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Classical lambda calculus in modern dress

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Recent developments in the categorical foundations of universal algebra have given an impetus to an understanding of the lambda calculus coming from categorical logic: an interpretation is a semi-closed algebraic theory. Scott's representation theorem is then completely natural and leads to a precise Fundamental Theorem showing the essential equivalence between the categorical and more familiar notions.

1. Introduction

The λ-calculus is one of the great discoveries of logic in the 20th century, but the question of its semantics has proved vexed. Barendregt's monumental text (Barendregt 1981) offered a variety of approaches and it is telling that under the influence of Scott (1980) the treatment was largely rewritten for the revised edition (Barendregt 1984). I offer a completely new approach to the semantics. I believe that it shows that the λ-calculus is a simple natural object of mathematical study.

The definition explained here is that an interpretation of the lambda calculus is an algebraic theory equipped with semi-closed structure: I call such a structure a λ-theory. In discussing semantics, I use the neutral term 'interpretation' in order to respect as far as possible established usage of the terms 'λ-algebra' and 'λ-model.' One benefit of the view which I propose is that the inductive definition of an interpretation is done once and for all in the abstract setting: there is no further need for it in individual cases. That is characteristic of categorical logic. An algebraic theory is admittedly a slightly more complicated mathematical structure than one expects in semantics, but interpreting abstraction involves handling free variables and the algebraic theory makes them explicit. Moreover the definition can be used effectively in giving examples: common interpretations are naturally presented as semi-closed algebraic theories so there is a real gain. A further benefit of making the notion of theory central is that it leads quickly to fundamental results. Specifically I show that Scott's interpretation by reflexive objects in cartesian closed categories arises naturally.

The initial λ-theory Λ has a presentation by the syntax of the λ-calculus. As an algebraic theory it has algebras, and a Λ-algebra is a clean version of a valuation interpretation or environment model of the λ-calculus, bypassing the issue of weak extensionality. The initial Λ-algebra is the closed term interpretation traditionally written Λ_0. Remarkably every λ-theory is the theory of extensions of some Λ-algebra. This gives a very tight equivalence between the categories of λ-theories and that of Λ-algebras. I want to call this the Fundamental Theorem of the Lambda Calculus.

I started writing this paper with a section on Combinatory Algebra and λ-calculus, justifying the λ-theory definition. Though there is a close link with the Fundamental Theorem, I have cut that material for reasons of space. I hope to present an account soon. Hints as to what is involved are in Freyd (1989) and especially (Selinger 2002), where what I call the λ-theory of a Λ-algebra is treated albeit from a completely different point of view. There are other omissions. In particular, I am sorry not to make more links with the universal algebra aspects of the λ-calculus pioneered by Antonino Salibra and his co-workers, see Manzonetto and Salibra (2006).

The notion of interpretation of the lambda calculus which I propose derives from an approach to semantics pioneered long ago by Andrew Pitts: one systematically takes contexts seriously. This had a major unrecognised influence on early thinking about categorical models for Linear Logic (Benton *et al.* 1993) and the idea to consider theories with structure dates to that era. However, only more recently have I realised how well that works for the λ-calculus. The multi-category theory perspective now seems compelling in the light of the foundations for notions of algebra (Hyland 2014) provided by Kleisli Bicategories (Fiore *et al.* 1999). I refrain from discussing that background and also say nothing about the related theory of variable binding initiated in Fiori *et al.* (1999) and extended variously for example in Tanaka and Power (2006). While this is important, it is unnecessary for a first appreciation of the λ-calculus.

This paper was refereed in an intelligent and helpful way. I hope that I have profited from advice to clarify the exposition and to give more precise details. Even so my account is very abstract and more condensed than I would have liked. I have written it to honour Corrado Böhm, the creator of the technique (Böhm 1968) which most influenced my understanding of the subject. I hope that readers will find it suggestive for further research. In this foundational treatment, I cannot get as far as Böhm's Theorem but I close the paper with suggestions for the future including reflections on Böhm's legacy.

2. Algebraic theories

2.1. *Algebraic theories as cartesian operads*

An algebraic theory is a theory of equality on terms. A clean mathematical expression of the idea is via multicategories or operads, so abstractly via monads in some Kleisli bicategory (Fiore *et al.* 1999). I only need the concrete version of a one object cartesian multicategory or cartesian operad. Write **Sets** for the category of sets and **F** for a standard skeleton of finite sets.

Definition 2.1. An *algebraic theory*. $\mathcal{T}$ is first a functor $\mathcal{T} : \mathbf{F} \to \mathbf{Sets}$: so we have sets $\mathcal{T}(n)$ of n-ary multimaps with variable renamings. In addition, $\mathcal{T}$ is equipped with projections $\mathbf{pr}_1, \dots, \mathbf{pr}_n \in \mathcal{T}(n)$ including as special case the identity $\mathbf{id} \in \mathcal{T}(1)$. Finally, there are compositions $\mathcal{T}(n) \times \mathcal{T}(m)^n \to \mathcal{T}(m)$ which are associative, unital, compatible with projections and natural in n and m (or dinatural in m). A *map* $F : \mathcal{S} \to \mathcal{T}$

of algebraic theories is a natural transformation with components $F_n : \mathcal{S}(n) \to \mathcal{T}(n)$ preserving projections and composition.

Clearly from the definition, we get a category of algebraic theories. It is easily seen to be locally finitely presentable but I shall not need that fact.

I have defined an algebraic theory in modern style as what could be called a cartesian operad. Experts will be aware of other formulations in the operads tradition, and I comment later on the relation with Lawvere theories and monads. But in essence, the notion of algebraic theory is the same as that of abstract clone (Taylor 1993) familiar in universal algebra. Abstract clones appear without the name already in Taylor (1973). The functorial perspective is simply a clean way to handle variable reindexing. Note in particular the elements of $\mathcal{T}(0)$ which are the constants of the theory. (These are often omitted in clone theory.) We have unique maps $0 \to n$, so we can identify the constants within each $\mathcal{T}(n)$. I shall take that kind of thing for granted hereafter.

I make a brief remark on the syntactic point of view. Writing $\Gamma \vdash t$ for t a term with variable declaration Γ, the definition of algebraic theory encapsulates the basic principles of term formation

$$\frac{}{\Gamma \vdash x} \qquad\qquad \frac{\Gamma \vdash t \quad \Delta \vdash s_1, \dots \ \Delta \vdash s_n}{\Delta \vdash t(\mathbf{s})},$$

with x declared in Γ and $t(\mathbf{s}) = t(s_1, \dots s_n)$ the result of substituting the string of terms $\mathbf{s}$ in t. An algebraic theory or abstract clone can be presented by allowing equations between terms. The basic syntactic rule of equality, the substitution of equals for equals, is implicit. I imagine that readers will have no problem with algebraic theories arising thus from concrete syntax. However, it is good to be aware of another source of examples. Suppose that $\mathcal{C}$ is a category with products and X an object of $\mathcal{C}$. Then, using the evident composition $\mathcal{C}(X^n, X) \times \mathcal{C}(X^m, X)^n \to \mathcal{C}(X^m, X)$, one has an algebraic theory, the *endomorphism theory* of X, with underlying functor $\mathcal{C}(X^n, X)$.

2.2. *Algebras for theories*

Very general notions of interpretation for an algebraic theory are supported by Fiore *et al.* (1999) but we only need the basic interpretation in the category **Sets** of sets.

Definition 2.2. An algebra for an algebraic theory $\mathcal{T}$ is a set A with an associative unital action $\mathcal{T}(n) \times A^n \to A$ of $\mathcal{T}$, natural in n. If A and B are $\mathcal{T}$-algebras then a homomorphism from A to B consists of a map $f : A \to B$ respecting the actions in the sense that the evident diagram commutes.

Concretely, one can take unital associative to mean first that the projections $\mathbf{pr}_i \in \mathcal{T}(n)$ act as projections and that the two evident maps $\mathcal{T}(n) \times \mathcal{T}(m)^n \times A^m \to A$ are equal.

From the definition, we get a category $\mathcal{T}$-algebras which I write $\mathrm{Alg}(\mathcal{T})$. I give some background on this category. Further details can be extracted from Adámek *et al.* (2011). First note that the compositions $\mathcal{T}(m) \times \mathcal{T}(n)^m \to \mathcal{T}(n)$ give each $\mathcal{T}(n)$ the structure of a $\mathcal{T}$-algebra.

Proposition 2.3. The algebra $\mathcal{T}(n)$ is the free $\mathcal{T}$-algebra on n generators; that is, for $A \in \mathrm{Alg}(\mathcal{T})$, we have $\mathrm{Alg}(\mathcal{T})(\mathcal{T}(n), A) \cong A^n$, natural in A and n.

Proof. This is a form of the Yoneda Lemma: the coordinates in A^n come from the n projections $\mathbf{pr}_1, \ldots, \mathbf{pr}_n \in \mathcal{T}(n)$. □

Corollary 2.4. $\mathcal{T}(0)$ is the initial algebra and $\mathcal{T}(n) + \mathcal{T}(m) \cong \mathcal{T}(n+m)$ gives binary coproducts.

The main closure properties of $\mathrm{Alg}(\mathcal{T})$ follow as in Adámek *et al.* (2011).

Proposition 2.5. The category $\mathrm{Alg}(\mathcal{T})$ of $\mathcal{T}$-algebras is complete and cocomplete.

Proof. Completeness is evident as the forgetful functor to **Sets** creates limits. It also creates sifted colimits so it suffices to define finite coproducts. The corollary above gives the initial algebra and also helps justify the description of the coproduct $A+B$ as a sifted colimit $A + B = \int^{m,n} \mathrm{Alg}(\mathcal{T})(\mathcal{T}(m), A) \times \mathrm{Alg}(\mathcal{T})(\mathcal{T}(n), B) \times \mathcal{T}(m+n)$. □

It is clear that if $F : \mathcal{S} \to \mathcal{T}$ is a map of algebraic theories then composition gives a functor $F^* : \mathrm{Alg}(\mathcal{T}) \to \mathrm{Alg}(\mathcal{S})$. I shall not need its left adjoint, though there are hints of it in what follows. For each n there is a map of $\mathcal{S}$-algebras $\mathcal{S}(n) \to F^*\mathcal{T}(n)$ with underlying map F_n carrying the n generators to the n generators. Let B be a $\mathcal{T}$-algebra. It is easy to see that composing the action of F^* with the map induced by $\mathcal{S}(n) \to F^*\mathcal{T}(n)$ in

$$B^n \cong \mathbf{Alg}(\mathcal{T})(\mathcal{T}(n), B) \longrightarrow \mathbf{Alg}(\mathcal{S})(F^*\mathcal{T}(n), F^*B) \longrightarrow \mathbf{Alg}(\mathcal{S})(\mathcal{S}(n), F^*B) \cong B^n,$$

gives the identity on the set B^n.

Proposition 2.6. Suppose that the functor $F^* : \mathrm{Alg}(\mathcal{T}) \to \mathrm{Alg}(\mathcal{S})$ is an equivalence of categories. Then, F is an isomorphism of algebraic theories.

Proof. The first arrow above is an isomorphism as F^* is full and faithful. F^* is essentially surjective on objects so we can put any $\mathcal{S}$-algebra A in place of F^*B. Hence, $\mathbf{Alg}(\mathcal{S})(F^*\mathcal{T}(n), A) \cong \mathbf{Alg}(\mathcal{S})(\mathcal{S}(n), A)$ and F^* full and faithful gives naturality in A. Thus the $\mathcal{S}(n) \to F^*\mathcal{T}(n)$ are isomorphisms. Hence, so are the $F_n : \mathcal{S}(n) \to \mathcal{T}(n)$, that is, F is an isomorphism. □

Note that in this situation F^* is in fact necessarily an isomorphism of categories.

Returning now to coproducts, $A[n] = A + \mathcal{T}(n)$ gives the free extension of a $\mathcal{T}$-algebra A by n indeterminates. Let A be a model of an algebraic theory $\mathcal{S}$. There is an algebraic theory $\mathcal{S}_A$ whose algebras are $\mathcal{S}$-algebras equipped with an $\mathcal{S}$-algebra map from A. As sets $\mathcal{S}_A(n) = A[n] = A + \mathcal{S}(n)$. Giving the structure of an algebraic theory is routine. (In case $A = \mathcal{S}(p)$, Corollary 2.4 gives the simple description $\mathcal{S}_{\mathcal{S}(p)}(n) = \mathcal{S}(n+p)$ with action not affecting the parameters in p.) We have an evident map $\mathcal{S} \to \mathcal{S}_A$ inducing a functor $\mathrm{Alg}(\mathcal{S}_A) \to \mathrm{Alg}(\mathcal{S})$ and an isomorphism between A and $\mathcal{S}_A(0)$ regarded as an $\mathcal{S}$-algebra. A concrete way to think about $\mathcal{S}_A$ is that it is obtained from $\mathcal{S}$ by adding constants for the new elements of A and equations coming from A but no other equations.

Let $F : \mathcal{S} \to \mathcal{T}$ be a map of algebraic theories. Set $A = F^*\mathcal{T}(0)$ and observe that F factors through $\mathcal{S} \to \mathcal{S}_A$ via a comparison $\mathcal{S}_A \to \mathcal{T}$. If the latter is an isomorphism $\mathcal{T}$ is *the theory of extensions of a model of* $\mathcal{S}$. Proposition 2.6 implies the following.

Proposition 2.7. $F : \mathcal{S} \to \mathcal{T}$ is a theory of extensions of a model if and only if $\mathcal{S}_A \to \mathcal{T}$ induces an equivalence between $\mathrm{Alg}(\mathcal{T})$ and the coslice category $A/\mathrm{Alg}(\mathcal{S})$.

Whenever $F : \mathcal{S} \to \mathcal{T}$ is a map of theories and B a $\mathcal{T}$-algebra, we get a map $\mathcal{S}_{F^*B} \to \mathcal{T}_B$ of theories. If $\mathcal{T}$ is a theory of extensions of a model of $\mathcal{S}$, then this will be an isomorphism.

2.3. *The presheaf topos*

A $\mathcal{T}$-algebra is a set A equipped with an action of $\mathcal{T}$ on the left. But $\mathcal{T}$ can also act on the right. In the operad literature one talks of a module: I prefer to say presheaf.

Definition 2.8. A *presheaf* X over an algebraic theory $\mathcal{T}$ is a functor $X : \mathbf{F} \to \mathbf{Sets}$ equipped with an action $X(m) \times \mathcal{T}(n)^m \to X(n)$ compatible with the operations of $\mathcal{T}$. A *map of presheaves* is a functor commuting with the action of $\mathcal{T}$.

We get a category of presheaves over $\mathcal{T}$ which I write $P\mathcal{T}$. Evidently, $\mathcal{T}$ is itself a presheaf. It is natural to call it the universal presheaf though generally $P\mathcal{T}$ is not the classifying topos for $\mathcal{T}$. It is easy to see that $P\mathcal{T}$ is a category with products (indeed limits) defined pointwise. So, we have the finite powers $\mathcal{T}^m$ of the universal with $\mathcal{T}^m(n) = \mathcal{T}(n)^m$. The Yoneda Lemma is then the following.

Proposition 2.9. We have an isomorphism $P\mathcal{T}(\mathcal{T}^m, X) \cong X(m)$ natural in $X \in P\mathcal{T}$.

For presheaves, a Yoneda Lemma should lead to a Yoneda embedding. This takes a particularly vivid form for the category $P\mathcal{T}$. The composition $\mathcal{T}(n) \times \mathcal{T}(m)^n \to \mathcal{T}(m)$ corresponds by transpose to a map $\mathcal{T}(n) \to (\mathcal{T}(m)^n \Rightarrow \mathcal{T}(m))$ whose image is clearly natural in m. So, we have $\mathcal{T}(n) \to P\mathcal{T}(\mathcal{T}^n, \mathcal{T})$ and the Yoneda Lemma says in particular that this is an isomorphism. But returning to the composition $\mathcal{T}(n) \times \mathcal{T}(m)^n \to \mathcal{T}(m)$, we check directly that the diagram

$$\begin{tikzcd}
\mathcal{T}(n) \times \mathcal{T}(m)^n \arrow[r] \arrow[d, "\cong"'] & \mathcal{T}(m) \arrow[d, "\cong"] \\
P\mathcal{T}(\mathcal{T}(n), \mathcal{T}) \times P\mathcal{T}(\mathcal{T}(m), \mathcal{T})^n \arrow[r] & P\mathcal{T}(\mathcal{T}(m), \mathcal{T})
\end{tikzcd}$$

commutes. The result is the following form of an embedding theorem.

Proposition 2.10. The Yoneda Lemma induces an isomorphism between an algebraic theory $\mathcal{T}$ and the endomorphism theory of the universal object $\mathcal{T}$ in $P\mathcal{T}$.

Proposition 2.9 shows inter alia that the products of the universal are dense, and so familiar arguments allow one to deduce more structure.

Proposition 2.11. $P\mathcal{T}$ is a topos; in particular it is locally cartesian closed.

For the λ-calculus, we are interested in function spaces. The following is essentially an old observation of Lawvere's, the proof of which is an easy computation.

Proposition 2.12. For any presheaf X, the function space $\mathcal{T}^p \Rightarrow X$ is given by the presheaf $(\mathcal{T}^p \Rightarrow X)(m) = X(m+p)$ with action $X(m+p) \times \mathcal{T}(n)^m \to X(n+p)$ leaving the parameters p undisturbed.

The function spaces $\mathcal{T}^p \Rightarrow \mathcal{T}$ with $(\mathcal{T}^p \Rightarrow \mathcal{T})(m) = \mathcal{T}(m+p)$ will be of particular interest for the λ-calculus.

2.4. *Monads and Lawvere theories*

In this paper, I use a multi-category theory approach to algebraic theories. I regard it as the fundamental one and it is particularly suited to the λ-calculus. However, for those who may prefer them I sketch the more traditional categorical approaches to algebra.

An algebraic theory $\mathcal{T}$ induces a monad T on **Sets** whose functor part is given by the coend formula $T(A) = \int^n \mathcal{T}(n) \times A^n$. Concretely, $T(A)$ is the set of terms from $\mathcal{T}$ with constants from A replacing variables. The unit and multiplication of the monad correspond to constants and substitution. Conversely from a monad T we get an algebraic theory $\mathcal{T}$ where $\mathcal{T}(n) = T(n)$, since terms in n variables form the underlying set of the free algebra on a set of size n. The monads which arise from algebraic theories are exactly the finitary monads and there is an equivalence of categories between algebraic theories and finitary monads. The notions of algebra correspond. On the other hand, the presheaf topos which is fundamental to my approach to the λ-calculus is less easy to handle from the monad point of view.

Lawvere theories introduced in Lawvere (1963) provide the other categorical approach to algebra. The equivalence with finitary monads was evident early but Lawvere theories have been comparatively neglected. That seems a mistake. Hyland and Power (2007) compares the two approaches with applications to computer science in mind. With that background, it is clear that the Lawvere Theory corresponding to an algebraic theory is essentially (I suppress the identity on objects functor) the free category $\mathbf{T}$ with products generated by $\mathcal{T}$: one has simply $\mathbf{T}(n,m) = \mathcal{T}(n)^m$ and the categorical composition is then easy. Algebras are given by product-preserving functors $\mathbf{T} \to \mathbf{Sets}$ and almost by definition this notion coincides with the one above. For the presheaf categories, there is the following easy consequence of our Yoneda Lemma, Proposition 2.9.

Proposition 2.13. The categories $P\mathcal{T}$ and $P\mathbf{T} = [\mathbf{T}^{\mathrm{op}}, \mathbf{Sets}]$ are equivalent.

Thus, one could very readily rewrite this paper from the Lawvere theory perspective.

3. The lambda calculus

3.1. *λ-theories*

The underlying philosophy is to take seriously the basic syntax of term formation in context first made prominent in Martin-Löf's treatment of Type Theory. The term formation rules are of form

$$\frac{\Gamma \vdash t \in a \Rightarrow b \quad \Gamma \vdash u \in a}{\Gamma \vdash tu \in b} \qquad \frac{\Gamma,\ x \in a \vdash s \in b}{\Gamma \vdash \lambda x.s \in a \Rightarrow b},$$

while the computation rule is $(\lambda x.s)u = s[u/x]$. The rules force one to handle terms in context and it is best to do that directly. We can capture the rules for λ-calculus by saying that an interpretation is a cartesian multicategory with semi-closed structure. The pure lambda calculus corresponds to the one-object version.

Definition 3.1. To equip an algebraic theory $\mathcal{L}$ with *semi-closed structure* is to give retractions $\mathcal{L}(n+1) \lhd \mathcal{L}(n)$, $\rho : \mathcal{L}(n) \to \mathcal{L}(n+1)$ and $\lambda : \mathcal{L}(n+1) \to \mathcal{L}(n)$ the *retraction* and *section*, natural in n and compatible with the actions $\mathcal{L}(m) \times \mathcal{L}(n)^m \to \mathcal{L}(n)$ and $\mathcal{L}(m+1) \times \mathcal{L}(n)^m \to \mathcal{L}(n+1)$. A *$\lambda$-theory* is an algebraic theory $\mathcal{L}$ equipped with semi-closed structure. Let $\mathcal{L}$ and $\mathcal{M}$ be λ-theories. A *map $\mathcal{L} \to \mathcal{M}$ of λ-theories* is a map of algebraic theories which commutes with retraction ρ and section λ.

We get a category of λ-theories. It is clearly locally finitely presentable; but by the Fundamental Theorem 4.11 it is in fact equivalent to a category of algebras.

To start with one should understand the definition of λ-theory concretely. The image of the identity $\mathbf{id} \in \mathcal{L}(1)$ under $\rho : \mathcal{L}(1) \to \mathcal{L}(2)$ is a binary operation $\mathbf{app} \in \mathcal{L}(2)$ of application, traditionally denoted by concatenation $\mathbf{app}(x, y) = xy$. Naturality implies that the retraction $\rho : \mathcal{L}(n) \to \mathcal{L}(n+1)$ is given by $a \to az$ where I adopt now and henceforth the convention to use z as the new extra variable in this and similar cases. Starting with $\mathbf{id} \in \mathcal{L}(1)$ and retracting n times gives iterated applications $\mathbf{app}_n \in \mathcal{L}(n+1)$ where we can write suggestively $\mathbf{app}_{n+1} = \mathbf{app}(\mathbf{app}_n, z)$ or spelling out the variables $\mathbf{app}_{n+1}(x, z_1, \ldots, z_{n+1}) = \mathbf{app}(\mathbf{app}_n(x, z_1, \ldots, z_n), z_{n+1})$. By convention the concatenation notation for application associates so that we could write $xz_1 \ldots z_n$.

The naturality of the section $\lambda : \mathcal{L}(n+1) \to \mathcal{L}(n)$ for λ-abstraction is more subtle. The critical fact is that $\mathcal{L}(2) \to \mathcal{L}(1)$ does not generally take application $\mathbf{app}$ to the identity $\mathbf{id}$. Rather it goes to an element $\mathbf{1}z$ where $\mathbf{1} \in \mathcal{L}(0)$ is the image of $\mathbf{app}$ in $\mathcal{L}(0)$. Naturality then gives the following. Suppose $s \in \mathcal{L}(n+1)$ has image $\lambda s \in \mathcal{L}(n)$. Then we have $s = (\lambda s)z$ and $\lambda s = \mathbf{1}(\lambda s)$, and these conditions determine λs. So, while maps of λ-theories are not mere maps of the algebraic theories and the preservation of the semi-closed structure is essential, we have the following.

Proposition 3.2. Suppose $\mathcal{L}$ and $\mathcal{M}$ are λ-theories. If a map $\mathcal{L} \to \mathcal{M}$ of algebraic theories preserves the binary operation $\mathbf{app}$ and constant $\mathbf{1}$, then it is a map of λ-theories.

Of course one can iterate the section. The following will prove essential.

Proposition 3.3. Suppose that $s \in \mathcal{L}(n)$ has image $\hat{s} = \lambda^n s \in \mathcal{L}(0)$. Then, s can be recovered from $\hat{s}$ and iterated application via the identity $s = \mathbf{app}_n(\lambda^n s, z_1, \ldots, z_n)$.

Unsurprisingly, the characterisation of λs extends. Rewriting the Proposition 3.3, we have $s = (\lambda^n s)z_1 \ldots z_n$, and we also have $\mathbf{1}_n(\lambda^n s) = \lambda^n s$ where $\mathbf{1}_n = \lambda^{n+1}(\mathbf{app}_n)$ is the image in $\mathcal{L}(0)$ of iterated application n arguments.

3.2. *Interpreting the λ-calculus*

I briefly sketch the interpretation of the lambda calculus in a λ-theory. For clarity, I shall here use semantic brackets $[\![t]\!]$ to indicate the interpretation of a term t, though in accord with general mathematical practice I shall dispense with them as soon as I prudently can. Let $\mathcal{L}$ be a λ-theory. Suppose that $\Gamma \vdash t$ is a term t in context Γ of length n. Then t has an interpretation $[\![t]\!] \in \mathcal{L}(n)$, which is defined inductively as follows. For the ith variable $\Gamma \vdash x_i$, let $[\![x_i]\!] = \mathbf{pr}_i$, the ith projection. Application tu of terms is interpreted by the application so that $[\![tu]\!] = \mathbf{app}([\![t]\!], [\![u]\!])$; and λ-abstraction of terms is defined using the section $\lambda : \mathcal{L}(n+1) \to \mathcal{L}(n)$ so that $[\![\lambda z.r]\!] = \lambda([\![r]\!])$. Of course the point is not the definition but the fact that it works in the sense that β-equality is satisfied. The main point of the proof concerns substitution.

Lemma 3.4. Suppose $\Gamma, z \vdash r$ and $\Gamma \vdash s$, so we have interpretations $[\![r]\!] \in \mathcal{L}(n+1)$ and $[\![s]\!] \in \mathcal{L}(n)$. Then, $\Gamma \vdash r[s/z]$ has interpretation $[\![r[s/z]]\!] = [\![r]\!](\mathbf{pr}_1, \ldots, \mathbf{pr}_n, [\![s]\!])$.

Proof. By induction on the structure of r. The crucial λ-abstraction step is just the compatibility of the sections $\lambda : \mathcal{L}(n+1) \to \mathcal{L}(n)$ with the action. $\square$

There is a series of results which follow by easy induction on the structure of λ-terms.

Proposition 3.5. The interpretation of the λ-calculus in a λ-theory respects β-equality in the sense that $[\![(\lambda z.s)u]\!] = [\![s[u/z]]\!]$.

That provides moral support for the definition of λ-theory. Next we consider maps of λ-theories.

Proposition 3.6. Suppose that $F : \mathcal{L} \to \mathcal{M}$ is a map of λ-theories. Then F preserves the interpretation of λ-terms: for every s we have $F[\![s]\!]_{\mathcal{L}} = [\![s]\!]_{\mathcal{M}}$.

Next, we show that the λ-calculus presents the initial λ-theory. Let $\Lambda(n)$ be the terms of the λ-calculus in context of n variables, factored out by β-equality: write $[s] \in \Lambda(n)$ for the equivalence class of $\Gamma \vdash s$, the terms s in context Γ of length n. Identities and projections in Λ are evident and composition is given by substitution, so we have an algebraic theory. The retraction of $\Lambda(n)$ onto $\Lambda(n+1)$ is $t \in \Lambda(n) \mapsto t.z \in \Lambda(n+1)$. The section is $s \in \Lambda(n+1) \mapsto \lambda z.s \in \Lambda(n)$. We do indeed have a retraction because $(\lambda z.s)z = s$ in the λ-calculus. Compatibility with the action is trivial in the case of the retraction, while for the section it is the basic syntactic lemma $(\lambda z.s)[r/x] = \lambda z(s[r/x])$. So, we get a λ-theory Λ. The last of our results proved by structural induction is the following.

Proposition 3.7. The interpretation $[\![s]\!] \in \Lambda(n)$ of a λ-term $\Gamma \vdash s$ in context is given by its equivalence class $[s]$.

From this essentially routine series of propositions, one gets the following basic result.

Theorem 3.8. The Λ is the initial λ-theory.

Henceforth, I shall use standard λ-calculus notation to define elements in λ-theories. We have already seen $\mathbf{1} = \lambda xy.xy$ (and more generally $\mathbf{1}_n = \lambda z_1 \ldots z_{n+1}.z_1 \ldots z_{n+1}$).

Note that the definition of λ-theory involves an unfamiliar use of the notion of algebraic theory. The initial algebraic theory is the pure theory of equality and usually we extend that with constants and function symbols and equation between them. Here, nothing of that kind is involved yet what we get is highly nontrivial. It is perhaps just worth drawing attention to a tension with usual examples of algebra. Operations like $x \mapsto x+1$ generally have no fixed points. But the λ-calculus has fixed point operators. So for example, the only semi-closed theory of rings is the terminal theory with $0 = 1$.

3.3. *Extensionality*

I turn aside for a moment to comment on two separate issues termed extensionality. What is called weak extensionality caused much concern in early developments; note the faintly apologetic remark 'In spite of not being weakly extensional λ-algebras are worth studying' on page 87 of Barendregt (1984). The category theoretic reading of weak extensionality is the question whether one has enough points (Scott 1980), and it is the same for λ-theories. An algebraic theory $\mathcal{T}$ *has enough points* just when equality on each $\mathcal{T}(n)$ is reflected in the action $\mathcal{T}(n) \times \mathcal{T}(0)^n \to \mathcal{T}(0)$ on constants. λ-theories with enough points are essentially the Frege Structures of Aczel (1980) but λ-theories may not have enough points: indeed the initial λ-theory does not. One does not need to worry about that. For those who do, there is this general fact.

Proposition 3.9. Any algebraic theory $\mathcal{T}$ or λ-theory $\mathcal{L}$ embeds in an algebraic theory or λ-theory with enough points.

Proof. Take $\mathcal{T}(\omega)$ to be the free algebra on countable many generators, constructed in the obvious way as a direct limit of free algebras $\mathcal{T}(n)$. Then $\mathcal{T}$ embeds in $\mathcal{T}_{\mathcal{T}(\omega)}$, the theory of extensions of $\mathcal{T}(\omega)$, which clearly has enough points. A λ-theory $\mathcal{L}$ embeds in $\mathcal{L}_{\mathcal{L}(\omega)}$, and the λ-theory structure is easy. $\square$

Of course this observation is not new for the λ-calculus: it is essentially in Section 4 of Barendregt and Koymans (1980). A special case is the move from the initial closed term to the open term interpretation.

Another quite different aspect of extensionality is the η-rule $\lambda x.tx = t$ when x is not free in t. This corresponds exactly to the requirement that the semi-closed structure be closed, in the sense that the retractions $\rho : \mathcal{L}(n) \to \mathcal{L}(n+1)$ are isomorphisms. In terms of the analysis above, this is equivalent to application **app** being taken to the identity **id**. The theory outlined in this paper is not much affected by adding this condition but doing so confuses rather than helps. There can be no doubt that it is the basic calculus with just the β-rule which is fundamental.

3.4. *Reflexive objects*

Scott's elegant categorical approach to the semantics of the λ-calculus stands out as clean mathematics. It is not flexible as a definition, but it captures much of importance. The most obvious element is a Representation Theorem. I give a new proof: the λ-theory perspective makes Scott's idea easy to understand. Suppose that $\mathbf{C}$ is a cartesian closed category and U an object of $\mathbf{C}$ equipped with a retraction onto the function space $U^U = U \Rightarrow U$. Set $\mathcal{U}(n) = \mathbf{C}(U^n, U)$. This is automatically an algebraic theory, the endomorphism theory of U. Moreover since $\mathbf{C}(U^n, U^U) \cong \mathbf{C}(U^{n+1}, U)$, we get retractions $\mathcal{U}(n+1) \lhd \mathcal{U}(n)$ manifestly natural in n. So we get a λ-theory $\mathcal{U}$, the *endomorphism λ-theory* of the reflexive object U.

The essence of Scott's wonderful insight was that any λ-theory can be so represented. His proof with discussion of the significance of the result is in Scott (1980). It involves serious coding which we shall not need until Section 4.2.

Theorem 3.10. (Scott's Representation Theorem). Any λ-theory is isomorphic to the endomorphism λ-theory of a reflexive object in a cartesian closed category.

Proof. We prepared for this in Section 2.3. Given $\mathcal{L}$ take the presheaf topos $P\mathcal{L}$ and let U be the universal object $\mathcal{L}$ itself. Proposition 2.12 gives the function space U^U: it is the presheaf $\mathcal{L}(n+1)$. By definition, a λ-theory consists of a retraction of $\mathcal{L}(n)$ onto $\mathcal{L}(n+1)$ and the naturality conditions say that the maps are in $P\mathcal{L}$. So, we have a retraction of U onto U^U and U is a reflexive object. It remains to consider the λ-theory $\mathcal{U}$ obtained from U. We have $\mathcal{U}(n) = P\mathcal{L}(U^n, U) \cong \mathcal{L}(n)$, an isomorphism of algebraic theories by Proposition 2.10, and then evidently an isomorphism of λ-algebras. □

The proof (Scott 1980) in the Curry Festschrift gives the reflexive object in the cartesian closed category of retracts of a λ-algebra. For a λ-theory version, take the monoid $\mathcal{L}(1)$ as a one object category. Any category $\mathbf{C}$ has a category of retracts, its Karoubi envelope or Cauchy completion. The objects are idempotents $e : A \to A$ with $e \circ e = e$ in $\mathbf{C}$; and maps between idempotents $e : A \to A$ and $f : B \to B$ are maps $v : A \to B$ such that $f \circ v \circ e = v$. Composition is inherited from $\mathbf{C}$ and each idempotent e is its own identity. Scott's category $\mathbf{R}$ of retracts is the Cauchy completion of $\mathcal{L}(1)$. Scott showed by explicit calculation that it is cartesian closed. I shall use the presheaf category $P\mathcal{L}$ to give a new proof of this, see Corollary 3.13.

3.5. *The Taylor fibration*

In his PhD thesis (Taylor 1986), Paul Taylor extended Scott's analysis in a very remarkable way. Taylor observed that the category $\mathbf{R}$ of retracts is not just cartesian closed but is relatively cartesian closed. I briefly explain what that means. Taylor gave a notion of a category $\mathcal{C}$ *equipped with display maps* $\mathcal{D}$ with the idea that the display maps model dependent or indexed types for some type theory. This approach is now standard: under the influence of homotopy type theory display maps are often now called fibrations but I shall stick with Taylor's terminology. Basic properties of a type theory are given by closure properties. Taylor required that $\mathcal{D}$ be closed by pullback along all maps, so indexed types are closed under arbitrary substitution; that $\mathcal{D}$ be closed under composition, so that types are closed under indexed sums; and finally that $\mathcal{D}$ contain all terminal projections, a condition which is best thought of as avoiding redundancy. With that in place, Taylor defined a category $\mathcal{C}$ with displays $\mathcal{D}$ to be *relatively cartesian closed* if the display maps are closed by products along display maps.

For each object $E \in \mathbf{R}$, Taylor localised the construction of $\mathbf{R}$. Let $U \in \mathbf{R}$ be the generating object given by the identity idempotent. Then over each $E \in \mathbf{R}$, we have $\Delta_E(U) = (U \times E \to E)$ which is a reflexive object in the slice $\mathbf{R}/E$. Taylor considered the subfibration $\mathbf{R}(E)$ of $\mathbf{R}/E$ consisting of retracts $A \to E$ of $\Delta_E(U)$. His result is that for every $\alpha : F \to E$ in $\mathbf{R}(E)$ the pullback functor $\alpha^* : \mathbf{R}(E) \to \mathbf{R}(F)$, which is evident from the definition of $\mathbf{R}(E)$, comes equipped with a right adjoint $\Pi_\alpha : \mathbf{R}(F) \to \mathbf{R}(E)$ and left adjoint $\Sigma_\alpha : \mathbf{R}(F) \to \mathbf{R}(E)$. That $\mathbf{R}$ is cartesian closed is a simple consequence.

In the spirit of Scott (1980), Taylor simply wrote down the various combinators and calculated to show that they work. I shall give a more abstract proof using the presheaf category $P(\mathcal{L})$ with universal object U. We already saw the retract from U to U^U. We need also a retract from U to $U \times U$. It relies on familiar λ-calculus: we have the section

$$\mathcal{L}(n) \times \mathcal{L}(n) \to \mathcal{L}(n); \quad (a, b) \to \lambda x.xab\,,$$

and the retraction

$$\mathcal{L}(n) \to \mathcal{L}(n) \times \mathcal{L}(n); \qquad c \to (c\mathbf{T}, c\mathbf{F})\,,$$

where $\mathbf{T} = \lambda xy.x$ and $\mathbf{F} = \lambda xy.y$.

Now consider Taylor's fibration over the whole of $P(\mathcal{L})$. For $X \in P(\mathcal{L})$, take $\mathbf{R}(X)$ to be the category of retracts of the reflexive object $\Delta_X(U) = (U \times X \to X)$ in $P(\mathcal{L})/X$.

Theorem 3.11. Take $X \in P(\mathcal{L})$. Let $\alpha : Y \to X$ be in $\mathbf{R}(X)$. Then for any $Q \to Y$ in $\mathbf{R}(Y)$, the standard indexed product $\Pi_\alpha Q \to X$ and standard indexed sum $\Sigma_\alpha Q \to X$ over $X \in P(\mathcal{L})$ both lie in $\mathbf{R}(X)$.

Proof. The aim is to reduce to a special case. First $Q \to Y$ is a retract of $\Delta_Y(U)$, so $\Pi_\alpha Q \to X$ and $\Sigma_\alpha Q \to X$ are retracts of $\Pi_\alpha \Delta_Y(U)$ and $\Sigma_\alpha \Delta_Y(U)$, respectively; so it suffices to show that these lie in $\mathbf{R}(X)$. Now, consider a diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "\alpha"'] & U \times X \arrow[r, "r"] \arrow[d, "q"] & Y \arrow[dl, "\alpha"] \\ & X & \end{tikzcd}$$

displaying $\alpha : Y \to X$ as a retract of $\Delta_X U$. From it, we get maps

$$\Pi_\alpha \Delta_Y U \to \Pi_\alpha \Pi_r \Delta_r \Delta_Y U \cong \Pi_q \Delta_{U \times X} U \to \Pi_q \Pi_i \Delta_i \Delta_{U \times X} U \cong \Pi_\alpha \Delta_Y U$$

$$\Sigma_\alpha \Delta_Y U \cong \Sigma_q \Sigma_i \Delta_i \Delta_{U \times X} U \to \Sigma_q \Delta_{U \times X} U \cong \Sigma_\alpha \Sigma_r \Delta_r \Delta_Y U \to \Sigma_\alpha \Delta_Y U,$$

displaying retracts $\Pi_\alpha \Delta_Y U \lhd \Pi_q \Delta_{U \times X} U$ and $\Sigma_\alpha \Delta_Y U \lhd \Sigma_q \Delta_{U \times X} U$. So, it suffices for products to consider $\Pi_q \Delta_{U \times X} U \cong \Delta_X(U \Rightarrow U)$; but we know $(U \Rightarrow U) \lhd U$, and so $\Delta_X(U \Rightarrow U) \lhd \Delta_X U$. Similarly for sums, we consider $\Sigma_q \Delta_{U \times X} U \cong \Delta_X(U \times U)$; but we saw above that $U \times U \lhd U$, and so $\Delta_X(U \times U) \lhd \Delta_X U$. That completes the proof. $\square$

Corollary 3.12. (Paul Taylor) The category of retracts of a λ-theory is relatively cartesian closed.

Proof. The category $\mathbf{R}$ of retracts of U is a subcategory of $P(\mathcal{L})$. Furthermore if $X \in \mathbf{R}$ then so are the objects of $\mathbf{R}(X)$. So, the result is immediate by restricting to $\mathbf{R}$. □

Corollary 3.13. (Dana Scott) The category of retracts of a λ-theory is cartesian closed.

Proof. Immediate by restricting to the fibre $\mathbf{R} = \mathbf{R}(1)$ over 1. □

In Section 2.4, I remarked that for a general algebraic theory $\mathcal{T}$, the presheaf category $P\mathcal{T}$ is equivalent to the standard presheaf category $P\mathbf{T}$ on the Lawvere theory $\mathbf{T}$ generated by $\mathcal{T}$. For λ-theories, $\mathcal{L}$ we have much more. Since the category $\mathbf{R}$ of retracts is closed under products we have a functor $\mathbf{L} \to \mathbf{R}$ from the Lawvere theory $\mathbf{L}$ generated by $\mathcal{L}$. Moreover, the monoid $\mathcal{L}(1)$ is embedded in the λ-theory $\mathcal{L}$. Restriction gives functors $P\mathbf{R} \to P\mathbf{L} \to P\mathcal{L} \to P\mathcal{L}(1)$.

Proposition 3.14. The functors $P\mathbf{R} \to P\mathbf{L} \to P\mathcal{L} \to P\mathcal{L}(1)$ are equivalences.

Proof. $\mathcal{L}(1)$ embeds in $\mathbf{L}$ which embeds in $\mathbf{R}$ the category of retracts of $\mathcal{L}(1)$, so $P\mathbf{R} \simeq P\mathbf{L} \simeq P\mathcal{L}(1)$ by Morita theory. The equivalence $P\mathcal{L} \simeq P\mathbf{L}$ from Section 2.4 is trivial. □

I shall make use of the equivalence $P\mathcal{L} \to P\mathcal{L}(1)$ at the beginning of Section 4.5. It takes the explicit form

$$\big(X(n) \times \mathcal{L}(m)^n \to X(m)\big) \;\mapsto\; \big(X(1) \times \mathcal{L}(1) \to X(1)\big)\,.$$

4. Algebras

4.1. *Algebras for λ-theories*

Consider any λ-theory $\mathcal{L}$ simply as an algebraic theory, and then we have a category $\mathrm{Alg}(\mathcal{L})$ of $\mathcal{L}$-algebras. A $\mathcal{L}$-algebra is a set A equipped with actions $\mathcal{L}(n) \times A^n \to A$. Concretely that means that for each term $t(\mathbf{x})$ in $\mathcal{L}(n)$ and each n-tuple $\mathbf{a} \in A^n$, we get an interpretation $t(\mathbf{a})$ in A; and this behaves as expected on variables and respects substitution and β-equality. It turns out that one can focus almost entirely on Λ, the initial λ-theory.

Definition 4.1. A Λ-*algebra* is an algebra for the initial λ-theory Λ.

This is a clean definition close in spirit to environment or valuation models but avoiding the explicit interpretation of abstraction and so the issue of weak extensionality. Giving Λ-algebra structure amounts to giving λ-algebra structure as in Barendregt (1981), but to underline the difference in perspective, I use the category theoretic terminology. Maps of Λ-algebras are maps in the category $\mathrm{Alg}(\Lambda)$. This is so smooth that one might wonder why one should not take this as the basic definition. The answer is that one never directly shows that one has a Λ-algebra. One needs an induction over λ-terms with free variables and that amounts to considering λ-theories.

I say a bit more about what the definition means concretely. Think of $s \in \Lambda(n)$ as a λ-term with n free variables. A Λ-algebra A is equipped with an interpretation $s(\mathbf{a}) \in A$

of the λ-term for every term s with constant $\mathbf{a} \in A^n$ substituted for the free variables. In particular, there is an interpretation $(a, b) \mapsto ab$ of the application **app** as a binary operation and also interpretations of all constant $s \in \Lambda(0)$ as $s \in A$. That determines the structure in the sense of the following.

Lemma 4.2. Suppose that A and B are Λ-algebras and $f : A \to B$ a map preserving the binary operation of application and the λ-definable constants. Then, f is a map of Λ-algebras.

Proof. This comes from Proposition 3.3. Write $s(\mathbf{x})$ as $\mathbf{app}_{n+1}(\hat{s}, \mathbf{x})$ where $\hat{s} = \lambda^n s$ is the constant obtained by iterated λ-abstraction. The interpretation of constants and iterated application are preserved by f. Hence, so is the interpretation of s. □

Any λ-theory $\mathcal{L}$ gives rise to a Λ-algebra in a straightforward way. $\mathcal{L}(0)$ is the initial $\mathcal{L}$-algebra and composition with the unique $\Lambda \to \mathcal{L}$ makes it a Λ-algebra. Here, it is only the map $\Lambda \to \mathcal{L}$ which matters and $\mathcal{L}$ need not be a λ-theory. But that generality is of no significance. The following is trivial.

Proposition 4.3. The operation $\mathcal{L} \mapsto \mathcal{L}(0)$ gives a functor from λ-theories to Λ-algebras.

4.2. *Presheaves on the monoid*

Let A be a Λ-algebra. On $A(1) = \{a \in A \mid \mathbf{1}a = a\}$ take the monoid structure with multiplication $(a, b) \mapsto a \circ b = \lambda x.a(bx)$ representing composition. Write M_A for this monoid. The underlying set is a retract of A and another way to give M_A is as formal elements of the form az factored out by $a \sim b$ if and only if $\mathbf{1}a = \mathbf{1}b$ and with composition $(az, bz) \mapsto a(bz) = (a \circ b)z$. Thus for a λ-theory $\mathcal{L}$, the monoid $M_{\mathcal{L}(0)}$ is isomorphic to the monoid $\mathcal{L}(1)$. Motivated by Section 3.5, consider $PA = PM_A$ the category of presheaves on the one object category M_A. The universal object $U = U_A$ is $A(1) = M_A$ as underlying set with the evident right action of $b \in M_A$ by composition, $(a, b) \mapsto a \circ b$.

The association $A \mapsto PA$ extends to maps. Suppose $f : A \to B$ is a map of Λ-algebras. As f preserves application and $\mathbf{1}$, it maps $A(1)$ to $B(1)$. Moreover since composition is implemented by $\lambda xyz.x(yz)$, we get a map $M_f : M_A \to M_B$ of monoids. Now, composition with M_f gives an easy functor $PB = PM_B \to PM_A = PA$ and this has a left adjoint $Pf : PA \to PB$ given by left Kan extension. I give the following now as a warm-up: some of the calculations appear in Koymans (1984).

Proposition 4.4. For a map of Λ-algebras $f : A \to B$, the induced $Pf : PA \to PB$ preserves finite products.

Proof. It is sufficient to show preservation of the terminal object and preservation of products of representables. These correspond to conditions on categories of elements as follows.

Terminal object. Consider the category with objects $b \in M_B$ and maps $b \to \bar{b}$ being given by $a \in M_A$ such that $f(a) \circ b = \bar{b}$. We need to show that this is connected. But for every $b \in M_B$, $\lambda x.\mathbf{I} \in M_A$ gives a map from b to $\lambda x.\mathbf{I} \in M_B$, so the latter is weakly terminal. (I use $\mathbf{I} = \lambda x.x$, the identity combinator but any λ-definable constant will do.)

Binary Products. For $b_1, b_2 \in M_B$ consider the category with objects given by $c \in M_B$ and $a_1, a_2 \in M_A$ with $f(a_1) \circ c = b_1$ and $f(a_2) \circ c = b_2$. Maps from $c \in M_B$ and $a_1, a_2 \in M_A$ to $\bar{c} \in M_B$ and $\bar{a}_1, \bar{a}_2 \in M_A$ are given by elements $\hat{a}$ in M_A with $f(\hat{a}) \circ c = \bar{c}$ and. We need to show this category is connected. Take $s = \lambda w(\lambda x.x(b_1 w)(b_2 w))$ in M_B and $p = \lambda x.x\mathbf{T}$ and $q = \lambda x.x\mathbf{F}$ in M_A. Since

$$p \circ s = \lambda w.p(\lambda x.x(b_1 w)(b_2 w)) = \lambda w((\lambda x.x(b_1 w)(b_2 w))\mathbf{T}) = \lambda w.b_1 w = b_1 ,$$

and similarly $q \circ s = b_2$, we have an object of our category. Moreover, given any object $c \in M_B$ and $a_1, a_2 \in M_A$ as above we can take $r = \lambda w(\lambda x.x(a_1 w)(a_2 w))$. Similar calculations to those just given show $p \circ r = a_1$, $q \circ r = a_2$ and finally one checks $f(r) \circ c = s$. So, again we have found a weakly terminal object. □

4.3. *The function space analysis*

For A a Λ-algebra, we study the function space U^U of the universal $U \in PA$.

First, I give a general categorical analysis. Let M be a monoid and $X, Y \in PM$ presheaves on M. The function space Y^X can be represented as the set $PM(M \times X, Y)$ of M-equivariant maps, that is of $\phi : M \times X \to Y$ such that $\phi(m.m', x.m') = \phi(m, x).m'$, with action given by $\phi.\bar{m}(m, x) = \phi(\bar{m}.m, x)$. This makes sense since

$$(\phi.\bar{m}(m, x)).m' = \phi(\bar{m}.m.m', x.m') = \phi.\bar{m}(m.m', x.m'),$$

and so $\phi.\bar{m}$ is again M-equivariant. Finally, the evaluation map $Y^X \times X \to Y$ is given in this representation by

$$PM(M \times X, Y) \times X \to Y \; ; \quad (\phi, x) \mapsto \phi(I, x),$$

where, here I is the unit of the monoid M.

For the monoid M_A of a λ-algebra A, there is a more concrete representation of the function space U^U. Let $A(2) = \{d \in A \,|\, \mathbf{1}_2 d = d\} = \{d \in M_A \,|\, \mathbf{1} \circ d = d\}$. The second characterization shows that $A(2)$ has an action of M_A by composition on the right. Now given $d \in A(2)$, we define a corresponding $\phi : M_A \times M_A \to M_A$ by $\phi(a, b) = \lambda y.d(ay)(by)$. Clearly

$$\phi(a \circ c, b \circ c) = \lambda y.d(a(cy))(b(cy)) = \phi(a, b) \circ c,$$

so ϕ is equivariant. Thus, $d \mapsto \phi$ is a map of sets $A(2) \to PA(U \times U, U)$. Furthermore for $c \in M_A$, $d \circ c = \lambda xy.d(cx)y$ maps to the function $\lambda y.d(c(ay))(by) = \phi.c(a, b)$ of a, b. Thus, $d \mapsto \phi$ gives a map in PA from $A(2)$ to U^U.

Now, let $p = \lambda x.x\mathbf{T}$ and $q = \lambda x.x\mathbf{F}$. Reflecting on the argument of Section 3.5 leads one to think that $\phi(p, q)$ is in some sense generic. I exploit that thought.

Proposition 4.5. The map $A(2) \to U^U$ above is an isomorphism in PA.

Proof. Given a map $\phi : A(1) \times A(1) \to A(1)$, set $d = \lambda yz.\phi(p, q)(\lambda x.xyz) \in A(2)$. This provides an inverse to our map $A(2) \to U^U$ in PA. For first starting with d passing to ϕ

and back gives

$$\begin{aligned}\lambda yz.\big(\lambda w.d(pw)(qw)\big)(\lambda x.xyz) &= \lambda yz.\big(\lambda w.d(w\mathbf{T})(w\mathbf{F})\big)(\lambda x.xyz)\\ &= \lambda yz.d(\mathbf{T}yz)(\mathbf{F}yz)\\ &= \lambda yz.dyz = d.\end{aligned}$$

On the other hand starting with ϕ passing to d and back gives a map taking (a,b) to

$$\begin{aligned}\lambda w.\big(\lambda yz.(\phi(p,q)(\lambda x.xyz)\big)(aw)(bw) &= \lambda w.\phi(p,q)(\lambda x.x(aw)(bw))\\ &= \phi(p,q)\circ\lambda w.(\lambda x.x(aw)(bw))\\ &= \phi\big(p\circ\lambda w.(\lambda x.x(aw)(bw)), q\circ\lambda w.(\lambda x.x(aw)(bw))\big)\\ &= \phi\big(\lambda w.p(\lambda x.x(aw)(bw)), \lambda w.q(\lambda x.x(aw)(bw))\big)\\ &= \phi(a,b),\end{aligned}$$

as $\lambda w.p(\lambda x.x(aw)(bw)) = \lambda w.\mathbf{T}(aw)(bw) = \lambda w.aw = a$, and $\lambda w.q(\lambda x.x(aw)(bw)) = b$ similarly. So, our map is bijective on underlying sets and so an isomorphism. □

Again, we see calculations from Koymans (1984). We shall also need the precise form of evaluation arising from the identification of $A(2)$ with U^U. It is

$$A(2)\times A(1)\to A(1)\,;\quad (d,a)\mapsto \lambda y.d(\mathbf{I}y)(ay) = \lambda y.dy(ay)\,,$$

as the identity combinator $\mathbf{I} = \lambda x.x$ is the unit in M_A.

For the record, there is an easy extension of the analysis of U^U. Each $U^n \Rightarrow U$ can be represented by $A(n+1) = \{d|\mathbf{1}_n\circ d = d\}$ with again the obvious action. Under evaluation $d\in A(n+1)$ corresponds to $(a_1,\dots,a_n)\mapsto \lambda y.dy(a_1y)\dots(a_ny)$ as a map $U^n\to U$. Observe that generally $\{d\in M_A|\mathbf{1}_n\circ d = d\} = \{d\in A|\mathbf{1}_{n+1}d = d\}$, where the left-hand side has a clear action on the right by the monad M_A.

4.4. *The λ-theory of a Λ-algebra*

In the previous section, we saw that the function space U^U of the universal object U in PA is given by $A(2) = \{d\in M_A|\mathbf{1}\circ d = d\}$ with the action of M_A on the right. Evidently, composition on the left with $\mathbf{1}$ gives a retract from U to U^U and the generic U is a reflexive object in the presheaf category PA.

Definition 4.6. The λ-theory $\mathcal{U}_A$ of a Λ-algebra A is the theory of the reflexive universal $U\in P(A)$.

Take $f : A\to B$ a map of Λ-algebras with induced functor $Pf : PA\to PB$. The left Kan extension Pf takes the universal U_A in $P(A)$ to U_B in $P(B)$ with a specified isomorphism $Pf(U_A)\cong U_B$. So, $A\mapsto (PA,U_A)$ is pseudofunctorial with (PA,U_A) considered as a category with specified object. By Proposition 4.4, Pf preserves finite products and so gives maps $P(A)(U_A^n,U_A)\to P(B)(U_B^n,U_B)$ which taken together give a map of algebraic theories $\mathcal{U}_A\to\mathcal{U}_B$.

Proposition 4.7. The operation $A\mapsto\mathcal{U}_A$ gives a functor from Λ-algebras to λ-theories.

Proof. Given a map $f : A \to B$ of Λ-algebras, we first check that the induced map $\mathcal{U}_A \to \mathcal{U}_B$ is a map of λ-theories. But f preserves $\mathbf{1}$ which determines the function space as a retract of the universal. So, Pf preserves the retract $U^U \lhd U$ and the result follows. Furthermore $A \mapsto M_A$ is functorial in A, so $A \mapsto (PA, U_A)$ is pseudofunctorial, but then $A \mapsto \mathcal{U}_A$ is functorial as we have a mere category of algebraic theories. □

4.5. *The fundamental theorem*

Let $\mathcal{L}$ be a λ-theory. Composing the Yoneda with the equivalence $P\mathcal{L} \to P\mathcal{L}(1)$ of Section 3.5 gives isomorphisms $\mathcal{L}(n) \to P\mathcal{L}(1)(U^n, U)$ taking $a \in \mathcal{L}(n)$ to the map $\mathcal{L}(1)^n \to \mathcal{L}(1)$ given by $(b_1, \ldots, b_n) \mapsto a(b_1, \ldots, b_n)$. This gives an isomorphism between $\mathcal{L}$ and the endomorphism λ-theory of $U \in P\mathcal{L}(1)$. Furthermore, we have a canonical isomorphism $M_{\mathcal{L}(0)} \cong \mathcal{L}(1)$ of monads and so an isomorphism $P\mathcal{L}(1) \cong P\mathcal{L}(0)$. Thus, we get isomorphisms $\mathcal{L}(n) \to P\mathcal{L}(0)(U^n, U) = \mathcal{U}_{\mathcal{L}(0)}(n)$ taking $a \in \mathcal{L}(n)$ to the map $\mathcal{L}(0)^n \to \mathcal{L}(1)$ given by $(c_1, \ldots, c_n) \mapsto \lambda x.a(c_1 x, \ldots, c_n x)$. This gives an isomorphism of λ-theories $\eta_{\mathcal{L}} : \mathcal{L} \to \mathcal{U}_{\mathcal{L}(0)}$. The following is immediate.

Proposition 4.8. The $\eta_{\mathcal{L}} : \mathcal{L} \to \mathcal{U}_{\mathcal{L}(0)}$ are λ-theory isomorphisms natural in $\mathcal{L}$.

Now take a Λ-algebra A, pass to the universal λ-theory $\mathcal{U}_A$ and then take the induced Λ-algebra $\mathcal{U}_A(0)$. Scott (1980) notes that these are isomorphic, and details of a syntactic argument are given in Barendregt (1984) and Koymans (1984). One considers open terms in the λ-calculus with constants from A and shows inductively that each term $t(\mathbf{x})$ with n free variables is interpreted in $\mathcal{U}_A(n) \cong A(n)$ as above by the interpretation in A of its closure $\lambda\mathbf{x}.t(\mathbf{x})$. The spirit of categorical logic is to engage in as few syntactic inductions as possible, and I present an alternative. The Λ-algebra $\mathcal{U}_A(0) = PA(1, U)$ consists of the fixed points of $A(1)$ under the composition action. There is evidently a map of sets $\varepsilon_A : \mathcal{U}_A(0) \to A; a \mapsto a\mathbf{I}$. (Since a is fixed any constant will do.)

Lemma 4.9. The maps $\varepsilon_A : \mathcal{U}_A(0) \to A$ preserve the application.

Proof. Internal application $U \times U \to U$ is given by evaluation. It follows that it takes $(a, b) \in A(1)^2$ to $\lambda y.ay(by)$. But $\varepsilon_A(\lambda y.ay(by)) = a\mathbf{I}(b\mathbf{I}) = \varepsilon_A(a)\varepsilon_A(b)$. □

We do not yet know that the ε_A are maps of Λ-algebras, but for $f : A \to B$ a map of Λ-algebras the naturality diagram

$$\begin{tikzcd}
\mathcal{U}_A(0) \arrow[r, "\mathcal{U}_f(0)"] \arrow[d, "\varepsilon_A"'] & \mathcal{U}_B(0) \arrow[d, "\varepsilon_B"'] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

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commutes because $f(a\mathbf{I}) = f(a)\mathbf{I}$. It is an equally trivial calculation to show that for $\mathcal{L}$, a λ-theory the familiar triangle identity diagram

$$\begin{tikzcd} \mathcal{L}(0) \arrow[r, "\eta_L(0)"] \arrow[dr, equal] & \mathcal{U}_{\mathcal{L}(0)}(0) \arrow[d, "\varepsilon_{\mathcal{L}(0)}"] \\ & \mathcal{L}(0) \end{tikzcd}$$

commutes: specifically the composite is $a \mapsto \lambda x.a \mapsto (\lambda x.a)\mathbf{I} = a$.

Proposition 4.10. The $\varepsilon_A : \mathcal{U}_A(0) \to A$ are Λ-algebra isomorphisms natural in A.

Proof. To show that ε_A is a map of Λ-algebras, consider the commuting diagram

$$\begin{tikzcd} \Lambda(0) \arrow[r, "\eta_\Lambda(0)"] \arrow[dr, equal] & \mathcal{U}_{\Lambda(0)}(0) \arrow[r] \arrow[d, "\varepsilon_{\Lambda(0)}"] & \mathcal{U}_A(0) \arrow[d, "\varepsilon_A"] \\ & \Lambda(0) \arrow[r] & A\,. \end{tikzcd}$$

We have unique Λ-algebra maps $\Lambda(0) \to \mathcal{U}_A(0)$ and $\Lambda(0) \to A$. It follows that ε_A preserves λ-definable constants. By Lemma 4.9, it preserves application, so by Proposition 4.2 it is indeed a Λ-algebra map. We saw the trivial naturality above and $a \mapsto a\mathbf{I}$ has inverse $c \mapsto \lambda x.c$ so we have a natural isomorphism. □

It follows from Propositions 4.8 and 4.10 that the functors $\mathcal{L} \mapsto \mathcal{L}(0)$ and $A \mapsto \mathcal{U}_A$ give an equivalence (in fact an adjoint equivalence) between the categories of λ-theories and of Λ-algebras. That is almost but not quite what I want to call the Fundamental Theorem: there is a little bit more. Consider Λ_A the algebraic theory of extensions of A as described in Section 2.2. It is not *a priori* obvious that it is a λ-theory but we can identify it with $\mathcal{U}_A$. There are a number of ways to see this in terms of the equivalence. The following seems down to earth. For any Λ-algebra A, we have the unique map of λ-theories $\Lambda \to \mathcal{U}_A$. Using the isomorphism $A \cong \mathcal{U}_A(0)$, we get a factorization $\Lambda \to \Lambda_A \to \mathcal{U}_A$ of algebraic theories. We get an induced functor $\mathrm{Alg}(\mathcal{U}_A) \to \mathrm{Alg}(\Lambda_A)$. To see that this is surjective on objects take an extension $A \to B$ of Λ-algebras; from the induced $\mathcal{U}_A \to \mathcal{U}_B$ we get $\mathcal{U}_A$-algebra structure on $B \cong \mathcal{U}_B(0)$. The functor is evidently faithful. To see it is full, suppose $B \to C$ is a Λ-algebra map between extensions $A \to B$ and $A \to C$ coming from $\mathcal{U}_A$-algebras B and C; we get $\mathcal{U}_B \to \mathcal{U}_C$ under $\mathcal{U}_A$ and so have a corresponding $\mathcal{U}_A$-algebra map. Since $\mathrm{Alg}(\mathcal{U}_A) \to \mathrm{Alg}(\Lambda_A)$ is an equivalence Proposition 2.7 shows that $\Lambda_A \to \mathcal{U}_A$ is an isomorphism of algebraic theories. So, in particular there is a canonical λ-theory structure on Λ_A. Putting all that together with Propositions 4.8 and 4.10 gives the following.

Theorem 4.11. (Fundamental Theorem of the λ-Calculus) There is an adjoint equivalence $\mathcal{L} \mapsto \mathcal{L}(0)$, $A \to \Lambda_A$ between λ-theories and Λ-algebras: for $\mathcal{L}$ a λ-theory and A a Λ-

algebra, there are natural isomorphisms $\mathcal{L} \cong \Lambda_{\mathcal{L}(0)}$ and $\Lambda_A(0) \cong A$. In particular, each λ-theory $\mathcal{L}$ is isomorphic to the theory of extensions of its initial algebra $\mathcal{L}(0)$.

There are many straightforward consequences of the Fundamental Theorem. Note that for every λ-theory $\mathcal{L}$, the $\mathcal{L}(n)$ are not just the free extensions of $\mathcal{L}(0)$ as $\mathcal{L}$-algebras but also as Λ-algebras. Much the same thought is expressed in the following.

Proposition 4.12. Suppose that A is a Λ-algebra. Then, there is a canonical Λ-algebra structure on the retracts $A(n) = \{a \in A | \mathbf{1}_n a = a\}$ making them the free Λ-algebra extending A by n indeterminates.

Proof. Immediate given the remark at the end of Section 4.3. □

This last result is folklore mentioned in passing in Freyd (1989) and spelled out in Selinger (2002). It arises very naturally in the approach which I have laid out, and itself suggests the development of connections with combinatory logic.

4.6. *An alternative approach*

I can readily imagine that syntactically minded colleagues will not be comfortable with my approach to the Fundamental Theorem, so I briefly sketch an alternative. Let us try to construct Λ_A syntactically. Given a λ-algebra A, take an extension of the syntax of the λ-calculus with constants from A. Let $\Lambda_A(n)$ be the terms with n variables factored out by the equality generated by β-equality in the λ-calculus and by the equalities given by the actions $\Lambda(m) \times A^m \to A$. Extending the argument for the initial λ-theory Λ, one can show that the resulting Λ_A is a λ-theory. Functoriality of the operation $A \mapsto \Lambda_A$ is straightforward, but after that things get delicate as in this approach we do not know that Λ_A is the algebraic theory of extensions of the Λ-algebra A. Even the isomorphism $A \cong \Lambda_A(0)$ is not obvious: how do we know that the syntactic theory does not produce a proper quotient of A? There are direct arguments for that but probably it is easiest to use Proposition 3.3 to identify the syntactic Λ_A with the theory of extensions.

Next what happens if we start with a λ-theory $\mathcal{L}$ and form $\Lambda_{\mathcal{L}(0)}$? We now know Λ_A as the theory of extensions and so we get a factorization $\Lambda \to \Lambda_{\mathcal{L}(0)} \to \mathcal{L}$ of algebraic theories. By Proposition 3.2, we deduce that $\Lambda_{\mathcal{L}(0)} \to \mathcal{L}$ is a map of λ-theories. We want to show that this is an isomorphism but we cannot exploit Proposition 2.7 as we have no handle on $\mathcal{L}$-algebras. It seems best to use Proposition 3.3 again and argue directly that any λ-theory $\mathcal{L}$ is the theory of extensions of the Λ-algebra $\mathcal{L}(0)$. That can be done but subtleties which appear in Section 4.5 cannot be avoided. Overall, the syntactic approach is not as straightforward as it seems.

5. Conclusions and vistas

I want to stress that understanding λ-theories comes before understanding Λ-algebras or any other equivalent notion. Though the case for the theory approach is foundational, I hope that this paper will encourage new research. For example customary questions about the syntactic theories represented by concrete cartesian closed categories seem narrow:

one should ask about the λ-theories represented. What can one say about them? How for example to compare them within and between categories? The Taylor Fibration of Section 3.5 exhibits for every λ-theory $\mathcal{L}$ a large family of interpretations corresponding to the slices of $P\mathcal{L}$. But there are many more endomorphism λ-theories of reflexive objects in $P\mathcal{L}$. Can one characterize them?

Another particularly interesting set of questions concerns cartesian closed categories arising from models of the differential lambda calculus. A form of the Approximation Theorem holds automatically and it follows that the quotients of the syntactic theories obtained are very restricted. These issues are intimately tied up with Böhm trees. For background see Ehrhard and Regnier (2003), Ehrhard and Regnier (2006) and Carraro (2011). However it appears that notwithstanding the restrictions, there is still a wide variety of interpretations. Is this impression true? How can one make sense of it? What general tools are there for telling differences in such cases?

I close with some remarks about the potential wider significance of the techniques discovered by Corrado Böhm and presented in his seminal paper (Böhm 1968). This is surely a cornerstone of our understanding of the λ-calculus and should come early in any account of fundamental ideas. But how well do we understand what is involved? Prima facie the techniques are syntactic. The original applications are to quotients of the initial $\beta\eta$ theory Λ_η and concern what are usually thought of as the limits of consistency: what can you or can you not do before such a quotient becomes the terminal (trivial) theory? There is Böhm's original point that identifying distinct $\beta\eta$ normal forms collapses a theory and the closely related result of Hyland (1976) that there is a unique maximal non-trivial quotient of the theory extending Λ_η by setting all unsolvables equal. But I think there is a broader set of semantic principles at stake. For example, when analysing models such as Scott's $P\omega$ in which η does not hold one exploits non-syntactic variants of Böhm's combinators. (This is well explained for Plotkin's T^ω model in Section 3 of Barendregt and Longo (1980).) There must surely be more to be said on a broader canvas.

In honouring Böhm, I want to stress the future. There is still much to discover about the pure λ-calculus. Here, I have presented it in modern dress with all the trappings of current categorical research. My intention is to convey the clear message that the λ-calculus is a remarkable and exceptionally elegant area of abstract mathematics.

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