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Models of the Lambda Calculus

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INTRODUCTION

In 1969 Scott constructed "mathematical" models for the λ-calculus; see Scott (1972). It took some time, however, before a general definition of the notion of a λ-calculus model was given. This was done independently in Barendregt (1977, 1981), Berry (1981), Hindley and Longo (1980), Meyer (1980), Obtulowicz (1979), and Scott (1980). All of these definitions except Berry's are reviewed in Cooperstock (1981).

There seemed to be some disagreement on the notion of a λ-calculus model. Barendregt introduced two classes of models, viz. the λ-algebras and the λ-models. Berry's models coincide essentially with the λ-algebras, whereas the models of Hindley–Longo, Meyer, Obtulowicz, and Scott all coincide with the λ-models.

Barendregt was inspired by proof theoretic considerations (ω-incompleteness, see Plotkin, 1974) for introducing both λ-algebras and λ-models. He did this both in a syntactical and a first order way. We will replace his syntactical method by the so-called environment models. (These are in fact also syntactical but somewhat easier to handle.) Moreover, inspired by Berry (1981) and Meyers (1974) (for the typed λ-calculus) we give a unified categorical description of both λ-algebras and λ-models. By methods taken from Scott (1980), it will be proved that the structures thus obtained consist of all λ-algebras and λ-models. The categorical description gives a convincing argument that the two kinds of models form a natural class of interpretations of the λ-calculus.

In the meantime there seemed to have formed a consensus about the need for both λ-algebras and λ-models. The revised version Meyer (1981) includes also λ-algebras. Scott (1980) constructs Cartesian closed categories (ccc's) from λ-theories; but this construction essentially goes via a λ-algebra (λ-theory $\rightarrow$ term model (which is a λ-algebra) $\rightarrow$ ccc). We prefer this way of describing Scott's construction, because different λ-algebras may have the same theory, but yet different ccc's.

Now we will give a short description of the three ways of introducing the

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λ-calculus models, i.e., as environmental models, as first order models, and as categorical models.

1. *Environment models* (following Hindley and Longo, 1980, Meyer, 1980, and Koymans, 1979). These are structures $\mathfrak{M} = (|\mathfrak{M}|, \cdot)$ with maps $[\![\ ]\!]_\rho$: λ-terms (with constants from $|\mathfrak{M}|) \to |\mathfrak{M}|$ satisfying some natural conditions (e.g., $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$). Such a structure is a *λ-algebra* if

$$\lambda \vdash M = N \Rightarrow [\![M]\!]_\rho = [\![N]\!]_\rho.$$

$\mathfrak{M}$ is a *λ-model* if $\mathfrak{M}$ is a λ-algebra and moreover

$$\forall d \in |\mathfrak{M}|, \qquad [\![M]\!]_{\rho(x/d)} = [\![N]\!]_{\rho(x/d)} \Rightarrow [\![\lambda x \cdot M]\!]_\rho = [\![\lambda x \cdot N]\!]_\rho.$$

2. *First order models* (following Barendregt, 1981, for λ-algebras and Meyer, 1980, and Scott, 1980, for λ-models). These are structures $\mathfrak{M} = (|\mathfrak{M}|, \cdot, k, s)$. Such a structure is a *λ-algebra* if $\mathfrak{M}$ satisfies the well-known axioms for the combinators K and S, e.g.,

$$\mathfrak{M} \models sxyz = xz(yz)$$

and the four Curry axioms, e.g.,

$$\mathfrak{M} \models s(ks)(s(kk)) = s(kk)(s(s(ks)(s(kk)\,i))(ki))$$

with $i = skk$ $(=\lambda x \cdot x)$; these imply, for example, $\mathfrak{M} \models skk = sks$. A *$\lambda$-model* is a λ-algebra in which moreover

$$\mathfrak{M} \models \forall x(ax = bx) \to 1a = 1b$$

with $1 = s(ki)$ $(=\lambda xy \cdot xy)$.

3. *Categorical models* (following Berry, 1981, for λ-algebras and Meyers, 1974, and Obtulowicz, 1979, for λ-models). Now the description of a *λ-algebra* consists of a Cartesian closed category with a special object U such that U^U is a retract of U. If moreover U has enough points, then the λ-algebra becomes a *λ-model.*

For each of the three approaches it is easy to describe extensional models.

All three ways of introducing λ-calculus models have their advantages and disadvantages. The environment models are simple to define but rather syntactical. To show that some structures are models (as, e.g., term models, models consisting of type filters, see Barendregt, Coppo, and Dezani-ciancaglini) it is best to take the environmental definition. The first order definitions have as advantage that they indicate the model theoretic status of the λ-calculus models, but as disadvantage that it is hard to show that some structure is a model. The categorical definitions are important because they unify the notions. Moreover for the mathematical structures such as $\mathbb{D}_\infty$ and $\mathbb{P}\omega$ it is best to prove that they are models via the categorical definition. One disadvantage is that we do not immediately see what the interpretation is of a λ-term in such models.

One last remark: If the cardinality of the domain of a model equals one, the model will be called trivial. Formally these models are included, but it may be implicitly understood that we are only interested in nontrivial models.

1. Combinatory Versions of the λ-Calculus

In this first section the characterization theorem for λ-calculus in terms of combinatory logic (via the standard translations) will be reviewed. The extra finite set of axioms that must be added to combinatory logic to give full equivalence to the λ-calculus is called the set of Curry axioms. We consider the slight variant of calculi with an arbitrary set of constants, to be used in Section 2.

1.1. Definition. Let C be a set of constants.

(i) $\lambda(C)$ is the usual $\lambda\beta$-calculus, using constants from C.

(ii) CL(C) is combinatory logic over S, K, and constants from C.

(iii) A_β is the set of Curry axioms, see Barendregt (1981, 7.3.15).

1.2. Definition (The standard translation). Let $\Lambda(C)$ and $\mathscr{C}(C)$ denote the terms of $\lambda(C)$ and CL(C), respectively.

(i) $\lambda: \mathscr{C}(C) \to \Lambda(C)$ is defined inductively by

$$\begin{aligned}
\lambda(x) &= x \qquad \text{for} \quad x \text{ a variable},\\
\lambda(c) &= c \qquad \text{for} \quad c \in C,\\
\lambda(S) &= \lambda xyz \cdot xz(yz),\\
\lambda(K) &= \lambda xy \cdot x,\\
\lambda(PQ) &= \lambda(P)\,\lambda(Q).
\end{aligned}$$

(ii) (Abstraction in CL) $\langle x\rangle$: $\mathscr{C}(C) \to \mathscr{C}(C)$ is defined inductively (for every variable x) by

$$\begin{aligned}
\langle x\rangle\, x &= SKK,\\
\langle x\rangle\, P &= KP \qquad &&\text{if} \quad x \notin P,\\
\langle x\rangle(PQ) &= S(\langle x\rangle\, P)(\langle x\rangle\, Q) \qquad &&\text{if} \quad x \in PQ.
\end{aligned}$$

(iii) CL: $\Lambda(C) \to \mathscr{C}(C)$ is defined inductively by

$$\begin{aligned} \mathrm{CL}(x) &= x \qquad \text{for} \quad x \text{ a variable}, \\ \mathrm{CL}(c) &= c \qquad \text{for} \quad c \in C, \\ \mathrm{CL}(MN) &= \mathrm{CL}(M)\, \mathrm{CL}(N), \\ \mathrm{CL}(\lambda x \cdot M) &= \langle x \rangle\, \mathrm{CL}(M). \end{aligned}$$

1.3. *Notation.* $\lambda(P) = P_\lambda$ for $P \in \mathscr{C}(C)$, and $\mathrm{CL}(M) = M_{\mathrm{CL}}$ for $M \in \Lambda(C)$.

1.4. FACT. (i) $\lambda(C) \vdash M = M_{\mathrm{CL},\lambda}$,

(ii) $\mathrm{CL}(C) + A_\beta \vdash P = P_{\lambda,\mathrm{CL}}$,

(iii) $\lambda(C) \vdash M = N \Leftrightarrow \mathrm{CL}(C) + A_\beta \vdash M_{\mathrm{CL}} = N_{\mathrm{CL}}$,

(iv) $\mathrm{CL}(C) + A_\beta \vdash P = Q \Leftrightarrow \lambda(C) \vdash P_\lambda = Q_\lambda$.

Proof. See Barendregt (1981, Chap. 7, Sect. 3). There is no difficulty in taking C into account. ∎

1.5. *Remark.* This fact enables us to replace the theory $\lambda(C)$, with the troublesome variable-binding operator λ, by the purely equational theory $\mathrm{CL}(C) + A_\beta$, for which the standard equational model theory can be developed. It is the main tool for proving the equivalence of pseudomodels and λ-algebras in Section 2.

1.6. *Notation.* (i) If $\mathfrak{M} = (X, ...)$ is a structure of any kind, where X is the domain, we write

$$|\mathfrak{M}| = X,$$

$\Lambda(\mathfrak{M}) = \Lambda(C)$, where $C = \{c_a \mid a \in |\mathfrak{M}|\}$ (and similarly for $\lambda(\mathfrak{M})$, $\mathrm{CL}(\mathfrak{M})$ and $\mathscr{C}(\mathfrak{M})$).

(ii) $\mathrm{Vars} = \{v_0, v_1, v_2, ...\}$ is a countably infinite set of variables.

(iii) If $\rho \in X^{\mathrm{Vars}}$, $x \in \mathrm{Vars}$, $d \in X$, then

$$\rho(x/d)(x) = d \qquad \text{and} \qquad \rho(x/d)(y) = \rho(y), \qquad y \neq x.$$

2. PSEUDOMODELS AND λ-ALGEBRAS

Now we will state the first two definitions of λ-calculus models as mentioned in the Introduction: the environmental and first-order approach. Their equivalence will be proved and weak extensionality will be taken into account.

2.1. DEFINITION (See Hindley and Longo, 1980). Let $\mathfrak{M} = (X, \cdot, [\![\cdot]\!].)$, where $\cdot : X^2 \to X$ and $[\![\cdot]\!]. : \Lambda(\mathfrak{M}) \times X^{\text{Vars}} \to X$.

(i) $\mathfrak{M}$ is a *pseudostructure* if

$$\begin{aligned} &[\![x]\!]_\rho = \rho(x) \qquad \text{for} \quad x \text{ a variable},\\ &[\![c_a]\!]_\rho = a \qquad \text{for} \quad a \in X,\\ &[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho,\\ &[\![\lambda x \cdot M]\!]_\rho \cdot d = [\![M]\!]_{\rho(x/d)},\\ &\rho \restriction FV(M) = \sigma \restriction FV(M) \Rightarrow [\![M]\!]_\rho = [\![M]\!]_\sigma,\\ &[\![\lambda x \cdot M]\!]_\rho = [\![\lambda y \cdot M[x/y]]\!]_\rho \qquad \text{if} \quad y \notin M. \end{aligned}$$

(ii) $\mathfrak{M}, \rho \models M = N \Leftrightarrow [\![M]\!]_\rho = [\![N]\!]_\rho$,

$$\mathfrak{M} \models M = N \Leftrightarrow \forall \rho \in X^{\text{Vars}}, \qquad \mathfrak{M}, \rho \models M = N.$$

(iii) $\mathfrak{M}$ is a *pseudomodel* if

$$\lambda(\mathfrak{M}) \vdash M = N \Rightarrow \mathfrak{M} \models M = N \qquad (\text{for all } M, N \in \Lambda(\mathfrak{M})).$$

Sometimes we will leave out $\cdot$ and write ab in stead of $a \cdot b$. Terms will be associated to the left, so abc means $(ab)\,c$.

2.2. DEFINITION (cf. Barendregt, 1981). Let $\mathfrak{M} = (X, \cdot, s, k)$, $\cdot : X^2 \to X$, $s, k \in X$. $\mathfrak{M}$ is a *λ-algebra* if

$$\mathrm{CL}(\mathfrak{M}) + A_\beta \vdash M = N \Rightarrow \mathfrak{M} \models M = N$$

(satisfaction here as in equational logic; $S^{\mathfrak{M}} = s$, $K^{\mathfrak{M}} = k$). That is,

$$\begin{aligned} &\mathfrak{M} \models Kxy = x,\\ &\mathfrak{M} \models Sxyz = xz(yz) \end{aligned}$$

and

$$\mathfrak{M} \models A_\beta.$$

2.3. DEFINITION. (i) Let $\mathfrak{M}$ be a pseudostructure. Define $\mathfrak{M}' = (X, \cdot, s, k)$, where

$$s = [\![\lambda xyz \cdot xz(yz)]\!]_\rho^{\mathfrak{M}}, \qquad k = [\![\lambda xy \cdot x]\!]_\rho^{\mathfrak{M}}.$$

(This does not depend on ρ.)

(ii) Let $\mathfrak{M}$ be a λ-algebra. Define $\mathfrak{M}^+ = (X, \cdot, [\![\cdot]\!].)$, where

$$[\![M]\!]_\rho = (M_{\mathrm{CL}})_\rho \qquad \text{(in the sense of equational logic).}$$

2.4. THEOREM. (i) *Let $\mathfrak{M}$ be a pseudomodel. Then $\mathfrak{M}'$ is a λ-algebra and $\mathfrak{M}'^+ \equiv \mathfrak{M}$.*

(ii) *Let $\mathfrak{M}$ be a λ-algebra. Then $\mathfrak{M}^+$ is a pseudomodel and $\mathfrak{M}^{+\prime} \equiv \mathfrak{M}$.*

Proof. (i) We claim $(P)_\rho^{\mathfrak{M}'} = [\![P_\lambda]\!]_\rho^{\mathfrak{M}}$ for $P \in \mathrm{CL}(\mathfrak{M})$. Proof: Induction on the structure of P.

Now assume $\mathrm{CL}(\mathfrak{M}') + A_\beta \vdash P = Q$. Then

$$\lambda(\mathfrak{M}) \vdash P_\lambda = Q_\lambda,$$

so

$$\forall \rho\ [\![P_\lambda]\!]_\rho^{\mathfrak{M}} = [\![Q_\lambda]\!]_\rho^{\mathfrak{M}}.$$

By the claim $\mathfrak{M}' \models P = Q$. Therefore $\mathfrak{M}'$ is a λ-algebra. This leaves us with $[\![M]\!]_\rho^{\mathfrak{M}'^+} = [\![M]\!]_\rho^{\mathfrak{M}}$, but that is an easy consequence of the definition and the claim.

(ii) It is easy to prove that $\mathfrak{M}^+$ is a pseudostructure, using the well-known properties of $\langle x \rangle$ in CL. Now assume $\lambda(\mathfrak{M}^+) \vdash M = N$. Then $\mathrm{CL}(\mathfrak{M}) + A_\beta \vdash M_{\mathrm{CL}} = N_{\mathrm{CL}}$. So $\mathfrak{M} \models M_{\mathrm{CL}} = N_{\mathrm{CL}}$. Therefore $[\![M]\!]_\rho^{\mathfrak{M}^+} = (M_{\mathrm{CL}})_\rho^{\mathfrak{M}} = (N_{\mathrm{CL}})_\rho^{\mathfrak{M}} = [\![N]\!]_\rho^{\mathfrak{M}^+}$. Claim:

$$(P)_\rho^{\mathfrak{M}} = [\![P_\lambda]\!]_\rho^{\mathfrak{M}^+}.$$

Proof: $[\![P_\lambda]\!]_\rho^{\mathfrak{M}^+} = (P_{\lambda,\mathrm{CL}})_\rho^{\mathfrak{M}} = (P)_\rho^{\mathfrak{M}}$. Consequently $s = (S)_\rho^{\mathfrak{M}} = [\![\lambda xyz \cdot xz(yz)]\!]_\rho^{\mathfrak{M}^+}$ and similarly for k. So $\mathfrak{M}^{+\prime} \equiv \mathfrak{M}$. ∎

2.5. DEFINITION. (i) Let $\mathfrak{M}$ be a pseudomodel. $\mathfrak{M}$ is a *model* if it satisfies

$$\forall d([\![M]\!]_{\rho(x/d)} = [\![N]\!]_{\rho(x/d)}) \Rightarrow [\![\lambda x \cdot M]\!]_\rho = [\![\lambda x \cdot N]\!]_\rho. \tag{ξ}$$

(ii) Let $\mathfrak{M}$ be a λ-algebra. $\mathfrak{M}$ is a *λ-model* if it satisfies

$$\forall d(d_0 \cdot d = d_1 \cdot d) \to 1 \cdot d_0 = 1 \cdot d_1 \qquad \text{for all } d_0, d_1 \in |\mathfrak{M}|. \tag{ξ_1}$$

Here $1 =_{\mathrm{def}} s(ki)$, where $i =_{\mathrm{def}} skk$.

2.6. THEOREM. *Let $\mathfrak{M}$ be a pseudomodel. $\mathfrak{M}$ is a model $\Leftrightarrow \mathfrak{M}'$ is a λ-model.*

Proof. $\Rightarrow$ Assume $\forall d(d_0 \cdot d = d_1 \cdot d)$. An application of (ξ) gives

$$[\![\lambda x \cdot c_{d_0} x]\!] = [\![\lambda x \cdot c_{d_1} x]\!].$$

But $\lambda(\mathfrak{M}) \vdash \lambda x \cdot c_{d_0} x = S_\lambda(K_\lambda I_\lambda)\, c_{d_0}$, where $I = SKK$. So $[\![\lambda x \cdot c_{d_0} x]\!] = 1 \cdot d_0$ and similarly for d_1.

$\Leftarrow$ We work in $\mathfrak{M}'^+ \equiv \mathfrak{M}$. $\lambda(\mathfrak{M}) \vdash S_\lambda(K_\lambda I_\lambda)(\lambda x \cdot M) = \lambda x \cdot M$, so $[\![\lambda x \cdot M]\!]_\rho = 1 \cdot [\![\lambda x \cdot M]\!]_\rho$. Assume $\forall d([\![M]\!]_{\rho(x/d)} = [\![N]\!]_{\rho(x/d)})$, equivalently $\forall d([\![\lambda x \cdot M]\!]_\rho \cdot d = [\![\lambda x \cdot N]\!]_\rho \cdot d)$. By (ξ_1) $1 \cdot [\![\lambda x \cdot M]\!]_\rho = 1 \cdot [\![\lambda x \cdot N]\!]_\rho$. So by the above $[\![\lambda x \cdot M]\!]_\rho = [\![\lambda x \cdot N]\!]_\rho$. ∎

We refer to (ξ), or equivalently (ξ_1), as (the axiom of) weak extensionality. Of course, by Theorem 2.4, Theorem 2.6 has a dual: A λ-algebra $\mathfrak{M}$ is a λ-model iff $\mathfrak{M}^+$ is a model.

3. Categorical Models

From now on C denotes an arbitrary Cartesian closed category (ccc), U an object in C and i, j maps in C such that

$$U^U \underset{j}{\overset{i}{\rightleftarrows}} U, \qquad j \circ i = id_{U^U}.$$

We say that U^U is a retract of U via i and j.

Now, $\langle\, ,\rangle$ denotes the usual pairing function(s), p_1, p_2 are the projections on the first and second factor, respectively. As usual, for maps f, g, we define

$$f \times g = \langle f \circ p_1, g \circ p_2 \rangle.$$

Here T is the terminal object of C, $!_A$ is the unique arrow in $\mathrm{Hom}(A, T)$. In addititon, ev denotes the evaluation map (indexed if necessary). If $f: A \times B \to C$, $\Lambda(f): A \to C^B$ is the exponential adjoint of f.

We note some equations:

$$ev \circ \Lambda(f) \times \mathrm{id}_B = f, \tag{1}$$

$$\Lambda(h \circ g \times \mathrm{id}_B) = \Lambda(h) \circ g, \tag{2}$$

$$\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle, \tag{3}$$

$$f \times g \circ \langle h, k \rangle = \langle f \circ h, g \circ k \rangle. \tag{4}$$

Now we will show how this setting enables us to define a pseudomodel with domain all "elements" of U: arrows from T to U.

3.1. DEFINITION. (i) $U^0 = T$, $U^{n+1} = U^n \times U$. (Note that $U^1 \equiv T \times U \cong U$, but $U^1 \not\equiv U$. This is convenient for a uniform treatment of interpretation.)

(ii) For $f_1,\dots,f_n$ maps with a common domain A, $[f_1,\dots,f_n[$ and $]f_1,\dots,f_n]$ are defined by induction on n:

$$[\,[=]\,] = !_A,$$
$$[f_1,\dots,f_n,f_{n+1}[= \langle [f_1,\dots,f_n[, f_{n+1}\rangle,$$
$$]f_1,\dots,f_n,f_{n+1}] = \langle f_1,]f_2,\dots,f_{n+1}]\rangle.$$

So $[\cdots[$ denotes association to the left and $]\cdots]$ association to the right. We have

$$\begin{aligned}[f_1,\dots,f_n[\circ h &= [f_1 \circ h,\dots,f_n \circ h[,\\ ]f_1,\dots,f_n] \circ h &=]f_1 \circ h,\dots,f_n \circ h].\end{aligned} \tag{5}$$

(iii) Let $\Delta = x_1,\dots,x_n$ be a sequence of different variables.

$$\#\Delta = n, \qquad X\Delta = U^{\#\Delta}.$$

$\pi^{\Delta}_{x_i}: X\Delta \to U$ is defined by

$$\pi^{\Delta}_{x_n} = p_2,$$
$$\pi^{x_1,\dots,x_n}_{x_i} = \pi^{x_1,\dots,x_{n-1}}_{x_i} \circ p_1 \qquad (i \neq n).$$

(iv) Let $\Delta = x_1,\dots,x_n$.

$$\begin{aligned}\Delta\backslash x &= x_1,\dots,x_{i-1},x_{i+1},\dots,x_n, && \text{if} \quad x \equiv x_i,\\ &= \Delta, && \text{if} \quad x \notin \Delta;\end{aligned}$$
$$\Delta; x = \Delta\backslash x, x;$$
$$|\Delta| = \{x_1,\dots,x_n\}.$$

(v) Let $|\Gamma| \subseteq |\Delta|$. Then define $\Pi^{\Delta}_{\Gamma}: X\Delta \to X\Gamma$ by

$$\Pi^{\Delta}_{y_1,\dots,y_m} = [\pi^{\Delta}_{y_1},\dots,\pi^{\Delta}_{y_m}[.$$

If $x \notin \Delta, \Gamma$ we have

$$\Pi^{\Delta,x}_{\Gamma,x} = \Pi^{\Delta}_{\Gamma} \times \mathrm{id}, \tag{6}$$

$$\Pi^{\Delta,x}_{\Delta} = p_1, \qquad \Pi^{\Delta,x}_{x} = p_2, \qquad \Pi^{\Delta}_{\Delta} = \mathrm{id}. \tag{7}$$

(vi) For any object A, let $\cdot_A$: $\mathrm{Hom}(A, U)^2 \to \mathrm{Hom}(A, U)$ be defined by $f \cdot_A g = \mathrm{ev} \circ \langle j \circ f, g\rangle$. It is easy to see that for any $h: B \to A$

$$(f \cdot_A g) \circ h = (f \circ h) \cdot_B (g \circ h). \tag{8}$$

Let $X = \mathrm{Hom}(T, U)$; $\cdot = \cdot_T$; $\mathfrak{M} = (X, \cdot)$.

(vii) Let $M \in \Lambda(\mathfrak{M})$, $|\Delta| \supseteq FV(M)$. Define $[\![M]\!]_\Delta: X\Delta \to U$ by induction on M

$$[\![x]\!]_\Delta = \pi^\Delta_x,$$
$$[\![c_a]\!]_\Delta = a \circ \Pi^\Delta_{\langle\rangle},$$
$$[\![MN]\!]_\Delta = [\![M]\!]_\Delta \cdot_{X\Delta} [\![N]\!]_\Delta,$$
$$[\![\lambda x \cdot M]\!]_\Delta = i \circ \Lambda([\![M]\!]_{\Delta;x}) \circ \Pi^\Delta_{\Delta\backslash x}.$$

(viii) If $\rho \in X^{\mathrm{Vars}}$ we put $\rho^\Delta = [\rho(x_1),..., \rho(x_n)[: T \to X\Delta$.

(ix) $[\![M]\!]_\rho = [\![M]\!]_{FV(M)} \circ \rho^{FV(M)}$, where

$M \in \Lambda(\mathfrak{M})$, $\rho \in X^{\mathrm{Vars}}$ and $FV(M)$ is the set of free variables of M (say, in alphabetical order).

(x) All this information defines a structure

$$\mathfrak{M}_C = (X, \cdot, [\![\cdot]\!].).$$

3.2. LEMMA. (i) $\pi^{x_1,\dots,x_n}_{x_i} \circ [f_1,...,f_n[= f_i$.

(ii) $\Pi^{x_1,\dots,x_n}_{x_i} \circ [f_1,...,f_n[= [f_i[= \langle !, f_i\rangle$.

(iii) $\Pi^\Delta_\Theta = \Pi^\Gamma_\Theta \circ \Pi^\Delta_\Gamma$, *where*

$$|\Theta| \subseteq |\Gamma| \subseteq |\Delta|.$$

(iv) $\rho^\Gamma = \Pi^\Delta_\Gamma \circ \rho^\Delta$, *where* $|\Gamma| \subseteq |\Delta|$.

Proof. (i) Induction on $n - i$.

(ii) Immediate by (i) and the definitions.

(iii) Using (i), we see that

$$\pi^\Gamma_{y_i} \circ \Pi^\Delta_\Gamma = \pi^\Delta_{y_i}. \tag{9}$$

Then, writing $\Theta = y_1,..., y_k$,

$$\Pi^\Gamma_\Theta \circ \Pi^\Delta_\Gamma \overset{(5)}{=} [\pi^\Gamma_{y_1} \circ \Pi^\Delta_\Gamma,..., \pi^\Gamma_{y_k} \circ \Pi^\Delta_\Gamma[= [\pi^\Delta_{y_1},..., \pi^\Delta_{y_k}[\overset{(9)}{=} \Pi^\Delta_\Theta.$$

(iv) Similarly. ∎

3.3. LEMMA. (i) *If* $|\Delta| \supseteq |\Gamma| \supseteq FV(M)$, *then*

$$[\![M]\!]_\Delta = [\![M]\!]_\Gamma \circ \Pi^\Delta_\Gamma.$$

(ii) $[\![M]\!]_\rho = [\![M]\!]_\Delta \circ \rho^\Delta$ *for all* $\Delta \supseteq FV(M)$.

(iii) *Let* $|\Delta| \equiv \{x_1,..., x_n\} \supseteq FV(M)$ *and* $|\Gamma| \supseteq FV(N_1) \cup \cdots \cup FV(N_n)$. *Suppose* $\boldsymbol{N}$ *is substitutable (simultaneously) for* $\boldsymbol{x}$ *in* M, *then*

$$[\![M[\boldsymbol{x} := \boldsymbol{N}]]\!]_\Gamma = [\![M]\!]_\Delta \circ [\,[\![\boldsymbol{N}]\!]_\Gamma[.$$

Proof. (i) Induction on $M \in \Lambda(\mathfrak{M})$:

$$[\![x]\!]_\Delta = \pi^\Delta_x \overset{(9)}{=} \pi^\Gamma_x \circ \Pi^\Delta_\Gamma = [\![x]\!]_\Gamma \circ \Pi^\Delta_\Gamma,$$

$$[\![c_a]\!]_\Delta = a \circ \Pi^\Delta_{\langle\rangle} = a \circ \Pi^\Gamma_{\langle\rangle} \circ \Pi^\Delta_\Gamma = [\![c_a]\!]_\Gamma \circ \Pi^\Delta_\Gamma,$$

$$\begin{aligned}[\![MN]\!]_\Delta &= [\![M]\!]_\Delta \cdot [\![N]\!]_\Delta = ([\![M]\!]_\Gamma \circ \Pi^\Delta_\Gamma) \cdot ([\![N]\!]_\Gamma \circ \Pi^\Delta_\Gamma)\\ &\overset{(8)}{=} ([\![M]\!]_\Gamma \cdot [\![N]\!]_\Gamma) \circ \Pi^\Delta_\Gamma = [\![MN]\!]_\Gamma \circ \Pi^\Delta_\Gamma,\end{aligned}$$

$$\begin{aligned}[\![\lambda x \cdot M]\!]_\Delta &= i \circ \Lambda([\![M]\!]_{\Delta;x}) \circ \Pi^\Delta_{\Delta\backslash x}\\ &\overset{(\mathrm{IH})}{=} i \circ \Lambda([\![M]\!]_{\Gamma;x} \circ \Pi^{\Delta;x}_{\Gamma;x}) \circ \Pi^\Delta_{\Delta\backslash x}\\ &\overset{(6)}{=} i \circ \Lambda([\![M]\!]_{\Gamma;x} \circ \Pi^{\Delta\backslash x}_{\Gamma\backslash x} \times \mathrm{id}) \circ \Pi^\Delta_{\Delta\backslash x}\\ &\overset{(2)}{=} i \circ \Lambda([\![M]\!]_{\Gamma;x}) \circ \Pi^{\Delta\backslash x}_{\Gamma\backslash x} \circ \Pi^\Delta_{\Delta\backslash x}\\ &\overset{(3.2(\mathrm{iii}))}{=} i \circ \Lambda([\![M]\!]_{\Gamma;x}) \circ \Pi^\Gamma_{\Gamma\backslash x} \circ \Pi^\Delta_\Gamma = [\![\lambda x \cdot M]\!]_\Gamma \circ \Pi^\Delta_\Gamma.\end{aligned}$$

(ii) $[\![M]\!]_\rho = [\![M]\!]_{FV(M)} \circ \rho^{FV(M)} = [\![M]\!]_{FV(M)} \circ \Pi^\Delta_{FV(M)} \circ \rho^\Delta = [\![M]\!]_\Delta \circ \rho^\Delta$, according to Lemmas 3.2(iv) and 3.3(i).

(iii) By induction on $M \in \Lambda(\mathfrak{M})$. The only difficult case is when $M \equiv \lambda y \cdot P$. Let $|\Delta'| \equiv FV(M) \subseteq |\Delta|$, say $\Delta' \equiv \boldsymbol{x}'$. Let $\boldsymbol{N}'$ be the corresponding subsequence of $\boldsymbol{N}$. Note that $y \notin \Delta'$ and $y \notin FV(\boldsymbol{N}')$ (this because of substitutability of $\boldsymbol{N}$ for $\boldsymbol{x}$ in M).

$$\begin{aligned}[\![M[\boldsymbol{x} := \boldsymbol{N}]]\!]_\Gamma &= [\![\lambda y \cdot P[\boldsymbol{x}' := \boldsymbol{N}', y := y]]\!]_\Gamma\\ &= i \circ \Lambda([\![P[\boldsymbol{x}' := \boldsymbol{N}', y := y]]\!]_{\Gamma;y}) \circ \Pi^\Gamma_{\Gamma\backslash y}\\ &\overset{(\mathrm{IH})}{=} i \circ \Lambda([\![P]\!]_{\Delta';y} \circ [\,[\![\boldsymbol{N}']\!]_{\Gamma;y}, [\![y]\!]_{\Gamma;y}[) \circ \Pi^\Gamma_{\Gamma\backslash y}\\ &\overset{(7),\,3.3(\mathrm{i})}{=} i \circ \Lambda([\![P]\!]_{\Delta';y} \circ \langle [\,[\![\boldsymbol{N}']\!]_{\Gamma\backslash y}[\circ p_1, p_2\rangle) \circ \Pi^\Gamma_{\Gamma\backslash y}\\ &\overset{(4)}{=} i \circ \Lambda([\![P]\!]_{\Delta';y} \circ [\,[\![\boldsymbol{N}']\!]_{\Gamma\backslash y}[\times \mathrm{id}) \circ \Pi^\Gamma_{\Gamma\backslash y}\end{aligned}$$

$$\overset{(2),3.3(i)}{=} i \circ \Lambda([\![P]\!]_{\Delta';y}) \circ [\,[\![N']\!]_{\Gamma}[$$

$$\overset{(5),3.2(ii)}{=} i \circ \Lambda([\![P]\!]_{\Delta';y}) \circ \Pi^{\Delta'}_{\Delta'\backslash y} \circ \Pi^{\Delta}_{\Delta'} \circ [\,[\![N]\!]_{\Gamma}[$$

$$= [\![\lambda y \cdot P]\!]_{\Delta'} \circ \Pi^{\Delta}_{\Delta'} \circ [\,[\![N]\!]_{\Gamma}[$$

$$= [\![M]\!]_{\Delta} \circ [\,[\![N]\!]_{\Gamma}[. \quad \blacksquare$$

From now on the use of $M[x := N]$ will imply that N is substitutable for x in M.

3.4. COROLLARY. (i) *Let* $|\Delta| \supseteq FV(\lambda x \cdot M)$ *and* $|\Gamma| \supseteq FV(N) \cup FV(\lambda x \cdot M)$ *and* $|\Gamma| \supseteq |\Delta\backslash x|$. *Then* $[\![M[x := N]]\!]_{\Gamma} = [\![M]\!]_{\Delta;x} \circ \langle \Pi^{\Gamma}_{\Delta\backslash x}, [\![N]\!]_{\Gamma}\rangle$.

(ii) *Let* $|\Delta| \supseteq FV(\lambda x \cdot M)$, $x, y \notin \Delta$. *Then* $[\![M[x := y]]\!]_{\Delta;y} = [\![M]\!]_{\Delta;x}$.

Proof. (i) Apply Lemma 3.3(iii) to $\Delta' \equiv FV(\lambda x \cdot M)$, x and Γ.

(ii) Apply Corollary 3.4(i) to Δ and $\Gamma' \equiv \Delta; y$. ▮

3.5. THEOREM. *Let* $|\Delta| \supseteq FV(M) \cup FV(N)$, *then*

$$\lambda(\mathfrak{M}) \vdash M = N \Rightarrow [\![M]\!]_{\Delta} = [\![N]\!]_{\Delta}.$$

Proof. By induction on the length of proof. We only need to check axioms (α) and (β), because the rest is trivial.

(α) Let $|\Delta| \equiv FV(\lambda x \cdot M)$ (then $x, y \notin \Delta$).

$$[\![\lambda y \cdot M[x := y]]\!]_{\Delta} = i \circ \Lambda([\![M[x := y]]\!]_{\Delta;y}) \circ \Pi^{\Delta}_{\Delta\backslash y}$$

$$\overset{3.4(ii)}{=} i \circ \Lambda([\![M]\!]_{\Delta;x}) \circ \mathrm{id} = [\![\lambda x \cdot M]\!]_{\Delta}.$$

(β) Let $|\Delta| \equiv FV(\lambda x \cdot M) \cup FV(N)$.

$$[\![(\lambda x \cdot M)\, N]\!]_{\Delta} = (i \circ \Lambda([\![M]\!]_{\Delta;x}) \circ \Pi^{\Delta}_{\Delta\backslash x}) \cdot [\![N]\!]_{\Delta}$$

$$= \mathrm{ev} \circ \langle j \circ i \circ \Lambda([\![M]\!]_{\Delta;x}) \circ \Pi^{\Delta}_{\Delta\backslash x}, [\![N]\!]_{\Delta}\rangle$$

$$\overset{(4),\, j\circ i = \mathrm{id}}{=} \mathrm{ev} \circ \Lambda([\![M]\!]_{\Delta;x}) \times \mathrm{id} \circ \langle \Pi^{\Delta}_{\Delta\backslash x}, [\![N]\!]_{\Delta}\rangle$$

$$\overset{(1)}{=} [\![M]\!]_{\Delta;x} \circ \langle \Pi^{\Delta}_{\Delta\backslash x}, [\![N]\!]_{\Delta}\rangle$$

$$= [\![M[x := N]]\!]_{\Delta} \qquad \text{(using Corollary 3.4(i) with } \Gamma = \Delta).$$

This establishes the results for $|\Delta| \equiv FV(\lambda x \cdot M)$ (resp. $|\Delta| \equiv FV(\lambda x \cdot M) \cup FV(N)$).

For $\Delta' \supseteq \Delta$, use Lemma 3.3(i). ▮

3.6. COROLLARY. *$\mathfrak{M}_C$ is a pseudomodel.*

We call $\mathfrak{M}_C$ the pseudomodel generated by $\langle C, U, i, j\rangle$.

This shows how to obtain a pseudomodel from special data in a ccc. In Section 4 we will show that essentially every λ-algebra can be obtained that way.

4. CATEGORICAL MODELS INDUCED BY λ-ALGEBRAS

In this section $\mathfrak{M} = (X, \cdot, s, k)$ is a fixed λ-algebra. We identify $\mathfrak{M}$ with its associated pseudomodel $\mathfrak{M}^+$. Put $b = [\![\lambda xyz \cdot x(yz)]\!]$ and

$$u \circ v = b \cdot u \cdot v, \qquad \text{so} \quad \circ : X^2 \to X.$$

4.1. DEFINITION (See Scott, 1980). $C_{\mathfrak{M}}$, the *category associated with* $\mathfrak{M}$, is defined by

Objects	$\{u \in X \mid u \circ u = u\}$,
Arrows	$\mathrm{Hom}(u, v) = \{f \in X \mid v \circ f \circ u = f\}$,
Identities	$\mathrm{id}_u = u$,
Composition	$\mathrm{Comp}: \mathrm{Hom}(u, v) \times \mathrm{Hom}(v, w) \to \mathrm{Hom}(u, w)$
	$(f, g) \mapsto g \circ f$.

4.2. LEMMA. *$C_{\mathfrak{M}}$ is a* ccc *with a special object U, of which U^U is a retract.*

Proof. We supply the relevant definitions:

Terminal object	$T = [\![\lambda xy \cdot y]\!] = [\![F]\!]$,
Products	$u \times v = [\![\lambda xy \cdot y(c_u(xK))(c_v(xF))]\!]$,
Exponents	$v^u = [\![\lambda x \cdot c_v \circ x \circ c_u]\!]$,
Special object	$U = skk = [\![\lambda x \cdot x]\!]$,
Embedding	$U^U \xrightarrow{i} U, \qquad i = [\![\lambda xy \cdot xy]\!] = [\![1]\!]$,
Projection	$U \xrightarrow{j} U^U, \qquad j = [\![\lambda xy \cdot xy]\!] = [\![1]\!]$.

These constructions will be analysed further in Section 6 and 7, where fully detailed proofs are given: see the remarks after Lemma 7.2.

4.3. DEFINITION. Let $|\mathfrak{M}| \xrightarrow{\phi_{\mathfrak{M}}} |\mathfrak{M}_{C_{\mathfrak{M}}}|$ be the canonical map from $\mathfrak{M}$ to the λ-algebra generated by the category associated with $\mathfrak{M}$, defined by $\phi_{\mathfrak{M}}(u) = k \cdot u$.

4.4. DEFINITION. (i) Let $\mathfrak{M}_i = (X_i, \cdot_i, s_i, k_i)$ be λ-algebras $i = 0, 1$. $\phi: |\mathfrak{M}_0| \to |\mathfrak{M}_1|$ is a *homomorphism* if

$$\phi(u \cdot_0 v) = \phi(u) \cdot_1 \phi(v),$$
$$\phi(s_0) = s_1, \qquad \phi(k_0) = k_1.$$

(ii) Let $\mathfrak{M}_i = (X_i, \cdot_i, [\![\cdot]\!]^i_\cdot)$ be pseudostructures. $\phi: |\mathfrak{M}_0| \to |\mathfrak{M}_1|$ is a *homomorphism* if

$$\phi(u \cdot_0 v) = \phi(u) \cdot_1 \phi(v),$$
$$\phi([\![M]\!]^0_\rho) = [\![M^\phi]\!]^1_{\rho^\phi},$$

where

$$M^\phi \equiv M \qquad \text{with all } c_u \text{ replaced by } c_{\phi(u)},$$
$$\rho^\phi \equiv \phi \circ \rho.$$

(iii) *Isomorphisms* are bijective homomorphisms (as a consequence: ϕ isomorphism $\Rightarrow \phi^{-1}$ isomorphism).

4.5. LEMMA. *Let $\mathfrak{M}$, $\mathfrak{N}$ be λ-algebras and let $\phi: |\mathfrak{M}| \to |\mathfrak{N}|$. Then ϕ is a homomorphism $\mathfrak{M} \to \mathfrak{N} \Leftrightarrow \phi$ is a homomorphism $\mathfrak{M}^+ \to \mathfrak{N}^+$.*

Proof. Trivial. ∎

Again we also have the dual of Lemma 4.5.

4.6. THEOREM. *$\phi_{\mathfrak{M}}$ is an isomorphism.*

Proof. By Lemma 4.5 it suffices to prove

(a) $\phi_{\mathfrak{M}}(u \cdot v) = \phi_{\mathfrak{M}}(u) \underset{\sim}{\cdot} \phi_{\mathfrak{M}}(v)$.

(b) $\phi_{\mathfrak{M}}(s) = \underset{\sim}{s}$, $\phi_{\mathfrak{M}}(k) = \underset{\sim}{k}$.

Here $\underset{\sim}{\cdot}$, $\underset{\sim}{s}$, $\underset{\sim}{k}$ are operators in $\mathfrak{M}_{C_{\mathfrak{M}}}$, e.g.,

$$\underset{\sim}{s} = [\![\lambda xyz \cdot xz(yz)]\!]^{\mathfrak{M}_{C_{\mathfrak{M}}}}.$$

(c) $\phi_{\mathfrak{M}}$ is bijective.

Let us first note some equations in $C_{\mathfrak{M}}$:

Projections $\quad u \times v \xrightarrow{p_1} u, \qquad p_1 = [\![\lambda x \cdot c_u(xK)]\!],$

$u \times v \xrightarrow{p_2} v, \qquad p_2 = [\![\lambda x \cdot c_v(xF)]\!].$

Pairing $\quad$ Let $u \xrightarrow{f} v, \qquad u \xrightarrow{g} w,$

then $\langle f, g\rangle = [\![\lambda xy \cdot y(c_f x)(c_g x)]\!]$.

Products $\quad$ Let $u_i \xrightarrow{f_i} v_i, \qquad i = 1, 2,$

then $f_1 \times f_2 = [\![\lambda xy \cdot y(c_{f_1}(c_{u_1}(xK)))(c_{f_2}(c_{u_2}(xF)))]\!]$.

Evaluation $\quad$ ev: $v^u \times u \to v,$

$\text{ev} = [\![\lambda x \cdot c_v((xK)(c_u(xF)))]\!]$.

Exponentional adjoint: let $u \times v \xrightarrow{f} w$, then

$$\Lambda(f) = [\![\lambda xy \cdot c_f(\lambda z \cdot zxy)]\!] : u \to w^v.$$

(a) $\phi_{\mathfrak{M}}(u) \underset{-}{\cdot} \phi_{\mathfrak{M}}(v) = (k \cdot u) \underset{-}{\cdot} (k \cdot v) = \text{ev} \circ \langle j \circ ku, kv\rangle = [\![\lambda x \cdot xK(xF)]\!] \circ [\![\lambda xy \cdot y(1(Kc_u x))(Kc_v x)]\!] = [\![\lambda x \cdot 1c_u(c_v)]\!] = k \cdot (u \cdot v) = \phi_{\mathfrak{M}}(u \cdot v)$.

(b) Let us calculate $\underset{-}{s} = [\![\lambda xyz \cdot xz(yz)]\!]^{\mathfrak{M} c_{\mathfrak{M}}} = [\![\lambda xyz \cdot xz(yz)]\!]$.

$$\begin{aligned} [\![x]\!]_{x,y,z} &= p_2 \circ p_1 \circ p_1 = [\![\lambda x \cdot xKKF]\!]^{\mathfrak{M}}, \\ [\![y]\!]_{x,y,z} &= \quad p_2 \circ p_1 \quad = [\![\lambda x \cdot xKF]\!]^{\mathfrak{M}}, \\ [\![z]\!]_{x,y,z} &= \qquad p_2 \qquad = [\![\lambda x \cdot xF]\!]^{\mathfrak{M}}. \end{aligned}$$

$\text{ev} \circ j \times \text{id} = [\![\lambda x \cdot xK(xF)]\!]^{\mathfrak{M}}$, therefore $u \underset{-}{\cdot} v = [\![\lambda x \cdot c_u x(c_v x)]\!]^{\mathfrak{M}}$, so

$$\begin{aligned} [\![xz]\!]_{x,y,z} &= [\![\lambda x \cdot xKKF(xF)]\!]^{\mathfrak{M}}, \\ [\![yz]\!]_{x,y,z} &= [\![\lambda x \cdot xKF(xF)]\!]^{\mathfrak{M}}, \\ m &= [\![xz(yz)]\!]_{x,y,z} = [\![\lambda x \cdot xKKF(xF)[xKF(xF)]]\!]^{\mathfrak{M}}, \\ \underset{-}{s} &= (i \circ \Lambda)^3\,(m) = (i \circ \Lambda)^2\,([\![\lambda yz \cdot yKFz[yFz]]\!]^{\mathfrak{M}}) \\ &= i \circ \Lambda([\![\lambda xyz \cdot xFz(yz)]\!]^{\mathfrak{M}}) = [\![\lambda x'xyz \cdot xz(yz)]\!]^{\mathfrak{M}}. \end{aligned}$$

So $\underset{-}{s} = k \cdot s = \phi_{\mathfrak{M}}(s)$. In an analogous way $\underset{-}{k} = \phi_{\mathfrak{M}}(k)$.

(c) Clearly $\phi_{\mathfrak{M}}$ is injective, so it suffices to prove: for every $v \in \text{Hom}(T, U)$ there exists $u \in |\mathfrak{M}|$ with $v = k \cdot u$. Let $v \in \text{Hom}(T, U)$, so $v = [\![\lambda x \cdot I(c_v(Fx))]\!]$. That is, $v = k \cdot [\![c_v I]\!]$. Choose $u = [\![c_v I]\!]$. ∎

5. Weak Extensionality and Extensionality

Now we will investigate how weakly extensional and extensional models behave from the categorical viewpoint.

5.1. Definition. Let $\mathfrak{M} = (X, \cdot, \ldots)$ be a structure.

$\mathfrak{M}$ is *extensional* (wrt $\cdot$) if $\forall d \in X(d_0 \cdot d = d_1 \cdot d) \Rightarrow d_0 = d_1$.

Note that this is a sharpening of (ξ_1) in Definition 2.5.

5.2. Lemma. *Let $\mathfrak{M} = (X, \cdot, s, k)$ be a λ-algebra. Then $\mathfrak{M}$ is extensional iff $\mathfrak{M}$ satisfies weak extensionality and $\mathfrak{M} \models 1 = I$* (i.e., $1 = s \cdot k \cdot k$).

Proof. *only if* $1 \cdot d \cdot e = d \cdot e = s \cdot k \cdot k \cdot d \cdot e$.

if Suppose $\forall d(d_0 \cdot d = d_1 \cdot d)$. By (ξ_1) $1 \cdot d_0 = 1 \cdot d_1$. So $skkd_0 = skkd_1$ or $d_0 = d_1$. ∎

5.3. Theorem. *Let (C, U, i, j) be given.*

(i) *If C has enough points at U, that is,*

$$\forall U \underset{g}{\overset{f}{\rightrightarrows}} U \, \exists T \xrightarrow{x} U(f \neq g \Rightarrow f \circ x \neq g \circ x),$$

then $\mathfrak{M}_C$ is a λ-model.

(ii) *If $i \circ j = \mathrm{id}_U$, then $\mathfrak{M}_C \models 1 = I$.*

Proof. First calculate $[\![\lambda x \cdot Mx]\!]_\Delta$, $|\Delta| \supseteq FV(M) \not\ni x$.

$$\begin{aligned}
[\![\lambda x \cdot Mx]\!]_\Delta &= i \circ \Lambda([\![Mx]\!]_{\Delta;x}) \circ \Pi^{\Delta}_{\Delta\backslash x} \\
&= i \circ \Lambda(\mathrm{ev} \circ \langle j \circ [\![M]\!]_{\Delta;x}, p_2\rangle) \circ \Pi^{\Delta}_{\Delta\backslash x} \\
&= i \circ \Lambda(\mathrm{ev} \circ (j \circ [\![M]\!]_{\Delta\backslash x}) \times \mathrm{id}) \circ \Pi^{\Delta}_{\Delta\backslash x} \\
&= i \circ j \circ [\![M]\!]_\Delta .
\end{aligned}$$

(i) C has also enough points at $U^1 \cong U$. Every $g\colon T \to U^1$ is of the form $g = \langle \mathrm{id}, f\rangle$ for some $f\colon T \to U$.

Now assume $\forall d(d_0 \cdot d = d_1 \cdot d)$. Then $\forall d(\mathrm{ev} \circ \langle j \circ d_0, d\rangle = \mathrm{ev} \circ \langle j \circ d_1, d\rangle)$. So $\forall d(\mathrm{ev} \circ (j \circ d_0) \times \mathrm{id} \circ \langle \mathrm{id}, d\rangle = \mathrm{ev} \circ (j \circ d_1) \times \mathrm{id} \circ \langle \mathrm{id}, d\rangle)$. By the above $\mathrm{ev} \circ (j \circ d_0) \times \mathrm{id} = \mathrm{ev} \circ (j \circ d_1) \times \mathrm{id}$. By adjunction and composition with i

$$[\![\lambda x \cdot c_{d_0} x]\!] = [\![\lambda x \cdot c_{d_1} x]\!] \qquad \text{id est}$$
$$1 \cdot d_0 = 1 \cdot d_1 .$$

(ii) It follows that $[\![\lambda x \cdot Mx]\!]_\Delta = [\![M]\!]_\Delta$ for all M. In particular we have $\mathfrak{M} \models 1 = I$. ∎

5.4. THEOREM. (i) *If $\mathfrak{M}$ is a λ-model, then $C_\mathfrak{M}$ has enough points at U.*

(ii) *If $\mathfrak{M} \models 1 = I$, then in $C_\mathfrak{M}$ we have $i \circ j = \mathrm{id}_U$.*

Proof. (i) We show that $C_\mathfrak{M}$ has enough points everywhere. Assume $a \rightrightarrows^f_g b$ are given maps. Now note that for all $u \in |\mathfrak{M}|$ $k(au)$ is a map from T to a. So assume $\forall u \in |\mathfrak{M}|$ $(f \circ k(au) = g \circ k(au))$. This means $\forall u \in |\mathfrak{M}|$ $(k(f(au)) = k(g(au)))$.

Equivalently $\forall u \in |\mathfrak{M}|$ $((f \circ a)\, u = (g \circ a)\, u)$. Using (ξ_1), $1 \cdot (f \circ a) = 1 \cdot (g \circ a)$. Now note that $f \circ a = f$, $g \circ a = g$ and moreover f is a function $(f = b \circ f = [\![\lambda x \cdot c_b(c_f x)]\!]$, so $1 \cdot f = f$ and the same for g. So $f = g$.

(ii) $i \circ j = [\![1]\!] \circ [\![1]\!] = [\![1]\!] = [\![I]\!] = \mathrm{id}_U$. ∎

As a corollary we see, that $\mathfrak{M}$ is extensional if and only if $C_\mathfrak{M}$ has enough points and $i \circ j = \mathrm{id}_U$, so that U^U is in fact isomorphic to U, not only a retract.

6. Cartesian Closed Monoids

In Section 4 we introduced the category $C_\mathfrak{M}$ associated with a given λ-algebra $\mathfrak{M}$. Now we are going to analyse this process a little further.

It appears that the category $C_\mathfrak{M}$ is of a very special nature and can be conveniently described in terms of a certain algebraic structure, called a Cartesian closed monoid (CCM). The idea of associating a monoid with a given λ-theory (see Section 7) and showing that this monoid is Cartesian closed goes back to Scott.

Let us first associate a category to any monoid $M = (X, \circ, I)$.

6.1. DEFINITION. Let $M = (X, \circ, I)$ be a monoid. Define a category C_M by

Objects	$\{u \in X; u \circ u = u\}$,
Arrows	$\{f \in X; v \circ f \circ u = f\} = \mathrm{Hom}(u, v)$,
Identity	$\mathrm{id}_u = u$,
Composition	$\mathrm{Hom}(u, v) \times \mathrm{Hom}(v, w) \to \mathrm{Hom}(u, w)$
	$(f, g) \mapsto g \circ f$.

6.2. LEMMA. (i) $f \in \mathrm{Hom}(u, v) \Leftrightarrow v \circ f = f \wedge f \circ u = f$.

(ii) *C_M is indeed a category.*

(iii) *I is a universal object in C_M.*

Proof. (i) Trivial.

(ii) Composition is well defined: use (i). Associative and identity laws for composition follow from the associative law in M and the definition of arrow.

(iii) For every object u we have $u \rightleftarrows^u_u I$ represents u as a retract of I. ∎

So in the category C_M we get the special object I for free. Therefore we can start interpreting λ-calculus as soon as we know C_M is Cartesian closed. Essentially this is the case when M is a CCM as defined as follows:

6.3. DEFINITION. Let $M = (X, \circ, I)$ be a monoid; let $p, q, \varepsilon \in M$ be constants; let $\Lambda: M \to M$, $\langle \cdot, \cdot \rangle: M^2 \to M$ be operations. $\underline{M} = (M, p, q, \langle \cdot, \cdot \rangle, \varepsilon, \Lambda)$ is a (CCM) if it satisfies the following axioms (for all $u, v, w \in X$):

$$\text{(I)} \qquad p \circ \langle u, v \rangle = u, \qquad q \circ \langle u, v \rangle = v,$$

$$\text{(II)} \qquad \langle u, v \rangle \circ w = \langle u \circ w, v \circ w \rangle,$$

$$\text{(III)} \qquad \varepsilon \circ \langle p, q \rangle = \varepsilon,$$

$$\text{(IV)} \qquad \Lambda(\varepsilon) \circ \Lambda(u) = \Lambda(u),$$

$$\text{(V)} \qquad \varepsilon \circ \langle \Lambda(u) \circ p, q \rangle = u \circ \langle p, q \rangle,$$

$$\text{(VI)} \qquad \Lambda(\varepsilon \circ \langle u \circ p, q \rangle) = \Lambda(\varepsilon) \circ u.$$

Notations for a CCM: $\underline{M}$, M, or even X.

What we want to show is that the category C_M associated with a monoid M is Cartesian closed if and only if we can consider M as a CCM. To separate the arguments we first consider the case of products.

6.4. DEFINITION. $M = (X, \circ, I, p, q, \langle \cdot, \cdot \rangle)$ is a *monoid with pairing* if M satisfies 6.3(I) and 6.3(II).

6.5. THEOREM. *Let M be a monoid. Then C_M has products $\Leftrightarrow$ There exist $p, q, \langle \cdot, \cdot \rangle$ making M into a monoid with pairing.*

Proof. $\Rightarrow$ Let us define p and q as the first and second projection, respectively, from $I \times I$ onto I. To define $\langle u, v \rangle$ for $u, v \in X$, consider u and v as arrows in $\mathrm{Hom}(I, I)$ and construct the usual diagram

$$\begin{tikzcd} & I \\ I \arrow[ur, "u"] \arrow[r, "\langle u,v\rangle"] \arrow[dr, "v"'] & I \times I. \arrow[u, "p"'] \arrow[d, "q"] \\ & I \end{tikzcd} \tag{$*$}$$

Here, $\langle u, v\rangle$ is the unique arrow making $(*)$ commute. So 6.3(I) follows immediately. To prove $\langle u, v\rangle \circ w = \langle u \circ w, v \circ w\rangle$, consider

$$\begin{tikzcd} & & I \\ I \arrow[r, "w"] \arrow[urr, "u \circ w"] \arrow[drr, "v \circ w"'] & I \arrow[ur, "u"] \arrow[r, "\langle u,v\rangle"] \arrow[dr, "v"'] & I \times I. \arrow[u, "p"'] \arrow[d, "q"] \\ & & I \end{tikzcd}$$

Here $\langle u, v\rangle \circ w$ makes the outside triangles commute, so 6.3(II) holds.

$\Leftarrow$ For any objects $u, v \in C_M$, define

$$u \times v = \langle u \circ p, v \circ q\rangle, \tag{1}$$

$$(p_{uv}\colon u \times v \to u) = u \circ p, \ (q_{uv}\colon u \times v \to v) = v \circ q. \tag{2}$$

We will show that this defines a product in C_M.

(i) $u \times v$ is a well-defined object

$$\begin{aligned} u \times v \circ u \times v &= \langle u \circ p, v \circ q\rangle \circ u \times v \\ &\overset{(\mathrm{II})}{=} \langle u \circ p \circ u \times v, v \circ q \circ u \times v\rangle \\ &\overset{(\mathrm{I})}{=} \langle u \circ u \circ p, v \circ v \circ q\rangle \\ &= \langle u \circ p, v \circ q\rangle = u \times v. \end{aligned}$$

(ii) p_{uv} is a well-defined arrow in $\mathrm{Hom}(u \times v, u)$

$$\begin{aligned} u \circ p_{uv} \circ u \times v = u \circ u \circ p \circ u \times v &\overset{(\mathrm{I})}{=} u \circ u \circ u \circ p \\ &= u \circ p = p_{uv}. \end{aligned}$$

Similarly $q_{uv} \in \mathrm{Hom}(u \times v, v)$.

(iii) Now let $f: w \to u$, $g: w \to v$ be given

$$\begin{tikzcd} & u \\ w \arrow[ur, "f"] \arrow[r, dashed] \arrow[dr, "g"'] & u \times v \arrow[u, "p_{uv}"'] \arrow[d, "q_{uv}"] \\ & v \end{tikzcd}$$

We claim $\langle f, g\rangle$ is the unique arrow making this diagram commute: $\langle f, g\rangle \in \mathrm{Hom}(w, u \times v)$, for

$$u \times v \circ \langle f, g\rangle \circ w \overset{(1)(II)}{=} \langle u \circ p \circ \langle f, g\rangle \circ w, v \circ q \circ$$
$$\langle f, g\rangle \circ w\rangle \overset{(I)}{=} \langle u \circ f \circ w, v \circ g \circ w\rangle = \langle f, g\rangle.$$

Here $\langle f, g\rangle$ makes the diagram commute, because of (I). For uniqueness, assume h makes the diagram commute. $h \in \mathrm{Hom}(w, u \times v)$, so: $h = (u \times v) \circ h =^{(1)} \langle u \circ p, v \circ q\rangle \circ h =^{(II)} \langle p_{uv} \circ h, q_{uv} \circ h\rangle = \langle f, g\rangle$. ∎

Note that if the product is defined as in Theorem 6.5 ($\Leftarrow$) we have:

For any *arrows* $f, g{:}\, f \times g = \langle f \circ p, g \circ q\rangle$. (3)

(Formally the same as for objects; this allows confusion of objects and identity arrows.)

In particular $I \times I = \mathrm{id}_I \times \mathrm{id}_I = \mathrm{id}_{I \times I} = \langle p, q\rangle$. (4)

We also have $f \times g \circ h \times k = f \circ h \times g \circ k$. (5)

When using (1) as an abbreviation we can restate Definition 6.3(III)–(VI) by

(III′) $\varepsilon \circ I \times I = \varepsilon$,

(IV′) $\Lambda(\varepsilon) \circ \Lambda(u) = \Lambda(u)$,

(V′) $\varepsilon \circ \Lambda(u) \times I = u \circ I \times I$,

(VI′) $\Lambda(\varepsilon \circ u \times I) = \Lambda(\varepsilon) \circ u$.

6.6. THEOREM. *Assume $M = (X, \circ, I)$ is a monoid. Then C_M is Cartesian closed $\Leftrightarrow$ There exist p, q, $\langle\cdot, \cdot\rangle$, Λ, ε, making M into a* CCM.

Proof. We did the "productpart" in Theorem 6.5. So now our concern is about exponentiation in C_M and Λ, ε in M. As products and exponentiation are only defined up to isomorphism, we may assume that the "productstructure" in C_M is defined by (1) and (2) in the proof of Theorem 6.5.

$\Rightarrow$ Define ε as the evaluation

$$I^I \times I \xrightarrow{\mathrm{ev}_{I,I} = \varepsilon} I.$$

For any $u \in X$, $u \circ I \times I \in \mathrm{Hom}(I \times I, I)$. Then let $\Lambda(u)$ be the exponential adjoint of $u \circ \langle p, q \rangle$

$$\begin{tikzcd} I^I \times I \arrow[r, "\varepsilon"] & I \\ I \times I \arrow[u, "\Lambda(u)\times \mathrm{id}_I"] \arrow[ur, "u \circ I \times I"'] & \end{tikzcd} \tag{**}$$

(Warning: Note that we use Λ here differently as in Section 3, where it denoted exponential adjoint; this use of Λ can be defended by noting that the two notions coincide when f is a map with a product as domain, as we shall see in 6.6($\Leftarrow$)(iv)).

Now 6.3(V′) follows from the commutativity of ($**$). Then $\varepsilon \in \mathrm{Hom}(I^I \times I, I)$ implies $\varepsilon = \varepsilon \circ I^I \times I =^{(5)} \varepsilon \circ I^I \times I \circ I \times I = \varepsilon \circ I \times I$ and this is Definition 6.3(III′).

As we just saw: $\varepsilon \circ I^I \times \mathrm{id}_I = \varepsilon \circ I \times I$ and $I^I \in \mathrm{Hom}(I, I^I)$. So I^I makes ($**$) (where $u = \varepsilon$) commute in place of $\Lambda(\varepsilon)$.

$$\text{By uniqueness } \Lambda(\varepsilon) = I^I. \tag{6}$$

Now $\Lambda(u) \in \mathrm{Hom}(I, I^I) =^{(6)} \mathrm{Hom}(I, \Lambda(\varepsilon))$, so $\Lambda(\varepsilon) \circ \Lambda(u) = \Lambda(u)$ and this is 6.3(IV′). It remains to prove 6.3(VI′). Consider the diagram

$$\begin{tikzcd} I^I \times I \arrow[r, "\varepsilon"] & I \\ I \times I \arrow[u, "\Lambda(\varepsilon)\times \mathrm{id}_I"] \arrow[ur, "\varepsilon = \varepsilon \circ I \times I"] & \\ I \times I \arrow[u, "u \times \mathrm{id}_I"] \arrow[uur, "\varepsilon \circ u \times I = \varepsilon \circ u \times I \circ I \times I."'] & \end{tikzcd}$$

Now $\Lambda(\varepsilon) \times I \circ u \times I$ makes the outside triangle commute, so by uniqueness of exponential adjoint, using (5), we see that

$$\Lambda(\varepsilon) \circ u = \Lambda(\varepsilon \circ u \times I).$$

This is 6.3(VI′).

$\Leftarrow$ Define $\mathrm{ev}_{u,v} = v \circ \varepsilon \circ I \times u$, $v^u = \Lambda(\mathrm{ev}_{u,v})$. Then (leaving out subscripts)

(i) v^u is a well-defined object

$$v^u \circ v^u \overset{(\mathrm{IV}')}{=} \Lambda(\varepsilon) \circ v^u \circ v^u \overset{(\mathrm{VI}')}{=} \Lambda(\varepsilon \circ (v^u \circ v^u) \times I)$$

$$\overset{(5)}{=} \Lambda(\varepsilon \circ v^u \times I \circ v^u \times I) \overset{(\mathrm{V}')}{=} \Lambda(\mathrm{ev} \circ I \times I \circ v^u \times I)$$

$$\overset{(5)}{=} \Lambda(v \circ \varepsilon \circ v^u \times I \circ I \times u) \overset{(\mathrm{V}')}{=} \Lambda(v \circ \mathrm{ev} \circ I \times I \circ I \times u)$$

$$= \Lambda(v \circ \varepsilon \circ I \times u) = \Lambda(\mathrm{ev}) = v^u.$$

(ii) $\mathrm{ev} \in \mathrm{Hom}(v^u \times u, v)$, for

$$v \circ \mathrm{ev} \circ v^u \times u = v \circ v \circ \varepsilon \circ I \times u \circ v^u \times u$$

$$\overset{(5)}{=} v \circ \varepsilon \circ v^u \times I \circ I \times u$$

$$\overset{(\mathrm{V}')}{=} v \circ \mathrm{ev} \circ I \times u$$

$$= v \circ v \circ \varepsilon \circ I \times u \circ I \times u$$

$$= v \circ \varepsilon \circ I \times u = \mathrm{ev}.$$

(iii) There is a more general version of (VI′):

(VI$^{\mathrm{S}}$) $\Lambda(v \circ u \times I) = \Lambda(v) \circ u$, for

$$\Lambda(v \circ u \times I) \overset{(5)}{=} \Lambda(v \circ I \times I \circ u \times I)$$

$$\overset{(\mathrm{V}')}{=} \Lambda(\varepsilon \circ \Lambda(v) \times I \circ u \times I)$$

$$\overset{(5)}{=} \Lambda(\varepsilon \circ (\Lambda(v) \circ u) \times I)$$

$$\overset{(\mathrm{VI}')}{=} \Lambda(\varepsilon) \circ \Lambda(v) \circ u \overset{(\mathrm{IV}')}{=} \Lambda(v) \circ u.$$

(iv) Now let $w \times u \xrightarrow{f} v$. Then $\Lambda(f) \in \mathrm{Hom}(w, v^u)$

$$v^u \circ \Lambda(f) \circ w \overset{(\mathrm{VI}^{\mathrm{S}})}{=} \Lambda(\mathrm{ev} \circ (\Lambda(f) \circ w) \times I)$$

$$\overset{(5)}{=} \Lambda(v \circ \varepsilon \circ \Lambda(f) \times I \circ w \times u)$$

$$\overset{(\mathrm{V}')}{=} \Lambda(v \circ f \circ I \times I \circ w \times u)$$

$$\overset{(5)}{=} \Lambda(v \circ f \circ w \times u) = \Lambda(f).$$

Then

$$\begin{aligned}
\mathrm{ev} \circ \Lambda(f) \times \mathrm{id}_u &= \mathrm{ev} \circ (\Lambda(f) \circ w) \times u \\
&\overset{(5)}{=} v \circ \varepsilon \circ \Lambda(f) \times I \circ w \times u \\
&\overset{(\mathrm{V}')}{=} v \circ f \circ I \times I \circ w \times u \\
&\overset{(5)}{=} v \circ f \circ w \times u = f.
\end{aligned}$$

So $\Lambda(f)$ is the exponential adjoint of f if it is the unique $h\colon w \to v^u$ such that

$$\mathrm{ev} \circ (h \times \mathrm{id}_u) = f.$$

Let h have these properties, then

$$\begin{aligned}
h = v^u \circ h = \Lambda(\mathrm{ev}) \circ h &\overset{(\mathrm{VI}^\mathrm{S})}{=} \Lambda(\mathrm{ev} \circ h \times I) \\
&= \Lambda(\mathrm{ev} \circ I \times u \circ h \times I) \\
&\overset{(5)}{=} \Lambda(\mathrm{ev} \circ h \times u) = \Lambda(f).
\end{aligned}$$

(v) What about the terminal object? (VI^S) implies $\Lambda(q) \circ u = \Lambda(q \circ u \times I) =^{(1)} \Lambda(q)$ for all $u \in X$. By taking $u = \Lambda(q)$ we see that $\Lambda(q)$ is an object and moreover $u \in \mathrm{Hom}(a, \Lambda(q))$ iff $\Lambda(q) \circ u \circ a = u$ iff $\Lambda(q) = u$, so $\Lambda(q)$ is a terminal object. ∎

7. The CCM of a λ-Algebra

In Section 4 we associated a CCM $C_{\mathfrak{M}}$ with any λ-algebra $\mathfrak{M}$. Now we show that $C_{\mathfrak{M}}$ in fact equals $C_{M_{\mathfrak{M}}}$ (as defined in Section 6), where $M_{\mathfrak{M}}$ is the CCM associated with the λ-algebra $\mathfrak{M}$ as will be defined now.

7.1. Definition. Let $\mathfrak{M}$ be a λ-algebra. $M_{\mathfrak{M}} = M = (X, \circ, I, p, q, \langle \cdot, \cdot \rangle, \varepsilon, \Lambda(\cdot))$ is the CCM defined as follows:

$$X = \{a \in |\mathfrak{M}| \mid a = [\![\lambda x \cdot c_a x]\!]\}$$

(i.e., X consists of the "function-elements" in $\mathfrak{M}$)

$$\begin{aligned}
a \circ b &= [\![\lambda x \cdot c_a(c_b x)]\!], \\
I &= [\![\lambda x \cdot x]\!],
\end{aligned}$$

$$p = [\![\lambda x \cdot xK]\!],$$
$$q = [\![\lambda x \cdot xF]\!],$$
$$\langle a, b\rangle = [\![\lambda x \cdot [c_a x, c_b x]]\!],$$
$$\varepsilon = [\![\lambda x \cdot xK(xF)]\!],$$
$$\Lambda(a) = [\![\lambda xy \cdot c_a[x, y]]\!].$$

Here $[M, N] =_{\text{def}} \lambda z \cdot zMN$ is λ-calculus pairing.

7.2. LEMMA. *$M_{\mathfrak{M}}$ as defined in* 7.1 *is indeed a* CCM.

Proof. All defined objects are "function"-elements in $\mathfrak{M}$ because they start with λx. Furthermore:

$$a \circ I = [\![\lambda x \cdot c_a(c_I x)]\!] = [\![\lambda x \cdot c_a x]\!] = a,$$
$$I \circ b = [\![\lambda x \cdot c_I(c_b x)]\!] = [\![\lambda x \cdot c_b x]\!] = b,$$
$$(a \circ b) \circ c = [\![\lambda x \cdot c_a(c_b(c_c x))]\!] = a \circ (b \circ c),$$
$$\begin{aligned} p \circ \langle a, b\rangle &= [\![\lambda x \cdot (\lambda y \cdot yK)[c_a x, c_b x]]\!] \\ &= [\![\lambda x \cdot c_a x]\!] = a. \end{aligned}$$

Also $q \circ \langle a, b\rangle = b$.

$$\begin{aligned} \langle a, b\rangle \circ c &= [\![\lambda x \cdot (\lambda y \cdot [c_a y, c_b y])(c_c x)]\!] \\ &= [\![\lambda x \cdot [c_a(c_c x), c_b(c_c x)]]\!] \\ &= [\![\lambda x \cdot [c_{a\circ c} x, c_{b\circ c} x]]\!] = \langle a \circ c, b \circ c\rangle. \\ \varepsilon \circ \langle p, q\rangle &= [\![\lambda x \cdot (\lambda y \cdot yK(yF))[c_p x, c_q x]]\!] \\ &= [\![\lambda x \cdot c_p x(c_q x)]\!] \\ &= [\![\lambda x \cdot xK(xF)]\!] = \varepsilon. \\ \Lambda(\varepsilon) \circ \Lambda(a) &= [\![\lambda x \cdot (\lambda yz \cdot c_\varepsilon[y, z])(\lambda y \cdot c_a[x, y])]\!] \\ &= [\![\lambda x \cdot (\lambda yz \cdot yz)(\lambda y \cdot c_a[x, y])]\!] \\ &= [\![\lambda x \cdot \lambda z \cdot c_a[x, z]]\!] = \Lambda(a). \\ \varepsilon \circ \langle \Lambda(a) \circ p, q\rangle &= [\![\lambda x \cdot c_{\Lambda(a)\circ p} x(c_q x)]\!] \\ &= [\![\lambda x \cdot (\lambda yz \cdot c_a[y, z])(xK)(xF)]\!] \\ &= [\![\lambda x \cdot c_a[xK, xF]]\!] \\ &= [\![\lambda x \cdot c_a(c_{\langle p,q\rangle} x)]\!] = a \circ \langle p, q\rangle. \end{aligned}$$

$$\begin{aligned} A(\varepsilon \circ \langle a \circ p, q\rangle) &= [\![\lambda xy \cdot (\lambda z \cdot zK(zF))[c_a(c_p[x,y]), c_q[x,y]]]\!] \\ &= [\![\lambda xy \cdot c_a xy]\!] \\ &= [\![\lambda x \cdot (\lambda zy \cdot zy)(c_a x)]\!] \\ &= [\![\lambda zy \cdot zy]\!] \circ a = A(\varepsilon) \circ a. \quad \blacksquare \end{aligned}$$

Comparing Definitions 4.1 and 6.1 we see, that $C_{\mathfrak{M}} \equiv C_{M\mathfrak{M}}$. Furthermore we see that the definitions in Lemma 4.2 are consistent with the constructions in Theorem 6.5($\Leftarrow$).

Now that we have an algebraically simple form for the category $C_{\mathfrak{M}}$ we are working with, it is interesting to see whether we can simplify and uniformize the corresponding interpretation. In order to do this we first define a variant of the interpretation in an arbitrary (C, U, i, j) as in Section 3, that suits our purposes.

To state the definition of interpretation, we need a "permutationoperator" $\$_n$:

7.3. Definition. Let $A \in \mathrm{Ob}(C)$.

(i) U_n^A is defined by

$$U_0^A = U \times A, \qquad U_{n+1}^A = U \times U_n^A.$$

(ii) $\$_n^A \colon U_n^A \to U_n^A$ is defined by

$$\begin{aligned} \$_0^A &= \langle p_2, p_2 \circ p_1\rangle, \\ \$_{n+1}^A &= \langle p_1 \circ p_1, \$_n^A \circ \langle p_2 \circ p_1, p_2\rangle\rangle. \end{aligned}$$

It immediately follows that

$$\$_n^A \circ \langle]f_0,..., f_n, g], h\rangle =]f_0,..., f_{n-1}, h, g].$$

Now we proceed by defining $[\![M]\!]^A$ for $A \in \mathrm{Ob}(C)$. $[\![M]\!]^A \colon U_n^A \to U$, where

$$n = \max\{i \mid v_i \text{ occurs (free or bound) in } M\}.$$

(Convention $\max \varnothing = -1$.)

For notational simplicity we suppress mention of A. So in the next definition all $[\![M]\!]$'s should be correctly "typed."

7.4. Definition. $[\![M]\!]$ is defined by induction on M

$$\begin{aligned} [\![v_n]\!] &= p_1 \circ p_2^n, \\ [\![c_a]\!] &= a \circ !, \end{aligned}$$

$$[\![MN]\!] = [\![M]\!] \cdot [\![N]\!],$$

$$[\![\lambda v_n \cdot M]\!] = i \circ \Lambda([\![M]\!] \circ \$_n).$$

7.5. FACT. *We have the following correspondence of this new and the old interpretation:*

Let $|\Delta| \supseteq FV(M)$, *then* $[\![M]\!]_\Delta = [\![M]\!]^T \circ \mathfrak{L}^\Delta_M$. *Here* $\mathfrak{L}^\Delta_M =]\pi^\Delta_{v_0}, \pi^\Delta_{v_1}, ..., \pi^\Delta_{v_n}, !]$ *with* $n = \max\{i \mid v_i \in \mathrm{Var}(M)\}$ *and* $\pi^\Delta_{v_i}$ *is defined to be an arbitrary map if* $v_i \notin \Delta$.

7.6. *Remark.* In the case of a CCM we have the interesting situation that $[\![M]\!]$ really does not depend on the chosen object A. This means that in this case Definition 7.4 is a completely rigorous definition without paying any attention to the object A.

8. CONCLUDING REMARKS

As we have seen there are essentially three different ways to look at a λ-calculus model. Any of these has its own advantages and disadvantages.

We have the environmental structures, such as pseudomodels and models, which are very convenient when considering "termmodels." On the other hand the definition is quite syntactical.

In the second place we have the algebraic models: λ-algebras and λ-models. These models bring out the first order character of λ-calculus and its relation to combinatory logic very clearly. But in general it is extremely hard to show directly that a certain structure satisfies the Curry-axioms for a λ-algebra. Up till now there is no good explanation for the particular forms of these axioms.

Third, we considered categorical models. There are two good reasons for considering these: first, λ-calculus is a theory of functions and categories model some generalized ideas about functions; second, λ-calculus has an intensional character which is a feature of categories too. In general the "modeldefinition" is quite technical and complex, but in the special case of concrete categories the definitions reduce considerably as will be illustrated below. The categorical approach has been shown fruitful for the construction of the mathematical models, known up till now.

Now let us consider the concrete case.

8.1. DEFINITION. A ccc C is called *strictly concrete* if there exists a functor $F: C \to \mathrm{Set}$ such that

(i) F is faithful.

(ii) F preserves the terminal object, products and projections.

(iii) For all A, $B \in \mathrm{Ob}(C)$:

$$F(B^A) \subseteq F(B)^{F(A)}$$
$$F(\mathrm{ev}_{AB}) = \mathrm{ev}_{FA,FB} \upharpoonright F(B^A) \times F(A).$$

(iv) F is full on every $\mathrm{Hom}_C(T, A)$.

Note that we have $F(\Lambda(f)) = \Lambda(F(f))$. Now let C be strictly concrete via F. Let (U, i, j) be a special object in C. Define $\hat{U} = F(U)$, $\square = F(i)$

$$a \cdot b = F(j)(a)(b) \qquad \text{for} \quad a, b \in \hat{U}.$$

Let $[\![M]\!]^F_\rho = F([\![M]\!]_{\hat{\rho}})(*) \in \hat{U}$, where $\mathscr{T} = \{*\}$ in Set, and $F(\hat{\rho}(x))(*) = \rho(x)$, ρ: Vars $\to \hat{U}$. Because of 8.1(i), (iv) this new interpretation is equivalent to the original one. But now we have the following easy inductive definition for $[\![M]\!]^F_\rho$.

8.2. THEOREM. *In the concrete case we have*

(i) $[\![v_n]\!]^F_\rho = \rho(v_n)$

(ii) $[\![c_a]\!]^F_\rho = a$

(iii) $[\![MN]\!]^F_\rho = [\![M]\!]^F_\rho \cdot [\![N]\!]^F_\rho$

(iv) $[\![\lambda v_n \cdot M]\!]^F_\rho = \square(\boldsymbol{\lambda} d \cdot [\![M]\!]^F_{\rho(v_n/d)})$.

Proof. Easy calculations; let us do (iv):

Write $\hat{d}$ for the element in $\mathrm{Hom}_C(T, U)$ satisfying $F(\hat{d})(*) = d$, where $d \in \hat{U}$.

$$\begin{aligned}
[\![\lambda v_n \cdot M]\!]^F_\rho &= F(i \circ \Lambda([\![M]\!]_{\Delta; v_n}) \circ \hat{\rho}^\Delta)(*) \\
&= \square(\Lambda(F([\![M]\!]_{\Delta; v_n}))(\rho^\Delta)) \\
&= \square(\boldsymbol{\lambda} d \cdot F([\![M]\!]_{\Delta; v_n})(\rho(v_n/d)^{\Delta; v_n})) \\
&= \square(\boldsymbol{\lambda} d \cdot F([\![M]\!]_{\Delta; v_n} \circ \hat{\rho}(v_n/\hat{d})^{\Delta; v_n})(*)) \\
&= \square(\boldsymbol{\lambda} d \cdot F([\![M]\!]_{\hat{\rho}(v_n/\hat{d})})(*)) \\
&= \square(\boldsymbol{\lambda} d \cdot [\![M]\!]^F \rho(v_n/d)). \quad \blacksquare
\end{aligned}$$

8.3. EXAMPLES. Consider the category CPO of complete partial orders. This is a strictly concrete category via the usual forgetful functor. In this category there are several interesting objects

(i) $\mathbb{P}\omega = \langle P\omega, \text{graph}, \text{fun} \rangle$, the graph-model, see Scott (1976).

(ii) $\mathbb{D}_\infty = \langle D_\infty, \phi, \psi \rangle$, see Scott (1972).

(iii) $\mathbb{T}\omega = \langle P\omega^2, \text{graph}, \text{fun} \rangle$, see Plotkin (1978), Barendregt and Longo (1980).

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