Problem 1. Consider a matrix $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ -1 & 3 & 2 & 1 \end{bmatrix}$.

(a) Find the null space of A.

(b) Find the dimension of $\ker(T_A)$ and the dimension of $\operatorname{ran}(T_A)$, where T_A is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ with the standard matrix A.

Problem 2. Let T be a linear operator on $\mathbb{R}^3$ that is the orthogonal projection onto the plane $y = z$ in $\mathbb{R}^3$. In other words, T maps a vector $\mathbf{x}$ to a vector $T(\mathbf{x})$ on the plane $y = z$ that minimizes $\|\mathbf{x} - T(\mathbf{x})\|$.

(a) Find the standard matrix for the transformation T.

(b) Find a vector $\mathbf{a}$ which is orthogonal to the $\operatorname{ran}(T)$.

(c) Find $\dim(\mathbf{a}^\perp)$.

Problem 3. (a) Show that any intersection of subspaces of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

(b) Let S be the intersection of two planes $x + y + 2z = 0$ and $x - y + z = 0$ in $\mathbb{R}^3$. Find a basis $\{v_1, \dots, v_d\}$ for S.

(c) Find a basis for $v_1^\perp \cap v_2^\perp \cap \cdots \cap v_d^\perp$, where $v_1, \dots, v_d$ are the vectors you found in (b).

Problem 4. Let $A = \begin{bmatrix} 2 & 4 & t \\ t & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$. Find all values of t that makes the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ positive. Find the dimension for each such t.

Problem 5. (a) Show that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ ia a basis for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -7 \\ 8 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Express $\mathbf{v} = \begin{bmatrix} -9 \\ -15 \\ -3 \end{bmatrix}$ as a linear combination of the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Problem 6. Prove each of the following statements.

(a) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an one-to-one linear operator and if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ is also a basis for $\mathbb{R}^n$.

(b) $\mathbf{x}$ is an eigenvector of an $n \times n$ matrix A if and only if the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$ and $A\mathbf{x}$ has dimension 1.

Problem 7. (MATLAB programming) In this problem, we make a function file `CheckBasis.m` to check that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ form a basis of $\mathbb{R}^4$ using the equivalent statements (a), (g), (h), and (o) of Theorem 7.2.7 in the textbook.

(a) Complete the shadow part (/////) of the m-file given below referring to the comments and the execution results.

```
%----------------- your function file -----------------%
function [Result]=CheckBasis(v1, v2, v3, v4, case_num)
  %  if case_num=1, check the statement (a),
  %  if case_num=2, check the statement (g),
  %  if case_num=3, check the statement (h).

  % Construct the matrix V.
  ////////////////////////

  % Use the switch statement to check
  % whether one of the statements (a), (g), and (h) holds.
  switch case_num
    case 1
      fprintf('* You enter %d: statement (a) *\n', case_num);
      ////////////////////////
      if ////////////////////////
        disp('  Given vectors form a basis of 4 dimensional space.');
      else
        disp('  Given vectors do not form a basis of 4 dimensional space.');
      end
    case 2
      fprintf('* You enter %d: statement (g) *\n', case_num);
      Result=det(V);
      if Result~=0
        disp('  Given vectors form a basis of 4 dimensional space.');
      else
        disp('  Given vectors do not form a basis of 4 dimensional space.');
      end
    ///////////////////////////  % check statement (h)
        ////////////////////////
```

```
        ////////////////////////
        ////////////////////////
        ////////////////////////
        ////////////////////////
        ////////////////////////
        ////////////////////////
  end
end
```

The execution results will be as follows:

```
>> v1=[1 0 0 0]'; v2=[0 2 0 0]'; v3=[0 0 4 5]'; v4=[0 0 0 -1]'; v5=[0 0 0 1]';
>> C=CheckBasis(v1, v2, v3, v4,3)
* You enter 3: statement (h) *
  Given vectors are basis of 4 dimensional space.

C =

     1
     2
    -1
     4

>> CheckBasis(v1, v2, v4, v5, 1);
* You enter 1: statement (a) *
  Given vectors do not form a basis of 4 dimensional space.

>> determinant=CheckBasis(v1, v2, v3, v5, 2)
* You enter 2: statement (g) *
  Given vectors form a basis of 4 dimensional space.

determinant =
     8
```

(b) Using `CheckBasis.m` from (a), check whether

i. $\mathbf{v}_1 = (-1, 0, 1, 0)^T$, $\mathbf{v}_2 = (2, 3, -2, 6)^T$, $\mathbf{v}_3 = (0, -1, 2, 0)^T$ and $\mathbf{v}_4 = (0, 0, 1, 5)^T$ form a basis of $\mathbb{R}^4$.

ii. $\mathbf{v}_1 = (a, b, c, d)^T$, $\mathbf{v}_2 = (-b, a, d, -c)^T$, $\mathbf{v}_3 = (-c, -d, a, b)^T$ and $\mathbf{v}_4 = (-d, c, -b, a)^T$ form a basis of $\mathbb{R}^4$. (Do not use the statement (h). Guess why not?)

Problem 8. Using the MATLAB command *sym* and *magic*, generate an 8×8 magic matrix A of symbolic numbers and find the basis for the null space Z using the command *null*. Verify that the result is correct by checking $A * Z = 0$.