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An optimized Schwarz waveform relaxation method for the unsteady convection diffusion equation in two dimensions

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Abstract

We are interested in solving time dependent problems using domain decomposition methods. In the classical approach, one discretizes first the time dimension and then one solves a sequence of steady problems by a domain decomposition method. In this paper, we treat directly the time dependent problem and we study a Schwarz waveform relaxation algorithm for the convection diffusion equation in two dimensions. We introduce the operators on the interfaces which minimize the convergence rate, resulting in an efficient method: numerical results illustrate the performances and show that the corresponding algorithms converge much faster than the classical one.

Keywords: Domain decomposition methods; Schwarz waveform relaxation algorithm; Convection diffusion equation

1. Introduction

When discretizing a partial differential equation, one obtains an enormous number of unknowns, often too many to be managed by a single computer. Domain decomposition methods propose solutions by partitioning the initial domain into several subdomains. Each subproblem is then solved by one processor and an iterative exchange of informations on the common interfaces leads to the global solution. The choice of interface conditions is important for the efficiency of the method and is the subject of many works.

Classical methods were applied to stationary problems and the first idea was introduced by Schwarz [17] who used Dirichlet conditions on the interfaces. For this method the overlap between the subdomains

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is necessary. Recently, new types of transmission conditions have been introduced which also apply to non-overlapping problems (see [13,3,16,14,11]). In particular, in [14] absorbing boundary conditions are used which lead to convergence in a finite number of iterations (equal to the number of subdomains). These conditions are however difficult to use and can be replaced by simpler conditions obtained by solving an optimization problem related to the convergence rate. This strategy proved to be very useful for many steady problems as, for instance, convection diffusion [10], Euler [2] or Helmholtz equations [8].

When solving time dependent problems, classical methods discretize the time dimension first and then use the domain decomposition methods above described at each time step. However this method implies a uniform time step whereas in some cases it can be interesting to refine the time dimension in only one subdomain (the one where we have refined in space, for example). Some recent works propose a domain decomposition method for evolution problems quite different from the classical one: they apply the iterative algorithm directly to the time dependent problem (Schwarz waveform relaxation method). This method allows thus to solve independently each time dependent subdomain problem leading to en efficient way to simulate multiphysic phenomena. The first interface conditions introduced were of Dirichlet type (see [7] for the heat equation) then more appropriate conditions using absorbing boundary conditions theory and optimization of the convergence rate have been introduced in [5].

In this paper, we design Schwarz waveform relaxation methods for the convection diffusion equation in $\Omega = \mathbb{R}^2$ (the corresponding operator is denoted by $\mathcal{L}$). We decompose Ω into two subdomains $\Omega^- = (-\infty, L) \times \mathbb{R}$ and $\Omega^+ = (0, +\infty) \times \mathbb{R}$ ($L \geqslant 0$) and we introduce the interfaces $\Gamma_L = \{y \in \mathbb{R},\ x = L\}$ and $\Gamma_0 = \{y \in \mathbb{R},\ x = 0\}$ (see Fig. 1).

The purpose of this paper is to study the algorithm

$$\begin{cases} \mathcal{L}u^{n+1} = f & \text{in } \Omega^- \times \mathbb{R}^+, \\ u^{n+1}(\cdot,\cdot,0) = w_0 & \text{in } \Omega^-, \\ \mathcal{B}^- u^{n+1} = \mathcal{B}^- v^n & \text{on } \Gamma_L \times \mathbb{R}^+, \end{cases} \qquad \begin{cases} \mathcal{L}v^{n+1} = f & \text{in } \Omega^+ \times \mathbb{R}^+, \\ v^{n+1}(\cdot,\cdot,0) = w_0 & \text{in } \Omega^+, \\ \mathcal{B}^+ v^{n+1} = \mathcal{B}^+ u^n & \text{on } \Gamma_0 \times \mathbb{R}^+, \end{cases} \tag{1}$$

with the interface operators $\mathcal{B}^\pm$ chosen such that the subproblem in each subdomain is well posed and such that the corresponding algorithm converges rapidly even without overlap ($L = 0$).

In Section 2, we recall some existence, uniqueness and regularity results on the convection diffusion equation in $\mathbb{R}^2$. In Section 3, we study an overlapping Schwarz waveform relaxation algorithm: we prove the well-posedness and the convergence of algorithm (1) when $\mathcal{B}^\pm$ are identity. Then, in Section 4, we introduce optimal transmission conditions which coincide with the transparent boundary conditions and we obtain an algorithm which converges in two iterations even without overlap. However these conditions are not easy to compute and cannot be used directly in a numerical algorithm. A solution is to approximate the corresponding operators by differential operators in the time direction and in the tangential direction on the interface. In fact this is equivalent to approximate the Fourier symbol of the transparent operators

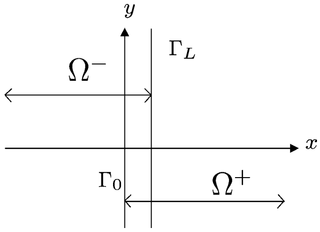

Fig. 1. Decomposition with overlap of $\mathbb{R}^2$.

by a polynomial. In Sections 5 and 6 we study the case where this polynomial is of degree 0 or 1. In each case we write the corresponding algorithm and we show that it is well-posed. Then, in Section 7, we prove that these algorithms converge toward the global solution. In Section 8 we choose the differential operators of order 0 and 1 in order to get fast convergence. In Section 9 we show numerical results which illustrate the efficiency of the method.

2. The convection diffusion equation in $\mathbb{R}^2$

We are interested in solving the scalar convection diffusion equation in $\Omega = \mathbb{R}^2$

$$\begin{cases} \frac{\partial w}{\partial t} + \vec{\mathbf{b}} \cdot \nabla w - \nu \triangle w + cw = f & \text{in } \Omega \times \mathbb{R}^+, \\ w(\cdot,\cdot,0) = w_0 & \text{in } \Omega, \end{cases} \tag{2}$$

where $\vec{\mathbf{b}} = (a, b)$ is a constant velocity field, $\nu > 0$ is the viscosity and the constant c is strictly positive. We denote by $\mathcal{L} \equiv \partial_t + \vec{\mathbf{b}} \cdot \nabla - \nu\triangle + c$ the convection diffusion operator.

Remark 1. Let $\mathcal{L}_t(u)(\sigma + i\omega) = \int_0^{+\infty} e^{-(\sigma+i\omega)t} u(t)\,\mathrm{d}t$, $\sigma > 0$, $\omega \in \mathbb{R}$ denote the Laplace transform of u and $\hat{u}(\omega) = (\sqrt{2\pi}\,)^{-1} \int_{\mathbb{R}} e^{-i\omega t} u(t)\,\mathrm{d}t$ the Fourier transform. Performing a Laplace transform of w is equivalent to performing a Fourier transform of the extension by 0 for $t < 0$ of $v = we^{-\sigma t}$ and v is solution of $\frac{\partial v}{\partial t} + \vec{\mathbf{b}} \cdot \nabla v - \nu\triangle v + (c + \sigma)v = fe^{-ct}$. Therefore, we can equivalently study (2) using a Laplace transform or a Fourier transform but in the last case (this is the choice in this paper), the number c in (2) is strictly positive.

2.1. Existence and uniqueness of the solution

In this paper $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$. Let w be a weak solution of (2), i.e., w is defined by

$$\begin{cases} \text{For any } v \text{ in } H^1(\Omega), \\ \frac{\mathrm{d}}{\mathrm{d}t}(w, v) + \nu(\nabla w, \nabla v) + \frac{1}{2}\big((\vec{\mathbf{b}} \cdot \nabla w, v) - (\vec{\mathbf{b}} \cdot \nabla v, w)\big) + c(w, v) = (f, v), \\ w(\cdot,\cdot,0) = w_0. \end{cases}$$

Theorem 2. *If w_0 is in $L^2(\Omega)$ and f in $L^2(\mathbb{R}^+; L^2(\Omega))$, then there exists a unique weak solution w of* (2) *in $L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(\mathbb{R}^+; H^1(\Omega))$.*

Proof of this theorem is obtained by a Galerkin method. See, for example, [1].

2.2. Regularity of weak solutions

Working with time dependent problems we need to introduce specific spaces (regularity for the space and time dimension is different). We remind the reader of the definition of the anisotropic Sobolev spaces. More information can be found in [12]. Let Ω be an open set of $\mathbb{R}^2$ and $H^{r,s}(\Omega \times (0, T))$ be the anisotropic Sobolev space defined by $H^{r,s}(\Omega \times (0, T)) = L^2(0, T; H^r(\Omega)) \cap H^s(0, T; L^2(\Omega))$. If Ω denotes the half space $\{y \in \mathbb{R}^2,\ x > 0\}$ and $T = +\infty$, we introduce the space

$$F = \Bigg\{ (f_k, g_j) \in \prod_{k<s-1/2} H^{p_k}\big(\mathbb{R}^+ \times \mathbb{R}\big) \times \prod_{j<r-1/2} H^{\mu_j,\nu_j}\big(\mathbb{R} \times (0,+\infty)\big),$$
$$\frac{\mu_j}{r} = \frac{\nu_j}{s} = \frac{r-j-1/2}{r},\ p_k = \frac{r}{s}(s-k-1/2),\ k, j \text{ integers}$$
$$\text{such that } \int_0^\infty \int_{\mathbb{R}} \left| \frac{\partial^j f_k}{\partial x^j}\big(\sigma^s, y\big) - \frac{\partial^k g_j}{\partial t^k}\big(y, \sigma^r\big) \right|^2 \mathrm{d}y \frac{\mathrm{d}\sigma}{\sigma} < \infty \text{ if } \frac{j}{r} + \frac{k}{s} = 1 - \frac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right),$$
$$\frac{\partial^k g_j}{\partial t^k}(\cdot, 0) = \frac{\partial^j f_k}{\partial x^j}(0, \cdot) \text{ if } \frac{j}{r} + \frac{k}{s} < 1 - \frac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right) \Bigg\},$$

and we give the following trace theorem (see [12]).

Theorem 3. *Let $r, s > 0$, such that $1 - \frac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right) > 0$. Then the map*

$$u \to (f_k, g_j) = \begin{cases} \left(\frac{\partial^k u}{\partial t^k}(\cdot,\cdot,0)\right)_{k<s-1/2}, \\ \left(\frac{\partial^j u}{\partial x^j}(0,\cdot,\cdot)\right)_{j<r-1/2}, \end{cases}$$

is linear, continuous and onto from $H^{r,s}(\mathbb{R}^+ \times \mathbb{R} \times (0,+\infty))$ to F.

We also remind the reader of regularity results (see [12]).

Theorem 4. *If f is in $L^2(\mathbb{R}^+; L^2(\Omega))$ and if w_0 is in $H^1(\Omega)$, then the weak solution of* (2) *is in $H^{2,1}(\Omega \times \mathbb{R}^+)$.*

Theorem 5. *If f is in $H^{1,1/2}(\Omega \times \mathbb{R}^+)$ and if w_0 is in $H^2(\Omega)$, then the weak solution of* (2) *is in $H^{3,3/2}(\Omega \times \mathbb{R}^+)$.*

In the framework of domain decomposition methods, we may be led to define the trace on the interface of the solution and of its first derivatives. According to the trace Theorem 3 and with the assumptions of Theorem 4, we obtain under some compatibility conditions, a bijection between w in $H^{2,1}(\Omega^\pm \times \mathbb{R}^+)$ and traces of w and $\partial_x w$ in $H^{3/2,3/4}(\partial\Omega^\pm \times \mathbb{R}^+)$ and $H^{1/2,1/4}(\partial\Omega^\pm \times \mathbb{R}^+)$. With the assumptions of Theorem 5 we have a bijection between w in $H^{3,3/2}(\Omega^\pm \times \mathbb{R}^+)$ and traces of w, $\partial_x w$ et $\partial_t w$ in $H^{5/2,5/4}(\partial\Omega^\pm \times \mathbb{R}^+)$, $H^{3/2,3/4}(\partial\Omega^\pm \times \mathbb{R}^+)$ and $H^{1/2,1/4}(\partial\Omega^\pm \times \mathbb{R}^+)$.

3. Overlapping Schwarz waveform relaxation algorithm

The classical Schwarz algorithm was first introduced and studied for stationary problems in [17]. It consists in exchanging Dirichlet conditions on the interface between the two subdomains. In [7] this method was extended to time dependent problems for the heat equation in $\mathbb{R}^d$, $d \geqslant 1$. Applying this method for the convection diffusion equation, it consists in choosing $\mathcal{B}^\pm = \mathcal{B}^\pm_{\mathrm{dir}}$ in (1) with

$$\mathcal{B}^-_{\mathrm{dir}} = Id \quad \text{and} \quad \mathcal{B}^+_{\mathrm{dir}} = Id. \tag{3}$$

We study the well-posedness of each subdomain problem involved in this algorithm in Section 3.1. This allows us to introduce the precise definition of the algorithm in Section 3.2. We prove the convergence in Section 3.3.

3.1. Well-posedness of the subproblems

We study the initial boundary value problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega^- \times \mathbb{R}^+, \\ u(\cdot,\cdot,0) = w_0 & \text{in } \Omega^-, \\ u = g & \text{on } \Gamma_L \times \mathbb{R}^+. \end{cases} \tag{4}$$

The following theorem gives a result of existence, uniqueness and regularity for the solution of (4). Proof can be found in [12].

Theorem 6. *Let f be in $L^2(\mathbb{R}^+; L^2(\Omega))$ and w_0 in $H^1(\Omega)$. If g belongs to $H^{3/2,3/4}(\mathbb{R} \times \mathbb{R}^+)$ and if the compatibility condition $g(\cdot, 0) = w_0(L, \cdot)$ on Γ holds, then problem* (4) *has a unique solution u in $H^{2,1}(\Omega^- \times \mathbb{R}^+)$.*

3.2. The algorithm

If $\mathcal{B}^\pm = \mathcal{B}^\pm_{\mathrm{dir}}$, we initialize algorithm (1) by

$$\begin{aligned} &\mathcal{L}u^0 = f \quad \text{in } \Omega^- \times \mathbb{R}^+, && u^0(\cdot,\cdot,0) = w_0 \quad \text{in } \Omega^-, && u^0(L,\cdot,\cdot) = g_L \quad \text{on } \Gamma_L \times \mathbb{R}^+, \\ &\mathcal{L}v^0 = f \quad \text{in } \Omega^+ \times \mathbb{R}^+, && v^0(\cdot,\cdot,0) = w_0 \quad \text{in } \Omega^+, && v^0(0,\cdot,\cdot) = g_0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \end{aligned} \tag{5}$$

with g_L in $H^{3/2,3/4}(\mathbb{R} \times \mathbb{R}^+)$ such that $g_L(\cdot,0) = w_0(L,\cdot)$ on Γ and g_0 in $H^{3/2,3/4}(\mathbb{R} \times \mathbb{R}^+)$ such that $g_0(\cdot,0) = w_0(0,\cdot)$ on Γ.

Theorem 7. *Let f be in $L^2(\mathbb{R}^+; L^2(\Omega))$ and w_0 in $H^1(\Omega)$. If $\mathcal{B}^\pm = \mathcal{B}^\pm_{\mathrm{dir}}$, then algorithm* (1) *initialized by* (5) *defines a unique sequence of iterates (u^n, v^n) in $H^{2,1}(\Omega^- \times \mathbb{R}^+) \times H^{2,1}(\Omega^+ \times \mathbb{R}^+)$.*

Proof. By Theorem 6 and the corresponding theorem for Ω^+, problem (5) defines a unique (u^0, v^0) in $H^{2,1}(\Omega^- \times \mathbb{R}^+) \times H^{2,1}(\Omega^+ \times \mathbb{R}^+)$. On the other hand if the solution of (1) in Ω^+, v^n, is in $H^{2,1}(\Omega^+ \times \mathbb{R}^+)$, according to the trace Theorem 3, $v^n(L,\cdot,\cdot)$ is in $H^{3/2,3/4}(\mathbb{R} \times \mathbb{R}^+)$ with $v^n(L,\cdot,0) = w_0(L,\cdot)$. Thus problem (1) in Ω^- is well-posed in $H^{2,1}(\Omega^- \times \mathbb{R}^+)$. The same holds in Ω^+ so that we deduce the result by induction. □

3.3. The convergence result for the algorithm

In the following, we use the Fourier transform with respect to the variables y and t of a function v in $L^2(\Omega^\pm \times \mathbb{R})$ defined by

$$\hat{v}(x,k,\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} v(x,y,t) e^{-i(ky+\omega t)} \,\mathrm{d}y \,\mathrm{d}t. \tag{6}$$

Theorem 8. *Let f be in $L^2(\mathbb{R}^+; L^2(\Omega))$ and w_0 in $H^1(\Omega)$. If $\mathcal{B}^\pm = \mathcal{B}^\pm_{\mathrm{dir}}$, then algorithm* (1) *initialized by* (5) *converges in $H^{2,1}(\Omega^- \times \mathbb{R}^+) \times H^{2,1}(\Omega^+ \times \mathbb{R}^+)$.*

Proof. We introduce the errors $e^n_\pm$ at step $n \geqslant 0$ in domains $\Omega^\pm$, i.e., $e^n_- = w_{|\Omega^-} - u^n$ and $e^n_+ = w_{|\Omega^+} - v^n$ where w, the solution of (2), is in $H^{2,1}(\Omega \times \mathbb{R}^+)$ (see Theorem 4). These errors satisfy algorithm (1) with $f = 0$ and $w_0 = 0$ and the following relations are satisfied:

$$e^{n+1}_- = e^n_+ \quad \text{on } \Gamma_L \times \mathbb{R}^+ \quad \text{and} \quad e^{n+1}_+ = e^n_- \quad \text{on } \Gamma_0 \times \mathbb{R}^+.$$

We extend $e^n_\pm$ by 0 for $t < 0$ (we denote these extensions by $\mathsf{e}^n_\pm$). We perform the Fourier transform in y and t (defined in (6)) of the convection diffusion equation in $\Omega^- \times \mathbb{R}^+$ and $\Omega^+ \times \mathbb{R}^+$. Since $e^n_\pm(\cdot,\cdot,0) = 0$ we have $\widehat{\partial_t \mathsf{e}^n_\pm} = i\omega \hat{\mathsf{e}}^n_\pm$ and we get

$$\begin{aligned} &\widehat{\mathcal{L}}\,\hat{\mathsf{e}}^n_-(x,k,\omega) = 0 \quad \text{for } (x,k,\omega) \in (-\infty, L) \times \mathbb{R} \times \mathbb{R}, \\ &\widehat{\mathcal{L}}\,\hat{\mathsf{e}}^n_+(x,k,\omega) = 0 \quad \text{for } (x,k,\omega) \in (0,+\infty) \times \mathbb{R} \times \mathbb{R}, \end{aligned}$$

where $\widehat{\mathcal{L}} = -\nu\partial_{xx} + a\partial_x + (i(\omega + bk) + c + \nu k^2)$. We are thus led to solve an ordinary differential equation in each subdomain. The roots of the corresponding characteristic polynomial are

$$\sigma^+ = \frac{1}{2\nu}\bigl(a + \delta^{1/2}\bigr) \quad \text{and} \quad \sigma^- = \frac{1}{2\nu}\bigl(a - \delta^{1/2}\bigr), \tag{7}$$

where $\delta^{1/2} = (R + \sqrt{R^2 + I^2})^{1/2}/\sqrt{2} + i(-R + \sqrt{R^2 + I^2})^{1/2} I/|I|\sqrt{2}$, with $R = a^2 + 4\nu c + 4\nu^2 k^2$ and $I = 4\nu(\omega + bk)$. Since $\mathcal{R}e(\sigma^-) < 0$ and $\mathcal{R}e(\sigma^+) > 0$, the solutions that do not increase exponentially at infinity are

$$\begin{aligned} &\hat{\mathsf{e}}^n_-(x,k,\omega) = \alpha^n(k,\omega)e^{\sigma^+ x} \quad \text{for } (x,k,\omega) \in (-\infty, L) \times \mathbb{R} \times \mathbb{R}, \\ &\hat{\mathsf{e}}^n_+(x,k,\omega) = \beta^n(k,\omega)e^{\sigma^- x} \quad \text{for } (x,k,\omega) \in (0,+\infty) \times \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{8}$$

where α^n and β^n will be computed using the boundary conditions on Γ_0 and Γ_L. We thus get

$$\hat{\mathsf{e}}^{n+1}_- = e^{(\sigma^- - \sigma^+)L}\hat{\mathsf{e}}^{n-1}_- \quad \text{and} \quad \hat{\mathsf{e}}^{n+1}_+ = e^{(\sigma^- - \sigma^+)L}\hat{\mathsf{e}}^{n-1}_+.$$

We can now define the convergence rate of algorithm (1) with $\mathcal{B}^\pm = \mathcal{B}^\pm_{\mathrm{dir}}$, by $\rho_{\mathrm{dir}} = |\hat{\mathsf{e}}^{n+1}_-|/|\hat{\mathsf{e}}^{n-1}_-| = e^{\mathcal{R}e(\sigma^- - \sigma^+)L}$. In order to find an upper bound on ρ_{dir}, we use the relations $\mathcal{R}e(\sigma^- - \sigma^+) = \mathcal{R}e(-\delta^{1/2}/\nu) \leqslant -\alpha$, where α is equal to $(a^2 + 4\nu c)^{1/2}/\sqrt{2}\nu > 0$ and is independent of the Fourier variables k and ω. We therefore obtain

$$\bigl\|\mathsf{e}^{n+1}_-\bigr\|_{H^{2,1}} \leqslant e^{-\alpha L}\bigl\|\mathsf{e}^{n-1}_-\bigr\|_{H^{2,1}} \quad \text{and} \quad \bigl\|\mathsf{e}^{n+1}_+\bigr\|_{H^{2,1}} \leqslant e^{-\alpha L}\bigl\|\mathsf{e}^{n-1}_+\bigr\|_{H^{2,1}}, \tag{9}$$

and by induction $\|\mathsf{e}^n_\pm\|_{H^{2,1}} \leqslant e^{-\lfloor n/2 \rfloor \alpha L}(\|\mathsf{e}^0_\pm\|_{H^{2,1}} + \|\mathsf{e}^0_\pm\|_{H^{2,1}})$, where $\lfloor\cdot\rfloor$ denotes the floor function. Letting n tend to infinity completes the proof. □

From (9) we see that the larger the overlap L, the faster the algorithm converges. If $L = 0$ then $\rho_{\mathrm{dir}} = 1$ and the algorithm does not converge. In fact, the convergence of the overlapping Schwarz waveform relaxation algorithm is slow and the overlap between the two subdomains is essential. In the next section we will introduce more appropriate transmission conditions which can be used even without overlap and can be chosen such that convergence is fast.

4. The optimal Schwarz waveform relaxation algorithm

Using absorbing boundary conditions theory (see [9] for the convection diffusion equation) we propose in this section an algorithm which converges in two iterations even without overlap (see, for example, [14] for stationary problems or [5] for time dependent problems). We denote now by Γ the common interface, i.e., $\Gamma = \{y \in \mathbb{R},\ x = 0\}$ which corresponds to the case $L = 0$.

4.1. Optimal transmission conditions

We consider the Schwarz waveform relaxation algorithm (1) with interface operators $\mathcal{B}^{\pm} = \mathcal{B}^{\pm}_{\text{opt}}$ given by

$$\mathcal{B}^{-}_{\text{opt}} = \frac{\partial}{\partial x} - \mathcal{S}^{-} \quad \text{and} \quad \mathcal{B}^{+}_{\text{opt}} = \frac{\partial}{\partial x} - \mathcal{S}^{+}, \tag{10}$$

with $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$ to be defined. They are operators which depend on the tangential and time variables (y and t).

Theorem 9. *If $\mathcal{B}^{\pm} = \mathcal{B}^{\pm}_{\text{opt}}$, then algorithm* (1) *converges in two iterations to the solution of* (2) *provided σ^{+} and σ^{-}, the symbols of $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$, are given by* (7).

Proof. We introduce the errors $e^n_{\pm}$ at step n in the domains $\Omega^{\pm}$, i.e., $e^n_{-} = w|_{\Omega^-} - u^n$ and $e^n_{+} = w|_{\Omega^+} - v^n$. These errors satisfy algorithm (1) with $f = 0$ and $w_0 = 0$. As in the previous section $\hat{e}^n_{-}(x,k,\omega) = \alpha^n(k,\omega)e^{\sigma^{+}x}$ for $(x,k,\omega) \in (-\infty,0) \times \mathbb{R} \times \mathbb{R}$ and $\hat{e}^n_{+}(x,k,\omega) = \beta^n(k,\omega)e^{\sigma^{-}x}$ for $(x,k,\omega) \in (0,+\infty) \times \mathbb{R} \times \mathbb{R}$ where the coefficients σ^{+} and σ^{-} are defined in (7). Since $\partial_x \hat{e}^n_{\pm} - \sigma^{+}\hat{e}^n_{\pm} = 0$, we obtain with $\sigma^{\pm}$ the Fourier symbols of $\mathcal{S}^{\pm}$, that as soon as $n = 2$ the error in each subdomain satisfies a homogeneous problem with zero initial and boundary data and thus vanishes. Therefore $u^2 = w|_{\Omega^-}$ and $v^2 = w|_{\Omega^+}$. □

This choice of operators $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$ leads to convergence in two iterations but $\mathcal{S}^{\pm}$ are not differential operators and thus are not easy to use in an algorithm. Like for stationary problems (see, for example, [15] or [10]) we introduce instead approximated differential operators of order 0 and 1 such that their symbols are polynomials.

4.2. The Schwarz waveform relaxation algorithm with conditions of order zero or one

We cannot compute explicitly the operators whose Fourier symbol are $\sigma^{\pm}$ because of the square root that appears in (7). In order to replace it, we introduce the following polynomials of order 0 and 1 in the Fourier variables: $\sigma^{\pm}_0 = \frac{a \pm p}{2\nu}$ and $\sigma^{\pm}_1 = \frac{a \pm p}{2\nu} \pm i\omega q \pm ibkq$, where p and q are coefficients which will be chosen in the following. We have chosen the same coefficients before $i\omega$ and ibk because of the expression of $\delta^{1/2}$ (see terms of order 1), therefore we have only two parameters (p and q) to choose in the following. The corresponding differential operators are

$$\mathcal{S}^{\pm}_0 = \frac{a \pm p}{2\nu}, \qquad \mathcal{S}^{\pm}_1 = \frac{a \pm p}{2\nu} \pm q\frac{\partial}{\partial t} \pm bq\frac{\partial}{\partial y}. \tag{11}$$

We then replace the operators $\mathcal{S}^{\pm}$ in (10) by the differential operators $\mathcal{S}^{\pm}_0$ or $\mathcal{S}^{\pm}_1$ (note that $\sigma^{+} + \sigma^{-} = \sigma^{+}_j + \sigma^{-}_j = \frac{a}{\nu}$, $j = 0, 1$). In both cases, we study the associated initial boundary value problems in each subdomain and prove their well-posedness (Sections 5 and 6). We prove also convergence of the corresponding algorithms (Section 7). Finally we show how to choose the parameters p and q in order to get the best transmission conditions in the sense that the convergence rate is optimized (Section 8).

5. Optimized Schwarz waveform relaxation algorithm with interface conditions of order zero

In this section we propose to replace $\mathcal{S}^{\pm}$ in (10) by $\mathcal{S}_0^{\pm}$, the operators of order zero defined in (11), i.e., we introduce the interface operators

$$\mathcal{B}_0^{\pm} = \frac{\partial}{\partial x} - \frac{a \pm p}{2\nu}. \tag{12}$$

5.1. *Well posedness of the subdomain problems*

In this part we study the initial boundary value problem in Ω^- from algorithm (1) with $\mathcal{B}^{\pm} = \mathcal{B}_0^{\pm}$ defined in (12). The problem reads

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega^- \times \mathbb{R}^+, \\ u(\cdot,\cdot,0) = w_0 & \text{in } \Omega^-, \\ \frac{\partial u}{\partial x} - \frac{a-p}{2\nu} u = g & \text{on } \Gamma \times \mathbb{R}^+. \end{cases} \tag{13}$$

Theorem 10. *Suppose that f is in $L^2(\mathbb{R}^+; L^2(\Omega))$, w_0 is in $H^1(\Omega)$ and p is strictly positive. If g is in $H^{1/2,1/4}(\mathbb{R} \times \mathbb{R}^+)$, then problem* (13) *has a unique solution u in $L^\infty(\mathbb{R}^+; L^2(\Omega^-)) \cap L^2(\mathbb{R}^+; H^1(\Omega^-))$. Furthermore u is in $H^{2,1}(\Omega^- \times \mathbb{R}^+)$.*

Proof. We introduce g_0 in $H^{3/2,3/4}(\mathbb{R} \times \mathbb{R}^+)$ such that $g_0(\cdot, 0) = w_0(0,\cdot)$ on Γ and define $g_1 = g + (a-p)g_0/2\nu$, i.e., g_1 is in $H^{1/2,1/4}(\mathbb{R} \times \mathbb{R}^+)$. Applying the extension Theorem 3 we find that there is a U in $H^{2,1}(\Omega^- \times \mathbb{R}^+)$ such that $U(\cdot,\cdot,0) = w_0$ in Ω^-, $U(0,\cdot,\cdot) = g_0$ on $\Gamma \times \mathbb{R}^+$ and $\partial_x U(0,\cdot,\cdot) = g_1$ on $\Gamma \times \mathbb{R}^+$. We define $\tilde{u} = u - U$ and $F = f_{|\Omega^-} - \mathcal{L}U$. F is in $L^2(\mathbb{R}^+; L^2(\Omega))$ and the problem becomes

$$\begin{cases} \mathcal{L}\tilde{u} = F & \text{in } \Omega^- \times \mathbb{R}^+, \\ \tilde{u}(\cdot,\cdot,0) = 0 & \text{in } \Omega^-, \\ \frac{\partial \tilde{u}}{\partial x} - \frac{a-p}{2\nu} \tilde{u} = 0 & \text{on } \Gamma \times \mathbb{R}^+. \end{cases} \tag{14}$$

We multiply the equation $\mathcal{L}\tilde{u} = F$ by v in $H^1(\Omega^-)$, integrate in space and use the boundary condition on $\Gamma \times \mathbb{R}^+$. If $(\cdot,\cdot)_{\Omega^-}$ denotes the scalar product in $L^2(\Omega^-)$ and $\|\cdot\|_{\Omega^-}^2$ the associated norm, we get the variational formulation

$$\begin{cases} \text{Find } \tilde{u} \text{ such that for any } v \text{ in } H^1(\Omega^-) \\ \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}, v)_{\Omega^-} + \nu(\nabla\tilde{u}, \nabla v)_{\Omega^-} + \frac{1}{2}\Big(\big(\vec{\mathbf{b}} \cdot \nabla\tilde{u}, v\big)_{\Omega^-} - \big(\vec{\mathbf{b}} \cdot \nabla v, \tilde{u}\big)_{\Omega^-}\Big) \\ \qquad + c(\tilde{u}, v)_{\Omega^-} + \frac{p}{2}\int_\Gamma \tilde{u} v \,\mathrm{d}y = (F, v)_{\Omega^-}, \\ \tilde{u}(\cdot,\cdot,0) = 0. \end{cases}$$

Multiplying the convection diffusion equation by $\tilde{u}$, integrating on Ω^- and using the boundary conditions, we first get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{u}\|^2_{\Omega^-} + \nu\|\nabla\tilde{u}\|^2_{\Omega^-} + \frac{p}{2}\int_\Gamma |\tilde{u}|^2\,\mathrm{d}y + c\|\tilde{u}\|^2_{\Omega^-} = (F,\tilde{u})_{\Omega^-}. \tag{15}$$

We recall the following relation for any a, b and α strictly positive

$$ab \leqslant \frac{1}{2\alpha}a^2 + \frac{\alpha}{2}b^2. \tag{16}$$

Applying the Cauchy–Schwarz inequality and the inequality (16) to the right side of (15) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{u}\|^2_{\Omega^-} + \nu\|\nabla\tilde{u}\|^2_{\Omega^-} + \frac{p}{2}\int_\Gamma |\tilde{u}|^2\,\mathrm{d}y + \frac{c}{2}\|\tilde{u}\|^2_{\Omega^-} \leqslant \frac{1}{2c}\|F\|^2_{\Omega^-}. \tag{17}$$

We integrate this estimate on $(0,t)$, $t>0$, and we use a Galerkin method to show that there is a unique solution of (14) in the space $L^\infty(\mathbb{R}^+; L^2(\Omega^-)) \cap L^2(\mathbb{R}^+; H^1(\Omega^-))$. The same holds obviously for the solution of (13).

We now prove the regularity result. We multiply the equation $\mathcal{L}\tilde{u} = F$ by $-(\partial_{xx}\tilde{u} + \gamma_1\partial_{yy}\tilde{u})$ (with γ_1 a positive number to be chosen) and integrate on Ω^-. We obtain after integrations by parts

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial\tilde{u}}{\partial x}\right\|^2_{\Omega^-} + \frac{\gamma_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-} - \int_\Gamma \frac{\partial\tilde{u}}{\partial t}\frac{\partial\tilde{u}}{\partial x}\mathrm{d}y - \frac{a}{2}\int_\Gamma \left|\frac{\partial\tilde{u}}{\partial x}\right|^2\mathrm{d}y + \frac{a\gamma_1}{2}\int_\Gamma\left|\frac{\partial\tilde{u}}{\partial y}\right|^2\mathrm{d}y \\
&\quad + c\left\|\frac{\partial\tilde{u}}{\partial x}\right\|^2_{\Omega^-} + c\gamma_1\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-} - c\int_\Gamma \tilde{u}\frac{\partial\tilde{u}}{\partial x}\mathrm{d}y + \nu\left\|\frac{\partial^2\tilde{u}}{\partial x^2}\right\|^2_{\Omega^-} \\
&\quad + \nu\gamma_1\left\|\frac{\partial^2\tilde{u}}{\partial y^2}\right\|^2_{\Omega^-} + \nu(1+\gamma_1)\left(\frac{\partial^2\tilde{u}}{\partial x^2}, \frac{\partial^2\tilde{u}}{\partial y^2}\right)_{\Omega^-} = -\left(F, \frac{\partial^2\tilde{u}}{\partial x^2} + \gamma_1\frac{\partial^2\tilde{u}}{\partial y^2}\right)_{\Omega^-}.
\end{aligned}$$

Using the boundary condition $\partial_x\tilde{u} = (a-p)\tilde{u}/2\nu$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial\tilde{u}}{\partial x}\right\|^2_{\Omega^-} + \frac{\gamma_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-} - \frac{1}{2}\frac{a-p}{2\nu}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma |\tilde{u}|^2\,\mathrm{d}y \\
&\quad - \frac{a-p}{2\nu}\left(c + \frac{a}{2}\frac{a-p}{2\nu}\right)\int_\Gamma |\tilde{u}|^2\,\mathrm{d}y + \frac{a\gamma_1}{2}\int_\Gamma\left|\frac{\partial\tilde{u}}{\partial y}\right|^2\mathrm{d}y + c\left\|\frac{\partial\tilde{u}}{\partial x}\right\|^2_{\Omega^-} + c\gamma_1\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-} \\
&\quad + \nu\left\|\frac{\partial^2\tilde{u}}{\partial x^2}\right\|^2_{\Omega^-} + \nu\gamma_1\left\|\frac{\partial^2\tilde{u}}{\partial y^2}\right\|^2_{\Omega^-} + \nu(1+\gamma_1)\left(\frac{\partial^2\tilde{u}}{\partial x^2}, \frac{\partial^2\tilde{u}}{\partial y^2}\right)_{\Omega^-} = -\left(F, \frac{\partial^2\tilde{u}}{\partial x^2} + \gamma_1\frac{\partial^2\tilde{u}}{\partial y^2}\right)_{\Omega^-}.
\end{aligned} \tag{18}$$

On the other hand, we use integration by parts first in y then in x and we obtain $(\partial_{xx}\tilde{u}, \partial_{yy}\tilde{u})_{\Omega_-} = -(\partial_{xx}(\partial_y\tilde{u}), \partial_y\tilde{u})_{\Omega_-} = \|\partial_{xy}\tilde{u}\|^2_{\Omega^-} - \int_\Gamma \partial_y(\partial_x\tilde{u})\partial_y\tilde{u}\,\mathrm{d}y = \|\partial_{xy}\tilde{u}\|^2_{\Omega^-} - ((a-p)/2\nu)\int_\Gamma |\partial_y\tilde{u}|^2\,\mathrm{d}y$. We introduce this result in (18) and using the Cauchy–Schwarz inequality and (16) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\frac{\partial\tilde{u}}{\partial x}\right\|^2_{\Omega^-} + \gamma_1\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-} - \frac{a-p}{2\nu}\int_\Gamma |\tilde{u}|^2\,\mathrm{d}y\right] - \frac{a-p}{2\nu}\left(c + \frac{a}{2}\frac{a-p}{2\nu}\right)\int_\Gamma |\tilde{u}|^2\,\mathrm{d}y$$

$$+\left(-\frac{a}{2}+\frac{p}{2}(1+\gamma_1)\right)\int\limits_{\Gamma}\left|\frac{\partial\tilde{u}}{\partial y}\right|^2\mathrm{d}y+c\left\|\frac{\partial\tilde{u}}{\partial x}\right\|^2_{\Omega^-}+c\gamma_1\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-}+\frac{\nu}{2}\left\|\frac{\partial^2\tilde{u}}{\partial x^2}\right\|^2_{\Omega^-}$$

$$+\frac{\nu\gamma_1}{2}\left\|\frac{\partial^2\tilde{u}}{\partial y^2}\right\|^2_{\Omega^-}+\nu(1+\gamma_1)\left\|\frac{\partial^2\tilde{u}}{\partial x\partial y}\right\|^2_{\Omega^-}\leqslant\frac{1}{2\nu}(1+\gamma_1)\|F\|^2_{\Omega^-}.$$

We multiply now (17) by a constant $\gamma_2>0$ to be defined, and add the result to the previous equation, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma_2\|\tilde{u}\|^2_{\Omega^-}+\left\|\frac{\partial\tilde{u}}{\partial x}\right\|^2_{\Omega^-}+\gamma_1\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-}-\frac{a-p}{2\nu}\int\limits_{\Gamma}|\tilde{u}|^2\,\mathrm{d}y\right]$$

$$+\left(\gamma_2\frac{p}{2}-\frac{a-p}{2\nu}\left(c+\frac{a}{2}\frac{a-p}{2\nu}\right)\right)\int\limits_{\Gamma}|\tilde{u}|^2\,\mathrm{d}y+\left(-\frac{a}{2}+\frac{p}{2}(1+\gamma_1)\right)\int\limits_{\Gamma}\left|\frac{\partial\tilde{u}}{\partial y}\right|^2\mathrm{d}y$$

$$+\frac{c\gamma_2}{2}\|\tilde{u}\|^2_{\Omega^-}+(c+\nu\gamma_2)\left\|\frac{\partial\tilde{u}}{\partial x}\right\|^2_{\Omega^-}+(c\gamma_1+\nu\gamma_2)\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-}+\frac{\nu}{2}\left\|\frac{\partial^2\tilde{u}}{\partial x^2}\right\|^2_{\Omega^-}$$

$$+\frac{\nu\gamma_1}{2}\left\|\frac{\partial^2\tilde{u}}{\partial y^2}\right\|^2_{\Omega^-}+\nu(1+\gamma_1)\left\|\frac{\partial^2\tilde{u}}{\partial x\partial y}\right\|^2_{\Omega^-}\leqslant\left(\frac{\gamma_2}{2c}+\frac{1}{2\nu}(1+\gamma_1)\right)\|F\|^2_{\Omega^-}. \tag{19}$$

Since $p>0$, by choosing $\gamma_1\geqslant a/p-1$ and $\gamma_2\geqslant(a-p)(c+a(a-p)/4\nu)/\nu p$ the coefficients in front of $\int_\Gamma|\tilde{u}|^2\,\mathrm{d}y$ and $\int_\Gamma|\partial_y\tilde{u}|^2\,\mathrm{d}y$ in (19) are positive. Integrating (19) over $(0,t)$ we get

$$\frac{1}{2}\left(\gamma_2\|\tilde{u}\|^2_{\Omega^-}(t)+\left\|\frac{\partial\tilde{u}}{\partial x}\right\|^2_{\Omega^-}(t)-\frac{a-p}{2\nu}\int\limits_{\Gamma}|\tilde{u}|^2(0,y,t)\,\mathrm{d}y\right)$$

$$+\int\limits_0^t\left(\frac{c\gamma_2}{2}\|\tilde{u}\|^2_{\Omega^-}(\sigma)+(c+\nu\gamma_2)\left\|\frac{\partial\tilde{u}}{\partial x}\right\|^2_{\Omega^-}(\sigma)+(c\gamma_1+\nu\gamma_2)\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-}(\sigma)\right)\mathrm{d}\sigma$$

$$+\int\limits_0^t\left(\frac{\nu}{2}\left\|\frac{\partial^2\tilde{u}}{\partial x^2}\right\|^2_{\Omega^-}(\sigma)+\frac{\nu\gamma_1}{2}\left\|\frac{\partial^2\tilde{u}}{\partial y^2}\right\|^2_{\Omega^-}(\sigma)+\nu(1+\gamma_1)\left\|\frac{\partial^2\tilde{u}}{\partial x\partial y}\right\|^2_{\Omega^-}(\sigma)\right)\mathrm{d}\sigma$$

$$\leqslant\left(\frac{\gamma_2}{2c}+\frac{1}{2\nu}(1+\gamma_1)\right)\int\limits_0^t\|F\|^2_{\Omega^-}(\sigma)\,\mathrm{d}\sigma. \tag{20}$$

From these estimates we get that if $p\geqslant a$ then $\tilde{u}$ is in $L^2(\mathbb{R}^+;H^2(\Omega^-))$. If not, we use the relation $\int_\Gamma|\tilde{u}|^2\,\mathrm{d}y\leqslant\alpha\|\tilde{u}\|^2_{\Omega_-}+(1/\alpha)\|\partial_x\tilde{u}\|^2_{\Omega_-}$ for $\alpha>0$, to find a lower bound for the first three terms of (20): $(\gamma_2-\alpha(a-p)/2\nu)\|\tilde{u}\|^2_{\Omega^-}+(1-(a-p)/2\nu\alpha)\|\partial_x\tilde{u}\|^2_{\Omega^-}$. Choosing γ_2 and α large enough, we obtain that the coefficients in front of $\|\tilde{u}\|^2_{\Omega^-}$ and $\|\partial_x\tilde{u}\|^2_{\Omega^-}$ are positive, which leads to the conclusion that when $0<p<a$ we also have $\tilde{u}$ in $L^2(\mathbb{R}^+;H^2(\Omega^-))$. Finally using the equation we get $\partial_t\tilde{u}$ in $L^2(\mathbb{R}^+;L^2(\Omega^-))$ and thus we have $\tilde{u}$ in $H^1(\mathbb{R}^+;L^2(\Omega^-))$. Therefore $\tilde{u}$ is in $H^{2,1}(\Omega^-\times\mathbb{R}^+)$. And since $\tilde{u}=u-U$ with U in $H^{2,1}(\Omega^-\times\mathbb{R}^+)$ we get the announced regularity result. □

5.2. The algorithm

If $\mathcal{B}^{\pm} = \mathcal{B}_0^{\pm}$, we initialize algorithm (1) by

$$\begin{array}{llllll} \mathcal{L}u^0 = f & \text{in } \Omega^- \times \mathbb{R}^+, & u^0(\cdot,\cdot,0) = w_0 & \text{in } \Omega^-, & \mathcal{B}_0^- u^0 = g & \text{on } \Gamma \times \mathbb{R}^+, \\ \mathcal{L}v^0 = f & \text{in } \Omega^+ \times \mathbb{R}^+, & v^0(\cdot,\cdot,0) = w_0 & \text{in } \Omega^+, & \mathcal{B}_0^+ v^0 = h & \text{on } \Gamma \times \mathbb{R}^+, \end{array} \tag{21}$$

where g is in $H^{1/2,1/4}(\mathbb{R} \times \mathbb{R}^+)$ and h in $H^{1/2,1/4}(\mathbb{R} \times \mathbb{R}^+)$.

Theorem 11. *Suppose that f is in $L^2(\mathbb{R}^+; L^2(\Omega))$, w_0 is in $H^1(\Omega)$ and p is strictly positive. If $\mathcal{B}^{\pm} = \mathcal{B}_0^{\pm}$, then algorithm* (1) *initialized by* (21) *defines a unique sequence of iterates (u^n, v^n) in $H^{2,1}(\Omega^- \times \mathbb{R}^+) \times H^{2,1}(\Omega^+ \times \mathbb{R}^+)$.*

Proof. Problem (21) has by Theorem 10 a unique solution (u^0, v^0) in $H^{2,1}(\Omega^- \times \mathbb{R}^+) \times H^{2,1}(\Omega^+ \times \mathbb{R}^+)$. On the other hand, if the solution of (1) in Ω^+, v^n, is in $H^{2,1}(\Omega^+ \times \mathbb{R}^+)$, then according to the trace Theorem 3 we have $v^n(0,\cdot,\cdot)$ in $H^{3/2,3/4}(\mathbb{R} \times \mathbb{R}^+)$ and $\partial_x v^n(0,\cdot,\cdot)$ in $H^{1/2,1/4}(\mathbb{R} \times \mathbb{R}^+)$. Thus $g = \partial_x v^n - (a-p)v^n/2\nu$ satisfies the assumptions of Theorem 10 and therefore, u^{n+1} is defined in $H^{2,1}(\Omega^- \times \mathbb{R}^+)$. The same result holds in Ω^+ and by induction, the proof is complete. □

6. Optimized Schwarz waveform relaxation algorithm with first order interface conditions

We now propose to replace $\mathcal{S}^{\pm}$ in (10) by $\mathcal{S}_1^{\pm}$, the operators of order one defined in (11). We introduce the interface operators

$$\mathcal{B}_1^{\pm} = \frac{\partial}{\partial x} - \frac{a \pm p}{2\nu} \mp q\frac{\partial}{\partial t} \mp bq\frac{\partial}{\partial y}. \tag{22}$$

The next sections are devoted to prove the well-posedness of each subdomain problem of algorithm (1) when $\mathcal{B}^{\pm} = \mathcal{B}_1^{\pm}$. In Section 6.1, we recall a theorem and give a lemma. Both of them will be useful when we prove uniqueness and regularity of the solution in each subdomain (Section 6.2). In Section 6.3 we define the first iterate (u^0, v^0) of the algorithm.

6.1. Technical lemma and theorem

We first introduce for $s > 1/2$ the space H_0^s defined by $H_0^s(0,+\infty) = \{u \in H^s(0,+\infty),\ \frac{\mathrm{d}^j u}{\mathrm{d}t^j}(0) = 0$ for $0 \leqslant j < s - 1/2\}$. Let u be a function defined on $\mathbb{R}^+$ and $\tilde{u}$ its extension by 0 for $t < 0$. The following theorem gives the regularity of $\tilde{u}$. A proof of this theorem can be found in [12].

Theorem 12. *The map $u \to \tilde{u}$, with $\tilde{u}$ the extension of u by 0 for $t < 0$, is continuous from $H^s(0,+\infty)$ to $H^s(\mathbb{R})$ if and only if $0 \leqslant s < \frac{1}{2}$. And it is continuous from $H_0^s(0,+\infty)$ to $H^s(\mathbb{R})$ for $s > \frac{1}{2}$ if and only if $s \neq$ integer $+1/2$.*

In order to establish a regularity result, we need a technical result which gives a bound to some functions of k and ω. We recall that σ^+ and $\delta^{1/2}$ are defined in (7).

Lemma 13. *Let p and q be strictly positive. There exist constants C_i, $i = 1, 7$ such that for any $(\omega, k) \in \mathbb{R}^2$ we have*

$$\phi_1(k,\omega) = \frac{(1+\omega^2)^{1/8}}{(\mathcal{R}e(\sigma^+))^{1/2}} \leqslant C_1, \qquad \phi_2(k,\omega) = \frac{(1+k^2)^{1/4}}{(\mathcal{R}e(\sigma^+))^{1/2}} \leqslant C_2,$$
$$\phi_3(k,\omega) = \frac{(1+\omega^2)^{1/2}}{|p+\delta^{1/2}+2\nu i(\omega+bk)q|} \leqslant C_3, \qquad \phi_4(k,\omega) = \frac{(1+k^2)^{1/2}}{|p+\delta^{1/2}+2\nu i(\omega+bk)|} \leqslant C_4,$$
$$\phi_5(k,\omega) = \frac{|\frac{a-p}{2\nu}+iq(\omega+bk)|}{|p+\delta^{1/2}+2\nu iq(\omega+bk)|} \leqslant C_5,$$
$$\phi_6(k,\omega) = \frac{|\sigma^+|^3}{(\mathcal{R}e(\sigma^+))^{1/2}} \frac{1}{(1+\omega^2)^{5/8}+(1+k^2)^{5/4}} \leqslant C_6,$$
$$\phi_7(k,\omega) = \frac{(1+\omega^2)^{5/8}+(1+k^2)^{5/4}}{|p+\delta^{1/2}+2\nu i(\omega+bk)q)|} \frac{1}{((1+\omega^2)^{3/8}+(1+k^2)^{3/4})} \leqslant C_7.$$

Proof. For each function ϕ_i, the method consists in showing that ϕ_i is continuous and bounded at infinity. On the one hand we have $\mathcal{R}e(\delta^{1/2}) \geqslant \sqrt{a^2+4\nu c}$, therefore $\mathcal{R}e(\sigma^+) \geqslant (a+\sqrt{a^2+4\nu c})/2\nu > 0$ and $\mathcal{R}e(p+\delta^{1/2}+2\nu i(\omega+bk)) = p+\mathcal{R}e(\delta^{1/2}) \geqslant p+\sqrt{a^2+4\nu c} > 0$. Thus each ϕ_i is continuous. On the other hand, we consider three cases:

(i) $|\omega|/|k| \to l$, $l \geqslant 0$ when $|k|, |\omega| \to +\infty$.
In this case we have $\mathcal{R}e(\sigma^+) \sim |k|$, $\mathcal{I}m(\sigma^+) \sim c_1$ (c_1 a positive constant) and $\delta^{1/2} \sim 2\nu|k|$. Therefore $|\sigma^+|^3 \sim |k|^3$ and $|p+\delta^{1/2}+2\nu i(\omega+bk)q| \sim c_2|k|$ (c_2 a positive constant). We can thus deduce that each ϕ_i, $i = 1, 7$ is bounded at infinity when $|\omega|/|k| \to l$.

(ii) $|k|/|\omega| \to 0$ when $|k|, |\omega| \to +\infty$ with $|\omega| \gg k^2$.
In this case we have $\mathcal{R}e(\sigma^+) \sim |\omega|^{1/2}/(2\nu)^{1/2}$, $\mathcal{I}m(\sigma^+) \sim |\omega|^{1/2}/(2\nu)^{1/2}$ and $\delta^{1/2} \sim c_3|\omega|^{1/2}$ (c_3 a positive constant). Therefore $|\sigma^+|^3 \sim |\omega|^{3/2}/\nu^{3/2}$ and $|p+\delta^{1/2}+2\nu i(\omega+bk)q| \sim 2\nu|\omega|$. We thus deduce that each ϕ_i, $i = 1, 7$ is bounded at infinity when $|k|/|\omega| \to 0$ with $|\omega| \gg k^2$.

(iii) $|k|/|\omega| \to 0$ when $|k|, |\omega| \to +\infty$ with $|\omega| \ll k^2$.
In this case we have $\mathcal{R}e(\sigma^+) \sim |k|$, $\mathcal{I}m(\sigma^+) \sim c_1$ and $\delta^{1/2} \sim 2\nu|k|$. Therefore $|\sigma^+|^3 \sim |k|^3$ and $|p+\delta^{1/2}+2\nu i(\omega+bk)q| \sim c_4|\omega|$. We thus deduce that each ϕ_i, $i = 1, 7$ is bounded when $|k|/|\omega| \to 0$ with $|\omega| \ll k^2$.

Thus each ϕ_i is continuous and bounded at infinity. We deduce that each ϕ_i is bounded on $\mathbb{R}^2$. □

6.2. Well-posedness of the subdomain problems

We now study the initial boundary value problem in Ω^- involved in algorithm (1) when interface operators are given by (22). It reads

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega^- \times \mathbb{R}^+, \\ u(\cdot,\cdot,0) = w_0 & \text{in } \Omega^-, \\ \frac{\partial u}{\partial x} - \frac{a-p}{2\nu}u + q\frac{\partial u}{\partial t} + bq\frac{\partial u}{\partial y} = g & \text{on } \Gamma \times \mathbb{R}^+. \end{cases} \tag{23}$$

According to Theorem 5 and to give a meaning to the trace of $\partial_t u$ on Γ, we make the assumption that f is in $H^{1,1/2}(\Omega \times \mathbb{R}^+)$ and w_0 in $H^2(\Omega)$. Therefore the solution in the whole domain is in $H^{3,3/2}(\Omega \times \mathbb{R}^+)$. We also make the assumption that g can be written as

$$g = g_1 + \left(-\frac{a-p}{2\nu} + q\frac{\partial}{\partial t} + bq\frac{\partial}{\partial y}\right)g_0, \tag{24}$$

with g_0 and g_1 such as

$$g_0 \in H^{5/2,5/4}\big(\mathbb{R} \times \mathbb{R}^+\big) \quad \text{and} \quad g_1 \in H^{3/2,3/4}\big(\mathbb{R} \times \mathbb{R}^+\big), \tag{25}$$

$$g_0(\cdot,0) = w_0(0,\cdot) \quad \text{and} \quad g_1(\cdot,0) = \frac{\partial w_0}{\partial x}(0,\cdot) \quad \text{on } \Gamma. \tag{26}$$

Theorem 14 gives existence and uniqueness of the solution of each subdomain problem and Theorem 18 gives the regularity.

6.2.1. *Existence and uniqueness of the solution*

Theorem 14. *Suppose that f is in $H^{1,1/2}(\Omega \times \mathbb{R}^+)$ and w_0 in $H^2(\Omega)$. Let p and q be strictly positive. If g is given as in* (24)–(26)*, then problem* (23) *has a unique solution in $L^\infty(\mathbb{R}^+; L^2(\Omega^-)) \cap L^2(\mathbb{R}^+; H^1(\Omega^-))$.*

Proof. In order to prove existence and uniqueness, we introduce the problem corresponding to (23) with homogeneous boundary conditions. We thus apply Theorem 3 which gives a U in $H^{3,3/2}(\Omega^- \times \mathbb{R}^+)$ such that $U(\cdot,\cdot,0) = w_0$ in Ω^-, $U(0,\cdot,\cdot) = g_0$ on $\Gamma \times \mathbb{R}^+$, $\partial_x U(0,\cdot,\cdot) = g_1$ on $\Gamma \times \mathbb{R}^+$. Using relation (24), we obtain that U satisfies

$$\begin{cases} U(\cdot,\cdot,0) = w_0 & \text{in } \Omega^-, \\ \frac{\partial U}{\partial x} - \frac{a-p}{2\nu}U + q\frac{\partial U}{\partial t} + bq\frac{\partial U}{\partial y} = g & \text{on } \Gamma \times \mathbb{R}^+. \end{cases}$$

If we introduce $\tilde{u} = u - U$ and $F = f|_{\Omega^-} - \mathcal{L}U \in H^{1,1/2}(\Omega^- \times \mathbb{R}^+)$, then we are led to solve the problem

$$\begin{cases} \mathcal{L}\tilde{u} = F & \text{in } \Omega^- \times \mathbb{R}^+, \\ \tilde{u}(\cdot,\cdot,0) = 0 & \text{in } \Omega^-, \\ \frac{\partial \tilde{u}}{\partial x} - \frac{a-p}{2\nu}\tilde{u} + q\frac{\partial \tilde{u}}{\partial t} + bq\frac{\partial \tilde{u}}{\partial y} = 0 & \text{on } \Gamma \times \mathbb{R}^+. \end{cases} \tag{27}$$

We first write the variational formulation of problem (27).

$$\begin{cases} \text{Find } \tilde{u} \text{ such that for any } v \text{ in } H^1(\Omega^-) \\ \frac{\mathrm{d}}{\mathrm{d}t}\big[(\tilde{u},v)_{\Omega^-} + q\nu\int_\Gamma \tilde{u}v\,\mathrm{d}y\big] + \frac{1}{2}\big((\vec{\mathbf{b}}\cdot\nabla\tilde{u},v)_{\Omega^-} - (\vec{\mathbf{b}}\cdot\nabla v,\tilde{u})_{\Omega^-}\big) \\ \qquad + \nu(\nabla\tilde{u},\nabla v)_{\Omega^-} + c(\tilde{u},v)_{\Omega^-} + \int_\Gamma \frac{p}{2}\tilde{u}\,\mathrm{d}y + \frac{bq\nu}{2}\int_\Gamma\big(\frac{\partial\tilde{u}}{\partial y}v - \frac{\partial v}{\partial y}\tilde{u}\big)\,\mathrm{d}y = (F,v)_{\Omega^-}, \\ \tilde{u}(\cdot,\cdot,0) = w_0. \end{cases} \tag{28}$$

We write now two *a priori* estimates for the solution of (27). The first one is obtained by multiplying the equation $\mathcal{L}\tilde{u} = F$ by $\tilde{u}$ and integrating the result on Ω^-. We get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{u}\|^2_{\Omega^-} + \nu\|\nabla\tilde{u}\|^2_{\Omega^-} - \nu\int_\Gamma\left(\frac{\partial\tilde{u}}{\partial x} - \frac{a}{2\nu}\tilde{u}\right)\tilde{u}\,\mathrm{d}y + c\|\tilde{u}\|^2_{\Omega^-} = (F,\tilde{u})_{\Omega^-}.$$

We use the boundary condition on $\Gamma \times \mathbb{R}^+$, the Cauchy Schwarz inequality and the inequality (16). We therefore obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\tilde{u}\|^2_{\Omega^-} + q\nu\int_\Gamma |\tilde{u}|^2\,\mathrm{d}y\right] + \nu\|\nabla\tilde{u}\|^2_{\Omega^-} + \frac{p}{2}\int_\Gamma |\tilde{u}|^2\,\mathrm{d}y + c\|\tilde{u}\|^2_{\Omega^-} \leqslant \frac{1}{2c}\|F\|^2_{\Omega^-} + \frac{c}{2}\|\tilde{u}\|^2_{\Omega^-}.$$

If we integrate this last equation on $(0,t)$, we get

$$\frac{1}{2}\left[\|\tilde{u}\|^2_{\Omega^-}(t) + q\nu\int_\Gamma |\tilde{u}|^2(0,y,t)\,\mathrm{d}y\right] + \nu\int_0^t \|\nabla\tilde{u}\|^2_{\Omega^-}(\sigma)\,\mathrm{d}\sigma$$
$$+\frac{p}{2}\int_0^t\int_\Gamma |\tilde{u}|^2(0,y,\sigma)\,\mathrm{d}y\,\mathrm{d}\sigma + \frac{c}{2}\int_0^t \|\tilde{u}\|^2_{\Omega^-}(\sigma)\,\mathrm{d}\sigma \leqslant \frac{1}{2c}\int_0^t \|F\|^2_{\Omega^-}(\sigma)\,\mathrm{d}\sigma. \tag{29}$$

We now will obtain a second *a priori* estimate by multiplying the equation by $-\partial_{yy}\tilde{u}$ and by integrating the result on Ω^-. Note that using integration by parts we obtain $-a(\partial_x\tilde{u}, \partial_{yy}\tilde{u})_{\Omega^-} = a(\partial_x(\partial_y\tilde{u}), \partial_y\tilde{u})_{\Omega^-} = (a/2)\int_\Gamma |\partial_y\tilde{u}|^2\,\mathrm{d}y$ and we also have $\nu(\partial_{xx}\tilde{u}, \partial_{yy}\tilde{u})_{\Omega^-} = -\nu(\partial_{xx}(\partial_y\tilde{u}), \partial_y\tilde{u})_{\Omega^-} = \nu\|\partial_{xy}\tilde{u}\|^2_{\Omega^-} - \nu\int_\Gamma \partial_y(\partial_x\tilde{u})\partial_y\tilde{u}\,\mathrm{d}y$. If we use the boundary conditions we get $\nu(\partial_{xx}\tilde{u}, \partial_{yy}\tilde{u})_{\Omega^-} = \nu\|\partial_{xy}\tilde{u}\|^2_{\Omega^-} - \frac{a-p}{2}\int_\Gamma |\partial_y\tilde{u}|^2\,\mathrm{d}y + \frac{q\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma |\partial_y\tilde{u}|^2\,\mathrm{d}y$. After integration on $(0,t)$ we use Cauchy–Schwarz inequality and (16) and we obtain

$$\frac{1}{2}\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-}(t) + \frac{q\nu}{2}\int_\Gamma\left|\frac{\partial\tilde{u}}{\partial y}\right|^2(0,y,t)\,\mathrm{d}y + \frac{\nu}{2}\int_0^t\left\|\frac{\partial^2\tilde{u}}{\partial y^2}\right\|^2_{\Omega^-}(\sigma)\,\mathrm{d}\sigma + \nu\int_0^t\left\|\frac{\partial^2\tilde{u}}{\partial x\partial y}\right\|^2_{\Omega^-}(\sigma)\,\mathrm{d}\sigma$$
$$+ c\int_0^t\left\|\frac{\partial\tilde{u}}{\partial y}\right\|^2_{\Omega^-}(\sigma)\,\mathrm{d}\sigma + \frac{p}{2}\int_0^t\int_\Gamma\left|\frac{\partial\tilde{u}}{\partial y}\right|^2(0,y,\sigma)\,\mathrm{d}y\,\mathrm{d}\sigma \leqslant \frac{1}{2\nu}\int_0^t \|F\|^2_{\Omega^-}(\sigma)\,\mathrm{d}\sigma. \tag{30}$$

Using a Galerkin method, estimates (29) and (30) give existence and uniqueness of the solution of problem (27), $\tilde{u}$, in $L^\infty(\mathbb{R}^+; L^2(\Omega^-)) \cap L^2(\mathbb{R}^+; H^1(\Omega^-))$ with $\tilde{u}|_\Gamma$ in $L^\infty(\mathbb{R}^+; L^2(\Gamma)) \cap L^2(\mathbb{R}^+; H^1(\Gamma))$. Since $\tilde{u} = u - U$, we obtain the proof of the theorem. □

6.2.2. *Regularity of the solution*

For the regularity result we work on the homogeneous equation with zero as initial condition. If $w \in H^{3,3/2}(\Omega \times \mathbb{R}^+)$ is the solution in the whole domain, we introduce $e_- = w_{|\Omega^-} - u$ and the problem becomes

$$\begin{cases} \mathcal{L}e_- = 0 & \text{in } \Omega^- \times \mathbb{R}^+,\\ e_-(\cdot,\cdot,0) = 0 & \text{in } \Omega^-,\\ \frac{\partial e_-}{\partial x} - \frac{a-p}{2\nu}e_- + q\frac{\partial e_-}{\partial t} + bq\frac{\partial e_-}{\partial y} = \tilde{g} & \text{on } \Gamma \times \mathbb{R}^+, \end{cases} \tag{31}$$

where $\tilde{g} = g - (\frac{\partial w}{\partial x} - \frac{a-p}{2\nu}w + q\frac{\partial w}{\partial t} + bq\frac{\partial w}{\partial y})|_\Gamma$ is in $H^{1/2,1/4}(\mathbb{R} \times \mathbb{R}^+)$.

In order to prove that e_- is in $H^{3,3/2}(\Omega^- \times \mathbb{R}^+)$, we split the result in three parts. In Lemma 15 we prove that e_- is in $L^2(\mathbb{R}^-; H^{3/2}(\mathbb{R}^+; L^2(\mathbb{R})))$. In Lemma 16 we prove that e_- is in $L^2(\mathbb{R}^-; L^2(\mathbb{R}^+; H^3(\mathbb{R})))$. Then we get that e_- is in $H^3(\mathbb{R}^-; L^2(\mathbb{R} \times \mathbb{R}^+))$ in Lemma 17. Finally, Theorem 18 sums up and gives the result: u is in $H^{3,3/2}(\Omega^- \times \mathbb{R}^+)$. For more details about regularity results for time dependent equations using Fourier transforms the reader is referred to [12].

Lemma 15. *If $\tilde{g}$ belongs to $H^{1/4}(\mathbb{R}^+; L^2(\mathbb{R}))$, then for p and q strictly positive the solution of* (31), *e_-, is in $L^2(\mathbb{R}^-; H^{3/2}(\mathbb{R}^+; L^2(\mathbb{R})))$.*

Proof. We extend e_- and $\tilde{g}$ for $t < 0$ by 0 (we denote those extensions by e_- and $\tilde{\mathsf{g}}$) and we perform the Fourier transform of the equation. Using the same notation as in Section 3, we have $\hat{\mathsf{e}}_-(x,k,\omega) = \alpha(k,\omega)e^{\sigma^+ x}$, where α is computed thanks to the boundary condition of (31). We thus get

$$\hat{\mathsf{e}}_-(x,k,\omega) = \frac{2\nu}{p + \delta^{1/2} + 2\nu i(\omega + bk)q}\hat{\tilde{\mathsf{g}}}(k,\omega)e^{\sigma^+ x}.$$

Thus, in order to show that e_- is in $L^2(\mathbb{R}^-; H^{3/2}(\mathbb{R}; L^2(\mathbb{R})))$, we can prove that

$$\iiint\limits_{\mathbb{R}^-\ \mathbb{R}\ \mathbb{R}} \frac{4\nu^2(1+\omega^2)^{3/2}}{|p + \delta^{1/2} + 2\nu i(\omega + bk)q|^2} e^{2\mathcal{R}e(\sigma^+)x}\left|\hat{\tilde{\mathsf{g}}}(k,\omega)\right|^2 \mathrm{d}x\,\mathrm{d}k\,\mathrm{d}\omega < \infty, \text{ i.e.,}$$

$$\frac{(1+\omega^2)^{3/4}}{(\mathcal{R}e(\sigma^+))^{1/2}|p + \delta^{1/2} + 2\nu i(\omega + bk)q|}\left|\hat{\tilde{\mathsf{g}}}\right| \in L^2(\mathbb{R}\times\mathbb{R}). \tag{32}$$

By Lemma 13, there are two positive constants C_1 and C_3 such that $\phi_1 \leqslant C_1$ and $\phi_3 \leqslant C_3$, so we deduce that

$$\frac{(1+\omega^2)^{3/4}|\hat{\tilde{\mathsf{g}}}(k,\omega)|}{|p + \delta^{1/2} + 2\nu i(\omega + bk)q|(\mathcal{R}e(\sigma^+))^{1/2}} \leqslant C_1C_3\bigl(1+\omega^2\bigr)^{1/8}\left|\hat{\tilde{\mathsf{g}}}(k,\omega)\right|.$$

Since $\tilde{g}$ is in $H^{1/4}(\mathbb{R}^+; L^2(\mathbb{R}))$ we have $\tilde{\mathsf{g}}$ in $H^{1/4}(\mathbb{R}; L^2(\mathbb{R}))$ (see Theorem 12) and we obtain (32). Therefore e_- is in $L^2(\mathbb{R}^-; H^{3/2}(\mathbb{R}; L^2(\mathbb{R})))$. □

Lemma 16. *If g is given as in* (24)–(26), *then for p and q strictly positive the solution of* (31), *e_-, is in $L^2(\mathbb{R}^-; L^2(\mathbb{R}^+; H^3(\mathbb{R})))$.*

Proof. We have to prove that

$$\iiint\limits_{\mathbb{R}^-\ \mathbb{R}\ \mathbb{R}} \frac{4\nu^2(1+k^2)^3}{|p + \delta^{1/2} + 2\nu i(\omega + bk)q|^2} e^{2\mathcal{R}e(\sigma^+)x}\left|\hat{\tilde{\mathsf{g}}}(k,\omega)\right|^2 \mathrm{d}x\,\mathrm{d}k\,\mathrm{d}\omega < \infty, \text{ i.e.,}$$

$$\frac{(1+k^2)^{3/2}}{(\mathcal{R}e(\sigma^+))^{1/2}|p + \delta^{1/2} + 2\nu i(\omega + bk)q|}\left|\hat{\tilde{\mathsf{g}}}\right| \in L^2(\mathbb{R}\times\mathbb{R}). \tag{33}$$

Using decomposition (24), we rewrite $\tilde{g}$ as

$$\tilde{g} = \left(-\frac{a-p}{2\nu} + q\frac{\partial}{\partial t} + bq\frac{\partial}{\partial y}\right)h_0 + h_1, \tag{34}$$

where $h_0 = g_0 - w_{|\Gamma}$ is in $H^{5/2,5/4}(\mathbb{R}\times\mathbb{R}^+)$ and $h_1 = g_1 - \partial_x w|\Gamma$ is in $H^{3/2,3/4}(\mathbb{R}\times\mathbb{R}^+)$. Since the compatibility conditions (26) are satisfied, h_0 and h_1 vanish at $t = 0$. On the one hand we conclude that the extension by 0 for $t < 0$ of h_0 and h_1 (denoted by h_0 and h_1) are respectively in $H^{5/2,5/4}(\mathbb{R}\times\mathbb{R})$

and $H^{3/2,3/4}(\mathbb{R}\times\mathbb{R})$ (see Theorem 12). On the other hand Fourier transform of $(-(a-p)/2\nu+q\partial_t+bq\partial_y)h_0$ is $(-(a-p)/2\nu+iq(\omega+bk))\hat{\mathsf{h}}_0$. Then we can write the relation $|\hat{\hat{\mathsf{g}}}|\leqslant|-(a-p)/2\nu+iq(\omega+bk)||\hat{\mathsf{h}}_0|+|\hat{\mathsf{h}}_1|$ and we prove (33) in two steps.

(i) Using Lemma 13 we have two strictly positive constants C_2 and C_5 such that $\phi_2\leqslant C_2$, and $\phi_5\leqslant C_5$, and thus we get

$$\frac{(1+k^2)^{3/2}|\frac{-a+p}{2\nu}+iq(\omega+bk)||\hat{\mathsf{h}}_0(k,\omega)|}{(\mathcal{R}e(\sigma^+))^{1/2}|p+\delta^{1/2}+2\nu i(\omega+bk)q|}\leqslant C_2C_5\big(1+k^2\big)^{5/4}\big|\hat{\mathsf{h}}_0(k,\omega)\big|.$$

Since h_0 is in $H^{5/2,5/4}(\mathbb{R}\times\mathbb{R})$ we get the first part of the result.

(ii) Similarly there are two constants C_2 and C_4 such as $\phi_2\leqslant C_2$ and $\phi_4\leqslant C_4$ and then we get

$$\frac{(1+k^2)^{3/2}|\hat{\mathsf{h}}_1(k,\omega)|}{(\mathcal{R}e(\sigma^+))^{1/2}|p+\delta^{1/2}+2\nu i(\omega+bk)q|}\leqslant C_2C_4\big(1+k^2\big)^{3/4}\big|\hat{\mathsf{h}}_1(k,\omega)\big|.$$

Since h_1 is in $H^{3/2,3/4}(\mathbb{R}\times\mathbb{R})$ we get the second part of the result.

Points (i) and (ii) prove (33) and therefore e_- is in $L^2(\mathbb{R}^-;L^2(\mathbb{R};H^3(\mathbb{R})))$. $\square$

Lemma 17. *If g is given as in* (24)–(26)*, then for p and q strictly positive the solution of* (31)*, e_-, is in* $H^3(\mathbb{R}^-;L^2(\mathbb{R}\times\mathbb{R}^+))$.

Proof. We have to show that e_-, $\partial_x e_-$, $\partial_{xx}e_-$ and $\partial_{x^3}e_-$ are in $L^2(\mathbb{R}^-;L^2(\mathbb{R}\times\mathbb{R}^+))$, which resumes to prove that

$$\frac{|\sigma^+|^j|\hat{\hat{\mathsf{g}}}|}{(\mathcal{R}e(\sigma^+))^{1/2}|p+\delta^{1/2}+2\nu i(\omega+bk)q|}\in L^2(\mathbb{R}\times\mathbb{R})\quad\text{for } j=1,2\text{ and }3. \tag{35}$$

We prove the result for $j=3$. As in the proof of Lemma 16, we use (34) and prove the result in two steps.

(i) Lemma 13 gives existence of two constants C_6 and C_5 such that $\phi_6\leqslant C_6$ and $\phi_5\leqslant C_5$ and thus we get

$$\frac{|\sigma^+|^3|\frac{a+p}{2\nu}+iq(\omega+bk)||\hat{\mathsf{h}}_0|}{(\mathcal{R}e(\sigma^+))^{1/2}|p+\delta^{1/2}+2\nu i(\omega+bk)q|}\leqslant C_6C_5\Big(\big(1+k^2\big)^{5/4}+\big(1+\omega^2\big)^{5/8}\Big)\big|\hat{\mathsf{h}}_0\big|.$$

Since h_0 is in $H^{5/2,5/4}(\mathbb{R}\times\mathbb{R})$, we get the first part of the result.

(ii) Lemma 13 gives existence of two constants C_6 and C_7 such that $\phi_6\leqslant C_6$ and $\phi_7\leqslant C_7$ and we get

$$\begin{aligned}&\frac{|\sigma^+|^3|\hat{\mathsf{h}}_1|}{(\mathcal{R}e(\sigma^+))^{1/2}|p+\delta^{1/2}+2\nu i(\omega+bk)q|}\\&\quad\leqslant C_6\frac{((1+k^2)^{5/4}+(1+\omega^2)^{5/8})}{|p+\delta^{1/2}+2\nu i(\omega+bk)q|}\big|\hat{\mathsf{h}}_1\big|\leqslant C_6C_7\Big(\big(1+k^2\big)^{3/4}+\big(1+\omega^2\big)^{3/8}\Big)\big|\hat{\mathsf{h}}_1\big|.\end{aligned}$$

Since h_1 is in $H^{3/2,3/4}(\mathbb{R}\times\mathbb{R})$ we get the second part of the result.

Applying the same idea for e_-, $\partial_x e_-$ and $\partial_{xx} e_-$, we get that the solution e_- is in $H^3(\mathbb{R}^-; L^2(\mathbb{R} \times \mathbb{R}^+))$. $\square$

Theorem 18. *Suppose that f is in $H^{1,1/2}(\Omega \times \mathbb{R}^+)$, w_0 in $H^2(\Omega)$ and $p, q > 0$. If g is given as in* (24)–(26)*, then the solution of* (23)*, u, is in $H^{3,3/2}(\Omega^- \times \mathbb{R}^+)$.*

Proof. The proof follows by induction from Lemmas 15, 16 and 17 which give $e_- = w|_{\Omega^-} - u$ in $H^{3,3/2}(\Omega^- \times \mathbb{R}^+)$ with $w|_{\Omega^-}$ in $H^{3,3/2}(\Omega^- \times \mathbb{R}^+)$. $\square$

6.3. The algorithm

If $\mathcal{B}^\pm = \mathcal{B}_1^\pm$, algorithm (1) is initialized by

$$\begin{aligned} &\mathcal{L}u^0 = f \quad \text{in } \Omega^- \times \mathbb{R}^+, \qquad u^0(\cdot,\cdot,0) = w_0 \quad \text{in } \Omega^-, \qquad \mathcal{B}_1^- u^0 = g \quad \text{on } \Gamma \times \mathbb{R}^+, \\ &\mathcal{L}v^0 = f \quad \text{in } \Omega^+ \times \mathbb{R}^+, \qquad v^0(\cdot,\cdot,0) = w_0 \quad \text{in } \Omega^+, \qquad \mathcal{B}_1^+ v^0 = h \text{ on } \Gamma \times \mathbb{R}^+, \end{aligned} \tag{36}$$

where g satisfies (24)–(26) and h satisfies the corresponding conditions in Ω^+.

Theorem 19. *Suppose that f is in $H^{1,1/2}(\Omega \times \mathbb{R}^+)$, w_0 in $H^2(\Omega)$ and $p, q > 0$. If $\mathcal{B}^\pm = \mathcal{B}_1^\pm$, then algorithm* (1) *initialized by* (36) *defines a unique sequence of iterates (u^n, v^n) in $H^{3,3/2}(\Omega^- \times \mathbb{R}^+) \times H^{3,3/2}(\Omega^+ \times \mathbb{R}^+)$.*

Proof. Problem (36) defines by Theorems 14 and 18 a unique solution (u^0, v^0) in $H^{3,3/2}(\Omega^- \times \mathbb{R}^+) \times H^{3,3/2}(\Omega^+ \times \mathbb{R}^+)$. On the other hand, if the solution of (1) in Ω^+, v^n, is in $H^{3,3/2}(\Omega^- \times \mathbb{R}^+)$, then according to the trace Theorem 3 we have $g_0 = v^n(0,\cdot,\cdot)$ in $H^{5/2,5/4}(\mathbb{R} \times \mathbb{R}^+)$, $g_1 = \partial_x v^n(0,\cdot,\cdot)$ in $H^{3/2,3/4}(\mathbb{R} \times \mathbb{R}^+)$ with $g_0(\cdot,0) = w_0(0,\cdot)$ and $g_1(\cdot,0) = \partial_x w_0(0,\cdot)$. Thus $g = g_1 + (-(a-p)/2\nu + q\partial_t + bq\partial_y)g_0$ satisfies the assumptions of Theorems 14 and 18. As a consequence, u^{n+1} is defined in $H^{3,3/2}(\Omega^- \times \mathbb{R}^+)$. The same holds in Ω^+. Iterating this process completes the proof. $\square$

Remark 20. To start the algorithm, it suffices to introduce a w^{init} in $H^{3,3/2}(\Omega \times \mathbb{R}^+)$ such that $w^{\text{init}}(\cdot,\cdot,0) = w_0$ in Ω. Right sides $g = \mathcal{B}_1^- w^{\text{init}}(0,\cdot,\cdot)$ and $h = \mathcal{B}_1^+ w^{\text{init}}(0,\cdot,\cdot)$ satisfy then assumptions of Theorems 14 and 18.

7. Convergence of the algorithms

In this section we prove the convergence of algorithm (1) when $\mathcal{B}^\pm$ are $\mathcal{B}_0^\pm$ or $\mathcal{B}_1^\pm$. In order to get this convergence result, we generalize the proof of [15] to the unsteady case.

In the following we make use of the operator

$$\mathcal{S}^s = -\frac{1}{\nu}\left(\frac{\partial}{\partial t} + b\frac{\partial}{\partial y} - \nu\frac{\partial^2}{\partial y^2} + c\right). \tag{37}$$

We reformulate the convection diffusion operator $\mathcal{L}$ using the differential operators defined in (11). Note that $\mathcal{S}_j^\pm$ are differential operators with constant coefficients, so they commute. Note also that $\mathcal{S}_j^+ + \mathcal{S}_j^- = a/\nu$. We have

$$\mathcal{L} = -\nu \frac{\partial^2}{\partial x^2} + a\frac{\partial}{\partial x} - \nu \mathcal{S}^s,$$
$$\mathcal{L} = -\nu\left(\frac{\partial}{\partial x} - \mathcal{S}_j^+\right)\left(\frac{\partial}{\partial x} - \mathcal{S}_j^-\right) + \nu\left(\mathcal{S}_j^+\mathcal{S}_j^- - \mathcal{S}^s\right), \tag{38}$$
$$\mathcal{L} = -\nu\left(\frac{\partial}{\partial x} - \mathcal{S}_j^-\right)\left(\frac{\partial}{\partial x} - \mathcal{S}_j^+\right) + \nu\left(\mathcal{S}_j^+\mathcal{S}_j^- - \mathcal{S}^s\right). \tag{39}$$

We first introduce a lemma that will be useful in the proof of Theorems 22 and 23.

Lemma 21. *If u is a solution of $\mathcal{L}u = 0$ in $\Omega^- \times \mathbb{R}^+$, then for $j = 0$ and $j = 1$, we have the estimate*

$$\frac{\nu}{2}\int_\Gamma \left|\frac{\partial u}{\partial x} - \mathcal{S}_j^+ u\right|^2 \mathrm{d}y - \frac{\nu}{2}\int_\Gamma \left|\frac{\partial u}{\partial x} - \mathcal{S}_j^- u\right|^2 \mathrm{d}y + \nu\left(\left(\mathcal{S}_j^+ - \mathcal{S}_j^-\right)\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right)_{\Omega^-}$$
$$+ a\left(\frac{\partial u}{\partial x}, \mathcal{S}_j^+ u\right)_{\Omega^-} - a\left(\mathcal{S}_j^+\frac{\partial u}{\partial x}, u\right)_{\Omega^-} - \nu\left(\mathcal{S}^s u, \left(\mathcal{S}_j^+ - \mathcal{S}_j^-\right)u\right)_{\Omega^-} = 0. \tag{40}$$

Proof. Multiplying the equation $\mathcal{L}u = 0$ in $\Omega^- \times \mathbb{R}^+$ by $\partial_x u - \mathcal{S}_j^- u$ using (38) and integrating on Ω^- yields

$$-\frac{\nu}{2}\int_\Gamma \left|\frac{\partial u}{\partial x} - \mathcal{S}_j^- u\right|^2 \mathrm{d}y + \nu\left(\mathcal{S}_j^+\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right)_{\Omega^-} - \nu\left(\mathcal{S}_j^+\mathcal{S}_j^- u, \frac{\partial u}{\partial x}\right)_{\Omega^-}$$
$$- \nu\left(\mathcal{S}_j^+\frac{\partial u}{\partial x}, \mathcal{S}_j^- u\right)_{\Omega^-} + \nu\left(\mathcal{S}_j^+\mathcal{S}_j^- u, \mathcal{S}_j^- u\right)_{\Omega^-} + \nu\left(\left(\mathcal{S}_j^+\mathcal{S}_j^- u - \mathcal{S}^s u\right), \frac{\partial u}{\partial x}\right)_{\Omega^-}$$
$$- \nu\left(\left(\mathcal{S}_j^+\mathcal{S}_j^- u - \mathcal{S}^s u\right), \mathcal{S}_j^- u\right)_{\Omega^-} = 0. \tag{41}$$

Note that the sum of terms with $\mathcal{S}_j^+\mathcal{S}_j^-$ vanishes. In the same way we multiply the equation by $-(\partial_x u - \mathcal{S}_j^+ u)$ using the expression (39) and we integrate on Ω^-. Adding the result to (41) yields

$$\frac{\nu}{2}\int_\Gamma \left|\frac{\partial u}{\partial x} - \mathcal{S}_j^+ u\right|^2 \mathrm{d}y - \frac{\nu}{2}\int_\Gamma \left|\frac{\partial u}{\partial x} - \mathcal{S}_j^- u\right|^2 \mathrm{d}y + \nu\left(\left(\mathcal{S}_j^+ - \mathcal{S}_j^-\right)\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right)_{\Omega^-}$$
$$+ \nu\left(\mathcal{S}_j^-\frac{\partial u}{\partial x}, \mathcal{S}_j^+ u\right)_{\Omega^-} - \nu\left(\mathcal{S}_j^+\frac{\partial u}{\partial x}, \mathcal{S}_j^- u\right)_{\Omega^-} - \nu\left(\mathcal{S}^s u, \left(\mathcal{S}_j^+ - \mathcal{S}_j^-\right)u\right)_{\Omega^-} = 0. \tag{42}$$

On the other hand, we use the relation $\mathcal{S}_j^+ + \mathcal{S}_j^- = a/\nu$ and we obtain the result. □

7.1. Algorithm with interface conditions of order zero

The following theorem proves convergence of algorithm (1) with interface conditions (12).

Theorem 22. *Suppose that f is in $L^2(\mathbb{R}^+; L^2(\Omega))$ and w_0 in $H^1(\Omega)$. If $\mathcal{B}^\pm = \mathcal{B}_0^\pm$, then for $p > 0$ algorithm* (1) *initialized by* (21) *converges in $L^\infty(\mathbb{R}^+; L^2(\Omega^-)) \cap L^2(\mathbb{R}^+; H^1(\Omega^-)) \times L^\infty(\mathbb{R}^+; L^2(\Omega^+)) \cap L^2(\mathbb{R}^+; H^1(\Omega^+))$ to the solution of* (2).

Proof. The error at step $n+1$ and in Ω^- of algorithm (1), e_-^{n+1}, satisfies the homogeneous equation $\mathcal{L}e_-^{n+1}=0$ in $\Omega^-\times\mathbb{R}^+$. We can therefore apply Lemma 21 and the variational formulation (40) is satisfied by e_-^{n+1}. Note that $(\partial_x u, \mathcal{S}_0^+ u)_{\Omega^-}=(\mathcal{S}_0^+\partial_x u, u)_{\Omega^-}$ and $\mathcal{S}_0^+-\mathcal{S}_0^-=p/\nu$. Using (37), (40) becomes

$$\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_-^{n+1}}{\partial x}-\mathcal{S}_0^+e_-^{n+1}\right|^2\mathrm{d}y-\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_-^{n+1}}{\partial x}-\mathcal{S}_0^-e_-^{n+1}\right|^2\mathrm{d}y$$
$$+p\left\|\frac{\partial e_-^{n+1}}{\partial x}\right\|_{\Omega^-}^2+\frac{p}{2\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left\|e_-^{n+1}\right\|_{\Omega^-}^2+p\left\|\frac{\partial e_-^{n+1}}{\partial y}\right\|_{\Omega^-}^2+\frac{pc}{\nu}\left\|e_-^{n+1}\right\|_{\Omega^-}^2=0. \tag{43}$$

On the other hand, if e_+^n is the error in Ω^+ at step n then e_-^{n+1} satisfies the boundary condition on Γ: $\partial_x e_-^{n+1}-\mathcal{S}_0^-e_-^{n+1}=\partial_x e_+^n-\mathcal{S}_0^-e_+^n$. We therefore obtain

$$\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_-^{n+1}}{\partial x}-\mathcal{S}_0^+e_-^{n+1}\right|^2\mathrm{d}y+p\left\|\nabla e_-^{n+1}\right\|_{\Omega^-}^2$$
$$+\frac{p}{2\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left\|e_-^{n+1}\right\|_{\Omega^-}^2+\frac{pc}{\nu}\left\|e_-^{n+1}\right\|_{\Omega^-}^2=\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_+^n}{\partial x}-\mathcal{S}_0^-e_+^n\right|^2\mathrm{d}y. \tag{44}$$

Noting that the same holds in Ω^+, we add (44) to the corresponding equality in Ω^+ for $n\in\{0,\dots,N\}$. After simplifications we find

$$\sum_{n=0}^N\Bigg[\frac{p}{2\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left\|e_-^{n+1}\right\|_{\Omega^-}^2+\frac{p}{2\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left\|e_+^{n+1}\right\|_{\Omega^+}^2+\frac{pc}{\nu}\left\|e_-^{n+1}\right\|_{\Omega^-}^2+\frac{pc}{\nu}\left\|e_+^{n+1}\right\|_{\Omega^+}^2$$
$$+p\left\|\nabla e_-^{n+1}\right\|_{\Omega^-}^2+p\left\|\nabla e_+^{n+1}\right\|_{\Omega^+}^2\Bigg]$$
$$+\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_-^{N+1}}{\partial x}-\mathcal{S}_0^+e_-^{N+1}\right|^2\mathrm{d}y+\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_+^{N+1}}{\partial x}-\mathcal{S}_0^-e_+^{N+1}\right|^2\mathrm{d}y$$
$$=\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_-^0}{\partial x}-\mathcal{S}_0^+e_-^0\right|^2\mathrm{d}y+\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_+^0}{\partial x}-\mathcal{S}_0^-e_+^0\right|^2\mathrm{d}y. \tag{45}$$

After integration on $(0,t)$, (45) becomes $\sum_{n=0}^N(F_-^n+F_+^n)\leqslant C$, where C is $(\nu/2)(\int_0^t\int_\Gamma|\partial_x e_-^0-\mathcal{S}_0^+e_-^0|^2(0,y,\sigma)\,\mathrm{d}y\,\mathrm{d}\sigma+\int_0^t\int_\Gamma|\partial_x e_+^0-\mathcal{S}_0^-e_+^0|^2(0,y,\sigma)\,\mathrm{d}y\,\mathrm{d}\sigma)$, and $F_\pm^n=\min(\frac{p}{2\nu},\frac{pc}{\nu},p)(\|e_\pm^{n+1}\|_{\Omega^-}^2(t)+\int_0^t(\|e_\pm^{n+1}\|_{\Omega^-}^2(\sigma)+\|\nabla e_\pm^{n+1}\|_{\Omega^-}^2(\sigma))\,\mathrm{d}\sigma)$. Since C does not depend on N, both series $\sum_{n=0}^\infty F_-^n$ and $\sum_{n=0}^\infty F_+^n$ are convergent, and therefore F_-^n and F_+^n tend to 0. We get convergence of the algorithm in the spaces specified by the theorem. □

7.2. *Algorithm with first order interface conditions*

The following theorem proves convergence of algorithm (1) with interface conditions (22).

Theorem 23. *Suppose that f is in $H^{1,1/2}(\Omega\times\mathbb{R}^+)$ and w_0 in $H^2(\Omega)$. If $\mathcal{B}^\pm=\mathcal{B}_1^\pm$ and if $q>0$ and $p-a^2q/2>0$ then algorithm* (1) *initialized by* (36)*, converges in $L^\infty(\mathbb{R}^+;H^1(\Omega^-))\cap L^2(\mathbb{R}^+;H^1(\Omega^-))\times L^\infty(\mathbb{R}^+;H^1(\Omega^+))\cap L^2(\mathbb{R}^+;H^1(\Omega^+))$ to the solution of* (2).

Proof. We first write the relations

$$\left(\frac{\partial u}{\partial x},\mathcal{S}_1^+u\right)_{\Omega^-}-\left(\mathcal{S}_1^+\frac{\partial u}{\partial x},u\right)_{\Omega^-}=2q\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)_{\Omega^-}+2bq\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)_{\Omega^-}-\frac{q}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma|u|^2\,\mathrm{d}y,$$

and

$$\begin{aligned}-\nu\big(\mathcal{S}^su,\big(\mathcal{S}_1^+-\mathcal{S}_1^-\big)u\big)_{\Omega^-}={}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{p}{2\nu}+cq\right)\|u\|^2_{\Omega^-}+q\nu\left\|\frac{\partial u}{\partial y}\right\|^2_{\Omega^-}\right]+2q\left\|\frac{\partial u}{\partial t}\right\|^2_{\Omega^-}\\&+\big(p+2b^2q\big)\left\|\frac{\partial u}{\partial y}\right\|^2_{\Omega^-}+4bq\left(\frac{\partial u}{\partial y},\frac{\partial u}{\partial t}\right)_{\Omega^-}+\frac{pc}{\nu}\|u\|^2_{\Omega^-}.\end{aligned}$$

Since the error of algorithm (1) in Ω^+, e_-^{n+1}, satisfies the homogeneous equation $\mathcal{L}e_-^{n+1}=0$, we can apply Lemma 21 and e_-^{n+1} satisfies (40) which reads

$$\begin{aligned}&\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_-^{n+1}}{\partial x}-\mathcal{S}_1^+e_-^{n+1}\right|^2\mathrm{d}y-\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_-^{n+1}}{\partial x}-\mathcal{S}_1^-e_-^{n+1}\right|^2\mathrm{d}y\\&\quad+\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{p}{2\nu}+cq\right)\|e_-^{n+1}\|^2_{\Omega^-}+q\nu\left\|\frac{\partial e_-^{n+1}}{\partial x}\right\|^2_{\Omega^-}+q\nu\left\|\frac{\partial e_-^{n+1}}{\partial y}\right\|^2_{\Omega^-}-\frac{aq}{2}\int_\Gamma\left|e_-^{n+1}\right|^2\mathrm{d}y\right]\\&\quad+p\left\|\frac{\partial e_-^{n+1}}{\partial x}\right\|^2_{\Omega^-}+\big(p+2b^2q\big)\left\|\frac{\partial e_-^{n+1}}{\partial y}\right\|^2_{\Omega^-}+2q\left\|\frac{\partial e_-^{n+1}}{\partial t}\right\|^2_{\Omega^-}+\frac{pc}{\nu}\|e_-^{n+1}\|^2_{\Omega^-}\\&\quad+2aq\left(\frac{\partial e_-^{n+1}}{\partial x},\frac{\partial e_-^{n+1}}{\partial t}\right)_{\Omega^-}+2abq\left(\frac{\partial e_-^{n+1}}{\partial x},\frac{\partial e_-^{n+1}}{\partial y}\right)_{\Omega^-}+4bq\left(\frac{\partial e_-^{n+1}}{\partial y},\frac{\partial e_-^{n+1}}{\partial t}\right)_{\Omega^-}=0.\end{aligned}\tag{46}$$

The last three terms in the left side of (46) can be written as

$$\begin{aligned}&\left\|\sqrt{2q}\frac{\partial e_-^{n+1}}{\partial t}+b\sqrt{2q}\frac{\partial e_-^{n+1}}{\partial y}+\frac{a}{2}\sqrt{2q}\frac{\partial e_-^{n+1}}{\partial x}\right\|^2_{\Omega^-}-2q\left\|\frac{\partial e_-^{n+1}}{\partial t}\right\|^2_{\Omega^-}\\&\quad-2qb^2\left\|\frac{\partial e_-^{n+1}}{\partial y}\right\|^2_{\Omega^-}-\frac{a^2}{2}q\left\|\frac{\partial e_-^{n+1}}{\partial x}\right\|^2_{\Omega^-}.\end{aligned}$$

Finally, we get

$$\begin{aligned}&\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_-^{n+1}}{\partial x}-\mathcal{S}_1^+e_-^{n+1}\right|^2\mathrm{d}y+\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{p}{2\nu}+cq\right)\|e_-^{n+1}\|^2_{\Omega^-}+q\nu\|\nabla e_-^{n+1}\|^2_{\Omega^-}-\frac{aq}{2}\int_\Gamma\left|e_-^{n+1}\right|^2\mathrm{d}y\right]\\&\quad+\left(p-\frac{a^2q}{2}\right)\left\|\frac{\partial e_-^{n+1}}{\partial x}\right\|^2_{\Omega^-}+p\left\|\frac{\partial e_-^{n+1}}{\partial y}\right\|^2_{\Omega^-}+\frac{pc}{\nu}\|e_-^{n+1}\|^2_{\Omega^-}\\&\leqslant\frac{\nu}{2}\int_\Gamma\left|\frac{\partial e_-^{n+1}}{\partial x}-\mathcal{S}_1^-e_-^{n+1}\right|^2\mathrm{d}y.\end{aligned}$$

On the other hand, if e_+^n is the error in Ω^+, e_-^{n+1} satisfies the boundary condition $\partial_x e_-^{n+1} - \mathcal{S}_1^- e_-^{n+1} = \partial_x e_+^n - \mathcal{S}_1^- e_+^n$ on $\Gamma \times \mathbb{R}^+$. Then, after integration in time over $(0,t)$, we get

$$
\begin{aligned}
&\frac{\nu}{2}\int_0^t \int_\Gamma \left| \frac{\partial e_-^{n+1}}{\partial x} - \mathcal{S}_1^+ e_-^{n+1} \right|^2 (0,y,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}y \\
&\quad + \left(\frac{p}{2\nu} + cq\right) \left\| e_-^{n+1} \right\|_{\Omega^-}^2 (t) + q\nu \left\| \nabla e_-^{n+1} \right\|_{\Omega^-}^2 (t) - \frac{aq}{2}\int_\Gamma \left| e_-^{n+1} \right|^2 (0,y,t)\,\mathrm{d}y \\
&\quad + \left(p - \frac{a^2 q}{2}\right) \int_0^t \left\| \frac{\partial e_-^{n+1}}{\partial x} \right\|_{\Omega^-}^2 (\sigma)\,\mathrm{d}\sigma + p \int_0^t \left\| \frac{\partial e_-^{n+1}}{\partial y} \right\|_{\Omega^-}^2 (\sigma)\,\mathrm{d}\sigma + \frac{pc}{\nu}\int_0^t \left\| e_-^{n+1} \right\|_{\Omega^-}^2 (\sigma)\,\mathrm{d}\sigma \\
&\leqslant \frac{\nu}{2}\int_0^t \int_\Gamma \left| \frac{\partial e_+^n}{\partial x} - \mathcal{S}_1^- e_+^n \right|^2 (0,y,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}y.
\end{aligned} \tag{47}
$$

Proceeding the same way in domain Ω^+, we obtain the relation

$$
\begin{aligned}
&\frac{\nu}{2}\int_0^t \int_\Gamma \left| \frac{\partial e_+^{n+1}}{\partial x} - \mathcal{S}_1^- e_+^{n+1} \right|^2 (0,y,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}y \\
&\quad + \left(\frac{p}{2\nu} + cq\right) \left\| e_+^{n+1} \right\|_{\Omega^+}^2 (t) + q\nu \left\| \nabla e_+^{n+1} \right\|_{\Omega^+}^2 (t) + \frac{aq}{2}\int_\Gamma \left| e_+^{n+1} \right|^2 (0,y,t)\,\mathrm{d}y \\
&\quad + \left(p - \frac{a^2 q}{2}\right) \int_0^t \left\| \frac{\partial e_+^{n+1}}{\partial x} \right\|_{\Omega^+}^2 (\sigma)\,\mathrm{d}\sigma + p \int_0^t \left\| \frac{\partial e_+^{n+1}}{\partial y} \right\|_{\Omega^+}^2 (\sigma)\,\mathrm{d}\sigma + \frac{pc}{\nu}\int_0^t \left\| e_+^{n+1} \right\|_{\Omega^+}^2 (\sigma)\,\mathrm{d}\sigma \\
&\leqslant \frac{\nu}{2}\int_0^t \int_\Gamma \left| \frac{\partial e_-^n}{\partial x} - \mathcal{S}_1^+ e_-^n \right|^2 (0,y,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}y.
\end{aligned} \tag{48}
$$

By assumption we have $p - a^2q/2 > 0$. We use then the Sobolev inequality for any $\alpha > 0$: $\int_\Gamma |e_-|^2\,\mathrm{d}y \leqslant \alpha \|e_-\|_{\Omega^-}^2 + (1/\alpha)\|\partial_x e_-\|_{\Omega^-}^2$, in order to give a lower bound to $-(aq/2)\int_\Gamma |e_-^n|^2(0,y,t)\,\mathrm{d}y$ if $a > 0$ (or to $(aq/2)\int_\Gamma |e_+^n|^2(0,y,t)\,\mathrm{d}y$ if $a < 0$). Note that we can find $\alpha > 0$ such that $p/2\nu + cq - \alpha(aq/2) \geqslant 0$ and $q\nu - aq/2\alpha \geqslant 0$ since $p - a^2q/2 > 0$.

Then adding inequalities (47) and (48) for $n \in \{0,\dots,N\}$ and letting N tend to infinity, we get that the error e_-^n tends to 0 in $L^\infty(\mathbb{R}^+; H^1(\Omega^-)) \cap L^2(\mathbb{R}^+; H^1(\Omega^-))$ and e_+^n tends to 0 in $L^\infty(\mathbb{R}^+; H^1(\Omega^+)) \cap L^2(\mathbb{R}^+; H^1(\Omega^+))$ (see proof of Theorem 22). □

8. The choice of interface conditions

In Section 4.1 we have introduced an optimal domain decomposition algorithm: we have defined the operators $\mathcal{S}^\pm$ on the interface which lead to convergence in two iterations. But due to their non-local nature, we have replaced them by the differential operators $\mathcal{S}_j^{\pm}$, $j = 0, 1$. This section is devoted to the choice of constants p and q in order to get fast convergence; we present here two strategies.

First, in [15,3], for example, the symbol of the Dirichlet-to-Neumann operator has been approximated by a Taylor approximation for low frequencies. Applied to our equation, the expansion of $\delta^{1/2}$ (see (7)) leads to $p = \sqrt{a^2 + 4\nu c}$ for the zeroth order approximation and to $p = \sqrt{a^2 + 4\nu c}$ and $q = \frac{1}{\sqrt{a^2+4\nu c}}$ for the first order operator. This strategy may however not be the best one for domain decomposition methods when also high frequencies are present, or when there is no overlap. Thus, in order to get a better approximation, we prefer to use a strategy which takes all the frequencies into account. A rapid method means that the convergence rate of the algorithm is small, so we compute numerically the p and q which minimize the convergence rate. For the steady convection diffusion equation in two dimensions this strategy has been applied in [10]: the convergence rate was optimized over the spatial frequencies (corresponding to the tangential variable), resulting in a very efficient method. For time dependent equations in one dimension, the optimization was applied to the time frequencies in [5]. We propose here an optimization over both variables: time and spatial frequencies.

8.1. Convergence rate

In Section 3.3, we have introduced the convergence rate of the overlapping Schwarz waveform relaxation algorithm. Similarly we introduce the convergence rate of algorithm (1) with interface operators (12) or (22). To begin with, we consider the algorithm satisfied by the errors. The error in Ω^-, e_-^n, and the error in Ω^+, e_+^n, satisfy the homogeneous equation and as a consequence they can be written as in (8). We introduce then (8) in the boundary conditions of algorithm (1) with $\mathcal{B}^{\pm} = \mathcal{B}_j^{\pm}$. We find $\alpha^{n+1}(\sigma^+ - \sigma_j^-) = \beta^n(\sigma^- - \sigma_j^-)$ and $\beta^{n+1}(\sigma^- - \sigma_j^+) = \alpha^n(\sigma^+ - \sigma_j^+)$. After simplifications and use of the relation $\sigma^+ + \sigma^- = \sigma_j^+ + \sigma_j^- = a/\nu$, we get the convergence rate

$$\rho_j = \left|\frac{\alpha^{n+1}}{\alpha^{n-1}}\right| = \left|\frac{\sigma^- - \sigma_j^-}{\sigma^+ - \sigma_j^-}\right|\left|\frac{\sigma^+ - \sigma_j^+}{\sigma^- - \sigma_j^+}\right| = \left|\frac{\sigma^+(k,\omega) - \sigma_j^+}{\sigma^-(k,\omega) - \sigma_j^+}\right|^2.$$

8.2. Optimization of the convergence rate

As an example we study the case when the approximation is of order 0 (i.e., $\sigma_0^{\pm} = \frac{a \pm p}{2\nu}$) and where $\nu = 0.01$, $a = b = 1$, $c = 0$. Time and space steps are 10^{-2}. Fig. 2 shows the maximum over all the

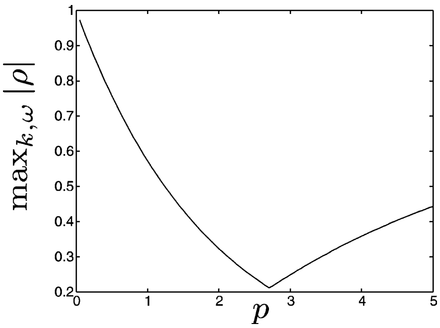

Fig. 2. $\max_{k,\omega} |\rho|$ as a function of p.

frequencies of ρ as a function of the parameter p. We can see that there is a $p = p_{\mathrm{opt}}$ for which the convergence rate is minimum. With the aim to get fast convergence (that means the lowest convergence rate) for all frequencies, we choose then this p_{opt} as a value of p. So in the case of approximation of order 0, we are led to solve the problem $\min_{p>0} \max_{k,\omega\in D} \rho_0$, and we solve $\min_{q>0,\, p-\frac{a^2}{2}q>0} \max_{k,\omega\in D} \rho_1$, in the case of first order approximation. D is the domain of discrete frequencies: in the discretized problem not all spatial frequencies are represented; they are bounded by the dimension L of the domain and the size $\mathrm{d}x$ of a cell. Similarly, time frequencies are bounded by the time step $\mathrm{d}t$ and the final time T. In fact it is shown in [4] that $|k| \in [\pi/L, \pi/\mathrm{d}x]$ and $|\omega| \in [\pi/T, \pi/\mathrm{d}t]$. In [6] the minimization problems have been solved analytically. Our problem in two dimensions is more difficult, thus we solve the minmax problems with a numerical method of type Nelder–Mead simplex (we used a Matlab function).

9. Numerical results

In the following Ω denotes the unit square $\Omega = \{(x, y),\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$. The scheme used to solve the equation in each subdomain is the Crank Nicolson scheme. Time and space steps are 10^{-2}.

9.1. Comparison Dirichlet/optimized methods

We solve here the equation $\mathcal{L}u = 0$ on $\Omega \times\,]0, 1[$ for the boundary problem described on Fig. 3. The initial condition is $u_0 = \exp(-100((x - 0.25)^2 + (y - 0.5)^2))$ and the physical data are $\nu = 0.01$, $a = 0.1$, $b = 0.1$, $c = 0$. The interface between the two subdomains is at $x = 0.5$ and there is an overlap of length Δx (the space step).

Fig. 4 shows sections of the three first iterates of the algorithm using either optimized of order 0 or Dirichlet interface conditions. The reason of the slow convergence of the overlapping Schwarz waveform relaxation algorithm is clearly seen on the figure: the convection and diffusion of the information are inhibited across the interface. See [5] for first results in one dimension.

In the two next sections we compare the methods with Taylor or optimized conditions, in the case where the velocity is constant or rotating and when there is no overlap between the two subdomains. The

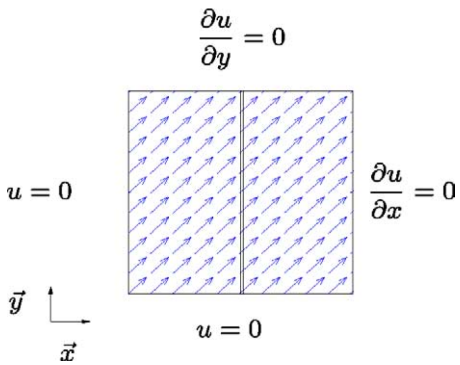

Fig. 3. Boundary problem.

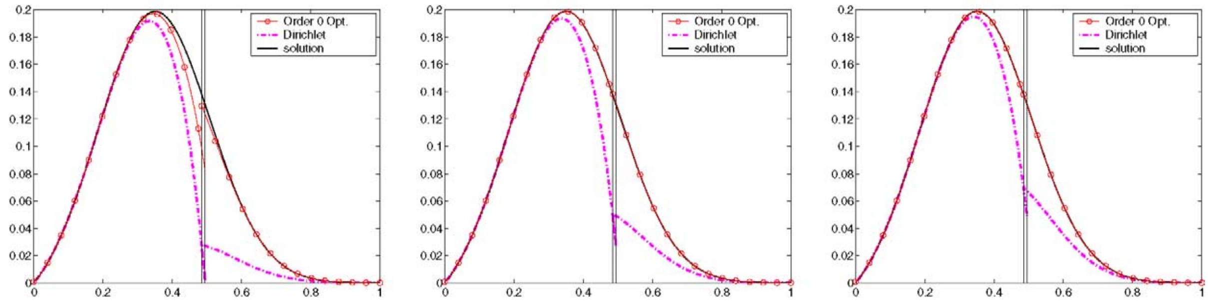

Fig. 4. Section of the three first iterates at the end of the time interval of the solution with zeroth order optimized interface condition and Dirichlet interface condition.

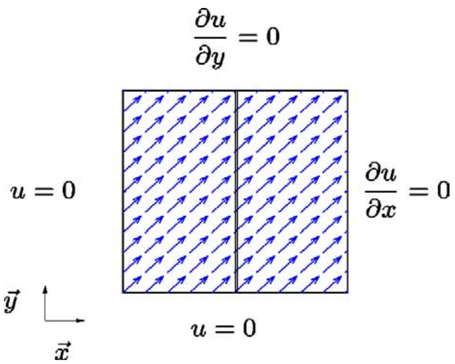

Fig. 5. Boundary problem.

Table 1
Optimized coefficients of order 0 and 1. $T = [0, 1]$

	Order 0				Order 1		
	$\nu = 0.01$	$\nu = 0.1$	$\nu = 1$		$\nu = 0.01$	$\nu = 0.1$	$\nu = 1$
p	2.7082	8.6642	63.5418	p	2.5371	8.6557	63.5413
				q	0.2278	0.07459	0.0079159

Taylor approximation for small values of k and ω is equivalent in some way to the approximation for small ν. We will thus see the influence of this parameter in the convergence of the algorithm.

9.2. *Optimized algorithm with constant velocity*

We consider the homogeneous convection diffusion equation $\mathcal{L}u = 0$ on $\Omega \times]0, 1[$ with the boundary conditions described on Fig. 5; the velocity is constant with $a = b = 1$ and $c = 0$. The initial condition is identically equal to 0, so the solution of this problem is 0 on Ω and thus computing the norm of the nth iterate of the algorithm is equivalent to computing the error at step n. The initial guess satisfies the compatibility conditions and the interface between the two non-overlapping subdomains is at $x = 0.5$. Table 1 gives the optimized coefficients for the methods of order 0 and 1 and for different values of ν.

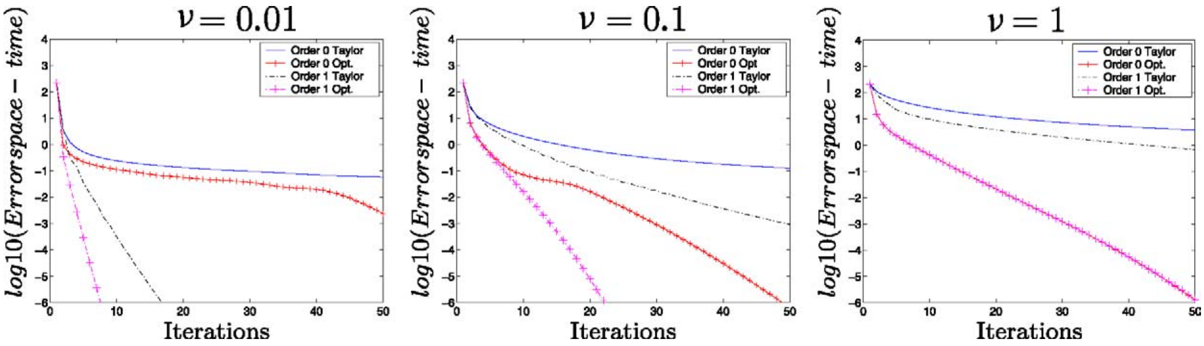

Fig. 6. Evolution of the error with respect to the iterations. Case of constant velocity.

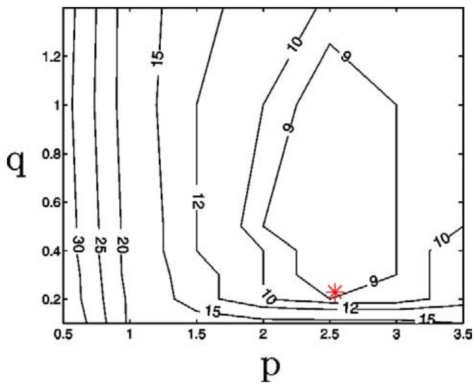

Fig. 7. Numerical and real optima compared.

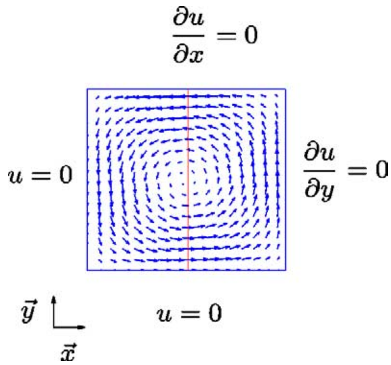

Fig. 8. Boundary problem.

Fig. 6 shows the error of the different algorithms with respect to the iterations for different values of ν. We can see that for a small viscosity ($\nu = 0.01$) the two best methods are the Taylor and optimized methods of order one. Then the larger the value of ν, the better are the optimized methods compared with the Taylor ones. The reason is that for large values of ν the Taylor expansion is not justified. We also remark that when $\nu = 1$ the optimized method of order zero and one tend to have the same performances. This is coherent with the fact that the optimized q tends toward 0 when ν increases (see Table 1). The optimized method of order 2 could be considered for high viscosities.

In Fig. 7 we varied the parameters p and q of the algorithm with transmission conditions of order 1, when $\nu = 0.01$ and compute the number of iterations needed to reach an error of 10^{-6}. The level lines represent the number of iterations and the red star represents the optimized parameters computed as described in Section 8.2. We can see how close is this value of the numerical optimum.

9.3. *Optimized algorithm with rotating velocity*

We now solve $\mathcal{L}u = 0$ on $\Omega \times]0, 1[$ in the case where the velocity is rotating: $a = -\cos(\pi(x - 0.5))\sin(\pi(y - 0.5))$, $b = \cos(\pi(y - 0.5))\sin(\pi(x - 0.5))$ and $c = 0$ (see Fig. 8). The decomposition of the domain Ω is the same as in the previous section. For the analysis we had assumed that the velocity field was constant. Here the velocity is a function of the ordinate and as a consequence the theorems

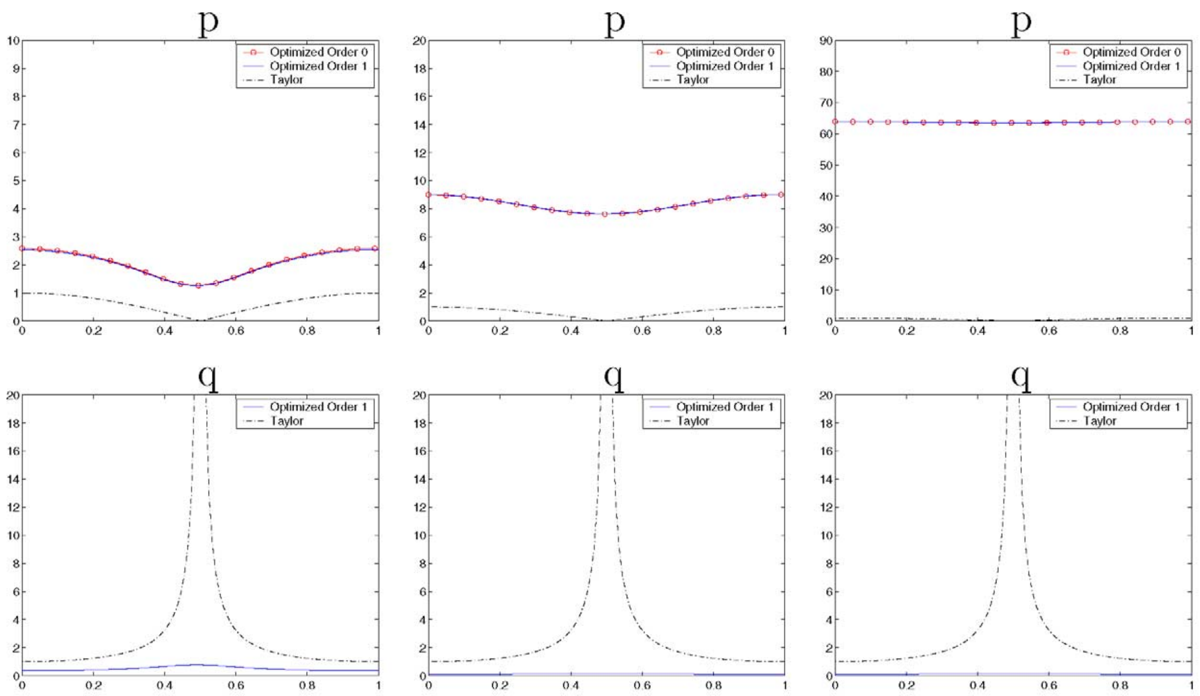

Fig. 9. Optimized and Taylor parameters as a function of the interface variable. From left to right: $\nu = 0.01$, $\nu = 0.1$ and $\nu = 1$.

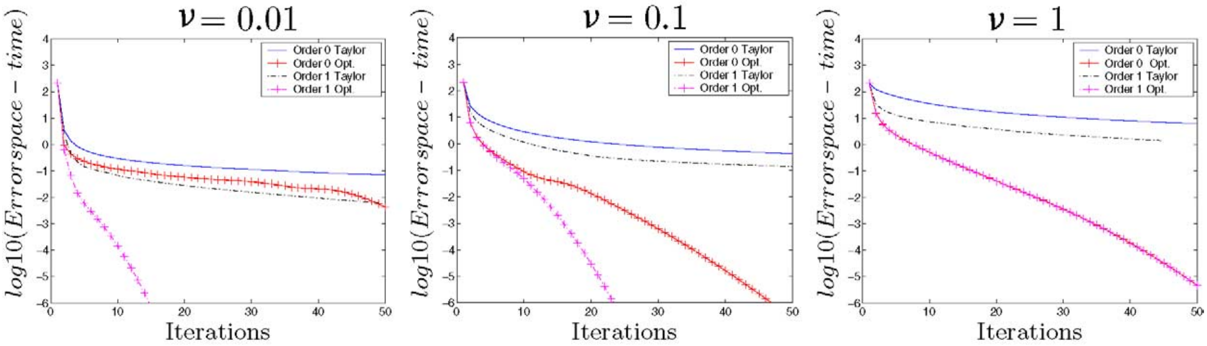

Fig. 10. Evolution of the error with respect to the iterations. Case of the rotating velocity.

are not valid. However, we can still apply our method to the rotating case. For the optimization of the coefficients we compute an optimized set of coefficients at each point of the interface. Fig. 9 shows these parameters: since p and q vary smoothly with the interface variables, a more economical solution would be to interpolate. On this figure we can see that the optimized parameters p for the method of order 0 and 1 are very closed. We also see that p increases with the viscosity whereas q decreases. See the singularity at $x = 0.5$ of the Taylor parameter q.

In Fig. 10 we can see again that the optimized methods are best by far when the value of ν is large, but still for small ν as well: for $\nu = 0.01$ the optimized method of order one is very efficient but conversely to the constant case, Taylor of order one is very bad. For high viscosities optimized method of order 2 could be considered.

10. Conclusion

In this paper we have proposed Schwarz waveform relaxation methods for the convection diffusion equation. When the velocity field is constant, we have proved well-posedness and convergence for our algorithms. Considering the convergence rate and minimizing it has led to very efficient methods. Numerical results have illustrated how well the optimized transmission conditions have improved the speed of convergence comparing to the classical methods. They have also shown that this method applies even when the velocity varies; in this case we compute a coefficient for each node of the interface and the numerical results have shown to be good. Optimized methods of order 2 could be considered for high viscosities.

For the sake of simplicity, we have applied this method to the case of two subdomains (two half-spaces) but it can be generalized to the multiple domains case, when Ω is decomposed into several strips. Well-posedness proofs are similar and convergence is obtained in adding the errors of each subdomain.

We can also treat more general interfaces. If $\mathbf{n}_1$ (respectively $\mathbf{n}_2$) denotes the outgoing normal vector to the interface in Ω^- (respectively Ω^+) and $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ the tangential vectors, then the interface operators become

$$\mathcal{B}^- = \left(\frac{\partial}{\partial \mathbf{n}_1} - \frac{\vec{\mathbf{b}}\cdot\mathbf{n}_1 - p}{2\nu} + q\left(\frac{\partial}{\partial t} + \vec{\mathbf{b}}\cdot\boldsymbol{\tau}_1\frac{\partial}{\partial \boldsymbol{\tau}_1}\right)\right),$$

$$\mathcal{B}^+ = \left(\frac{\partial}{\partial \mathbf{n}_2} - \frac{\vec{\mathbf{b}}\cdot\mathbf{n}_2 + p}{2\nu} - q\left(\frac{\partial}{\partial t} + \vec{\mathbf{b}}\cdot\boldsymbol{\tau}_2\frac{\partial}{\partial \boldsymbol{\tau}_2}\right)\right),$$

where p and q depend on the physical parameters and on $\vec{\mathbf{b}}\cdot\mathbf{n}_i$ and $\vec{\mathbf{b}}\cdot\boldsymbol{\tau}_i$.

References

[1] R. Dautray, J.-L. Lions, Analyse Mathématique et Calcul Numérique Pour les Sciences et les Techniques, Masson, Paris, 1987–1988.

[2] V. Dolean, S. Lanteri, F. Nataf, Optimized interface conditions for domain decomposition methods in fluid dynamics, in: LMS Workshop on Domain Decomposition Methods in Fluid Mechanics, London, 2001, Internat. J. Numer. Methods Fluids 40 (12) (2002) 1539–1550.

[3] B. Engquist, H.K. Zhao, Absorbing boundary conditions for domain decomposition, Appl. Numer. Math. 13 (3) (1998) 189–203.

[4] C.A.J. Flechter, Computational Techniques for Fluid Dynamics, second ed., Springer Series in Computational Physics, Springer, Berlin, 1991.

[5] M.J. Gander, L. Halpern, F. Nataf, Optimal convergence for overlapping and non-overlapping Schwarz waveform relaxation, in: C.-H. Lai, P. Bjørstad, M. Cross, O. Widlund (Eds.), Eleventh International Conferences on Domain Decomposition Methods, 1999, pp. 27–36.

[6] M.J. Gander, L. Halpern, Méthodes de relaxation d'ondes pour l'équation de la chaleur en dimension 1, C. R. Math. Acad. Sci. Paris 336 (6) (2003) 519–524.

[7] M.J. Gander, H. Zhao, Overlapping Schwarz waveform relaxation for parabolic problems in higher dimension, in: A. Handlovičová, M. Komorníkova, K. Mikula (Eds.), Proceedings of Algoritmy, vol. 14, 1997, pp. 42–51.

[8] M.J. Gander, F. Magoulès, F. Nataf, Optimized Schwarz methods without overlap for the Helmholtz equation, SIAM J. Sci. Comput. 24 (1) (2002) 38–60.
[9] L. Halpern, Artificial boundary conditions for the linear advection diffusion equation, Math. Comp. 46 (1986) 425–439.
[10] C. Japhet, Méthode de décomposition de domaine et conditions aux limites artificielles en mécanique des fluides : méthode optimisée d'ordre 2, Ph.D. Thesis, Université Paris XIII Villetaneuse, 1998.
[11] C. Japhet, F. Nataf, F. Rogier, The optimized order 2 method. Application to convection–diffusion problems, Future Generat. Comput. Systems 18 (2001) 17–30.
[12] J.L. Lions, E. Magenes, Problèmes aux Limites non Homogènes et Applications, Dunod, Paris, 1968.
[13] P.L. Lions, On the Schwarz alternating method III: A variant for non-overlapping subdomains, in: T.F. Chan, R. Glowinski, J. Périaux, O. Widlund (Eds.), Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, 1990, pp. 202–223.
[14] F. Nataf, F. Rogier, E. Sturler, Domain decomposition methods for fluid dynamics, in: A. Sequeira (Ed.), Navier–Stokes Equations and Related Nonlinear Analysis, Plenum, New York, 1995, pp. 367–376.
[15] F. Nataf, F. Rogier, Factorization of the convection–diffusion operator and the Schwarz algorithm, Math. Models Methods Appl. Sci. 5 (1) (1995) 67–93.
[16] A. Quarteroni, A. Valli, Domain Decomposition Methods for Partial Differential Equations, Oxford Science Publications, Oxford, 1999.
[17] H.A. Schwarz, Über einen Grenzübergang durch alternierendes Verfahren, Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, 1870.