URN MODELS

Report
for
M. Stat. Dissertation

Gourab Ray
Roll Number: MB0804

under supervision of
Krishanu Maulik

1. Introduction

Consider an urn model with balls of p colors. The model evolves according to a non-random $p \times p$ matrix $R = ((r_{ij}))$ known as the replacement matrix. If a ball of ith color is drawn, then r_{ij} balls of jth color are added. We assume that the number of balls to be nonnegative real numbers and the row sums to be same. The second property of the replacement matrix is called balanced. We further assume, without loss of generality, the row sums to be 1, that is we make R stochastic. The results extend to the case with same general row sum by an obvious rescaling.

Let $\boldsymbol{C}_n$ be the row vector denoting the number of balls after the nth trial. So, the row vector $\boldsymbol{C}_0$ denotes the number of balls to start with. At each trial balls are drawn randomly. So at n-th trial, i-th ball appears with probability $C_{i,n-1}/n$. The urn model is well studied in the literature.

We shall use the notations $\mathbf{1}$ and $\mathbf{0}$ for the column vectors of relevant dimensions with all co-ordinates 1 or 0. For any vector $\boldsymbol{\xi}$, we denote by $\boldsymbol{\xi}^2$, each of whose elements are square of those of $\boldsymbol{\xi}$. We shall also use the filtration $\{\mathcal{F}_n\}_{n\geq 1}$, where $\mathcal{F}_n$ is the σ-field generated by $\{\boldsymbol{C}_1, \ldots, \boldsymbol{C}_n\}$ and study martingales adapted to this filtration. Further, for any sequence of random variables X_n, we define, $\Delta X_n = X_{n+1} - X_n$.

In this dissertation, we shall study the rates of strong/weak convergence of the counts of balls of each color, as well as, their linear combinations under suitable scaling for different reducible replacement matrices. If the fraction of colors converge almost surely to a vector $\boldsymbol{\pi}$, then the nonzero co-ordinates of $\boldsymbol{\pi}$ correspond to, what we call, the *dominant* colors. In Section 2 of the dissertation, we review some of the existing results in the literature. In Section 3, we classify the urn models with balls of three colors based on the number of communicating classes. In Sections 4 and 5, we consider a three color urn model with block triangular replacement matrix and only one dominant color. The submatrix of the replacement matrix corresponding to the other two colors are taken to be irreducible, but not necessarily balanced. In Section 4, we study the strong/weak convergence of the count of the non-dominant colors. In fact, the results are known when the submatrix is balanced. In Section 5, we consider the L^p convergence of the color counts. In Section 6, we consider another urn model with balls of three colors, where two of them are dominant and the submatrix corresponding to the dominant colors is an identity matrix. We provide a representation for the limiting distribution for the fraction of the dominant colors. In Sections 7 and 8, we consider urn models with balls of p colors, where the replacement matrix is triangular. In this case, the rates of strong convergence of the color counts are known from Bose et al. (2009b) under a technicality condition. We extend the result by removing the technicality condition in Section 7. In Section 8, we study the linear combination of the color counts in this model. Finally, in Section 9, we consider a particular urn model with infinitely many colors and do some simulations for them and make some conjectures.

For two sequences of real numbers $\{x_n\}$ and $\{y_n\}$, we shall use the following notations:

(i) $x_n \sim y_n$ means $x_n/y_n \to 1$.
(ii) $x_n \asymp y_n$ means $x_n/y_n \to c$, for some non-zero real c.
(iii) $O(n)$ and $o(n)$ will carry their usual meanings.

The following continued product and its asymptotic relation with gamma function due to Euler will also be useful. We define:

$$\Pi_n(s) = \prod_{i=0}^{n-1}\left(1 + \frac{s}{i+1}\right) \sim n^s/\Gamma(s+1), \tag{1.1}$$

where s real but not a negative integer.

2. Review of the literature

In this section, we review some of the results regarding the urn models available in the literature. In Subsection 2.1, we consider the urn models with irreducible replacement matrices. In Subsection 2.2, we consider an urn model with balls of p colors and a block diagonal reducible replacement matrix. The case of irreducible replacement matrix can be obtained as a special case. In Subsection 2.3, we consider the urn model with balls of two colors and identity replacement matrix. In Subsection 2.4, we consider a two color urn model, where only one color is dominant. In Subsections 2.5 and 2.6, we review some results about urn models with balls of three colors, of which one and two are dominant respectively. Finally, in Subsection 2.7, we consider urn models with triangular replacement matrix.

2.1. **Irreducible replacement matrix.** This is possibly the most well studied case among the urn models. We shall call any $p \times p$ matrix R to be irreducible, if for any $1 \leq i, j \leq p$, there exists some $d \equiv d(i,j)$, such that the (i,j)-th entry of R^d is nonzero. If the replacement matrix R of a p color urn model is irreducible, then $\boldsymbol{C}_n/n$ converges almost surely and in L^2 to $\boldsymbol{\pi}_R$, where $\boldsymbol{\pi}_R R = \boldsymbol{\pi}_R$. It follows easily from the result of Gouet (1997) quoted in Subsection 2.2. Note that, $\boldsymbol{\pi}_R$ is a left eigenvector of R for the eigenvalue 1. Since R is assumed to be stochastic, 1 is always an eigenvalue. In fact, it is the eigenvalue with the largest modulus and sometimes referred as the Perron-Frobenius eigenvalue in the literature and the corresponding eigenvector is called Perron-Frobenius eigenvector. Since, it is known that $\boldsymbol{\pi}_R$ has all entries positive, all colors become dominant.

For any irreducible (but not necessarily balanced) matrix, there will be a positive eigenvalue (larger than the modulus of any other eigenvalue) called Perron-Frobenius eigenvalue and the corresponding eigenvectors will have all coordinates positive. Such eigenvalues and left eigenvectors play an important role in the study of the urn models, as they do in the irreducible case.

2.2. **A reducible model with balls of p colors.** Gouet (1997) considered an interesting urn model with balls of p colors and reducible replacement matrix. In particular, he considered a replacement matrix of the form:

$$R = \begin{pmatrix} R_{11} & 0 & 0\dots & 0 \\ 0 & R_{22} & 0\dots & 0 \\ \vdots & \ddots & & \\ & & R_{rr} & 0 \\ 0 & \dots & 0 & Q \end{pmatrix} \tag{2.1}$$

where

$$Q = \begin{pmatrix} Q_{11} & \cdots & Q_{1q} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_{qq} \end{pmatrix}$$

Let $R_{11}, \dots, R_{rr}, Q_{11}, \dots, Qqq$ be irreducible diagonal submatrices of R. We also further have row sums of $Q_{11}, \dots, Q_{q-1q-1}$ to be less than 1. Let $\boldsymbol{\pi}_i$, $i = 1, \dots, r$ denote the $1 \times p$ vector with all entries 0 except those corresponding to the submatrix R_{ii} where we have the stationary distribution $\boldsymbol{\pi}_{R_{ii}}$. Similarly we define $\boldsymbol{\pi}_{qq}$ to have all entries 0 except those corresponding to the submatrix Q_{qq} where we have the stationary distribution $\boldsymbol{\pi}_{Q_{qq}}$. Then we have the following theorem:

Theorem 2.1. *Suppose we have a p color urn with the replacement matrix* (2.1)*. Then* $\boldsymbol{C}_n/\boldsymbol{C}_n\mathbf{1} \to \sum_{i=1}^{r} Y_i\boldsymbol{\pi}_i + Z_q\boldsymbol{\pi}_{qq}$ *almost surely where* $(Y_1, Y_2, \dots, Z_q)$ *is a* $1 \times (r+1)$ *random vector with Dirichlet density*

$$\Gamma(\sum_{j=1}^{r+1} \theta_j) \prod_{j=1}^{r+1} \frac{y_j^{\theta_j - 1}}{\Gamma(\theta_j)},$$

where $\theta_j = T_{0,C_j}$ $j = 1, \dots, r$ *where* T_{0,C_j} *is the total number of balls initially taken for colors corresponding to the communicating class* R_{jj} *and* $\theta_{r+1} = \boldsymbol{C}_0\mathbf{1} - \sum_{j=1}^{r} \theta_j$.

2.3. **Two color urn model with identity replacement matrix.** The urn models, with balls of two colors and a replacement matrix which is not irreducible, can be of two types. In this subsection, we consider the case where the replacement matrix is the identity matrix. This is the original urn model studied by **? give reference**. We shall denote the colors by white and black and accordingly write $\boldsymbol{C}_n = (W_n, B_n)$. Let $\mathcal{F}^{(n)}$ be the σ-field spanned by $\{(W_j, B_j) : j \geq n+1\}$. The tail σ-field of $(W_n, B_n) : n \geq 0$ is $\mathcal{F}^{(\infty)} = \cup_{n=0}^{\infty} \mathcal{F}^{(n)}$. The fraction of color count converges to a random vector with beta distribution in this case. We quote the following result from Freedman (1965).

Theorem 2.2. *The process* $\{(W_n + B_n)^{-1}W_n : n \geq 0\}$ *is a martingale and converges with probability* 1 *to a limiting random variable* Z*. The tail* σ*-field of* $(W_n, B_n) : n \geq 0$ *is equivalent to the* σ*-field spanned by* Z*. Given this* σ*-field, the* $\{X_n = W_{n+1} - W_n : n \geq 0\}$ *are independent and identically distributed, being 1 with probability* Z *and* 0 *with probability* $1 - Z$*.* Z *has beta distribution with parameters* (W_0, B_0)*.*

Note that both the colors are dominant in this model. It is also easy to obtain the limiting distribution of the linear combination of the fractions of the color counts. The result can easily be extended to the case with more than two colors with the limiting random variable having Dirichlet distribution with the parameters given by the initial configuration $\boldsymbol{C}_0$.

2.4. **Two color urn model with one dominant color.** We now consider the other two color urn model with reducible replacement matrix. Here the replacement matrix is upper triangular, but not diagonal. In particular, we consider a replacement matrix of the form

$$R = \begin{pmatrix} s & 1-s \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

We shall again denote the colors by white and black and write $\boldsymbol{C}_n = (W_n, B_n)$. We quote the following result from Bose et al. (2009a), which shows only one color to be dominant.

Theorem 2.3. *Suppose we have the replacement matrix of the form* (2.2)*.Then,*

(i) $\boldsymbol{C}_n/(n+1) \stackrel{a.s}{\to} (0,1)$
(ii) $\boldsymbol{W}_n/\Pi_n(s)$ *is an* L^2 *bounded martingale and hence converges almost surely and in* L^2*. The limit is a non-degenerate random variable. By Euler's formula,* W_n/n^s *converges almost surely and in* L^2 *to a non-degenerate random variable.*

2.5. **Three color reducible urn models with one dominant color.** In this subsection and the following one, we consider urn models with balls of three colors labeled green, white and black. Accordingly, we shall write $\boldsymbol{C}_n = (G_n, W_n, B_n)$. The replacement matrix is of the following form:

$$R = \begin{pmatrix} sQ & \begin{matrix} 1-sc \\ 1-s \end{matrix} \\ \boldsymbol{0} & 1 \end{pmatrix}, \tag{2.3}$$

where $0 < c \leq 1$ and $0 < s \leq 1$ with $sc < 1$ and Q is a 2×2 irreducible matrix. Note that Q need not be balanced, unless $c = 1$. In this subsection, we quote some results from Bose et al. (2009a), where $c = 1$, that is Q is balanced. To state the results, denote the count vector of the first two colors (G_n, W_n) by $\boldsymbol{S}_n$. Since Q is irreducible, it has Perron-Frobenius eigenvalue, denoted by λ_1, which is positive and is larger than the modulus of other eigenvalue denoted by λ_2. The corresponding right eigenvectors are denoted by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ respectively, of which the first one has both coordinates positive. When $c = 1$ and Q is balanced, in fact stochastic, we have $\lambda_1 = 1$ and $\boldsymbol{\xi}_1 = \boldsymbol{1}$.

Theorem 2.4. *Consider a three color urn model with reducible replacement matrix* R *given by* (2.3) *with* $c = 1$ *(balanced case). Then the following hold:*

(i) $\boldsymbol{C}_n\boldsymbol{1}/(n+1) = 1$.
(ii) $\boldsymbol{C}_n/(n+1) \to (0,0,1)$ *almost surely and in* L^2*.*
(iii) $\boldsymbol{S}_n\boldsymbol{1}/(n+1)^s$ *converges almost surely as well as in* L^2*, to a non-degenerate random variable* U*.*
(iv) $\boldsymbol{S}_n/(n+1)^s \to \boldsymbol{\pi}_Q U$ *almost surely and in* L^2*, where* $\boldsymbol{\pi}_Q$ *is the left eigenvector of* Q *corresponding to its Perron-Frobenius eigenvalue* 1*.*
(v) *If* $\lambda_2 < 1/2$*, then* $\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s/2} \Rightarrow N\left(0, \frac{s^2\lambda_2^2}{s(1-2\lambda_2)}U\boldsymbol{\pi}_Q\boldsymbol{\xi}^2\right)$.
(vi) *If* $\lambda_2 = 1/2$*, then* $\boldsymbol{S}_n\boldsymbol{\xi}_2/\sqrt{n^s \log n} \Rightarrow N(0, s^2\lambda_2^2 U\boldsymbol{\pi}_Q\boldsymbol{\xi}^2)$
(vii) *If* $\lambda_2 > 1/2$*, then* $\boldsymbol{S}_n\boldsymbol{\xi}_2/\Pi_n(s\lambda_2)$ *is an* L^2 *bounded martingale and converges almost surely, as well as in* L^2 *and* $\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_2} \to V$ *where* V *is a non-degenerate random variable.*

Clearly, black alone is the dominant color in this model. We can obtain the asymptotic behavior of the linear combination of the color counts from the above result as well. We shall treat the case where $c < 1$, that is, Q is unbalanced in Section 4 in detail.

2.6. **Three color reducible urn models with two dominant colors.** In this subsection, we consider other three color reducible urn models, namely, those with two dominant colors. We shall continue to denote the colors by green, white and black respectively and $\boldsymbol{C}_n = (G_n, W_n, B_n)$. The replacement matrix is taken to be of the form

$$R = \begin{pmatrix} s & (1-s)\boldsymbol{p} \\ 0 & P \end{pmatrix}, \tag{2.4}$$

where $\boldsymbol{p}$ is a probability row vector and P is a 2×2 stochastic matrix. Thus 1 is always an eigenvalue of P with eigenvector $\mathbf{1}$. Denote the other eigenvalue of P by λ with corresponding eigenvector $\boldsymbol{\xi}$. Note $|\lambda| < 1$. Also note s and 1 are two eigenvalues of R with eigenvectors $(1, 0, 0)'$ and $\mathbf{1}$ respectively. Denote the left eigenvector of P corresponding to 1 by $\boldsymbol{\pi}_P$. We quote the following results from Bose et al. (2009a).

Theorem 2.5. *Consider the three color urn model with the replacement matrix given by* (2.4)*. Then the following hold:*

(i) $\boldsymbol{C}_n\mathbf{1}/(n+1) = 1$.
(ii) $\boldsymbol{C}_n/n \to (0, \boldsymbol{\pi}_P)$ *almost surely and in* L^2.
(iii) $G_n/n^s \to V$ *almost surely as well as in* L^2. *Here* V *has same distribution as the limit distribution obtained in Theorem 2.3 (ii), when we start with the initial vector* $(G_0, W_0 + B_0)$.

The result above shows that white and black are dominant colors. The asymptotic behavior of two independent linear combinations corresponding to $\mathbf{1}$ and $(1, 0, 0)'$ are given above. The asymptotic behavior of the linear combination of the color counts corresponding to one remaining vector is more complex and depends on whether R is diagonalizable or not. If R is diagonalizable, then there is a complete set of eigenvectors and let $\boldsymbol{v}$ be the third independent eigenvector.

Theorem 2.6. *Consider a three color urn model with replacement matrix* R *given by* (2.4) *where* R *is diagonalizable. Then*

(i) *If* $\lambda < 1/2$, *then* $\boldsymbol{C}_n\boldsymbol{v}/\sqrt{n} \Rightarrow N\left(0, \frac{\lambda^2}{1-2\lambda}\boldsymbol{\pi}_P\boldsymbol{\xi}^2\right)$.
(ii) *If* $\lambda = 1/2$, *then* $\boldsymbol{C}_n\boldsymbol{v}/\sqrt{n \log n} \Rightarrow N(0, \lambda^2\boldsymbol{\pi}_P\boldsymbol{\xi}^2)$.
(iii) *If* $\lambda > 1/2$, *then* $\boldsymbol{C}_n\boldsymbol{v}/\Pi_n(\lambda)$ *is an* L^2 *bounded martingale and* $\boldsymbol{C}_n\boldsymbol{v}_2/n^\lambda$ *converges almost surely to a non-degenerate random variable.*

If R is not diagonalizable, then one of the eigenvalue must repeat and we have $\lambda = s$. Consider the Jordan decomposition of R as $RT = TJ$ where J is given by

$$J = \begin{pmatrix} s & 1 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and first and third columns of T can be chosen to be $\boldsymbol{t}_1 = (1, 0, 0)'$ and $\boldsymbol{t}_3 = \mathbf{1}$. Let $\boldsymbol{t}_2$ be the second column. We have the following theorem:

Theorem 2.7. *Consider a three color urn model with a replacement matrix given by* (2.4)*. Then we have*

(i) *If* $s < 1/2$, *then* $\boldsymbol{C}_n\boldsymbol{t}_2/\sqrt{n} \Rightarrow N(0, \frac{s^2}{1-2s}\boldsymbol{\pi}_P\boldsymbol{\xi}^2)$.
(ii) *If* $s \geq 1/2$, *then* $\boldsymbol{C}_n\boldsymbol{t}_2/n^s \log n$ *converges almost surely and in* L^2 *to* V *as obtained in Theorem 2.5 (iii).*

2.7. **Urn models with triangular replacement matrix.** Bose et al. (2009b) considered urn models with upper triangular replacement matrix and obtained the rate of almost sure convergence of all the color counts. The final identification of the rates and the relations among the limit random variables depend on an algorithmic rearrangement and relabeling of the colors, which we describe first.

As in Bose et al. (2009b), we denote total number of colors as $p = K + 1$. Then the replacement matrix $R = ((r_{i,j}))$ is a balanced $(K+1) \times (K+1)$ triangular replacement matrix with row sums as 1. Denote the diagonal elements of R as r_k, $1 \leq k \leq K+1$. Let $1 = i_1 < i_2 < \cdots < i_J < i_{J+1}(= K+1)$ denote the indices of running maxima of diagonals, namely, $r_1 = r_{i_1} \leq r_{i_2} \leq \dots r_{i_J} \leq r_{i_{J+1}} = r_{K+1}$ and, for $i_j < k < i_{j+1}$, we have $r_k < r_{i_j}$ for $j = 1, 2, \ldots J$. It is clear from the definition that $(K+1)$ will always be an index of running maxima of diagonals.

Definition 2.8. For $j = 1, 2, \ldots J$, the colors indexed by $i_j, i_j + 1, \ldots i_{j+1} - 1$ constitute the *jth block* of colors, i_j is called the *leading index*, and the corresponding color is called the *leading color* of the jth block.

The colors of the urn model can be rearranged in a canonical fashion described in the next definition.

Definition 2.9. The colors are said to be arranged in *increasing* order if $\boldsymbol{R}$ satisfies the following: with $1 = i_1 < i_2 < \cdots < i_J < i_{J+1} (= K+1)$ as the indices of the running maxima of diagonals, for $i_j < k < i_{(j+1)}$, $j = 1, 2, \ldots J$, we have

$$\sum_{m=i_j}^{k-1} r_{m,k} > 0. \tag{2.5}$$

Bose et al. (2009b) showed that the colors can be rearranged in increasing order. Henceforth, in this subsection, we shall assume the colors to be in increasing order.

Note that if $r_{i_j} = r_{i_{(j+1)}}$ and $r_{m,i_{(j+1)}} = 0$ for $m = i_j, i_j + 1, \ldots i_{(j+1)} - 1$, then we can reshuffle the colors to bring the i_{j+1}th color ahead of the i_jth one, yet maintaining the triangular structure of the replacement matrix and the increasing order of colors. Hence the rearrangement will not be unique and to make this unique, an assumption has to be made, which is:

$$\sum_{m=i_j}^{i_{(j+1)}-1} r_{m,i_{(j+1)}} > 0 \quad \text{whenever} \quad r_{i_j} = r_{i_{(j+1)}} \tag{2.6}$$

The assumption (2.6) is the condition (2.5) for the leading color. To state the results from Bose et al. (2009b), we need some more definitions.

Definition 2.10. Let $R^{(j)}$ be the submatrix formed by the rows and columns corresponding to the indices of the jth block. We will write $\lambda_j = r_{i_j} = r_1^{(j)}$. The part of the vector $\boldsymbol{C}_n$ corresponding to the jth block will be denoted by $\boldsymbol{C}_n^{(j)}$. Also, denote by $\boldsymbol{\rho}^{(j)}$ the part of the i_{j+1}th column corresponding to the jth block.

So $R^{(j)}$ has $r_{i_j} = \lambda_j$ as the largest eigenvalue. So there exists a unique left eigenvector $\boldsymbol{\pi}^{(j)}$ corresponding to λ_j normalized so that its first element is 1, that is $\boldsymbol{\pi}_1^{(j)} = 1$. The following result from Bose et al. (2009b) can be proved by induction and is used to show the non-degeneracy of the limit.

Lemma 2.11. *If the colors are in increasing order and the replacement matrix R is triangular, then all co-ordinates of the vector $\boldsymbol{\pi}^{(j)}$ are positive.*

The following index is important to obtain the rates of convergence.

Definition 2.12. For the jth block with leading color index i_j, let

$$\nu_j = \#\{m : r_m = \lambda_j, m < i_j\}$$

So, $\lambda_{j-1} = \lambda_j$ if and only if $\nu_j > 0$, and in this case $\nu_{j-1} = \nu_j - 1$ holds.

Now, we are ready to state the rates of convergence from Bose et al. (2009b).

Theorem 2.13. *Suppose that $\boldsymbol{R}$ is a $(K+1) \times (K+1)$ balanced triangular matrix with row sums equal to 1 and $(J+1)$ blocks, and that the colors are in increasing order, satisfying condition (2.6). Then for $j = 1, 2, \ldots J+1$,*

$$\frac{\boldsymbol{C}_N^{(j)}}{N^{\lambda_j} (\log N)^{\nu_j}} \to \boldsymbol{\pi}^{(j)} V_j$$

almost surely as well as in L^2, where $V_{J+1} = 1$. If $r_1 = 0$ then $V_1 = \boldsymbol{C}_{01}$. If $r_1 > 0$ then V_1 is a nondegenerate random variable. For $j = 2, 3, \ldots J$, if $\nu_j = 0$ then V_j is also a nondegenerate random variable. If $\nu_j > 0$, we further have

$$V_j = \frac{1}{\nu_j} \boldsymbol{\pi}^{(j-1)} \boldsymbol{\rho}^{(j-1)} V_{j-1}.$$

The rearrangement helps us to identify the limits in Theorem 2.13 under the assumption (2.6). However, Bose et al. (2009b) further showed that the rates of convergence of the color counts can be obtained even without the rearrangement. We describe the method, which is stated inductively, below:

For the first color,

$$\frac{C_{N1}}{N^{r_1}} \to W_1$$

almost surely and in L^2, for some non-degenerate random variable W_1. Assume that, for all $1 \leq j \leq k$, there exists s_j, δ_j, and random variables W_j such that

$$\frac{C_{Nj}}{N^{s_j}(\log N)^{\delta_j}} \to W_j$$

almost surely and in L^2, for some non-degenerate random variable W_1.

Next, we consider the $(k+1)$-th color. If the part of the $(k+1)$-th column of R above the diagonal has all entries zero, then, for some random variable W_{K+1}, we have

$$\frac{\boldsymbol{C}_{N,(k+1)}}{N^{r_{(k+1)}}} \to W_{k+1}$$

almost surely and in L^2. However if $r_{j,(k+1)} > 0$ for some $j = 1, 2, \ldots k$, consider all the colors indexed by j such that $r_{j,(k+1)} > 0$. Let the highest rate of convergence for such color counts be $n^s(\log n)^\delta$. Then

$$\frac{C_{N,(k+1)}}{a_n} \to W_{k+1}$$

almost surely and in L^2, for some non-degenerate random variable W_{k+1}, where

$$a_n = \begin{cases} n^s(\log n)^\delta & \text{if} \quad r_{k+1} < s \\ n^s(\log n)^{\delta+1} & \text{if} \quad r_{k+1} = s \\ n^{r_{k+1}} & \text{if} \quad r_{k+1} > s. \end{cases}$$

While the above algorithm gives the rates of convergence, it is not possible to identify the relations between the limiting random variables in absence of the assumption (2.6). We address this issue in Section 7.

3. Classification of three-color urns

Suppose we consider the colors as states and the replacement matrix as the transition matrix. We assume that each state communicates with itself that is $r_{ii} > 0$ for all i. We define a equivalence relation as i is equivalent to j if both $r_{ij} > 0$ and $r_{ji} > 0$. Each equivalence class formed by this relation is what we call a *communicating class*. Again we call a communicating class *essential* if no state of the class communicate with any state outside the class. Otherwise we call the class *inessential*.

We present here a full classification of the 3-color urns according to the number of communicating classes and according to essentialness of each class.

(i) Number of communicating classes:3

(a) All classes essential. In this case we have the replacement matrix as I_3. The result is done in Theorem 2.1 which shows that $\boldsymbol{C}_n/n+1$ converges to a Dirichlet random vector. So, the distribution of linear combinations of the colors is also known.

(b) One class inessential remaining two classes essential. Replacement matrix looks like this:

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $p_{11} < 1$.

If $p_{13} = 0$ then the replacement matrix takes the form:

$$\begin{pmatrix} p & 1-p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

where $p < 1$. Denote by τ_n the succesive times when either red or green balls are drawn. Then it is easy to see that $\tau_n \to \infty$ almost surely. Then $\boldsymbol{S}_{\tau_n}/\boldsymbol{S}_{\tau_n}\mathbf{1} \stackrel{a.s}{\to} (0,1)$. from Theorem 2.3 Again, $\boldsymbol{S}_n\mathbf{1}/n \stackrel{a.s}{\to} \text{Beta}(R_0 + G_0, B_0)$ from Gouet (1997). Again, $\frac{\boldsymbol{S}_n}{\boldsymbol{S}_n\mathbf{1}}$ remains constant for $\tau_k < n < \tau_{k+1}$. So $G_n/n \xrightarrow{\text{a.s.}} \text{Beta}(R_0+G_0, B_0)$. Again $B_n/n \xrightarrow{\text{a.s.}} \text{Beta}(B_0, R_0+G_0)$ from Gouet (1997) . Again from Theorem 2.3 we have $R_n/n^p \xrightarrow{\text{a.s.}} W$ for some non-degenerate random variable W. So we have the individual convergence rates. Also, note that the asymptotic distributions of the linear combinations of the colors also follow from the above results.
For $p_{13} > 0$, the refer to Bose et al. (2009b). The results are stated there.

(c) One class essential, remaining two inessential. We have the replacement matrix as follows

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ 0 & p_{22} & p_{23} \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

where $p_{11} < 1$, $p_{22} < 1$.
We have the following Theorem Bose et al. (2009b):

Theorem 3.1. *Suppose in the three color urn model we have replacement matrix given by* (3.2)*. If we further have $p_{12} > 0$ whenever $p_{11} = p_{22}$ then there exists non-degenerate random variables V_1, V_2, V_3 such that the following assertions hold:*

(i) $\boldsymbol{C}_{n,3}/n \to 1$.
(ii) *If $p_{11} = 0$, then $\boldsymbol{C}_{n,1}$ stays unchanged at $\boldsymbol{C}_{0,1}$. If $p_{11} > 0$ then $\boldsymbol{C}_{n,1}/n^{p_{11}} \to V_1$.*
(iii) *If $p_{22} > p_{11}$ then $\boldsymbol{C}_{n,2}/n^{p_{22}} \to V_2$.*
(iv) *If $p_{22} = p_{11}$ and $p_{12} > 0$ then $\boldsymbol{C}_{n,2}/(n^{p_{22}} \log n) \to p_{12}V_1$.*
(v) *If $p_{22} < p_{11}$ and $p_{12} > 0$ then $\boldsymbol{C}_{n,2}/n^{p_{11}} \to p_{12}V_1/(p_{11} - p_{22})$. If $0 < p_{22} < p_{11}$ and $p_{12} = 0$, then $\boldsymbol{C}_{n,2}/n^{p_{22}} \to V_3$.*
(vi) *If $p_{12} = p_{22} = 0$ then $\boldsymbol{C}_{n,2}$ stays unchanged at $\boldsymbol{C}_{0,2}$.*

The above convergences happen almost surely as well as in L^2.

To deal with the linear combinations we note that the eigenvectors are $\boldsymbol{\xi}_1 = (1,1,1)'$ and $\boldsymbol{\xi}_2 = (1,0,0)'$ corresponding to the eigenvalues 1 and p_{11}. Clearly $\boldsymbol{C}_n\boldsymbol{\xi}_1/(n+1) = 1$ for all n and $\boldsymbol{C}_n\boldsymbol{\xi}_2 = \boldsymbol{C}_{n,1}$ whose asymptotic properties are given by Theorem 3.1. If $p_{11} \neq p_{22}$ then the replacement matrix has another eigenvector $\boldsymbol{\xi}_2 = (p_{12}, p_{22} - p_{11}, 0)'$. Then we have the following Theorem:

Theorem 3.2. *suppose we have a replacement matrix given by 3.2. Assume $0 < p_{22} < p_{11}$ and $p_{12} > 0$. Let V_1 be the almost sure limit of $\boldsymbol{C}_{n,1}/n^{p_{11}}$ obtained in Theorem 3.1. Then the following assertions hold:*

(i) *If $p_{22} < p_{11}/2$ then*

$$\frac{\boldsymbol{C}_n\boldsymbol{\xi}_2}{\sqrt{n^{p_{11}}}} \Rightarrow N\left(0, \frac{p_{12}p_{22}^2(p_{12} + p_{11} - p_{22})}{p_{11} - 2p_{22}} V_1\right).$$

(ii) *If $p_{22} = p_{11}/2$ then*

$$\frac{\boldsymbol{C}_n\boldsymbol{\xi}_2}{\sqrt{n^{p_{11}} \log n}} \Rightarrow N\left(0, p_{12}p_{22}^2(p_{12} + p_{11} - p_{22})V_1\right)$$

(iii) *If $p_{22} > p_{11}/2$ then $\boldsymbol{C}_n\boldsymbol{\xi}_2/n^{p_{22}}$ converges almost surely as well as in L^2 to a non-degenerate random variable.*

Now, if the above condition of the Theorem 3.1 does not hold the matrix reduces to:

$$\begin{pmatrix} p & 0 & 1-p \\ 0 & p & 1-p \\ 0 & 0 & 1 \end{pmatrix}$$

But, in this case if we proceed in the same way as (3.1), then defining τ_n in the same way, we have $\boldsymbol{S}_{\tau_k}/\boldsymbol{S}_{\tau_k}\boldsymbol{1} \xrightarrow{\text{a.s.}} (V_1, V_2)$ where $V_1 \sim \text{Beta}(R_0/p, G_0/p)$ and $V_2 \sim \text{Beta}(G_0/p, R_0/p)$Freedman (1965), and $\boldsymbol{S}_n\boldsymbol{1}/n^p \xrightarrow{\text{a.s.}} V$ for some non-degenerate V (Theorem 2.3). Hence $\boldsymbol{S}_n/n^p \xrightarrow{\text{a.s.}} V(V_1, V_2)$. Also note, that in this case we know the marginal distributions of V and (V_1, V_2) only.

(ii) No of communicating classes: 2.

(a) Both classes essential. We have the replacement matrix of the following form:

$$\begin{pmatrix} p_1 & 1-p_1 & 0 \\ 1-p_2 & p_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $p_1 < 1 \quad p_2 < 1$.
In this case the results follow from Theorem 2.1. The results of linear combinations are also well known.

(b) Larger class essential, smaller class inessential. We have the replacement matrix of the form:

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ 0 & p_1 & 1-p_1 \\ 0 & 1-p_2 & p_2 \end{pmatrix}.$$

where $p_{12} > 0$ and $p_1 < 1 \quad p_2 < 1$. This is a three-color case with two dominant colors. So refer to subsection 2.6.

(c) Larger class inessential, smaller class essential. The replacement matrix takes the form:

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

where $p_{12} > 0 \quad p_{21} > 0 \quad p_{13} \vee p_{23} > 0$. This is the matrix that we are going to study in the subsequent articles. Note that the matrix reduces to the form (2.3). For $c = 1$ the results are stated in subsection 2.5.

(iii) Number of communicating classes: 1
This is the irreducible case. We refer to Gouet (1997) for comprehensive discussion.

4. Strong convergence for a three color model with one dominating color

The crucial result used for the balanced three-color model is that $\lambda_1 = 1$ and $\boldsymbol{\xi}_1 = (1, 1)'$. But this is not the case in the unbalanced case. So we cannot use the result for convergence of color proportions as used in Gouet's paper. We take a slightly different route. Let $\boldsymbol{\pi}_Q$ be the left eigenvector corresponding to λ_1 for Q. Note that such an eigenvector exists and has all non-negative entries by Perron-Frobenius. Also without loss of generality we can choose $\boldsymbol{\pi}_Q$ such that $\boldsymbol{\pi}_Q\boldsymbol{\xi}_1 = 1$.

Theorem 4.1. *Consider the three color model with replacement matrix as in* (2.3) *with* $c < 1$.

(i) $\boldsymbol{S}_n\boldsymbol{\xi}_1/n^{s\lambda_1} \to U$ *almost surely as well as in* L^2 *where* U *is a non degenerate random variable.*

(ii) $\boldsymbol{S}_n/n^{s\lambda_1} \xrightarrow{a.s.} \boldsymbol{\pi}_Q U$

Proof. Define $\boldsymbol{\chi}_{n+1}$ be the random variable takine values $(1,0)/(0,1)/(0,0)$ according as green, white or black is drawn at the $(n+1)$th draw. Then we have

$$\begin{aligned}
E(\boldsymbol{S}_{n+1}\boldsymbol{\xi}_1 \mid \mathcal{F}_n) &= E(\boldsymbol{S}_n\boldsymbol{\xi}_1 + s\boldsymbol{\chi}_{n+1}Q\boldsymbol{\xi}_1|\mathcal{F}_n)\\
&= \boldsymbol{S}_n\boldsymbol{\xi}_1 + \lambda_1 s\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\\
&= \boldsymbol{S}_n\boldsymbol{\xi}_1\left(1+\frac{\lambda_1 s}{n+1}\right)
\end{aligned}$$

Hence $T_n = \boldsymbol{S}_n\boldsymbol{\xi}_1/\Pi_n(s)$ is a positive martingale since $\boldsymbol{\xi}_1$ has all entries positive. Hence $T_n \xrightarrow{\text{a.s.}} U$ for some random variable U (say). Now since T_n is a martingale we have

$$E(T_{n+1}^2) = E(T_n^2) + E(T_{n+1}-T_n)^2$$

But an easy calculation shows that,

$$\begin{aligned}
E(T_{n+1}-T_n)^2 &= \frac{s^2\lambda_1^2}{(\Pi_{n+1}(s\lambda_1))^2}E\left[\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1^2}{n+1} - \left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^2\right] \qquad (4.1)\\
&\le \frac{Ks^2\lambda_1^2}{(n+1)\Pi_n^2(s\lambda_1)}E(\boldsymbol{S}_n\boldsymbol{\xi}_1)
\end{aligned}$$

where $K = \xi_{11} \vee \xi_{12}$ as $\boldsymbol{\xi}_1$ has non-negative entries

$$= \frac{c}{(n+1)\Pi_n(s\lambda_1)}E(T_n)$$

where $c = Ks^2\lambda_1^2$

$$\le \frac{c}{2(n+1)\Pi_n(s\lambda_1)}\left(1+E(T_n^2)\right)$$

So, we finally have,

$$\begin{aligned}
1+E(T_{n+1}^2) &\le (1+E(T_n^2))\left(1+O\left(\frac{1}{n^{1+s\lambda_1}}\right)\right)\\
&\le (1+E(T_1^2))\prod_{i=1}^{n}\left[1+O\left(\frac{1}{i^{1+s\lambda_1}}\right)\right]
\end{aligned}$$

Now as $s\lambda_1 > 0$ the product is finite. Hence $E(T_n^2)$ is bounded. This implies L^2 convergence holds. Hence from (1.1) we have

$$\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n^{s\lambda_1}} \xrightarrow{\text{a.s.}} U$$

as well as in L^2. Now note that if $\boldsymbol{S}_0 = (0,0)$ then only black balls would have been drawn and added which is an uninteresting case. So barring that we have (from Fatou's lemma) $E(U) \le \liminf E(T_n)$. So,

$$\begin{aligned}
Var(U) &= E(U^2) - (E(U))^2\\
&\ge \lim_n E(T_n^2) - \liminf_n (E(T_n))^2
\end{aligned}$$

$$\geq E(T_1)^2 - (E(T_1))^2$$

as T_n^2 is a submartingale and T_n is a martingale,

$$> 0. \tag{4.2}$$

This implies U is non-degenerate. So we have proved (i).

To prove (ii) our main strategy is to somehow prove : $\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_1} \xrightarrow{\text{a.s.}} 0$. For then we shall have

$$\frac{\boldsymbol{S}_n}{n^{s\lambda_1}} \xrightarrow{\text{a.s.}} (U,0)(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)^{-1}$$

which implies

$$\frac{\boldsymbol{S}_n}{n^{s\lambda_1}} \xrightarrow{\text{a.s.}} \boldsymbol{\pi}_Q U$$

and our proof is complete. We divide it into several cases:

Case 1 ($\lambda_2 > \lambda_1/2$):

Redefine $T_n = \boldsymbol{S}_n\boldsymbol{\xi}_2/\Pi_n(s\lambda_2)$. It is easy to see that this is a martingale. So we have:

$$\begin{aligned}
E(T_{n+1}^2) &= E(T_n^2) + E(T_{n+1} - T_n)^2 \\
&= E(T_n^2) + \frac{s^2\lambda_2^2}{[\Pi_{n+1}(s\lambda_2)]^2} E\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_2^2}{n+1} - \left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_2}{n+1}\right)^2\right) \text{(follows similarly from (4.1))} \\
&\leq E(T_n^2) + \frac{s^2\lambda_2^2 K \Pi_n(s\lambda_1)}{[\Pi_n(s\lambda_2)]^2(n+1)} E\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{\Pi_n(s\lambda_1)}\right)
\end{aligned} \tag{4.3}$$

where $K = \frac{\xi_{21}^2}{\xi_{11}} \vee \frac{\xi_{22}^2}{\xi_{12}}$ as both ξ_{11} and ξ_{12} are positive.

Now, from part (i), $E\left(\boldsymbol{S}_n\boldsymbol{\xi}_1/\Pi_n(s\lambda_1)\right)$ is bdd since, $\boldsymbol{S}_n\boldsymbol{\xi}_1/\Pi_n(s\lambda_1)$ is L^2 bdd, hence L^1 bdd. Now from (1.1) we have

$$\begin{aligned}
E(T_{n+1}^2) &\leq E(T_n^2) + O\left(\frac{n^{s\lambda_1}}{n^{2s\lambda_2+1}}\right) \\
&= E(T_n^2) + O\left(\frac{1}{n^{1+s(2\lambda_2-\lambda_1)}}\right)
\end{aligned}$$

Now, 2nd term on the right is summable since $2\lambda_2 > \lambda_1$. Hence T_n is L^2 bdd. Hence, for some non-degenerate random variable V, we have (Non degeneracy follows from (4.2)).

$$T_n \to V \tag{4.4}$$

almost surely as well as in L^2. This implies,

$$\frac{\boldsymbol{S}_n\boldsymbol{\xi}_2}{n^{s\lambda_1}} \xrightarrow{\text{a.s.}} 0$$

Case 2: ($\lambda_2 < \lambda_1/2$)

Define $X_n = \boldsymbol{S}_n \boldsymbol{\xi}_2 / n^{s\lambda_1/2}$. Then we have,

$$X_{n+1} = E(X_{n+1}|\mathcal{F}_n) + X_{n+1} - E(X_{n+1}|\mathcal{F}_n)$$

Now,

$$\begin{aligned} E(X_{n+1}|\mathcal{F}_n) &= \frac{\boldsymbol{S}_n \boldsymbol{\xi}_2}{(n+1)^{s\lambda_1/2}} + \frac{s\lambda_2}{(n+1)^{s\lambda_1/2}} \frac{\boldsymbol{S}_n \boldsymbol{\xi}_2}{(n+1)} \\ &= X_n \left(1 + \frac{1}{n}\right)^{-s\lambda_1/2} + \frac{s\lambda_2}{(n+1)} X_n \left(1 + \frac{1}{n}\right)^{-s\lambda_1/2} \\ &= X_n \left[1 - \left(\frac{s\lambda_1/2 - s\lambda_2}{n}\right)\right] + X_n O\left(\frac{1}{n^2}\right) \end{aligned} \tag{4.5}$$

We have the martigale difference as:

$$\begin{aligned} M_{n+1} &= X_{n+1} - E(X_{n+1}|\mathcal{F}_n) \\ &= \frac{s\lambda_2}{(n+1)^{s\lambda_1/2}} \left[\boldsymbol{\chi}_{n+1} - \frac{\boldsymbol{S}_n}{(n+1)}\right] \boldsymbol{\xi}_2 \end{aligned}$$

So, we have the recursion relation:

$$\begin{aligned} X_{n+1} &= X_n \left[1 - \frac{s}{n}\left(\frac{\lambda_1}{2} - \lambda_2\right)\right] + X_n O\left(\frac{1}{n^2}\right) + M_{n+1} \\ &= X_1 \prod_{i=1}^{n} \left[1 - \frac{s}{i}\left(\frac{\lambda_1}{2} - \lambda_2\right)\right] + \sum_{j=1}^{n} X_j O(j^{-2}) \prod_{i=j+1}^{n} \left[1 - \frac{s}{i}\left(\frac{\lambda_1}{2} - \lambda_2\right)\right] \\ &\qquad + \sum_{j=1}^{n} M_{j+1} \prod_{i=j+1}^{n} \left[1 - \frac{s}{i}\left(\frac{\lambda_1}{2} - \lambda_2\right)\right] \end{aligned} \tag{4.6}$$

Define

$$b_n = \prod_{i=1}^{n} \left[1 - \frac{s}{i}\left(\frac{\lambda_1}{2} - \lambda_2\right)\right] = \Pi_n(-s(\lambda_1/2 - \lambda_2)). \tag{4.7}$$

Note that, (1.1) gives

$$b_n \asymp n^{-s(\lambda_1/2 - \lambda_2)} \to 0$$

because ($\lambda_1/2 > \lambda_2$). Therefore first term goes almost surely to 0. Now,

$$|X_j| = \left|\frac{\boldsymbol{S}_j \boldsymbol{\xi}_2}{j.j^{s\lambda_1/2-1}}\right| = O(j^{-(s\lambda_1/2-1)})$$

almost everywhere. Therefore 2nd term is almost everywhere equal to

$$\begin{aligned} \sum_{j=1}^{n} X_j O(j^{-2}) \frac{b_n}{b_j} &= b_n \sum_{j=1}^{n} O\left(\frac{1}{j^{s\lambda_1/2+1-s(\lambda_1/2-\lambda_2)}}\right) \\ &= b_n \sum_{j=1}^{n} O(j^{-(s\lambda_2+1)}) \\ &\asymp n^{-s\lambda_1/2} \end{aligned} \tag{4.8}$$

So, 2nd term goes to 0 almost surely. Now, note that we want to prove:

$$\frac{X_n}{n^{s\lambda_1/2}} \to 0,$$

almost surely. So, it suffices to show:

$$\sum_{j=1}^{n} \frac{M_{j+1}}{n^{s\lambda_1/2}} \frac{b_n}{b_j} \xrightarrow{\text{a.s.}} 0$$

or $K_n D_n \xrightarrow{\text{a.s.}} 0$ where define

$$K_n = \sum_{j=1}^{n} \frac{M_{j+1}}{b_j}, \quad D_n = \frac{b_n}{n^{s\lambda_1/2}}.$$

We wish to show $K_n D_n \to 0$ in L^2 and $K_n D_n$ converges almost surely to some random variable. These two together will prove our claim.

To show $K_n D_n \to 0$ in L^2, we note that M_{j+1} is a martingale difference. Hence M_{j+1}/b_j is also a martingale difference. So, K_n is a martingale. So,

$$E(K_n^2) = \sum_{j=1}^{n} E\left(\frac{M_{j+1}^2}{b_j^2}\right).$$

Now,

$$\begin{aligned} E(M_{j+1}^2) &= \frac{s^2\lambda_2^2}{(j+1)^{s\lambda_1}} E\left[\frac{\boldsymbol{S}_j\boldsymbol{\xi}_2^2}{j+1} - \left(\frac{\boldsymbol{S}_j\boldsymbol{\xi}_2}{j+1}\right)^2\right] \\ &\leq K\frac{s^2\lambda_2^2}{(j+1)^{s\lambda_1}(j+1)} E(\boldsymbol{S}_j\boldsymbol{\xi}_1) \end{aligned} \tag{4.9}$$

where $K = \frac{\xi_{21}^2}{\xi_{11}} \vee \frac{\xi_{22}^2}{\xi_{12}}$ as $\xi_{11} \wedge \xi_{12} > 0$

$$= O(j^{-1})$$

Also $\boldsymbol{S}_j\boldsymbol{\xi}_1/j^{s\lambda_1}$ is L^2 bounded which implies L^1 bounded.

$$\begin{aligned} \sum_{j=1}^{n} E\left(\frac{M_{j+1}^2}{b_j^2}\right) &= \sum_{j=1}^{n} O\left(\frac{1}{j^{1-2s(\lambda_1/2-\lambda_2)}}\right) \quad \text{a.e.} \\ &\asymp n^{2s(\lambda_1/2-\lambda_2)} \end{aligned} \tag{4.10}$$

Now note that

$$D_n \asymp n^{-s(\lambda_1-\lambda_2)}. \tag{4.11}$$

So we have

$$\begin{aligned} E(K_n^2 D_n^2) &\asymp n^{2s(\lambda_1/2-\lambda_2)}.n^{-2s(\lambda_1-\lambda_2)} \\ &= n^{-s\lambda_1} \end{aligned}$$

This proves our claim of L^2 convergence.
We now show, $K_n D_n$ converges almost surely. We consider

$$\Delta(K_n D_n) = K_{n+1}D_{n+1} - K_n D_n$$

$$= K_{n+1}D_{n+1} - K_{n+1}D_n + K_{n+1}D_n - K_nD_n$$
$$= K_{n+1}\Delta D_n + \Delta K_n D_n \tag{4.12}$$

We first show, the 1st term of (4.12) is absolutely almost surely summable. Define $\alpha = s(\lambda_1 - \lambda_2)$. So we have:

$$\begin{aligned}
\Delta D_n &= D_{n+1} - D_n \\
&= \frac{b_{n+1}}{(n+1)^{s\lambda_1/2}} - \frac{b_n}{n^{s\lambda_1/2}} \\
&= \frac{b_{n+1} - b_n}{(n+1)^{s\lambda_1/2}} + b_n\left[\frac{1}{(n+1)^{s\lambda_1/2}} - \frac{1}{n^{s\lambda_1/2}}\right] \\
&\asymp \frac{n^{-s(\frac{\lambda_1}{2}-\lambda_2)}}{(n+1)^{s\lambda_1/2+1}} + \frac{1}{n^{s(\lambda_1-\lambda_2)}}\left[1 - \frac{s\lambda_1}{n} + O(n^{-2}) - 1\right] \\
&= O(n^{-(\alpha+1)})
\end{aligned} \tag{4.13}$$

From (4.10) and Cauchy-Schwarz inequality we have$E(|K_n|) = O(n^{s(\lambda_1/2-\lambda_2)})$. So, $E(|K_n\Delta D_n|) = O(n^{-(1+s\lambda_1/2)})$ which is summable. . So, $\sum_1^\infty |K_n\Delta D_n| < \infty$ almost surely and hence $\sum_1^N K_n\Delta D_n$ converges almost surely as N goes to infinity. This takes care of the first term.
For the second term we note that K_n is a martingale. So, $D_n\Delta K_n$ is a martingale difference. So, $\sum_1^N D_n\Delta K_n$ is a martingale. So, we have

$$E[(\sum_1^N D_n\Delta K_n)^2] = \sum_1^N E(D_n^2\Delta K_n^2)$$

. Now,

$$\begin{aligned}
E(\Delta K_n^2) &= E\left(\frac{M_{n+2}^2}{b_{n+1}^2}\right) \\
&= O(n^{-1+2s(\lambda_1/2-\lambda_2)})
\end{aligned} \tag{4.14}$$

so, from (4.11), we have

$$E(D_n^2\Delta K_n^2) = O(n^{-1-2s(\lambda_1-\lambda_2)+2s(\lambda_1/2-\lambda_2)}) \tag{4.15}$$

which is summable. So, $\sum_1^N D_n\Delta K_n$ is a martingale which is L^2 bdd hence, converges almost surely. So, both terms on the right hand side converges almost surely as N goes to infinity. This concludes K_nD_n converges almost surely.

Case 3: $\lambda_2 = \lambda_1/2$

Define $X_n = \boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_2}\sqrt{(\log n)} \quad n \geq 2$
We have

$$X_{n+1} = E(X_{n+1}|\mathcal{F}_n) + X_{n+1} - E(X_{n+1}|\mathcal{F}_n)$$

Now,

$$E(X_{n+1}|\mathcal{F}_n) = E\left(\frac{\boldsymbol{S}_{n+1}\boldsymbol{\xi}_2}{(n+1)^{s\lambda_2}\sqrt{\log(n+1)}}|\mathcal{F}_n\right)$$

$$= \frac{\boldsymbol{S}_n\boldsymbol{\xi}_2}{(n+1)^{s\lambda_2}\sqrt{\log(n+1)}} + \frac{s\lambda_2\boldsymbol{S}_n\boldsymbol{\xi}_2}{(n+1)^{s\lambda_2+1}\sqrt{\log(n+1)}}$$

$$= X_n\left(1+\frac{1}{n}\right)^{-s\lambda_2}\left[\frac{\log n}{\log(n+1)}\right]^{1/2}\left[1+\frac{s\lambda_2}{(n+1)}\right] \tag{4.16}$$

We use approximations like $\log(n+1) = \log n + 1/n + O(n^{-2})$
So, (4.16) reduces to

$$X_n\left(1-\frac{s\lambda_2}{n}+O(n^{-2})\right)\left(1+\frac{1}{n\log n}+O\left(\frac{1}{n^2\log n}\right)\right)^{-1/2}\left[1+\frac{s\lambda_2}{(n+1)}\right]$$
$$= X_n\left(1-\frac{s\lambda_2}{n}+O(n^{-2})\right)\left(1-\frac{1}{2n\log n}+O\left(\frac{1}{n^2\log n}\right)\right)\left[1+\frac{s\lambda_2}{(n+1)}\right]$$
$$= X_n\left(1-\frac{1}{2n\log n}-\frac{s\lambda_2}{n}+O(n^{-2})\right)\left[1+\frac{s\lambda_2}{(n+1)}\right]$$
$$= X_n\left(1-\frac{1}{2n\log n}\right)+X_nO(n^{-2})$$

Now the martingale difference is

$$M_{n+1} = X_{n+1} - E(X_{n+1}|\mathcal{F}_n)$$
$$= \frac{s\lambda_2}{(n+1)^{s\lambda_2}\sqrt{\log(n+1)}}\left(\boldsymbol{\chi}_{n+1}-\frac{\boldsymbol{S}_n}{n+1}\right)\boldsymbol{\xi}_2 \tag{4.17}$$

Therefore, we have similar type of recursion as in the previous case as follows:

$$X_2\prod_{i=2}^{n}\left[1-\frac{1}{2ilogi}\right]+\sum_{j=2}^{n}X_jO(j^{-2})\prod_{i=j+1}^{n}\left[1-\frac{1}{2ilogi}\right]+\sum_{j=2}^{n}M_{j+1}\prod_{i=j+1}^{n}\left[1-\frac{1}{2ilogi}\right] \tag{4.18}$$

Now, redefine

$$b_n = \prod_{i=2}^{n}(1-(2ilogi)^{-1}).$$

It is easy to see that $b_n \asymp (\log n)^{-1/2}$. So 1st term of (4.18) converges to zero almost surely. Now, $|X_j| = O(1/j^{s\lambda_2-1})$. Therefore second term of (4.18) is $\sum_{j=2}^{n} O(j^{s\lambda_2+1})$ almost surely. But, the product term goes to zero. So by D.C.T, second term almost surely converges to zero. But, we want to show:

$$\frac{X_n\sqrt{\log n}}{n^{s\lambda_2}} \xrightarrow{\text{a.s.}} 0$$

. That is, we are left to show:

$$\left(\frac{\sqrt{\log n}}{n^{s\lambda_2}}\right)\sum_{j=2}^{n}M_{j+1}\prod_{i=j+1}^{n}\left[1-\frac{\lambda_2}{ilogi}\right]$$

goes to zero almost surely. We proceed in similar way as case 2. We redefine : $D_n = b_n\left(\sqrt{\log n}/n^{s\lambda_2}\right)$ and $K_n = \sum_{j=1}^{n} M_{j+1}/b_j$. We first show : $K_nD_n \to 0$ in L^2. As before K_n is a martingale. So,

$$E(K_n^2) = \sum_{j=2}^{n} E(\frac{M_{j+1}^2}{b_j^2})$$
$$\leq \sum_{j=2}^{n}\frac{s^2\lambda_2^2}{b_j^2(j+1)^{2s\lambda_2+1}\log(j+1)}E(\boldsymbol{S}_j\boldsymbol{\xi}_2^2)$$

$$= \sum_{j=1}^{n} \frac{1}{b_j^2} O\left(\frac{1}{(j+1)\log(j+1)}\right) \tag{4.19}$$

(4.19) follows because of L^1 boundedness of $\boldsymbol{S}_j\boldsymbol{\xi}_1/j^{s\lambda_1}$.Now,

$$E(D_n^2 K_n^2) \leq \frac{\log n}{n^{2s\lambda_2}} \sum_{j=2}^{n} O((j\log j)^{-1}) \tag{4.20}$$

since $b_n^2/b_j^2 \leq 1$

$$\sim \frac{\log(n)\log(\log n)}{n^{2s\lambda_2}} \to 0 \tag{4.21}$$

Therefore $E(D_n^2 K_n^2)$ converges to 0 in L^2.
Now we show $D_n K_n$ converges almost surely to some random variable. We again break up as (4.12). So we are similarly done if we show $\sum_1^N D_n \Delta K_n$ is L^2 bounded and $\sum_1^\infty E|K_{n+1}\Delta D_n|$ is finite. Now,

$$\begin{aligned} E(\sum_1^N D_n \Delta K_n)^2 &= \sum_1^N D_n^2 E(\Delta K_n)^2 \\ &= \sum_1^N b_n^2 \left(\frac{\log n}{n^{2s\lambda_2}}\right) E\left(\frac{M_{n+2}^2}{b_{n+1}^2}\right) \\ &\leq \sum_1^N b_n^2 \left(\frac{\log n}{n^{2s\lambda_2}}\right) \frac{s^2\lambda_2^2}{b_{n+1}^2 (n+2)^{2s\lambda_2+1}\log(n+2)} E(\boldsymbol{S}_j \boldsymbol{\xi}_2^2) \\ &= \sum_1^N O\left(\frac{1}{n^{2s\lambda_2+1}}\right) \\ &\leq \sum_1^\infty O\left(\frac{1}{n^{2s\lambda_2+1}}\right) < \infty \end{aligned} \tag{4.22}$$

Now to show $\sum_1^\infty E \mid K_{n+1} | \Delta D_n$ is finite.

$$\begin{aligned} |\Delta D_n| &= \left| b_{n+1}\left(\frac{\sqrt{\log(n+1)}}{(n+1)^{s\lambda_2}}\right) - b_n \left(\frac{\sqrt{\log(n)}}{n^{s\lambda_2}}\right)\right| \\ &= \left|(b_{n+1}-b_n)\frac{\sqrt{\log(n+1)}}{(n+1)^{s\lambda_2}} + b_n \left(\frac{\sqrt{\log(n+1)}}{(n+1)^{s\lambda_2}} - \frac{\sqrt{\log n}}{n^{s\lambda_2}}\right)\right| \\ &\leq \left| b_n \frac{\sqrt{\log(n+1)}}{2n\sqrt{\log n}(n+1)^{s\lambda_2}} + O\left(\frac{1}{n^{s\lambda_2+1}}\right)\right| \end{aligned} \tag{4.23}$$

as $b_n \asymp (\log n)^{-1/2}$

$$= O(n^{-s\lambda_2 - 1}) \tag{4.24}$$

Now, from (4.21)

$$
\begin{aligned}
E(K_n^2 D_n^2) &= O\left(\frac{\log n \log(\log n)}{n^{2s\lambda_2}}\right) \\
\Rightarrow E(K_n^2) &= O\left(\frac{\log(\log n)}{b_n^2}\right) \\
\Rightarrow E \mid K_n| &= O(\frac{\sqrt{(\log(\log n))}}{b_n}) \\
\Rightarrow E|K_n \Delta D_n| &= \frac{\sqrt{\log n \log(\log n)}}{n^{s\lambda_2+1}}
\end{aligned}
$$

Now, this is summable. So in all cases we have proved $\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_1} \xrightarrow{\text{a.s.}} 0$. So from (i) and the fact that $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are linearly independent we have (ii). □

We now try to find out the weak limits of $\boldsymbol{S}_n\boldsymbol{\xi}_2$ with proper scaling.

Theorem 4.2. *With the same replacement matrix as in* (4.1) *we have the following*

(i) *If* $\lambda_2 < \lambda_1/2$ *then* $\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_1/2} \Rightarrow N(0, \frac{s^2\lambda_2^2 U\boldsymbol{\pi}_Q\xi_2^2}{s(\lambda_1-2\lambda_2)})$

(ii) *If* $\lambda_2 = \lambda_1/2$ *then* $\boldsymbol{S}_n\boldsymbol{\xi}_2/(n^{s\lambda_1/2}\sqrt{\log n}) \Rightarrow N(0, s^2\lambda_2^2 U\boldsymbol{\pi}_Q\xi_2^2)$.

(iii) *If* $\lambda_2 > \lambda_1/2$ *then* $\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_1/2}$ *converge almost surely as well as in* L^2 *to a non- degenerate random variable.*

where U *is the non-degenerate random variable where* $\boldsymbol{S}_n\boldsymbol{\xi}_1/n^{s\lambda_1}$ *converge almost surely .*

Proof. First note (iii) follows from (4.5). To prove (i) We take $X_n = \boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_1/2}$. Then we get a similar recursion like (4.6). The first two terms go to 0 almost surely as shown in the previous theorem. We redefine b_n as in (4.7). Checking Lyapunov condition

$$
\sum_{j=1}^{n} E(|M_{j+1}|^k|\mathcal{F}_j) \prod_{i=j+1}^{n} \left[1 - \frac{s}{i}(\lambda_1/2 - \lambda_2)\right]^k = \sum_{j=1}^{n} O\left(\frac{1}{j^{s\lambda_1 k/2}}\right)
$$

which is finite if we choose $k > 2/s\lambda_1$. Now we calculate the asymptotic variance :

$$
E(M_{n+1}^2|F_n) = \frac{s^2\lambda_2^2}{(n+1)^{s\lambda_1}}\left[\frac{\boldsymbol{S}_n\boldsymbol{\xi}_2^2}{(n+1)} - \left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_2}{n+1}\right)^2\right] \tag{4.25}
$$

$$
\sim \frac{s^2\lambda_2^2 U\boldsymbol{\pi}_Q\xi_2^2}{n+1} \tag{4.26}
$$

from (4.1). Now

$$
\prod_{i=j+1}^{n} \left[1 - \frac{s}{i}(\lambda_1/2 - \lambda_2)\right]^2 \sim \frac{n^{-s(\lambda_1-2\lambda_2)}}{j^{-s(\lambda_1-2\lambda_2)}}
$$

from (1.1). So

$$
\begin{aligned}
&\sum_{j=1}^{n} E(|M_{j+1}|^2|\mathcal{F}_j) \prod_{i=j+1}^{n} \left[1 - \frac{s}{i}(\lambda_1/2 - \lambda_2)\right]^2 \\
&\sim \sum_{j=1}^{n} \frac{s^2\lambda_2^2 U\boldsymbol{\pi}_Q\xi_2^2}{n^{s(\lambda_1-2\lambda_2)} j^{1-s(\lambda_1-2\lambda_2)}}
\end{aligned}
$$

$$\sim \frac{s^2\lambda_2^2 U\boldsymbol{\pi}_Q\xi_2^2 n^{s(\lambda_1-2\lambda_2)}}{n^{s(\lambda_1-2\lambda_2)}s(\lambda_1-2\lambda_2)} \tag{4.27}$$

So we have

$$\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_1/2} \Rightarrow N(0, \frac{s^2\lambda_2^2 U\boldsymbol{\pi}_Q\xi_2^2}{s(\lambda_1-2\lambda_2)})$$

the required variance mixture of normal.
To prove (ii) we note that if we take $X_n = \boldsymbol{S}_n\boldsymbol{\xi}_2/(n^{s\lambda_1/2}\sqrt{\log n})$ then we have the recursion as in (4.16) and the first two terms go to zero and the Lyapunov condition is satisfied as in (i). So to check the asymptotic variance. We redefine

$$b_n = \prod_{i=2}^{n}(1-(2ilogi)^{-1}).$$

and then we have

$$\begin{aligned}
&\sum_{j=1}^{n} E(M_{j+1}^2|\mathcal{F}_j)\frac{b_n^2}{b_j^2} \\
&= \sum_{j=1}^{n} \frac{s^2\lambda_2^2 b_n^2}{(j+1)^{2s\lambda_2}\log(j+1)b_j^2}\left[\frac{\boldsymbol{S}_j\boldsymbol{\xi}_2^2}{(j+1)} - \left(\frac{\boldsymbol{S}_j\boldsymbol{\xi}_2}{j+1}\right)^2\right] \\
&\sim \sum_{j=1}^{n} \frac{s^2\lambda_2^2 U\boldsymbol{\pi}_Q\xi_2^2}{(j+1)\log(j+1)}\frac{b_n^2}{b_j^2} \\
&\sim (\log n)^{-1} s^2\lambda_2^2 U\boldsymbol{\pi}_Q\xi_2^2 \sum_{j=1}^{n}\frac{1}{j} \\
&\sim s^2\lambda_2^2 U\boldsymbol{\pi}_Q\xi_2^2
\end{aligned}$$

So we have (ii). $\square$

5. L^p CONVERGENCE FOR A THREE COLOR MODEL WITH ONE DOMINATING COLOR

We denote by U the same random variable as in the previous section. In this section we seek the L^p convergence properties of the random variable $\boldsymbol{S}_n\boldsymbol{\xi}_1/n^{s\lambda_1}$ as well as $\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_2}$. We make the following claim:

Lemma 5.1. $\sup_n E\left(\boldsymbol{S}_n\boldsymbol{\xi}_1/\Pi_n(s\lambda_1)\right)^p < \infty$ *for* $p > 0$.

Proof. Let $V_n = \frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{\Pi_n(s\lambda_1)}$.For $p = 2$ the result is true from Theorem 4.1. Now

$$V_{n+1} - V_n = \frac{s\lambda_1}{\Pi_{n+1}(s\lambda_1)}\left[\boldsymbol{\chi}_{n+1}\boldsymbol{\xi}_1 - \frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right]$$

We will use induction. Assume true for $j = 2, 3, \ldots p-1$

$$\begin{aligned}
E(V_{n+1}^p|\mathcal{F}_n)) &= E[(V_n + V_{n+1} - V_n)^p|\mathcal{F}_n] \\
&= E\left[\sum_{k=0}^{p}\left(\binom{p}{k}V_n^k(V_{n+1}-V_n)^{p-k}\right)|\mathcal{F}_n\right] \\
&= V_n^p + E\left[\sum_{k=0}^{p-2}\binom{p}{k}V_n^k(V_{n+1}-V_n)^{p-k}|\mathcal{F}_n\right]
\end{aligned}$$

$$= V_n^p + E\left[\sum_{k=0}^{p-2}\binom{p}{k}V_n^k(V_{n+1}-V_n)^{p-k}|\mathcal{F}_n\right] \tag{5.1}$$

because $(p-1)$th term is equal to zero by the martingale property. Now for $2 \le l \le p$

$$E\left[(V_{n+1}-V_n)^l|\mathcal{F}_n\right] = E\left[\frac{(s\lambda_l)^l}{[\Pi_{n+1}(s\lambda_1)]^l}\left(\boldsymbol{\chi}_{n+1}\boldsymbol{\xi}_1 - \frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^l|\mathcal{F}_n\right]$$

Now note $\boldsymbol{\xi}_1$ has all entries positive. So let $b = \xi_{11} \vee \xi_{12}$ and $c = \xi_{11} \wedge \xi_{12}$. Here $c > 0$. Then we have,

$$\begin{aligned}
E\left[\left(\boldsymbol{\chi}_{n+1}\boldsymbol{\xi}_1 - \frac{\boldsymbol{S}_n\mathbf{1}}{n+1}\right)^l|\mathcal{F}_n\right] &= \left(\xi_{11} - \frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^l\frac{G_n}{n+1} + \left(\xi_{12} - \frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^l\frac{W_n}{n+1} + \left(-\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^l\frac{B_n}{n+1} \\
&\le \left(b + \frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^l\frac{\boldsymbol{S}_n\mathbf{1}}{n+1} + \left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^l\frac{B_n}{n+1} \\
&= \sum_{m=0}^{l} b^{l-m}\binom{l}{m}\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^m\frac{\boldsymbol{S}_n\mathbf{1}}{n+1} + \left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^l\left(1 - \frac{\boldsymbol{S}_n\mathbf{1}}{n+1}\right) \\
&\le \left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^l\left(1+\frac{bl}{c}\right) + \sum_{m=0}^{l-2}\binom{l}{m}\frac{b^{l-m}}{c}\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^{m+1} \qquad (5.2)
\end{aligned}$$

Putting it in (5.1) we have:

$$\begin{aligned}
&EV_{n+1}^p \\
&\le EV_n^p + \sum_{k=0}^{p-2}\binom{p}{k}E\left[V_n^k\frac{(s\lambda_1)^{p-k}}{\Pi_{n+1}(s\lambda_1)^{p-k}}\left(\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^{p-k}\left(1+\frac{b(p-k)}{c}\right) + \sum_{m=0}^{p-k-2}\binom{p-k}{m}\frac{b^{p-k-m}}{c}\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^{m+1}\right)\right] \\
&\le EV_n^p + \sum_{k=0}^{p-2}\binom{p}{k}E\left[\frac{V_n^kV_n^{p-k}}{(n+1)^{p-k}}\left(1+\frac{b(p-k)}{c}\right) + \sum_{m=0}^{p-k-2}\binom{p-k}{m}\frac{b^{p-k-m}}{c}\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^{m+1}\frac{V_n^k}{\Pi_{n+1}(s\lambda_1)^{p-k}}\right] \\
&= EV_n^p + \sum_{k=0}^{p-2}\binom{p}{k}\frac{EV_n^p}{(n+1)^{p-k}}\left(1+\frac{b(p-k)}{c}\right) + \sum_{k=0}^{p-2}\sum_{m=0}^{p-k-2}\binom{p}{k}\binom{p-k}{m}\frac{b^{p-k-m}}{c}\frac{EV_n^{m+1+k}}{(n+1)^{m+1}\Pi_n(s\lambda_1)^{p-k-m-1}}
\end{aligned}$$

Now note that $0 \le m \le p-k-2 \Rightarrow m+k+1 \le p-1$. So $\sup_n EV_n^{m+k+1} < \infty$ by induction processes. And also note that $0 \le k \le p-2 \Rightarrow p-k \ge 2$. Now using (1.1) we have

$$\begin{aligned}
EV_{n+1}^p &\le EV_n^p + EV_n^pO(n^{-2}) + \sum_{k=0}^{p-2}\sum_{m=0}^{p-k-2} O\left(\frac{1}{n^{m+1+s\lambda_1(p-k-m-1)}}\right) \\
&\le EV_n^p + EV_n^pO(n^{-2}) + \sum_{k=0}^{p-2}\sum_{m=0}^{p-k-2} O\left(\frac{1}{n^{1+(1-s\lambda_1)m+s\lambda_1(p-k-1)}}\right)
\end{aligned}$$

Now since $0 \le k \le p-2 \quad (p-k-1)s \ge s$ and $(1-s\lambda_1) > 0$. So we have

$$1 + EV_{n+1}^p \le (1+EV_n^p)\left[1 + O\left(\frac{1}{n^{1+s\lambda_1}}\right)\right] \tag{5.3}$$

This proves $\sup_n EV_n^p < \infty$ for $p \ge 2 \quad p \in \mathcal{N}$, hence true for all real $p > 0$. □

Corollary 5.2. $\boldsymbol{S}_n\boldsymbol{\xi}_1/n^{s\lambda_1} \to U$ *in* L^p *for all* $p > 0$.

Corollary 5.3. $\boldsymbol{S}_n/n^{s\lambda_1} \to U\boldsymbol{\pi}_Q$ *in* L^p *for all* $p > 0$.

Proof. Fix $p > 0$. We know $\boldsymbol{S}_n\boldsymbol{\xi}_1/n^{s\lambda_1} \to U$ in L^p as well as almost surely. Again, $\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_1} \to 0$ almost surely from the results obtained in the proof of Theorem 4.1. Again, as the elements of $\boldsymbol{\xi}_1$ are strictly positive, we have $\left|\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_1}\right| \le M\boldsymbol{S}_n\boldsymbol{\xi}_1/n^{s\lambda_1}$ for some constant $M > 0$. So, from Pratt's lemma, $\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_1} \to 0$ in L^p. So, $\boldsymbol{S}_n/n^{s\lambda_1} \to U\boldsymbol{\pi}_Q$ in L^p as the transformation $\boldsymbol{z} \mapsto \boldsymbol{z}(\boldsymbol{\xi}_1\boldsymbol{\xi}_2)^{-1}$ is linear and lipschitz. □

If $\lambda_2 > \lambda_1/2$ then we saw $\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_2}$ converges almost everywhere as well as in L^2 to some non-degenerate random variable U. We prove the general result for L^p here.

Theorem 5.4. *$\boldsymbol{S}_n\boldsymbol{\xi}_2/n^{s\lambda_2} \to U$ in L^p provided $p_e < \lambda_1/(\lambda_1 - \lambda_2)$ where p_e is the smallest even integer greater that p.*

Proof. Let $T_n = \boldsymbol{S}_n\boldsymbol{\xi}_2/\Pi_n(s\lambda_2)$. We know T_n is L^2 bounded. Suppose it is L^j bounded for $j = 2, 4, \dots 2p-2$. Then we proceed by induction. Note that in this case

$$T_{n+1} - T_n = \frac{s\lambda_2}{\Pi_{n+1}(s\lambda_2)}\left[\boldsymbol{\chi}_{n+1}\boldsymbol{\xi}_2 - \frac{\boldsymbol{S}_n\boldsymbol{\xi}_2}{n+1}\right]$$

Now again we have

$$\begin{aligned}
E(T_{n+1}^{2p}|\mathcal{F}_n)) &= E[(T_n + T_{n+1} - T_n)^{2p}|\mathcal{F}_n] \\
&= E\left[\sum_{k=0}^{2p}\left(\binom{2p}{k}T_n^k(T_{n+1} - T_n)^{2p-k}\right)|\mathcal{F}_n\right] \\
&= T_n^{2p} + E\left[\sum_{k=0}^{2p-2}\binom{2p}{k}T_n^k(T_{n+1} - T_n)^{2p-k}|\mathcal{F}_n\right] \\
&\le T_n^{2p} + E\left[\sum_{k=0}^{2p-2}\binom{2p}{k}|T_n|^k|T_{n+1} - T_n|^{2p-k}|\mathcal{F}_n\right]
\end{aligned}$$

Now, for $2 \le l \le p$

$$\begin{aligned}
E\left[\left|\boldsymbol{\chi}_{n+1}\xi_2 - \frac{\boldsymbol{S}_n\boldsymbol{\xi}_2}{n+1}\right|^l \mid \mathcal{F}_n\right] &= E\left[\sum_{m=0}^{l}\binom{l}{m}|\boldsymbol{\chi}_{n+1}\xi_2|^m\left|\frac{\boldsymbol{S}_n\boldsymbol{\xi}_2}{n+1}\right|^{l-m}|\mathcal{F}_n\right] \\
&\le E\left[c\sum_{m=0}^{l}\binom{l}{m}\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^{l-m+1}|\mathcal{F}_n\right]
\end{aligned}$$

For some large c. Now again note:

$$\begin{aligned}
|T_n|^k &= \left|\frac{\boldsymbol{S}_n\boldsymbol{\xi}_2}{\Pi_n(s\lambda_2)}\right|^k \\
&\le c'\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{\Pi_n(s\lambda_1)}\right)^k\left(\frac{\Pi_n(s\lambda_1)}{\Pi_n(s\lambda_2)}\right)^k
\end{aligned}$$

for some large c'. So we have

$$\begin{aligned}
ET_{n+1}^{2p} &\le ET_n^{2p} + cE\left[\sum_{k=0}^{2p-2}\binom{2p}{k}|T_n|^k\sum_{m=0}^{2p-k}\binom{2p-k}{m}\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{n+1}\right)^{2p-k-m+1}\frac{(s\lambda_2)^{2p-k}}{[\Pi_{n+1}(s\lambda_2)]^{2p-k}}\right] \\
&\le ET_n^{2p} + cc'E\left[\sum_{k=0}^{2p-2}\binom{2p}{k}\left(\frac{\Pi_n(s\lambda_1)}{\Pi_n(s\lambda_2)}\right)^k\sum_{m=0}^{2p-k}\binom{2p-k}{m}\left(\frac{\boldsymbol{S}_n\boldsymbol{\xi}_1}{\Pi_n(s\lambda_1)}\right)^{2p-m+1}\frac{(\Pi_n(s\lambda_1))^{2p-k-m+1}(s\lambda_2)^{2p-k}}{(n+1)^{2p-k-m+1}(\Pi_{n+1}(s\lambda_2))^{2p-k}}\right] \\
&\le ET_n^{2p} + M^*\sum_{k=0}^{2p-2}\sum_{m=0}^{2p-k}O\left(\frac{1}{n^{(1-s\lambda_1)(2p-k-m+1)+(2p-k)s\lambda_2+ks(\lambda_2-\lambda_1)}}\right)
\end{aligned}$$

for some large number M^*

Now consider the exponent of n. It is enough to show that this is strictly greater than 1 for all values of k and m.

$$2p - k - m + 1 + s[2p\lambda_2 - \lambda_1(2p - m + 1)]$$

Now note that $0 \leq k \leq 2p-2$ implies that $2p-k+1 \geq 3$. Again $0 \leq m \leq 2p-k$ implies that $2p-k-m+1 \geq 1$. So the exponent is at least

$$1 + \lambda_1 s(m-2) + s[2p\lambda_2 - \lambda_1(2p-1)]$$

So if $m \geq 2$ we are done. If $m = 1$, the exponent reduces to $2p - k + s[2p\lambda_2 - \lambda_1 2p]$ Now as $2p - k \geq 2$ and $2p\lambda_2 - 2p\lambda_1 > -\lambda_1 > -1$ (condition required as per the theorem), we are done. Now if $m = 0$ exponent reduces to $2p - k + 1 + s[2p\lambda_2 - \lambda_1(2p+1)]$. Now $2p - k + 1 \geq 3$ and $2p\lambda_2 - (2p+1)\lambda_1 > -2\lambda_1 > -2$ gives us our result. $\square$

6. Limit distribution for a three color urn model with two dominating colors

Suppose we have a 3-color urn with colors green, white and black and the initial number of balls (G_0, W_0, B_0), with the following replacement matrix:

$$R = \begin{pmatrix} \gamma & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.1}$$

where $\gamma + \alpha + \beta = 1$ and $\alpha \neq \beta$. We have $\boldsymbol{C}_n = (G_n, W_n, B_n)$ We know from Subection 2.7 the rates of convergence of the individual colors. So we have

$$\frac{G_n}{n^\gamma} \to G$$
$$\frac{W_n}{n} \to W$$
$$\frac{B_n}{n} \to (1 - W)$$

almost surely as well as in L^2. Our aim is to find the distribution of W.

Let $\tau_1, \tau_2, \ldots$ be the succesive times a green ball is drawn. Note that once a green ball is drawn, α white and β black balls are added. If we mark the white and black balls added at the kth drawing (say) then because of the structure of the replacement matrix they sort of evolve among themselves without adding any green ball to the urn. With this motivation we arrive at the following definition:

Definition 6.1. Consider an urn model with the replacement matrix given by (6.1). Mark the white balls put in the urn due to p th drawing of green with a mark $'p'$. In the subsequent drawings if we draw a $'p'$ marked white ball we add a white ball and mark it with $'p'$. $W_n^{(p)}$ is defined as the number of $'p'$ marked balls in the urn at the n th step. So, formally

$$W_n^{(p)} = \begin{cases} 0 & \text{if} \quad n < \tau_p \\ \alpha & \text{if} \quad n = \tau_p \\ W_{n-1}^{(p)} + \chi_n^{(p)} & \text{if} \quad n > \tau_p \end{cases}$$

where $\chi_n^{(p)}$ is 1 or 0 according as a $'p'$ marked ball is drawn at the n th drawing or not.

Define the sigma-field $\mathcal{F}_p^{(n)} = \sigma(W_j^{(p)}, j \geq n+1)$ and $X_n^{(p)} = W_{n+1}^{(p)} - W_n^{(p)}$

Now note that,

$$P(X_j^{(p)} = \epsilon_j, l \le j \le l+k, \quad \tau_p = l|\mathcal{F}_p^{(n)}) = P(\tau_p = l|\mathcal{F}_p^{(n)}) \frac{\binom{n-l-k}{W_{n+1}^{(p)} - \alpha - \sum_{j=l}^{l+k} \epsilon_j}}{\binom{n-l}{W_{n+1}^{(p)} - \alpha}} \tag{6.2}$$

Here ϵ_j, $j = 1, \dots l-k$ takes values 1 or 0. (6.2) holds because of some sort of conditional exchangeability of the event $\{X_j^{(p)} = \epsilon_j, l \le j \le l+k, \quad \tau_p = l\}$ which can be clearly seen from the following explicit calculation:

$$\begin{aligned} & P(X_j^{(p)} = \epsilon_j, l \le j \le n, \quad \tau_p = l|\mathcal{F}_p^{(n)}) \\ = & \sum_{i_1, i_2, \dots i_l \in A_l^{(p)}} \frac{i_1 i_2 \dots i_l}{(G_0 + W_0 + B_0)(G_0 + W_0 + B_0 + 1) \dots (G_0 + W_0 + B_0 + l - 1)} \\ & \times \frac{\delta_l(\delta_l + 1) \dots (\delta_l + (n-l) - \sum_{j=l}^{n} \epsilon_j - 1)) \alpha(\alpha+1) \dots (\alpha + \sum_{j=l}^{n} \epsilon_j - 1)}{(l + G_0 + W_0 + B_0) \dots (G_0 + W_0 + B_0 + n - 1)} \end{aligned}$$

where $\delta_l = (G_0 + W_0 + B_0 + l - \alpha)$ and $A_l^{(p)}$ is the set of all $\{i_1, i_2, \dots i_l\}$ which satisfy the following conditions

(i) $i_l = G_0 + (p-1)\gamma$

(ii) $\exists \{k_1, k_2, \dots k_{p-1}\} \subseteq \{1, 2, \dots l\} \ni i_{k_1} = G_0, \dots i_{k_{p-1}} = G_0 + (p-2)\gamma$

(iii) for $i_{k_s} < i_j < i_{k_{s+1}}, i_j = (G_0 + W_0 + B_0 + j - 1 - (G_0 + (s-1)\gamma))$ for all $s = 1(1)(p-2)$

As the expression depends on ϵ_j only through $\sum_{j=l}^{n} \epsilon_j$, (6.2) is valid. Also note that the first term in the expression is precisely $P(\tau_p = l|\mathcal{F}_p^{(n)})$, that is,

$$P(\tau_p = l|\mathcal{F}_p^{(n)}) = \sum_{i_1, i_2, \dots i_l \in A_l^{(p)}} \frac{i_1 i_2 \dots i_l}{(G_0 + W_0 + B_0)(G_0 + W_0 + B_0 + 1) \dots (G_0 + W_0 + B_0 + l - 1)}$$

Now define $(G_0 + W_0 + B_0) = I_0$

$$M_{j+1}^{(p)} = \frac{W_{j+1}^{(p)}}{(I_0 + 1 + j)} I_{(j \ge \tau_p)}$$

Define the filtration $\mathcal{G}_j = \sigma(\mathcal{F}_{j,p}; G_{k, 0 \le k \le j})$ where $\mathcal{F}_{j,p} = \sigma(W_i^{(p)}, 0 \le i \le j)$. so we have

$$\begin{aligned} E(M_{j+1}|\mathcal{G}_j) &= E\left(\frac{W_{j+1}^{(p)}}{j+1+I_0} I_{(j \ge \tau_p)} |\mathcal{G}_j \right) \\ &= E\left(\frac{W_j^{(p)} + \chi_{j+1}^{(p)}}{j+1+I_0} I_{(j \ge \tau_p)} |\mathcal{G}_j \right) \\ &= \frac{W_j^{(p)} + W_j^{(p)}/(j+I_0)}{j+1+I_0} I_{(j \ge \tau_p)} \\ &= \frac{W_j^{(p)}}{j+I_0} I_{(j \ge \tau_p)} \qquad (6.3) \\ &\ge \frac{W_j^{(p)}}{j+I_0} I_{(j-1 \ge \tau_p)} \end{aligned}$$

So, $M_j^{(p)}$ is a submartingale which is obviously bounded by 1. Hence it converges to some random variable, say $Z^{(p)}$. Now as G_n goes to infinity almost surely, we conclude: $I_{(j \ge \tau_p)}$ goes to 1 almost surely. Hence we have

$$\frac{W_j^{(p)}}{j+I_0} \to Z^{(p)} \tag{6.4}$$

almost surely.

Now we have the following theorem regarding the joint distribution of $(Z^{(p)}, Z^{(p+1)})$ for some integer $p \geq 0$.

Theorem 6.2.

$$(Z^{(p)}, Z^{(p+1)}) \sim Dirichlet(W^{(p)}_{\tau_{p+1}}, \alpha, \tau_{p+1} + G_0 + B_0 + W_0 - W^{(p)}_{\tau_{p+1}} - \alpha)$$

Proof. Take $\mathcal{F}^{(n)}_{p,p+1} = \mathcal{F}^{(n)}_p \cap \mathcal{F}^{(n)}_{p+1}$ for $n > p+1$. So we can clearly see that,

$$\begin{aligned}
&P(X^{(i)}_j = \epsilon^{(i)}_j, i = p, p+1; l \leq j \leq n; \tau_{p+1} = l; W^{(p)}_l = b^{(p)} | \mathcal{F}^{(n)}_{p,p+1}) \\
&= P(\tau_{p+1} = l, W^{(p)}_l = b^{(p)} | \mathcal{F}^{(n)}_{p,p+1}) \\
&\times b^{(p)}(b^{(p)} + 1) \ldots (b^{(p)} + \sum_{j=l}^{n} \epsilon^{(p)}_j - 1) \\
&\times \alpha(\alpha+1) \ldots (\alpha + \sum_{j=l}^{n} \epsilon^{(p+1)}_j - 1) \delta(\delta+1) \ldots (\delta + n - l - \sum_{i=p}^{p+1} \sum_{j=l}^{n} \epsilon^{(i)}_j - 1) \\
&\times \frac{1}{(l + G_0 + W_0 + B_0)(l + G_0 + W_0 + B_0 + 1) \ldots (n - 1 + G_0 + W_0 + B_0)}
\end{aligned} \tag{6.5}$$

where $\delta = l + G_0 + W_0 + B_0 - \alpha - b^{(p)}$. Hence we can write the following probability as

$$\begin{aligned}
&P(X^{(i)}_j = \epsilon^{(i)}_j, i = p, p+1; l \leq j \leq l + k - 1; \tau_{p+1} = l; W^{(p)}_l = b^{(p)} | \mathcal{F}^{(n)}_{p,p+1}) \\
&= P(\tau_{p+1} = l, W^{(p)}_l = b^{(p)} | \mathcal{F}^{(n)}_{p,p+1}) \frac{(n-l-k+1)!}{(n-l+1)!} \\
&\times \frac{(W^{(p)}_{n+1} - b^{(p)})!(W^{(p+1)}_{n+1} - b^{(p+1)})!(n - l - W^{(p)}_{n+1} - W^{(p+1)}_{n+1} + b^{(p)} + b^{(p+1)})!}{(W^{(p)}_{n+1} - b^{(p)} - \sigma^{(p)})!(W^{(p+1)}_{n+1} - b^{(p+1)} - \sigma^{(p+1)})!(n - l - k + 1 - W^{(p)}_{n+1} - W^{(p+1)}_{n+1} + b^{(p)} + b^{(p+1)} + \sigma^{(p)} + \sigma^{(p+1)})!}
\end{aligned}$$

where $\sigma^{(i)} = \sum_{j=l}^{l+k-1} \epsilon^{(i)}_j$, $i = p, p+1$. So we have,

$$\begin{aligned}
&P(X^{(i)}_j = \epsilon^{(i)}_j, i = p, p+1, l \leq j \leq l + k - 1 | \tau_{p+1} = l; W^{(p)}_l = b^{(p)}, \mathcal{F}^{(n)}_{p,p+1}) \\
&= \frac{a_n! c'^{(p)}_n! c'^{(p+1)}_n! d'_n!}{a'_n! c^{(p)}_n! c^{(p+1)}_n! d_n!}
\end{aligned} \tag{6.6}$$

where

$$\begin{aligned}
a_n &= (n - l - k + 1), \quad a'_n = (n - l + 1) \\
c^{(i)}_n &= (W^{(i)}_{n+1} - b^{(i)} - \sigma^{(i)}), \quad c'^{(i)}_n = (W^{(i)}_{n+1} - b^{(i)}) \quad i = p, p+1 \\
d_n &= \left(n - l - k + 1 - W^{(p)}_{n+1} - W^{(p+1)}_{n+1} + b^{(p)} + b^{(p+1)} + \sigma^{(p)} + \sigma^{(p+1)}\right) \\
d'_n &= \left(n - l - W^{(p)}_{n+1} - W^{(p+1)}_{n+1} + b^{(p)} + b^{(p+1)}\right)
\end{aligned}$$

We want to find the limit of (6.6) as n goes to infinity. Because of (6.4) we know that each term in factorials go to infinity hence we can apply stirling's approximation. Also note that

$$c^{(p)}_n + c^{(p+1)}_n + d_n = a_n \tag{6.7}$$

$$c'^{(p)}_n + c'^{(p+1)}_n + d'_n = a'_n \tag{6.8}$$

So from (6.4), (6.7), (6.8), we reduce (6.6) to

$$\frac{a_n^{(a_n+1/2)} \left(c'^{(p)}_n\right)^{(c'^{(p)}_n + 1/2)} \left(c'^{(p+1)}_n\right)^{(c'^{(p+1)}_n + 1/2)}}{a'^{a'_n+1/2}_n \left(c^{(p)}_n\right)^{(c^{(p)}_n + 1/2)} \left(c^{(p+1)}_n\right)^{(c^{(p+1)}_n + 1/2)} d_n^{(d_n + 1/2)}} \tag{6.9}$$

But, we observe

$$\begin{aligned} a_n/a'_n &\to 1 \\ c'^{(i)}_n/c^{(i)}_n &\to 1 \\ d_n/d'_n &\to 1 \end{aligned}$$

almost surely for $i = p, p+1$. Hence (6.9) reduces to

$$\begin{aligned} &\frac{a_n^{a_n}(c'^{(p)}_n)^{c'^{(p)}_n}(c'^{(p+1)}_n)^{(c'^{(p+1)}_n}d'^{d'_n}_n}{a'^{a'_n}_n(c^{(p)}_n)^{c^{(p)}_n}(c^{(p+1)}_n)^{c^{(p+1)}_n}d_n^{d_n}} \\ =&\frac{(c'^{(p)}_n/a'_n)^{c'^{(p)}_n}(c'^{(p+1)}_n/a'_n)^{c'^{p+1}_n}(d'_n/a'_n)^{d'_n}}{(c^{(p)}_n/a_n)^{c^{(p)}_n}(c^{(p+1)}_n/a_n)^{c^{(p+1)}_n}(d_n/a_n)^{d_n}} \end{aligned}$$

The last equality follows from (6.7) and (6.8). The calculation of the limit follows from the following two lemmas.

Lemma 6.3.

$$\frac{(c'^{(i)}_n/a'_n)^{c'^{(i)}_n}}{(c^{(i)}_n/a_n)^{c^{(i)}_n}} \to (Z^{(i)})^{\sigma^{(i)}}e^{\sigma^{(i)}-kZ^{(i)}}$$

almost surely. Here $i = p, p+1$.

Proof. We observe from (6.4)

$$\left(\frac{W^{(i)}_{n+1} - b^{(i)} - \sigma^{(i)}}{n-l-k+1}\right)^{\sigma^{(i)}} \to (Z^{(i)})^{\sigma^{(i)}}$$

almost surely. Taking log of the remaining part we get,

$$\begin{aligned} &\log\left(\frac{(W^{(i)}_{n+1} - b^{(i)})/(n-l+1)}{(W^{(i)}_{n+1} - b^{(i)} - \sigma^{(i)})/(n-l-k+1)}\right)^{(W^{(i)}_{n+1}-b^{(i)})} \\ &= (W^{(i)}_{n+1} - b^{(i)})\left[\log\left(1 - \frac{k}{n-l+1}\right) - \log\left(1 - \frac{\sigma^{(i)}}{(W^{(i)}_{n+1} - b^{(i)})}\right)\right] \\ &= \frac{(W^{(i)}_{n+1} - b^{(i)})}{n-l+1}(n-l+1)\log\left(1 - \frac{k}{n-l+1}\right) - (W^{(i)}_{n+1} - b^{(i)})\log\left(1 - \frac{\sigma^{(i)}}{(W^{(i)}_{n+1} - b^{(i)})}\right) \\ &\xrightarrow{\text{a.s.}} -Z^{(i)}k + \sigma^{(i)} \end{aligned} \tag{6.10}$$

This proves the lemma. ☐

Lemma 6.4.

$$\frac{(d'_n/a'_n)^{d'_n}}{(d_n/a_n)^{d_n}} \to \left(1 - Z^{(p)} - Z^{(p+1)}\right)^{(k-\sigma^{(p)}-\sigma^{(p+1)})}e^{(k(Z^{(p)}+Z^{(p+1)})-\sigma^{(p)}-\sigma^{(p+1)})}$$

almost surely.

Proof. From (6.4) we have

$$\left(1 - \frac{(W^{(p)}_{n+1} + W^{(p+1)}_{n+1}) - (b^{(p)} + b^{(p+1)}) - (\sigma^{(p)} + \sigma^{(p+1)})}{n-l-k+1}\right)^{(k-\sigma^{(p)}-\sigma^{(p+1)})} \xrightarrow{\text{a.s.}} (1-Z^{(p)}-Z^{(p+1)})^{(k-\sigma^{(p)}-\sigma^{(p+1)})}$$

Remaining calculations are similar to the previous lemma. ☐

So applying Lemma 6.4 and Lemma 6.3 we get the limit of (6.6) to be

$$(Z^{(p)})^{\sigma^{(p)}}(Z^{(p+1)})^{\sigma^{(p+1)}}e^{(\sigma^{(p)}+\sigma^{(p+1)}-k(Z^{(p)}+Z^{(p+1)}))}(1-Z^{(p)}-Z^{(p+1)})^{(k-\sigma^{(p)}-\sigma^{(p+1)})}e^{(k(Z^{(p)}+Z^{(p+1)})-\sigma^{(p)}-\sigma^{(p+1)})}$$

so we have,

$$\begin{aligned}&(Z^{(p)})^{\sigma^{(p)}}(Z^{(p+1)})^{\sigma^{(p+1)}}(1-Z^{(p)}-Z^{(p+1)})^{(k-\sigma^{(p)}-\sigma^{(p+1)})}\\&=P(X_j^{(i)}=\epsilon_j^{(i)}, i=p,p+1; l\le j\le l+k-1|\tau_{p+1}=l; W_l^{(p)}=b^{(p)},\mathcal{F}^{(\infty)}_{p,p+1})\end{aligned}$$

taking limit on the left hand side of (6.6).

So now fix k_1, k_2. Then take $k=k_1+k_2$. Then take

$$\epsilon_j^{(p)}=\begin{cases}1 & \text{for } l\le j\le l+k_1-1\\ 0 & \text{otherwise.}\end{cases}$$

$$\epsilon_j^{(p+1)}=\begin{cases}1 & \text{for } l+k_1\le j\le l+k-1\\ 0 & \text{otherwise.}\end{cases}$$

Now consider

$$P(X_j^{(i)}=\epsilon_j^{(i)}, i=p,p+1; \tau_{p+1}\le j\le \tau_{p+1}+k-1) \tag{6.11}$$

Taking conditional expectation (6.11) is equal to

$$\begin{aligned}&E[P(X_j^{(i)}=\epsilon_j^{(i)}, i=p,p+1; \tau_{p+1}\le j\le\tau_{p+1}+k-1|\tau_{p+1}=l, W_l^{(p)}=b^{(p)},\mathcal{F}^{(\infty)}_{p,p+1})]\\&=E[(Z^{(p)})^{k_1}(Z^{(p+1)})^{k_2}]\end{aligned} \tag{6.12}$$

Now again we decompose (6.11) in a different way. In the following calculations the sum is over all possible values of $\{l, b^{(1)}\}$. We have,

$$\begin{aligned}&\sum P(X_j^{(i)}=\epsilon_j^{(i)}, i=p,p+1; \tau_{p+1}\le j\le\tau_{p+1}+k-1|\tau_{p+1}=l, W_l^{(p)}=b^{(p)})\\&\qquad\qquad\times P(\tau_{p+1}=l, W_l^{(p)}=b^{(p)})\\&=\sum\frac{b^{(p)}(b^{(p)}+1)\dots(b^{(p)}+k_1-1)\alpha(\alpha+1)\dots(\alpha+k_2-1)}{(l+G_0+B_0+W_0)(l+G_0+B_0+W_0+1)\dots(l+G_0+B_0+W_0+k-1)}\\&\qquad\qquad\times P(\tau_{p+1}=l, W_l^{(p)}=b^{(p)})\end{aligned}$$

But this is equal to $E(U_1^{k_1}U_2^{k_2})$ where

$$(U_1,U_2)\sim\text{Dirichlet}(W^{(p)}_{\tau_{p+1}},\alpha,\tau_{p+1}+G_0+B_0+W_0-W^{(p)}_{\tau_{p+1}}-\alpha)$$

So, we have

$$(Z^{(p)},Z^{(p+1)})\sim\text{Dirichlet}(W^{(p)}_{\tau_{p+1}},\alpha,\tau_{p+1}+G_0+B_0+W_0-W^{(p)}_{\tau_{p+1}}-\alpha)$$

This proves the theorem. □

Theorem 6.2 can be easily generalized to the following theorem. Proof is essentially the same.

Theorem 6.5.

$$(Z^{(p)},Z^{(p+1)},\dots Z^{(p+r)})\sim\textit{Dirichlet}(W^{(p)}_{\tau_{p+r}},W^{(p+1)}_{\tau_{p+r}},\dots W^{(p+r-1)}_{\tau_{p+r}},\alpha,\tau_{p+r}+G_0+B_0+W_0-\sum_{i=p}^{p+r-1}W^{(i)}_{\tau_{p+r}}-\alpha)$$

Remark 6.6. Note that we can find the marginal of $Z^{(p)}$ from the above theorem which turns out to be Beta$(\alpha,\tau_p+G_0+B_0+W_0-\alpha)$.

We were initially interested in finding the limiting distribution of W_n/n. Now as we observe $W_n/n=\sum_{p=0}^{\infty}W_n^{(p)}/n$, it can be intuitively guessed that the limit should be $\sum_{p=0}^{\infty}Z^{(p)}$, and we already know from Theorem 6.2 the finite dimensional distributions of $Z^{(p)}$. Hence we need to prove the following theorem:

Theorem 6.7.

$$\frac{W_n}{n} \to \sum_{p=0}^{\infty} Z^{(p)}$$

almost surely.

Proof. Consider $G_{\tau_k}/(\tau_k)^\gamma$. Now observe that $G_{\tau_k}/(\tau_k)^\gamma \le \sup_n (G_n/n^\gamma)$ for all k. As $G_n/\Pi_n(\gamma)$ is an L^2 bounded martingale it follows from doob's L^2 inequality that $E[\sup_n (G_n/\Pi_n(\gamma)]$ is bounded. Hence $E[\sup_n G_n/n^\gamma]$ is bounded. Hence $E[G_{\tau_k}/(\tau_k)^\gamma] \le M$ for some $M > 0$ for all k. Now note that this implies

$$\begin{aligned} &E\left[\frac{G_0 + k\gamma}{(\tau_k)^\gamma}\right] \le M \\ \Rightarrow &E\left[\frac{k}{(\tau_k)^\gamma}\right] \le M' \end{aligned} \tag{6.13}$$

for some M'.

Now fix $\varepsilon > 0$. Now for any two positive integers m and n, we have

$$\begin{aligned} P\left(\sum_{m\le p\le n} Z^{(p)} > \varepsilon\right) &\le \sum_{m\le p\le n} E\left(\frac{\alpha}{\alpha + \tau_p + G_0 + W_0 + B_0}\right)\frac{1}{\varepsilon} \\ &\le \sum_{m\le p\le n} E\left(\frac{\alpha}{\tau_p}\right)\frac{1}{\varepsilon} \\ &\le \sum_{m\le p\le n} \alpha\left[E\left(\frac{1}{(\tau_p)^\gamma}\right)\right]^{1/\gamma}\frac{1}{\varepsilon} \\ &\le c\sum_{m\le p\le n}\frac{1}{p^{1/\gamma}} \end{aligned} \tag{6.14}$$

for some constant $c > 0$. The last inequality follows from (6.13) and Jensen's inequality. Hence as $\gamma < 1$, we see that $\sum_{p=0}^{n} Z^{(p)}$ is cauchy in probability. Hence it converges in probability. Hence it converges almost surely to $\sum_{p=0}^{\infty} Z^{(p)}$ because of monotonicity. Now observe that it also follows from the above calculation that

$$\sum_{k=1}^{\infty} E(1/\tau_k) < \infty \tag{6.15}$$

.

Note that it suffices to show that W_n/n converges in probability to $\sum_{p=0}^{\infty} Z^{(p)}$, as the almost sure convergence is already guaranteed. Now fix $\varepsilon > 0$ and $\eta > 0$. Choose N such that $\sum_{k=N+1}^{\infty} E(\alpha/\tau_k) < \eta\varepsilon/8$. Choose N_0 such for all $n \ge N_0$

$$P\left(\left|\frac{W_n^{(k)}}{n} - Z^{(k)}\right| > \frac{\varepsilon}{2(N+1)}\right) < \frac{\eta}{2(N+1)}$$

for all $k = 1, \dots N$. Now, for all $n \ge N_0$

$$\begin{aligned} P\left(\left|\frac{W_n}{n} - \sum_{p=0}^{\infty} Z^{(p)}\right| > \varepsilon\right) &\le P\left(\left|\sum_{k=0}^{N}\frac{W_n^{(k)}}{n} - \sum_{p=0}^{N} Z^{(p)}\right| > \varepsilon/2\right) + P\left(\left|\sum_{k=N+1}^{\infty}\frac{W_n^{(k)}}{n} - \sum_{p=N+1}^{\infty} Z^{(p)}\right| > \varepsilon/2\right) \\ &\le \sum_{k=0}^{N} P\left(\left|\frac{W_n^{(k)}}{n} - Z^{(k)}\right| > \frac{\varepsilon}{2(N+1)}\right) + \left(\sum_{k=N+1}^{\infty} E\left(\frac{W_n^{(k)}}{n}\right) + \sum_{p=N+1}^{\infty} E(Z^{(p)})\right)\frac{2}{\varepsilon} \end{aligned} \tag{6.16}$$

Now first term of (6.16) is less than $\eta/2$ because of the choice of n. Now note that

$$\begin{aligned}
E\left(\frac{W_n^{(k)}}{n}\right) &= E\left(\frac{W_n^{(k)}}{n} I_{(n\geq\tau_k)}\right) \\
&= E\left(\frac{W_n^{(k)}}{n} I_{(n-1\geq\tau_k)} + \frac{W_n^{(k)}}{n} I_{(n=\tau_k)}\right) \\
&= E\left(\frac{W_n^{(k)}}{n} I_{(n-1\geq\tau_k)}\right) + \frac{\alpha}{n} P(\tau_k = n)
\end{aligned}$$

Take the filtration $\mathcal{G}_{n-1,k} = \sigma(\mathcal{F}_{n-1,k}, G_{i;i\geq 0})$. We then have,

$$\begin{aligned}
E\left(\frac{W_n^{(k)}}{n} I_{(n-1\geq\tau_k)} | \mathcal{G}_{n-1,k}\right) &= I_{(n-1\geq\tau_k)} E\left(\frac{W_n^{(k)}}{n} | \mathcal{G}_{n-1,k}\right) \\
&= I_{(n-1\geq\tau_k)} \left[\frac{W_{n-1}^{(k)} + W_{n-1}^{(k)}/(n-1+G_0+W_0+B_0)}{n} I_{(n-1\geq\tau_k)} + \frac{\alpha}{n} I_{(\tau_k=n)}\right] \\
&= \frac{W_{n-1}^{(k)}}{(n-1)} \frac{(n+G_0+W_0+B_0)(n-1)}{n(n-1+G_0+W_0+B_0)} I_{(n-1\geq\tau_k)} \\
&\leq \frac{W_{n-1}^{(k)}}{(n-1)} I_{(n-1\geq\tau_k)}
\end{aligned}$$

So, we have,

$$\begin{aligned}
E\left(\frac{W_n^{(k)}}{n}\right) &\leq E\left(\frac{W_{n-1}^{(k)}}{(n-1)} I_{(n-1\geq\tau_k)}\right) + \frac{\alpha}{n} P(\tau_k = n) \\
&= E\left(\frac{W_{n-1}^{(k)}}{(n-1)}\right) + \frac{\alpha}{n} P(\tau_k = n)
\end{aligned}$$

which is a recursion relation. So simplifying we get,

$$\begin{aligned}
E\left(\frac{W_n^{(k)}}{n}\right) &= \sum_{i=1}^{n} \frac{\alpha}{i} P(\tau_k = i) \\
&\leq \alpha E(1/\tau_k)
\end{aligned}$$

which is summable because of (6.15). So because of (6.14) second term of (6.16) is bounded above by

$$\left(\sum_{k=N+1}^{\infty} E\left(\frac{\alpha}{\tau_k}\right) + \sum_{p=N+1}^{\infty} E\left(\frac{\alpha}{\tau_k}\right)\right) \frac{2}{\varepsilon} < \eta/2$$

because of choice of N. This completes the proof of the theorem. $\square$

7. Urn models with upper triangular replacement matrices

We first introduce the following definitions. We assume that we have applied the algorithmic procedure of Bose et al. (2009b) and obtained some arrangement of colors (which may not be unique) so that some non decreasing arrangement of the leading colors is obtained.

Definition 7.1. Let the leading colors be $r_1 = r_{i_1} \leq r_{i_2} \leq \ldots r_{i_J} \leq r_{i_J+1} = 1$. Let $S_1 = \{j > 1 \ni r_{i_j} = r_{i_1}\}$ Let β_1 be the maximum index in S_1. We inductively define S_k as follows: after having defined $S_1, S_2, \ldots S_k$ and getting the corresponding $\beta_1, \beta_2, \ldots \beta_k$, we define $S_{k+1} = \{j > \beta_k \ni r_{i_j} = r_{i_{\beta_k+1}}\}$. Continuing this way we get $S_1, S_2, \ldots S_I, S_{I+1}$. So, obviously we have $S_{I+1} = \{j : r_{i_j} = 1\}$.

We next inductively label each block in S_k for $k = 1, 2, \dots I + 1$. Call the block corresponding to $\beta_{k-1} + 1$ th color (that is first element in S_k) a *type 0* block. Having labelled $\beta_{k-1} + 1, \dots, p$ th block for $p \in S_k$ consider $r_{i_{p+1}}$ and the column above it, that is consider $D = \{r_{m,i_{p+1}} : m = i_{\beta_{k-1}+1}, \dots i_{p+1} - 1\}$. If all the elements are zero, we call the m th block a type 0 block. Otherwise consider the non-zero elements of D. Consider the blocks of $\{r_m : r_{m,i_{p+1}} \in D, r_{m,i_{p+1}} \neq 0\}$. Consider the maximum type of such blocks. Let it be γ. Then we label the i_{p+1} th block as *type* $\gamma + 1$.

Call the colors corresponding to the blocks in S_k together kth *superblock*. So there are $I + 1$ superblocks. There are $\beta_k - \beta_{k-1}$ blocks in the k th superblock.

We claim that the colors can always be arranged so that the blocks in a superblock are in non-decreasing order of types.

Theorem 7.2. *Suppose R is a replacement matrix with row sums equal to 1. The colors can be arranged in increasing order such that the blocks in each superblock are in non-decreasing order of types keeping the triangular structure of the matrix unchanged.*

Proof. We apply the algorithmic procedure of Bose et al. (2009b) to R to get colors in increasing order. Note that this order might not be unique but taking any one of them will serve the purpose. We need to arrange the order of types in each superblock so that the types are in non-decreasing order and the colors remain arranged in increasing order by some row and column permutation. Note that row and column permutation in one superblock does not affect any other superblock. If we do it for any one of the superblocks we are done. So without loss of generality let $I = 1$. Let it contain p blocks each having type $\gamma_i, i = 1, \dots p$. We shall reshuffle the colors so that the types are in non-decreasing order keeping the triangular structure unaltered. We keep the first block (always of type 0) unaltered. Induction hypothesis: Suppose we have reshuffled colors in the first j blocks so that the types in the first j blocks are in non-decreasing order, the colors are in increasing order and the triangular structure remain unaltered. Also reshuffling is done among colors in the first j blocks only except block 1, so that the columns $i_{j+1}, i_{j+1} + 1, \dots, i_{J+1}$ of R suffers only some row wise permutation and the rows $i_{j+1}, i_{j+1} + 1, \dots, i_{J+1}$ do not undergo any permutation among themselves.

Let the types for the reshuffled blocks be $0 = \gamma_1 = \gamma'_1 \leq \gamma'_2 \leq \dots \leq \gamma'_j$. If $\gamma_{j+1} \geq \gamma'_j$ put $\gamma'_{j+1} = \gamma_{j+1}$ and proceed to the next block. If $\gamma_{j+1} < \gamma'_j$, let $\gamma_{j+1} < \gamma'_j, \gamma_{j+1} < \gamma'_{j-1}, \dots, \gamma_{j+1} < \gamma'_{j-k}, \gamma_{j+1} \geq \gamma'_{j-k-1}$ for some k. (Note: We shall always get such k as $\gamma_1 = 0$). Also, always we shall have $\gamma_{j+1} = \gamma_{j-k-1}$. Note that $r_{m,i_{j+1}} = 0$ for $m = i_{j-k}, i_{j-k} + 1, \dots i_{j+1} - 1$ because of the type of the $j + 1$ th block. Hence we can re shuffle the colors to bring the i_j th color just ahead of the i_{j-k} th color. The types of the first j blocks do not change due to such an operation. Let the leading colors after re shuffling be renamed as $1 = i_1 \leq i_2 \leq \dots i_{J+1}$.

Note that non-leading colors of first j blocks now still satisfy condition (2.5) because of the induction hypothesis. Consider elements in the new $j + 1$ th block. Having obtained k colors from $j + 1$ th block (including the leading color) so that the increasing order criterion is satisfied the new index of the leading color be i'_{j+1} so that last considered color was $i'_{j+1} + k - 1$. Also assume that the construction has been such that columns $i'_{j+1} + k, \dots i_{J+1}$ of R has undergone only row wise permutation and the rows $i'_{j+1} + k, \dots i_{J+1}$ are undisturbed. If $i'_{j+1} + k = i_{j+2}$ the construction is over.

If not, consider $i'_{j+1} + k$ th color. Consider $r_{m,i'_{j+1}+k}$, $i'_{j+1} + k - 1 \geq m \geq 1$. One of these numbers must be non-zero as this column has only undergone row wise permutation and initially R was arranged. let m_0 be the maximum of such m's. Let $r_{m_0,i'_{j+1}+k+1}$ be in the q th block. If $q = j + 1$ proceed to the next color. If $q < j + 1$. Then reshuffle the $i'_{j+1} + k$ th color to bring it above i_{q+1} th color. After this the index of colors in the $j + 1$ th block will increase and the new leading index is $i'_{j+1} + 1$. Note that this does not hamper the triangular structure and the columns $i'_{j+1} + k + 1, \dots i_{J+1}$ only suffer row wise permutation and the rows $i'_{j+1} + k + 1, \dots i_{J+1}$ remains undisturbed.

After the whole procedure, rename the leading colors to be $1 = i_1 \leq \dots \leq i_{J+1}$. After this procedure, the first $j + 1$ blocks will have increasing order of types. Also the triangular structure is maintained and the colors remain in increasing order. Also the columns $i_{j+2}, \dots i_{J+1}$ has undergone only row wise permutation and the rows $i_{j+2}, \dots i_{J+1}$ do not undergo any permutation among themselves.

We then proceed to the $j+2$ th block. Thus by induction such an arrangement can be obtained. □

It is clear from the discussion following Theorem 2.13 that the rate of the convergence of the colors in the j th block is $n^{r_{i_j}}(\log n)^{\nu_j}$ where ν_j denotes the type of the j th block. However the relationship between the random variables where they converge is also not clear. These questions are answered in the next theorem which claims that the dimension of the limiting random variables where the colors of a particular superblock converge is equal to the number of blocks of type 0 present in the superblock.

First we fix some notations. Let α_i denote number of blocks in the ith superblock. Let d_j denote number of colors in the jth block and ${}^{(i)}d_j$ denote the number of colors in the jth block of ith superblock. Let ${}^{(i)}\nu_j$ be the type of the jth block of the ith superblock.

Let ${}^{(i)}C_n^{(j)}$ denote the colors of the j th block of the ith superblock at nth step and ${}^{(i)}R^{(j)}$ denote the corresponding matrix part of the replacement matrix. Let ${}^{(i)}r_l^{(j)}$ denote the lth color of the jth block of the ith superblock. Here, $j = 1, \dots \alpha_i$, $i = 1, \dots I+1$, $l = 1, \dots {}^{(i)}d_j$. So we have

$$ {}^{(i)}r_l^{(j)} = r_{(\sum_{v=1}^{i-1}\sum_{u=1}^{\alpha_v} {}^{(v)}d_u + \sum_{u=1}^{j-1} {}^{(i)}d_u + l)} $$

Define, $\phi(i,j,l) = (\sum_{v=1}^{i-1}\sum_{u=1}^{\alpha_v} {}^{(v)}d_u + \sum_{u=1}^{j-1} {}^{(i)}d_u + l)$. Note that we have previously fixed the R matrix so ϕ is a one-one function of i,j and l.

Also, let ${}^{(i)}r_1^{(1)} = {}^{(i)}r_1^{(2)} = \dots {}^{(i)}r_1^{(\alpha_i)} = \lambda_i$

So, we have the following theorem:

Theorem 7.3. *Suppose that R is a $(K+1)\times(K+1)$ balanced triangular matrix with row sums equal to 1 and $J+1$ blocks and $I+1$ superblocks. Suppose the colors are in increasing order and the types of blocks in each superblock are in non-decreasing order. Then if $\phi^{-1}(l) = (i,j,k)$ then*

$$ \frac{\boldsymbol{C}_{N,l}}{n^{\lambda_i}(\log n)^{{}^{(i)}\nu_j}} \to V_l $$

almost surely as well as in L^2, where we have,

(i) *If $j = J+1$, $V_j = 1$*

(ii) *If $k = 1$, and ${}^{(i)}\nu_j = 0$*

$$ V_l = {}^{(i)}V_j $$

for some non-degenerate random variable ${}^{(i)}V_j$.

(iii) *If $k = 1$, and ${}^{(i)}\nu_j > 0$*

$$ V_l = \frac{1}{{}^{(i)}\nu_j}\sum_{B_{i,j}} V_{\phi(u,v,w)} r_{\phi(u,v,w),l} $$

(iv) *If $k > 1$*

$$ V_l = \frac{1}{(\lambda_i - r_l)}\sum_{A_{i,j,l}} V_{\phi(u,v,w)} r_{\phi(u,v,w),l}. $$

Here $B_{i,j} = \{(u,v,w) : u = i, {}^{(i)}\nu_v = {}^{(i)}\nu_j - 1, w = 1,2,\dots {}^{(i)}d_v\}$ and $A_{i,j,l} = \{(i,v,w) : {}^{(i)}\nu_v = {}^{(i)}\nu_j, v \neq j, w = 1,2,\dots {}^{(i)}d_v\} \cup \{(i,j,w) : w = 1,2,\dots l-1\}$.

Remark 7.4. Note that if in a superblock there are q blocks of type 0, the random variables where the colors converge has at most q possibly independent variables. The others are just some linear combination of them.

Proof. Let $\boldsymbol{\chi}_n$ be the row vector of order $(K+1)$ whose mth entry is 1 if mth color is drawn at the nth draw and whose other entries are all 0. Let $\mathcal{F}_n$ denote $\sigma\{\boldsymbol{\chi}_k : 1 \le k \le n\}$.

For $l = 1$, the result is easily verified from proposition 2.2(iii) of Bose et al. (2009a).

Suppose the result is true for the first $l-1$ colors. we prove the result for lth color. Let $\phi^{-1}(l) = (i,j,k)$. We make the following observations from the induction hypothesis. For $m < l$,

(i) If $j = 1, k = 1$,
$$E(\boldsymbol{C}_{n,m}) = O(n^{\lambda_{i-1}}(\log n)^{{}^{(i-1)}\nu_{\alpha_{i-1}}}) \tag{7.1}$$
(ii) If $j \neq 1, k = 1$
$$E(\boldsymbol{C}_{n,m}) = O(n^{\lambda_i}(\log n)^{{}^{(i)}\nu_{j-1}}) \tag{7.2}$$
(iii) If $k > 1$,
$$E(\boldsymbol{C}_{n,m}) = O(n^{\lambda_i}(\log n)^{{}^{(i)}\nu_j}) \tag{7.3}$$
case 1: ${}^{(i)}\nu_j = 0, k = 1$. We can choose a right eigenvector $\boldsymbol{\zeta}$ corresponding to λ_i such that $\boldsymbol{\zeta}_{\phi(i,j,1)} = 1$ and $\boldsymbol{\zeta}_k = 0$ for $k \geq \phi(i,1,1)$, $k \neq \phi(i,j,1)$ and $\zeta_k \geq 0 \quad \forall k$. Observe that because of the triangular structure of the R matrix and the nature of the columns of the leading colors of blocks of the same type, the above eigenvector can be obtined by solving a triangular structure of equations. By standard calculations it can be easily shown that $U_n = \boldsymbol{C}_n\boldsymbol{\zeta}/\Pi_n(\lambda_i)$ is an L^2 bounded martingale. So it converges to some random variable ${}^{(i)}V_j$ (say).

Note $U_1 = \boldsymbol{C}_1\boldsymbol{\zeta}/(1+\lambda_i) = (\boldsymbol{C}_0\boldsymbol{\zeta} + \lambda_i\boldsymbol{\chi}_1\boldsymbol{\zeta})/(1+\lambda_i)$. Without loss of generality $\boldsymbol{C}_0$ has all entries positive, hence $\boldsymbol{\chi}_1$ takes all coordinate vectors as values with positive probability. Thus $\boldsymbol{\chi}_1\boldsymbol{\zeta}$ is constant if and only if all co-ordinates of $\boldsymbol{\zeta}$ has the same value. Then $j = J + 1$. So, $U_n = 1 = V_{J+1}$. Hence we are done.

If $j \leq J$ then U_1 is non-degenerate, hence, has positive variance. Since U_n is a martingale, variances are non-decreasing. hence, ${}^{(i)}V_j$ is non-degenerate. Thus, we have $\boldsymbol{C}_n\boldsymbol{\zeta}/n^{\lambda_i}$ has non-degenerate limit. Because of the structure of the vector $\boldsymbol{\zeta}$ and (7.1) we have $\boldsymbol{C}_{n,l}/n^{\lambda_i} \to^{(i)} V_j$ almost surely as well as in L^2.

The other two cases are carried out in two steps. First we show L^1 boundedness of $\boldsymbol{C}_{n,l}/n^{\lambda_i}(\log n)^{{}^{(i)}\nu_j}$, and then the required almost sure and L^2 convergence by constructing appropriate martingales.

Step 1: L^1 bound. We have
$$\boldsymbol{C}_{N+1,l} = \boldsymbol{C}_{N,l} + \sum_{m=1}^{l} \boldsymbol{\chi}_{N+1,m} r_{m,l}$$
So, taking conditional expectation,
$$E[\boldsymbol{C}_{N+1,l}|\mathcal{F}_n] = \boldsymbol{C}_{N,l} + \sum_{m=1}^{l} \frac{\boldsymbol{C}_{N,m}}{N+1} r_{m,l}$$
which leads to
$$E(\boldsymbol{C}_{N+1,l}) = E(\boldsymbol{C}_{N,l})\left(1 + \frac{r_l}{(N+1)}\right) + \sum_{m=1}^{l-1} E\left(\frac{\boldsymbol{C}_{N,m}}{N+1}\right) r_{m,l}$$
So, iterating, we have,
$$E(\boldsymbol{C}_{N+1,l}) = E(\boldsymbol{C}_{0,l})\Pi_{N+1}(r_l) + \sum_{m=1}^{l-1}\sum_{n=0}^{N} \frac{E(\boldsymbol{C}_{n,m})\Pi_{N+1}(r_l)}{(n+1)\Pi_{n+1}(r_l)} r_{m,l} \tag{7.4}$$
Case 2: ${}^{(i)}\nu_j = \delta > 0, \quad k = 1$.
Let q be the minimum of all j such that ${}^{(i)}r_1^{(j)}$ is the leading color of a block of type δ. Then (7.4) reduces to
$$E(\boldsymbol{C}_{N+1,l}) = E(\boldsymbol{C}_{0,l})\Pi_{N+1}(r_l) + \sum_{m=1}^{{}^{(i)}r_1^{(q)}-1}\sum_{n=0}^{N} \frac{E(\boldsymbol{C}_{n,m})\Pi_{N+1}(r_l)}{(n+1)\Pi_{n+1}(r_l)} r_{m,l}$$
because of the structure of the R matrix. Now, for $m <^{(i)} r_1^{(q)}$, we have from induction hypothesis and the choice of q, $E(\boldsymbol{C}_{N,m}) = O(N^{\lambda_i}(\log N)^{{}^{(i)}\nu_j - 1})$. So, we have
$$\begin{aligned}\frac{E(\boldsymbol{C}_{N+1,l})}{\Pi_{N+1}(\lambda_i)} &= E(\boldsymbol{C}_{0,l}) + \sum_{m=1}^{{}^{(i)}r_1^{(q)}-1} r_{m,l} \sum_{n=0}^{N} \frac{E(\boldsymbol{C}_{n,m})}{n^{\lambda_i}(\log n)^{{}^{(i)}\nu_j - 1}} \frac{n^{\lambda_i}(\log n)^{{}^{(i)}\nu_j - 1}}{(n+1)\Pi_{n+1}(r_l)} \\ &= O((\log N)^{{}^{(i)}\nu_j})\end{aligned}$$

This follows from the fact that $r_l = \lambda_i$. So L^1 boundedness is established in this case.
Case 3: $k > 1$. From (7.3) and (7.4), we have

$$\begin{aligned}\frac{E(\boldsymbol{C}_{N+1,l})}{\Pi_{N+1}(r_l)} &= E(\boldsymbol{C}_{0,l}) + \sum_{m=1}^{l-1} r_{m,l} \sum_{n=0}^{N} \frac{E(\boldsymbol{C}_{n,m})}{n^{\lambda_i}(\log n)^{{}^{(i)}\nu_j}} \frac{n^{\lambda_i}(\log n)^{{}^{(i)}\nu_j}}{(n+1)\Pi_{n+1}(r_l)} \\ &= O(N^{\lambda_i - r_l}(\log N)^{{}^{(i)}\nu_j})\end{aligned}$$

This establishes the L^1 boundedness.
Step 2: convergence. We construct the following martingale:

$$M_N = \frac{\boldsymbol{C}_{N,l}}{\Pi_N(r_l)} - \sum_{m=1}^{l-1}\sum_{n=0}^{N-1} \frac{r_{m,l}}{n+1+r_l}\frac{\boldsymbol{C}_{n,m}}{\Pi_n(r_l)} \tag{7.5}$$

The martingale difference is given by

$$M_{N+1} - M_N = \frac{1}{\Pi_{N+1}(r_l)} \sum_{m=1}^{l} \left(\boldsymbol{\chi}_{N+1,m} - \frac{\boldsymbol{C}_{N,m}}{N+1}\right) r_{m,l}$$

So, using (7.2), (7.3) and L^1 boundedness of $\boldsymbol{C}_{N,l}/N^{\lambda_i}(\log N)^{{}^{(i)}\nu_j}$, we have

$$\begin{aligned}E[(M_{N+1} - M_N)^2] &= \frac{1}{\Pi^2_{N+1}(r_l)} E\left[\sum_{m=1}^{l} \frac{\boldsymbol{C}_{N,m}}{N+1} r^2_{m,l} - \left(\sum_{m=1}^{l} \frac{\boldsymbol{C}_{N,m}}{N+1} r_{m,l}\right)^2\right] \\ &\le \frac{1}{\Pi^2_{N+1}(r_l)} E\left(\sum_{m=1}^{l} \frac{\boldsymbol{C}_{N,m}}{N+1} r^2_{m,l}\right) &(7.6)\\ &= O\left(\frac{(\log N)^{{}^{(i)}\nu_j}}{N^{1+2r_l-\lambda_i}}\right) &(7.7)\end{aligned}$$

Note here that if $r_l = \lambda_j = 0$, then, since r_l is the leading color, we have $r_m = 0$ for all $m \le l$. Hence we have first l diagonal entries as l blocks. Hence from (7.6), and the fact that $r_l = 0$ we get,

$$E[(M_{N+1} - M_N)^2] = O\left(\frac{(\log N)^{{}^{(i)}\nu_j - 1}}{N}\right) \tag{7.8}$$

case 2: $k = 1$, ${}^{(i)}\nu_j > 0$.
Proof in this case is very similar to that done in Bose et al. (2009b). We give here a brief sketch. If $r_l = \lambda_j > 0$, then right hand side of (7.7) is summable. hence M_N is an L^2 bounded martingale, so it converges almost surely as well as in L^2. So we have $M_N/(\log N)^{{}^{(i)}\nu_j}$ converging almost surely as well as in L^2 to 0 since ${}^{(i)}\nu_j > 0$. If $r_l = 0$ then from (7.8) $M_N/(\log N)^{{}^{(i)}\nu_j/2}$ is L^2 bounded. Hence

$$Y_N = \frac{M_N}{(\log N)^{{}^{(i)}\nu_j}} \to 0 \qquad \text{in } L^2. \tag{7.9}$$

We shall show Y_N converges almost surely to some random variable. So using standard procedure we break up as follows,

$$Y_{N+1} - Y_N = M_{N+1}(\Delta_{N+1} - \Delta_N) + \Delta_N(M_{N+1} - M_N) \tag{7.10}$$

Now, we can see that $|\Delta_{N+1} - \Delta_N| \sim \nu_j/N(\log N)^{\nu_j+1}$. Now from the fact that $\nu_j > 0$ and $M_N/(\log N)^{{}^{(i)}\nu_j/2}$ is L^2 bounded it can be easily inferred that $E[|M_{N+1}(\Delta_{N+1} - \Delta_N)|]$ is summable. Hence first term of (7.10) is almost surely absolutely summable. The second term of (7.10) is a martingale difference and from (7.8) we have

$$E[\Delta^2_N(M_{N+1} - M_N)^2] = O\left(\frac{1}{(\log N)^{2{}^{(i)}\nu_j}} \frac{(\log N)^{{}^{(i)}\nu_j - 1}}{N}\right)$$

which is summable. Hence the second term on the right hand side of (7.10) is the a martingale difference where the martingale converges almost surely as well as in L^2. Thus even when $r_l = 0$, Y_N converges almost surely as well as in L^2 to 0. So, from (7.5) we have,

$$\lim_{N\to\infty} \frac{\boldsymbol{C}_{N,l}}{\Pi_N(r_l)(\log N)^{{}^{(i)}\nu_j}} = \lim_{N\to\infty} \frac{1}{(\log N)^{{}^{(i)}\nu_j}} \sum_{m=1}^{l-1}\sum_{n=0}^{N-1} \frac{r_{m,l}}{n+1+r_l} \frac{\boldsymbol{C}_{n,m}}{\Pi_n(r_l)}. \tag{7.11}$$

Where the limit is in almost sure as well as L^2 sense. Note that in (7.11), $r_{m,l} = 0$ if m belongs to a block of ith superblock whose type is ${}^{(i)}\nu_j$. Recall that, $B_{i,j} = \{(u,v,w) : u = i, {}^{(i)}\nu_v = {}^{(i)}\nu_j - 1, w = 1,2,\dots {}^{(i)}d_v\}$. Thus we have,

$$\begin{aligned}
\lim_{N\to\infty} \frac{\boldsymbol{C}_{N,l}}{\Pi_N(r_l)(\log N)^{{}^{(i)}\nu_j}} &= \lim_{N\to\infty} \frac{1}{(\log N)^{{}^{(i)}\nu_j}} \sum_{B_{i,j}}\sum_{n=0}^{N-1} \frac{r_{\phi(u,v,w),l}}{n+1+\lambda_i} \frac{\boldsymbol{C}_{n,\phi(u,v,w)}}{\Pi_n(\lambda_i)(\log n)^{{}^{(i)}\nu_j-1}} (\log n)^{{}^{(i)}\nu_j-1} \\
&= \frac{1}{{}^{(i)}\nu_j} \sum_{B_{i,j}} V_{\phi(u,v,w)} r_{\phi(u,v,w),l}
\end{aligned}$$

because of induction hypothesis. Thus induction step for case 2 is proved.
case 3: $k > 1$.
We follow the procedure given in Bose et al. (2009b). Here we have $r_l < \lambda_j$. If $r_l > \lambda_j/2$ then we have from (7.7) M_N is an L^2 bounded martingale. Hence

$$W_N = \frac{M_N}{N^{\lambda_i - r_l}(\log N)^{{}^{(i)}\nu_j}} \to 0$$

almost surely as well as in L^2. If $r_l = \lambda_j/2$, then from (7.7), we have $M_N/(\log N)^{({}^{(i)}\nu_j+1)/2}$ is L^2 bounded. Hence W_N converges to 0 almost surely as well as in L^2. If $r_l < \lambda_j/2$ it can be shown that $W_N \to 0$ almost surely as well as in L^2 following the exact similar method as in case 2 when $r_l > 0$. Thus we have W_N converging to 0 almost surely as well as in L^2 in all cases. Recall we have $A_{i,j,l} = \{(i,v,w) : {}^{(i)}\nu_v = {}^{(i)}\nu_j, v \neq j, w = 1,2,\dots {}^{(i)}d_v\} \cup \{(i,j,w) : w = 1,2,\dots l-1\}$. So we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{\boldsymbol{C}_{N,l}}{N^{\lambda_i}(\log N)^{{}^{(i)}\nu_j}} &= \lim_{N\to\infty} \frac{1}{N^{\lambda_i - r_l}(\log N)^{{}^{(i)}\nu_j}} \sum_{m=1}^{l-1}\sum_{n=0}^{N-1} \frac{r_{m,l}}{n+1+r_l} \frac{\boldsymbol{C}_{n,m}}{n^{\lambda_i}(\log n)^{{}^{(i)}\nu_j}} \frac{n^{\lambda_i}(\log n)^{{}^{(i)}\nu_j}}{\Pi_n(r_l)} \\
&= \frac{1}{\lambda_i - r_l} \sum_{A_{i,j,l}} V_{\phi(u,v,w)} r_{\phi(u,v,w),l}
\end{aligned}$$

because of induction hypothesis. This completes the proof of the theorem. $\square$

8. Linear combinations of the color counts for urn models with triangular replacement matrices

In this section we try to find out the rate of the linear combinations for triangular urns. If we find out the rate for linear combinations with $K+1$ linearly independent $(K+1)\times 1$ vectors then we are done. The vectors we search for will be mostly like Jordan or eigenvectors. We will find out for each i a vector which has positive entry in the ith place and 0 for all greater than i, where $i = 1,2,\dots K+1$. Note that such a set will be linearly independent. So figuring out the rates for linear combination with such a set of vectors will serve the purpose.

8.1. Finding out the vectors.

We will focus on a particular diagonal entry μ, and for each occurrence of μ in the diagonal we first try to figure out an eigenvector for μ.

Definition 8.1. Let $r_{\sigma_1} = r_{\sigma_2} = \dots r_{\sigma_p} = \mu$, where $\sigma_i \in \{1,2,\dots K\}$ for $i = 1,\dots,p$. We call the points r_{σ_i} as *joint* with *position* σ_i. Let $r_{\sigma_1}, r_{\sigma_2}, \dots r_{\sigma_{p'}}$ be leading colors and the rest non leading. Let $r_{\sigma_1^{(i)}}, r_{\sigma_2^{(i)}}, \dots r_{\sigma_{p'_i}^{(i)}}$ be the leading colors of type i, $i = 1,2,\dots$.

We first focus on $r_{\sigma_1^{(0)}}$. We choose $\boldsymbol{\xi}$ such that $\xi_k = 0$ for $k > \sigma_1^{(0)}$. We try to solve

$$R\boldsymbol{\xi} = \mu\boldsymbol{\xi} \tag{8.1}$$

We end up with a triangular system of equations

$$\begin{aligned}
\mu\boldsymbol{\xi}_{\sigma_1^{(0)}} &= \mu\boldsymbol{\xi}_{\sigma_1^{(0)}}\\
r_{\sigma_1^{(0)}-1}\boldsymbol{\xi}_{\sigma_1^{(0)}-1} + r_{\sigma_1^{(0)}-1,\sigma_1^{(0)}}\boldsymbol{\xi}_{\sigma_1^{(0)}} &= \mu\boldsymbol{\xi}_{\sigma_1^{(0)}-1}\\
&\vdots\\
r_1\boldsymbol{\xi}_1 + \cdots + r_{1,\sigma_1^{(0)}}\boldsymbol{\xi}_{\sigma_1^{(0)}} &= \mu\boldsymbol{\xi}_1
\end{aligned}$$

It is easy to observe that we can choose any positive value of our choice for $\boldsymbol{\xi}_{\sigma_1^{(0)}}$ and arrive at a solution for $\boldsymbol{\xi}$ whose all entries are non-negative. Hence we get the required eigenvector.

Now for any μ of type 0 leading color, that is, for $r_{\sigma_k^{(0)}}$ for any $k \in \{1, 2, \ldots p_0'^{(0)}\}$ we try to solve (8.1). Again choose $\boldsymbol{\xi}_j = 0$ for $j > \sigma_k^{(0)}$. For any $k > 1$ we again end up with a triangular system of equation similar to previous calculation. The first equation is

$$\mu\boldsymbol{\xi}_{\sigma_k^{(0)}} = \mu\boldsymbol{\xi}_{\sigma_k^{(0)}}$$

We can choose any value of $\boldsymbol{\xi}_{\sigma_k^{(0)}}$ of our choice to satisfy this equation. Now note that $r_{\sigma_k^{(0)}-1,\sigma_k^{(0)}} = 0$. Hence, next equation is like

$$r_{\sigma_k^{(0)}-1}\boldsymbol{\xi}_{\sigma_k^{(0)}-1} = \mu\boldsymbol{\xi}_{\sigma_k^{(0)}-1}$$

which has the solution $\boldsymbol{\xi}_{\sigma_k^{(0)}-1} = 0$. If we continue this way we observe that we will end up with $\boldsymbol{\xi}_{\sigma_{k-1}^{(0)}+1} = \ldots = \boldsymbol{\xi}_{\sigma_k^{(0)}-1} = 0$. So we have

$$\mu\boldsymbol{\xi}_{\sigma_{k-1}^{(0)}} = \mu\boldsymbol{\xi}_{\sigma_{k-1}^{(0)}}$$

for which we can choose any non negative value of $\xi_{\sigma_{k-1}^{(0)}}$ for a solution. Again note that $r_{\sigma_{k-1}^{(0)}-i,\sigma_{k-1}^{(0)}} = r_{\sigma_{k-1}^{(0)}-i,\sigma_k^{(0)}} = 0$ for $i = 1, 2, \ldots \sigma_{k-1}^{(0)} - \sigma_{k-2}^{(0)}$. Hence it is easy to see that we shall have $\boldsymbol{\xi}_{\sigma_{k-1}^{(0)}-1} = \ldots = \boldsymbol{\xi}_{\sigma_{k-2}^{(0)}+1} = 0$. Hence, the next equation is like

$$\mu\boldsymbol{\xi}_{\sigma_{k-2}^{(0)}} = \mu\boldsymbol{\xi}_{\sigma_{k-2}^{(0)}}$$

for which we can again choose any value of $\boldsymbol{\xi}_{\sigma_{k-2}^{(0)}}$ to be a solution.

So we have the following lemma:

Lemma 8.2. *There exists an eigenvector for μ in such a way that $\boldsymbol{\xi}_j = 0$ for $j > \sigma_k^{(0)}$, all entries non-negative and we can have any non-negative value of our choice for $\boldsymbol{\xi}_{\sigma_{k-i}^{(0)}}$ for $i = 1, 2, \ldots k-1$, any positive value of our choice for $i = 0$ and $k = 1, 2, \ldots p_0'^{(0)}$.*

Assume for each $i = 0, \ldots, s-1$ and $k = 1, 2, \ldots p_i'^{(i)}$, we have a vector $\boldsymbol{\xi}^{(\sigma_k^{(i)})}$ such that $\boldsymbol{\xi}_j^{(\sigma_k^{(i)})} = 0$ for $j > \sigma_k^{(i)}$, and we can have any value of our choice for all joints whose positions are less than $\sigma_k^{(i)}$ and any positive value of our choice for joint with position $\sigma_k^{(i)}$ such that

$$R\boldsymbol{\xi}^{(\sigma_k^{(i)})} = \mu\boldsymbol{\xi}^{(\sigma_k^{(i)})}$$

or

$$R\boldsymbol{\xi}^{(\sigma_k^{(i)})} = \boldsymbol{\xi}^{(\sigma_{k'}^{(i')})} + \mu\boldsymbol{\xi}^{(\sigma_k^{(i)})}$$

for some joint position $\sigma_{k'}^{(i')}$ less than $\sigma_k^{(i)}$. For $i = 0$ the assumption is true because of Lemma 8.2. Now for $\sigma_1^{(s)}$, we again try to solve (8.1). Again we set $\boldsymbol{\xi}_k = 0$ for $k > \sigma_1^{(s)}$. Now observe that if we try to solve the triangular system of equations we shall face consistency problems for the equations at the joints which are precisely the following sets of equations

$$\mu\boldsymbol{\xi}_{\sigma_1^{(s)}} = \mu\boldsymbol{\xi}_{\sigma_1^{(s)}}$$
$$r_{\sigma_k^{(i)}}\boldsymbol{\xi}_{\sigma_k^{(i)}} + \cdots + r_{\sigma_k^{(i)},\sigma_1^{(s)}}\boldsymbol{\xi}_{\sigma_1^{(s)}} = \mu\boldsymbol{\xi}_{\sigma_k^{(i)}}$$

for $i = 1, 2, \ldots s-1$ and $k = 1, 2, \ldots p_i'^{(i)}$.

$\boldsymbol{\xi}_{\sigma_1^{(s)}}$ can be chosen to be any positive value and the remaining equations are consistent if and only if

$$r_{\sigma_k^{(i)},\sigma_k^{(i)}+1}\boldsymbol{\xi}_{\sigma_k^{(i)}+1} + \cdots + r_{\sigma_k^{(i)},\sigma_1^{(s)}}\boldsymbol{\xi}_{\sigma_1^{(s)}} = 0 \tag{8.2}$$

for $i = 1, 2, \ldots s-1$ and $k = 1, 2, \ldots p_i'^{(i)}$. Look at this equation for $i = s-1$ and $k = p_i'^{(i)}$. Since $\boldsymbol{\xi}_{\sigma_1^{(s)}} > 0$, we have non-negative solutions for $\boldsymbol{\xi}_{\sigma_k^{(i)}+1}, \boldsymbol{\xi}_{\sigma_1^{(s)}+2}, \ldots, \boldsymbol{\xi}_{\sigma_1^{(s)}-1}$ and also $r_{\sigma_k^{(i)},\sigma_1^{(s)}} = 0$. Now note that $r_{\sigma_k^{(i)},\sigma_k^{(i)}+1} \neq 0$ because that would violate (2.5). Hence, $\boldsymbol{\xi}_{\sigma_k^{(i)}+1} = 0$. So, from the equation

$$r_{\sigma_k^{(i)}+1}\boldsymbol{\xi}_{\sigma_k^{(i)}+1} + \cdots + r_{\sigma_k^{(i)}+1,\sigma_1^{(s)}}\boldsymbol{\xi}_{\sigma_1^{(s)}} = \mu\boldsymbol{\xi}_{\sigma_k^{(i)}+1}$$

we have

$$r_{\sigma_k^{(i)}+1,\sigma_k^{(i)}+2}\boldsymbol{\xi}_{\sigma_k^{(i)}+2} + \cdots + r_{\sigma_k^{(i)}+1,\sigma_1^{(s)}}\boldsymbol{\xi}_{\sigma_1^{(s)}} = 0$$

So we have, $r_{\sigma_k^{(i)}+1,\sigma_1^{(s)}} = 0$ and $\boldsymbol{\xi}_{\sigma_k^{(i)}+2} = 0$ since if not we would have $r_{\sigma_k^{(i)}+1,\sigma_k^{(i)}+2} = r_{\sigma_k^{(i)},\sigma_k^{(i)}+2} = 0$ which would violate (2.5). So continuing this way we would have $r_{j,\sigma_1^{(s)}} = 0$ for $j = \sigma_k^{(i)}, \sigma_k^{(i)} + 1, \ldots \sigma_{k+1}^{(i)} - 1$. Similarly proceeding backwards we would have $r_{j,\sigma_1^{(s)}} = 0$ for $j \geq \sigma_1^{(s-1)}$ we arrive at a contradiction as $s > 0$. Hence these equations are never consistent. So, for some k and $i = s-1$ (8.2) is violated. Choose the maximum $\sigma_{k_0}^{(i_0)}$ for which the equation (8.2) is violated. From the induction hypothesis, choose the vector $\boldsymbol{\xi}^{(\sigma_{k_0}^{(i_0)})}$ whose values at the joints whose positions are less than or equal to $\sigma_{k_0}^{(i_0)}$ make the equation $R\boldsymbol{\xi} = \boldsymbol{\xi}^{(\sigma_{k_0}^{(i_0)})} + \mu\boldsymbol{\xi}$ consistent. So we get the required vector. Note that after plugging in the values of $\boldsymbol{\xi}^{(\sigma_{k_0}^{(i_0)})}$ at the joint points to make the equation consistent we are free to choose the values of $\boldsymbol{\xi}$ at the joint points. Hence the conditions of induction hypothesis are satisfied for $\sigma_1^{(s)}$.

Suppose for $j = 1, 2, \ldots k-1$ we have got a vector $\boldsymbol{\xi}^{(\sigma_j^{(s)})}$ satisfying the conditions of the induction hypothesis. Similarly for $\sigma_k^{(s)}$ if we try to solve the equation (8.1) by similar calculation we can choose a possible solution $\boldsymbol{\xi}$ with $\boldsymbol{\xi}_j = 0$ for all $j \geq \sigma_1^{(s)}$ except for $j = \sigma_k^{(s)}$.

The remaining equations at the joints (again these are the equations where we shall face possible consistency problems) are as follows

$$r_{\sigma_k^{(i)}}\boldsymbol{\xi}_{\sigma_k^{(i)}} + \cdots + r_{\sigma_k^{(i)},\sigma_1^{(s)}-1}\boldsymbol{\xi}_{\sigma_1^{(s)}-1} + r_{\sigma_k^{(i)},\sigma_1^{(s)}}\boldsymbol{\xi}_{\sigma_1^{(s)}} = \mu\boldsymbol{\xi}_{\sigma_k^{(i)}}$$

for $i = 1, 2, \ldots s-1$ and $k = 1, 2, \ldots p_i'^{(i)}$.

from which we again get,

$$r_{\sigma_k^{(i)},\sigma_k^{(i)}+1}\boldsymbol{\xi}_{\sigma_k^{(i)}+1} + \cdots + r_{\sigma_k^{(i)},\sigma_1^{(s)}-1}\boldsymbol{\xi}_{\sigma_1^{(s)}-1} + r_{\sigma_k^{(i)},\sigma_1^{(s)}}\boldsymbol{\xi}_{\sigma_1^{(s)}} = 0$$

for $i = 1, 2, \ldots s-1$ and $k = 1, 2, \ldots p_i'^{(i)}$. Now from these equations by similar arguments we can conclude that $r_{j,\sigma_k^{(s)}} = 0$ for $j = \sigma_k^{(i)}, \sigma_k^{(i)}+1, \ldots \sigma_{k+1}^{(i)}-1$. So noting these equations for $i = s-1$ and $k = 1, 2, \ldots, p_{s-1}'^{(s-1)}$ we arrive at a contradiction as $s > 0$. So again for some k and i (8.2) is violated. Choose the maximum $\sigma_{k_0}^{(i_0)}$

for which the equation (8.2) is violated. From the induction hypothesis, choose a vector $\boldsymbol{\xi}^{(\sigma_{k_0}^{(i_0)})}$ whose values at the joints whose positions are less than or equal to $\sigma_{k_0}^{(i_0)}$ make the equation $R\boldsymbol{\xi} = \boldsymbol{\xi}^{(\sigma_{k_0}^{(i_0)})} + \mu\boldsymbol{\xi}$ consistent.

For each $\sigma_k^{(i)}$ call the corresponding vector that we have worked out to be $\boldsymbol{\xi}^{\sigma_k^{(i)}}$. Also from the construction we have the following theorem

Theorem 8.3. *For every diagonal element μ which are in leading positions we get a set of linearly independent vectors $\boldsymbol{\xi}^{\sigma_k^{(i)}}$, $i = 1, 2, \ldots p'$ and $k = 1, 2, \ldots p_i'^{(i)}$ such that for every vector $\boldsymbol{\xi}^{\sigma_k^{(i)}}$ which is not an eigenvector for μ, we have $\boldsymbol{\xi}_j^{\sigma_k^{(i)}} = 0$ for $j > \sigma_1^{(i)}$ if $k = 1$ and for $j \geq \sigma_1^{(i)}$ except $\sigma_k^{(i)}$ if $k > 1$. Also for all joints with positions less than $\sigma_1^{(i)}$ we can put any value of our choice.*

For the first non-leading color, $r_{\sigma_{p'+1}}$, we again put $\boldsymbol{\xi}_j = 0$ for $j > \sigma_{p'+1}$. We have the equations at the joints to be

$$\begin{aligned} \mu\boldsymbol{\xi}_{\sigma_{p'+1}} &= \mu\boldsymbol{\xi}_{\sigma_{p'+1}} \\ r_{\sigma_k^{(i)}}\boldsymbol{\xi}_{\sigma_k^{(i)}} + \cdots + r_{\sigma_k^{(i)},\sigma_{p'+1}}\boldsymbol{\xi}_{\sigma_{p'+1}} &= \mu\boldsymbol{\xi}_{\sigma_k^{(i)}} \end{aligned}$$

for $i = 1, 2, \ldots p'$ and $k = 1, 2, \ldots p_i'^{(i)}$.

Either we obtain an eigenvector or we have

$$r_{\sigma_k^{(i)},\sigma_k^{(i)}+1}\boldsymbol{\xi}_{\sigma_k^{(i)}+1} + \cdots + r_{\sigma_k^{(i)},\sigma_{p'+1}}\boldsymbol{\xi}_{\sigma_{p'+1}} \neq 0 \tag{8.3}$$

for some k and i, Hence we take the vector corresponding to the maximum joint point where (8.3) holds with suitable values at the joint positions to get the Jordan vector $\boldsymbol{\xi}^{(p'+1)}$. Here also we are free to choose the values at the previous joint positions.

Now for r_{σ_k} for $k > p' + 1$, we put $\boldsymbol{\xi}_j = 0$ for $j > k$. We get similar equations at the joints

$$\begin{aligned} \mu\boldsymbol{\xi}_{\sigma_k} &= \mu\boldsymbol{\xi}_{\sigma_k} \\ r_{\sigma_{k'}}\boldsymbol{\xi}_{\sigma_{k'}} + \cdots + r_{\sigma_{k'},\sigma_k}\boldsymbol{\xi}_{\sigma_k} &= \mu\boldsymbol{\xi}_{\sigma_{k'}} \end{aligned}$$

for $k' = p' + 1, p' + 2, \ldots k - 1$

$$r_{\sigma_k^{(i)}}\boldsymbol{\xi}_{\sigma_k^{(i)}} + \cdots + r_{\sigma_k^{(i)},\sigma_k}\boldsymbol{\xi}_{\sigma_k} = \mu\boldsymbol{\xi}_{\sigma_k^{(i)}}$$

for $i = 1, 2, \ldots p'$ and $k = 1, 2, \ldots p_i'^{(i)}$.

Suppose none of the following holds.

$$r_{\sigma_{k'},\sigma_{k'}+1}\boldsymbol{\xi}_{\sigma_{k'}} + \cdots + r_{\sigma_{k'},\sigma_k}\boldsymbol{\xi}_{\sigma_k} \neq 0 \tag{8.4}$$

for $k' = p' + 1, p' + 2, \ldots k - 1$

$$r_{\sigma_k^{(i)},\sigma_k^{(i)}+1}\boldsymbol{\xi}_{\sigma_k^{(i)}+1} + \cdots + r_{\sigma_k^{(i)},\sigma_k}\boldsymbol{\xi}_{\sigma_k} \neq 0 \tag{8.5}$$

for $i = 1, 2, \ldots p'$ and $k = 1, 2, \ldots p_i'^{(i)}$.

Then we get an eigenvector. Otherwise we take the vector at the maximum joint point where the above relation holds. By similar methods we get our required vector .So for every non-leading μ also we get hold of the vectors with the required properties. We call them $\boldsymbol{\xi}^{(j)}$, for $j = p' + 1, \ldots, p$.

We introduce the following definition here which will be used later to find out rates of convergence.

Definition 8.4. If the maximum joint point for $\boldsymbol{\xi}^{(j)}$, for some $j \in \{p'+1, \dots, p\}$ where the inequalities (8.4) or (8.5) hold be at a leading position for μ, then we call the joint point with position j to be a *non-leading position with leading support*.

8.2. **Rates of Convergence.** We shall now consider the rates of convergence for the linear combinations. First of all note that for any eigenvector ξ for μ, $\boldsymbol{C}_n\boldsymbol{\xi}/\Pi_n(\mu)$ is always an L^2 bounded martingale. So the rate of convergence is n^{μ}.

For the vectors $\boldsymbol{\xi}^{\sigma_k^{(i)}}$, for some k and i, note that from Theorem 8.3 and Theorem 7.3

$$\begin{aligned}\lim_{n\to\infty}\frac{\boldsymbol{C}_n\boldsymbol{\xi}^{\sigma_k^{(i)}}}{n^{\mu}(\log n)^i} &= \lim_{n\to\infty}\frac{\boldsymbol{C}_{n,\sigma_k^{(i)}}\boldsymbol{\xi}^{\sigma_k^{(i)}}_{\sigma_k^{(i)}}}{n^{\mu}(\log n)^i} + \sum_{j<\sigma_1^{(i)}}\lim_{n\to\infty}\frac{\boldsymbol{C}_{n,j}\boldsymbol{\xi}_j^{\sigma_k^{(i)}}}{n^{\mu}(\log n)^i}\\ &= \lim_{n\to\infty}\frac{\boldsymbol{C}_{n,\sigma_k^{(i)}}\boldsymbol{\xi}^{\sigma_k^{(i)}}_{\sigma_k^{(i)}}}{n^{\mu}(\log n)^i}\\ &= V_{\sigma_k^{(i)}}\boldsymbol{\xi}^{\sigma_k^{(i)}}_{\sigma_k^{(i)}}\end{aligned}$$

where the limits are both in almost sure as well as L^2 sense. We are left to find out the rates of convergence for linear combinations with the vectors $\boldsymbol{\xi}^{(j)}$ for $j \ge p'+1$.

If j be in a non-leading position with leading support then we have

$$R\boldsymbol{\xi}^{(j)} = \boldsymbol{\eta} + \mu\boldsymbol{\xi}^{(j)} \tag{8.6}$$

for some vector $\boldsymbol{\eta} = \boldsymbol{\xi}^{\sigma_k^{(i)}}$ for some k and i, $k = 1, 2, \dots p_i'^{(i)}$, $i = 1, 2, \dots p'$. Let $r_{\sigma(j)}$ be in the block with leading color ν of type γ. Of course $\mu < \nu$. Let the rate for $\boldsymbol{C}_n\boldsymbol{\eta}$ be $n^{\mu}(\log n)^{\delta}$.

Case 1: $\mu < \nu/2$.

We follow standard procedures for calculation to find out a weak limit. We take

$$X_n = \frac{\boldsymbol{C}_n\boldsymbol{\xi}^{(j)}}{n^{\nu/2}(\log n)^{\gamma/2}}$$

Define $M_{j+1} = X_{j+1} - E(X_{j+1}|\mathcal{F}_j)$ where $\mathcal{F}_j = \sigma(\boldsymbol{C}_i; i \le j)$. Then we have

$$\begin{aligned}X_{n+1} &= X_n\left(1-\frac{\gamma}{2n\log n}\right)\left(1-\frac{(\nu/2-\mu)}{n}\right) + X_nO(n^{-2}) + \frac{\boldsymbol{C}_n\boldsymbol{\eta}}{(n+1)^{1+\nu/2}(\log(n+1))^{\gamma/2}} + M_{n+1}\\ &= X_1b_n + \sum_{j=1}^{n}X_jO(j^{-2})\frac{b_n}{b_j} + \sum_{j=1}^{n}\frac{\boldsymbol{C}_j\boldsymbol{\eta}}{(j+1)^{1+\nu/2}(\log(j+1))^{\gamma/2}}\frac{b_n}{b_j} + \sum_{j=1}^{n}M_{j+1}\frac{b_n}{b_j}\end{aligned} \tag{8.7}$$

where

$$b_n = \prod_{i=1}^{n}\left(1-\frac{\gamma}{2i\log i}\right)\left(1-\frac{(\nu/2-\mu)}{i}\right)$$

It clearly follows that the first two terms of (8.7) go to zero almost surely. For the third term it follows from the assumption on rate of $\boldsymbol{C}_n\boldsymbol{\eta}$ that

$$\begin{aligned}&\sum_{j=1}^{n}\frac{\boldsymbol{C}_j\boldsymbol{\eta}}{(j+1)^{1+\nu/2}(\log(j+1))^{\gamma/2}}\prod_{i=j+1}^{n}\left(1-\frac{\gamma}{2i\log i}\right)\left(1-\frac{(\nu/2-\mu)}{i}\right)\\ &\sim (\log n)^{-\gamma/2}n^{-(\nu/2-\mu)}\sum_{j=1}^{n}\frac{j^{\mu}(\log j)^{\delta}}{j^{1+\mu}}\\ &\sim (\log n)^{-\gamma/2}n^{-(\nu/2-\mu)}\frac{(\log n)^{\delta+1}}{\delta+1}\end{aligned}$$

which goes to 0 almost surely because $\mu < \nu/2$. Now for the fourth term the Lyapunov condition for Martingale CLT can be checked pretty easily. We are only left to find the asymptotic variance. Let $\boldsymbol{\alpha} = \boldsymbol{\eta} + \mu\boldsymbol{\xi}^{(j)}$. Then

$$M_{n+1} = \frac{1}{(n+1)^{\nu/2}(\log(n+1))^{\gamma/2}}\left[\boldsymbol{\chi}_{n+1} - \frac{\boldsymbol{C}_n}{n+1}\right]\boldsymbol{\alpha}$$

As $\boldsymbol{\alpha}$ has non-zero elements upto j which is in the block of ν which has type γ, we have

$$\begin{aligned} E(M_{n+1}^2|\mathcal{F}_n) &= \frac{1}{(n+1)^{\nu}(\log(n+1)^{\gamma}}\left[\frac{\boldsymbol{C}_n}{n+1}\boldsymbol{\alpha}^2 - \left(\frac{\boldsymbol{C}_n\boldsymbol{\alpha}}{n+1}\right)^2\right] \\ &\sim \frac{\boldsymbol{V}^{(\nu,\gamma)}\boldsymbol{\alpha}^2}{n} \end{aligned}$$

where define $\boldsymbol{V}^{(\nu,\gamma)}$ is the part of the vector $(V_1, V_2, \ldots, V_{K+1})$ obtained from Theorem 7.3 which corresponds to blocks of type γ with leading element ν and the remaining elements 0. So the asymptotic variance turns out to be like

$$\begin{aligned} &\sim n^{-(\nu-2\mu)}(\log n)^{-\gamma}\sum_{j=1}^{n}\frac{\boldsymbol{V}^{(j)}\boldsymbol{\alpha}^2}{j^{1-(\nu-2\mu)}(\log j)^{-\gamma}} \\ &\to \frac{\boldsymbol{V}^{(\nu,\gamma)}\boldsymbol{\alpha}^2}{\nu - 2\mu} \end{aligned}$$

Case 2: $\mu > \nu/2$.
We consider the following martingale:

$$M_n = \frac{\boldsymbol{C}_n\boldsymbol{\xi}^{(j)}}{\Pi_n(\mu)} - \sum_{j=1}^{n-1}\frac{\boldsymbol{C}_j\boldsymbol{\eta}}{(j+1)\Pi_{j+1}(\mu)}$$

Then we have

$$E((M_{n+1} - M_n)^2|\mathcal{F}_n) = \frac{1}{\Pi_{n+1}^2(\mu)}\left[\frac{\boldsymbol{C}_n}{n+1}\boldsymbol{\alpha}^2 - \left(\frac{\boldsymbol{C}_n\boldsymbol{\alpha}}{n+1}\right)^2\right]$$

where $\boldsymbol{\alpha} = \boldsymbol{\eta} + \mu\boldsymbol{\xi}^{(j)}$. Now $\boldsymbol{\alpha}$ has non-zero elements upto $\sigma^{(j)}$. Hence from the proof of Theorem 7.3 we have $E(\boldsymbol{C}_n\boldsymbol{\alpha}/n^{\nu}(\log n)^{\gamma})$ is bounded. So

$$E((M_{n+1} - M_n)^2) \le O\left(\frac{(\log j)^{\gamma}}{j^{2\mu-\nu+1}}\right)$$

which is summable as $\nu < 2\mu$. Thus M_n is an L^2 bounded martingale which converges almost surely as well as in L^2. So $M_n/(\log n)^{\delta+1}$ goes to 0 almost surely as well as in L^2. So we have

$$\begin{aligned} \lim_{n\to\infty}\frac{\boldsymbol{C}_n\boldsymbol{\xi}^{(j)}}{\Pi_n(\mu)(\log n)^{\delta+1}} &= \lim_{n\to\infty}\frac{1}{(\log n)^{\delta+1}}\sum_{j=1}^{n-1}\frac{\boldsymbol{C}_j\boldsymbol{\eta}}{(j+1)\Pi_{j+1}(\mu)} \\ &= \frac{U}{\delta+1} \end{aligned}$$

where

$$U = \lim_{n\to\infty}\frac{\boldsymbol{C}_n\boldsymbol{\eta}}{n^{\mu}(\log n)^{\delta}}$$

Now we look into the limiting distribution for $\boldsymbol{\xi}^{(k)}$, $k \ge p' + 1$ which do not have leading support. Let σ_k be in the block with leading color ν_k. Now observe that if $R\boldsymbol{\xi}^{(k)} = \boldsymbol{\eta} + \mu\boldsymbol{\xi}^{(k)}$ and $\boldsymbol{C}_n\boldsymbol{\eta}$ has rate $n^{\mu}(\log n)^{\delta}$ and $\mu > \nu_k/2$ holds then by similar arguments we can see that the rate of $\boldsymbol{C}_n\boldsymbol{\xi}^{(k)}$ is $n^{\mu}(\log n)^{\delta+1}$.

Now let the support joint for σ_k be σ_{β_1}. Again that of σ_{β_1} be σ_{β_2}. After finite number of steps the joint $\sigma_{\beta_{k_0}}$ will have leading support. Note that for each σ_{β_i}, if ν_{β_i} be the leading color of the corresponding blocks then $\mu > \nu_{\beta_i}/2$ holds because colors are arranged in increasing order. Hence $\boldsymbol{C}_n\boldsymbol{\xi}^{(\beta_{k_0})}$ will have rate

$n^{\mu}(\log n)^{\delta}$ for some δ. Continuing the argument through the sequence of $\{\sigma_{\beta_i}\}$, $i = 1, 2, \dots k$, we see that the rate of $\boldsymbol{C}_n\boldsymbol{\xi}^{(\sigma_k)}$ is $n^{\mu}(\log n)^{\delta'}$ where $\delta' = \varsigma - 1$ where $n^{\mu}(\log n)^{\varsigma}$ is the rate of $\boldsymbol{C}_n\boldsymbol{\eta}$.

Again if $\mu < \nu_k/2$ and its support joint be in a block with leading color $\nu_{k'}$ with $\mu > \nu_{k'}/2$, then $\boldsymbol{C}_n\boldsymbol{\eta}$ has rate $n^{\mu}(\log n)^{\delta}$ for some δ. Hence from similar arguments it follows that the weak limit rate of $\boldsymbol{C}_n\boldsymbol{\xi}^{(k)}$ is $n^{\nu/2}(\log n)^{\gamma/2}$ with appropriate normal limit.

Now let $\mu < \nu_{k'}/2$. Suppose that the support joint for σ_k be $\sigma_{k'}$ and suppose $\sigma_{k'}$ has leading support. Suppose we have

$$R\boldsymbol{\xi}^{(k)} = \boldsymbol{\xi}^{(k')} + \mu\boldsymbol{\xi}^{(k)}$$
$$R\boldsymbol{\xi}^{(k')} = \boldsymbol{\xi} + \mu\boldsymbol{\xi}^{(k')}$$

where the rate of $\boldsymbol{C}_n\boldsymbol{\xi}$ is $n^{\mu}(\log n)^{\varsigma}$ for some $\varsigma \geq 0$. Now let the types of ν_k and $\nu_{k'}$ be δ_k and $\delta_{k'}$. Define

$$X_n = \left(\frac{\boldsymbol{C}_n\boldsymbol{\xi}^{(k')}}{n^{\nu'_k/2}(\log n)^{\delta'_k/2}} : \frac{\boldsymbol{C}_n\boldsymbol{\xi}^{(k)}}{n^{\nu_k/2}(\log n)^{\delta_k/2}}\right) \tag{8.8}$$

Now define the functions

$$A_n(x) = \left(1 - \frac{x}{2n\log n}\right)$$
$$B_n(x) = \left(1 - \frac{x/2 - \mu}{n}\right)$$

Then simple algebra yields

$$E(X_{n+1}|\mathcal{F}_n) = (\Lambda_n : \Xi_n) + \left(\frac{\boldsymbol{C}_n\boldsymbol{\xi}^{(k')}}{n^{\nu'_k/2}(\log n)^{\delta'_k/2}}O(n^{-2}) : \frac{\boldsymbol{C}_n\boldsymbol{\xi}^{(k)}}{n^{\nu_k/2}(\log n)^{\delta_k/2}}O(n^{-2})\right)$$

where

$$\Lambda_n = \frac{\boldsymbol{C}_n\boldsymbol{\xi}^{(k')}}{n^{\nu'_k/2}(\log n)^{\delta'_k/2}}A_n(\delta_{k'})B_n(\nu_{k'}) + \frac{\boldsymbol{C}_n\boldsymbol{\xi}}{(n+1)n^{\nu'_k/2}(\log n)^{\delta'_k/2}}A_n(\delta_{k'})B_n(\nu_{k'})$$
$$\Xi_n = \frac{\boldsymbol{C}_n\boldsymbol{\xi}^{(k)}}{n^{\nu_k/2}(\log n)^{\delta_k/2}}A_n(\delta_k)B_n(\nu_k) + \frac{\boldsymbol{C}_n\boldsymbol{\xi}^{(k')}}{(n+1)n^{\nu_k/2}(\log n)^{\delta_k/2}}A_n(\delta_k)B_n(\nu_k)$$

So we have

$$E(X_{n+1}|\mathcal{F}_n) = X_n \begin{pmatrix} A_n(\delta_{k'})B_n(\nu_{k'}) & \frac{A_n(\delta_k)B_n(\nu_k)n^{\nu'_k/2}(\log n)^{\delta'_k/2}}{(n+1)n^{\nu_k/2}(\log n)^{\delta_k/2}} \\ 0 & A_n(\delta_k)B_n(\nu_k) \end{pmatrix} + \left(\frac{\boldsymbol{C}_n\boldsymbol{\xi}}{(n+1)n^{\nu'_k/2}(\log n)^{\delta'_k/2}} : 0\right)$$
$$+ X_n \begin{pmatrix} O(n^{-2}) & 0 \\ 0 & O(n^{-2}) \end{pmatrix}$$

Let

$$O_n = \begin{pmatrix} O(n^{-2}) & 0 \\ 0 & O(n^{-2}) \end{pmatrix}$$

Let

$$\Delta_n = \begin{pmatrix} A_n(\delta_{k'})B_n(\nu_{k'}) & \frac{A_n(\delta_k)B_n(\nu_k)n^{\nu'_k/2}(\log n)^{\delta'_k/2}}{(n+1)n^{\nu_k/2}(\log n)^{\delta_k/2}} \\ 0 & A_n(\delta_k)B_n(\nu_k) \end{pmatrix} \tag{8.9}$$

Then $\Delta_n = I_n + F_n$ where

$$I_n = \begin{pmatrix} A_n(\delta_{k'})B_n(\nu_{k'}) & 0 \\ 0 & A_n(\delta_k)B_n(\nu_k) \end{pmatrix} \tag{8.10}$$

and

$$F_n = \begin{pmatrix} 0 & \frac{A_n(\delta_k)B_n(\nu_k)n^{\nu'_k/2}(\log n)^{\delta'_k/2}}{(n+1)n^{\nu_k/2}(\log n)^{\delta_k/2}} \\ 0 & 0 \end{pmatrix} \tag{8.11}$$

Note that $F_iF_j = 0$ for any two integers i and j. Let I_0 to be the identity matrix of order 2. So we have

$$\prod_{i=p+1}^{n} \Delta_i = \prod_{i=p+1}^{n} I_i + \sum_{i=p+1}^{n} I_{p+1}I_{p+2}\ldots I_{i-1}F_iI_{i+1}\ldots I_n$$

Now note the following facts

$$\prod_{i=p+1}^{n} I_i = \begin{pmatrix} \prod_{i=p+1}^{n} A_i(\delta_{k'})B_i(\nu_{k'}) & 0 \\ 0 & \prod_{i=p+1}^{n} A_i(\delta_k)B_i(\nu_k) \end{pmatrix} \tag{8.12}$$

$$I_{p+1}I_{p+2}\ldots I_{i-1}F_iI_{i+1}\ldots I_n = \begin{pmatrix} 0 & \prod_{j=p+1}^{i-1} A_j(\delta_{k'})B_j(\nu_{k'}) \frac{A_i(\delta_k)B_i(\nu_k) i^{\nu'_k/2}(\log i)^{\delta'_k/2}}{(i+1)i^{\nu_k/2}(\log i)^{\delta_k/2}} \prod_{j=i+1}^{n} A_j(\delta_k)B_j(\nu_k) \\ 0 & 0 \end{pmatrix} \tag{8.13}$$

The non-zero term in (8.13) is

$$\begin{aligned} &\sim \frac{A_i(\delta_k)B_i(\nu_k) i^{\nu'_k/2}(\log i)^{\delta'_k/2}(\log(i-1))^{-\delta'_k/2}(i-1)^{-(\nu'_k/2-\mu)}(\log n)^{-\delta_k/2}n^{-(\nu_k/2-\mu)}}{(\log(p))^{-\delta'_k/2}(p)^{-(\nu'_k/2-\mu)}(i+1)i^{\nu_k/2}(\log i)^{\delta_k/2}(\log i)^{-\delta_k/2}i^{-(\nu_k/2-\mu)}} \\ &= \frac{(\log n)^{-\delta_k/2}n^{-(\nu_k/2-\mu)}}{(\log p)^{-\delta'_k/2}p^{-(\nu'_k/2-\mu)}}O(i^{-1}) \end{aligned} \tag{8.14}$$

So $(1,2)$ th term of $\sum_{i=p+1}^{n} I_{p+1}I_{p+2}\ldots I_{i-1}F_iI_{i+1}\ldots I_n$ is like

$$\frac{(\log n)^{1-\delta_k/2}n^{-(\nu_k/2-\mu)}}{(\log p)^{1-\delta'_k/2}p^{-(\nu'_k/2-\mu)}} \tag{8.15}$$

From (8.12) and (8.15), it is clear that

$$\prod_{i=1}^{n} \Delta_i \to 0 \tag{8.16}$$

if we define $M_{j+1} = X_{j+1} - E(X_{j+1}|\mathcal{F}_j)$, then we have the iterating equation

$$E(X_{n+1}) = X_1 \prod_{i=1}^{n} \Delta_i + \sum_{j=1}^{n} X_j O_j \prod_{i=j+1}^{n} \Delta_i + \sum_{j=1}^{n} \left(\frac{\boldsymbol{C}_j \boldsymbol{\xi}}{(j+1)j^{\nu'_k/2}(\log j)^{\delta'_k/2}} : 0 \right) \prod_{i=j+1}^{n} \Delta_i + \sum_{j=1}^{n} M_{j+1} \prod_{i=j+1}^{n} \Delta_i \tag{8.17}$$

From (8.15) and (8.12) we can easily see that the first three terms of (8.17) go to zero almost surely. now note that

$$M_{j+1} = \left(\frac{1}{(j+1)^{\nu'_k/2}(\log(j+1))^{\delta'_k/2}} \left(\boldsymbol{\chi}_{j+1} - \frac{\boldsymbol{C}_j}{j+1} \right) \boldsymbol{\alpha} : \frac{1}{(j+1)^{\nu_k/2}(\log(j+1))^{\delta_k/2}} \left(\boldsymbol{\chi}_{j+1} - \frac{\boldsymbol{C}_j}{j+1} \right) \boldsymbol{\alpha}' \right)$$

where $\boldsymbol{\alpha} = \boldsymbol{\xi} + \mu \boldsymbol{\xi}^{(k')}$ and $\boldsymbol{\alpha}' = \boldsymbol{\xi}^{(k')} + \mu \boldsymbol{\xi}^{(k)}$. So $E(M'_{j+1}M_{j+1}|\mathcal{F}_j)$ is the matrix

$$\begin{pmatrix} \frac{1}{(j+1)^{\nu'_k}(\log(j+1))^{\delta'_k}} \left(\frac{\boldsymbol{C}_j}{j+1}\boldsymbol{\alpha}^2 - \left(\frac{\boldsymbol{C}_j\boldsymbol{\alpha}}{j+1} \right)^2 \right) & \frac{1}{(j+1)^{\nu'_k/2+\nu_k/2}(\log(j+1))^{\delta'_k/2+\delta_k/2}} \left(\frac{\boldsymbol{C}_j}{j+1}\boldsymbol{\alpha}\boldsymbol{\alpha}' - \frac{\boldsymbol{C}_j\boldsymbol{\alpha}}{j+1}\frac{\boldsymbol{C}_j\boldsymbol{\alpha}'}{j+1} \right) \\ \frac{1}{(j+1)^{\nu'_k/2+\nu_k/2}(\log(j+1))^{\delta'_k/2+\delta_k/2}} \left(\frac{\boldsymbol{C}_j}{j+1}\boldsymbol{\alpha}\boldsymbol{\alpha}' - \frac{\boldsymbol{C}_j\boldsymbol{\alpha}}{j+1}\frac{\boldsymbol{C}_j\boldsymbol{\alpha}'}{j+1} \right) & \frac{1}{(j+1)^{\nu_k}(\log(j+1))^{\delta_k}} \left(\frac{\boldsymbol{C}_j}{j+1}\boldsymbol{\alpha}'^2 - \left(\frac{\boldsymbol{C}_j\boldsymbol{\alpha}'}{j+1} \right)^2 \right) \end{pmatrix} \tag{8.18}$$

From this matrix the Lyapunov condition for martingale CLT follows easily. We are left to fing the asymptotic variance. That is we have to find the limit of

$$\sum_{j=1}^{n} (\prod_{i=j+1}^{n} \Delta_i)' E(M'_{j+1}M_{j+1}|\mathcal{F}_j)(\prod_{i=j+1}^{n} \Delta_i) \tag{8.19}$$

The $(1,1)$ th term of (8.19) is the limit of

$$\sum_{j=1}^{n} \frac{(\log n)^{-\delta'_k}n^{-(\nu'_k-2\mu)}}{(\log j)^{-\delta'_k}j^{-(\nu'_k-2\mu)}} \frac{1}{(j+1)^{\nu'_k}(\log(j+1))^{\delta'_k}} \frac{\boldsymbol{C}_j\boldsymbol{\alpha}^2}{j+1}$$

$$\sim (\log n)^{-\delta'_k} n^{-(\nu'_k - 2\mu)} \sum_{j=1}^{n} \frac{\boldsymbol{V}^{(\nu'_k,\delta'_k)} j^{\nu'_k} (\log j)^{\delta'_k} \boldsymbol{\alpha}^2}{j^{1+2\mu}}$$

So, the $(1,1)$ th term of asymptotic variance turns out to be

$$\frac{\boldsymbol{V}^{(\nu'_k,\delta'_k)} \boldsymbol{\alpha}^2}{\nu'_k - 2\mu} \tag{8.20}$$

By symmetry the $(2,2)$ th term of asymptotic variance is

$$\frac{\boldsymbol{V}^{(\nu_k,\delta_k)} \boldsymbol{\alpha}'^2}{\nu_k - 2\mu} \tag{8.21}$$

The $(1,2)$ th term of (8.19) is the limit of

$$\sum_{j=1}^{n} \frac{(\log n)^{-(\delta'_k+\delta_k)/2} n^{-((\nu'_k+\nu_k)/2-2\mu)}}{(\log j)^{-(\delta'_k+\delta_k)/2} j^{-((\nu'_k+\nu_k)/2-2\mu)}} \frac{1}{(j+1)^{(\nu'_k+\nu_k)/2} (\log(j+1))^{(\delta'_k+\delta_k)/2}} \frac{\boldsymbol{C}_j \boldsymbol{\alpha}\boldsymbol{\alpha}'}{j+1}$$
$$\sim \frac{(\log n)^{-(\delta'_k+\delta_k)/2} n^{-((\nu'_k+\nu_k)/2-2\mu)}}{j^{1+2\mu}} \frac{\boldsymbol{V}^{(\nu'_k,\delta'_k)} \boldsymbol{\alpha}\boldsymbol{\alpha}' n^{(\nu'_k-2\mu)} (\log n)^{\delta'_k}}{\nu'_k - 2\mu}$$

whose limit is 0 if $\nu_k > \nu'_k$ or $\delta_k > \delta'_k$. If $\nu_k = \nu'_k$ and $\delta_k = \delta'_k$ then the $(1,2)$ th term of the asymptotic variance is

$$\frac{\boldsymbol{V}^{(\nu'_k,\delta'_k)} \boldsymbol{\alpha}\boldsymbol{\alpha}'}{\nu'_k - 2\mu} \tag{8.22}$$

So, asymptotically the linear combinations are not independent if and only if the μ's belong to the same superblock and blocks of same type. So we have the following Theorem.

Theorem 8.5. *Consider an urn model with a $(K+1)\times(K+1)$ triangular replacement matrix R. Let $\boldsymbol{\xi}^{\sigma_k^{(i)}}$ for $k = 1,2,\ldots,p_i'^{(i)}$ and $i = 1,2,\ldots p'$ and $\boldsymbol{\xi}^{(k)}$ for $k = p'+1,\ldots,p$ be defined as before. Let V_j, $j = 1,2,\ldots,K+1$ be the random variables obtained in Theorem 7.3. For $j \geq p'+1$, let ν_j be the leading diagonal entry of the block where μ belongs. Let its type be γ_j. also define $\boldsymbol{V}^{(\nu,\gamma)}$ to be the part of the vector $\boldsymbol{V} = (V_1, V_2, V_{K+1})$ which corresponds to the blocks of R with leading color ν and type γ. Then we have*

(i) *For any eigenvector $\boldsymbol{\varsigma}$, $\boldsymbol{C}_n\boldsymbol{\varsigma}/Pi_n(\mu)$ is an L^2 bounded martingale which converges almost surely as well as in L^2 to some non degenerate random variable.*

(ii) *For any vector $\boldsymbol{\xi}^{\sigma_k^{(i)}}$ we have $\boldsymbol{C}_n\boldsymbol{\xi}^{\sigma_k^{(i)}}/n^\mu(\log n)^i \to V_{\sigma_k^{(i)}} \boldsymbol{\xi}^{\sigma_k^{(i)}}_{\sigma_k^{(i)}}$ almost surely as well as in L^2.*

(iii) *If $\boldsymbol{\xi}^{(j)}$ be a vector with leading support and $\mu < \nu_j/2$ then $\boldsymbol{C}_n\boldsymbol{\xi}^{(j)}/n^{\nu_j/2}(\log n)^{\gamma_j/2} \Rightarrow N(0, \boldsymbol{V}^{(\nu_j,\gamma_j)}\boldsymbol{\alpha}^2/(\nu_j - 2\mu))$ where $\boldsymbol{\alpha} = \boldsymbol{\eta} + \mu\boldsymbol{\xi}^{(j)}$ where $R\boldsymbol{\xi}^{(j)} = \boldsymbol{\eta} + \mu\boldsymbol{\xi}^{(j)}$.*

(iv) *If $\boldsymbol{\xi}^{(j)}$ be a vector and $\mu > \nu_j/2$ then $\boldsymbol{C}_n\boldsymbol{\xi}^{(j)}/n^\mu(\log n)^{\delta+1} \to U/(\delta+1)$ almost surely as well as in L^2. Here $U = \lim_{n\to\infty} \boldsymbol{C}_n\eta/(n^\mu(\log n)^\delta)$ where $R\boldsymbol{\xi}^{(j)} = \boldsymbol{\eta} + \mu\boldsymbol{\xi}^{(j)}$.*

(v) *If $\boldsymbol{\xi}^{(j)}$ be a vector with non-leading support and $\mu < \nu_j/2$. Let the supporting μ be $\boldsymbol{\xi}^{(j')}$ where $\mu > \nu_{j'}$, then $\boldsymbol{C}_n\boldsymbol{\xi}^{(j)}/n^{\nu_j/2}(\log n)^{\gamma_j/2} \Rightarrow N(0, \boldsymbol{V}^{(\nu_j,\gamma_j)}\boldsymbol{\alpha}^2/(\nu_j - 2\mu))$ where $\boldsymbol{\alpha} = \boldsymbol{\xi}^{(j')} + \mu\boldsymbol{\xi}^{(j)}$.*

(vi) *If $\boldsymbol{\xi}^{(j)}$ be a Jordan vector with non-leading support and $\mu < \nu_j/2$. Let the supporting μ be $\boldsymbol{\xi}^{(j')}$ where $\mu < \nu_{j'}$ and also suppose j' has leading support. Let $R\boldsymbol{\xi}^{(j')} = \boldsymbol{\alpha}$ and $R\boldsymbol{\xi}^{(j)} = \boldsymbol{\alpha}'$. Then $(\boldsymbol{C}_n\boldsymbol{\xi}^{(j')}/n^{\nu_{j'}/2}(\log n)^{\gamma_{j'}/2}, \boldsymbol{C}_n\boldsymbol{\xi}^{(j)}/n^{\nu_j/2}(\log n)^{\gamma_j/2}) \Rightarrow N((0,0)', \Sigma_{j',j})$ where*

$$\Sigma_{j',j} = \begin{pmatrix} \boldsymbol{V}^{(\nu_{j'},\gamma_{j'})}\boldsymbol{\alpha}^2/(\nu_{j'} - 2\mu) & \boldsymbol{V}^{(\nu_{j'},\gamma_{j'})}\boldsymbol{\alpha}\boldsymbol{\alpha}'/(\nu_{j'} - 2\mu) \\ \boldsymbol{V}^{(\nu_{j'},\gamma_{j'})}\boldsymbol{\alpha}\boldsymbol{\alpha}'/(\nu_{j'} - 2\mu) & \boldsymbol{V}^{(\nu_j,\gamma_j)}\boldsymbol{\alpha}'^2/(\nu_j - 2\mu) \end{pmatrix} \tag{8.23}$$

if $\nu_j = \nu_{j'}$ and $\gamma_j = \gamma_{j'}$. If not then

$$\Sigma_{j',j} = \begin{pmatrix} \boldsymbol{V}^{(\nu_{j'},\gamma_{j'})}\boldsymbol{\alpha}^2/(\nu_{j'} - 2\mu) & 0 \\ 0 & \boldsymbol{V}^{(\nu_j,\gamma_j)}\boldsymbol{\alpha}'^2/(\nu_j - 2\mu) \end{pmatrix} \tag{8.24}$$

FIGURE 1. Density of π_0 for different values of α, (from left to right), $\alpha = 0.8, 1$.

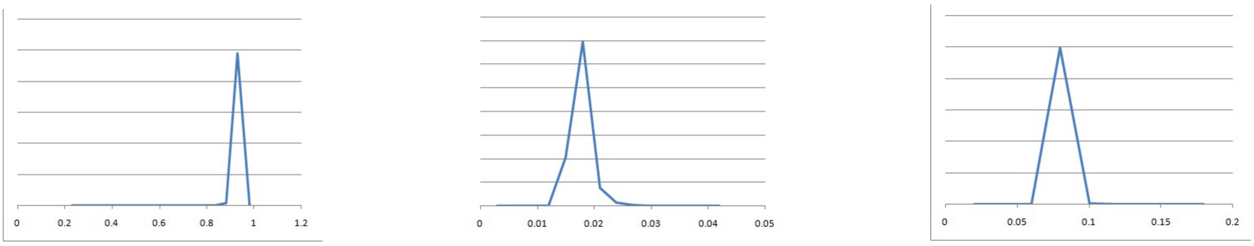

FIGURE 2. Degenerate density of π_0 for $\alpha = 3$ for colors (from left to right) $0, 1$ and 2.

9. An urn model with balls of infinitely many colors

In this final section, we consider an urn model which contains balls of infinitely many colors. The colors are labeled as $0, 1, 2, \ldots$. We consider a particular replacement matrix, which is irreducible. To describe the replacement matrix, consider a probability distribution $\boldsymbol{p} = \{p_0, p_1, p_2, \ldots\}$ on the set of non-negative integers $\{0, 1, 2, \ldots\}$. If a ball of the 0-th color is drawn, then p_i many balls of i-th color are added, for $i \geq 0$. For $i > 0$, if a ball of i-th color is drawn, then 1 ball of $(i-1)$-th color is added. Thus, we obtain the replacement matrix

$$R = \begin{pmatrix} p_0 & p_1 & p_2 \ldots & \\ 1 & 0 & 0 \ldots & \\ 0 & 1 & 0 \ldots & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{9.1}$$

The replacement matrix R above is clearly irreducible. The matrix is positive or null recurrent, according as the probability vector $\boldsymbol{p}$ has finite or infinite expectation, that is, $\sum ip_i$ is finite or infinite. Not much is known in the literature about the ratio of the vector of the color counts and we tried simulating such urns for some choices of the vector $\boldsymbol{p}$. When R is positive recurrent, the matrix has a stationary distribution, that is, a left eigenvector corresponding to the eigenvalue 1, the sum of whose coordinates is also 1. In analogy to the urn with balls of finitely many colors, we expect the vector of color fraction to converge to the stationary distribution. However, it is difficult to guess the limit, when R is null recurrent.

For simulation, we considered probability vectors $\boldsymbol{p}$, which are distributed as discrete Pareto with parameter α, that is, $p_i \propto i^{-(\alpha+1)}$, where $\alpha > 0$. This leads to positive recurrent R if and only if $\alpha > 1$. We ran the simulation for $\alpha = 0.8, 1,$. We assumed we initially had 1 ball of color 0. So in first draw we draw the 0th color ball surely. Hence we add p_i balls of color i for $i = 0, 1, 2, \ldots$. Then we simulate a uniform$[0, 1]$ random variable and find out from tracked proportion of color counts which color ball drawing is simulated. Then we add color according to matrix (9.1). We do this for 100000 iterations and obtained the final proportion of color counts. We plot the density of the simulated limiting fraction of the 0-th color π_0 in Figure 1.

When $\alpha = 3$ and R is positive recurrent, the limiting fraction for color 0 is degenerate at $\pi_0 = 0.90193$, that of color 1 is degenerate at $\pi_1 = .068$ and that of color 2 is degenerate at $\pi_2 = .016$ as expected (ref Figure 2). However, if $\alpha \leq 1$, the limiting fraction is non-degenerate which has a support $[0, \alpha/(\alpha+1)]$. Based on the simulation, we make the following conjecture.

Conjecture: Consider a replacement matrix R is as in (9.1).

(i) When R is positive recurrent, there exists a probability vector $\boldsymbol{\pi}$, such that $\boldsymbol{\pi} R = \boldsymbol{\pi}$, where $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$. The fraction $C_{n,i}/(n+1)$ converges to π_i almost surely for $i = 0, 1, 2$.

(ii) When R is null recurrent with $\pi.$ regularly varying of index $-(\alpha+1)$, the fraction $C_{n,0}/(n+1)$ converges weakly to a distribution which is supported on $[0, \alpha/(\alpha+1)]$.

References

Arup Bose, Amites Dasgupta, and Krishanu Maulik. (2009a) Multicolor urn models with reducible replacement matrices. *Bernoulli*, 15(1):279–295, 2009. ISSN 1350-7265.

Arup Bose, Amites Dasgupta, and Krishanu Maulik. Strong laws for balanced triangular urns. *J. Appl. Probab.*, 46(2):571–584, 2009. ISSN 0021-9002.

David A. Freedman. Bernard Friedman's urn. *Ann. Math. Statist*, 36:956–970, 1965. ISSN 0003-4851.

Raúl Gouet. Strong convergence of proportions in a multicolor Pólya urn. *J. Appl. Probab.*, 34(2):426–435, 1997. ISSN 0021-9002.