Air Quality Monitoring Wireless System

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Model

Advection-diffusion model equation: second order pde equation

$$\frac{\partial x}{\partial t} - \lambda \nabla^2 x + \mathrm{v}^T \nabla x = f \quad \text{in } \mathbb{R}^2 \tag{1}$$

where:
$x(\mathrm{p}, t)$ is the space-time dependent pollutant concentration field, defined over the space-time domain with an initial condition $x(\mathrm{p}, 0) = x_0$;
$\mathrm{p} \in \mathbb{R}^2$ denotes the 2-dimensional position vector;
$t \in \mathbb{R}^+$ denotes time;
λ is the diffusion coefficient;
$\mathrm{v}(\mathrm{p}, t)$ is the advection velocity vector;
$f(\mathrm{p}, t) = s(t)F(\mathrm{p})$ represents the internal sources of pollution, defined as the product of the source signal waveform $s(t)$ and the position-dependent function $F(\mathrm{p})$.
We need to truncate the physical domain to solve numerically the problem by using the finite element method (FEM): $\mathbb{R}^2 -> \Omega$.

Variational formulation of the problem

$$\int_\Omega \frac{\partial x}{\partial t}\varphi \, d\mathrm{p} - \lambda \int_\Omega \nabla^2 x \, \varphi \, d\mathrm{p} + \int_\Omega \mathrm{v}^T \, \nabla x \, \varphi \, d\mathrm{p} = \int_\Omega f \varphi \, d\mathrm{p}$$

where $\varphi(\mathrm{p})$ is a generic space-dependent weight function.

Green's identity substitution
$\varphi \nabla^2 x = \nabla.(\varphi \nabla x) - \nabla \varphi . \nabla x$

$$\int_\Omega \frac{\partial x}{\partial t}\varphi \, d\mathrm{p} - \lambda \int_\Omega \nabla.(\varphi \nabla x) \, d\mathrm{p} + \lambda \int_\Omega \nabla \varphi . \nabla x \, d\mathrm{p} + \int_\Omega \mathrm{v}^T \, \nabla x \, \varphi \, d\mathrm{p} = \int_\Omega f \varphi \, d\mathrm{p}$$

$$\int_\Omega \frac{\partial x}{\partial t}\varphi \, d\mathrm{p} - \lambda \int_{\partial\Omega} \varphi \nabla x . \mathrm{n} \, d\mathrm{p} + \lambda \int_\Omega \nabla \varphi . \nabla x \, d\mathrm{p} + \int_\Omega \mathrm{v}^T \, \nabla x \, \varphi \, d\mathrm{p} = s(t) \int_\Omega F(\mathrm{p}) \varphi \, d\mathrm{p}$$

FEM derivation

FEM approximation:

$$x(\mathrm{p}, t) \simeq \sum_{j=1}^{n} \phi_j(\mathrm{p})\, x_j(t) \;=\; \phi^T(\mathrm{p})\, \mathrm{x}(t)$$

FEM derivative model: a set of n inhomogeneus differential equations

$$\underbrace{\left[\int_\Omega \phi(\mathrm{p})\, \phi^T(\mathrm{p}) d\mathrm{p}\right]}_{\mathrm{M}} \dot{\mathrm{x}}(t) + \underbrace{\left[\lambda \int_\Omega \nabla\phi^T(\mathrm{p})\, \nabla\phi(\mathrm{p}) d\mathrm{p}\right]}_{\mathrm{S}_\lambda} \mathrm{x}(t)$$

$$+ \underbrace{\left[\int_\Omega \phi(\mathrm{p})\, (\mathrm{v}^T(\mathrm{p})\, \nabla\phi(\mathrm{p}))\, d\mathrm{p}\right]}_{\mathrm{S}_\mathrm{v}} \mathrm{x}(t) - \underbrace{\left[\lambda \int_{\partial\Omega} \phi(\mathrm{p})(\mathrm{n}^T \nabla\phi(\mathrm{p}))\, d\mathrm{p}\right]}_{\mathrm{Q}_\lambda} \mathrm{x}(t) =$$

$$= \underbrace{\left[\int_\Omega \phi(\mathrm{p}) F(\mathrm{p}) d\mathrm{p}\right]}_{\mathrm{f}} s(t) \quad (2)$$

Source terms

example1:
single source
$f(\mathrm{p},t) = s(t)\delta(\mathrm{p}-\mathrm{p}_s)$

$$s(t)\int_\Omega \phi(\mathrm{p})F(\mathrm{p})d\mathrm{p} = s(t)\int_\Omega \phi(\mathrm{p})\delta(\mathrm{p}-\mathrm{p}_s)d\mathrm{p} = s(t)\phi(\mathrm{p}_s)$$

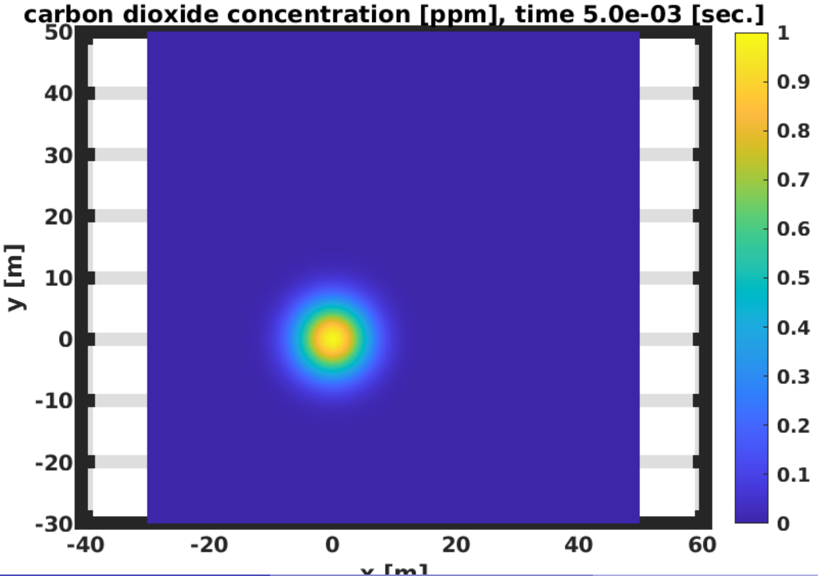
example2:
multiple point source
$f(\mathrm{p},t) = \sum_{s=1}^{N_s} s_s(t)\delta(\mathrm{p}-\mathrm{p}_s)$

$$\int_\Omega \phi(\mathrm{p})f(\mathrm{p},t)d\mathrm{p} = \sum_{s=1}^{N_s} s_s(t)\int_\Omega \phi(\mathrm{p})\delta(\mathrm{p}-\mathrm{p}_s)d\mathrm{p} = \sum_{s=1}^{N_s} s_s(t)\phi(\mathrm{p}_s)$$

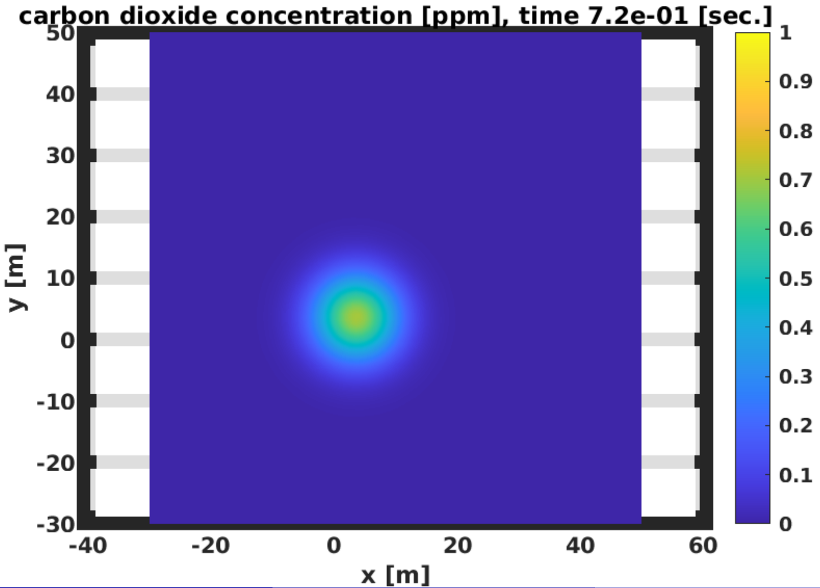
Advection Diffusion Model without source

advection velocity vector $= [5 \;\; 5]$ m/sec.
diffusion coefficient $= 5$ m^2/sec

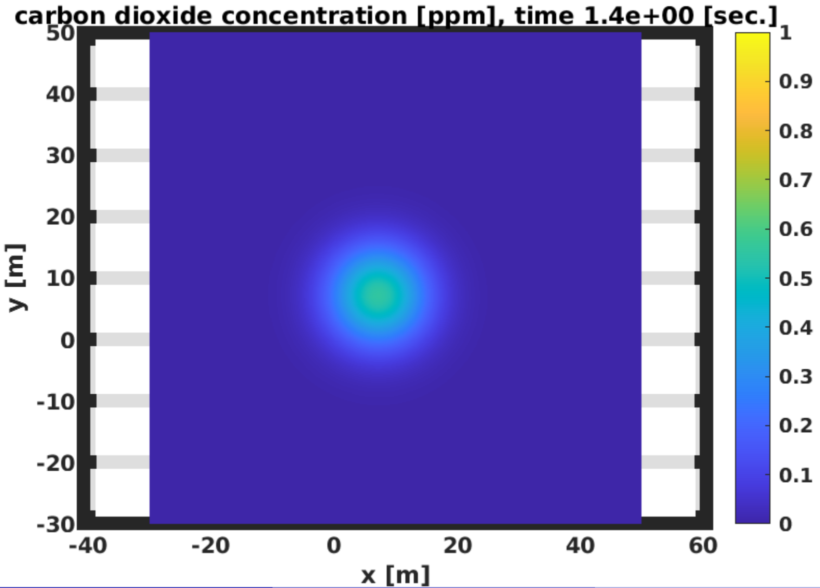
Advection Diffusion Model without source



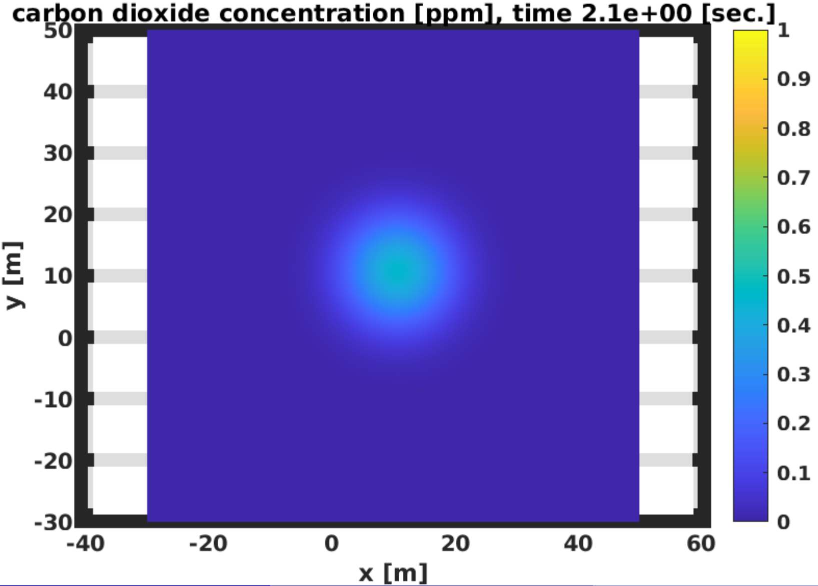
Advection Diffusion Model without source



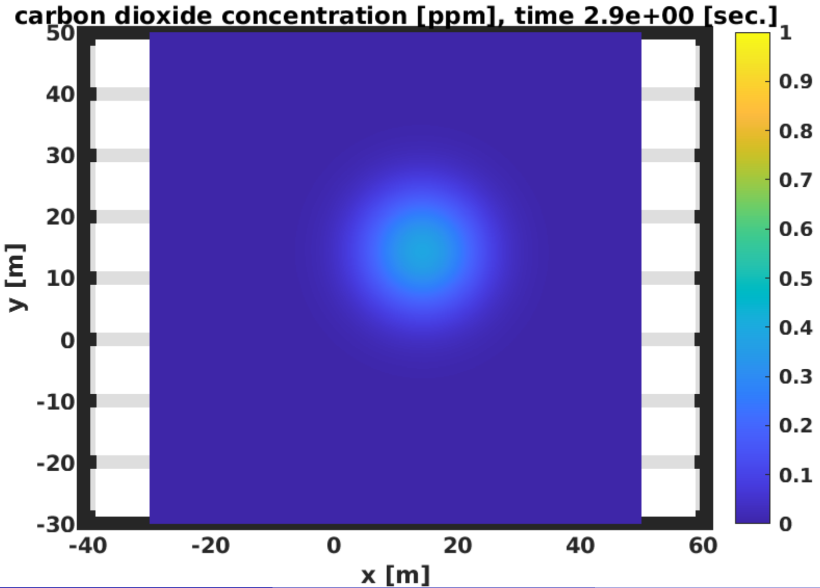
Advection Diffusion Model without source



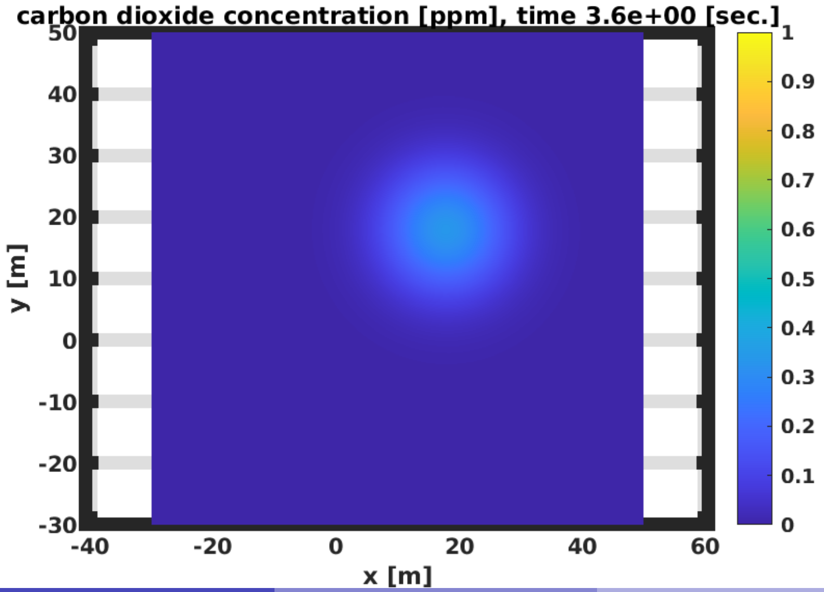
Advection Diffusion Model without source



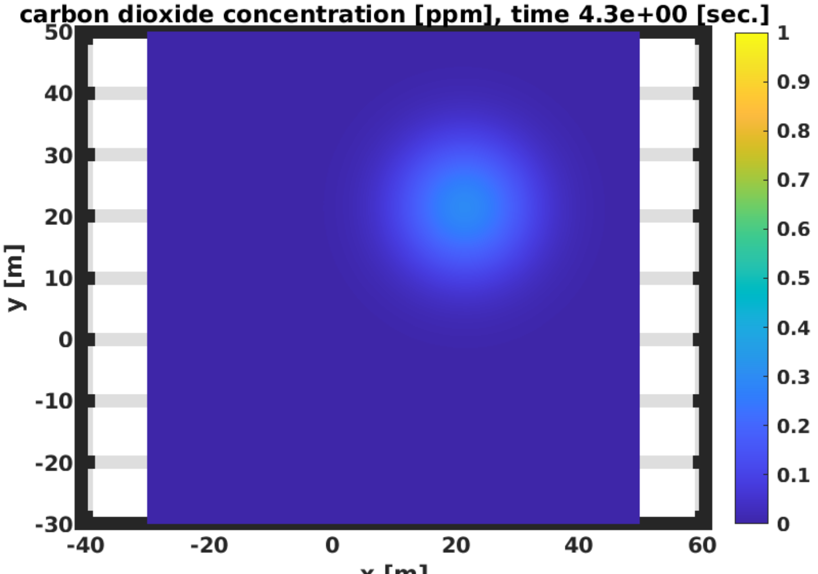
Advection Diffusion Model without source



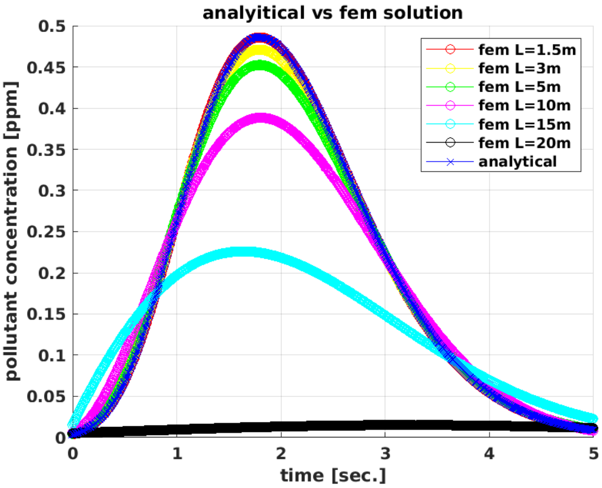
Advection Diffusion Model without source



Advection Diffusion Model without source



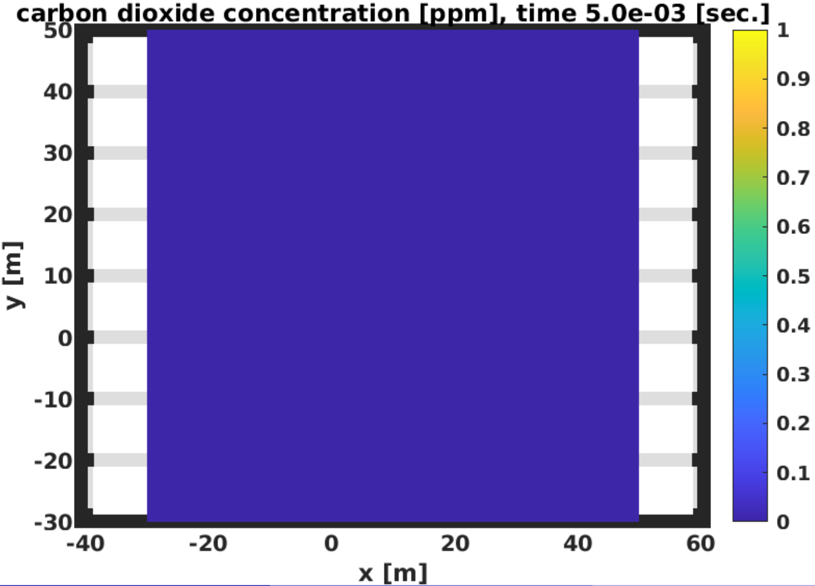
Analytical vs FEM comparison



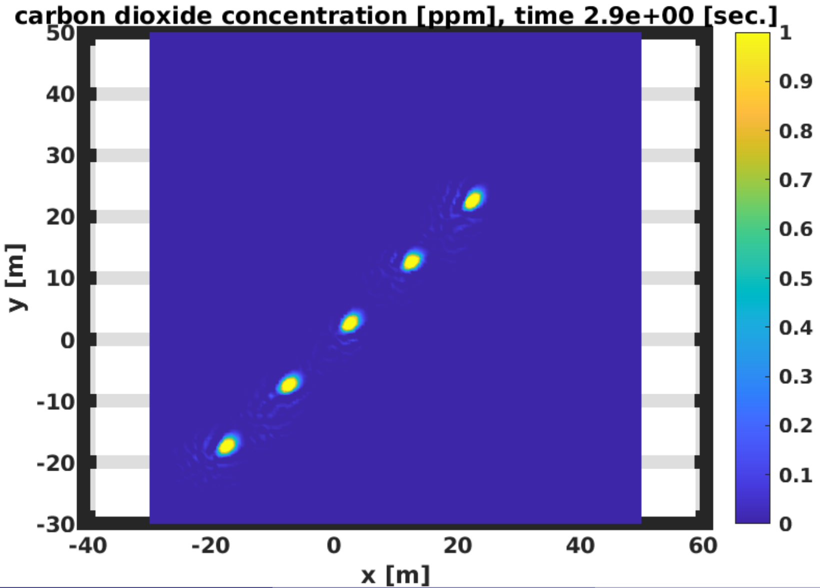
Advection Diffusion Model with sources

advection velocity vector = $[2 \;\; 2]$ m/sec.
diffusion coefficient = 0.1 m^2/sec

Advection Diffusion Model with sources



Advection Diffusion Model with sources



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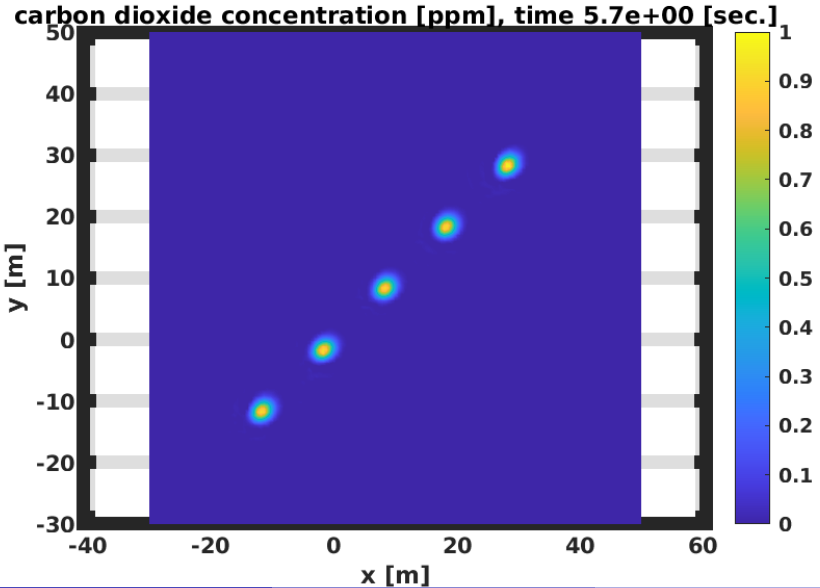

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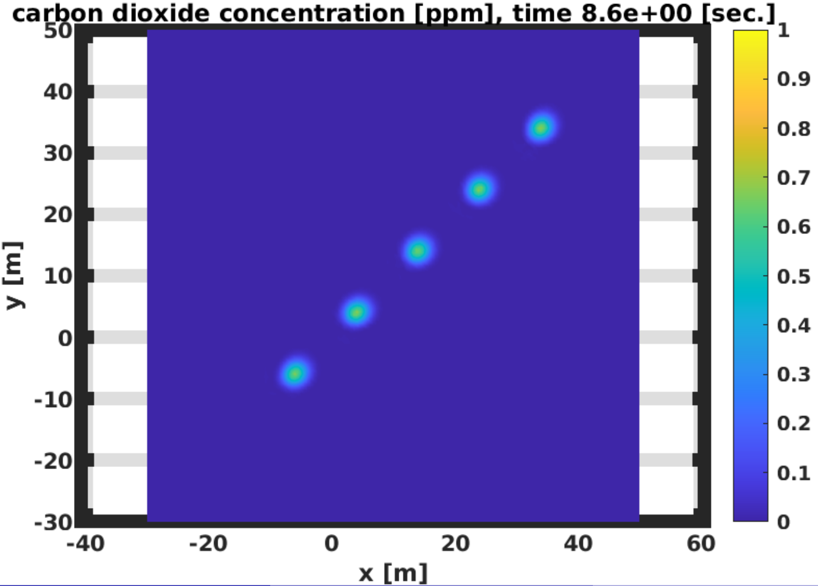

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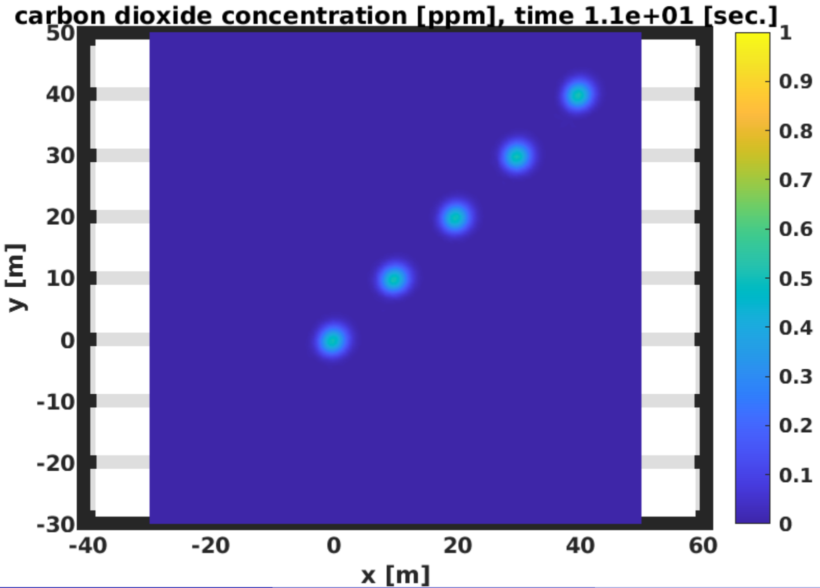

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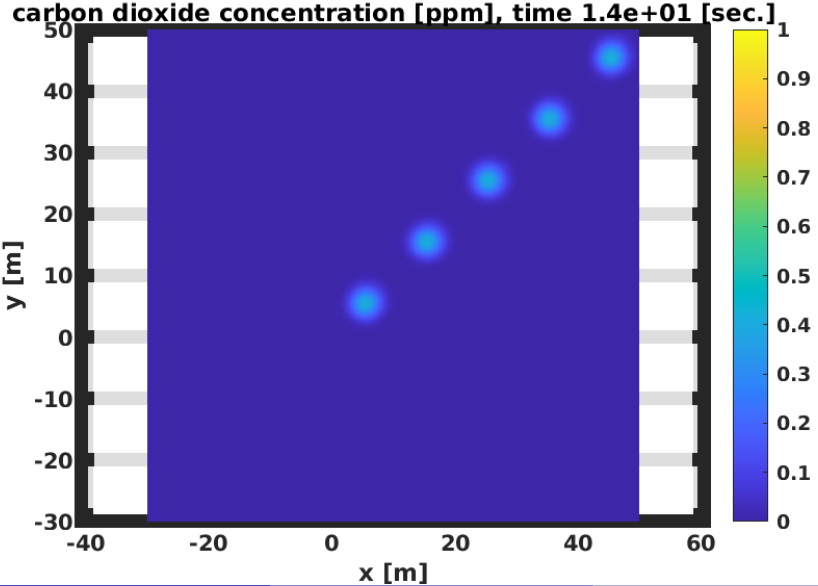
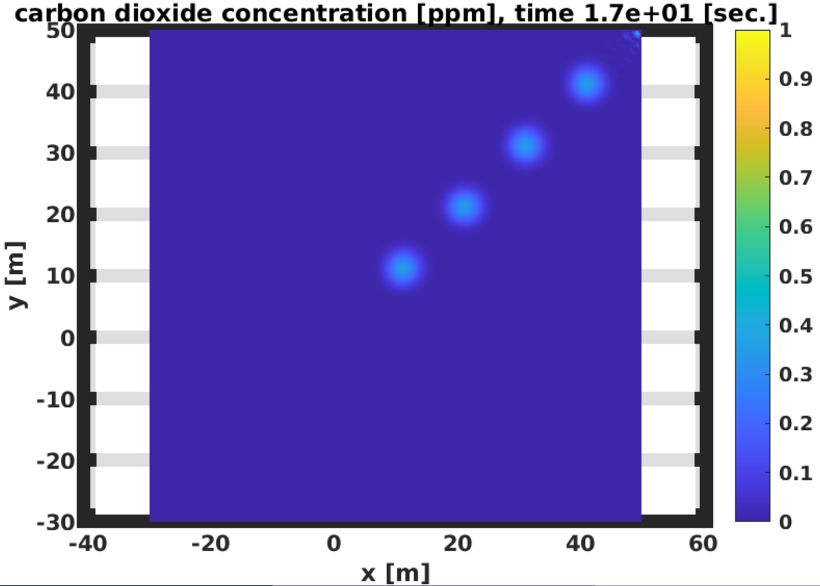
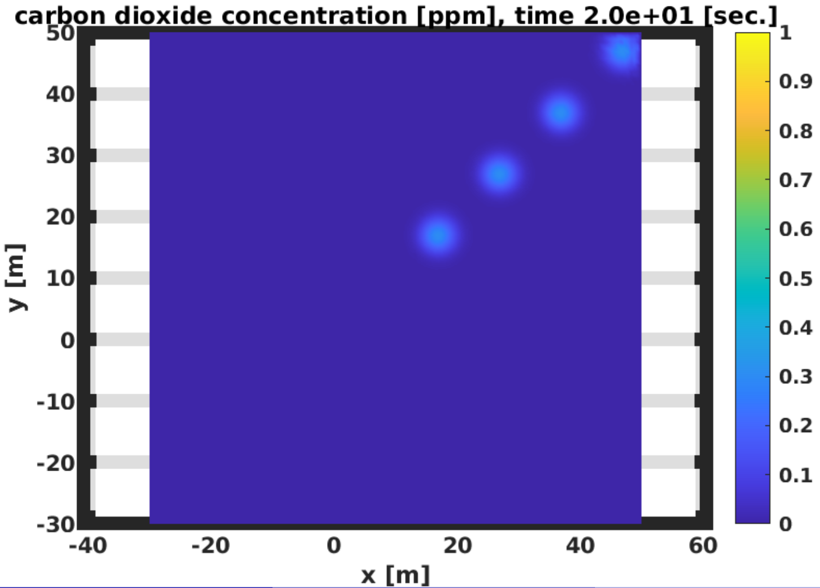


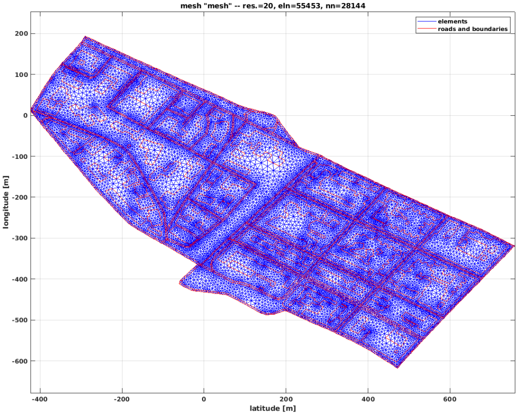
Advection Diffusion Model with sources



Advection Diffusion Model with sources



Domain Mesh



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roads and boundaries
longitude [m]
latitude [m]
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100
0
-100
-200
-300
-400
-500
-600
-400
-200
0
200
400
600

Domain source points