' part of [3, Theorem 6.6.]) is a direct consequence of Theorems 2.3 and 2.6. He also proved the inverse of Scott embedding ('only if' part of [3, Theorem 6.6]). Theorem 2.7 generalizes it. If a CCM is regarded as a semi-ccc, then the natural transformation corresponding to λ-abstraction is normal. Hence, the interpretation of λ-terms in a CCM can be achieved by its Karoubi envelope as was observed in Remark 1.12 (see [3, 9]).

The weak cartesian closed monoid of Lambek and Scott [6] can be defined as a CCM that need not satisfy condition (6) of Theorem 2.3. (So it is a category with just one object equipped with the second and third semiadjunctions of the definition of semi-ccc. If these semiadjunctions are adjunctions of *functors*, then it is a cartesian closed monoid in the sense of [6].) But this condition is superfluous as was noted in Remark 2.5. So the notion of weak cartesian closed category is essentially equivalent to the notion of CCM.

Yokouchi's C-domain is another description of CCM (or weak cartesian closed monoid) with condition (5), but without condition (6). See [11] for a discussion on the equivalence of C-domain and CCM.

2.3.2. Semi-ccat and Church algebraic theory

A semi cartesian closed algebraic theory (semi-ccat) is an algebraic theory A in the sense of Lawvere with the following semiadjunction (with a parameter n):

$$A(m+n, p) \underset{\varepsilon_n}{\overset{\lambda_n}{\rightleftarrows}} A(m, p^n)$$

satisfying

$$p^n = p,$$

$$\lambda_m \circ \lambda_n = \lambda_{m+n}, \quad \lambda_0 = \mathrm{id},$$

$$\varepsilon_m \circ \varepsilon_n = \varepsilon_{m+n}, \quad \varepsilon_0 = \mathrm{id}.$$

Hence, a semi-ccat is a semi-ccc. It is easy to check that the notion of semi-ccat is essentially equivalent to the notion of Church algebraic theory of Obtułowicz and Wiweger [8]. A semi-ccat or a Church algebraic theory is a categorical description of a $\lambda\beta$-theory (in the sense of Barendregt [2]).

Let (C, U, i, j) be a categorical model of λ-calculus in the sense of Koymans [3, Section 3]. Then the full subcategory $\{U^m \mid m \in N\}$ of C is an algebraic theory. With the aid of morphisms i and j the algebraic theory turns out to be a semi-ccat, say $T(C, U, i, j)$. A *model* of semi-ccat A in (C, U, i, j) is a semi-ccc morphism from A to $T(C, U, i, j)$. Then, by Theorem 2.6 and Proposition 1.11, there is an identical model in $\tilde{A}$ for any semi-ccat A (completeness theorem of semi-ccat).

3. Typed $\lambda\beta$-theory and semi-ccc

As was shown in the previous section semi-ccc is a generalization of some categorical or algebraic system corresponding to nonextensional λ-calculus. We will introduce the notion of *typed $\lambda\beta$-theory* (with pairing) and relate it to semi-ccc. A similar but extensional typed λ-theory can be found in [6, 11]. We will follow the way of [6].

3.1. *Definition and notation*

A typed $\lambda\beta$-theory is a typed equational theory equipped with the following data:

(1) The set of types is closed under cartesian product $A\times B$ and exponential A^B. There is a special type 1.

(2) If t_1 and t_2 are terms of types A and B respectively, then $\langle t_1, t_2\rangle$ is a term of the type $A\times B$. If t is a term of type $A\times B$ then $\pi(t)$ and $\pi'(t)$ are terms of types A and B, respectively. There is a constant $*$ of type 1.

(3) If x is a variable of a type A and t is a term of a type B, then $\lambda x.t(x)$ is a term of the type B^A. If t_1 and t_2 are terms of types A^B and B respectively, then t_1t_2 is a term of the type A.

(4) Substitution $t_1[x:=t_2]$ is defined as usual. Note that

$$\langle t_1, t_2\rangle[x:=t_3]=\langle t_1[x:=t_3], t_2[x:=t_3]\rangle.$$

(5) The following equations are the postulates:

$$\pi(\langle t_1, t_2\rangle)=t_1,$$

$$\pi'(\langle t_1, t_2\rangle)=t_2,$$

$$(\lambda x.t_1(x))t_2=t_1[x:=t_2].$$

3.2. *Interpretation of the typed $\lambda\beta$-theory*

Koymans [3] gives an interpretation of $\lambda\beta$-theory in a reflexive domain in a ccc. By a similar method, we can give an interpretation of our typed $\lambda\beta$-theory in a semi-ccc. The point is how to interpret a constant and a variable in an environment (assignment). This problem is solved to fix a product of n objects in a systematic way (see [3, Sections 3.1–3.4, 7.3–7.5] and [10, Section 2.2] for the type-free case). By Lindenbaum–Tarski construction, a typed $\lambda\beta$-theory has a semi-ccc with an identical interpretation (see [2, Section 5.3.13] for the type-free case). Hence, the notion of typed $\lambda\beta$-theory is essentially equivalent to the notion of semi-ccc.

Addendum

After sending the manuscript of this paper to the editor, I learned from Dr. A. Obtułowicz that Wiweger [10] had already defined the concept of preadjunction. Preadjunction and semiadjunction are different, although they are quite similar. The main difference of this paper from Wiweger's work is as follows. Wiweger defined preadjunction only for functors and he did not consider the generalization of Scott embedding and monoid representations of λ-calculus like CCM either. But his treatment of λ-algebraic theory by means of preadjunction generalizes our treatment of ccat.

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