Getting Pitted in the Pipe: Skateboarding Through the Lens of Lagrangian Mechanics

Grayson King (49390438)

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1 Introduction

Skateboarding is a sport that requires significant control and awareness of one's body. Whether it is bending and extending one's legs or shifting forward and backward over the board, changing the location of one's center of mass in a controlled manor is a critical part of navigating obstacles. One of the more common obstacles found in skate parks is the half pipe.

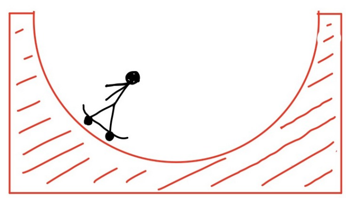


Figure 1: Sketch of a simple half pipe

As the name suggests this obstacle resembles a half cylinder or ellipse, where the skaters can transfer their weight in a "pumping" motion, gaining momentum, to ride up on to increasingly vertical portions of the ramp. This so called pumping motion turns out to be fairly periodic when done correctly, and will be the primary focus of investigation in this paper.

1.1 Motivations

One of the first questions that should arise when trying to tackle the half pipe is: what sort of pumping motion will get me to the top of this ramp quickest? After all, the vertical portions of the ramp is where the majority of tricks are executed! In order to answer this question, the period of oscillations of a skateboarders center of mass as a function of their angle on the ramp will need to be investigated.

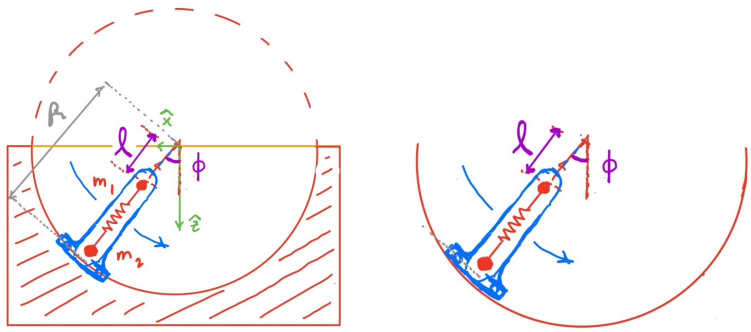
1.2 Next Steps

To analyze this problem, a simplified model of the physical situation will first need to be created. After this, a Lagrangian can be determined for the system, and different analysis techniques can be used to extract information from the Lagrangian. First a small oscillations approximation will be attempted to gain information on the system's resonant behaviour. After this, an exact solution will be solved for and visualized using python. Listed below are links to the results uncovered over the course of the paper (for the purposes of following along while reading).

- Link to GitHub page containing all referenced documents.
- Link to video of Small Oscillations visualization with python.
- Link to video of medium, Coupled Oscillations visualization with python.
- Link to video of Large Oscillations visualization with python.

2 The Model and Setup

To model this system, the total distributed mass of a person and a skateboard can be shrunken down to two points of interest: the skateboarder's center of mass and the location where the skateboard touches the ramp. After placing point masses at these two locations, a spring can be added between the two, to simulated storage and release of energy when one bends and extends their legs.



(a) Modelled System (b) Generalized Coordinates

Figure 2: Sketch of Modelled System

To further simplify the situation and to place focus the analysis of pumping in the half pipe, the assumption can be made that the skateboarder can keep their center of mass laterally centered over the board at all times (this is close to the ideal operating conditions for riding a half pipe). As a result, the motion of the mass on the top of the spring can be assumed to move directly towards the center of the circular profile of the half pipe.

2.1 Lagrangian

The Lagrangian for this system can be seen below. All scratch work done to arrive at this answer can be seen in Subsection 1 of the Appendix.

$$\mathcal{L} = \frac{m_2(\dot{l}^2 + l^2\dot{\varphi})}{2} + \frac{m_1\dot{l}^2}{2} + m_2gRcos(\varphi) + m_1gRcos(\varphi) - \frac{k(R-l-l_0)^2}{2}$$

Where m_1, m_2, k, l_0, R are constants labelled in Figure 2a, and l and φ are the generalized coordinates of this modelled system, labelled in Figure 2b. This result is quite complicated, and would be a hassle to go about solving by hand. In particularly, the $cos(\varphi)$ terms and the $l^2\dot{\varphi}$ term contribute to pretty nasty Euler-Lagrange equations. A first attempt at getting around this complexity, can be approximating this Lagrangian about a stable equilibrium point of l and φ.

3 Small Oscillations Approximation

The stable equilibrium points about which the system's Lagrangian can be Taylor Expanded are:

$$l_e = R - l_0 + \frac{m_1 g}{k} \quad \text{and} \quad \varphi_e = 0$$

Work done to find these critical points can be found in Subsection 2 of the Appendix. Now the Lagrangian can be approximated close to these points.

3.1 Approximating the Lagrangian

The approximated Lagrangian can be seen below:

$$\mathcal{L} \approx \frac{(m_1+m_2)\dot{l}^2}{2} + \frac{m_2 l_e^2 \dot{\varphi}^2}{2} - \frac{(m_2+m_2)g\varphi^2}{2} - \frac{k(l-l_e)^2}{2}$$

The simplified Euler-Lagrange Equations are now:

$$\ddot{l} = \frac{-k}{m_1 + m_2}(l - l_e) \quad \text{and} \quad \ddot{\varphi} = \frac{-(m_1 + m_2)g}{m_2(R - l_0 - \frac{m_1 g}{k})}\varphi$$

From this result we can gather the normal frequencies of small oscillations for φ and l.

$$\omega_l = \sqrt{\frac{k}{m_1 + m_2}} \quad \text{and} \quad \omega_\varphi = \sqrt{\frac{(m_1 + m_2)g}{m_2(R - l_0 + \frac{m_1 g}{k})^2}}$$

3.2 Results and Evaluation

From the normal frequencies obtained above, it can be determined that, during small oscillations, the modelled system of a skateboarder on a half pipe behaves according to two separate, uncoupled functions l and φ that vary in t. This is an interesting result, but it is not particularly revealing as to the ideal technique required to build up height while riding the half pipe. After all, this approximation is only valid at values near the stable equilibrium points.

Instead, it is expected that the exact solutions to the Lagrangian for this system would exhibit coupled behaviour, where expressions for l and φ share some dependence on each other or some common frequency of oscillations. This is expected in contrast to the results of the small oscillations approximation, as the changing location of the center of mass of the system should contribute to a changing oscillation frequency, and thus a changing φ in t. To proceed with the analysis of this system, the next step is to get a numerical solution for valid for larger angles.

4 Numerical Solution

Python was used to achieve a numerical solution to the second order Euler-Lagrange equations (source code can be found Here).

In order to obtain numbers for the expressions listed throughout this paper, values for the constants defined in the exact Lagrangian will need to be determined. Based on the approximate height and weight of a person, the natural length of the spring $l+0$ and the mass m_1 and m_2 were determined to be 1.5m, 60kg, and 3kg respectively. The radius of the ramp was chosen based on an approximation of the size of the half pipe in the UBC skate park, and the spring constant k was reasonably stiff starting point for the mass m_1.

- $m_1 = 60$kg
- $m_2 = 3$kg
- $k = 800$N/m
- $g = 9.8$m/s^2
- $R = 3$m
- $l_0 = 1.5$m

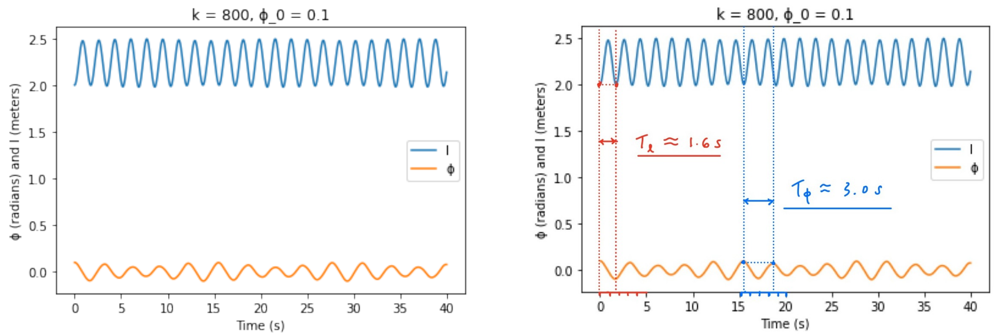
Where the values of the spring constant of the system will be of particular interest, as changing this value can be used to model different techniques of pumping on the half pipe. With a larger spring constant, it is expected that the oscillations of m_1 will occur at higher frequency (although this will also be impacted by the location of the system on the half pipe).

The upshot is that by tuning the spring constant and the starting angle of the masses and spring on the ramp, one can get a better idea of the actual behaviour of the system at small and large oscillations. In particular, graphical techniques and a simple visualization will be used.

4.1 Results and Evaluation

A good first step to checking the validity of the numerical solution constructed is to compare its results at small oscillations with the period and frequency achieved in the small oscillations approximation.

Figure 3a shows the plot of values of φ and l when the system starts at an approximate angle of $5°$. It can be seen that both functions seem to be operating in a fairly independent manor, with two different oscillation periods. This seems to match the results obtained from the small oscillations approximation.



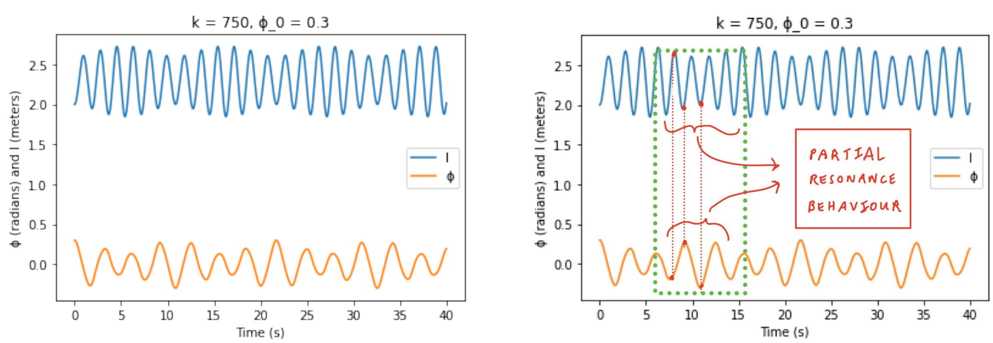
(a) φ and l Small Oscillations (b) Estimating Period

Figure 3: Small Oscillations Graph

From the equations for the angular frequency for small oscillations, it can be seen that approximate period of oscillation for φ and l follow the numerical results in Figure 3b (obtained by plugging in the constant values defined above).

$$\omega_l \approx \sqrt{\frac{k}{m_1 + m_2}} = 1.76s \quad \text{and} \quad \omega_\varphi \approx \sqrt{\frac{(m_1 + m_2)g}{m_2(R - l_0 + \frac{m_1 g}{k})^2}} = 3.2s$$

These results show that the numerical solutions seem to be working well (link to visualized result for Small Oscillations)! Now, by changing the starting angle of system and the spring constant, more heavily couple behaviour can be realized.



(a) φ and l Coupled Oscillations (b) Investigating Resonance

Figure 4: Coupled Oscillation Graphs

Figure 4a and 4b show the coupled behaviour produced by running the model with a spring constant of 750 and an initial angle of $15°$ (link to visualized result for Coupled Oscillations). By taking a closer look at this graph, one can start to deduce what sort of pumping technique is ideal for gaining height on the half pipe (increasing φ the quickest).

Inside green box in Figure 4b, a type of partial resonance behaviour is present between the oscillations of φ and l. Tracing along the two left-most red vertical lines, it can be seen that as the angle of oscillation starts to build, troughs of the graph for φ begin to line up with peaks in the graph of l. These are precisely the conditions required for optimal angle gain when riding the half pipe!

The red line third to the left shows a break in this resonance behaviour causing a loss in amplitude. This comes as the system has not achieved perfect resonance conditions with these particular values of k and φ_0, thus oscillations don't increase indefinitely.

5 Conclusion

Although perfect resonance conditions seem unreachable with this simplified model, the partial resonance behaviour at distinct moments illustrates how the oscillations of m_1 seem to drive the increase or decrease in amplitude of the oscillations of φ. Moreover, if the value of the generalized coordinate l can be maximized as the value of φ is minimized, the amplitude of oscillations of φ will increase most rapidly.

Changing focus to what this would look like for a skateboarder riding the ramp, it can be said that the ideal pumping behaviour follows a pattern where the skateboarder is crouched at the bottom of the ramp and gradually extends their legs, reaching full extension at their turning point on the vertical portion of the ramp. Adhering to this movement while their φ value increases will result in the quickest climb to the top!

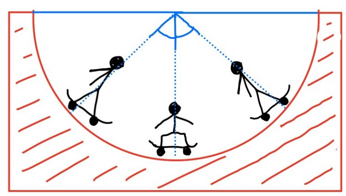


Figure 5: Sketch of Ideal Riding Technique

5.1 Evaluation of Model

Choosing to model a skateboarder on a ramp as two point masses with a spring between them was certainly a useful for investigating a skateboarder's movement on an idealized, half-circle shaped ramp. But, this model is far from perfect, and contains inaccuracies that make it fairly poor at generalizing to different operating conditions.

One such case can be found in the fact that the majority of half pipes are not simply half circles, rather two quarter circles stitched together by a flat portion at the bottom. In reality, this construction offers skateboarders some time to get ready for their next move up the vertical of the ramp; however, such a configuration would ruin the resonance behaviour visible between oscillations in l and φ in the modelled situation.

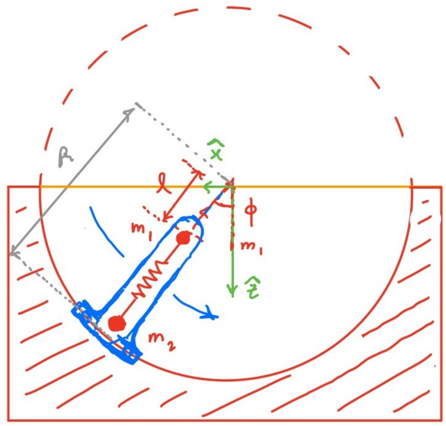
Another short coming of the model is the fact that a perfect resonance condition was not reachable by simply varying k and φ_0 with the other constants fixed. This means that the ideal motion of a skateboarder gaining height on a half pipe could not be modelled with the assumptions made in this paper. One way to achieve a better model would be to add an external driving force to the oscillation of m_1. In this way, the frequency of the driving force could be tuned to perfectly match the increasing period of oscillations in φ.

5.2 Final Words

Although this model was an oversimplification of the true physics of the problem, it was still useful in gaining information about the idealized operating conditions for a skateboarder on half pipe. Through first performing a small oscillations approximation, it was determined that φ and l were expected to behave independently. This is not the behaviour that one would expect from the model, so a more robust solution to the equations of the motion was obtained using python. With these equations of motion, graphs of φ and l as well as physical visualizations were produced. By analyzing the plots, it was determined that lowering one's center of mass at the bottom of the ramp and raising it gradually while ascending up the vertical is the most effective means of gaining height while riding a half pipe.

6 Appendix

6.1 Determination of Lagrangian



$$x_1 = \ell \sin(\phi) \qquad \dot{x}_1 = \dot{\ell}\sin(\phi) + \ell\cos(\phi)\dot{\phi}$$

$$y_1 = \ell \cos(\phi) \qquad \dot{y}_1 = \dot{\ell}\cos(\phi) - \ell\sin(\phi)\dot{\phi}$$

$$x_2 = R \sin(\phi) \qquad \dot{x}_2 = R\cos(\phi)\dot{\phi}$$

$$y_2 = R \cos(\phi) \qquad \dot{y}_2 = -R\sin(\phi)\dot{\phi}$$

$$T = T_1 + T_2$$

$$T_1 = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right)$$

$$\dot{x}_1^2 = \dot{\ell}^2 \sin^2(\phi) + \ell^2\cos^2(\phi)\dot{\phi}^2 + 2\dot{\ell}\dot{\phi}\cancel{\ell\cos(\phi)\sin(\phi)}$$

$$\dot{y}_1^2 = \dot{\ell}^2 \cos^2(\phi) + \ell^2\sin^2(\phi)\dot{\phi}^2 - 2\dot{\ell}\dot{\phi}\cancel{\ell\cos(\phi)\sin(\phi)}$$

$$T_1 = \frac{1}{2} m_1 \left(\dot{\ell}^2 + \ell^2\dot{\phi}^2\right)$$

$$T_2 = \frac{1}{2} m_2 \left(R^2\dot{\phi}^2\right)$$

$$T = \frac{1}{2} m_1 \left(\dot{\ell}^2\right) + \frac{1}{2} m_2 \left(\dot{\ell}^2 + \ell^2\dot{\phi}^2\right)$$

$$U = -\underbrace{m_2 g R\cos(\phi)} - \underbrace{m_1 g \ell\cos(\phi)} + \frac{1}{2}k\left(R - \ell - \ell_0\right)^2$$

APPROXIMATE THESE, COMPLICATED...

Figure 6: Determination of Lagrangian

6.2 Small Angle Approximation Calculations

FIND STABLE EQ. IN ϕ & ℓ ...

$$\frac{\partial U}{\partial \ell} = -m_1 g \cos(\phi) - k(R - \ell - \ell_0) \longrightarrow \text{LETTING } \phi = 0 \longrightarrow \frac{-m_1 g}{k} = R - \ell - \ell_0$$

$$\frac{\partial U}{\partial \phi} = m_2 g \sin(\phi) + m_1 g \ell \sin(\phi) \longrightarrow \phi = 0 \text{ EQUILIBRIUM POINT.}$$

$$\ell = \left[R - \ell_0 + \frac{m_1 g}{k} \right] = \ell_{Eq.} \longrightarrow \ell = \ell_{Eq.} \text{ EQUILIBRIUM POINT.}$$

Figure 7: Determining Stable Equilibrium Points

USE:

$$U \approx U(\ell = \ell_{Eq.}, \phi = 0) + \sum_{i,j} \frac{1}{2} \frac{\partial U}{\partial q_i} \frac{\partial U}{\partial q_j} \Bigg|_{q_{0i}, q_{0j}} (q_j - q_{0j})(q_i - q_{0i})$$

$$U \approx \left[-m_1 g R - m_2 g \ell_{Eq.} + \frac{1}{2} k \left(R - \ell_{Eq.} - \ell_0 \right)^2 \right] + (\dots)$$

$$\frac{\partial^2 U}{\partial \phi^2} \Bigg|_{\phi = 0,\ \ell = \ell_{Eq.}} = \underbrace{\boxed{m_2 g \cos(\phi) + m_1 g \cos(\phi)}}_{k_{11}} = (m_1 + m_2) g > 0$$

$$\frac{\partial^2 U}{\partial \ell^2} \Bigg|_{\phi = 0,\ \ell = \ell_{Eq.}} = \underbrace{\boxed{k}}_{k_{22}} > 0$$

$\therefore$ $\ell_{Eq.}$ & 0 ARE STABLE EQ. POINTS...

$$\frac{\partial^2 U}{\partial \phi \partial \ell} = m_1 g \sin(\phi) \Bigg|_{\ell_{Eq.}, 0} = 0$$

$$\frac{\partial^2 U}{\partial \phi \partial \ell} = m_1 g \sin(\phi) \Bigg|_{\ell_{Eq.}, 0} = 0$$

$$\therefore U \approx U(\ell_{Eq.}, 0) + \frac{1}{2} k_{11} \phi^2 + \frac{1}{2} k_{22} (\ell - \ell_{Eq})^2$$

$$T \approx T(\ell_{Eq.}, 0) = \frac{1}{2} m_1 \dot{\ell}^2 + \frac{1}{2} m_2 \dot{\ell}^2 + \frac{1}{2} m_2 \ell_{Eq}^2 \dot{\phi}^2$$

$$T \approx \frac{1}{2} \underbrace{(m_1 + m_2)}_{\tilde{\alpha}_{22}} \dot{\ell}^2 + \frac{1}{2} \underbrace{m_2 \ell_{Eq.}^2}_{\tilde{\alpha}_{11}} \dot{\phi}^2$$

$$\mathcal{L} \approx \tilde{T} - \tilde{U}$$

E.L. EQUATIONS FOR APPROXIMATION:

$$\text{EL \#1:} \quad (m_1 + m_2) \ddot{\ell} = -k_{22} (\ell - \ell_{Eq.})$$

$$\text{EL \#2:} \quad (m_2 \ell_{Eq}^2) \ddot{\phi} = -k_{11} (\phi)$$

INDEPENDENTLY SOLVABLE...!

$$\boxed{\Omega_1 = \sqrt{\frac{k_{11}}{m_2 \ell_{Eq.}^2}}, \quad \Omega_2 = \sqrt{\frac{k_{22}}{m_1 + m_2}}}$$

$$= \frac{(m_1 + m_2) g}{m_2 \left(R - \ell_0 + \frac{m_1 g}{k} \right)^2} \qquad = \frac{k}{(m_1 + m_2)}$$

Figure 8: Approximating Lagrangian