Air Quality Monitoring Wireless System

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Advection-diffusion-reaction model equation: second order pde equation

$$\frac{\partial x}{\partial t} - \lambda \nabla^2 x + \mathrm{v}^T \nabla x \;=\; f \quad \text{in } \mathbb{R}^2 \tag{1}$$

where:
$x(\mathrm{p}, t)$ is the space-time dependent pollutant concentration field, defined over the space-time domain with an initial condition $x(\mathrm{p}, 0) = x_0$;
$\mathrm{p} \in \mathbb{R}^2$ denotes the 2-dimensional position vector;
$t \in \mathbb{R}^+$ denotes time;
λ is the diffusion coefficient;
$\mathrm{v}(\mathrm{p}, t)$ is the advection velocity vector;
$f(\mathrm{p}, t)$ represents the internal sources of pollution.
$f(\mathrm{p}, t) = s(t)F(\mathrm{p})$

We need to truncate the physical domain to solve numerically the problem by using the finite element method (FEM).

variational formulation of the problem:

$$\int_\Omega \frac{\partial x}{\partial t}\varphi \, dp \; - \; \lambda \int_\Omega \; \nabla^2 x \; \varphi \, dp + \int_\Omega \mathrm{v}^T \; \nabla x \; \varphi \, dp = \int_\Omega f \varphi \, dp$$

where $\varphi(\mathrm{p})$ is a generic space-dependent weight function.
$\varphi \nabla^2 x = \nabla.(\varphi \nabla x) - \nabla \varphi . \nabla x$

$$\int_\Omega \frac{\partial x}{\partial t}\varphi \, dp - \lambda \int_\Omega \nabla.(\varphi \nabla x) \, dp + \lambda \int_\Omega \nabla \varphi . \nabla x \, dp + \int_\Omega \mathrm{v}^T \; \nabla x \, \varphi \, dp = \int_\Omega f \varphi \, dp$$

$$\int_\Omega \frac{\partial x}{\partial t}\varphi \, dp \; - \; \lambda \int_{\partial\Omega} \varphi \nabla x . \mathrm{n} \, dp + \lambda \int_\Omega \nabla \varphi . \nabla x \, dp + \int_\Omega \mathrm{v}^T \; \nabla x \; \varphi \, dp =$$
$$s(t) \int_\Omega F(\mathrm{p}) \varphi \, dp$$

FEM approximation:

$$x(\mathrm{p}, t) = \sum_{j=1}^n \phi_j(\mathrm{p}) \, x_j(t) \; = \; \phi^T(\mathrm{p}) \, \mathrm{x}(t)$$

FEM derivative model:

$$\underbrace{\left[\int_{\Omega} \phi(\mathrm{p})\, \phi^T(\mathrm{p}) d\mathrm{p}\right]}_{\mathrm{M}} \dot{\mathrm{x}}(t) + \underbrace{\left[\lambda \int_{\Omega} \nabla\phi(\mathrm{p})\, \nabla\phi^T(\mathrm{p}) d\mathrm{p}\right]}_{\mathrm{S}_\lambda} \mathrm{x}(t)$$

$$+ \underbrace{\left[\int_{\Omega} \phi(\mathrm{p})\, \mathrm{v}^T(\mathrm{p})\, \nabla\phi^T(\mathrm{p})\, d\mathrm{p}\right]}_{\mathrm{G}} \mathrm{x}(t) + \underbrace{\left[\lambda \int_{\partial\Omega} \beta(\mathrm{p})\, \phi(\mathrm{p})\, \phi^T(\mathrm{p})\, d\mathrm{p}\right]}_{\mathrm{Q}_\beta} \mathrm{x}(t) =$$

$$= \underbrace{\left[\int_{\Omega} \phi(\mathrm{p}) F(\mathrm{p}) d\mathrm{p}\right]}_{\mathrm{Q}_f} s(t) \quad (2)$$

example1:
$F(\mathrm{p}) = \delta(\mathrm{p} - \mathrm{p}_s)$

$$\int_\Omega \phi(\mathrm{p}) F(\mathrm{p}) d\mathrm{p} = \int_\Omega \phi(\mathrm{p}) \delta(\mathrm{p} - \mathrm{p}_s) d\mathrm{p} = \phi(\mathrm{p}_s)$$

example2: multiple point source
$f(\mathrm{p}, t) = \frac{1}{N_s} \sum_{s=1}^{N_s} s_s(t) \delta(\mathrm{p} - \mathrm{p}_s)$

$$\int_\Omega \phi(\mathrm{p}) F(\mathrm{p}) d\mathrm{p} = \frac{1}{N_s} \sum_{s=1}^{N_s} \int_\Omega \phi(\mathrm{p}) \delta(\mathrm{p} - \mathrm{p}_s) d\mathrm{p} = \frac{1}{N_s} \sum_{s=1}^{N_s} \phi(\mathrm{p}_s)$$

Advection model:
advection velocity vector $= [55]$ m/sec.

Diffusion model:
diffusion coefficient $= 5$ ppm/m^2