Finite element method applied to 2D dynamic wetting problems with modelisation of interface formation effects

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These notes are a guide into the model here considered and were written for a beginer in the FEM who has certain familiartity with fluid mechanics. The notes were edited several times as improvements were made to the code implementation. It is possible that some portions of the text were not fully updated to be an exact match to the latest version of the code (version 43). Still, they constitute the roadmap used to code the FEM implementation, and the code that accompanies them is largely based on the notation here used. Questions or comments about these notes can be directed to the email of the author.

Key words: Dynamic wetting, finite element method, spine method.

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1. Introduction

We consider the 2-dimensional flow in a droplet of incompressible fluid, as it spreads over a solid surface. The fluid domain Ω is enclosed by a free surface $\partial\Omega^1$, a solid surface $\partial\Omega^2$ and the axis of symmetry $\partial\Omega^3$ (see figure 1). We model the interface formation process following Shikhmurzaev (1993, 1997, 2007, 2020) and we closely follow Sprittles

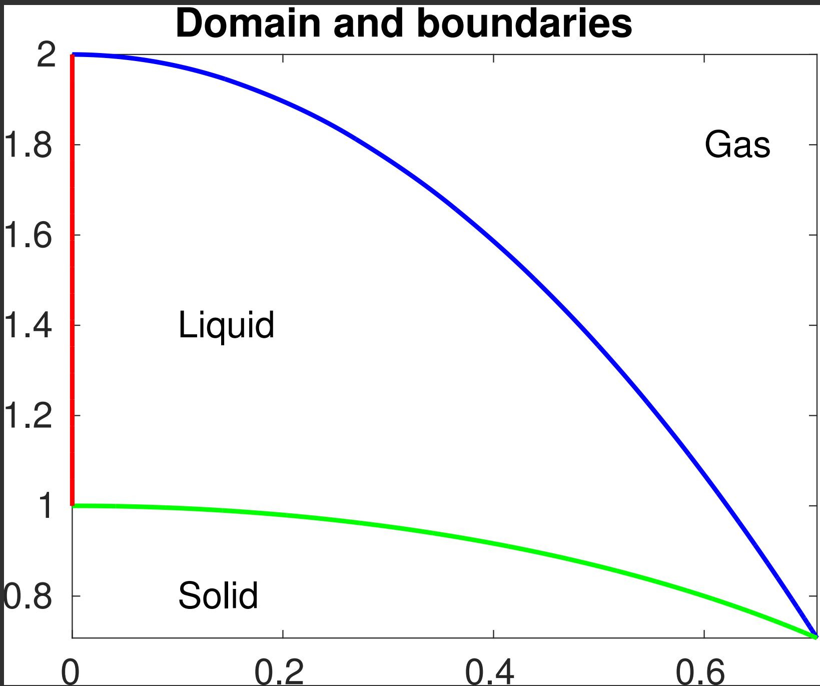

FIGURE 1. The fluid domain Ω is enclosed by the free surface $\partial\Omega^1$, shown here in blue; the solid surface $\partial\Omega^2$ along the r axis, shown in green; and the axis of symmetry $\partial\Omega^3$ along the $r = 0$ axis, shown in red. The contact line lies at a distance equal to f from the origin (here $f = 3$). The liquid-solid line is curved here, as the method can in principle be easily adaptable for a curved solid surface; however, from here on we assume a flat solid surface.

& Shikhmurzaev (2013, 2012a) and adapt their finite-element-method framework to the case at hand.

2. Governing equations

The equations for conservation of momentum in an arbitrary Lagrangian-Eulerian (ALE) frame of reference are given by

$$\partial_t \boldsymbol{u}' + (\boldsymbol{u}' - \boldsymbol{c}') \cdot \nabla' \boldsymbol{u}' = \frac{\nabla' \cdot \boldsymbol{P}'}{\rho} + g\hat{\boldsymbol{g}}, \tag{2.1}$$

where $\boldsymbol{u}' = (u', w')$ is the velocity of the fluid, $\boldsymbol{c}'$ is the velocity of the ALE coordinates, ρ is the fluid density (which we assume is uniform and constant), g is the modulus of the acceleration of gravity, $\hat{\boldsymbol{g}}$ is the unit vector that points in the direction of the gravitational acceleration, and $\boldsymbol{P}'$ is the stress tensor defined by

$$\boldsymbol{P}' = \left\{ -p'\boldsymbol{I} + \rho\nu \left[\nabla' \boldsymbol{u}' + (\nabla' \boldsymbol{u}')^T \right] \right\}, \tag{2.2}$$

where ν is the kinematic viscosity of the fluid. All dimensional variables and differential operators are denoted with a $'$.

Conservation of mass is given by

$$\nabla' \cdot \boldsymbol{u}' = 0. \tag{2.3}$$

2.1. Boundary conditions at the axis of symmetry

Flow at the axis of symmetry is subject to impermeability

$$\boldsymbol{u}' \cdot \boldsymbol{n}^3 = 0, \tag{2.4}$$

where $\boldsymbol{n}^3$ is the unit normal to boundary 3 which points into the domain, and to no tangential stress

$$\boldsymbol{n}^3 \cdot \boldsymbol{P}' \cdot \left(\boldsymbol{I} - \boldsymbol{n}^3\boldsymbol{n}^3\right) = 0. \tag{2.5}$$

2.2. Interface formation boundary conditions

We treat the liquid-gas interface and the liquid-solid interface as separate 2-dimensional phases, whose velocities and 2-dimensional densities are represented by $\boldsymbol{v}^{s_1\prime}$, $\boldsymbol{v}^{s_2\prime}$, $\rho^{s_1\prime}$ and $\rho^{s_2\prime}$, respectively; where the sub-index 1 refers to the liquid-gas interface, and sub-index 2 to the liquid-solid interface.

2.2.1. The free surface

On the free surface, the flow is subject to the kinematic boundary condition (KBC)

$$\left(\boldsymbol{v}^{s_1\prime} - \boldsymbol{c}'\right) \cdot \boldsymbol{n}^1 = 0, \tag{2.6}$$

where $\boldsymbol{n}^1$ is the normal to boundary 1 that points into Ω; and to tangential and normal dynamic boundary conditions (DBC), given respectively by

$$\boldsymbol{n}^1 \cdot \boldsymbol{P}' \cdot (\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) = -\nabla^{s\prime}\sigma^{1\prime}, \tag{2.7}$$

and

$$\left(\boldsymbol{n}^1 \cdot \boldsymbol{P} \cdot \boldsymbol{n}^1\right)\boldsymbol{n}^1 = \left(\sigma^{1\prime}\nabla^{s\prime} \cdot \boldsymbol{n}^1\right)\boldsymbol{n}^1 - p^{g\prime}\boldsymbol{n}^1. \tag{2.8}$$

where $\sigma^{1\prime}$ is the surface tension coefficient along surface $\partial\Omega^1$, $p^{g\prime}$ is the pressure of the gas on the free surface, and $\nabla^{s\prime}$ is the surface ∇ (nabla) operator†.

The latter two equations can be combined into a single one (henceforth called DBC1) by noticing that

$$\begin{aligned}
\nabla^{s\prime} \cdot \left[\sigma'\left(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1\right)\right] &= \nabla^{s\prime} \cdot (\sigma'\boldsymbol{I}) - \nabla^{s\prime} \cdot \left[\sigma'\left(\boldsymbol{n}^1\boldsymbol{n}^1\right)\right] \\
&= (\nabla^{s\prime}\sigma')\,\boldsymbol{I} - \sigma'\underbrace{\nabla^{s\prime} \cdot \boldsymbol{I}}_{0} - \underbrace{(\nabla^{s\prime}\sigma') \cdot \boldsymbol{n}^1\boldsymbol{n}^1}_{0} - \sigma'\left(\nabla^{s\prime} \cdot \boldsymbol{n}^1\right)\boldsymbol{n}^1 \\
&= (\nabla^{s\prime}\sigma')\,\boldsymbol{I} - \sigma'\left(\nabla^{s\prime} \cdot \boldsymbol{n}^1\right)\boldsymbol{n}^1,
\end{aligned} \tag{2.9}$$

and that adding equations (2.7) and (2.8) we have the DBC1

$$\boldsymbol{P}' \cdot \boldsymbol{n}^1 = -p^{g\prime}\boldsymbol{n}^1 - \nabla^{s\prime} \cdot \left[\sigma^{1\prime}(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)\right]. \tag{2.10}$$

Four further conditions are needed for the free surface. These are

$$\left(\boldsymbol{v}^{s_1\prime} - \boldsymbol{u}'\right) \cdot \left(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1\right) = \frac{1 + 4\alpha_g\beta_g}{4\beta_g}\nabla^s\sigma^{1\prime}, \tag{2.11}$$

which states that the tangential velocity between the free-surface phase and the bulk

† We recall that $\nabla^s f$, where f is a scalar function defined on the surface, is simply the weighted sum of each of the unit vectors tangent to the curvilinear coordinates, with the derivatives of f with respect to the corresponding curvilinear coordinate as the weights. Moreover, $\nabla^s \cdot \boldsymbol{v}$ (where $\boldsymbol{v}$ is a vector function defined on the surface) is the sum of the inner products of each unit vector tangent to the curvilinear coordinate by the derivative of $\boldsymbol{v}$ with respect to the corresponding coordinate. We highlight that the inner products require the use of the metric tensor if the inner products are to be computed with respect to intrinsic coordinates.

phase at the same location is proportional to the surface gradient of surface tension (henceforth referred to a the SC1);

$$\sigma^1 = \gamma_g \left({\rho^{s}}'_{(0)} - {\rho^{s_1}}' \right), \tag{2.12}$$

which controls the changes in surface tension as a function of the changes in the density of the 2-dimensional free-surface phase (henceforth referred to as TDC1); the mass exchange between surface and bulk condition

$$\rho \left(\boldsymbol{u}' - {\boldsymbol{v}^{s_1}}' \right) \cdot \boldsymbol{n}^1 = \frac{\left({\rho^{s_1}}' - \rho^{s_1,e}, \right)}{\tau_g}, \tag{2.13}$$

which quantifies the mass transfer between the surface phase and the bulk (henceforth referred to as MEC1); and the mass transport equations on the free surface

$$\partial_{t'} {\rho^{s_1}}' - \boldsymbol{c}' \cdot {\nabla^{s}}' {\rho^{s_1}}' + {\nabla^{s}}' \cdot \left({\rho^{s_1}}' {\boldsymbol{v}^{s_1}}' \right) = \frac{\rho^{s_1,e} - {\rho^{s_1}}'}{\tau_g}, \tag{2.14}$$

which models density transport within the surface (notice that the second term on the left-hand side appears due to our adoption of an ALE reference system). This condition will be referred to as the DTC1 from here on.

We recall the vector calculus identity

$$\nabla^s \cdot (\phi \boldsymbol{A}) = \boldsymbol{A} \cdot \nabla^s \phi + \phi \nabla^s \cdot \boldsymbol{A}. \tag{2.15}$$

We take $\phi = {\rho^{s_1}}'$ and $A = \boldsymbol{c}'$ and obtain

$${\nabla^{s}}' \cdot \left({\rho^{s_1}}' \boldsymbol{c}' \right) = \boldsymbol{c} \cdot {\nabla^{s}}' {\rho^{s_1}}' + {\rho^{s_1}}' {\nabla^{s}}' \cdot \boldsymbol{c}'. \tag{2.16}$$

i.e.

$$-\boldsymbol{c} \cdot {\nabla^{s}}' {\rho^{s_1}}' = -{\nabla^{s}}' \cdot \left({\rho^{s_1}}' \boldsymbol{c}' \right) + {\rho^{s_1}}' {\nabla^{s}}' \cdot \boldsymbol{c}'. \tag{2.17}$$

Taking this result into condition DTC1, we have

$$\partial_t {\rho^{s_1}}' + {\rho^{s_1}}' {\nabla^{s}}' \cdot \boldsymbol{c}' - {\nabla^{s}}' \cdot \left({\rho^{s_1}}' \boldsymbol{c}' \right) + {\nabla^{s}}' \cdot \left({\rho^{s_1}}' {\boldsymbol{v}^{s_1}}' \right) = \frac{\rho^{s_1,e} - {\rho^{s_1}}'}{\tau_g}, \tag{2.18}$$

i.e.

$$\partial_t {\rho^{s_1}}' + {\rho^{s_1}}' {\nabla^{s}}' \cdot \boldsymbol{c}' + {\nabla^{s}}' \cdot \left[{\rho^{s_1}}' \left({\boldsymbol{v}^{s_1}}' - \boldsymbol{c}' \right) \right] = \frac{\rho^{s_1,e} - {\rho^{s_1}}'}{\tau_g}. \tag{2.19}$$

In the equations above, α_g, β_g, γ_g and τ_g (the relaxation time of the surface) are constants that depend on the liquid being considered (and possibly the gas as well). Moreover, $\rho^s_{(0)}$ is the surface density for which surface tension is null; which is a property of the liquid. Moreover, $\rho^{s_1,e}$ is the equilibrium surface density for the liquid-gas interface.

2.2.2. **The liquid-solid surface**

On the liquid-solid interface, we have the impermeability condition (IC)

$$\left({\boldsymbol{v}^{2}}' - {\boldsymbol{u}^{s}}' \right) \cdot \boldsymbol{n}^2 = 0, \tag{2.20}$$

where ${\boldsymbol{u}^{s}}'$ is the velocity of the solid; the slip condition between the surface phase, the bulk and the solid (SC2)

$$\left[{\boldsymbol{v}^{s_2}}' - \frac{1}{2} \left(\boldsymbol{u}' + {\boldsymbol{u}^{s}}' \right) \right] \cdot \left(\boldsymbol{I} - \boldsymbol{n}^2 \boldsymbol{n}^2 \right) = \alpha_s {\nabla^{s}}' {\sigma^{2}}'; \tag{2.21}$$

the Generalised Navier Slip Condition (GNSC), which plays the role of tangential DBC,

$$\boldsymbol{n}^2 \cdot \boldsymbol{P}' \cdot (\boldsymbol{I} - \boldsymbol{n}^2 \boldsymbol{n}^2) + \frac{1}{2} {\nabla^{s}}' {\sigma^{2}}' = \beta_s (\boldsymbol{u}' - {\boldsymbol{u}^{s}}') \cdot (\boldsymbol{I} - \boldsymbol{n}^2 \boldsymbol{n}^2), \tag{2.22}$$

where $\boldsymbol{n}^2$ is the inward-pointing unit normal to $\partial\Omega^2$. We substitute condition SC1 into the GNSC and obtain

$$\boldsymbol{n}^2 \cdot \boldsymbol{P}' \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2) + \frac{1}{2\alpha_s}\left[\boldsymbol{v}^{s_2\prime} - \frac{1}{2}\left(\boldsymbol{u}' + \boldsymbol{u}^{s\prime}\right)\right] \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2) = \beta_s(\boldsymbol{u}' - \boldsymbol{u}^{s\prime}) \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2), \tag{2.23}$$

the dependence of surface tension on surface density for boundary 2 (henceforth called TDC2)

$$\sigma^{2\prime} = \gamma_s\left(\rho^s_{(0)} - \rho^{s_2\prime}\right); \tag{2.24}$$

the condition for mass exchange between interface 2 and the bulk (henceforth called MEC2)

$$\rho\left(\boldsymbol{u}' - \boldsymbol{v}^{s_2\prime}\right) \cdot \boldsymbol{n}^2 = \frac{\rho^{s_2\prime} - \rho^{s_2,e}}{\tau_s}; \tag{2.25}$$

and the condition for the transport of density along boundary 2 (henceforth called DTC2)

$$\partial_t\rho^{s_2\prime} - \boldsymbol{c}' \cdot \nabla'\rho^{s_2\prime} + \nabla^{s\prime} \cdot \left(\rho^{s_2\prime}\boldsymbol{v}^{s_2\prime}\right) = \frac{\rho^{s_2,e} - \rho^{s_2\prime}}{\tau_s}, \tag{2.26}$$

which can be reformulated (using identical arguments as were used above for boundary 1) into

$$\partial_t\rho^{s_2\prime} + \rho^{s_2\prime}\nabla^{s\prime} \cdot \boldsymbol{c}' + \nabla^{s\prime} \cdot \left[\rho^{s_2\prime}\left(\boldsymbol{v}^{s_2\prime} - \boldsymbol{c}'\right)\right] = \frac{\rho^{s_2,e} - \rho^{s_2\prime}}{\tau_s}. \tag{2.27}$$

2.3. **Boundary conditions at contact line**

DBC1 on the free surface and the GNSC on the liquid-solid interface require their own boundary conditions for the surface-tension variables, given that they involve terms in which the surface tension is differentiated along the respective interfaces. The condition for these two equations is given by Young's equation for the contact angle, i.e.

$$\sigma^{2\prime} + \sigma^{1\prime}\cos\theta_c = \sigma^{g-s}, \tag{2.28}$$

where σ^{g-s} is the surface tension between the gas and solid phases.

Similarly, DTC1 and DTC2 both need boundary conditions on the flux $(\rho\boldsymbol{v})$, for the same reason a stated above. The necessary condition is given by the mass balance (henceforth referred to as MBC) at the contact line, i.e.

$$\rho^{s_1\prime}\left(\boldsymbol{v}_{\|}^{s_1\prime} - \boldsymbol{u}_c{}'\right) \cdot \boldsymbol{m}^1 + \rho^{s_2\prime}\left(\boldsymbol{v}_{\|}^{s_2\prime} - \boldsymbol{u}_c{}'\right) \cdot \boldsymbol{m}^2 = 0, \tag{2.29}$$

where $\boldsymbol{m}^1$ and $\boldsymbol{m}^2$ are unit vectors that are normal to the contact line, tangent to the free surface and the liquid-solid interface, respectively, and that point into the corresponding surface; and $\boldsymbol{u}_c$ is the bulk velocity at the contact line.

Since $\boldsymbol{m}^1$ is given by a rotation of θ_c of $\boldsymbol{m}^2$, it is more convenient to express $\boldsymbol{m}^1$ as

$$\boldsymbol{m}^1 = \cos(\theta_c)\boldsymbol{m}^2 + \sin(\theta_c)\boldsymbol{n}^2(c), \tag{2.30}$$

where $\boldsymbol{n}^2(c)$ is the normal to surface 2 at the contact line. Where, in turn,

$$\boldsymbol{m}^2 = (-\cos(-\theta_{s^c}), \sin(-\theta_{s^c})), \tag{2.31}$$

where

$$\theta_{s^c} = \arctan(\partial_r z^s). \tag{2.32}$$

Equation (2.30) implies that

$$\cos(\theta_c) = \boldsymbol{m}^1 \cdot \boldsymbol{m}^2. \tag{2.33}$$

2.4. The split-domain formulation

The formulation presented above is complete, provided initial conditions are given for the momentum (initial velocity) and the surface transport equations (initial surface densities); however, its solution by means of the standard finite element method has been shown to conduct to the correct solution only when the contact angle is acute (Sprittles & Shikhmurzaev 2011*b*). In the case of an obtuse contact angle, this formulation allows a spurious *eigen-solution* (i.e. a non-zero solution to the Stokes equation in a wedge, which is subject to impermeability and no-tangential-stress conditions on its boundary) to be captured by the standard finite element method when used to solve the problem above. This eigen-solution strongly disturbs the flow in the vicinity of the contact line resulting in the model predicting non-physical effects for the flow and the pressure distribution.

Consequently, when the contact angle is obtuse, we must resort to the methods developed in Sprittles & Shikhmurzaev (2011*b*), which are discussed in detail in the second half of this manuscript. The method there presented only modifies the treatment of the equations above in the vicinity of the contact line, so we can split the domain and use the standard method for a region that exclude the vicinity of the contact line and the obtuse-angle method only where needed (as it involves a more complex treatment of the flow).

As is well known, dynamic wetting flows may alternate between one case and the other; hence, when solving the equations above we need an algorithm that can seamlessly transition between one method and the other. This is achieved by always solving the problem in a split domain and matching the flow, stresses and pressure along their separatrix. When the angle is acute, both methods will solve the standard formulation; when it is obtuse, the near-field of the contact line will be solved with the special formulation.

We thus introduce one further boundary which is a separatrix of the domain for which we will also need boundary conditions. We will referred to this boundary as $\partial\Omega^4$, when we discus the far field formulation and it will be then understood that it is associated to its normal that points into the far-field portion of the split domain. In an entirely analogue manner we will refer to the same separatrix as $\partial\Omega^5$ when we are dealing with the near-field formulation and it will be then understood that the opposite normal is chosen by default.

Matching conditions along boundary 4 will be discussed when the near field formulation is introduced. For now it suffices to act as if stresses and velocities along this boundary are given.

On boundary 4 we thus have

$$\boldsymbol{n}^4 \cdot \boldsymbol{P'} \cdot \boldsymbol{n}^4 = \lambda^4, \tag{2.34}$$

i.e. the normal stress on boundary 4 is given by $\lambda^4 \boldsymbol{n}^4$.

Similarly

$$\boldsymbol{n}^4 \cdot \boldsymbol{P'} \cdot \left(\boldsymbol{I} - \boldsymbol{n}^4 \boldsymbol{n}^4\right) = \gamma^4 \boldsymbol{t}^4, \tag{2.35}$$

i.e. the tangential stress on boundary 4 is given by $\lambda^4 \boldsymbol{t}^4$, where $\boldsymbol{t}^4$ is the unit tangent to surface 4.

2.5. Non-dimensionalisation

2.5.1. Bulk equations

We choose U as the characteristic velocity, L as the characteristic length, and $\rho\nu U/L$ as the characteristic stress; and we non-dimensionalise the momentum equation as follows

$$\frac{U^2}{L}\partial_t \boldsymbol{u} + \frac{U^2}{L}\boldsymbol{u}(\boldsymbol{u}-\boldsymbol{c})\cdot\nabla\boldsymbol{u} = \frac{\cancel{\rho}\nu U}{L^2}\frac{\nabla\cdot\boldsymbol{P}}{\cancel{\rho}} + g\hat{\boldsymbol{g}}, \tag{2.36}$$

with

$$\boldsymbol{P} = \left\{-p\boldsymbol{I} + \left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right]\right\}, \tag{2.37}$$

where variables without a $'$ correspond to the dimensionless version of the respective variables with a $'$.

We multiply equation (2.36) by $\nu U/L^2$ to obtain

$$Re\,[\partial_t\boldsymbol{u} + (\boldsymbol{u}-\boldsymbol{c})\cdot\nabla\boldsymbol{u}] - \nabla\cdot\boldsymbol{P} + St\,\boldsymbol{e}_z = 0, \tag{2.38}$$

where

$$Re = UL/\nu \tag{2.39}$$

and

$$St = gL^2/(\nu U) \tag{2.40}$$

are the Reynolds and Stokes numbers, respectively; and the continuity equation retains its form

$$\nabla\cdot\boldsymbol{u} = 0. \tag{2.41}$$

2.5.2. The axis of symmetry

On the axis of symmetry, the boundary conditions are given by

$$\boldsymbol{u}\cdot\boldsymbol{n}^3 = 0, \tag{2.42}$$

and

$$\boldsymbol{n}^3\cdot\boldsymbol{P}\cdot\left(\boldsymbol{I} - \boldsymbol{n}^3\boldsymbol{n}^3\right) = 0. \tag{2.43}$$

2.5.3. The free surface

On the free surface, the kinematic boundary condition yields

$$(\boldsymbol{v}^{s_1} - \boldsymbol{c})\cdot\boldsymbol{n}^1 = 0, \tag{2.44}$$

and the dynamic boundary condition

$$\frac{\rho\nu U}{L}\left(p^{g\prime} + \boldsymbol{P}'\right)\cdot\boldsymbol{n}^1 = -\frac{\sigma_e^{1\prime}}{L}\nabla^s\cdot\left[\sigma^{1\prime}(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)\right], \tag{2.45}$$

results in

$$(p^g + \boldsymbol{P})\cdot\boldsymbol{n}^1 = -\frac{\nabla^s\cdot\left[\sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)\right]}{Ca}, \tag{2.46}$$

where $Ca = \rho\nu U/\sigma^{1,e}$, with $\sigma^{1,e}$ being the equilibrium surface tension of the gas-liquid interface and $\sigma^1 = \sigma^{1\prime}/\sigma^{1,e}$ (as opposed to what would result if we used the stress and lengths units to define a surface tension unit).

Moreover, the SC1 becomes

$$(\boldsymbol{v}^{s_1} - \boldsymbol{u})\cdot\left(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1\right) = \frac{1 + 4Eg\,Bg}{4Bg}\nabla^s\sigma^1, \tag{2.47}$$

where $Eg = \alpha\sigma_e^1/(UL)$, $Bg = \beta UL/\sigma^{1,e}$; the dependence of surface tension on boundary 1 is given by

$$\sigma^1 = Cg\,(1-\rho^{s_1}), \tag{2.48}$$

where $Cg = \gamma_g\rho^s_{(0)}/\sigma^{1,e}$, and $\rho^{s_1} = \rho^{s_1\prime}/\rho^s_{(0)}$ (as opposed to what would be if the surface-density unit were derived from the stress unit, the length unit and the time unit); and the mass exchange between the free surface and the bulk

$$(\boldsymbol{u}-\boldsymbol{v}^{s_1})\cdot\boldsymbol{n}^1 = Fg\,(\rho^{s_1}-Dg), \tag{2.49}$$

where $Fg = \rho^s_{(0)}/(\rho U\tau_g)$ and $Dg = \rho^{s_1,e}/\rho^s_{(0)}$. We highlight here that Fg is indeed dimensionless, as the units of ρ^{s_1} (2-dimensional density) are different from those of ρ (standard 3-dimensional density).

Furthermore, on boundary 1 we have the DTC1 which, in dimensionless form, is given by

$$Tg\,\{\partial_t\rho^{s_1} + \rho^{s_1}\nabla^s\cdot\boldsymbol{c} + \nabla^s\cdot[\rho^{s_1}(\boldsymbol{v}^{s_1}-\boldsymbol{c})]\} = Dg - \rho^{s_1}. \tag{2.50}$$

where $Tg = \tau_g U/L$.

2.5.4. The liquid-solid interface

On the solid surface we have the IC, which in dimensionless form is given by

$$\left(\boldsymbol{v}^2-\boldsymbol{u}^s\right)\cdot\boldsymbol{n}^2 = 0; \tag{2.51}$$

the SC2 becomes

$$\left[\boldsymbol{v}^{s_2}-\frac{1}{2}(\boldsymbol{u}+\boldsymbol{u}^s)\right]\cdot\left(\boldsymbol{I}-\boldsymbol{n}^2\boldsymbol{n}^2\right) = Es\,\nabla^s\sigma^2, \tag{2.52}$$

where we introduce $Es = \alpha_s\sigma^{1,e}/(UL)$. We highlight that $\sigma^2 = \sigma^{2\prime}/\sigma^{1,e}$, i.e. the free-surface equilibrium surface tension is taken as unit of surface tension and we note that if $\alpha_g = \alpha_s$, $Es = Eg$.

The GNSC is given by

$$\boldsymbol{n}^2\cdot\boldsymbol{P}\cdot(\boldsymbol{I}-\boldsymbol{n}^2\boldsymbol{n}^2) + \frac{1}{2Ca\,Es}\left[\boldsymbol{v}^{s_2}-\frac{1}{2}(\boldsymbol{u}+\boldsymbol{u}^s)\right]\cdot\left(\boldsymbol{I}-\boldsymbol{n}^2\boldsymbol{n}^2\right) = Be\,(\boldsymbol{u}-\boldsymbol{u}^s)\cdot(\boldsymbol{I}-\boldsymbol{n}^2\boldsymbol{n}^2), \tag{2.53}$$

where $Be = \beta_s L/(\rho\nu)$ and we recall that $Ca = \rho\nu U/\sigma_e^1$.

The dependence of surface tension on density (TDC2) becomes

$$\sigma^2 = Cs\,(1-\rho^{s_2}); \tag{2.54}$$

where $Cs = \gamma_s\rho^{s,0}/\sigma^{1,e}$ (we note that if $\gamma_g = \gamma_s$, Cs= Cg); and the mass exchange between the bulk and the surface is given by

$$(\boldsymbol{u}-\boldsymbol{v}^{s_2})\cdot\boldsymbol{n}^2 = Fs\,(\rho^{s_2}-Ds), \tag{2.55}$$

where $Ds = \rho^{s_2,e}/\rho^{s,0}$ and $Fs = \rho^{s,0}/(\rho U\tau_s)$.

Furthermore, we have the DTC2 which is given by

$$Ts\,\{\partial_t\rho^{s_2} + \rho^{s_2}\nabla^s\cdot\boldsymbol{c} + \nabla^s\cdot[\rho^{s_2}(\boldsymbol{v}^{s_2}-\boldsymbol{c})]\} = Ds - \rho^{s_2}. \tag{2.56}$$

where we recall that $Ts = \tau_s U/L$.

2.5.5. Contact-line conditions

For the stress condition at the contact line we have

$$\sigma^1\cos(\theta_c) + \sigma^2 = So, \tag{2.57}$$

where $So = \sigma^{g-s}/\sigma^{1,e}$.

Finally for the mass balance condition MBC between the two surfaces we have

$$\rho^{s_1} \left(\boldsymbol{v}^{s_1} - \boldsymbol{u}\right) \cdot \boldsymbol{m}^1 + \rho^{s_2} \left(\boldsymbol{v}^{s_2} - \boldsymbol{u}\right) \cdot \boldsymbol{m}^2 = 0, \tag{2.58}$$

which holds on the contact line.

The contact angle condition

$$\cos(\theta_c) = \boldsymbol{m}^1 \cdot \boldsymbol{m}^2 \tag{2.59}$$

remains unchanged.

3. Momentum, continuity and kinematic conditions in Cartesian coordinates

Defining $\boldsymbol{u} = (u, w)$, $\boldsymbol{c} = (u^c, w^c)$, $\hat{\boldsymbol{g}} = (\hat{g}_r, \hat{g}_z)$ and re-writing the governing equations by components we have the r-momentum equation

$$Re\,\partial_t u + Re\,u\partial_r u + Re\,w\partial_z u - Re\,u^c\partial_r u - Re\,w^c\partial_z u - St\,\hat{g}_r - \boldsymbol{e}_r\cdot\nabla\cdot\boldsymbol{P} = 0, \tag{3.1}$$

the z-momentum equation

$$Re\,\partial_t w + Re\,u\partial_r w + Re\,w\partial_z w - Re\,u^c\partial_r w - Re\,w^c\partial_z w - St\,\hat{g}_z - \boldsymbol{e}_z\cdot\nabla\cdot\boldsymbol{P} = 0, \tag{3.2}$$

and the continuity equation

$$\partial_r u + \partial_z w = 0; \tag{3.3}$$

which are subject to the KBC

$$(u - u^c)n_r^1 + (w - w^c)n_z^1 = 0 \tag{3.4}$$

on the free surface, where $\boldsymbol{n}^1 = (n_r^1, n_z^1)$, and to the IC

$$(u - u^s)n_r^2 + (w - w^s)n_z^2 = 0, \tag{3.5}$$

on the solid surface, $\boldsymbol{n}^2 = (n_r^2, n_z^2)$. For the time being we will not write the DBC (2.10) and the NSC (2.53) in components.

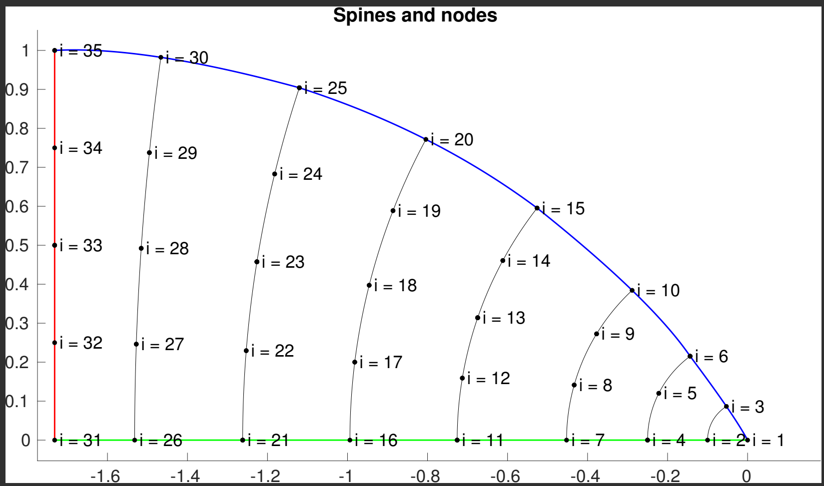

FIGURE 2. Schematics of the distribution of node in the domain. The lines connecting the horizontal solid surface to the free surface are the *spines*, whose lengths determine the shape of the domain and along which the nodes are evenly spaced out. The spine lengths will be variables in our equation system. Nodes are evenly spaced along spines which are arcs of circumferences of Apolonian circles in a bipolar coordinate system for which the contact line coincides with one of the coordinate system's foci, and boundary 3 is the mediatrix of the line determined by both foci. The set of spines to be used is chosen by setting the number of spines desired and the ratio of the distances between the"foo" of each spine (i.e. its intersection with boundary 2) to the foot of the two spines adjacent to it. That is to say, the distance between a given spine foot to the the first spine foot to its left over the distance from the the given spine foot to the first spine foot to its right (following Sprittles & Shikhmurzaev 2012*b*). The numbering convention of the nodes is also illustrated.

4. The r-momentum residuals

We define the i-th residuals of the r-momentum equation as

$$\begin{aligned} M_i^r = &Re \int_{\Omega^f} \phi_i \partial_t u + Re \int_{\Omega^f} \phi_i u \partial_r u + Re \int_{\Omega^f} \phi_i w \partial_z u - Re \int_{\Omega^f} \phi_i u^c \partial_r u \\ &- Re \int_{\Omega^f} \phi_i w^c \partial_z u - St \int_{\Omega^f} \phi_i \hat{g}_r - \int_{\Omega^f} \phi_i \boldsymbol{e}_r \cdot \nabla \cdot \boldsymbol{P}, \end{aligned} \tag{4.1}$$

where functions ϕ_i are chosen to be low-degree piece-wise polynomials with the property that their value is equal to 1 in a single point inside the domain. Moreover, the i index goes from 1 to n_v, where each index is also associated with a different point in the domain, i.e. a *velocity node* (see figure 2). Naturally, residuals must be identically zero for all i.

We recall the tensor identity†

$$\nabla \cdot (\boldsymbol{x} \cdot \boldsymbol{A}) = \boldsymbol{x} \cdot \nabla \cdot \boldsymbol{A} + \nabla \boldsymbol{x} : \boldsymbol{A}, \tag{4.2}$$

taking $x = \phi_i \boldsymbol{e}_r$ and $\boldsymbol{A} = \boldsymbol{P}$ we have

$$-\phi_i \boldsymbol{e}_r \cdot \nabla \cdot \boldsymbol{P} = -\nabla \cdot (\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P}) + \nabla (\phi_i \boldsymbol{e}_r) : \boldsymbol{P}, \tag{4.3}$$

† In the case of Cartesian coordinate, the : symbol can be thought of as the canonical inner product of matrices (sum of products of corresponding entries), when used between two tensors of second order.

which reduces M_i^r to

$$\begin{aligned} M_i^r = {} & Re\int\limits_{\Omega^f}\phi_i\partial_t u + Re\int\limits_{\Omega^f}\phi_i u\partial_r u + Re\int\limits_{\Omega^f}\phi_i w\partial_z u - Re\int\limits_{\Omega^f}\phi_i u^c\partial_r u \\ & - Re\int\limits_{\Omega^f}\phi_i w^c\partial_z u - St\int\limits_{\Omega^f}\phi_i\hat{g}_r + \int\limits_{\Omega^f}\nabla\left(\phi_i\boldsymbol{e}_r\right):\boldsymbol{P} - \int\limits_{\Omega^f}\nabla\cdot\left(\phi_i\boldsymbol{e}_r\cdot\boldsymbol{P}\right), \end{aligned} \tag{4.4}$$

we can now apply the divergence theorem to the last term on the right hand side above to obtain

$$\begin{aligned} M_i^r = {} & Re\int\limits_{\Omega^f}\phi_i\partial_t u + Re\int\limits_{\Omega^f}\phi_i u\partial_r u + Re\int\limits_{\Omega^f}\phi_i w\partial_z u - Re\int\limits_{\Omega^f}\phi_i u^c\partial_r u \\ & - Re\int\limits_{\Omega^f}\phi_i w^c\partial_z u - St\int\limits_{\Omega^f}\phi_i\hat{g}_r + \int\limits_{\Omega^f}\nabla\left(\phi_i\boldsymbol{e}_r\right):\boldsymbol{P} + \int\limits_{\partial\Omega}\phi_i\boldsymbol{e}_r\cdot\boldsymbol{P}\cdot\boldsymbol{n}, \end{aligned} \tag{4.5}$$

where $\partial\Omega$ is the boundary of Ω, and $\boldsymbol{n}$ is its unit normal, that points into Ω.

We notice that

$$\nabla(\phi_i\boldsymbol{e}_r):\boldsymbol{P} = \begin{bmatrix} \partial_r\phi_i & \partial_z\phi_i \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \boldsymbol{P}_{rr} & \boldsymbol{P}_{rz} \\ \boldsymbol{P}_{zr} & \boldsymbol{P}_{zz} \end{bmatrix} \tag{4.6}$$

i.e.

$$\nabla(\phi_i\boldsymbol{e}_r):\boldsymbol{P} = \begin{bmatrix} \partial_r\phi_i & \partial_z\phi_i \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} -p+2\partial_r u & \partial_z u+\partial_r w \\ \partial_r w+\partial_z u & -p+2\partial_z w \end{bmatrix}, \tag{4.7}$$

which is

$$\nabla(\phi_i\boldsymbol{e}_r):\boldsymbol{P} = \partial_r\phi_i\boldsymbol{P}_{rr} + \partial_z\phi_i\boldsymbol{P}_{rz} = -p\partial_r\phi_i + 2\partial_r u\partial_r\phi_i + \partial_z u\partial_z\phi_i + \partial_r w\partial_z\phi_i. \tag{4.8}$$

Therefore we have

$$\begin{aligned} M_i^r = {} & Re\int\limits_{\Omega^f}\phi_i\partial_t u + Re\int\limits_{\Omega^f}\phi_i u\partial_r u + Re\int\limits_{\Omega^f}\phi_i w\partial_z u - Re\int\limits_{\Omega^f}\phi_i u^c\partial_r u - Re\int\limits_{\Omega^f}\phi_i w^c\partial_z u \\ & - St\int\limits_{\Omega^f}\phi_i\hat{g}_r - \int\limits_{\Omega^f}p\partial_r\phi_i + 2\int\limits_{\Omega^f}\partial_r u\partial_r\phi_i + \int\limits_{\Omega^f}\partial_z u\partial_z\phi_i + \int\limits_{\Omega^f}\partial_r w\partial_z\phi_i + \int\limits_{\partial\Omega}\phi_i\boldsymbol{e}_r\cdot\boldsymbol{P}\cdot\boldsymbol{n}, \end{aligned} \tag{4.9}$$

We now consider the penultimate term in the RHS of the equation above, i.e.

$$\int\limits_{\Omega^f}\partial_r w\partial_z\phi_i. \tag{4.10}$$

We recall the multi-variable integration by parts formula given by

$$\int\limits_{\Omega} f\partial_{x_i}g = -\int\limits_{\Omega} g\partial_{x_i}f - \int\limits_{\partial\Omega} fgn^i, \tag{4.11}$$

where n^i is the i-th Cartesian component of the inward-pointing unit normal to Ω. †

† This expression can be derived from the Gauss-Green theorem, which is a scalar version of the Gauss divergence theorem that can, in turn, be derived from the standard vector version of the Gauss divergence theorem.

Taking $f = \partial_r w$ and $g = \phi_i$, we have

$$\int\limits_{\Omega^f} \partial_r w \partial_z \phi_i = -\int\limits_{\Omega^f} \phi_i \partial_z \partial_r w - \int\limits_{\partial\Omega^f} \phi_i n_z^1 \partial_r w. \tag{4.12}$$

We can then exchange the order of the derivatives of w in the first integral on the RHS above to obtain

$$\int\limits_{\Omega^f} \partial_r w \partial_z \phi_i = -\int\limits_{\Omega^f} \underbrace{\phi_i}_{f} \partial_r \underbrace{\partial_z w}_{g} - \int\limits_{\partial\Omega^f} \phi_i n_z^1 \partial_r w, \tag{4.13}$$

and taking $f = \phi_i$ and $g = \partial_z w$ above, we can apply integration by parts once more, obtaining

$$\int\limits_{\Omega^f} \partial_r w \partial_z \phi_i = \int\limits_{\Omega^f} \partial_r \phi_i \partial_z w + \int\limits_{\partial\Omega^f} \phi_i n_r^1 \partial_z w - \int\limits_{\partial\Omega^f} \phi_i n_z^1 \partial_r w. \tag{4.14}$$

We now recall equation 2.41, which implies that $\partial_z w = -\partial_r u$, and we substitute this expression into the second integral on the RHS above, obtaining

$$\int\limits_{\Omega^f} \partial_r w \partial_z \phi_i = \int\limits_{\Omega^f} \partial_r \phi_i \partial_z w - \int\limits_{\partial\Omega^f} \phi_i n_r^1 \partial_r u - \int\limits_{\partial\Omega^f} \phi_i n_z^1 \partial_r w. \tag{4.15}$$

We substitute this into equation 4.9 and we have

$$\begin{aligned} M_i^r = {} & Re \int\limits_{\Omega^f} \phi_i \partial_t u + Re \int\limits_{\Omega^f} \phi_i u \partial_r u + Re \int\limits_{\Omega^f} \phi_i w \partial_z u - Re \int\limits_{\Omega^f} \phi_i u^c \partial_r u \\ & - Re \int\limits_{\Omega^f} \phi_i w^c \partial_z u - St \int\limits_{\Omega^f} \phi_i \hat{g}_r - \int\limits_{\Omega^f} p \partial_r \phi_i + \int\limits_{\Omega^f} \partial_r u \partial_r \phi_i + \int\limits_{\Omega^f} \partial_z u \partial_z \phi_i \\ & + \int\limits_{\Omega^f} \partial_r \phi_i \underbrace{(\partial_r u + \partial_z w)}_{=0} - \int\limits_{\partial\Omega^f} \phi_i n_r^1 \partial_r u - \int\limits_{\partial\Omega^f} \phi_i n_z^1 \partial_r w + \int\limits_{\partial\Omega^f} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}, \end{aligned} \tag{4.16}$$

We now consider the last integral on the right hand side of the equation above

$$\begin{aligned} \int\limits_{\partial\Omega} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n} = {} & \int\limits_{\partial\Omega^{1,f}} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^1 + \int\limits_{\partial\Omega^{2,f}} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 \\ & + \int\limits_{\partial\Omega^3} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3 + \int\limits_{\partial\Omega^4} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4, \end{aligned} \tag{4.17}$$

where $\partial\Omega^1$ is the free surface, $\partial\Omega^2$ is the solid surface, $\partial\Omega^3$ is the axis of symmetry, and $\partial\Omega^4$ the separatrix.

Taking this expansion into equation (4.9) we have

$$M_i^r = M_i^{r,0} + M_i^{r,1} + M_i^{r,2} + M_i^{r,3} + M_i^{r,5} \tag{4.18}$$

where

$$\begin{aligned} M_i^{r,0} = {} & Re \int\limits_{\Omega^f} \phi_i \partial_t u + Re \int\limits_{\Omega^f} \phi_i u \partial_r u + Re \int\limits_{\Omega^f} \phi_i w \partial_z u - Re \int\limits_{\Omega^f} \phi_i u^c \partial_r u \\ & - Re \int\limits_{\Omega^f} \phi_i w^c \partial_z u - St \int\limits_{\Omega^f} \phi_i \hat{g}_r - \int\limits_{\Omega^f} p \partial_r \phi_i + \int\limits_{\Omega^f} \partial_r u \partial_r \phi_i + \int\limits_{\Omega^f} \partial_z u \partial_z \phi_i, \end{aligned} \tag{4.19}$$

$$M_i^{r,1} = -\int_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r u - \int_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r w + \int_{\partial\Omega^{1,f}} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^1, \tag{4.20}$$

$$M_i^{r,2} = -\int_{\partial\Omega^{2,f}} \phi_i n_r^2 \partial_r u - \int_{\partial\Omega^{2,f}} \phi_i n_z^2 \partial_r w + \int_{\partial\Omega^{2,f}} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2, \tag{4.21}$$

$$M_i^{r,3} = -\int_{\partial\Omega^{3}} \phi_i n_r^3 \partial_r u - \int_{\partial\Omega^{3}} \phi_i n_z^3 \partial_r w + \int_{\partial\Omega^{3}} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3, \tag{4.22}$$

$$M_i^{r,5} = -\int_{\partial\Omega^{4}} \phi_i n_r^4 \partial_r u - \int_{\partial\Omega^{4}} \phi_i n_z^4 \partial_r w + \int_{\partial\Omega^{4}} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4, \tag{4.23}$$

For equation (4.20) we have equation (2.46), which states that

$$\boldsymbol{P} \cdot \boldsymbol{n}^1 = -p^g \boldsymbol{n}^1 - \frac{\nabla^s \cdot \left[\sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)\right]}{Ca}, \tag{4.24}$$

and therefore

$$\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^1 = -\phi_i p^g \boldsymbol{e}_r \cdot \boldsymbol{n}^1 - \frac{1}{Ca} \phi_i \boldsymbol{e}_r \cdot \nabla^s \cdot \left[\sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)\right]. \tag{4.25}$$

Now, we have the following surface vector calculus identity

$$\nabla^s \cdot (\boldsymbol{x} \cdot \boldsymbol{A}) = \boldsymbol{A} : \nabla^s \boldsymbol{x} + \boldsymbol{x} \cdot \nabla^s \cdot \boldsymbol{A}, \tag{4.26}$$

and taking $\boldsymbol{x} = \phi_i \boldsymbol{e}_r$ and $\boldsymbol{A} = \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)$, we have

$$\nabla^s \cdot (\phi_i \boldsymbol{e}_r \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)) = \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) : \nabla^s (\phi_i \boldsymbol{e}_r) + \phi_i \boldsymbol{e}_r \cdot \nabla^s \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) \tag{4.27}$$

which yields

$$\phi_i \boldsymbol{e}_r \cdot \nabla^s \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) = \nabla^s \cdot (\phi_i \boldsymbol{e}_r \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)) - \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) : \nabla^s (\phi_i \boldsymbol{e}_r). \tag{4.28}$$

In this $1D$-surface case, we have

$$\nabla^s (\phi_i \boldsymbol{e}_r) = \begin{bmatrix} t_r^1 \partial_s \phi_i & 0 \\ t_z^1 \partial_s \phi_i & 0 \end{bmatrix}, \tag{4.29}$$

where $\boldsymbol{t}^1 = (t_r^1, t_z^1)$, and the tangent vector must be pointing in the direction of increasing arc-length s, therefore

$$(\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) : \nabla^s (\phi_i \boldsymbol{e}_r) = \begin{bmatrix} 1 - n_r^1 n_r^1 & -n_r^1 n_z^1 \\ -n_z^1 n_r^1 & 1 - n_z^1 n_z^1 \end{bmatrix} : \begin{bmatrix} t_r^1 \partial_s \phi_i & 0 \\ t_z^1 \partial_s \phi_i & 0 \end{bmatrix}, \tag{4.30}$$

where $\boldsymbol{n}^1 = (n_r^1, n_z^1)$, i.e.

$$(\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) : \nabla^s \phi_i \boldsymbol{e}_r = t_r^1 \partial_s \phi_i - (\boldsymbol{t}^1 \cdot \boldsymbol{n}^1) n_r^1 \partial_s \phi_i = t_r^1 \partial_s \phi_i. \tag{4.31}$$

We therefore have in equation (4.28)

$$\phi_i \boldsymbol{e}_r \cdot \nabla^s \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) = \nabla^s \cdot (\phi_i \boldsymbol{e}_r \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)) - \sigma^1 t_r^1 \partial_s \phi_i, \tag{4.32}$$

and consequently in equation (4.25)

$$\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^1 = -\phi_i p^g \boldsymbol{e}_r \cdot \boldsymbol{n}^1 - \frac{1}{Ca} \nabla^s \cdot (\phi_i \boldsymbol{e}_r \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)) + \frac{1}{Ca} \sigma^1 t_r^1 \partial_s \phi_i. \tag{4.33}$$

Taking this result into (4.20) we have

$$
\begin{aligned}
M_i^{r,1} = &- \int\limits_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r u - \int\limits_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r w - \int\limits_{\partial\Omega^{1,f}} \phi_i p^g \boldsymbol{e}_r \cdot \boldsymbol{n}^1 \\
&- \frac{1}{Ca} \int\limits_{\partial\Omega^{1,f}} \nabla^s \cdot (\sigma^1 \phi_i \boldsymbol{e}_r \cdot (\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)) + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,f}} t_r^1 \sigma^1 \partial_s \phi_i,
\end{aligned}
\tag{4.34}
$$

Using the surface divergence theorem and the definition of the surface divergence for a $1D$ surface, we have

$$
\begin{aligned}
M_i^{r,1} = &- \int\limits_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r u - \int\limits_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r w - \int\limits_{\partial\Omega^{1,f}} \phi_i p^g \boldsymbol{e}_r \cdot \boldsymbol{n}^1 \\
&+ \frac{1}{Ca} \int\limits_{C_1} \sigma^1 \phi_i \boldsymbol{e}_r \cdot \boldsymbol{m}^1 + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,f}} t_r^1 \sigma^1 \partial_s \phi_i,
\end{aligned}
\tag{4.35}
$$

where C_1 is actually the two points bounding the free surface, and $\boldsymbol{m}^1$ is the vector that is tangent to the free surface, normal to the contact line and points into the free surface. Above, we have also used that $\boldsymbol{n}^1 = (n_r^1, n_z^1)$.

Therefore we have

$$
\begin{aligned}
M_i^{r,1} = &- \int\limits_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r u - \int\limits_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r w - \int\limits_{\partial\Omega^{1,f}} \phi_i p^g \boldsymbol{e}_r \cdot \boldsymbol{n}^1 \\
&+ \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1})}{Ca} m_r^{1,f}(r_{J^1}, z_{J^1}) \\
&+ \frac{\sigma^1(r_a, z_a) \phi_i(r_a, z_a)}{Ca} m_r^1(r_a, z_a) + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,f}} t_r^1 \sigma^1 \partial_s \phi_i,
\end{aligned}
\tag{4.36}
$$

where the sub-index J^1 indicates the junction point of the two sub-domains along boundary 1. We recall that, for the case at hand $\boldsymbol{m}^{1,f}(r_{J^1}, z_{J^1})$ is given by the tangent to the far-field portion of the free surface. Furthermore,

$$
\boldsymbol{m}^1(r_a, z_a) = m_r^1(r_a, z_a)\boldsymbol{e}_r + m_z^1(r_a, z_a)\boldsymbol{e}_z = \sin(\theta_a)\boldsymbol{e}_r - \cos(\theta_a)\boldsymbol{e}_z, \tag{4.37}
$$

at the other end of the free surface (droplet apex). We highlight that in our case $\theta_a = \pi/2$.

We consider now the term

$$
\int\limits_{\partial\Omega^{2,f}} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2, \tag{4.38}
$$

in equation (4.21) where we have

$$
\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 = \phi_i \boldsymbol{e}_r \cdot \underbrace{\boldsymbol{n}^2 \cdot \boldsymbol{P} \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2)}_{Be\,(\boldsymbol{u}-\boldsymbol{u}^s)\cdot(\boldsymbol{I}-\boldsymbol{n}^2\boldsymbol{n}^2) - \frac{1}{2Ca\,Es}\left[\boldsymbol{v}^{s2} - \frac{1}{2}(\boldsymbol{u}+\boldsymbol{u}^s)\right]\cdot(\boldsymbol{I}-\boldsymbol{n}^2\boldsymbol{n}^2)} + \phi_i \boldsymbol{e}_r \cdot \underbrace{(\boldsymbol{n}^2 \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2)}_{\lambda^2} \boldsymbol{n}^2, \tag{4.39}
$$

where we have used equation (2.53) and we have introduced λ^2, i.e. the normal stress on boundary 2.

Hence, we have

$$\begin{aligned}\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 = Be\, \phi_i \left(\boldsymbol{u} \cdot \boldsymbol{t}^2\right) \boldsymbol{e}_r \cdot \boldsymbol{t}^2 - Be\, \phi_i \left(\boldsymbol{u}^s \cdot \boldsymbol{t}^2\right) \boldsymbol{e}_r \cdot \boldsymbol{t}^2 + \lambda^2 \phi_i n_r^2 \\ - \frac{1}{2Ca\, Es} \phi_i \left(\boldsymbol{v}^{s_2} \cdot \boldsymbol{t}^2 - \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{t}^2 - \frac{1}{2}\boldsymbol{u}^s \cdot \boldsymbol{t}^2\right) \boldsymbol{e}_r \cdot \boldsymbol{t}^2.\end{aligned} \tag{4.40}$$

Rewriting we have

$$\begin{aligned}\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 = Be\, \phi_i \left(u t_r^2 + w t_z^2\right) t_r^2 - Be\, \phi_i \left(u^s t_r^2 + w^s t_z^2\right) t_r^2 + \lambda^2 \phi_i n_r^2 \\ - \frac{1}{2Ca\, Es} \phi_i \left(u^{s_2} t_r^2 + w^{s_2} t_z^2 - \frac{1}{2} u t_r^2 - \frac{1}{2} w t_z^2 - \frac{1}{2} u^s t_r^2 - \frac{1}{2} w^s t_z^2\right) t_r^2,\end{aligned} \tag{4.41}$$

where $\boldsymbol{u}^s = (u^s, w^s)$. Equivalently

$$\begin{aligned}\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 = {} & \lambda^2 \phi_i n_r^2 \\ & + \left(\frac{1}{4Ca\, Es} + Be\right) \phi_i u t_r^2 t_r^2 + \left(\frac{1}{4Ca\, Es} + Be\right) \phi_i w t_z^2 t_r^2 \\ & + \left(\frac{1}{4Ca\, Es} - Be\right) \phi_i u^s t_r^2 t_r^2 + \left(\frac{1}{4Ca\, Es} - Be\right) \phi_i w^s t_z^2 t_r^2 \\ & - \frac{1}{2Ca\, Es} \phi_i u^{s_2} t_r^2 t_r^2 - \frac{1}{2Ca\, Es} \phi_i w^{s_2} t_z^2 t_r^2.\end{aligned} \tag{4.42}$$

We thus have

$$\begin{aligned}M_i^{r,2} = {} & - \int\limits_{\partial\Omega^{2,f}} \phi_i n_r^2 \partial_r u - \int\limits_{\partial\Omega^{2,f}} \phi_i n_z^2 \partial_r w + \int\limits_{\partial\Omega^{2,f}} \lambda^2 \phi_i n_r^2 \\ & + \left(\frac{1}{4Ca\, Es} + Be\right) \int\limits_{\partial\Omega^{2,f}} \phi_i u t_r^2 t_r^2 + \left(\frac{1}{4Ca\, Es} + Be\right) \int\limits_{\partial\Omega^{2,f}} \phi_i w t_r^2 t_z^2 \\ & + \left(\frac{1}{4Ca\, Es} - Be\right) \int\limits_{\partial\Omega^{2,f}} \phi_i u^s t_r^2 t_r^2 + \left(\frac{1}{4Ca\, Es} - Be\right) \int\limits_{\partial\Omega^{2,f}} \phi_i w^s t_r^2 t_z^2 \\ & - \frac{1}{2Ca\, Es} \int\limits_{\partial\Omega^{2,f}} \phi_i u^{s_2} t_r^2 t_r^2 - \frac{1}{2Ca\, Es} \int\limits_{\partial\Omega^{2,f}} \phi_i w^{s_2} t_r^2 t_z^2.\end{aligned} \tag{4.43}$$

We consider now the term

$$\int\limits_{\partial\Omega^3} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3, \tag{4.44}$$

in equation (4.22) where we have

$$\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3 = \phi_i \boldsymbol{e}_r \cdot \underbrace{\boldsymbol{n}^3 \cdot \boldsymbol{P} \cdot \left(\boldsymbol{I} - \boldsymbol{n}^3 \boldsymbol{n}^3\right)}_{\gamma^3 \boldsymbol{t}^3} + \phi_i \boldsymbol{e}_r \cdot \underbrace{\left(\boldsymbol{n}^3 \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3\right)}_{\lambda^3} \boldsymbol{n}^3, \tag{4.45}$$

where we have introduced γ^3 and λ^3, the tangential and normal stresses on boundary 2 (respectively).

Hence, we have

$$\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3 = \gamma^3 \phi_i \boldsymbol{e}_r \cdot \boldsymbol{t}^3 + \lambda^3 \phi_i \boldsymbol{e}_r \cdot \boldsymbol{n}^3, \tag{4.46}$$

which yields

$$M_i^{r,3} = - \int\limits_{\partial\Omega^3} \phi_i n_r^3 \partial_r u - \int\limits_{\partial\Omega^3} \phi_i n_z^3 \partial_r w + \int\limits_{\partial\Omega^3} \gamma^3 \phi_i t_r^3 + \int\limits_{\partial\Omega^3} \lambda^3 \phi_i n_r^3. \tag{4.47}$$

We consider now the term

$$\int\limits_{\partial\Omega^4} \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4, \tag{4.48}$$

in equation (4.23) where we have

$$\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4 = \phi_i \boldsymbol{e}_r \cdot \underbrace{\boldsymbol{n}^4 \cdot \boldsymbol{P} \cdot (\boldsymbol{I} - \boldsymbol{n}^4\boldsymbol{n}^4)}_{\gamma^4 \boldsymbol{t}^4} + \phi_i \boldsymbol{e}_r \cdot \underbrace{\left(\boldsymbol{n}^4 \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4\right)}_{\lambda^4} \boldsymbol{n}^4, \tag{4.49}$$

where we have introduced γ^4 and λ^4, the tangential and normal stresses on boundary 4 (respectively).

Hence, we have

$$\phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4 = \gamma^4 \phi_i \boldsymbol{e}_r \cdot \boldsymbol{t}^4 + \lambda^4 \phi_i \boldsymbol{e}_r \cdot \boldsymbol{n}^4, \tag{4.50}$$

which yields

$$M_i^{r,4} = -\int\limits_{\partial\Omega^4} \phi_i n_r^4 \partial_r u - \int\limits_{\partial\Omega^4} \phi_i n_z^4 \partial_r w + \int\limits_{\partial\Omega^5} \gamma^4 \phi_i t_r^4 + \int\limits_{\partial\Omega^5} \lambda^4 \phi_i n_r^4. \tag{4.51}$$

We now define $\Delta_t^n = t_n - t_{n-1}$ and $q_n = \Delta_t^n / \Delta_t^{n-1}$ and we make the following approximations

$$\partial_t u(t_n) \approx \frac{(1+2q_n)u(t_n) - (1+q_n)^2 u(t_{n-1}) + q_n^2 u(t_{n-2})}{(1+q_n)\Delta_t^n}, \tag{4.52}$$

$$\partial_t w(t_n) \approx \frac{(1+2q_n)w(t_n) - (1+q_n)^2 w(t_{n-1}) + q_n^2 w(t_{n-2})}{(1+q_n)\Delta_t^n}, \tag{4.53}$$

$$u^c(t_n) = \partial_t r^c(t_n) \approx \frac{(1+2q_n)r^c(t_n) - (1+q_n)^2 r^c(t_{n-1}) + q_n^2 r^c(t_{n-2})}{(1+q_n)\Delta_t^n}, \tag{4.54}$$

$$w^c(t_n) = \partial_t z^c(t_n) \approx \frac{(1+2q_n)z^c(t_n) - (1+q_n)^2 z^c(t_{n-1}) + q_n^2 z^c(t_{n-2})}{(1+q_n)\Delta_t^n}. \tag{4.55}$$

Substituting these into (4.18) and obtain

$$\begin{aligned}
\mathfrak{M}_i^{r,0} = {} & Re \int\limits_{\Omega^f} \phi_i \frac{(1+2q_n)u(t_n) - (1+q_n)^2 u(t_{n-1}) + q_n^2 u(t_{n-2})}{(1+q_n)\Delta_t^n} \\
& + Re \int\limits_{\Omega^f} \phi_i u \partial_r u + Re \int\limits_{\Omega^f} \phi_i w \partial_z u \\
& - Re \int\limits_{\Omega^f} \phi_i \frac{(1+2q_n)r^c(t_n) - (1+q_n)^2 r^c(t_{n-1}) + q_n^2 r^c(t_{n-2})}{(1+q_n)\Delta_t^n} \partial_r u \\
& - Re \int\limits_{\Omega^f} \phi_i \frac{(1+2q_n)z^c(t_n) - (1+q_n)^2 z^c(t_{n-1}) + q_n^2 z^c(t_{n-2})}{(1+q_n)\Delta_t^n} \partial_z u \\
& - St \int\limits_{\Omega^f} \phi_i \hat{g}_r - \int\limits_{\Omega^f} p \partial_r \phi_i + \int\limits_{\Omega^f} \partial_r u \partial_r \phi_i + \int\limits_{\Omega^f} \partial_z u \partial_z \phi_i,
\end{aligned} \tag{4.56}$$

$$
\begin{aligned}
\mathfrak{M}_i^{r,1} = &- \int\limits_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r u - \int\limits_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r w - \int\limits_{\partial\Omega^{1,f}} \phi_i p^g n_r^1 \\
&- \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1})}{Ca} m_r^{1,n}(r_{J^1}, z_{J^1}) \\
&+ \frac{\sigma^1(r_a, z_a) \phi_i(r_a, z_a)}{Ca} m_r^1(r_a, z_a) + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,f}} t_r^1 \sigma^1 \partial_s \phi_i.
\end{aligned} \tag{4.57}
$$

$$
\begin{aligned}
\mathfrak{M}_i^{r,2} = &- \int\limits_{\partial\Omega^{2,f}} \phi_i n_r^2 \partial_r u - \int\limits_{\partial\Omega^{2,f}} \phi_i n_z^2 \partial_r w + \int\limits_{\partial\Omega^{2,f}} \lambda^2 \phi_i n_r^2 \\
&+ \left(\frac{1}{4Ca\, Es} + Be \right) \int\limits_{\partial\Omega^{2,f}} \phi_i u t_r^2 t_r^2 + \left(\frac{1}{4Ca\, Es} + Be \right) \int\limits_{\partial\Omega^{2,f}} \phi_i w t_r^2 t_z^2 \\
&+ \left(\frac{1}{4Ca\, Es} - Be \right) \int\limits_{\partial\Omega^{2,f}} \phi_i u^s t_r^2 t_r^2 + \left(\frac{1}{4Ca\, Es} - Be \right) \int\limits_{\partial\Omega^{2,f}} \phi_i w^s t_r^2 t_z^2 \\
&- \frac{1}{2Ca\, Es} \int\limits_{\partial\Omega^{2,f}} \phi_i u^{s_2} t_r^2 t_r^2 - \frac{1}{2Ca\, Es} \int\limits_{\partial\Omega^{2,f}} \phi_i w^{s_2} t_r^2 t_z^2.
\end{aligned} \tag{4.58}
$$

$$
\mathfrak{M}_i^{r,3} = - \int\limits_{\partial\Omega^{3}} \phi_i n_r^3 \partial_r u - \int\limits_{\partial\Omega^{3}} \phi_i n_z^3 \partial_r w + \int\limits_{\partial\Omega^{3}} \gamma^3 \phi_i t_r^3 + \int\limits_{\partial\Omega^{3}} \lambda^3 \phi_i n_r^3, \tag{4.59}
$$

and

$$
\mathfrak{M}_i^{r,5} = - \int\limits_{\partial\Omega^{5}} \phi_i n_r^5 \partial_r u - \int\limits_{\partial\Omega^{5}} \phi_i n_z^5 \partial_r w + \int\limits_{\partial\Omega^{5}} \gamma^5 \phi_i t_r^5 + \int\limits_{\partial\Omega^{5}} \lambda^5 \phi_i n_r^5. \tag{4.60}
$$

Multiplying (4.56-4.60) by $2\Delta_t/3$ and re-arranging terms we have

$$
\begin{aligned}
\mathscr{M}_i^{r,0}(t_n) = {} & \underbrace{\frac{2(1+2q_n)}{3(1+q_n)}}_{a_n} Re \int\limits_{\Omega^f} \phi_i u(r,z,t_n) \\
& - \underbrace{\frac{2(1+q_n)Re}{3}}_{a_{n-1}} \int\limits_{\Omega^f} \phi_i u(r,z,t_{n-1}) + \underbrace{\frac{2q_n^2 Re}{3(1+q_n)}}_{a_{n-2}} \int\limits_{\Omega^f} \phi_i u(t_{n-2}) \\
& + \frac{2\Delta_t Re}{3} \int\limits_{\Omega^f} \phi_i u \partial_r u + \frac{2\Delta_t Re}{3} \int\limits_{\Omega^f} \phi_i w \partial_z u \\
& - \underbrace{\frac{2(1+2q_n)}{3(1+q_n)}}_{a_n} Re \int\limits_{\Omega^f} \phi_i r^c(t_n) \partial_r u \\
& + \underbrace{\frac{2(1+q_n)Re}{3}}_{a_{n-1}} \int\limits_{\Omega^f} \phi_i r^c(t_{n-1}) \partial_r u - \underbrace{\frac{2q_n^2 Re}{3(1+q_n)}}_{a_{n-2}} \int\limits_{\Omega^f} \phi_i r^c(t_{n-2}) \partial_r u \\
& - \underbrace{\frac{2(1+2q_n)}{3(1+q_n)}}_{a_n} Re \int\limits_{\Omega^f} \phi_i z^c(t_n) \partial_z u \\
& + \underbrace{\frac{2(1+q_n)Re}{3}}_{a_{n-1}} \int\limits_{\Omega^f} \phi_i z^c(t_{n-1}) \partial_z u - \underbrace{\frac{2q_n^2 Re}{3(1+q_n)}}_{a_{n-2}} \int\limits_{\Omega^f} \phi_i z^c(t_{n-2}) \partial_z u \\
& - \frac{2\Delta_t St}{3} \int\limits_{\Omega^f} \phi_i \hat{g}_r - \frac{2\Delta_t}{3} \int\limits_{\Omega^f} p \partial_r \phi_i + \frac{2\Delta_t}{3} \int\limits_{\Omega^f} \partial_r u \partial_r \phi_i + \frac{2\Delta_t}{3} \int\limits_{\Omega^f} \partial_z u \partial_z \phi_i,
\end{aligned}
\tag{4.61}
$$

$$
\begin{aligned}
\mathscr{M}_i^{r,1} = {} & -\frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r u - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r w - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{1,f}} \phi_i p^g n_r^1 \\
& - 2\Delta_t \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1})}{3Ca} m_r^{1,n}(r_{J^1}, z_{J^1}) \\
& + 2\Delta_t \frac{\sigma^1(r_a, z_a) \phi_i(r_a, z_a)}{3Ca} m_r^1(r_a, z_a) + \frac{2\Delta_t}{3Ca} \int\limits_{\partial\Omega^{1,f}} t_r^1 \sigma^1 \partial_s \phi_i,
\end{aligned}
\tag{4.62}
$$

$$\begin{aligned}\mathscr{M}_i^{r,2} = &-\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,f}}\phi_i n_r^2\partial_r u - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,f}}\phi_i n_z^2\partial_r w + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,f}}\lambda^2\phi_i n_r^2\\ &+\left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\int\limits_{\partial\Omega^{2,f}}\phi_i u t_r^2 t_r^2 + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\int\limits_{\partial\Omega^{2,f}}\phi_i w t_r^2 t_z^2\\ &+\left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\int\limits_{\partial\Omega^{2,f}}\phi_i u^s t_r^2 t_r^2 + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\int\limits_{\partial\Omega^{2,f}}\phi_i w^s t_r^2 t_z^2\\ &-\frac{\Delta_t}{3Ca\,Es}\int\limits_{\partial\Omega^{2,f}}\phi_i u^{s_2} t_r^2 t_r^2 - \frac{\Delta_t}{3Ca\,Es}\int\limits_{\partial\Omega^{2,f}}\phi_i w^{s_2} t_r^2 t_z^2 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^3}\gamma^3\phi_i t_r^3\\ &+\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^3}\lambda^3\phi_i n_r^3 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^4}\gamma^4\phi_i t_r^4 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\lambda^4\phi_i n_r^4,\end{aligned} \tag{4.63}$$

$$\mathscr{M}_i^{r,3} = -\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^3}\phi_i n_r^3\partial_r u - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^3}\phi_i n_z^3\partial_r w + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^3}\gamma^3\phi_i t_r^3 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^3}\lambda^3\phi_i n_r^3, \tag{4.64}$$

and

$$\mathscr{M}_i^{r,5} = -\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\phi_i n_r^5\partial_r u - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\phi_i n_z^5\partial_r w + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\gamma^5\phi_i t_r^5 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\lambda^5\phi_i n_r^5, \tag{4.65}$$

where all time dependent functions whose argument are not explicitly presented are to be evaluated at time t_n.

We now introduce the following approximations

$$u(r,z,t) \approx \sum_{j=1}^{n_v} u_j(t)\phi_j(r,z), \tag{4.66}$$

$$w(r,z,t) \approx \sum_{j=1}^{n_v} w_j(t)\phi_j(r,z), \tag{4.67}$$

$$r^c(r,z,t) \approx \sum_{j=1}^{n_v} r_j^c(t)\phi_j(r,z), \tag{4.68}$$

$$z^c(r,z,t) \approx \sum_{j=1}^{n_v} z_j^c(t)\phi_j(r,z), \tag{4.69}$$

$$p(r,z,t) \approx \sum_{j=1}^{n_p} p_j(t)\psi_j(r,z), \tag{4.70}$$

$$\sigma^1(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\sigma}_j^1(t)\phi_j^1(r,z), \tag{4.71}$$

$$\lambda^2(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\lambda}_j^2(t)\phi_j^2(r,z), \tag{4.72}$$

$$\lambda^3(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\lambda}_j^3(t)\phi_j^3(r,z), \tag{4.73}$$

$$\gamma^3(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\gamma}_j^3(t)\phi_j^3(r,z), \tag{4.74}$$

$$\lambda^4(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\lambda}_j^4(t)\phi_j^4(r,z), \tag{4.75}$$

$$\gamma^4(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\gamma}_j^4(t)\phi_j^4(r,z), \tag{4.76}$$

$$u^s(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{u}_j^s(t)\phi_j(r,z), \tag{4.77}$$

$$w^s(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{w}_j^s(t)\phi_j(r,z), \tag{4.78}$$

$$p^g(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{p}_j^g(t)\phi_j^1(r,z), \tag{4.79}$$

$$\sigma^2(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\sigma}_j^2(t)\phi_j^2(r,z); \tag{4.80}$$

where n_v is the number of nodes where velocity is calculated, n_p is the number of nodes where pressure is calculated, the j index indicates global node numbers that we will use in the Galerkin method (that is to say, ϕ_j is the hat function centred at the j-th node), and ϕ_j^k coincides on the k-th boundary with ϕ_j, and is identically null elsewhere. Moreover, functions $\tilde{\sigma}_j^k$, $\tilde{\lambda}_j^k$ and $\tilde{\gamma}_j^k$ are identically null for all j such that $\phi_j = 0$ on boundary k; and functions $\tilde{u}_j^s$ and $\tilde{w}_j^s$ are identically null away from the solid boundary. Furthermore, functions p_j are numbered following the pressure-node numbering, which are a subset of the global node numbering. That is to say, there is a subset of all nodes, that have two numbers, their velocity-node number and their pressure-node number, how we decide which subset of the nodes are simultaneously velocity and pressure nodes and which subset of the nodes consists just of velocity nodes will be described later.

We highlight that we are interpolating the gas pressure using the velocity-interpolating function rather than the pressure interpolating functions because what we actually need from the gas in the normal stress at the boundary, this just happens to be given by the gas pressure, had it been a non-ideal fluid, the correct quantity to insert here would be the normal stress which is interpolated by velocity-interpolating functions.

Substituting these approximations into (4.56-4.60) we define

$$\mathcal{M}_i^r = \mathcal{M}_i^{r,0} + \mathcal{M}_i^{r,1} + \mathcal{M}_i^{r,2} + \mathcal{M}_i^{r,3} \tag{4.81}$$

where

$$
\begin{aligned}
\mathcal{M}_i^{r;0}(t_n) &= a_n Re \int\limits_{\Omega^f} \phi_i \left(\sum_{j=1}^{n_v} u_j \phi_j \right) - a_{n-1} Re \int\limits_{\Omega^f} \phi_i \left(\sum_{j=1}^{n_v} u_j(t_{n-1}) \phi_j \right) \\
&+ a_{n-2} Re \int\limits_{\Omega^f} \phi_i \left(\sum_{j=1}^{n_v} u_j(t_{n-2}) \phi_j \right) + \frac{2\Delta_t Re}{3} \int\limits_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} u_k \phi_k \right) \partial_r \left(\sum_{j=1}^{n_v} u_j \phi_j \right) \\
&+ \frac{2\Delta_t Re}{3} \int\limits_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} w_k \phi_k \right) \partial_z \left(\sum_{j=1}^{n_v} u_j \phi_j \right) - a_n Re \int\limits_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} r_k^c \phi_k \right) \partial_r \left(\sum_{j=1}^{n_v} u_j \phi_j \right) \\
&+ a_{n-1} Re \int\limits_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} r_k^c(t_{n-1}) \phi_k \right) \partial_r \left(\sum_{j=1}^{n_v} u_j \phi_j \right) \\
&- a_{n-2} Re \int\limits_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} r_k^c(t_{n-2}) \phi_k \right) \partial_r \left(\sum_{j=1}^{n_v} u_j \phi_j \right) \\
&- a_n Re \int\limits_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} z_k^c \phi_k \right) \partial_z \left(\sum_{j=1}^{n_v} u_j \phi_j \right) \\
&+ a_{n-1} Re \int\limits_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} z_k^c(t_{n-1}) \phi_k \right) \partial_z \left(\sum_{j=1}^{n_v} u_j \phi_j \right) \\
&- a_{n-2} Re \int\limits_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} z_k^c(t_{n-2}) \phi_k \right) \partial_z \left(\sum_{j=1}^{n_v} u_j \phi_j \right) \\
&- \frac{2\Delta_t St}{3} \int\limits_{\Omega^f} \phi_i \hat{g}_r - \frac{2\Delta_t}{3} \int\limits_{\Omega^f} \left(\sum_{j=1}^{n_p} p_j \psi_j \right) \partial_r \phi_i \\
&+ \frac{2\Delta_t}{3} \int\limits_{\Omega^f} \partial_r \left(\sum_{j=1}^{n_v} u_j \phi_j \right) \partial_r \phi_i + \frac{2\Delta_t}{3} \int\limits_{\Omega^f} \partial_z \left(\sum_{j=1}^{n_v} u_j \phi_j \right) \partial_z \phi_i .
\end{aligned}
\tag{4.82}
$$

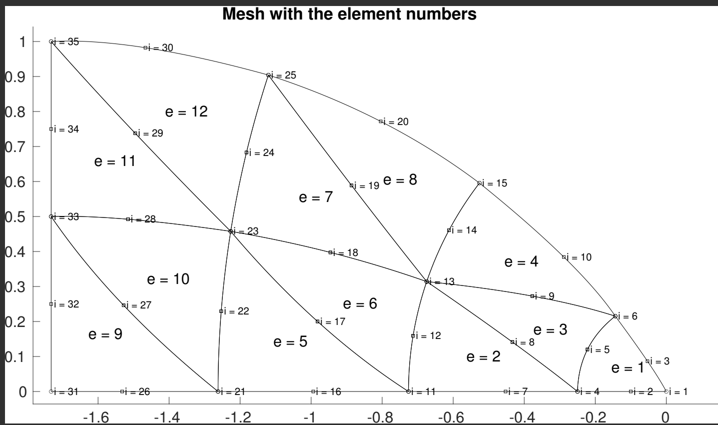

FIGURE 3. Example of domain partition into curve-sided triangular elements. Element and node numbering conventions used are also shown. The figure also shows that the number of elements along the first spines is increasing up to a given point, and then it is kept constant, so as to not make the resulting system of equations intractably large. The mesh used in this example has a single element at the contact line; however, in the implementation, 3 elements were used at the contact line.

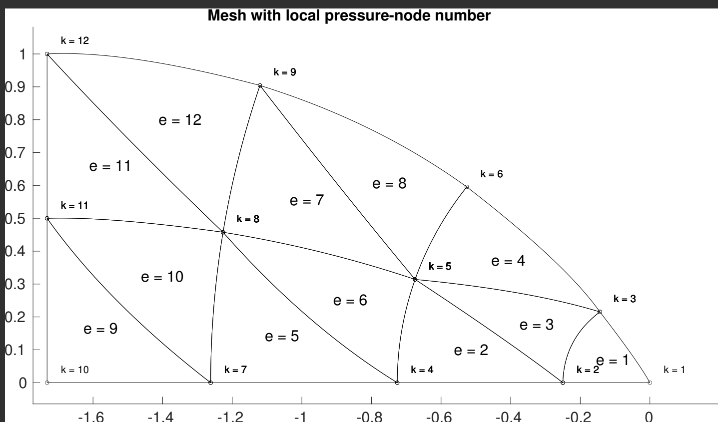

FIGURE 4. Element and pressure-node numbering conventions shown. Pressure nodes are a subset of the velocity nodes (see figure 3). These nodes only correspond to the corners of the curve-sided triangular elements. That is to say, we have fewer ψ_k functions than ϕ_i functions.

$$
\begin{aligned}
\mathcal{M}_i^{r,1} = & -\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r \left(\sum_{j=1}^{n_v} u_j\phi_j\right) - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r \left(\sum_{j=1}^{n_v} w_j\phi_j\right) \\
& - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,f}} \phi_i n_r^1 \left(\sum_{j=1}^{n_v} \tilde{p}_j^g \phi_j^1\right) \\
& - 2\Delta_t \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1})}{3Ca} m_r^{1,n}(r_{J^1}, z_{J^1}) \\
& + 2\Delta_t \frac{\sigma^1(r_a, z_a)\phi_i(r_a, z_a)}{3Ca} m_r^1(r_a, z_a) + \frac{2\Delta_t}{3Ca}\int\limits_{\partial\Omega^{1,f}} t_r^1 \left(\sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \phi_j^1\right) \partial_s \phi_i,
\end{aligned}
\tag{4.83}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,2} = &-\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,f}} \phi_i n_r^2 \partial_r \left(\sum_{j=1}^{n_v} u_j\phi_j\right) - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,f}} \phi_i n_z^2 \partial_r \left(\sum_{j=1}^{n_v} w_j\phi_j\right) \\
&+ \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,f}} \phi_i n_r^2 \left(\sum_{j=1}^{n_v} \tilde{\lambda}_j^2\phi_j^2\right) + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t \int\limits_{\partial\Omega^{2,f}} \phi_i t_r^2 t_r^2 \left(\sum_{j=1}^{n_v} u_j\phi_j\right) \\
&+ \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t \int\limits_{\partial\Omega^{2,f}} \phi_i t_r^2 t_z^2 \left(\sum_{j=1}^{n_v} w_j\phi_j\right) \\
&+ \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t \int\limits_{\partial\Omega^{2,f}} \phi_i t_r^2 t_r^2 \left(\sum_{j=1}^{n_v} \tilde{u}_j^s\phi_j\right) \\
&+ \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t \int\limits_{\partial\Omega^{2,f}} \phi_i t_r^2 t_z^2 \left(\sum_{j=1}^{n_v} \tilde{w}_j^s\phi_j\right) \\
&- \frac{\Delta_t}{3Ca\,Es}\int\limits_{\partial\Omega^{2,f}} \phi_i t_r^2 t_r^2 \left(\sum_{j=1}^{n_v} \tilde{u}_j^{s_2}\phi_j\right) - \frac{\Delta_t}{3Ca\,Es}\int\limits_{\partial\Omega^{2,f}} \phi_i t_r^2 t_z^2 \left(\sum_{j=1}^{n_v} \tilde{w}_j^{s_2}\phi_j\right),
\end{aligned}
\tag{4.84}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,3} = &-\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{3,f}} \phi_i n_r^3 \partial_r \left(\sum_{j=1}^{n_v} u_j\phi_j\right) - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{3,f}} \phi_i n_z^3 \partial_r \left(\sum_{j=1}^{n_v} w_j\phi_j\right) \\
&+ \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{3}} \left(\sum_{j=1}^{n_v} \tilde{\lambda}_j^3\phi_j^3\right)\phi_i n_r^3 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{3}} \left(\sum_{j=1}^{n_v} \tilde{\gamma}_j^3\phi_j^3\right)\phi_i t_r^3
\end{aligned}
\tag{4.85}
$$

and

$$
\begin{aligned}
\mathcal{M}_i^{r,5} = &-\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{5,f}} \phi_i n_r^5 \partial_r \left(\sum_{j=1}^{n_v} u_j\phi_j\right) - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{5,f}} \phi_i n_z^5 \partial_r \left(\sum_{j=1}^{n_v} w_j\phi_j\right) \\
&+ \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{5}} \left(\sum_{j=1}^{n_v} \tilde{\lambda}_j^5\phi_j^5\right)\phi_i n_r^5 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{5}} \left(\sum_{j=1}^{n_v} \tilde{\gamma}_j^5\phi_j^5\right)\phi_i t_r^5.
\end{aligned}
\tag{4.86}
$$

Moving the integrals into the sums and re-arranging terms we have

$$
\begin{aligned}
\mathcal{M}_i^{r,0} = {} & a_n Re \sum_{j=1}^{n_v} u_j \int\limits_{\Omega^f} \phi_i\phi_j - a_{n-1} Re \sum_{j=1}^{n_v} u_j(t_{n-1}) \int\limits_{\Omega^f} \phi_i\phi_j + a_{n-2} Re \sum_{j=1}^{n_v} u_j(t_{n-2}) \int\limits_{\Omega^f} \phi_i\phi_j \\
& + \frac{2\Delta_t Re}{3} \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} u_k \int\limits_{\Omega^f} \phi_i\phi_k\partial_r\phi_j + \frac{2\Delta_t Re}{3} \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} w_k \int\limits_{\Omega^f} \phi_i\phi_k\partial_z\phi_j \\
& - a_n Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} r_k^c \int\limits_{\Omega^f} \phi_i\phi_k\partial_r\phi_j + a_{n-1} Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} r_k^c(t_{n-1}) \int\limits_{\Omega^f} \phi_i\phi_k\partial_r\phi_j \\
& - a_{n-2} Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} r_k^c(t_{n-2}) \int\limits_{\Omega^f} \phi_i\phi_k\partial_r\phi_j - a_n Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} z_k^c \int\limits_{\Omega^f} \phi_i\phi_k\partial_z\phi_j \\
& + a_{n-1} Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} z_k^c(t_{n-1}) \int\limits_{\Omega^f} \phi_i\phi_k\partial_z\phi_j \\
& - a_{n-2} Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} z_k^c(t_{n-2}) \int\limits_{\Omega^f} \phi_i\phi_k\partial_z\phi_j - \frac{2\Delta_t St}{3} \int\limits_{\Omega^f} \phi_i\hat{g}_r \\
& - \frac{2\Delta_t}{3} \sum_{j=1}^{n_p} p_j \int\limits_{\Omega^f} \psi_j\partial_r\phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\Omega^f} \partial_r\phi_j\partial_r\phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\Omega^f} \partial_z\phi_j\partial_z\phi_i,
\end{aligned}
\tag{4.87}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,1} = {} & -\frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r\phi_j - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r\phi_j - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{p}_j^g \int\limits_{\partial\Omega^{1,f}} \phi_i n_r^1 \phi_j^1 \\
& - 2\Delta_t \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1})}{3Ca} m_r^{1,n}(r_{J^1}, z_{J^1}) \\
& + 2\Delta_t \frac{\sigma^1(r_a, z_a)\phi_i(r_a, z_a)}{3Ca} m_r^1(r_a, z_a) + \frac{2\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \int\limits_{\partial\Omega^{1,f}} t_r^1 \phi_j^1 \partial_s\phi_i,
\end{aligned}
\tag{4.88}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,2} = & -\frac{2\Delta_t}{3}\sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{2,f}} \phi_i n_r^1 \partial_r \phi_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{2,f}} \phi_i n_z^1 \partial_r \phi_j \\
& + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} \tilde{\lambda}_j^2 \int\limits_{\partial\Omega^{2,f}} \phi_i \phi_j^2 n_r^2 + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{2,f}} \phi_i \phi_j t_r^2 t_r^2 \\
& + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{2,f}} \phi_i \phi_j t_r^2 t_z^2 \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t \sum_{j=1}^{n_v} \tilde{u}_j^s \int\limits_{\partial\Omega^{2,f}} \phi_i \phi_j t_r^2 t_r^2 \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t \sum_{j=1}^{n_v} \tilde{w}_j^s \int\limits_{\partial\Omega^{2,f}} \phi_i \phi_j t_r^2 t_z^2 \\
& - \frac{\Delta_t}{3Ca\,Es}\sum_{j=1}^{n_v} \tilde{u}_j^{s_2} \int\limits_{\partial\Omega^{2,f}} \phi_i \phi_j t_r^2 t_r^2 - \frac{\Delta_t}{3Ca\,Es}\sum_{j=1}^{n_v} \tilde{w}_j^{s_2} \int\limits_{\partial\Omega^{2,f}} \phi_i \phi_j t_r^2 t_z^2,
\end{aligned}
\tag{4.89}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,3} = & -\frac{2\Delta_t}{3}\sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{3,f}} \phi_i n_r^1 \partial_r \phi_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{3,f}} \phi_i n_z^1 \partial_r \phi_j \\
& + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} \tilde{\lambda}_j^3 \int\limits_{\partial\Omega^{3}} \phi_j^3 \phi_i n_r^3 + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} \tilde{\gamma}_j^3 \int\limits_{\partial\Omega^{3}} \phi_j^3 \phi_i t_r^3
\end{aligned}
\tag{4.90}
$$

and

$$
\begin{aligned}
\mathcal{M}_i^{r,4} = & -\frac{2\Delta_t}{3}\sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{4,f}} \phi_i n_r^1 \partial_r \phi_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{4,f}} \phi_i n_z^1 \partial_r \phi_j \\
& + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} \tilde{\lambda}_j^4 \int\limits_{\partial\Omega^{4}} \phi_j^4 \phi_i n_r^4 + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} \tilde{\gamma}_j^4 \int\limits_{\partial\Omega^{4}} \phi_j^4 \phi_i t_r^4
\end{aligned}
\tag{4.91}
$$

Re-arranging terms we have

$$\begin{aligned}
\mathcal{M}_i^{r,0} = &-\frac{2\Delta_t St}{3}\int\limits_{\Omega^f}\phi_i\hat{g}_r \\
&+\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}u_j\int\limits_{\Omega^f}\partial_r\phi_j\partial_r\phi_i+\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}u_j\int\limits_{\Omega^f}\partial_z\phi_j\partial_z\phi_i+a_nRe\sum_{j=1}^{n_v}u_j\int\limits_{\Omega^f}\phi_i\phi_j \\
&-a_{n-1}Re\sum_{j=1}^{n_v}u_j(t_{n-1})\int\limits_{\Omega^f}\phi_i\phi_j+a_{n-2}Re\sum_{j=1}^{n_v}u_j(t_{n-2})\int\limits_{\Omega^f}\phi_i\phi_j \\
&+\frac{2\Delta_t Re}{3}\sum_{j=1}^{n_v}u_j\sum_{k=1}^{n_v}u_k\int\limits_{\Omega^f}\phi_i\phi_k\partial_r\phi_j+\frac{2\Delta_t Re}{3}\sum_{j=1}^{n_v}u_j\sum_{k=1}^{n_v}w_k\int\limits_{\Omega^f}\phi_i\phi_k\partial_z\phi_j \\
&-a_nRe\sum_{j=1}^{n_v}u_j\sum_{k=1}^{n_v}r_k^c\int\limits_{\Omega^f}\phi_i\phi_k\partial_r\phi_j+a_{n-1}Re\sum_{j=1}^{n_v}u_j\sum_{k=1}^{n_v}r_k^c(t_{n-1})\int\limits_{\Omega^f}\phi_i\phi_k\partial_r\phi_j \\
&-a_{n-2}Re\sum_{j=1}^{n_v}u_j\sum_{k=1}^{n_v}r_k^c(t_{n-2})\int\limits_{\Omega^f}\phi_i\phi_k\partial_r\phi_j \\
&-a_nRe\sum_{j=1}^{n_v}u_j\sum_{k=1}^{n_v}z_k^c\int\limits_{\Omega^f}\phi_i\phi_k\partial_z\phi_j+a_{n-1}Re\sum_{j=1}^{n_v}u_j\sum_{k=1}^{n_v}z_k^c(t_{n-1})\int\limits_{\Omega^f}\phi_i\phi_k\partial_z\phi_j \\
&-a_{n-2}Re\sum_{j=1}^{n_v}u_j\sum_{k=1}^{n_v}z_k^c(t_{n-2})\int\limits_{\Omega^f}\phi_i\phi_k\partial_z\phi_j \\
&-\frac{2\Delta_t}{3}\sum_{j=1}^{n_p}p_j\int\limits_{\Omega^f}\psi_j\partial_r\phi_i,
\end{aligned}\tag{4.92}$$

$$\begin{aligned}
\mathcal{M}_i^{r,1} = &-2\Delta_t\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})}{3Ca}m_r^{1,n}(r_{J^1},z_{J^1})+2\Delta_t\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)}{3Ca}m_r^1(r_a,z_a) \\
&+\frac{2\Delta_t}{3Ca}\sum_{j=1}^{n_v}\tilde{\sigma}_j^1\int\limits_{\partial\Omega^{1,f}}t_r^1\phi_j^1\partial_s\phi_i-\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{p}_j^g\int\limits_{\partial\Omega^{1,f}}\phi_i n_r^1\phi_j^1 \\
&-\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}u_j\int\limits_{\partial\Omega^{1,f}}\phi_i n_r^1\partial_r\phi_j-\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}w_j\int\limits_{\partial\Omega^{1,f}}\phi_i n_z^1\partial_r\phi_j,
\end{aligned}\tag{4.93}$$

$$
\begin{aligned}
\mathcal{M}_i^{r,2} = {} & \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\lambda}_j^2 \int\limits_{\partial\Omega^{2,f}} \phi_i\phi_j^2 n_r^2 + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\sum_{j=1}^{n_v}u_j \int\limits_{\partial\Omega^{2,f}} \phi_i\phi_j t_r^2 t_r^2 \\
& + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\sum_{j=1}^{n_v}w_j \int\limits_{\partial\Omega^{2,f}} \phi_i\phi_j t_r^2 t_z^2 \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\sum_{j=1}^{n_v}\tilde{u}_j^s \int\limits_{\partial\Omega^{2,f}} \phi_i\phi_j t_r^2 t_r^2 \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\sum_{j=1}^{n_v}\tilde{w}_j^s \int\limits_{\partial\Omega^{2,f}} \phi_i\phi_j t_r^2 t_z^2 \\
& - \frac{\Delta_t}{3Ca\,Es}\sum_{j=1}^{n_v}\tilde{u}_j^{s_2} \int\limits_{\partial\Omega^{2,f}} \phi_i\phi_j t_r^2 t_r^2 - \frac{\Delta_t}{3Ca\,Es}\sum_{j=1}^{n_v}\tilde{w}_j^{s_2} \int\limits_{\partial\Omega^{2,f}} \phi_i\phi_j t_r^2 t_z^2 \\
& - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}u_j \int\limits_{\partial\Omega^{2,f}} \phi_i n_r^1 \partial_r\phi_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}w_j \int\limits_{\partial\Omega^{2,f}} \phi_i n_z^1 \partial_r\phi_j ,
\end{aligned}
\tag{4.94}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,3} = {} & \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\lambda}_j^3 \int\limits_{\partial\Omega^3} \phi_j^3\phi_i n_r^3 + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\gamma}_j^3 \int\limits_{\partial\Omega^3} \phi_j^3\phi_i t_r^3 \\
& - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}u_j \int\limits_{\partial\Omega^{3,f}} \phi_i n_r^1 \partial_r\phi_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}w_j \int\limits_{\partial\Omega^{3,f}} \phi_i n_z^1 \partial_r\phi_j
\end{aligned}
\tag{4.95}
$$

and

$$
\begin{aligned}
\mathcal{M}_i^{r,4} = {} & \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\lambda}_j^4 \int\limits_{\partial\Omega^4} \phi_j^4\phi_i n_r^4 + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\gamma}_j^4 \int\limits_{\partial\Omega^4} \phi_j^4\phi_i t_r^4 \\
& - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}u_j \int\limits_{\partial\Omega^{4,f}} \phi_i n_r^1 \partial_r\phi_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}w_j \int\limits_{\partial\Omega^{4,f}} \phi_i n_z^1 \partial_r\phi_j
\end{aligned}
\tag{4.96}
$$

We now partition the domain into a series of closed curve-sided triangular *elements* (see figure 3), whose interiors are disjoint, and proceed to decompose the integrals above in a sum of integrals over each element. The boundary integrals, are in turn converted into a sum of integrals over line elements in the boundary, i.e. those portions of the boundary of the triangular elements that lie on the domain boundary $\partial\Omega$. Figure 4 shows that we have chosen only corner nodes of the elements to be pressure-and-velocity nodes, and illustrates the pressure-node numbering convention used.

This yields

$$
\mathcal{M}_i^r = \underbrace{\mathcal{M}_i^{r,0a} + \mathcal{M}_i^{r,0b} + \mathcal{M}_i^{r,0c} + \mathcal{M}_i^{r,0d}}_{\mathcal{M}_i^{r,0}} + \mathcal{M}_i^{r,1} + \mathcal{M}_i^{r,2} + \mathcal{M}_i^{r,3} + \mathcal{M}_i^{r,4}, \tag{4.97}
$$

where

$$
\mathcal{M}_i^{r,0a} = \sum_{e=1}^{n_{\mathrm{el}}}\left[-\frac{2\Delta_t St}{3}\int\limits_{\Omega_e}\phi_i\hat{g}_r\right], \tag{4.98}
$$

$$\mathcal{M}_i^{r,0b} = \sum_{e=1}^{n_{\mathrm{el}}} \left[\frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int_{\Omega_e} \partial_r\phi_j \partial_r\phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int_{\Omega_e} \partial_z\phi_j \partial_z\phi_i + a_n Re \sum_{j=1}^{n_v} u_j \int_{\Omega_e} \phi_i\phi_j \right.$$
$$\left. - a_{n-1} Re \sum_{j=1}^{n_v} u_j(t_{n-1}) \int_{\Omega_e} \phi_i\phi_j + a_{n-2} Re \sum_{j=1}^{n_v} u_j(t_{n-2}) \int_{\Omega_e} \phi_i\phi_j \right], \tag{4.99}$$

$$\mathcal{M}_i^{r,0c} = \sum_{e=1}^{n_{\mathrm{el}}} \left[\frac{2\Delta_t Re}{3} \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} u_k \int_{\Omega_e} \phi_i\phi_k\partial_r\phi_j + \frac{2\Delta_t Re}{3} \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} w_k \int_{\Omega_e} \phi_i\phi_k\partial_z\phi_j \right.$$
$$- a_n Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} r_k^c \int_{\Omega_e} \phi_i\phi_k\partial_r\phi_j + a_{n-1} Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} r_k^c(t_{n-1}) \int_{\Omega_e} \phi_i\phi_k\partial_r\phi_j$$
$$- a_{n-2} Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} r_k^c(t_{n-2}) \int_{\Omega_e} \phi_i\phi_k\partial_r\phi_j - a_n Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} z_k^c \int_{\Omega_e} \phi_i\phi_k\partial_z\phi_j$$
$$+ a_{n-1} Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} z_k^c(t_{n-1}) \int_{\Omega_e} \phi_i\phi_k\partial_z\phi_j$$
$$\left. - a_{n-2} Re \sum_{j=1}^{n_v} u_j \sum_{k=1}^{n_v} z_k^c(t_{n-2}) \int_{\Omega_e} \phi_i\phi_k\partial_z\phi_j \right], \tag{4.100}$$

$$\mathcal{M}_i^{r,0d} = \sum_{e=1}^{n_{\mathrm{el}}} \left[-\frac{2\Delta_t}{3} \sum_{j=1}^{n_p} p_j \int_{\Omega_e} \psi_j \partial_r\phi_i \right], \tag{4.101}$$

with Ω_e being element number e. Similarly, we have

$$\mathcal{M}_i^{r,1} = -2\Delta_t \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1})}{3Ca} m_r^{1,n}(r_{J^1}, z_{J^1}) + 2\Delta_t \frac{\sigma^1(r_a, z_a)\phi_i(r_a, z_a)}{3Ca} m_r^1(r_a, z_a)$$
$$+ \sum_{e_1=1}^{n_{\mathrm{el}}^{1,f}} \left[\frac{2\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \int_{\partial\Omega^{1,f}} t_r^1\phi_j^1\partial_s\phi_i - \sum_{j=1}^{n_v} \frac{2\Delta_t}{3} \tilde{p}_j^g \int_{\partial\Omega_{e_1}^1} \phi_i n_r^1 \phi_j^1 \right.$$
$$\left. - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r\phi_j - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r\phi_j \right], \tag{4.102}$$

and for boundary 2

$$
\begin{aligned}
\mathcal{M}_i^{r,2} = \sum_{e_2=1}^{n_{\mathrm{el}}^{2,f}} \Bigg[& \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}_j^2 \int\limits_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_j^2 n_r^2 + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_j t_r^2 t_r^2 \\
& + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_j t_r^2 t_z^2 \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{j=1}^{n_v} \tilde{u}_j^s \int\limits_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_j t_r^2 t_r^2 \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{j=1}^{n_v} \tilde{w}_j^s \int\limits_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_j t_r^2 t_z^2 - \frac{\Delta_t}{3Ca\,Es} \sum_{j=1}^{n_v} \tilde{u}_j^{s_2} \int\limits_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_j t_r^2 t_r^2 \\
& - \frac{\Delta_t}{3Ca\,Es} \sum_{j=1}^{n_v} \tilde{w}_j^{s_2} \int\limits_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_j t_r^2 t_z^2 - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{2,f}} \phi_i n_r^1 \partial_r \phi_j \\
& - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{2,f}} \phi_i n_z^1 \partial_r \phi_j \Bigg].
\end{aligned}
\tag{4.103}
$$

Similarly, for boundary 3

$$
\begin{aligned}
\mathcal{M}_i^{r,3} = \sum_{e_3=1}^{n_{n_{\mathrm{el}}}^{3,f}} \Bigg[& \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}_j^3 \int\limits_{\partial\Omega_{e_3}^3} \phi_j^3 \phi_i n_r^3 + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma}_j^3 \int\limits_{\partial\Omega_{e_3}^3} \phi_j^3 \phi_i t_r^3 \\
& - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{3,f}} \phi_i n_r^1 \partial_r \phi_j - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{3,f}} \phi_i n_z^1 \partial_r \phi_j \Bigg],
\end{aligned}
\tag{4.104}
$$

and

$$
\begin{aligned}
\mathcal{M}_i^{r,4} = \sum_{e_4=1}^{n_{\mathrm{el}}^4} \Bigg[& \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}_j^4 \int\limits_{\partial\Omega_{e_4}^4} \phi_j^4 \phi_i n_r^4 + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma}_j^4 \int\limits_{\partial\Omega_{e_4}^4} \phi_j^4 \phi_i t_r^4 \\
& - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{4,f}} \phi_i n_r^1 \partial_r \phi_j - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{4,f}} \phi_i n_z^1 \partial_r \phi_j \Bigg];
\end{aligned}
\tag{4.105}
$$

where n_{el} is the number of triangular elements, n_{el}^k is the number of line elements on the k-th boundary and $\partial\Omega_{e_k}^k$ is the part of $\partial\Omega^k$ that is contained in e_k.

Now, we impose that each function ϕ_j, ψ_j will only be supported on the elements that contain node j. Upon imposing this, we notice that the vast majority of the j and k indexed terms that are added in the sum on each element is identically null. This is, of course, because the integral of the product of these functions will be summing zero unless all functions involved are associated to (i.e. attain the value 1 in) some node on

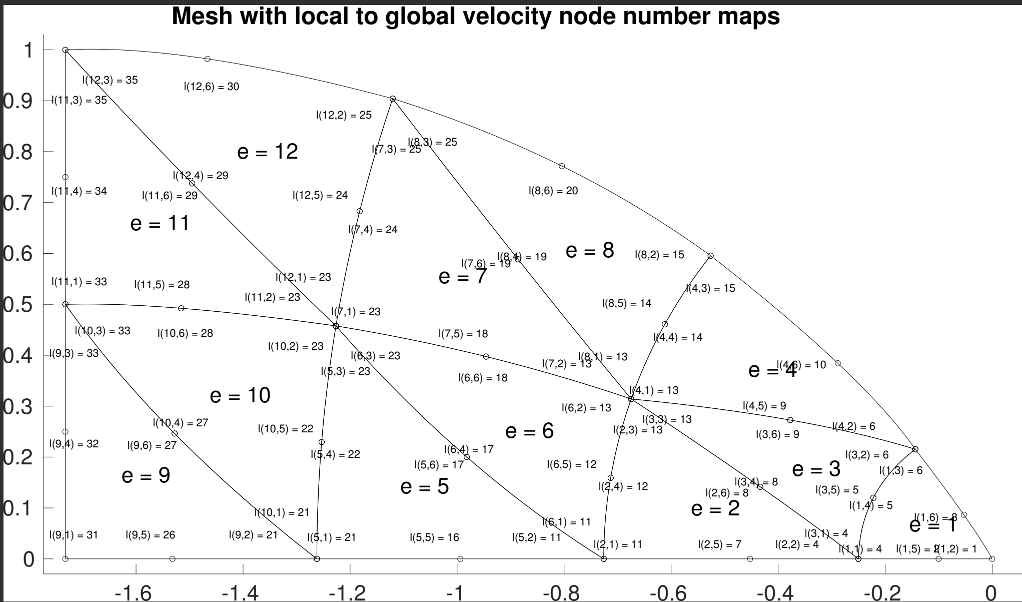

FIGURE 5. Mesh with local-to-global node number map l illustrated.

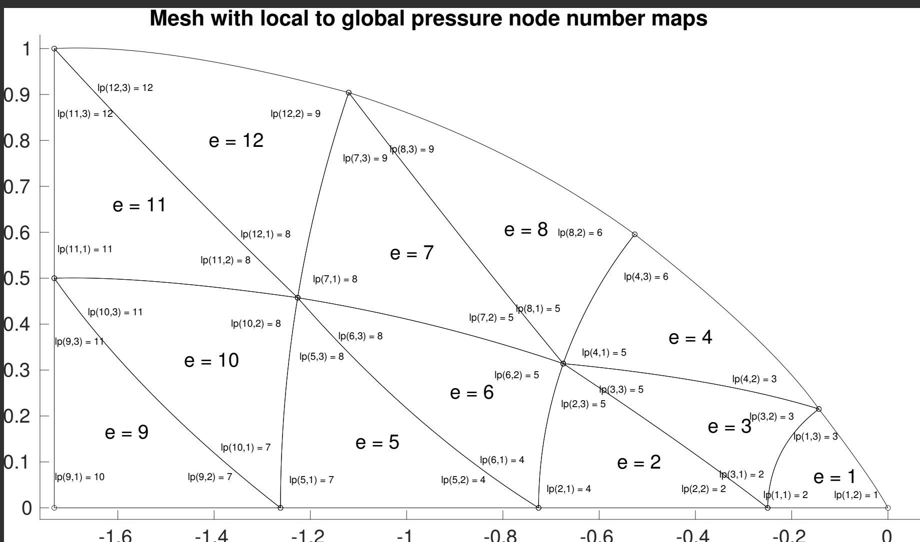

FIGURE 6. Mesh with local-to-global pressure-node number map l^p illustrated.

the element. Therefore, a more efficient way to express this sums is to resort to *local node numbering*. That is to say, when we are calculating the integral on each element, we know that non-zero contributions can only come from a basis function whose index corresponds to one of the node indices of the element at hand and it is therefore better to have the sums over k and j above to only go over the nodes contained in that element. In order to do this, we give each node another number for each element in which the node is contained. Hence, it is more convenient to re-write the sum above as

$$\mathcal{M}_i^{r,0a} = \sum_{e=1}^{n_{\text{el}}^f} \left[-\frac{2\Delta_t St}{3} \int\limits_{\Omega_e} \phi_i \hat{g}_r \right], \tag{4.106}$$

$$\begin{aligned}\mathcal{M}_i^{r,0b} = \sum_{e=1}^{n_{\text{el}}^f} \Bigg[& \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int\limits_{\Omega_e} \partial_r \phi_{l(e,jj)} \partial_r \phi_i + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int\limits_{\Omega_e} \partial_z \phi_{l(e,jj)} \partial_z \phi_i \\ & + a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,jj)} - a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-1}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,jj)} \\ & + a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-2}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,jj)} \Bigg],\end{aligned} \tag{4.107}$$

$$\begin{aligned}\mathcal{M}_i^{r,0c} = \sum_{e=1}^{n_{\text{el}}^f} \Bigg[& \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} u_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\ & + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} w_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\ & - a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\ & + a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\ & - a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\ & - a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\ & + a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\ & - a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \Bigg],\end{aligned} \tag{4.108}$$

$$\mathcal{M}_i^{r,0d} = \sum_{e=1}^{n_{\text{el}}^f} \left[-\frac{2\Delta_t}{3} \sum_{jj=1}^{n_p^e} p_{l^p(e,jj)} \int\limits_{\Omega_e} \psi_{l^p(e,jj)} \partial_r \phi_i \right], \tag{4.109}$$

$$
\begin{aligned}
\mathcal{M}_i^{r,1} = &-2\Delta_t \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1})}{3Ca} m_r^{1,n}(r_{J^1}, z_{J^1}) + 2\Delta_t \frac{\sigma^1(r_a, z_a)\phi_i(r_a, z_a)}{3Ca} m_r^1(r_a, z_a) \\
&+ \sum_{e_1=1}^{n_{\mathrm{el}}^{1,f}} \Bigg[\frac{2\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \int_{\partial\Omega^{1,f}} t_r^1 \phi_j^1 \partial_s \phi_i - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1}} \tilde{p}_{l_1(e_1,jj)}^g \int_{\partial\Omega_{e_1}^1} \phi_i^1 \phi_{l_1(e_1,jj)}^1 n_r^1 \\
&- \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r \phi_j - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r \phi_j \Bigg],
\end{aligned}
\tag{4.110}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,2} = &\sum_{e_2=1}^{n_{\mathrm{el}}^{2,f}} \Bigg[\frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2}} \tilde{\lambda}_{l_2(e_2,jj)}^2 \int_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_{l_2(e_2,jj)}^2 n_r^2 \\
&+ \left(\frac{1 + 4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} \int_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_{l_2(e_2,jj)}^2 t_r^2 t_r^2 \\
&+ \left(\frac{1 + 4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} \int_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_{l_2(e_2,jj)}^2 t_r^2 t_z^2 \\
&+ \left(\frac{1 - 4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} \tilde{u}_{l_2(e_2,jj)}^s \int_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_{l_2(e_2,jj)}^2 t_r^2 t_r^2 \\
&+ \left(\frac{1 - 4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} \tilde{w}_{l_2(e_2,jj)}^s \int_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_{l_2(e_2,jj)}^2 t_r^2 t_z^2 \\
&- \frac{\Delta_t}{3Ca\,Es} \sum_{jj=1}^{n_v^{2,e_2}} \tilde{u}_{l_2(e_2,jj)}^{s_2} \int_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_{l_2(e_2,jj)}^2 t_r^2 t_r^2 \\
&- \frac{\Delta_t}{3Ca\,Es} \sum_{jj=1}^{n_v^{2,e_2}} \tilde{w}_{l_2(e_2,jj)}^{s_2} \int_{\partial\Omega_{e_2}^{2,f}} \phi_i \phi_{l_2(e_2,jj)}^2 t_r^2 t_z^2 - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int_{\partial\Omega^{2,f}} \phi_i n_r^1 \partial_r \phi_j \\
&- \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int_{\partial\Omega^{2,f}} \phi_i n_z^1 \partial_r \phi_j \Bigg].
\end{aligned}
\tag{4.111}
$$

$$\mathcal{M}_i^{r,3} = \sum_{e_3=1}^{n_{\text{el}}^3} \left[\frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \tilde{\lambda}^3_{l_3(e_3,jj)} \int\limits_{\partial\Omega^3_{e_3}} \phi^3_{l_3(e_3,jj)} \phi_i n^3_r + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \tilde{\gamma}^3_{l_3(e_3,jj)} \int\limits_{\partial\Omega^3_{e_3}} \phi^3_{l_3(e_3,jj)} \phi_i t^3_r \right.$$
$$\left. - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v} u_{l_3(e_3,jj)} \int\limits_{\partial\Omega^{3,f}} \phi_i n^1_r \partial_r \phi_j - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{3,f}} \phi_i n^1_z \partial_r \phi_j \right], \tag{4.112}$$

and

$$\mathcal{M}_{e,ii}^{r,4} = \sum_{e_4=1}^{n_{\text{el}}^4} \left[\frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4}} \tilde{\lambda}^4_{l_4(e_4,jj)} \int\limits_{\partial\Omega^4_{e_4}} \phi^4_{l_4(e_4,jj)} \phi^4_i n^4_r + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4}} \tilde{\gamma}^4_{l_4(e_4,jj)} \int\limits_{\partial\Omega^4_{e_4}} \phi^4_{l_4(e_4,jj)} \phi_i t^4_r \right.$$
$$\left. - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{4,f}} \phi_i n^1_r \partial_r \phi_j - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{4,f}} \phi_i n^1_z \partial_r \phi_j \right]; \tag{4.113}$$

where double-letter indices are used to reference local node numbers, n_v^e is the number of nodes where velocity is calculated in element e, n_p^e is the number of nodes where pressure is calculated in element e, n_v^{i,e_i} is the number of nodes where velocity is to be calculated on line element e_i of boundary i, and $l(e,jj) = j$, i.e. $l(e,jj)$ maps the local number jj of a node in element e to its global number j (see figure 5), $l^p(e,jj) = j$ maps the local node number jj of element e onto its pressure-node number j (see figure 6), and similarly $l_k(e_k,jj) = j$ maps the local node number jj of line-element e_k in boundary k to its global node number j (see figures 7 and 8).

We note that the terms that involve derivatives with respect to r and z in equations (4.110), (4.111), (4.112) and (4.113) did not change the j index for the $l_x(e_x,jj)$. This is beacuase, in these terms, we need to go through the bulk indexing, rather than the surface indexes, as bulk functions contribute to the derivatives.

We now consider functions $\tilde{u}^s_j$, $\tilde{w}^s_j$, $\tilde{\sigma}^1_j$, $\tilde{\sigma}^2_j$, $\tilde{\lambda}^2_j$ and $\tilde{\lambda}^3_j$, and $\tilde{p}^g_j$. We recall that for all j indices that correspond to nodes outside their respective boundaries these functions are identically zero. It is therefore more convenient to introduce functions u^s_j, w^s_j, σ^1_j, σ^2_j, λ^2_j and λ^3_j, and p^g_j where j is *the boundary numbering*, i.e. a numbering of the nodes that lie on the corresponding boundary (boundary 2 in the case of functions with the super-index s, and boundary 1 in the case of the function with the super index g). We also introduce function $l^1_1(e_1,jj)$ which maps local node number jj on element e_1 on boundary 1 to its corresponding boundary-node number, and the analogue functions $l^2_2(e_2,jj)$ and $l^3_3(e_3,jj)$ (see figure 9, and compare it to 7). The only difference between functions $l_k(e_k,jj)$ and $l^k_k(e_k,jj)$ is that the image of the latter is the node number in the boundary numbering and in the former it is the node number in the global numbering. In other words, $\tilde{\sigma}^1_{l_1(e_1,jj)} = \sigma^1_{l^1_1(e_1,jj)}$, though the second notation avoids numbering identically null functions.

Re-writing the equations above under this new convention we have

$$\mathcal{M}_i^{r,0a} = \sum_{e=1}^{n_{\text{el}}^f} \left[-\frac{2\Delta_t St}{3} \int\limits_{\Omega_e} \phi_i \hat{g}_r \right], \tag{4.114}$$

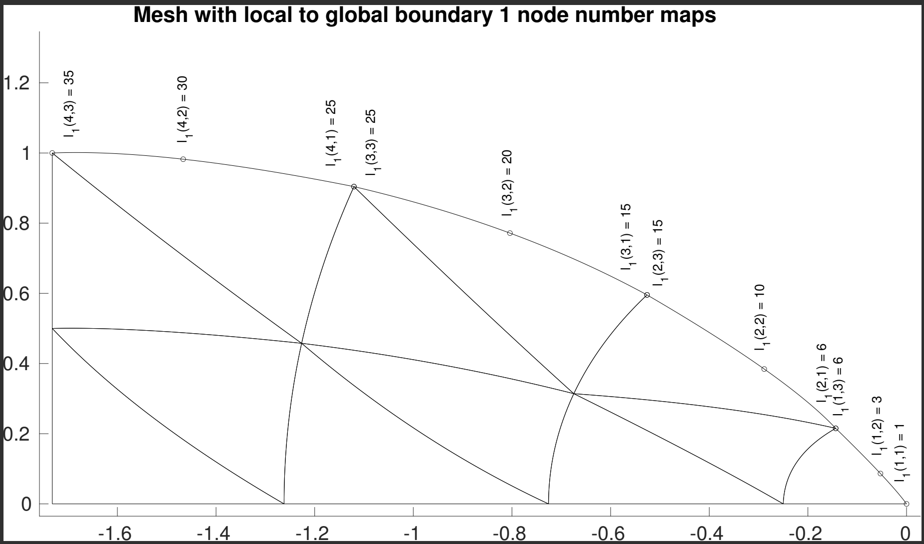

FIGURE 7. Local line-element to global-velocity-node number map l_1.

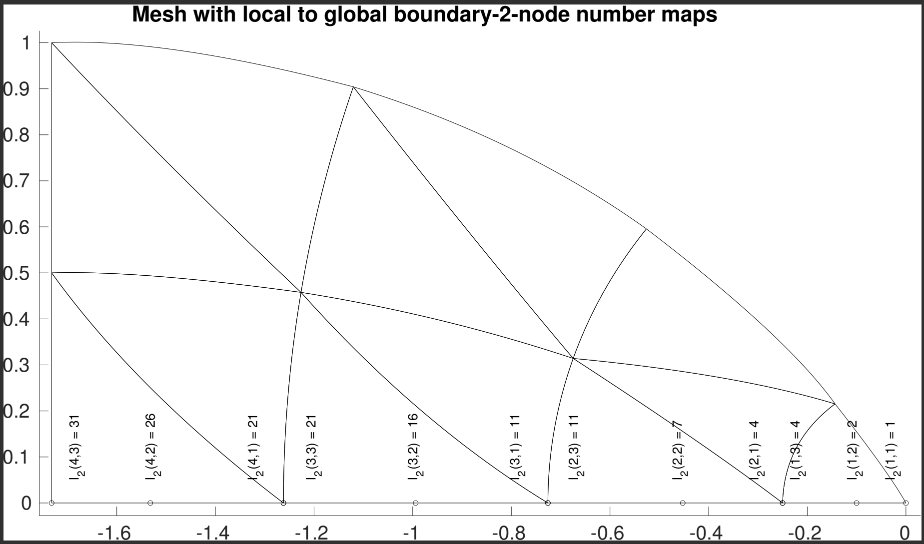

FIGURE 8. Local line-element to global-velocity-node number map l_2.

$$
\begin{aligned}
\mathcal{M}_i^{r,0b} = \sum_{e=1}^{n_{\mathrm{el}}^f} \Bigg[& \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int\limits_{\Omega_e} \partial_r \phi_{l(e,jj)} \partial_r \phi_i + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int\limits_{\Omega_e} \partial_z \phi_{l(e,jj)} \partial_z \phi_i \\
& + a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,jj)} - a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-1}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,jj)} \\
& + a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-2}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,jj)} \Bigg],
\end{aligned}
\tag{4.115}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,0c} = \sum_{e=1}^{n_{\mathrm{el}}^f} \Bigg[& \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} u_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} w_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
& - a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& + a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& - a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& - a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
& + a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
& - a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \Bigg],
\end{aligned} \tag{4.116}
$$

$$
\mathcal{M}_i^{r,0d} = \sum_{e=1}^{n_{\mathrm{el}}^f} \left[-\frac{2\Delta_t}{3} \sum_{jj=1}^{n_p^e} p_{l^p(e,jj)} \int\limits_{\Omega_e} \psi_{l^p(e,jj)} \partial_r \phi_i \right], \tag{4.117}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,1} = & -2\Delta_t \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1})}{3Ca} m_r^{1,n}(r_{J^1}, z_{J^1}) + 2\Delta_t \frac{\sigma^1(r_a, z_a)\phi_i(r_a, z_a)}{3Ca} m_r^1(r_a, z_a) \\
& + \sum_{e_1=1}^{n_{\mathrm{el}}^{1,f}} \Bigg[\frac{2\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \int\limits_{\partial\Omega^{1,f}} t_r^1 \phi_j^1 \partial_s \phi_i - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1}} p^g_{l_1^1(e_1,jj)} \int\limits_{\partial\Omega^1_{e_1}} \phi_i^1 \phi^1_{l_1(e_1,jj)} n_r^1 \\
& \qquad - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1^T}} u_{l_1(e_1^T,jj)} \int\limits_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r \phi_{l_1(e_1^T,jj)} \\
& \qquad - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1^T}} w_{l_1(e_1^T,jj)} \int\limits_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r \phi_{l_1(e_1^T,jj)} \Bigg],
\end{aligned} \tag{4.118}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,2} = \sum_{e_2=1}^{n_{\mathrm{el}}^{2,f}} \Bigg[& \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2}} \tilde{\lambda}^2_{l_2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} n_r^2 \\
& + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_r^2 \\
& + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_z^2 \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} \tilde{u}^s_{l_2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_r^2 \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} \tilde{w}^s_{l_2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_z^2 \\
& - \frac{\Delta_t}{3Ca\,Es} \sum_{jj=1}^{n_v^{2,e_2}} \tilde{u}^{s_2}_{l_2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_r^2 \\
& - \frac{\Delta_t}{3Ca\,Es} \sum_{jj=1}^{n_v^{2,e_2}} \tilde{w}^{s_2}_{l_2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_z^2 \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2^T}} u_{l_2(e_2,jj))} \int_{\partial\Omega^{2,f}} \phi_i n_r^1 \partial_r \phi_{l_2(e_2^T,jj))} \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2^T}} w_{l_2(e_2,jj))} \int_{\partial\Omega^{2,f}} \phi_i n_z^1 \partial_r \phi_{l_2(e_2^T,jj))} \Bigg].
\end{aligned}
\tag{4.119}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,3} = \sum_{e_3=1}^{n_{\mathrm{el}}^{3}} \Bigg[& \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \tilde{\lambda}^3_{l_3(e_3,jj)} \int_{\partial\Omega^{3}_{e_3}} \phi^3_{l_3(e_3,jj)} \phi_i n_r^3 + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \tilde{\gamma}^3_{l_3(e_3,jj)} \int_{\partial\Omega^{3}_{e_3}} \phi^3_{l_3(e_3,jj)} \phi_i t_r^3 \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} u_{l_3(e_3,jj)} \int_{\partial\Omega^{3,f}} \phi_i n_r^1 \partial_r \phi_{l_3(e_3^T,jj)} \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} w_{l_3(e_3,jj)} \int_{\partial\Omega^{3,f}} \phi_i n_z^1 \partial_r \phi_{l_3(e_3^T,jj)} \Bigg],
\end{aligned}
\tag{4.120}
$$

and

$$\mathcal{M}^{r,4}_{e,ii} = \sum_{e_4=1}^{n^4_{\mathrm{el}}} \left[\frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4}} \tilde{\lambda}^4_{l_4(e_4,jj)} \int\limits_{\partial\Omega^4_{e_4}} \phi^4_{l_4(e_4,jj)} \phi^4_i n^4_r + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4}} \tilde{\gamma}^4_{l_5(e_5,jj)} \int\limits_{\partial\Omega^4_{e_4}} \phi^4_{l_4(e_4,jj)} \phi_i t^4_r \right.$$

$$- \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4^T}} u_{l_4(e_4^T,jj))} \int\limits_{\partial\Omega^{4,f}} \phi_i n^4_r \partial_r \phi_{l_4(e_4^T,jj))}$$

$$\left. - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4^T}} w_{l_4(e_4^T,jj))} \int\limits_{\partial\Omega^{4,f}} \phi_i n^4_z \partial_r \phi_{l_4(e_4^T,jj))} \right]; \tag{4.121}$$

where $n_v^{1,e_x^{x,T}}$ stands for the number of velocity nodes in the triagular element that contains the line element over which we are integrating and e_x^T, use in $l_x(e_x^T, jj)$, stands for the traingular element wich contains line element e_x.

A form of index optimisation that is very similar to what was done with j can be done in terms of index i. For a given index i, only the integrals on the elements that contain this node can have a non-zero contribution to the i-th residual. Hence, its is more convenient to loop over each element's nodes and find the contribution to $\mathcal{M}_i^{r,l}$ for each of the is that are indices of the nodes in the element at hand, and to sum this contribution to each $\mathcal{M}_i^{r,l}$. Passing to local node number for index i, we have

$$\mathcal{M}^{r,0a}_i = \sum_{\substack{e=1 \\ i=l(e,ii)}}^{n^f_{\mathrm{el}}} \left[-\frac{2\Delta_t St}{3} \int\limits_{\Omega_e} \phi_{l(e,ii)} \hat{g}_r \right], \tag{4.122}$$

$$\mathcal{M}^{r,0b}_i = \sum_{\substack{e=1 \\ i=l(e,ii)}}^{n^f_{\mathrm{el}}} \left[\frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int\limits_{\Omega_e} \partial_r \phi_{l(e,jj)} \partial_r \phi_{l(e,ii)} \right.$$

$$+ \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int\limits_{\Omega_e} \partial_z \phi_{l(e,jj)} \partial_z \phi_{l(e,ii)} + a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,jj)}$$

$$- a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-1}) \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,jj)}$$

$$\left. + a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-2}) \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,jj)} \right], \tag{4.123}$$

$$
\begin{aligned}
\mathcal{M}_i^{r,0c} = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}^f} \Bigg[& \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} u_{l(e,kk)} \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)} \\
& + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} w_{l(e,kk)} \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)} \\
& - a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)} \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)} \\
& + a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)} \\
& - a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)} \\
& - a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)} \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)} \\
& + a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)} \\
& - a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)} \Bigg],
\end{aligned}
\tag{4.124}
$$

$$
\mathcal{M}_i^{r,0d} = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}^f} \left[-\frac{2\Delta_t}{3} \sum_{jj=1}^{n_p^e} p_{l^p(e,jj)} \int_{\Omega_e} \psi_{l^p(e,jj)}\partial_r\phi_{l(e,ii)} \right],
\tag{4.125}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,1} = & -2\Delta_t \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1})}{3Ca} m_r^{1,n}(r_{J^1}, z_{J^1}) + 2\Delta_t \frac{\sigma^1(r_a, z_a)\phi_i(r_a, z_a)}{3Ca} m_r^1(r_a, z_a) \\
& + \sum_{e_1=1}^{n_{\mathrm{el}}^{1,f}} \Bigg[\frac{2\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \int_{\partial\Omega^{1,f}} t_r^1 \phi_j^1 \partial_s \phi_i - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1}} p^g_{l_1^1(e_1,jj)} \int_{\partial\Omega^1_{e_1}} \phi_i^1 \phi^1_{l_1(e_1,jj)n_r^1} \\
& \qquad - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1^T}} u_{l_1(e_1^T,jj)} \int_{\partial\Omega^{1,f}} \phi_i n_r^1 \partial_r \phi_{l_1(e_1^T,jj)} \\
& \qquad - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1^T}} w_{l_1(e_1^T,jj)} \int_{\partial\Omega^{1,f}} \phi_i n_z^1 \partial_r \phi_{l_1(e_1^T,jj)} \Bigg],
\end{aligned}
\tag{4.126}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,2} = \sum_{e_2=1}^{n_{\mathrm{el}}^{2,f}} \Bigg[& \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2}} \lambda^2_{l_2^2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} n_r^2 \\
& + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_r^2 \\
& + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_z^2 \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} u^s_{l_2^2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_r^2 \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} w^s_{l_2^2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_z^2 \\
& - \frac{\Delta_t}{3Ca\,Es} \sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_r^2 \\
& - \frac{\Delta_t}{3Ca\,Es} \sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)} \int_{\partial\Omega^{2,f}_{e_2}} \phi_i \phi^2_{l_2(e_2,jj)} t_r^2 t_z^2 \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2^T}} u_{l_2(e_2,jj))} \int_{\partial\Omega^{2,f}} \phi_i n_r^1 \partial_r \phi_{l_2(e_2^T,jj))} \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2^T}} w_{l_2(e_2,jj))} \int_{\partial\Omega^{2,f}} \phi_i n_z^1 \partial_r \phi_{l_2(e_2^T,jj))} \Bigg].
\end{aligned}
\tag{4.127}
$$

$$
\begin{aligned}
\mathcal{M}_i^{r,3} = \sum_{e_3=1}^{n_{\mathrm{el}}^{3}} \Bigg[& \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \lambda^3_{l_3^3(e_3,jj)} \int_{\partial\Omega^{3}_{e_3}} \phi^3_{l_3(e_3,jj)} \phi_i n_r^3 + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \gamma^3_{l_3^3(e_3,jj)} \int_{\partial\Omega^{3}_{e_3}} \phi^3_{l_3(e_3,jj)} \phi_i t_r^3 \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} u_{l_3(e_3,jj)} \int_{\partial\Omega^{3,f}} \phi_i n_r^1 \partial_r \phi_{l_3(e_3^T,jj)} \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} w_{l_3(e_3,jj)} \int_{\partial\Omega^{3,f}} \phi_i n_z^1 \partial_r \phi_{l_3(e_3^T,jj)} \Bigg],
\end{aligned}
\tag{4.128}
$$

and

$$\mathcal{M}_i^{r,4} = \sum_{e_4=1}^{n_{\mathrm{el}}^4} \left[\frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4}} \lambda^4_{l_4^4(e_4,jj)} \int\limits_{\partial\Omega_{e_4}^4} \phi^4_{l_4(e_4,jj)} \phi_i^4 n_r^4 + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4}} \gamma^4_{l_4^4(e_4,jj)} \int\limits_{\partial\Omega_{e_4}^4} \phi^4_{l_4(e_4,jj)} \phi_i t_r^4 \right.$$
$$- \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4^T}} u_{l_4(e_4^T,jj))} \int\limits_{\partial\Omega^{4,f}} \phi_i n_r^1 \partial_r \phi_{l_4(e_4^T,jj))}$$
$$\left. - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4^T}} w_{l_4(e_4^T,jj))} \int\limits_{\partial\Omega^{4,f}} \phi_i n_z^1 \partial_r \phi_{l_4(e_4^T,jj))} \right]; \tag{4.129}$$

We now introduce the decomposition

$$\mathcal{M}_i^r = \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \left[\mathcal{M}_{e,ii}^{r,0a} + \mathcal{M}_{e,ii}^{r,0b} + \mathcal{M}_{e,ii}^{r,0c} + \mathcal{M}_{e,ii}^{r,0d} \right]}_{\mathcal{M}_i^{r,0}}$$
$$- 2\Delta_t \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1})}{3Ca} m_r^{1,n}(r_{J^1}, z_{J^1}) + 2\Delta_t \frac{\sigma^1(r_a, z_a)\phi_i(r_a, z_a)}{3Ca} m_r^1(r_a, z_a)$$
$$+ \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \mathcal{M}_{e_1,ii}^{r,1} + \underbrace{\sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \mathcal{M}_{e,ii}^{r,2}}_{\mathcal{M}_i^{r,2}} + \underbrace{\sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\mathrm{el}}^3} \mathcal{M}_{e_3,ii}^{r,3}}_{\mathcal{M}_i^{r,3}} + \underbrace{\sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\mathrm{el}}^4} \mathcal{M}_{e_4,ii}^{r,4}}_{\mathcal{M}_i^{r,4}}; \tag{4.130}$$

where

$$\mathcal{M}_i^{r,1} = -2\Delta_t \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1})}{3Ca} m_r^{1,n}(r_{J^1}, z_{J^1})$$
$$+ 2\Delta_t \frac{\sigma^1(r_a, z_a)\phi_i(r_a, z_a)}{3Ca} m_r^1(r_a, z_a) + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \mathcal{M}_{e_1,ii}^{r,1} \tag{4.131}$$

and

$$\mathcal{M}_{e,ii}^{r,0a} = -\frac{2\Delta_t St}{3} \underbrace{\int\limits_{\Omega_e} \phi_{l(e,ii)} \hat{g}_r}_{a_{ii,g_r}(e)}, \tag{4.132}$$

$$\mathcal{M}^{r,0b}_{e,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)} \underbrace{\int\limits_{\Omega_e} \partial_r\phi_{l(e,ii)}\partial_r\phi_{l(e,jj)}}_{a^{r,r}_{ii,jj}(e)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)} \underbrace{\int\limits_{\Omega_e} \partial_z\phi_{l(e,ii)}\partial_z\phi_{l(e,jj)}}_{a^{z,z}_{ii,jj}(e)}$$

$$+ a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \underbrace{\int\limits_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)} - a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-1}) \underbrace{\int\limits_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)}$$

$$+ a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-2}) \underbrace{\int\limits_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)}, \tag{4.133}$$

$$
\begin{aligned}
\mathcal{M}^{r,0c}_{e,ii} = \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} u_{l(e,kk)} \underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
+ \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} w_{l(e,kk)} \underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
-a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)} \underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
+a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) \underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
-a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) \underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
-a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)} \underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
+a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) \underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
-a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) \underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)},
\end{aligned} \tag{4.134}
$$

and

$$
\mathcal{M}^{0,d}_{e,ii} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{n_p^e} p_{l^p(e,jj)} \underbrace{\int_{\Omega_e} \psi_{l^p(e,jj)}\partial_r\phi_{l(e,ii)}}_{b^r_{jj,ii}(e)}; \tag{4.135}
$$

We recall that for boundary 1 we have

$$\mathcal{M}^{r,1}_{e,ii} = \frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{1,e_1}} \sigma^1_{l_1^1(e_1,jj)} \underbrace{\int_{\partial\Omega^{1,f}} t^1_r \phi^1_{l_1(e_1,jj)} \partial_s \phi_{l_1(e_1,ii)}}_{c^s_{jj,ii,t_r}(e_1)}$$

$$- \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1}} p^g_{l_1^1(e_1,jj)} \underbrace{\int_{\partial\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_r}_{c_{ii,jj,n_r}(e_1)} \tag{4.136}$$

$$- \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1^T}} u_{l_1(e_1^T,jj)} \underbrace{\int_{\partial\Omega^{1,f}} \phi_{l_1(e_1,ii)} n^1_r \partial_r \phi_{l_1(e_1^T,jj)}}_{c^r_{ii,jj,n_r}(e_1)}$$

$$- \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1^T}} w_j \underbrace{\int_{\partial\Omega^{1,f}} \phi_{l_1(e_1,ii)} n^1_z \partial_r \phi_{l_1(e_1^T,jj)}}_{c^r_{ii,jj,n_z}(e_1)};$$

for boundary 2

$$
\begin{aligned}
\mathcal{M}^{r,2}_{e,ii} = \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2}} \lambda^2_{l_2^2(e_2,jj)} & \underbrace{\int\limits_{\partial\Omega^{2,f}_{e_2}} \phi_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} n^2_r}_{d_{ii,jj,n_r}(e_2)} \\
& + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,f}_{e_2}} \phi_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_r}_{d_{ii,jj,t_r,t_r}(e_2)} \\
& + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,f}_{e_2}} \phi_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_z}_{d_{ii,jj,t_r.t_z}(e_2)} \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} u^s_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,f}_{e_2}} \phi_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_r}_{d_{ii,jj,t_r,t_r}(e_2)} \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \sum_{jj=1}^{n_v^{2,e_2}} w^s_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,f}_{e_2}} \phi_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_z}_{d_{ii,jj,t_r,t_z}(e_2)} \qquad (4.137) \\
& - \frac{\Delta_t}{3Ca\,Es} \sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,f}_{e_2}} \phi_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_r}_{d_{ii,jj,t_r,t_r}(e_2)} \\
& - \frac{\Delta_t}{3Ca\,Es} \sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,f}_{e_2}} \phi_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_z}_{d_{ii,jj,t_r,t_z}(e_2)} \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2^T}} u_j \underbrace{\int\limits_{\partial\Omega^{2,f}} \phi_{l_2(e_2,ii)} n^2_r \partial_r \phi_{l_2(e_2^T,jj)}}_{d^r_{ii,jj,n_r}(e_2)} \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2^T}} w_{l_2(e_2^T,jj)} \underbrace{\int\limits_{\partial\Omega^{2,f}} \phi_{l_2(e_2,ii)} n^2_z \partial_r \phi_{l_2(e_2^T,jj)}}_{d^r_{ii,jj,n_z}(e_2)};
\end{aligned}
$$

for boundary 3

$$\begin{aligned}\mathcal{M}^{r,3}_{e,ii} = {} & \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \lambda^3_{l_3^3(e_3,jj)} \underbrace{\int_{\partial\Omega^3_{e_3}} \phi^3_{l_3(e_3,ii)}\phi^3_{l_3(e_3,jj)} n^3_r}_{f_{ii,jj,n_r}(e_3)} \\ & + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \gamma^3_{l_3^3(e_3,jj)} \underbrace{\int_{\partial\Omega^3_{e_3}} \phi^3_{l_3(e_3,ii)}\phi^3_{l_3(e_3,jj)} t^3_r}_{f_{ii,jj,t_r}(e_3)} \\ & - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} u_j \underbrace{\int_{\partial\Omega^{3,f}} \phi_i n^1_r \partial_r \phi_j}_{f^r_{ii,jj,n_r}(e_3)} \\ & - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3^T}} w_{l_3(e_3^T,jj)} \underbrace{\int_{\partial\Omega^{3,f}} \phi_{l_3(e_3,ii)} n^1_z \partial_r \phi_{l_3(e_3^T,jj)}}_{f^r_{ii,jj,n_z}(e_3)};\end{aligned} \tag{4.138}$$

and for boundary 4

$$\begin{aligned}\mathcal{M}^{r,4}_{e,ii} = {} & \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \lambda^4_{l_4^4(e_3,jj)} \underbrace{\int_{\partial\Omega^4_{e_4}} \phi^3_{l_4(e_4,ii)}\phi^3_{l_4(e_4,jj)} n^4_r}_{e_{ii,jj,n_r}(e_4)} \\ & + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \gamma^4_{l_4^4(e_4,jj)} \underbrace{\int_{\partial\Omega^4_{e_4}} \phi^4_{l_4(e_4,ii)}\phi^4_{l_4(e_4,jj)} t^4_r}_{e_{ii,jj,t_r}(e_4)} \\ & - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4^T}} u_{l_4(e_4^T,jj)} \underbrace{\int_{\partial\Omega^{4,f}} \phi_{l_4(e_4,ii)} n^1_r \partial_r \phi_{l_4(e_4^T,jj)}}_{e^r_{ii,jj,n_r}(e_4)} \\ & - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4^T}} w_{l_4(e_4^T,jj)} \underbrace{\int_{\partial\Omega^{4,f}} \phi_{l_4(e_4,ii)} n^1_z \partial_r \phi_{l_4(e_4^T,jj)}}_{e^r_{ii,jj,n_z}(e_4)};.\end{aligned} \tag{4.139}$$

The names of the integral quantities above, are chosen so that a and b always stand for integrals in a triangular element, with a been integral of the velocity-interpolating functions ϕ and their spatial derivatives, and b been integrals of the pressure-interpolating functions ψ potentially multiplied by some ϕ functions and/or their derivatives. Similarly, integrals identified as c, d and f have a domain on the free, solid and inflow boundary, respectively. Moreover, sub-indices indicate the quantities been integrated and super-indices indicate which derivatives are taken of these quantities. Hence, the number of super-indices is always lower than the number of sub-indices. Furthermore, k super-indices indicate that the last k sub-indices correspond to differentiated variables and each one of the last k sub-indices is differentiated with respect to its matching number of k super-indices (e.g. if there are two super-indices, the last one indicates the variable with respect to which one is supposed to differentiate the function indicated in the last sub-index, and the first super-index indicates the variable with respect to which one must differentiate the function indicated in the sub-index the is just prior to the last sub-index).

Re-writing we have

$$\mathcal{M}^{r,0a}_{e,ii} = -\frac{2\Delta_t St}{3} a_{ii,g_r}(e), \tag{4.140}$$

$$\begin{aligned}\mathcal{M}^{r,0b}_{e,ii} = & \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)} a^{r,r}_{ii,jj}(e) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)} a^{z,z}_{ii,jj}(e) \\ & + a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} a_{ii,jj}(e) - a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-1}) a_{ii,jj}(e) \\ & + a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-2}) a_{ii,jj}(e),\end{aligned} \tag{4.141}$$

$$\begin{aligned}\mathcal{M}^{r,0c}_{e,ii} = & \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} u_{l(e,kk)} a^{r}_{ii,kk,jj}(e) \\ & + \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} w_{l(e,kk)} a^{z}_{ii,kk,jj}(e) \\ & - a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)} a^{r}_{ii,kk,jj}(e) \\ & + a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) a^{r}_{ii,kk,jj}(e) \\ & - a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) a^{r}_{ii,kk,jj}(e) \\ & - a_n Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)} a^{z}_{ii,kk,jj}(e) \\ & + a_{n-1} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) a^{z}_{ii,kk,jj}(e) \\ & - a_{n-2} Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) a^{z}_{ii,kk,jj}(e),\end{aligned} \tag{4.142}$$

$$\mathcal{M}^{0,d}_{e,ii} = -\frac{2\Delta_t}{3}\sum_{jj=1}^{n_p^e} p_{l^p(e,jj)} b^r_{jj,ii}(e); \tag{4.143}$$

For boundary 1 we have

$$\begin{aligned}\mathcal{M}^{r,1}_{e,ii} = &\frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{1,e_1}} \tilde{\sigma}^1_{l_1^1(e_1,jj)} c^s_{jj,ii,t_r}(e_1) - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1}} p^g_{l_1^1(e_1,jj)} c_{ii,jj,n_r}(e_1)\\ &- \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1^T}} u_{l_1(e_1^T,jj)} c^r_{ii,jj,n_r} - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1^T}} w_{l_1(e_1^T,jj)} c^r_{ii,jj,n_z};\end{aligned} \tag{4.144}$$

for boundary 2

$$\begin{aligned}\mathcal{M}^{r,2}_{e,ii} = &\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} \lambda^2_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} d_{ii,jj,t_r,t_r}(e_2)\\ &+ \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} d_{ii,jj,t_r.t_z}(e_2)\\ &+ \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\sum_{jj=1}^{n_v^{2,e_2}} u^s_{l_2^2(e_2,jj)} d_{ii,jj,t_r,t_r}(e_2)\\ &+ \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right)\Delta_t\sum_{jj=1}^{n_v^{2,e_2}} w^s_{l_2^2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2)\\ &- \frac{\Delta_t}{3Ca\,Es}\sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_r,t_r}(e_2) - \frac{\Delta_t}{3Ca\,Es}\sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2)\\ &- \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2^T}} u_{l_2(e_2^T,jj)} d^r_{ii,jj,n_r} - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2^T}} w_{l_2(e_2^T,jj)} d^r_{ii,jj,n_z};\end{aligned} \tag{4.145}$$

for boundary 3

$$\begin{aligned}\mathcal{M}^{r,3}_{e,ii} = &\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \lambda^3_{l_3^3(e_3,jj)} f_{ii,jj,n_r}(e_3) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \gamma^3_{l_3^3(e_3,jj)} f_{ii,jj,t_r}(e_3)\\ &- \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3^T}} u_{l_3(e_3^T,jj)} f^r_{ii,jj,n_r} - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3^T}} w_{l_3(e_3^T,jj)} f^r_{ii,jj,n_z}.\end{aligned} \tag{4.146}$$

and for boundary 4

$$\begin{aligned}\mathcal{M}^{r,4}_{e,ii} = &\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \lambda^4_{l_4^4(e_3,jj)} e_{ii,jj,n_r}(e_4) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \gamma^4_{l_4^4(e_4,jj)} e_{ii,jj,t_r}(e_4)\\ &- \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4^T}} u_{l_4(e_4^T,jj)} e^r_{ii,jj,n_r} - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4^T}} w_{l_4(e_4^T,jj)} e^r_{ii,jj,n_z}.\end{aligned} \tag{4.147}$$

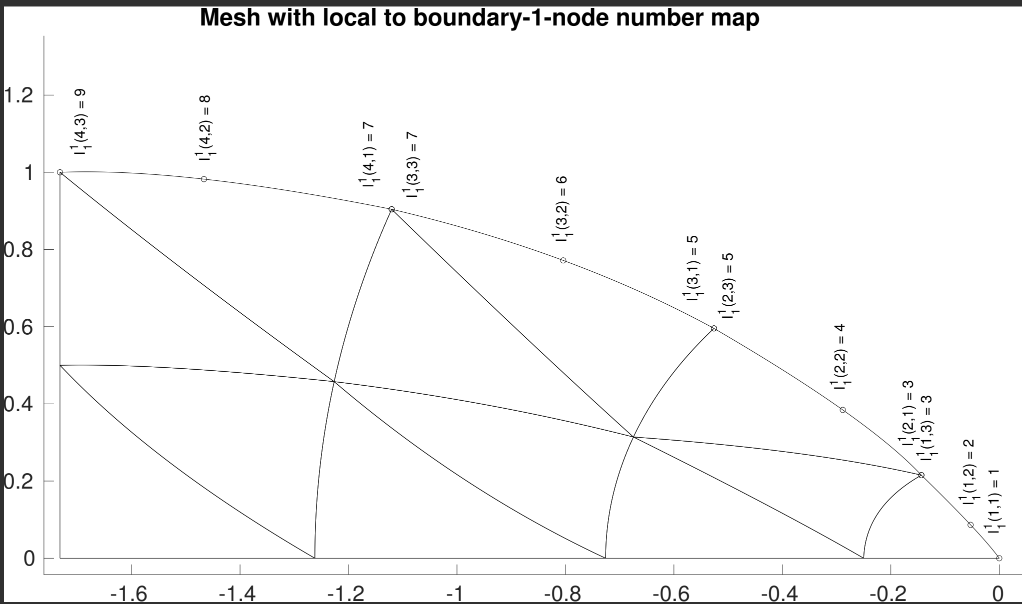

FIGURE 9. Local line-element-node number to boundary-1-node number map l_1^1.

Summarising and re-arranging terms, we have

$$\begin{aligned}
\mathcal{M}_i^r \;=& -\frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca}\\
&+ \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \mathcal{M}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \mathcal{M}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \mathcal{M}_{e,ii}^{r,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \mathcal{M}_{e,ii}^{r,0d}\\
&+ \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}^1_{\mathrm{el}}} \mathcal{M}_{e_1,ii}^{r,1} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}^2_{\mathrm{el}}} \mathcal{M}_{e,ii}^{r,2} + \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}^3_{\mathrm{el}}} \mathcal{M}_{e_3,ii}^{r,3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}^4_{\mathrm{el}}} \mathcal{M}_{e_4,ii}^{r,4};
\end{aligned} \tag{4.148}$$

where

$$\mathcal{M}_{e,ii}^{r,0a} = -\frac{2\Delta_t St}{3} a_{ii,g_r}(e), \tag{4.149}$$

$$\begin{aligned}
\mathcal{M}_{e,ii}^{r,0b} = \sum_{jj=1}^{n_v^e}\Bigg(&\frac{2\Delta_t}{3}\left\{w_{l(e,jj)}a_{ii,jj}^{z,r}(e) + u_{l(e,jj)}\left[2a_{ii,jj}^{r,r}(e) + a_{ii,jj}^{z,z}(e)\right]\right\}\\
&+ Re\, a_{ii,jj}(e)\left[a_n u_{l(e,jj)} - a_{n-1}u_{l(e,jj)}(t_{n-1}) + a_{n-2}u_{l(e,jj)}(t_{n-2})\right]\Bigg),
\end{aligned} \tag{4.150}$$

$$
\begin{aligned}
\mathcal{M}^{r,0c}_{e,ii} = \sum_{jj=1}^{n_v^e} Re\, u_{l(e,jj)} \Bigg\{ & \frac{2\Delta_t}{3} \sum_{kk=1}^{n_v^e} \underbrace{\left[u_{l(e,kk)} a^r_{ii,kk,jj}(e) + w_{l(e,kk)} a^z_{ii,kk,jj}(e)\right]}_{A_{ii,jj}(e)} \\
& - \underbrace{\sum_{kk=1}^{n_v^e} a^r_{ii,kk,jj}(e) \left[a_n r^c_{l(e,kk)} - a_{n-1} r^c_{l(e,kk)}(t_{n-1}) + a_{n-2} r^c_{l(e,kk)}(t_{n-2})\right]}_{B_{ii,jj}(e)} \\
& - \underbrace{\sum_{kk=1}^{n_v^e} a^z_{ii,kk,jj}(e) \left[a_n z^c_{l(e,kk)} - a_{n-1} z^c_{l(e,kk)}(t_{n-1}) + a_{n-2} z^c_{l(e,kk)}(t_{n-2})\right]}_{C_{ii,jj}(e)} \Bigg\},
\end{aligned}
\tag{4.151}
$$

$$
\mathcal{M}^{0,d}_{e,ii} = \sum_{jj=1}^{n_p^e} -\frac{2\Delta_t}{3} p_{l^p(e,jj)} b^r_{jj,ii}(e). \tag{4.152}
$$

For boundary 1 we have

$$
\begin{aligned}
\mathcal{M}^{r,1}_{e,ii} = \sum_{jj=1}^{n_v^{1,e_1}} \frac{2\Delta_t}{3} \Bigg\{ & \frac{2\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}^1_j c^s_{jj,ii,t_r}(e_1) - p^g_{l^1_1(e_1,jj)} c_{ii,jj,n_r}(e_1) \\
& - u_j c^r_{ii,jj,n_r} - w_j c^r_{ii,jj,n_z} \Bigg\};
\end{aligned}
\tag{4.153}
$$

for boundary 2

$$
\begin{aligned}
\mathcal{M}^{r,2}_{e,ii} = \sum_{jj=1}^{n_v^{2,e_2}} \Bigg\{ & \frac{2\Delta_t}{3} \lambda^2_{l^2_2(e_2,jj)} d_{ii,jj,n_r}(e_2) \\
& + \left(\frac{1+4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \left[u_{l_2(e_2,jj)} d_{ii,jj,t_r,t_r}(e_2) + w_{l_2(e_2,jj)} d_{ii,jj,t_r.t_z}(e_2)\right] \\
& + \left(\frac{1-4Be\,Ca\,Es}{6Ca\,Es}\right) \Delta_t \left[u^s_{l^2_2(e_2,jj)} d_{ii,jj,t_r,t_r}(e_2) + w^s_{l^2_2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2)\right] \\
& - \frac{\Delta_t}{3Ca\,Es} \left[u^{s_2}_{l^2_2(e_2,jj)} d_{ii,jj,t_r,t_r}(e_2) + w^{s_2}_{l^2_2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2)\right] \\
& - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j d^r_{ii,jj,n_r} - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j d^r_{ii,jj,n_z} \Bigg\};
\end{aligned}
\tag{4.154}
$$

for boundary 3

$$\mathcal{M}^{r,3}_{e,ii} = \sum_{jj=1}^{n_v^{3,e_3}} \frac{2\Delta_t}{3} \Bigg[\lambda^3_{l^3_3(e_3,jj)} f_{ii,jj,n_r}(e_3) + \gamma^3_{l^3_3(e_3,jj)} f_{ii,jj,t_r}(e_3) \\ - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j f^r_{ii,jj,n_r} - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j f^r_{ii,jj,n_z} \Bigg]. \tag{4.155}$$

and for boundary 5

$$\mathcal{M}^{r,5}_{e,ii} = \sum_{jj=1}^{n_v^{5,e_5}} \frac{2\Delta_t}{3} \Bigg[\lambda^5_{l^5_5(e_3,jj)} e_{ii,jj,n_r}(e_5) + \gamma^5_{l^5_5(e_5,jj)} e_{ii,jj,t_r}(e_5) \\ - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j e^r_{ii,jj,n_r} - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} w_j e^r_{ii,jj,n_z} \Bigg]. \tag{4.156}$$

In practice, we loop over the element nodes once again for index ii (i.e. the local index of the i-th residual component) defining and calculating $\mathcal{M}^r_{e,ii}$ for each local node ii on each element and then adding the contribution to the $\mathcal{M}^r$ vector at entry $i = (e, ii)$.

4.1. Jacobian terms

We now calculate the derivatives of $\mathcal{M}_i^r$ with respect to u_q, w_q, p_q, σ_q^1, σ_q^2, λ_q^2, λ_q^3, γ_q^3, λ_q^4, γ_q^4, and h_q.

4.1.1. Derivatives of $\mathcal{M}_i^r$ with respect to u_q

This derivative has contribution from the bulk terms and boundary 2 terms.
From equation (4.148)

$$\begin{aligned}
\partial_{u_q}\mathcal{M}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{u_q}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right] \\
& + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{u_q}\mathcal{M}_{e_1,ii}^{r,1} - \frac{2\Delta_t}{3}\partial_{u_q}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{u_q}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{u_q}\mathcal{M}_{e,ii}^{r,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\text{el}}^3} \partial_{u_q}\mathcal{M}_{e_3,ii}^{r,3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\text{el}}^4} \partial_{u_q}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned} \tag{4.157}$$

i.e.

$$\begin{aligned}
\partial_{u_q}\mathcal{M}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{u_q}\mathcal{M}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{u_q}\mathcal{M}_{e,ii}^{r,0c} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{u_q}\mathcal{M}_{e,ii}^{r,1} \\
& + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{u_q}\mathcal{M}_{e,ii}^{r,2} + \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\text{el}}^3} \partial_{u_q}\mathcal{M}_{e,ii}^{r,3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\text{el}}^4} \partial_{u_q}\mathcal{M}_{e,ii}^{r,4}.
\end{aligned} \tag{4.158}$$

Now, from equation (4.141) we have

$$\begin{aligned}
\partial_{u_q}\mathcal{M}_{e,ii}^{r,0b} = & +\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}^{r,r}(e)\partial_{u_q}u_{l(e,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}^{z,z}(e)\partial_{u_q}u_{l(e,jj)} \\
& + Re\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{u_q}u_{l(e,jj)} - \frac{4Re}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{u_q}u_{l(e,jj)}(t_{n-1}) \\
& + \frac{Re}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{u_q}u_{l(e,jj)}(t_{n-2}),
\end{aligned} \tag{4.159}$$

i.e.

$$\partial_{u_q}\mathcal{M}_{e,ii}^{r,0b} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}^{r,r}(e) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}^{z,z}(e) + Re\sum_{jj=1}^{n_v^e} a_{ii,jj}(e), \tag{4.160}$$

or equivalently

$$\partial_{u_q}\mathcal{M}^{r,0b}_{e,ii} = \sum_{jj=1}^{n_v^e}\left[\frac{2\Delta_t}{3}\left(a^{r,r}_{ii,jj}(e) + a^{z,z}_{ii,jj}(e)\right) + Re\, a_{ii,jj}(e)\right]. \tag{4.161}$$

Similarly, from equation (4.142) we have

$$\begin{aligned}
\partial_{u_q}\mathcal{M}^{r,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e}\partial_{u_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e}u_{l(e,kk)}a^{r}_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e}u_{l(e,jj)}\sum_{kk=1}^{n_v^e}a^{r}_{ii,kk,jj}(e)\partial_{u_q}u_{l(e,kk)} \\
& + \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e}\partial_{u_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e}w_{l(e,kk)}a^{z}_{ii,kk,jj}(e) \\
& - Re\sum_{jj=1}^{n_v^e}\partial_{u_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e}r^{c}_{l(e,kk)}a^{r}_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{jj=1}^{n_v^e}\partial_{u_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e}r^{c}_{l(e,kk)}(t_{n-1})a^{r}_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{jj=1}^{n_v^e}\partial_{u_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e}r^{c}_{l(e,kk)}(t_{n-2})a^{r}_{ii,kk,jj}(e) \\
& - Re\sum_{jj=1}^{n_v^e}\partial_{u_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e}z^{c}_{l(e,kk)}a^{z}_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{jj=1}^{n_v^e}\partial_{u_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e}z^{c}_{l(e,kk)}(t_{n-1})a^{z}_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{jj=1}^{n_v^e}\partial_{u_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e}z^{c}_{l(e,kk)}(t_{n-2})a^{z}_{ii,kk,jj}(e),
\end{aligned} \tag{4.162}$$

i.e.

$$\begin{aligned}\partial_{u_q}\mathcal{M}^{r,0c}_{e,ii} = \frac{2\Delta_t Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} u_{l(e,kk)} a^r_{ii,kk,jj}(e)\\
+\frac{2\Delta_t Re}{3} \sum_{\substack{jj=1\\ q=l(e,kk)}}^{n_v^e} u_{l(e,jj)} a^r_{ii,kk,jj}(e) + \frac{2\Delta_t Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} w_{l(e,kk)} a^z_{ii,kk,jj}(e)\\
- Re \sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} r^c_{l(e,kk)} a^r_{ii,kk,jj}(e) + \frac{4Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) a^r_{ii,kk,jj}(e)\\
- \frac{Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) a^r_{ii,kk,jj}(e)\\
- Re \sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} z^c_{l(e,kk)} a^z_{ii,kk,jj}(e) + \frac{4Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) a^z_{ii,kk,jj}(e)\\
-\frac{Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) a^z_{ii,kk,jj}(e),\end{aligned} \tag{4.163}$$

or, equivalently,

$$\begin{aligned}\partial_{u_q}\mathcal{M}^{r,0c}_{e,ii} = \sum_{\substack{jj=1\\ q=l(e,kk)}}^{n_v^e} \frac{2\Delta_t Re}{3} a^r_{ii,kk,jj}(e) u_{l(e,jj)}\\
+\frac{2\Delta_t Re}{3} \underbrace{\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} \left[u_{l(e,kk)} a^r_{ii,kk,jj}(e) + w_{l(e,kk)} a^z_{ii,kk,jj}(e)\right]}_{A_{ii,jj}(e)}\\
- Re \underbrace{\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} a^r_{ii,kk,jj}(e) \left[r^c_{l(e,kk)} - \frac{4}{3} r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,kk)}(t_{n-2})\right]}_{B_{ii,jj}(e)}\\
-Re \underbrace{\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} a^z_{ii,kk,jj}(e) \left[z^c_{l(e,kk)} - \frac{4}{3} z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,kk)}(t_{n-2})\right]}_{C_{ii,jj}(e)},\end{aligned} \tag{4.164}$$

From equation (4.144), we have

$$\partial_{u_q}\mathcal{M}^{r,1}_{e,ii} = \frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q}\sigma^1_{l_1^1(e_1,jj)} c^s_{jj,ii,t_r}(e_1) - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q} p^g_{l_1^1(e_1,jj)} c_{ii,jj,n_r}(e_1)$$
$$- \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} c^r_{ii,jj,n_r}\partial_{u_q}u_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\partial_{u_q}w_j c^r_{ii,jj,n_z}, \tag{4.165}$$

i.e.

$$\partial_{u_q}\mathcal{M}^{r,1}_{e,ii} = -\frac{2\Delta_t}{3} c^r_{ii,jj,n_r}|_{q=l_1(e_1,jj)}, \tag{4.166}$$

From equation (4.145), we have

$$\partial_{u_q}\mathcal{M}^{r,2}_{e,ii} = \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_r}(e_2)\partial_{u_q}u_{l_2(e,jj)} + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{u_q}w_{l_2(e,jj)}$$
$$- \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_r}(e_2)\partial_{u_q}u^s_{l_2^2(e,jj)}$$
$$- \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{u_q}w^s_{l_2^2(e,jj)}$$
$$+ \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,n_r}(e_2)\partial_{u_q}\lambda^2_{l_2^2(e_2,jj)} - \frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{2,e_2}} d^s_{ii,jj,t_r}(e_2)\partial_{u_q}\sigma^2_{l_2^2(e_2,jj)}$$
$$- \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} d^r_{ii,jj,n_r}\partial_{u_q}u_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\partial_{u_q}w_j d^r_{ii,jj,n_z}, \tag{4.167}$$

i.e.

$$\partial_{u_q}\mathcal{M}^{r,2}_{e,ii} = \frac{2\Delta_t}{3}\left[Be\, d_{ii,jj,t_r,t_r}(e_2) - d^r_{ii,jj,n_r}(e_2)\right]|_{q=l_2(e_2,jj)}. \tag{4.168}$$

From equation (4.146), we have

$$\partial_{u_q}\mathcal{M}^{r,3}_{e,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \partial_{u_q}\lambda^3_{l_3^3(e_3,jj)} f_{ii,jj,n_r}(e_3) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \partial_{u_q}\gamma^3_{l_3^3(e_3,jj)} f_{ii,jj,t_r}(e_3)$$
$$- \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} f^r_{ii,jj,n_r}\partial_{u_q}u_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\partial_{u_q}w_j f^r_{ii,jj,n_z} \tag{4.169}$$

i.e.

$$\partial_{u_q}\mathcal{M}^{r,3}_{e,ii} = -\frac{2\Delta_t}{3} f^r_{ii,jj,n_r}|_{q=l_3(e_3,jj)}. \tag{4.170}$$

From equation (4.147), we have

$$\partial_{u_q}\mathcal{M}^{r,4}_{e,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}}\partial_{u_q}\lambda^4_{l_4^4(e_3,jj)}e_{ii,jj,n_r}(e_4) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}}\partial_{u_q}\gamma^4_{l_4^4(e_4,jj)}e_{ii,jj,t_r}(e_4)$$
$$- \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}e^r_{ii,jj,n_r}\partial_{u_q}u_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\partial_{u_q}w_j e^r_{ii,jj,n_z} \tag{4.171}$$

i.e.

$$\partial_{u_q}\mathcal{M}^{r,4}_{e,ii} = -\frac{2\Delta_t}{3}e^r_{ii,jj,n_r}|_{q=l_4(e_4,jj)}. \tag{4.172}$$

4.1.2. **Derivatives of $\mathcal{M}_i^r$ with respect to w_q**

This derivative has contribution from the bulk terms and boundary 2 terms. From equation (4.148)

$$
\begin{aligned}
\partial_{w_q}\mathcal{M}_i^r &= \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{w_q}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right] \\
&+ \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}^1_{\text{el}}} \partial_{w_q}\mathcal{M}_{e_1,ii}^{r,1} - \frac{2\Delta_t}{3}\partial_{w_q}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
&+ \frac{2\Delta_t}{3}\partial_{w_q}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}^2_{\text{el}}} \partial_{w_q}\mathcal{M}_{e,ii}^{r,2} \\
&+ \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}^3_{\text{el}}} \partial_{w_q}\mathcal{M}_{e_3,ii}^{r,3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}^4_{\text{el}}} \partial_{w_q}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned}
\tag{4.173}
$$

i.e.

$$
\begin{aligned}
\partial_{w_q}\mathcal{M}_i^r &= \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{w_q}\mathcal{M}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{w_q}\mathcal{M}_{e,ii}^{r,0c} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}^1_{\text{el}}} \partial_{w_q}\mathcal{M}_{e,ii}^{r,1} \\
&+ \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}^2_{\text{el}}} \partial_{w_q}\mathcal{M}_{e,ii}^{r,2} + \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}^3_{\text{el}}} \partial_{w_q}\mathcal{M}_{e,ii}^{3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}^4_{\text{el}}} \partial_{w_q}\mathcal{M}_{e,ii}^{4}.
\end{aligned}
\tag{4.174}
$$

Now, from equation (4.141) we have

$$
\begin{aligned}
\partial_{w_q}\mathcal{M}_{e,ii}^{r,0b} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}^{r,r}(e)\partial_{w_q}u_{l(e,jj)} &+ \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}^{z,z}(e)\partial_{w_q}u_{l(e,jj)} \\
+Re\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{w_q}u_{l(e,jj)} &- \frac{4Re}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{w_q}u_{l(e,jj)}(t_{n-1}) \\
+\frac{Re}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{w_q}u_{l(e,jj)}(t_{n-2}),
\end{aligned}
\tag{4.175}
$$

i.e.

$$
\partial_{w_q}\mathcal{M}_{e,ii}^{r,0b} = 0. \tag{4.176}
$$

Similarly, from equation (4.142) we have

$$
\begin{aligned}
\partial_{w_q}\mathcal{M}^{r,0c}_{e,ii} = \frac{2\Delta_t Re}{3}\partial_{w_q}\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} u_{l(e,kk)}a^r_{ii,kk,jj}(e)\\
+\frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} a^z_{ii,kk,jj}(e)\partial_{w_q}w_{l(e,kk)}\\
-Re\sum_{jj=1}^{n_v^e}\partial_{w_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}a^r_{ii,kk,jj}(e)\\
+\frac{4Re}{3}\sum_{jj=1}^{n_v^e}\partial_{w_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1})a^r_{ii,kk,jj}(e)\\
-\frac{Re}{3}\sum_{jj=1}^{n_v^e}\partial_{w_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2})a^r_{ii,kk,jj}(e)\\
-Re\sum_{jj=1}^{n_v^e}\partial_{w_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}a^z_{ii,kk,jj}(e)\\
+\frac{4Re}{3}\sum_{jj=1}^{n_v^e}\partial_{w_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1})a^z_{ii,kk,jj}(e)\\
-\frac{Re}{3}\sum_{jj=1}^{n_v^e}\partial_{w_q}u_{l(e,jj)}\sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2})a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{4.177}
$$

i.e.

$$
\partial_{w_q}\mathcal{M}^{r,0c}_{e,ii} = \sum_{\substack{jj=1\\ q=l(e,kk)}}^{n_v^e} \frac{2\Delta_t Re}{3} u_{l(e,jj)}a^z_{ii,kk,jj}(e). \tag{4.178}
$$

From equation (4.145), we have

$$
\begin{aligned}
\partial_{w_q}\mathcal{M}^{r,1}_{e,ii} = \frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{w_q}\sigma^1_{l_1^1(e_1,jj)}c^s_{jj,ii,t_r}(e_1) - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{w_q}p^g_{l_1^1(e_1,jj)}c_{ii,jj,n_r}(e_1)\\
-\frac{2\Delta_t}{3}\sum_{j=1}^{n_v} c^r_{ii,jj,n_r}\partial_{w_q}u_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} c^r_{ii,jj,n_z}\partial_{w_q}w_j,
\end{aligned}
\tag{4.179}
$$

i.e.

$$
\partial_{w_q}\mathcal{M}^{r,1}_{e,ii} = -\frac{2\Delta_t}{3}c^r_{ii,jj,n_z}|_{q=l_1(e_1,jj)}, \tag{4.180}
$$

From equation (4.145), we have

$$\begin{aligned}
\partial_{w_q}\mathcal{M}^{r,2}_{e,ii} = {} & \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_r}(e_2)\partial_{w_q}u_{l_2(e,jj)} + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{w_q}w_{l_2(e,jj)} \\
& - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_r}(e_2)\partial_{w_q}u^s_{l_2^2(e,jj)} \\
& - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{w_q}w^s_{l_2^2(e,jj)} \\
& + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,n_r}(e_2)\partial_{w_q}\lambda^2_{l_2^2(e_2,jj)} - \frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{2,e_2}} d^s_{ii,jj,t_r}(e_2)\partial_{w_q}\sigma^2_{l_2^2(e_2,jj)} \\
& - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} d^r_{ii,jj,n_r}\partial_{w_q}u_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} d^r_{ii,jj,n_z}\partial_{w_q}w_j,
\end{aligned} \tag{4.181}$$

i.e.

$$\partial_{w_q}\mathcal{M}^{r,2}_{e,ii} = \frac{2\Delta_t}{3}\left[Be\, d_{ii,jj,t_r,t_z}(e_2) - d^r_{ii,jj,n_z}\partial_{w_q}w_j\right]|_{q=l(e,jj)}. \tag{4.182}$$

From equation (4.146), we have

$$\begin{aligned}
\partial_{w_q}\mathcal{M}^{r,3}_{e,ii} = {} & \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \partial_{w_q}\lambda^3_{l_3^3(e_3,jj)} f_{ii,jj,n_r}(e_3) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \partial_{w_q}\gamma^3_{l_3^3(e_3,jj)} f_{ii,jj,t_r}(e_3) \\
& - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} f^r_{ii,jj,n_r}\partial_{w_q}u_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} f^r_{ii,jj,n_z}\partial_{w_q}w_j,
\end{aligned} \tag{4.183}$$

i.e.

$$\partial_{w_q}\mathcal{M}^{r,3}_{e,ii} = -\frac{2\Delta_t}{3} f^r_{ii,jj,n_z}|_{q=l_3(e_3,jj)}, \tag{4.184}$$

From equation (**??**), we have

$$\begin{aligned}
\partial_{w_q}\mathcal{M}^{r,4}_{e,ii} = {} & \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \partial_{w_q}\lambda^4_{l_4^4(e_3,jj)} e_{ii,jj,n_r}(e_4) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \partial_{w_q}\gamma^4_{l_4^4(e_4,jj)} e_{ii,jj,t_r}(e_4) \\
& - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} e^r_{ii,jj,n_r}\partial_{w_q}u_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} e^r_{ii,jj,n_z}\partial_{w_q}w_j,
\end{aligned} \tag{4.185}$$

i.e.

$$\partial_{w_q}\mathcal{M}^{r,4}_{e,ii} = -\frac{2\Delta_t}{3} e^r_{ii,jj,n_z}|_{q=l_4(e_4,jj)}. \tag{4.186}$$

4.1.3. Derivatives of $\mathcal{M}_i^r$ with respect to p_q

From equation (4.148)

$$
\begin{aligned}
\partial_{p_q}\mathcal{M}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right] \\
& + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{p_q}\mathcal{M}_{e_1,ii}^{r,1} - \frac{2\Delta_t}{3}\partial_{p_q}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{p_q}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{p_q}\mathcal{M}_{e,ii}^{r,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\mathrm{el}}^3} \partial_{p_q}\mathcal{M}_{e_3,ii}^{r,3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{p_q}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned}
\tag{4.187}
$$

i.e.

$$
\partial_{p_q}\mathcal{M}_i^r \;=\; \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\mathcal{M}_{e,ii}^{r,0d}. \tag{4.188}
$$

From equation (4.152), we have

$$
\partial_{p_q}\mathcal{M}_{e,ii}^{0,d} = \sum_{jj=1}^{n_p^e} -\frac{2\Delta_t}{3} b_{jj,ii}^r(e)\partial_{p_q}p_{l^p(e,jj)}, \tag{4.189}
$$

i.e.

$$
\partial_{p_q}\mathcal{M}_{e,ii}^{0,d} = -\frac{2\Delta_t}{3} b_{jj,ii}^r(e)|_{q=l^p(e,jj)}. \tag{4.190}
$$

4.1.4. **Derivatives of $\mathcal{M}_i^r$ with respect to σ_q^1**

From equation (4.148)

$$\begin{aligned}\partial_{\sigma_q^1}\mathcal{M}_i^r \;=\; &\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\sigma_q^1}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right]\\ &+\sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1}\partial_{\sigma_q^1}\mathcal{M}_{e_1,ii}^{r,1}-\frac{2\Delta_t}{3}\partial_{\sigma_q^1}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca}\\ &+\frac{2\Delta_t}{3}\partial_{\sigma_q^1}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca}+\sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2}\partial_{\sigma_q^1}\mathcal{M}_{e,ii}^{r,2}\\ &+\sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\mathrm{el}}^3}\partial_{\sigma_q^1}\mathcal{M}_{e_3,ii}^{r,3}+\sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\mathrm{el}}^4}\partial_{\sigma_q^1}\mathcal{M}_{e_4,ii}^{r,4}.\end{aligned}\tag{4.191}$$

i.e.

$$\begin{aligned}\partial_{\sigma_q^1}\mathcal{M}_i^r = &\sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1}\partial_{\sigma_q^1}\mathcal{M}_{e_1,ii}^{r,1}-\frac{2\Delta_t}{3}\partial_{\sigma_q^1}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca}\\ &+\frac{2\Delta_t}{3}\partial_{\sigma_q^1}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca},\end{aligned}\tag{4.192}$$

or, equivalently,

$$\partial_{\sigma_q^1}\mathcal{M}_i^r = \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1}\partial_{\sigma_q^1}\mathcal{M}_{e_1,ii}^{r,1}-\frac{2\Delta_t}{3}\frac{m_r^{1,n}(r_{J^1},z_{J^1})}{Ca}\delta_{i,J^1}\delta_{q,J^1}+\frac{2\Delta_t}{3}\frac{m_r^1(r_a,z_a)}{Ca}\delta_{i,a}\delta_{q,a}.\tag{4.193}$$

From equation (4.144)

$$\partial_{\sigma_q^1}\mathcal{M}_{e,ii}^{r,1}=\frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{1,e_1}}c_{jj,ii,t_r}^s(e_1)\partial_{\sigma_q^1}\sigma_{l_1^1(e_1,jj)}^1-\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1}}c_{ii,jj,n_r}(e_1)\partial_{\sigma_q^1}p_{l_1^1(e_1,jj)}^g,\tag{4.194}$$

i.e.

$$\partial_{\sigma_q^1}\mathcal{M}_{e,ii}^{r,1}=\frac{2\Delta_t}{3Ca}c_{jj,ii,t_r}^s(e_1)|_{q=l_1^1(e_1,jj)}.\tag{4.195}$$

4.1.5. Derivatives of $\mathcal{M}_i^r$ with respect to σ_q^2

From equation (4.148)

$$\begin{aligned}
\partial_{\sigma_q^2}\mathcal{M}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\sigma_q^2}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right] \\
& + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{\sigma_q^2}\mathcal{M}_{e_1,ii}^{r,1} - \frac{2\Delta_t}{3}\partial_{\sigma_q^2}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{\sigma_q^2}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{\sigma_q^2}\mathcal{M}_{e,ii}^{r,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\text{el}}^3} \partial_{\sigma_q^2}\mathcal{M}_{e_3,ii}^{r,3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\text{el}}^4} \partial_{\sigma_q^2}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned} \tag{4.196}$$

i.e.

$$\partial_{\sigma_q^2}\mathcal{M}_i^r = \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{\sigma_q^2}\mathcal{M}_{e,ii}^{r,2}. \tag{4.197}$$

From equation (4.145) we have

$$\begin{aligned}
\partial_{\sigma_q^2}\mathcal{M}_{e,ii}^{r,2} = & \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_r}(e_2)\partial_{\sigma_q^2}u_{l_2(e,jj)} + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{\sigma_q^2}w_{l_2(e,jj)} \\
& - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_r}(e_2)\partial_{\sigma_q^2}u^s_{l_2^2(e,jj)} \\
& - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{\sigma_q^2}w^s_{l_2^2(e,jj)} \\
& + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,n_r}(e_2)\partial_{\sigma_q^2}\lambda^2_{l_2^2(e_2,jj)} - \frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{2,e_2}} d^s_{ii,jj,t_r}(e_2)\partial_{\sigma_q^2}\sigma^2_{l_2^2(e_2,jj)};
\end{aligned} \tag{4.198}$$

i.e.

$$\partial_{\sigma_q^2}\mathcal{M}_{e,ii}^{r,2} = -\frac{\Delta_t}{3Ca}d^s_{ii,jj,t_r}(e_2)|_{q=l_2^2(e_2,jj)}. \tag{4.199}$$

4.1.6. **Derivatives of $\mathcal{M}_i^r$ with respect to λ_q^2**

From equation (4.148)

$$\begin{aligned}
\partial_{\lambda_q^2}\mathcal{M}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\lambda_q^2}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right] \\
& + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{\lambda_q^2}\mathcal{M}_{e_1,ii}^{r,1} - \frac{2\Delta_t}{3}\partial_{\lambda_q^2}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{\lambda_q^2}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{\lambda_q^2}\mathcal{M}_{e,ii}^{r,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\text{el}}^3} \partial_{\lambda_q^2}\mathcal{M}_{e_3,ii}^{r,3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\text{el}}^4} \partial_{\lambda_q^2}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned} \tag{4.200}$$

i.e.

$$\partial_{\lambda_q^2}\mathcal{M}_i^r = \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{\lambda_q^2}\mathcal{M}_{e,ii}^{r,2}. \tag{4.201}$$

From equation (4.145) we have

$$\begin{aligned}
\partial_{\lambda_q^2}\mathcal{M}_{e,ii}^{r,2} = & \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_r}(e_2)\partial_{\lambda_q^2}u_{l_2(e,jj)} + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{\lambda_q^2}w_{l_2(e,jj)} \\
& - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_r}(e_2)\partial_{\lambda_q^2}u^s_{l_2^2(e,jj)} \\
& - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{\lambda_q^2}w^s_{l_2^2(e,jj)} \\
& + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,n_r}(e_2)\partial_{\lambda_q^2}\lambda^2_{l_2^2(e_2,jj)} - \frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{2,e_2}} d^s_{ii,jj,t_r}(e_2)\partial_{\lambda_q^2}\sigma^2_{l_2^2(e_2,jj)},
\end{aligned} \tag{4.202}$$

i.e.

$$\partial_{\lambda_q^2}\mathcal{M}_{e,ii}^{r,2} = \frac{2\Delta_t}{3}d_{ii,jj,n_r}(e_2)\big|_{q=l_2^2(e_2,jj)}. \tag{4.203}$$

4.1.7. Derivatives of $\mathcal{M}_i^r$ with respect to λ_q^3

From equation (4.148)

$$
\begin{aligned}
\partial_{\lambda_q^3}\mathcal{M}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^3}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right] \\
& + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\lambda_q^3}\mathcal{M}_{e_1,ii}^{r,1} - \frac{2\Delta_t}{3}\partial_{\lambda_q^3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{\lambda_q^3}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\lambda_q^3}\mathcal{M}_{e,ii}^{r,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\mathrm{el}}^3} \partial_{\lambda_q^3}\mathcal{M}_{e_3,ii}^{r,3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{\lambda_q^3}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned} \tag{4.204}
$$

i.e.

$$
\partial_{\lambda_q^3}\mathcal{M}_i^r = \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\mathrm{el}}^3} \partial_{\lambda_q^3}\mathcal{M}_{e_3,ii}^{r,3}. \tag{4.205}
$$

From equation (4.146), we have

$$
\partial_{\lambda_q^3}\mathcal{M}_{e,ii}^{r,3} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} f_{ii,jj,n_r}(e_3)\partial_{\lambda_q^3}\lambda_{l_3^3(e_3,jj)}^3 + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} f_{ii,jj,t_r}(e_3)\partial_{\lambda_q^3}\gamma_{l_3^3(e_3,jj)}^3, \tag{4.206}
$$

i.e.

$$
\partial_{\lambda_q^3}\mathcal{M}_{e,ii}^{r,3} = \frac{2\Delta_t}{3} f_{ii,jj,n_r}(e_3)\big|_{q=l_3^3(e_3,jj)}. \tag{4.207}
$$

4.1.8. **Derivatives of $\mathcal{M}_i^r$ with respect to γ_q^3**

From equation (4.148)

$$\begin{aligned}
\partial_{\gamma_q^3}\mathcal{M}_i^r &= \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\gamma_q^3}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right]\\
&+\sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{\gamma_q^3}\mathcal{M}_{e_1,ii}^{r,1} - \frac{2\Delta_t}{3}\partial_{\gamma_q^3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca}\\
&+\frac{2\Delta_t}{3}\partial_{\gamma_q^3}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca}+\sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{\gamma_q^3}\mathcal{M}_{e,ii}^{r,2}\\
&+\sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\text{el}}^3} \partial_{\gamma_q^3}\mathcal{M}_{e_3,ii}^{r,3}+\sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\text{el}}^4} \partial_{\gamma_q^3}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned} \tag{4.208}$$

i.e.

$$\partial_{\gamma_q^3}\mathcal{M}_i^r = \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\text{el}}^3} \partial_{\gamma_q^3}\mathcal{M}_{e_3,ii}^{r,3}. \tag{4.209}$$

From equation (4.146), we have

$$\partial_{\gamma^3}\mathcal{M}_{e,ii}^{r,3} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} f_{ii,jj,n_r}(e_3)\partial_{\gamma_q^3}\lambda_{l_3^3(e_3,jj)}^3 + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} f_{ii,jj,t_r}(e_3)\partial_{\gamma_q^3}\gamma_{l_3^3(e_3,jj)}^3, \tag{4.210}$$

i.e.

$$\partial_{\gamma_q^3}\mathcal{M}_{e,ii}^{r,3} = \frac{2\Delta_t}{3} f_{ii,jj,t_r}(e_3)|_{q=l_3^3(e_3,jj)}. \tag{4.211}$$

4.1.9. **Derivatives of $\mathcal{M}_i^r$ with respect to λ_q^4**

From equation (4.148)

$$\begin{aligned}
\partial_{\lambda_q^4}\mathcal{M}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^4}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right] \\
& + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\lambda_q^4}\mathcal{M}_{e_1,ii}^{r,1} - \frac{2\Delta_t}{3}\partial_{\lambda_q^4}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{\lambda_q^4}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\lambda_q^4}\mathcal{M}_{e,ii}^{r,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\mathrm{el}}^3} \partial_{\lambda_q^4}\mathcal{M}_{e_3,ii}^{r,3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{\lambda_q^4}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned} \tag{4.212}$$

i.e.

$$\partial_{\lambda_q^4}\mathcal{M}_i^r = \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{\lambda_q^4}\mathcal{M}_{e_4,ii}^{r,4}. \tag{4.213}$$

From equation (**??**), we have

$$\partial_{\lambda_q^4}\mathcal{M}_{e,ii}^{r,4} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} e_{ii,jj,n_r}(e_4)\partial_{\lambda_q^4}\lambda_{l_4^4(e_3,jj)}^4 + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} e_{ii,jj,t_r}(e_4)\partial_{\lambda_q^4}\gamma_{l_4^4(e_4,jj)}^4. \tag{4.214}$$

i.e.

$$\partial_{\lambda_q^4}\mathcal{M}_{e,ii}^{r,4} = \frac{2\Delta_t}{3}e_{ii,jj,n_r}(e_4)|_{q=l_4^4(e_4,jj)}. \tag{4.215}$$

4.1.10. **Derivatives of $\mathcal{M}_i^r$ with respect to γ_q^4**

From equation (4.148)

$$\begin{aligned}
\partial_{\gamma_q^4}\mathcal{M}_i^r &= \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\gamma_q^4}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right]\\
&+\sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}^1_{\rm el}} \partial_{\gamma_q^4}\mathcal{M}_{e_1,ii}^{r,1} - \frac{2\Delta_t}{3}\partial_{\gamma_q^4}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca}\\
&+\frac{2\Delta_t}{3}\partial_{\gamma_q^4}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}^2_{\rm el}} \partial_{\gamma_q^4}\mathcal{M}_{e,ii}^{r,2}\\
&+\sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}^3_{\rm el}} \partial_{\gamma_q^4}\mathcal{M}_{e_3,ii}^{r,3} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}^4_{\rm el}} \partial_{\gamma_q^4}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned} \tag{4.216}$$

i.e.

$$\partial_{\gamma_q^4}\mathcal{M}_i^r = \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}^4_{\rm el}} \partial_{\gamma_q^4}\mathcal{M}_{e_4,ii}^{r,4}. \tag{4.217}$$

From equation (**??**), we have

$$\partial_{\gamma_q^4}\mathcal{M}_{e,ii}^{r,4} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} e_{ii,jj,n_r}(e_4)\partial_{\gamma_q^4}\lambda^4_{l_4^4(e_3,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} e_{ii,jj,t_r}(e_4)\partial_{\gamma_q^4}\gamma^4_{l_4^4(e_4,jj)}. \tag{4.218}$$

i.e.

$$\partial_{\gamma_q^4}\mathcal{M}_{e,ii}^{r,4} = \frac{2\Delta_t}{3}e_{ii,jj,t_r}(e_4)|_{q=l_4^4(e_4,jj)}. \tag{4.219}$$

4.1.11. Derivatives of $\mathcal{M}_i^r$ with respect to h_q

From equation (4.148)

$$
\begin{aligned}
\partial_{h_q}\mathcal{M}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{h_q}\left[\mathcal{M}_{e,ii}^{r,0a}+\mathcal{M}_{e,ii}^{r,0b}+\mathcal{M}_{e,ii}^{r,0c}+\mathcal{M}_{e,ii}^{r,0d}\right] \\
&+\sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{h_q}\mathcal{M}_{e_1,ii}^{r,1}-\frac{2\Delta_t}{3}\partial_{h_q}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
&+\frac{2\Delta_t}{3}\partial_{h_q}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_r^1(r_a,z_a)}{Ca}+\sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2}\partial_{h_q}\mathcal{M}_{e,ii}^{r,2} \\
&+\sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\mathrm{el}}^3}\partial_{h_q}\mathcal{M}_{e_3,ii}^{r,3}+\sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\mathrm{el}}^4}\partial_{h_q}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned}
\tag{4.220}
$$

i.e.

$$
\begin{aligned}
\partial_{h_q}\mathcal{M}_i^r = & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}}\partial_{h_q}\mathcal{M}_{e,ii}^{r,0a}+\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}}\partial_{h_q}\mathcal{M}_{e,ii}^{r,0b}+\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}}\partial_{h_q}\mathcal{M}_{e,ii}^{r,0c}+\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}}\partial_{h_q}\mathcal{M}_{e,ii}^{r,0d} \\
&+\sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1}\partial_{h_q}\mathcal{M}_{e_1,ii}^{r,1}-\frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})}{Ca}\partial_{h_q}m_r^{1,n}(r_{J^1},z_{J^1}) \\
&+\frac{2\Delta_t}{3}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)}{Ca}\partial_{h_q}m_r^1(r_a,z_a)+\sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2}\partial_{h_q}\mathcal{M}_{e,ii}^{r,2} \\
&+\sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\mathrm{el}}^3}\partial_{h_q}\mathcal{M}_{e_3,ii}^{r,3}+\sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\mathrm{el}}^4}\partial_{h_q}\mathcal{M}_{e_4,ii}^{r,4}.
\end{aligned}
\tag{4.221}
$$

From equation (4.140) we have

$$\partial_{h_q}\mathcal{M}_{e,ii}^{r,0a} = -\frac{2\Delta_t St}{3}\partial_{h_q}a_{ii,g_r}(e). \tag{4.222}$$

We notice that in the sum by elements above, it is only those spines that contain nodes in these elements that are going to have an effect on each of the derivatives shown above. Put differently, the vast majority of the derivatives above will be identically null. Hence, we once again resort to a function that maps objects in the element to the global number of these elements. Here we define as the "local spines" of an element a those spines that contain nodes that are part of the element being considered, and we number those spines with a local spine number (from 1 to the number of spines that contain nodes from the element). We then introduce the local-spine-number to global-spine-number map $S(e,qq)=q$, which maps the qq-th local spine number on element e to its global spines number (previously referred to as simply *the spine number*) q. Similarly, we define

$S_i(e_i, qq) = q$, which maps the local spine number qq of element e_i on boundary i to its global spine number q.

Hence, using local spine numbers, we have

$$\partial_{h_q} \mathcal{M}^{r,0a}_{e,ii} = -\frac{2\Delta_t St}{3} \partial_{h_{S_{e,qq}}} a_{ii,g_r}(e)|_{q=S(e,qq)}. \tag{4.223}$$

Now, from equation (4.141), we have

$$\begin{aligned}\partial_{h_q} \mathcal{M}^{r,0b}_{e,ii} = &\frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \partial_{h_q} a^{r,r}_{ii,jj}(e) + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \partial_{h_q} a^{z,z}_{ii,jj}(e) \\ &+ Re \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \partial_{h_q} a_{ii,jj}(e) - \frac{4Re}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-1}) \partial_{h_q} a_{ii,jj}(e) \\ &+ \frac{Re}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)}(t_{n-2}) \partial_{h_q} a_{ii,jj}(e),\end{aligned} \tag{4.224}$$

i.e.

$$\begin{aligned}\partial_{h_q} \mathcal{M}^{r,0b}_{e,ii} = &\sum_{jj=1}^{n_v^e} \frac{2\Delta_t}{3} u_{l(e,jj)} \left(\partial_{h_q} a^{r,r}_{ii,jj}(e) + \partial_{h_q} a^{z,z}_{ii,jj}(e) \right) \\ &+ Re \sum_{jj=1}^{n_v^e} \partial_{h_q} a_{ii,jj}(e) \left[u_{l(e,jj)} - \frac{4}{3} u_{l(e,jj)}(t_{n-1}) + \frac{1}{3} u_{l(e,jj)}(t_{n-2}) \right],\end{aligned} \tag{4.225}$$

and using local spine numbers we have

$$\begin{aligned}\partial_{h_q} \mathcal{M}^{r,0b}_{e,ii} = &\sum_{\substack{jj=1 \\ q=S(e,qq)}}^{n_v^e} \frac{2\Delta_t}{3} u_{l(e,jj)} \left(\partial_{h_{S(e,qq)}} a^{r,r}_{ii,jj}(e) + \partial_{h_{S(e,qq)}} a^{z,z}_{ii,jj}(e) \right) \\ &+ Re \sum_{\substack{jj=1 \\ q=S(e,qq)}}^{n_v^e} \partial_{h_{S(e,qq)}} a_{ii,jj}(e) \left[u_{l(e,jj)} - \frac{4}{3} u_{l(e,jj)}(t_{n-1}) + \frac{1}{3} u_{l(e,jj)}(t_{n-2}) \right].\end{aligned} \tag{4.226}$$

From equation (4.142) we have

$$
\begin{aligned}
\partial_{h_q}\mathcal{M}^{r,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} u_{l(e,kk)}\partial_{h_q}a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} w_{l(e,kk)}\partial_{h_q}a^z_{ii,kk,jj}(e) \\
& - Re\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} a^r_{ii,kk,jj}(e)\partial_{h_q}r^c_{l(e,kk)} \\
& - Re\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}\partial_{h_q}a^r_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1})\partial_{h_q}a^r_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2})\partial_{h_q}a^r_{ii,kk,jj}(e) \\
& - Re\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} a^z_{ii,kk,jj}(e)\partial_{h_q}z^c_{l(e,kk)} \\
& - Re\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}\partial_{h_q}a^z_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1})\partial_{h_q}a^z_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2})\partial_{h_q}a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{4.227}
$$

i.e.

$$\partial_{h_q}\mathcal{M}^{r,0c}_{e,ii} = \sum_{jj=1}^{n_v^e} Re\, u_{l(e,jj)} \left\{ \frac{2\Delta_t}{3} \underbrace{\sum_{kk=1}^{n_v^e} \left[u_{l(e,kk)}\partial_{h_q} a^r_{ii,kk,jj}(e) + w_{l(e,kk)}\partial_{h_q} a^z_{ii,kk,jj}(e) \right]}_{\partial_{h_q} A_{ii,jj}(e)} \right. \tag{4.228}$$

$$\underbrace{-\sum_{kk=1}^{n_v^e} \left(a^r_{ii,kk,jj}(e)\partial_{h_q} r^c_{l(e,kk)} + \partial_{h_q} a^r_{ii,kk,jj}(e) \left[r^c_{l(e,kk)} - \frac{4}{3} r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,kk)}(t_{n-2}) \right] \right)}_{\partial_{h_q} B_{ii,jj}(e)}$$

$$\left. \underbrace{-\sum_{kk=1}^{n_v^e} \left(a^z_{ii,kk,jj}(e)\partial_{h_q} z^c_{l(e,kk)} + \partial_{h_q} a^z_{ii,kk,jj}(e) \left[z^c_{l(e,kk)} - \frac{4}{3} z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,kk)}(t_{n-2}) \right] \right)}_{\partial_{h_q} C_{ii,jj}(e)} \right\},$$

and using local spines numbers we have

$$\begin{aligned}
\partial_{h_{S(e,qq)}}\mathcal{M}^{r,0c}_{e,ii} &= \\
&\sum_{\substack{jj=1 \\ q=S(e,qq)}}^{n_v^e} Re\, u_{l(e,jj)} \left\{ \frac{2\Delta_t}{3} \underbrace{\sum_{kk=1}^{n_v^e} \left[u_{l(e,kk)}\partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) + w_{l(e,kk)}\partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \right]}_{\partial_{h_{S(e,qq)}} A_{ii,jj}(e)} \right. \\
&\qquad - \sum_{kk=1}^{n_v^e} \left(a^r_{ii,kk,jj}(e)\partial_{h_{S(e,qq)}} r^c_{l(e,kk)} \right. \\
&\qquad \left. + \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \left[r^c_{l(e,kk)} - \frac{4}{3} r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,kk)}(t_{n-2}) \right] \right) \\
&\qquad - \sum_{kk=1}^{n_v^e} \left(a^z_{ii,kk,jj}(e)\partial_{h_{S(e,qq)}} z^c_{l(e,kk)} \right. \\
&\qquad \left. \left. + \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \left[z^c_{l(e,kk)} - \frac{4}{3} z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,kk)}(t_{n-2}) \right] \right) \right\},
\end{aligned} \tag{4.229}$$

From equation (4.152), we have

$$\partial_{h_q}\mathcal{M}^{0,d}_{e,ii} = \sum_{jj=1}^{n_p^e} -\frac{2\Delta_t}{3} p_{l^p(e,jj)} \partial_{h_q} b^r_{jj,ii}(e), \tag{4.230}$$

and using local spine numbers

$$\partial_{h_q}\mathcal{M}^{0,d}_{e,ii} = \sum_{\substack{jj=1 \\ q=S(e,qq)}}^{n_p^e} -\frac{2\Delta_t}{3} p_{l^p(e,jj)} \partial_{h_{S(e,qq)}} b^r_{jj,ii}(e). \tag{4.231}$$

From equation (4.144)

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,1}_{e,ii} = {} & \frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{1,e_1}} \sigma^1_{l^1_1(e_1,jj)} \partial_{h_q} c^s_{jj,ii,t_r}(e_1) - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1}} p^g_{l^1_1(e_1,jj)} \partial_{h_q} c_{ii,jj,n_r}(e_1) \\ & - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} u_j \partial_{h_q} c^r_{ii,jj,n_r} - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} w_j \partial_{h_q} c^r_{ii,jj,n_z},\end{aligned} \tag{4.232}$$

i.e.

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,1}_{e,ii} = \sum_{jj=1}^{n_v^{1,e_1}} \frac{2\Delta_t}{3}\Bigg[& \frac{1}{Ca}\sigma^1_{l^1_1(e_1,jj)} \partial_{h_q} c^s_{jj,ii,t_r}(e_1) - p^g_{l^1_1(e_1,jj)} \partial_{h_q} c_{ii,jj,n_r}(e_1) \\ & - u_j \partial_{h_q} c^r_{ii,jj,n_r} - w_j \partial_{h_q} c^r_{ii,jj,n_z}\Bigg],\end{aligned} \tag{4.233}$$

and using local spine numbers

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,1}_{e,ii} = {} & \sum_{\substack{jj=1 \\ q=S_1(e_1,qq)}}^{n_v^{1,e_1}} \frac{2\Delta_t}{3}\Bigg[\frac{1}{Ca}\sigma^1_{l^1_1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c^s_{jj,ii,t_r}(e_1) \\ & - p^g_{l^1_1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_r}(e_1) - u_j \partial_{h_{S_1(e_1,qq)}} c^r_{ii,jj,n_r} - w_j \partial_{h_{S_1(e_1,qq)}} c^r_{ii,jj,n_z}\Bigg].\end{aligned} \tag{4.234}$$

From equation (4.145) we have

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,2}_{e,ii} = {} & \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e,jj)} \partial_{h_q} d_{ii,jj,t_r,t_r}(e_2) + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e,jj)} \partial_{h_q} d_{ii,jj,t_r,t_z}(e_2) \\ & - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} u^s_{l^2_2(e,jj)} \partial_{h_q} d_{ii,jj,t_r,t_r}(e_2) \\ & - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} w^s_{l^2_2(e,jj)} \partial_{h_q} d_{ii,jj,t_r,t_z}(e_2) \\ & + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} \lambda^2_{l^2_2(e_2,jj)} \partial_{h_q} d_{ii,jj,n_r}(e_2) - \frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{2,e_2}} \sigma^2_{l^2_2(e_2,jj)} \partial_{h_q} d^s_{ii,jj,t_r}(e_2) \\ & - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} u_j \partial_{h_q} d^r_{ii,jj,n_r} - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} w_j \partial_{h_q} d^r_{ii,jj,n_z},\end{aligned} \tag{4.235}$$

i.e.

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,2}_{e,ii} = \sum_{jj=1}^{n_v^{2,e_2}} \frac{2\Delta_t}{3}\Big[Be\,\Big(&u_{l_2(e,jj)}\partial_{h_q}d_{ii,jj,t_r,t_r}(e_2) + w_{l_2(e,jj)}\partial_{h_q}d_{ii,jj,t_r,t_z}(e_2)\\ &- u^s_{l_2^2(e,jj)}\partial_{h_q}d_{ii,jj,t_r,t_r}(e_2) - w^s_{l_2^2(e,jj)}\partial_{h_q}d_{ii,jj,t_r,t_z}(e_2)\Big)\\ &+ \lambda^2_{l_2^2(e_2,jj)}\partial_{h_q}d_{ii,jj,n_r}(e_2) - \frac{1}{2Ca}\sigma^2_{l_2^2(e_2,jj)}\partial_{h_q}d^s_{ii,jj,t_r}(e_2) - u_j\partial_{h_q}d^r_{ii,jj,n_r}\\ &- w_j\partial_{h_q}d^r_{ii,jj,n_z}\Big],\end{aligned} \tag{4.236}$$

or, equivalently

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,2}_{e,ii} = \sum_{jj=1}^{n_v^{2,e_2}} \frac{2\Delta_t}{3}\Big[Be\,\Big(&\partial_{h_q}d_{ii,jj,t_r,t_r}(e_2)\Big\{u_{l_2(e,jj)} - u^s_{l_2^2(e,jj)}\Big\}\\ &+ \partial_{h_q}d_{ii,jj,t_r,t_z}(e_2)\Big\{w_{l_2(e,jj)} - w^s_{l_2^2(e,jj)}\Big\}\Big)\\ &+ \lambda^2_{l_2^2(e_2,jj)}\partial_{h_q}d_{ii,jj,n_r}(e_2) - \frac{1}{2Ca}\sigma^2_{l_2^2(e_2,jj)}\partial_{h_q}d^s_{ii,jj,t_r}(e_2) - u_j\partial_{h_q}d^r_{ii,jj,n_r}\\ &- w_j\partial_{h_q}d^r_{ii,jj,n_z}\Big],\end{aligned} \tag{4.237}$$

and using local spine numbers

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,2}_{e,ii} = \sum_{\substack{jj=1\\ q=S_2(e_2,qq)}}^{n_v^{2,e_2}} \frac{2\Delta_t}{3}\Big[Be\,\Big(&\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_r,t_r}(e_2)\Big\{u_{l_2(e,jj)} - u^s_{l_2^2(e,jj)}\Big\}\\ &+ \partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_r,t_z}(e_2)\Big\{w_{l_2(e,jj)} - w^s_{l_2^2(e,jj)}\Big\}\Big)\\ &+ \lambda^2_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2) - \frac{1}{2Ca}\sigma^2_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d^s_{ii,jj,t_r}(e_2)\\ &- u_j\partial_{h_{S_2(e_2,qq)}}d^r_{ii,jj,n_r} - w_j\partial_{h_{S_2(e_2,qq)}}d^r_{ii,jj,n_z}\Big].\end{aligned} \tag{4.238}$$

From equation (4.146), we have

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,3}_{e,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}}\lambda^3_{l_3^3(e_3,jj)}\partial_{h_q}f_{ii,jj,n_r}(e_3) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}}\gamma^3_{l_3^3(e_3,jj)}\partial_{h_q}f_{ii,jj,t_r}(e_3)\\ - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}u_j\partial_{h_q}f^r_{ii,jj,n_r} - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}w_j\partial_{h_q}f^r_{ii,jj,n_z},\end{aligned} \tag{4.239}$$

i.e.

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,3}_{e,ii} = \sum_{jj=1}^{n_v^{3,e_3}}\frac{2\Delta_t}{3}\Big[&\lambda^3_{l_3^3(e_3,jj)}\partial_{h_q}f_{ii,jj,n_r}(e_3) + \gamma^3_{l_3^3(e_3,jj)}\partial_{h_q}f_{ii,jj,t_r}(e_3)\\ &- u_j\partial_{h_q}f^r_{ii,jj,n_r} - w_j\partial_{h_q}f^r_{ii,jj,n_z}\Big].\end{aligned} \tag{4.240}$$

and using local spine numbers

$$\partial_{h_q}\mathcal{M}^{r,3}_{e,ii} = \sum_{\substack{jj=1\\ q=S_3(e_3,qq)}}^{n_v^{3,e_3}} \frac{2\Delta_t}{3}\Big[\lambda^3_{l^3_3(e_3,jj)}\partial_{h_{S_3(e_3,qq)}} f_{ii,jj,n_r}(e_3) + \gamma^3_{l^3_3(e_3,jj)}\partial_{h_{S_3(e_3,qq)}} f_{ii,jj,t_r}(e_3) - u_j\partial_{h_{S_3(e_3,qq)}} f^r_{ii,jj,n_r} - w_j\partial_{h_{S_3(e_3,qq)}} f^r_{ii,jj,n_z}\Big]. \tag{4.241}$$

Finally, from equation (**??**), we have

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,4}_{e,ii} &= \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \lambda^4_{l^4_4(e_3,jj)}\partial_{h_q} e_{ii,jj,n_r}(e_4) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \gamma^4_{l^4_4(e_4,jj)}\partial_{h_q} e_{ii,jj,t_r}(e_4)\\ &- \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} u_j\partial_{h_q} e^r_{ii,jj,n_r} - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} w_j\partial_{h_q} e^r_{ii,jj,n_z}.\end{aligned} \tag{4.242}$$

i.e.

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{r,4}_{e,ii} = \sum_{jj=1}^{n_v^{4,e_4}} \frac{2\Delta_t}{3}\Big[&\lambda^4_{l^4_4(e_3,jj)}\partial_{h_q} e_{ii,jj,n_r}(e_4) + \gamma^4_{l^4_4(e_4,jj)}\partial_{h_q} e_{ii,jj,t_r}(e_4)\\ &- u_j\partial_{h_q} e^r_{ii,jj,n_r} - w_j\partial_{h_q} e^r_{ii,jj,n_z}\Big],\end{aligned} \tag{4.243}$$

and using local spine numbers we have

$$\partial_{h_q}\mathcal{M}^{r,4}_{e,ii} = \sum_{\substack{jj=1\\ q=S_4(e_4,qq)}}^{n_v^{4,e_4}} \frac{2\Delta_t}{3}\Big[\lambda^4_{l^4_4(e_3,jj)}\partial_{h_{S_4(e_4,qq)}} e_{ii,jj,n_r}(e_4) + \gamma^4_{l^4_4(e_4,jj)}\partial_{h_{S_4(e_4,qq)}} e_{ii,jj,t_r}(e_4) - u_j\partial_{h_{S_4(e_4,qq)}} e^r_{ii,jj,n_r} - w_j\partial_{h_{S_4(e_4,qq)}} e^r_{ii,jj,n_z}\Big]. \tag{4.244}$$

5. The z-momentum equation

Using equation (3.2), we define the i-th residuals of the z-momentum equation as

$$\begin{aligned} M_i^z = & Re\int_{\Omega^f}\phi_i\partial_t w + Re\int_{\Omega^f}\phi_i u\partial_r w + Re\int_{\Omega^f}\phi_i w\partial_z w - Re\int_{\Omega^f}\phi_i u^c\partial_r w \\ & - Re\int_{\Omega^f}\phi_i w^c\partial_z w - St\int_{\Omega^f}\phi_i\hat{g}_z - \int_{\Omega^f}\phi_i \boldsymbol{e}_z\cdot\nabla\cdot\boldsymbol{P}, \end{aligned} \tag{5.1}$$

which must be identically zero for all i.

Once again, we recall the tensor identity

$$\nabla\cdot(\boldsymbol{x}\cdot\boldsymbol{A}) = \boldsymbol{x}\cdot\nabla\cdot\boldsymbol{A} + \nabla\boldsymbol{x} : \boldsymbol{A}, \tag{5.2}$$

taking $\boldsymbol{x} = \phi_i\boldsymbol{e}_z$ and $\boldsymbol{A} = \boldsymbol{P}$, we have

$$-\phi_i\boldsymbol{e}_z\cdot\nabla\cdot\boldsymbol{P} = -\nabla\cdot(\phi_i\boldsymbol{e}_z\cdot\boldsymbol{P}) + \nabla(\phi_i\boldsymbol{e}_z) : \boldsymbol{P}, \tag{5.3}$$

which reduces M_i^z to

$$\begin{aligned} M_i^z = & Re\int_{\Omega^f}\phi_i\partial_t w + Re\int_{\Omega^f}\phi_i u\partial_r w + Re\int_{\Omega^f}\phi_i w\partial_z w - Re\int_{\Omega^f}\phi_i u^c\partial_r w \\ & - Re\int_{\Omega^f}\phi_i w^c\partial_z w - St\int_{\Omega^f}\phi_i\hat{g}_z + \int_{\Omega^f}\nabla(\phi_i\boldsymbol{e}_z) : \boldsymbol{P} - \int_{\Omega^f}\nabla\cdot(\phi_i\boldsymbol{e}_z\cdot\boldsymbol{P}), \end{aligned} \tag{5.4}$$

we can now apply the divergence theorem to the last integral on the right hand side above to obtain

$$\begin{aligned} M_i^z = & Re\int_{\Omega^f}\phi_i\partial_t w + Re\int_{\Omega^f}\phi_i u\partial_r w + Re\int_{\Omega^f}\phi_i w\partial_z w - Re\int_{\Omega^f}\phi_i u^c\partial_r w \\ & - Re\int_{\Omega^f}\phi_i w^c\partial_z w - St\int_{\Omega^f}\phi_i\hat{g}_z + \int_{\Omega^f}\nabla(\phi_i\boldsymbol{e}_z) : \boldsymbol{P} + \int_{\partial\Omega}\phi_i\boldsymbol{e}_z\cdot\boldsymbol{P}\cdot\boldsymbol{n}, \end{aligned} \tag{5.5}$$

where $\partial\Omega$ is the boundary of Ω and $\boldsymbol{n}$ is the normal to $\partial\Omega$, that points into Ω.

We notice that

$$\nabla(\phi_i\boldsymbol{e}_z) : \boldsymbol{P} = \begin{bmatrix} 0 & 0 \\ \partial_r\phi_i & \partial_z\phi_i \end{bmatrix} : \begin{bmatrix} \boldsymbol{P}_{rr} & \boldsymbol{P}_{rz} \\ \boldsymbol{P}_{zr} & \boldsymbol{P}_{zz} \end{bmatrix} \tag{5.6}$$

i.e.

$$\nabla(\phi_i\boldsymbol{e}_z) : \boldsymbol{P} = \begin{bmatrix} 0 & 0 \\ \partial_r\phi_i & \partial_z\phi_i \end{bmatrix} : \begin{bmatrix} -p+2\partial_r u & \partial_z u+\partial_r w \\ \partial_r w+\partial_z u & -p+2\partial_z w \end{bmatrix}, \tag{5.7}$$

which is

$$\nabla(\phi_i\boldsymbol{e}_z) : \boldsymbol{P} = \partial_r\phi_i\boldsymbol{P}_{zr} + \partial_z\phi_i\boldsymbol{P}_{zz} = \partial_r w\partial_r\phi_i + \partial_z u\partial_r\phi_i - p\partial_z\phi_i + 2\partial_z w\partial_z\phi_i. \tag{5.8}$$

Therefore we have

$$\begin{aligned} M_i^z = & Re\int_{\Omega^f}\phi_i\partial_t w + Re\int_{\Omega^f}\phi_i u\partial_r w + Re\int_{\Omega^f}\phi_i w\partial_z w - Re\int_{\Omega^f}\phi_i u^c\partial_r w - Re\int_{\Omega^f}\phi_i w^c\partial_z w \\ & - St\int_{\Omega^f}\phi_i\hat{g}_z + \int_{\Omega^f}\partial_r w\partial_r\phi_i + \int_{\Omega^f}\partial_z u\partial_r\phi_i - \int_{\Omega^f}p\partial_z\phi_i + 2\int_{\Omega^f}\partial_z w\partial_z\phi_i + \int_{\partial\Omega}\phi_i\boldsymbol{e}_r\cdot\boldsymbol{P}\cdot\boldsymbol{n}, \end{aligned} \tag{5.9}$$

We now consider the last integral in the right hand side of the equation above

$$\begin{aligned}\int_{\partial\Omega} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n} = \int_{\partial\Omega^{1,f}} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^1 + \int_{\partial\Omega^{2,f}} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 \\ + \int_{\partial\Omega^3} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4 + \int_{\partial\Omega^4} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4,\end{aligned} \tag{5.10}$$

where $\partial\Omega^1$ is the free surface, $\partial\Omega^2$ is the solid surface, and $\partial\Omega^3$ is the axis of symmetry.

For the free-surface we have equation (2.10), which states

$$\boldsymbol{P} \cdot \boldsymbol{n}^1 = -p^g \boldsymbol{n}^1 - \frac{\nabla^s \cdot \left[\sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)\right]}{Ca}, \tag{5.11}$$

and therefore

$$\phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^1 = -\phi_i p^g \boldsymbol{e}_z \cdot \boldsymbol{n}^1 - \frac{1}{Ca} \phi_i \boldsymbol{e}_z \cdot \nabla^s \cdot \left[\sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)\right]. \tag{5.12}$$

Now, we have the following surface vector calculus identity

$$\nabla^s \cdot (\boldsymbol{x} \cdot \boldsymbol{A}) = \boldsymbol{A} : \nabla^s \boldsymbol{x} + \boldsymbol{x} \cdot \nabla^s \cdot \boldsymbol{A} \tag{5.13}$$

taking $\boldsymbol{x} = \phi_i \boldsymbol{e}_z$ and $\boldsymbol{A} = \sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)$, we have

$$\nabla^s \cdot (\phi_i \boldsymbol{e}_z \cdot \sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)) = \sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) : \nabla^s(\phi_i \boldsymbol{e}_z) + \phi_i \boldsymbol{e}_z \cdot \nabla^s \cdot \sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) \tag{5.14}$$

which yields

$$-\phi_i \boldsymbol{e}_z \cdot \nabla^s \cdot \sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) = -\nabla^s \cdot (\phi_i \boldsymbol{e}_z \cdot \sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)) + \sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) : \nabla^s(\phi_i \boldsymbol{e}_z). \tag{5.15}$$

In this $1D$-surface case, we have

$$\nabla^s(\phi_i \boldsymbol{e}_z) = \begin{bmatrix} 0 & \partial_s \phi_i t_r^1 \\ 0 & \partial_s \phi_i t_z^1 \end{bmatrix} \tag{5.16}$$

and therefore

$$(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) : \nabla^s(\phi_i \boldsymbol{e}_z) = \begin{bmatrix} 1 - n_r^1 n_r^1 & -n_r^1 n_z^1 \\ -n_z^1 n_r^1 & 1 - n_z^1 n_z^1 \end{bmatrix} : \begin{bmatrix} 0 & \partial_s \phi_i t_r^1 \\ 0 & \partial_s \phi_i t_z^1 \end{bmatrix} \tag{5.17}$$

i.e.

$$(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) : \nabla^s(\phi_i \boldsymbol{e}_z) = \partial_s \phi_i t_z^1 - \partial_s \phi_i n_z^1 (\boldsymbol{t}^1 \cdot \boldsymbol{n}^1) = \partial_s \phi_i t_z^1. \tag{5.18}$$

We therefore have in equation (5.15)

$$-\phi_i \boldsymbol{e}_z \cdot \nabla^s \cdot \sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) = -\nabla^s \cdot (\phi_i \boldsymbol{e}_z \cdot \sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)) + \sigma^1 \partial_s \phi_i t_z^1. \tag{5.19}$$

Taking equation (5.19) into (5.12) and the result of that into (5.10) and finally taking that into the z-momentum residual (5.9) we have

$$\begin{aligned} M_i^z = & Re \int_{\Omega^f} \phi_i \partial_t w + Re \int_{\Omega^f} \phi_i u \partial_r w + Re \int_{\Omega^f} \phi_i w \partial_z w - Re \int_{\Omega^f} \phi_i u^c \partial_r w \\ & - Re \int_{\Omega^f} \phi_i w^c \partial_z w - St \int_{\Omega^f} \phi_i \hat{g}_z + \int_{\Omega^f} \partial_r w \partial_r \phi_i + \int_{\Omega^f} \partial_z u \partial_r \phi_i + 2 \int_{\Omega^f} \partial_z w \partial_z \phi_i \\ & - \int_{\Omega^f} p \partial_z \phi_i - \int_{\partial\Omega^{1,f}} \phi_i p^g \boldsymbol{e}_z \cdot \boldsymbol{n}^1 - \frac{1}{Ca} \int_{\partial\Omega^{1,f}} \nabla^s \cdot (\sigma^1 \phi_i \boldsymbol{e}_z \cdot (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)) \\ & + \frac{1}{Ca} \int_{\partial\Omega^{1,f}} \sigma^1 \partial_s \phi_i t_z^1 + \int_{\partial\Omega^{2,f}} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 + \int_{\partial\Omega^3} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3 + \int_{\partial\Omega^4} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4. \end{aligned} \tag{5.20}$$

Using the surface divergence theorem and the definition of the surface divergence for a $1D$ surface, we have

$$\begin{aligned} M_i^z = & Re \int_{\Omega^f} \phi_i \partial_t w + Re \int_{\Omega^f} \phi_i u \partial_r w + Re \int_{\Omega^f} \phi_i w \partial_z w - Re \int_{\Omega^f} \phi_i u^c \partial_r w \\ & - Re \int_{\Omega^f} \phi_i w^c \partial_z w - St \int_{\Omega^f} \phi_i \hat{g}_z + \int_{\Omega^f} \partial_r w \partial_r \phi_i + \int_{\Omega^f} \partial_z u \partial_r \phi_i + 2 \int_{\Omega^f} \partial_z w \partial_z \phi_i \\ & - \int_{\Omega^f} p \partial_z \phi_i - \int_{\partial\Omega^{1,f}} \phi_i p^g n_z^1 - \int_{\partial\Omega^{1,f}} \phi_i p^g \boldsymbol{e}_z \cdot \boldsymbol{n}^1 + \frac{1}{Ca} \int_{C_1} \sigma^1 \phi_i \boldsymbol{e}_z \cdot \boldsymbol{m}^1 \\ & + \frac{1}{Ca} \int_{\partial\Omega^{1,f}} \sigma^1 \partial_s \phi_i t_z^1 + \int_{\partial\Omega^{2,f}} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 + \int_{\partial\Omega^3} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3 + \int_{\partial\Omega^4} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4, \end{aligned} \tag{5.21}$$

where C_1 is actually the two points bounding the free surface, and $\boldsymbol{m}^1$ is the vector that is tangent to the free-surface, normal to the separatrix line or symmetry axis (accordingly) and points into the far-field free-surface. Therefore we have

$$\begin{aligned} M_i^z = & Re \int_{\Omega^f} \phi_i \partial_t w + Re \int_{\Omega^f} \phi_i u \partial_r w + Re \int_{\Omega^f} \phi_i w \partial_z w - Re \int_{\Omega^f} \phi_i u^c \partial_r w - Re \int_{\Omega^f} \phi_i w^c \partial_z w \\ & - St \int_{\Omega^f} \phi_i \hat{g}_z + \int_{\Omega^f} \partial_r w \partial_r \phi_i + \int_{\Omega^f} \partial_z u \partial_r \phi_i + 2 \int_{\Omega^f} \partial_z w \partial_z \phi_i - \int_{\Omega^f} p \partial_z \phi_i - \int_{\partial\Omega^{1,f}} \phi_i p^g n_z^1 \\ & + \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^{1,n}(r_{J^1}, z_{J^1})}{Ca} + \frac{\sigma^1(r_a, z_a) \phi_i(r_a, z_a) m_z^1(r_a, z_a)}{Ca} \\ & + \frac{1}{Ca} \int_{\partial\Omega^{1,f}} \sigma^1 \partial_s \phi_i t_z^1 + \int_{\partial\Omega^{2,f}} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 + \int_{\partial\Omega^3} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3 + \int_{\partial\Omega^4} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4, \end{aligned} \tag{5.22}$$

where $\boldsymbol{m}^{1,n}(r_{J^1}, z_{J^1})$ is tangent to the near-field portion of the free surface. This choice is important as it prevents formation of a corner on the free surface when solving the simulations.

We consider now the term

$$\int_{\partial\Omega^{2,f}} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2. \tag{5.23}$$

We have

$$\phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 = \phi_i \boldsymbol{e}_z \cdot \underbrace{\left(\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2\right) \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2}_{Be\,(\boldsymbol{u}-\boldsymbol{u}^s)\cdot(\boldsymbol{I}-\boldsymbol{n}^2\boldsymbol{n}^2) - \frac{1}{2Ca}\nabla^2\sigma^2} + \phi_i \boldsymbol{e}_z \cdot \underbrace{\left(\boldsymbol{n}^2 \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2\right)}_{\lambda^2} \boldsymbol{n}^2, \tag{5.24}$$

where λ^2 is the normal stress on surface 2 and we have used the GNSC (2.53), and therefore

$$\phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 = Be\,\phi_i \boldsymbol{e}_z \cdot (\boldsymbol{u} \cdot \boldsymbol{t}^2)\boldsymbol{t}^2 - Be\,\phi_i \boldsymbol{e}_z \cdot (\boldsymbol{u}^s \cdot \boldsymbol{t}^2)\boldsymbol{t}^2 - \frac{1}{2Ca}\phi_i \boldsymbol{e}_z \cdot \nabla^2\sigma^2 + \lambda^2 \phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^2, \tag{5.25}$$

i.e.

$$\phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 = Be\,\phi_i \boldsymbol{e}_z \cdot (ut_r^2 + wt_z^2)\boldsymbol{t}^2 - Be\,\phi_i \boldsymbol{e}_z \cdot (u^s t_r^2 + w^s t_z^2)\boldsymbol{t}^2 - \frac{1}{2Ca}\phi_i(\partial_s\sigma^2)\boldsymbol{e}_z \cdot \boldsymbol{t}^2 + \lambda^2 \phi_i n_z^2, \tag{5.26}$$

where we have used that $\nabla^2\sigma^2 = \partial_s\sigma^2\boldsymbol{t}^2$ (with $\boldsymbol{t}^2$ pointing in the direction of increasing arc-length s). Equivalently, we have

$$\begin{aligned} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 = {} & Be\,\phi_i u t_r^2 t_z^2 + Be\,\phi_i w t_z^2 t_z^2 - Be\,\phi_i u^s t_r^2 t_z^2 \\ & - Be\,\phi_i w^s t_z^2 t_z^2 - \frac{1}{2Ca}\phi_i \boldsymbol{t}_z^2 \partial_s\sigma^2 + \lambda^2 \phi_i n_z^2. \end{aligned} \tag{5.27}$$

Consequently we have

$$\begin{aligned} \int_{\partial\Omega^{2,f}} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^2 = {} & Be \int_{\partial\Omega^{2,f}} \phi_i u t_r^2 t_z^2 + Be \int_{\partial\Omega^{2,f}} \phi_i w t_z^2 t_z^2 - Be \int_{\partial\Omega^{2,f}} \phi_i u^s t_r^2 t_z^2 \\ & - Be \int_{\partial\Omega^{2,f}} \phi_i w^s t_z^2 t_z^2 - \frac{1}{2Ca} \int_{\partial\Omega^{2,f}} \phi_i \boldsymbol{t}_z^2 \partial_s\sigma^2 + \int_{\partial\Omega^{2,f}} \lambda^2 \phi_i n_z^2. \end{aligned} \tag{5.28}$$

Similarly, we have for the term

$$\int_{\partial\Omega^3} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3, \tag{5.29}$$

$$\boldsymbol{P} \cdot \boldsymbol{n}^3 = \boldsymbol{n}^3 \underbrace{\boldsymbol{n}^3 \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3}_{\lambda^3} + \underbrace{\left(\boldsymbol{I} - \boldsymbol{n}^3\boldsymbol{n}^3\right) \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3}_{\gamma^3\boldsymbol{t}^3}, \tag{5.30}$$

where λ^3 is the normal stress on surface 3 and γ^3 is the tangential stress on surface 3. We therefore have

$$\boldsymbol{P} \cdot \boldsymbol{n}^3 = \lambda^3\boldsymbol{n}^3 + \gamma^3\boldsymbol{t}^3. \tag{5.31}$$

Consequently we have

$$\int_{\partial\Omega^3} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3 = \int_{\partial\Omega^3} \phi_i \lambda^3 \boldsymbol{e}_z \cdot \boldsymbol{n}^3 + \int_{\partial\Omega^3} \phi_i \gamma^3 \boldsymbol{e}_z \cdot \boldsymbol{t}^3, \tag{5.32}$$

i.e.

$$\int_{\partial\Omega^3} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^3 = \int_{\partial\Omega^3} \lambda^3 n_z^3 \phi_i + \int_{\partial\Omega^3} \gamma^3 t_z^3 \phi_i. \tag{5.33}$$

Finally, for the term

$$\int\limits_{\partial\Omega^4} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4, \tag{5.34}$$

$$\boldsymbol{P} \cdot \boldsymbol{n}^4 = \boldsymbol{n}^4 \underbrace{\boldsymbol{n}^4 \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4}_{\lambda^4} + \underbrace{(\boldsymbol{I} - \boldsymbol{n}^4\boldsymbol{n}^4) \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4}_{\gamma^4 \boldsymbol{t}^4}, \tag{5.35}$$

where λ^4 is the normal stress on surface 4 and γ^4 is the tangential stress on surface 4. We therefore have

$$\boldsymbol{P} \cdot \boldsymbol{n}^4 = \lambda^4 \boldsymbol{n}^4 + \gamma^4 \boldsymbol{t}^4. \tag{5.36}$$

Consequently we have

$$\int\limits_{\partial\Omega^4} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4 = \int\limits_{\partial\Omega^4} \phi_i \lambda^4 \boldsymbol{e}_z \cdot \boldsymbol{n}^4 + \int\limits_{\partial\Omega^4} \phi_i \gamma^4 \boldsymbol{e}_z \cdot \boldsymbol{t}^4, \tag{5.37}$$

i.e.

$$\int\limits_{\partial\Omega^4} \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^4 = \int\limits_{\partial\Omega^4} \lambda^4 n_z^4 \phi_i + \int\limits_{\partial\Omega^4} \gamma^4 t_z^4 \phi_i. \tag{5.38}$$

Taking (5.28) and (5.33) into (5.22) we have

$$\begin{aligned}
M_i^z = {} & Re \int\limits_{\Omega^f} \phi_i \partial_t w + Re \int\limits_{\Omega^f} \phi_i u \partial_r w + Re \int\limits_{\Omega^f} \phi_i w \partial_z w - Re \int\limits_{\Omega^f} \phi_i u^c \partial_r w \\
& - Re \int\limits_{\Omega^f} \phi_i w^c \partial_z w - St \int\limits_{\Omega^f} \phi_i \hat{g}_z + \int\limits_{\Omega^f} \partial_r w \partial_r \phi_i + \int\limits_{\Omega^f} \partial_z u \partial_r \phi_i + 2 \int\limits_{\Omega^f} \partial_z w \partial_z \phi_i \\
& - \int\limits_{\Omega^f} p \partial_z \phi_i - \int\limits_{\partial\Omega^{1,f}} \phi_i p^g n_z^1 + \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^{1,n}(r_{J^1}, z_{J^1})}{Ca} \\
& + \frac{\sigma^1(r_a, z_a) \phi_i(r_a, z_a) m_z^1(r_a, z_a)}{Ca} + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,f}} \sigma^1 \partial_s \phi_i t_z^1 + Be \int\limits_{\partial\Omega^{2,f}} \phi_i u t_r^2 t_z^2 \\
& + Be \int\limits_{\partial\Omega^{2,f}} \phi_i w t_z^2 t_z^2 - Be \int\limits_{\partial\Omega^{2,f}} \phi_i u^s t_r^2 t_z^2 - Be \int\limits_{\partial\Omega^{2,f}} \phi_i w^s t_z^2 t_z^2 - \frac{1}{2Ca} \int\limits_{\partial\Omega^{2,f}} \phi_i \boldsymbol{t}_z^2 \partial_s \sigma^2 \\
& + \int\limits_{\partial\Omega^{2,f}} \lambda^2 \phi_i n_z^2 + \int\limits_{\partial\Omega^3} \lambda^3 n_z^3 \phi_i + \int\limits_{\partial\Omega^3} \gamma^3 t_z^3 \phi_i + \int\limits_{\partial\Omega^4} \lambda^4 n_z^4 \phi_i + \int\limits_{\partial\Omega^4} \gamma^4 t_z^4 \phi_i,
\end{aligned} \tag{5.39}$$

We recall approximations (4.52)-(4.55) and substitute them into (5.39) obtaining

$$
\begin{aligned}
\mathfrak{M}_i^z = {} & Re\int\limits_{\Omega^f}\phi_i\frac{3w(t_n)-4w(t_{n-1})+w(t_{n-2})}{2\Delta_t}+Re\int\limits_{\Omega^f}\phi_i u\partial_r w \\
& + Re\int\limits_{\Omega^f}\phi_i w\partial_z w - Re\int\limits_{\Omega^f}\phi_i\frac{3r^c(t_n)-4r^c(t_{n-1})+r^c(t_{n-2})}{2\Delta_t}\partial_r w \\
& - Re\int\limits_{\Omega^f}\phi_i\frac{3z^c(t_n)-4z^c(t_{n-1})+z^c(t_{n-2})}{2\Delta_t}\partial_z w - St\int\limits_{\Omega^f}\phi_i\hat{g}_z \\
& + \int\limits_{\Omega^f}\partial_r w\partial_r\phi_i+\int\limits_{\Omega^f}\partial_z u\partial_r\phi_i+2\int\limits_{\Omega^f}\partial_z w\partial_z\phi_i-\int\limits_{\Omega^f}p\partial_z\phi_i-\int\limits_{\partial\Omega^{1,f}}\phi_i p^g n_z^1 \\
& + \frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca}+\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} \\
& + \frac{1}{Ca}\int\limits_{\partial\Omega^{1,f}}\sigma^1\partial_s\phi_i t_z^1+Be\int\limits_{\partial\Omega^{2,f}}\phi_i u t_r^2 t_z^2+Be\int\limits_{\partial\Omega^{2,f}}\phi_i w t_z^2 t_z^2 \\
& - Be\int\limits_{\partial\Omega^{2,f}}\phi_i u^s t_r^2 t_z^2-Be\int\limits_{\partial\Omega^{2,f}}\phi_i w^s t_z^2 t_z^2-\frac{1}{2Ca}\int\limits_{\partial\Omega^{2,f}}\phi_i \boldsymbol{t}_z^2\partial_s\sigma^2 \\
& + \int\limits_{\partial\Omega^{2,f}}\lambda^2\phi_i n_z^2+\int\limits_{\partial\Omega^3}\lambda^3 n_z^3\phi_i+\int\limits_{\partial\Omega^3}\gamma^3 t_z^3\phi_i+\int\limits_{\partial\Omega^4}\lambda^4 n_z^4\phi_i+\int\limits_{\partial\Omega^4}\gamma^4 t_z^4\phi_i,
\end{aligned}
\tag{5.40}
$$

and multiplying (26.43) by $2\Delta_t/3$ we have

$$
\begin{aligned}
\mathscr{M}_i^z = {} & Re \int\limits_{\Omega^f} \phi_i w - \frac{4Re}{3} \int\limits_{\Omega^f} \phi_i w(t_{n-1}) + \frac{Re}{3} \int\limits_{\Omega^f} \phi_i w(t_{n-2}) + \frac{2\Delta_t Re}{3} \int\limits_{\Omega^f} \phi_i u \partial_r w \\
& + \frac{2\Delta_t Re}{3} \int\limits_{\Omega^f} \phi_i w \partial_z w - Re \int\limits_{\Omega^f} \phi_i r^c \partial_r w + \frac{4Re}{3} \int\limits_{\Omega^f} \phi_i r^c(t_{n-1}) \partial_r w \\
& - \frac{Re}{3} \int\limits_{\Omega^f} \phi_i r^c(t_{n-2}) \partial_r w - Re \int\limits_{\Omega^f} \phi_i z^c \partial_z w + \frac{4Re}{3} \int\limits_{\Omega^f} \phi_i z^c(t_{n-1}) \partial_z w \\
& - \frac{Re}{3} \int\limits_{\Omega^f} \phi_i z^c(t_{n-2}) \partial_z w - \frac{2\Delta_t St}{3} \int\limits_{\Omega^f} \phi_i \hat{g}_z + \frac{2\Delta_t}{3} \int\limits_{\Omega^f} \partial_r w \partial_r \phi_i \\
& + \frac{2\Delta_t}{3} \int\limits_{\Omega^f} \partial_z u \partial_r \phi_i + \frac{4\Delta_t}{3} \int\limits_{\Omega^f} \partial_z w \partial_z \phi_i - \frac{2\Delta_t}{3} \int\limits_{\Omega^f} p \partial_z \phi_i \\
& - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{1,f}} \phi_i p^g n_z^1 + \frac{2\Delta_t}{3Ca} \sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^{1,n}(r_{J^1}, z_{J^1}) \\
& + \frac{2\Delta_t}{3Ca} \sigma^1(r_a, z_a) \phi_i(r_a, z_a) m_z^1(r_a, z_a) + \frac{2\Delta_t}{3Ca} \int\limits_{\partial\Omega^{1,f}} \sigma^1 \partial_s \phi_i t_z^1 \\
& + \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,f}} \phi_i u t_r^2 t_z^2 + \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,f}} \phi_i w t_z^2 t_z^2 - \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,f}} \phi_i u^s t_r^2 t_z^2 \\
& - \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,f}} \phi_i w^s t_z^2 t_z^2 - \frac{\Delta_t}{3Ca} \int\limits_{\partial\Omega^{2,f}} \phi_i t_z^2 \partial_s \sigma^2 + \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,f}} \lambda^2 \phi_i n_z^2 \\
& + \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^3} \lambda^3 n_z^3 \phi_i + \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^3} \gamma^3 t_z^3 \phi_i + \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^4} \lambda^4 n_z^4 \phi_i + \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^4} \gamma^4 t_z^4 \phi_i,
\end{aligned}
\tag{5.41}
$$

where time dependent function whose time argument is not indicated are evaluated at time t_n.

We now substitute approximations (4.66)-(4.80) into (5.41) and define

$$
\begin{aligned}
\mathcal{M}_i^z =& Re \int_{\Omega^f} \phi_i \left(\sum_{j=1}^{n_v} w_j \phi_j \right) - \frac{4Re}{3} \int_{\Omega^f} \phi_i \left(\sum_{j=1}^{n_v} w_j(t_{n-1}) \phi_j \right) \qquad (5.42)\\
&+ \frac{Re}{3} \int_{\Omega^f} \phi_i \left(\sum_{j=1}^{n_v} w_j(t_{n-2}) \phi_j \right)\\
&+ \frac{2\Delta_t Re}{3} \int_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} u_k \phi_k \right) \partial_r \left(\sum_{j=1}^{n_v} w_j \phi_j \right) + \frac{2\Delta_t Re}{3} \int_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} w_k \phi_k \right) \partial_z \left(\sum_{j=1}^{n_v} w_j \phi_j \right)\\
&- Re \int_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} r_k^c \phi_k \right) \partial_r \left(\sum_{j=1}^{n_v} w_j \phi_j \right) + \frac{4Re}{3} \int_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} r_k^c(t_{n-1}) \phi_k \right) \partial_r \left(\sum_{j=1}^{n_v} w_j \phi_j \right)\\
&- \frac{Re}{3} \int_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} r_k^c(t_{n-2}) \phi_k \right) \partial_r \left(\sum_{j=1}^{n_v} w_j \phi_j \right)\\
&- Re \int_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} z_k^c \phi_k \right) \partial_z \left(\sum_{j=1}^{n_v} w_j \phi_j \right) + \frac{4Re}{3} \int_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} z_k^c(t_{n-1}) \phi_k \right) \partial_z \left(\sum_{j=1}^{n_v} w_j \phi_j \right)\\
&- \frac{Re}{3} \int_{\Omega^f} \phi_i \left(\sum_{k=1}^{n_v} z_k^c(t_{n-2}) \phi_k \right) \partial_z \left(\sum_{j=1}^{n_v} w_j \phi_j \right) - \frac{2\Delta_t St}{3} \int_{\Omega^f} \phi_i \hat{g}_z\\
&+ \frac{2\Delta_t}{3} \int_{\Omega^f} \partial_r \left(\sum_{j=1}^{n_v} w_j \phi_j \right) \partial_r \phi_i + \frac{2\Delta_t}{3} \int_{\Omega^f} \partial_z \left(\sum_{j=1}^{n_v} u_j \phi_j \right) \partial_r \phi_i + \frac{4\Delta_t}{3} \int_{\Omega^f} \partial_z \left(\sum_{j=1}^{n_v} w_j \phi_j \right) \partial_z \phi_i\\
&- \frac{2\Delta_t}{3} \int_{\Omega^f} \left(\sum_{j=1}^{n_p} p_j \psi_j \right) \partial_z \phi_i - \frac{2\Delta_t}{3} \int_{\partial\Omega^{1,f}} \phi_i \left(\sum_{j=1}^{n_v} \tilde{p}_j^g \phi_j^1 \right) n_z^1\\
&+ \frac{2\Delta_t}{3Ca} \sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^{1,n}(r_{J^1}, z_{J^1}) + \frac{2\Delta_t}{3Ca} \sigma^1(r_a, z_a) \phi_i(r_a, z_a) m_z^1(r_a, z_a)\\
&+ \frac{2\Delta_t}{3Ca} \int_{\partial\Omega^{1,f}} \left(\sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \phi_j^1 \right) t_z^1 \partial_s \phi_i + \frac{2\Delta_t Be}{3} \int_{\partial\Omega^{2,f}} \phi_i t_r^2 t_z^2 \left(\sum_{j=1}^{n_v} u_j \phi_j \right)\\
&+ \frac{2\Delta_t Be}{3} \int_{\partial\Omega^{2,f}} \phi_i t_z^2 t_z^2 \left(\sum_{j=1}^{n_v} w_j \phi_j \right) - \frac{2\Delta_t Be}{3} \int_{\partial\Omega^{2,f}} \phi_i \left(\sum_{j=1}^{n_v} \tilde{u}_j^s \phi_j \right) t_r^2 t_z^2\\
&- \frac{2\Delta_t Be}{3} \int_{\partial\Omega^{2,f}} \phi_i \left(\sum_{j=1}^{n_v} \tilde{w}_j^s \phi_j \right) t_z^2 t_z^2 - \frac{\Delta_t}{3Ca} \int_{\partial\Omega^{2,f}} \phi_i t_z^2 \partial_s \left(\sum_{j=1}^{n_v} \tilde{\sigma}_j^2 \phi_j^2 \right)\\
&+ \frac{2\Delta_t}{3} \int_{\partial\Omega^{2,f}} \left(\sum_{j=1}^{n_v} \tilde{\lambda}^2{}_j \phi_j \right) \phi_i n_z^2 + \frac{2\Delta_t}{3} \int_{\partial\Omega^3} \left(\sum_{j=1}^{n_v} \tilde{\lambda}^3{}_j \phi_j \right) n_z^3 \phi_i\\
&+ \frac{2\Delta_t}{3} \int_{\partial\Omega^3} \left(\sum_{j=1}^{n_v} \tilde{\gamma}^3{}_j \phi_j \right) t_z^3 \phi_i + \frac{2\Delta_t}{3} \int_{\partial\Omega^4} \left(\sum_{j=1}^{n_v} \tilde{\lambda}^4{}_j \phi_j \right) n_z^4 \phi_i + \frac{2\Delta_t}{3} \int_{\partial\Omega^4} \left(\sum_{j=1}^{n_v} \tilde{\gamma}^4{}_j \phi_j \right) t_z^4 \phi_i,
\end{aligned}
$$

where the numbering in functions p_j and ψ_j corresponds to pressure-node numbering (see figure 6).

Moving the integrals into the sums, we can re-write this as

$$
\begin{aligned}
\hat{M}_i^z = {} & Re \sum_{j=1}^{n_v} w_j \int\limits_{\Omega^f} \phi_i \phi_j - \frac{4Re}{3} \sum_{j=1}^{n_v} w_j(t_{n-1}) \int\limits_{\Omega^f} \phi_i \phi_j + \frac{Re}{3} \sum_{j=1}^{n_v} w_j(t_{n-2}) \int\limits_{\Omega^f} \phi_i \phi_j \\
& + \frac{2\Delta_t Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} u_k \int\limits_{\Omega^f} \phi_i \phi_k \partial_r \phi_j + \frac{2\Delta_t Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} w_k \int\limits_{\Omega^f} \phi_i \phi_k \partial_z \phi_j \\
& - Re \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} r_k^c \int\limits_{\Omega^f} \phi_i \phi_k \partial_r \phi_j + \frac{4Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} r_k^c(t_{n-1}) \int\limits_{\Omega^f} \phi_i \phi_k \partial_r \phi_j \\
& - \frac{Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} r_k^c(t_{n-2}) \int\limits_{\Omega^f} \phi_i \phi_k \partial_r \phi_j - Re \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} z_k^c \int\limits_{\Omega^f} \phi_i \phi_k \partial_z \phi_j \\
& + \frac{4Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} z_k^c(t_{n-1}) \int\limits_{\Omega^f} \phi_i \phi_k \partial_z \phi_j - \frac{Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} z_k^c(t_{n-2}) \int\limits_{\Omega^f} \phi_i \phi_k \partial_z \phi_j \\
& - \frac{2\Delta_t St}{3} \int\limits_{\Omega^f} \phi_i \hat{g}_z + \sum_{j=1}^{n_v} w_j \frac{2\Delta_t}{3} \int\limits_{\Omega^f} \partial_r \phi_j \partial_r \phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\Omega^f} \partial_z \phi_j \partial_r \phi_i \\
& + \frac{4\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\Omega^f} \partial_z \phi_j \partial_z \phi_i - \frac{2\Delta_t}{3} \sum_{j=1}^{n_p} p_j \int\limits_{\Omega^f} \psi_j \partial_z \phi_i - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{p}_j^g \int\limits_{\partial\Omega^{1,f}} \phi_i \phi_j^1 n_z^1 \\
& + \frac{2\Delta_t}{3Ca} \sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^{1,n}(r_{J^1}, z_{J^1}) + \frac{2\Delta_t}{3Ca} \sigma^1(r_a, z_a) \phi_i(r_a, z_a) m_z^1(r_a, z_a) \\
& + \frac{2\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \int\limits_{\partial\Omega^{1,f}} \phi_j^1 t_z^1 \partial_s \phi_i + \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{2,f}} \phi_i t_r^2 t_z^2 \phi_j \\
& + \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{2,f}} \phi_i t_z^2 t_z^2 \phi_j - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{u}_j^s \int\limits_{\partial\Omega^{2,f}} \phi_i \phi_j t_r^2 t_z^2 \\
& - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{w}_j^s \int\limits_{\partial\Omega^{2,f}} \phi_i \phi_j t_z^2 t_z^2 - \frac{\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^2 \int\limits_{\partial\Omega^{2,f}} \phi_i \boldsymbol{t}_z^2 \partial_s \phi_j^2 \\
& + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}^2{}_j \int\limits_{\partial\Omega^{2,f}} \phi_j \phi_i n_z^2 + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}^3{}_j \int\limits_{\partial\Omega^3} \phi_j n_z^3 \phi_i \\
& + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma}^3{}_j \int\limits_{\partial\Omega^3} t_z^3 \phi_j \phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}^4{}_j \int\limits_{\partial\Omega^4} \phi_j n_z^4 \phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma}^4{}_j \int\limits_{\partial\Omega^4} t_z^4 \phi_j \phi_i,
\end{aligned}
\tag{5.43}
$$

We now introduce the decomposition

$$\mathcal{M}_i^z = \mathcal{M}_i^{z,0} + \mathcal{M}_i^{z,1} + \mathcal{M}_i^{z,2} + \mathcal{M}_i^{z,3}, \tag{5.44}$$

where

$$\begin{aligned}\mathcal{M}_i^{z,0} = {} & Re \sum_{j=1}^{n_v} w_j \int\limits_{\Omega^f} \phi_i\phi_j - \frac{4Re}{3} \sum_{j=1}^{n_v} w_j(t_{n-1}) \int\limits_{\Omega^f} \phi_i\phi_j + \frac{Re}{3} \sum_{j=1}^{n_v} w_j(t_{n-2}) \int\limits_{\Omega^f} \phi_i\phi_j \\ & + \frac{2\Delta_t Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} u_k \int\limits_{\Omega^f} \phi_i\phi_k\partial_r\phi_j + \frac{2\Delta_t Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} w_k \int\limits_{\Omega^f} \phi_i\phi_k\partial_z\phi_j \\ & - Re \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} r_k^c \int\limits_{\Omega^f} \phi_i\phi_k\partial_r\phi_j + \frac{4Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} r_k^c(t_{n-1}) \int\limits_{\Omega^f} \phi_i\phi_k\partial_r\phi_j \\ & - \frac{Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} r_k^c(t_{n-2}) \int\limits_{\Omega^f} \phi_i\phi_k\partial_r\phi_j - Re \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} z_k^c \int\limits_{\Omega^f} \phi_i\phi_k\partial_z\phi_j \\ & + \frac{4Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} z_k^c(t_{n-1}) \int\limits_{\Omega^f} \phi_i\phi_k\partial_z\phi_j - \frac{Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} z_k^c(t_{n-2}) \int\limits_{\Omega^f} \phi_i\phi_k\partial_z\phi_j \\ & - \frac{2\Delta_t St}{3} \int\limits_{\Omega^f} \phi_i\hat{g}_z + \sum_{j=1}^{n_v} w_j \frac{2\Delta_t}{3} \int\limits_{\Omega^f} \partial_r\phi_j\partial_r\phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\Omega^f} \partial_z\phi_j\partial_r\phi_i \\ & + \frac{4\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\Omega^f} \partial_z\phi_j\partial_z\phi_i - \frac{2\Delta_t}{3} \sum_{j=1}^{n_p} p_j \int\limits_{\Omega^f} \psi_j\partial_z\phi_i,\end{aligned} \tag{5.45}$$

$$\begin{aligned}\mathcal{M}_i^{z,1} = {} & -\frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{p}_j^g \int\limits_{\partial\Omega^{1,f}} \phi_i\phi_j^1 n_z^1 + \frac{2\Delta_t}{3Ca} \sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1}) \\ & + \frac{2\Delta_t}{3Ca} \sigma^1(r_a, z_a)\phi_i(r_a, z_a) m_z^1(r_a, z_a) + \frac{2\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \int\limits_{\partial\Omega^{1,f}} \phi_j^1 t_z^1 \partial_s\phi_i,\end{aligned} \tag{5.46}$$

$$\begin{aligned}\mathcal{M}_i^{z,2} = {} & \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^{2,f}} \phi_i t_r^2 t_z^2 \phi_j + \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^{2,f}} \phi_i t_z^2 t_z^2 \phi_j \\ & - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{u}_j^s \int\limits_{\partial\Omega^{2,f}} \phi_i\phi_j t_r^2 t_z^2 - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{w}_j^s \int\limits_{\partial\Omega^{2,f}} \phi_i\phi_j t_z^2 t_z^2 \\ & - \frac{\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^2 \int\limits_{\partial\Omega^{2,f}} \phi_i \boldsymbol{t}_z^2 \partial_s\phi_j^2 + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda^2}_j \int\limits_{\partial\Omega^{2,f}} \phi_j\phi_i n_z^2,\end{aligned} \tag{5.47}$$

$$\mathcal{M}_i^{z,3} = \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda^3}_j \int\limits_{\partial\Omega^3} \phi_j n_z^3 \phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma^3}_j \int\limits_{\partial\Omega^3} t_z^3 \phi_j\phi_i. \tag{5.48}$$

and

$$\mathcal{M}_i^{z,4} = \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda^4}_j \int\limits_{\partial\Omega^4} \phi_j n_z^4 \phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma^4}_j \int\limits_{\partial\Omega^4} t_z^4 \phi_j\phi_i. \tag{5.49}$$

Re-arranging terms we have

$$
\begin{aligned}
\mathcal{M}_i^{z,0} = &-\frac{2\Delta_t St}{3}\int_{\Omega^f}\phi_i\hat{g}_z + Re\sum_{j=1}^{n_v}w_j\int_{\Omega^f}\phi_i\phi_j - \frac{4Re}{3}\sum_{j=1}^{n_v}w_j(t_{n-1})\int_{\Omega^f}\phi_i\phi_j\\
&+\frac{Re}{3}\sum_{j=1}^{n_v}w_j(t_{n-2})\int_{\Omega^f}\phi_i\phi_j + \sum_{j=1}^{n_v}w_j\frac{2\Delta_t}{3}\int_{\Omega^f}\partial_r\phi_j\partial_r\phi_i + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}u_j\int_{\Omega^f}\partial_z\phi_j\partial_r\phi_i\\
&+\frac{4\Delta_t}{3}\sum_{j=1}^{n_v}w_j\int_{\Omega^f}\partial_z\phi_j\partial_z\phi_i + \frac{2\Delta_t Re}{3}\sum_{j=1}^{n_v}w_j\sum_{k=1}^{n_v}u_k\int_{\Omega^f}\phi_i\phi_k\partial_r\phi_j\\
&+\frac{2\Delta_t Re}{3}\sum_{j=1}^{n_v}w_j\sum_{k=1}^{n_v}w_k\int_{\Omega^f}\phi_i\phi_k\partial_z\phi_j - Re\sum_{j=1}^{n_v}w_j\sum_{k=1}^{n_v}r_k^c\int_{\Omega^f}\phi_i\phi_k\partial_r\phi_j\\
&+\frac{4Re}{3}\sum_{j=1}^{n_v}w_j\sum_{k=1}^{n_v}r_k^c(t_{n-1})\int_{\Omega^f}\phi_i\phi_k\partial_r\phi_j - \frac{Re}{3}\sum_{j=1}^{n_v}w_j\sum_{k=1}^{n_v}r_k^c(t_{n-2})\int_{\Omega^f}\phi_i\phi_k\partial_r\phi_j\\
&- Re\sum_{j=1}^{n_v}w_j\sum_{k=1}^{n_v}z_k^c\int_{\Omega^f}\phi_i\phi_k\partial_z\phi_j + \frac{4Re}{3}\sum_{j=1}^{n_v}w_j\sum_{k=1}^{n_v}z_k^c(t_{n-1})\int_{\Omega^f}\phi_i\phi_k\partial_z\phi_j\\
&- \frac{Re}{3}\sum_{j=1}^{n_v}w_j\sum_{k=1}^{n_v}z_k^c(t_{n-2})\int_{\Omega^f}\phi_i\phi_k\partial_z\phi_j - \frac{2\Delta_t}{3}\sum_{j=1}^{n_p}p_j\int_{\Omega^f}\psi_j\partial_z\phi_i,
\end{aligned}
\tag{5.50}
$$

$$
\begin{aligned}
\mathcal{M}_i^{z,1} = &\frac{2\Delta_t}{3Ca}\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^1(r_{J^1},z_{J^1}) + \frac{2\Delta_t}{3Ca}\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)\\
&+\frac{2\Delta_t}{3Ca}\sum_{j=1}^{n_v}\tilde{\sigma}_j^1\int_{\partial\Omega^{1,f}}\phi_j^1 t_z^1\partial_s\phi_i - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{p}_j^g\int_{\partial\Omega^{1,f}}\phi_i\phi_j^1 n_z^1,
\end{aligned}
\tag{5.51}
$$

$$
\begin{aligned}
\mathcal{M}_i^{z,2} = &\frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v}u_j\int_{\partial\Omega^{2,f}}\phi_i t_r^2 t_z^2\phi_j + \frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v}w_j\int_{\partial\Omega^{2,f}}\phi_i t_z^2 t_z^2\phi_j\\
&-\frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v}\tilde{u}_j^s\int_{\partial\Omega^{2,f}}\phi_i\phi_j t_r^2 t_z^2 - \frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v}\tilde{w}_j^s\int_{\partial\Omega^{2,f}}\phi_i\phi_j t_z^2 t_z^2\\
&-\frac{\Delta_t}{3Ca}\sum_{j=1}^{n_v}\tilde{\sigma}_j^2\int_{\partial\Omega^{2,f}}\phi_i \boldsymbol{t}_z^2\partial_s\phi_j^2 + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\lambda}^2{}_j\int_{\partial\Omega^{2,f}}\phi_j\phi_i n_z^2,
\end{aligned}
\tag{5.52}
$$

$$
\mathcal{M}_i^{z,3} = \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\lambda}^3{}_j\int_{\partial\Omega^3}\phi_j n_z^3\phi_i + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\gamma}^3{}_j\int_{\partial\Omega^3}t_z^3\phi_j\phi_i. \tag{5.53}
$$

and

$$
\mathcal{M}_i^{z,4} = \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\lambda}^4{}_j\int_{\partial\Omega^4}\phi_j n_z^4\phi_i + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\gamma}^4{}_j\int_{\partial\Omega^4}t_z^4\phi_j\phi_i. \tag{5.54}
$$

We now use the same triangular domain partition partition and proceed to decompose the integrals above in a sum of integrals over each element. The boundary integrals, are in turn converted into a sum of integrals over line elements on the boundary, i.e. those portions of the boundary of the triangular elements that lie on the domain boundary $\partial\Omega$. This yields

$$\mathcal{M}_i^z = \underbrace{\mathcal{M}_i^{z,0a} + \mathcal{M}_i^{z,0b} + \mathcal{M}_i^{z,0c} + \mathcal{M}_i^{z,0d}}_{\mathcal{M}_i^{z,0}} + \mathcal{M}_i^{z,1} + \mathcal{M}_i^{z,2} + \mathcal{M}_i^{z,3} + \mathcal{M}_i^{z,4}, \quad (5.55)$$

where

$$\mathcal{M}_i^{z,0a} = \sum_{e=1}^{n_{\mathrm{el}}} \left[-\frac{2\Delta_t St}{3} \int\limits_{\Omega_e} \phi_i \hat{g}_z \right], \quad (5.56)$$

$$\begin{aligned}\mathcal{M}_i^{z,0b} = \sum_{e=1}^{n_{\mathrm{el}}} \Bigg[&+Re \sum_{j=1}^{n_v} w_j \int\limits_{\Omega_e} \phi_i\phi_j - \frac{4Re}{3} \sum_{j=1}^{n_v} w_j(t_{n-1}) \int\limits_{\Omega_e} \phi_i\phi_j + \frac{Re}{3} \sum_{j=1}^{n_v} w_j(t_{n-2}) \int\limits_{\Omega_e} \phi_i\phi_j \\ &+ \sum_{j=1}^{n_v} w_j \frac{2\Delta_t}{3} \int\limits_{\Omega_e} \partial_r\phi_j \partial_r\phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} u_j \int\limits_{\Omega_e} \partial_z\phi_j \partial_r\phi_i + \frac{4\Delta_t}{3} \sum_{j=1}^{n_v} w_j \int\limits_{\Omega_e} \partial_z\phi_j \partial_z\phi_i \Bigg],\end{aligned} \quad (5.57)$$

$$\begin{aligned}\mathcal{M}_i^{z,0c} = \sum_{e=1}^{n_{\mathrm{el}}} \Bigg[&\frac{2\Delta_t Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} u_k \int\limits_{\Omega_e} \phi_i\phi_k\partial_r\phi_j + \frac{2\Delta_t Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} w_k \int\limits_{\Omega_e} \phi_i\phi_k\partial_z\phi_j \\ &- Re \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} r_k^c \int\limits_{\Omega_e} \phi_i\phi_k\partial_r\phi_j + \frac{4Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} r_k^c(t_{n-1}) \int\limits_{\Omega_e} \phi_i\phi_k\partial_r\phi_j \\ &- \frac{Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} r_k^c(t_{n-2}) \int\limits_{\Omega_e} \phi_i\phi_k\partial_r\phi_j - Re \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} z_k^c \int\limits_{\Omega_e} \phi_i\phi_k\partial_z\phi_j \\ &+ \frac{4Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} z_k^c(t_{n-1}) \int\limits_{\Omega_e} \phi_i\phi_k\partial_z\phi_j - \frac{Re}{3} \sum_{j=1}^{n_v} w_j \sum_{k=1}^{n_v} z_k^c(t_{n-2}) \int\limits_{\Omega_e} \phi_i\phi_k\partial_z\phi_j \Bigg],\end{aligned} \quad (5.58)$$

$$\mathcal{M}_i^{z,0d} = \sum_{e=1}^{n_{\mathrm{el}}} \left[-\frac{2\Delta_t}{3} \sum_{j=1}^{n_p} p_j \int\limits_{\Omega_e} \psi_j \partial_z\phi_i \right], \quad (5.59)$$

and, as before,

$$\begin{aligned}\mathcal{M}_i^{z,1} = &\frac{2\Delta_t}{3Ca} \sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1}) + \frac{2\Delta_t}{3Ca} \sigma^1(r_a, z_a) \phi_i(r_a, z_a) m_z^1(r_a, z_a) \\ &+ \sum_{e_1=1}^{n_{\mathrm{el}}^1} \left[\frac{2\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \int\limits_{\partial\Omega_{e_1}^1} \phi_j^1 t_z^1 \partial_s\phi_i - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{p}_j^g \int\limits_{\partial\Omega_{e_1}^1} \phi_i\phi_j^1 n_z^1 \right],\end{aligned} \quad (5.60)$$

$$\mathcal{M}_i^{z,2} = \sum_{e_2=1}^{n_{\mathrm{el}}^2} \Bigg[\frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} u_j \int_{\partial\Omega^2_{e_2}} \phi_i t_r^2 t_z^2 \phi_j + \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} w_j \int_{\partial\Omega^2_{e_2}} \phi_i t_z^2 t_z^2 \phi_j$$
$$- \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{u}_j^s \int_{\partial\Omega^2_{e_2}} \phi_i \phi_j t_r^2 t_z^2 - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{w}_j^s \int_{\partial\Omega^2_{e_2}} \phi_i \phi_j t_z^2 t_z^2 \qquad (5.61)$$
$$- \frac{\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^2 \int_{\partial\Omega^2_{e_2}} \phi_i \boldsymbol{t}_z^2 \partial_s \phi_j^2 + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda^2}_j \int_{\partial\Omega^2_{e_2}} \phi_j \phi_i n_z^2 \Bigg],$$

$$\mathcal{M}_i^{z,3} = \sum_{e_3=1}^{n_{\mathrm{el}}^3} \left[\frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda^3}_j \int_{\partial\Omega^3_{e_3}} \phi_j n_z^3 \phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma^3}_j \int_{\partial\Omega^3_{e_3}} t_z^3 \phi_j \phi_i \right], \qquad (5.62)$$

and

$$\mathcal{M}_i^{z,4} = \sum_{e_4=1}^{n_{\mathrm{el}}^4} \left[\frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda^4}_j \int_{\partial\Omega^4_{e_4}} \phi_j n_z^4 \phi_i + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma^4}_j \int_{\partial\Omega^4_{e_4}} t_z^4 \phi_j \phi_i \right]; \qquad (5.63)$$

where Ω_e is the curved-sided triangular portion of the domain that defines element e and $\partial\Omega_{e_k}$ is the portion of the boundary $\partial\Omega_k$ that lies on the line element e_k.

We notice that the vast majority of the terms that are added in the sum on each element is identically null. This is, of course, because each one of the basis functions ϕ_i and ψ_i is chosen so that they are only supported on the elements that contain node i. Therefore, a more efficient way to express this sums is to resort to local node numbering. That is to say, when we are calculating the integral on each element, we know that non-zero contributions can only come from a basis function whose index corresponds to one of the node indices of the element at hand and its it therefore better so have the sums over k and j above to only go over the nodes contained in that element. We then give each node another number for each element in which the node is contained. Hence, it is better to re-write the sum above as

$$\mathcal{M}_i^{z,0} = \sum_{e=1}^{n_{\mathrm{el}}} \left[-\frac{2\Delta_t St}{3} \int_{\Omega_e} \phi_i \hat{g}_z \right], \qquad (5.64)$$

$$\mathcal{M}_i^{z,0b} = \sum_{e=1}^{n_{\mathrm{el}}} \Bigg[Re \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \int_{\Omega_e} \phi_i \phi_{l(e,jj)} - \frac{4Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)}(t_{n-1}) \int_{\Omega_e} \phi_i \phi_{l(e,jj)}$$
$$+ \frac{Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)}(t_{n-2}) \int_{\Omega_e} \phi_i \phi_{l(e,jj)} + \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \frac{2\Delta_t}{3} \int_{\Omega_e} \partial_r \phi_{l(e,jj)} \partial_r \phi_i$$
$$+ \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int_{\Omega_e} \partial_z \phi_{l(e,jj)} \partial_r \phi_i + \frac{4\Delta_t}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \int_{\Omega_e} \partial_z \phi_{l(e,jj)} \partial_z \phi_i \Bigg], \qquad (5.65)$$

$$
\begin{aligned}
\mathcal{M}_i^{z,0c} = \sum_{e=1}^{n_{\mathrm{el}}} \Bigg[& \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} u_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} w_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
& - Re \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& + \frac{4Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& - \frac{Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& - Re \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)} \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
& + \frac{4Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
& - \frac{Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) \int\limits_{\Omega_e} \phi_i \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \Bigg],
\end{aligned} \tag{5.66}
$$

$$
\mathcal{M}_i^{z,0d} = \sum_{e=1}^{n_{\mathrm{el}}} \left[-\frac{2\Delta_t}{3} \sum_{jj=1}^{n_p^e} p_{l^p(e,jj)} \int\limits_{\Omega_e} \psi_{l^p(e,jj)} \partial_z \phi_i \right], \tag{5.67}
$$

$$
\begin{aligned}
\mathcal{M}_i^{z,1} = & \frac{2\Delta_t}{3Ca} \sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1}) \\
& + \frac{2\Delta_t}{3Ca} \sigma^1(r_a, z_a) \phi_i(r_a, z_a) m_z^1(r_a, z_a) + \sum_{e_1=1}^{n_{\mathrm{el}}^1} \Bigg[\frac{2\Delta_t}{3Ca} \sum_{jj=1}^{n_v^{e_1}} \sigma^1_{l_1^1(e_1,jj)} \int\limits_{\partial\Omega^1_{e_1}} \phi_j^1 t_z^1 \partial_s \phi_i^1 \\
& - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{e_1}} p^g_{l_1^1(e_1,jj)} \int\limits_{\partial\Omega^1_{e_1}} \phi_i^1 \phi^1_{l_1(e_1,jj)} n_z^1 \Bigg],
\end{aligned} \tag{5.68}
$$

$$
\begin{aligned}
\mathcal{M}_i^{z,2} = \sum_{e_2=1}^{n_{\mathrm{el}}^2} \Bigg[& \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} \int\limits_{\partial\Omega^2_{e_2}} \phi_i t_r^2 t_z^2 \phi_{l_2(e_2,jj)} \\
& + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} \int\limits_{\partial\Omega^2_{e_2}} \phi_i t_z^2 t_z^2 \phi_{l_2(e_2,jj)} \\
& - \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} u^s_{l_2^2(e_2,jj)} \int\limits_{\partial\Omega^2_{e_2}} \phi_i \phi_{l_2(e_2,jj)} t_r^2 t_z^2 \\
& - \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} w^s_{l_2^2(e_2,jj)} \int\limits_{\partial\Omega^2_{e_2}} \phi_i \phi_{l_2(e_2,jj)} t_z^2 t_z^2 \\
& - \frac{\Delta_t}{3Ca} \sum_{jj=1}^{n_v^{e_2}} \sigma^2_{l_2^2(e_2,jj)} \int\limits_{\partial\Omega^2_{e_2}} \phi_i^2 \boldsymbol{t}_z^2 \partial_s \phi^2_{l_2(e_2,jj)} \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2}} \lambda^2_{l_2^2(e_2,jj)} \int\limits_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,jj)} \phi_i n_z^2 \Bigg],
\end{aligned}
\tag{5.69}
$$

$$
\begin{aligned}
&\mathcal{M}_i^{z,3} \\
&= \sum_{e_3=1}^{n_{\mathrm{el}}^3} \left[\frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \lambda^3_{l_3^3(e_3,jj)} \int\limits_{\partial\Omega^3_{e_3}} \phi_{l_3(e_3,jj)} n_z^3 \phi_i + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \gamma^3_{l_3^3(e_3,jj)} \int\limits_{\partial\Omega^3_{e_3}} t_z^3 \phi_{l_3(e_3,jj)} \phi_i \right],
\end{aligned}
\tag{5.70}
$$

and

$$
\begin{aligned}
&\mathcal{M}_i^{z,4} \\
&= \sum_{e_4=1}^{n_{\mathrm{el}}^4} \left[\frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4}} \lambda^4_{l_4^4(e_4,jj)} \int\limits_{\partial\Omega^4_{e_4}} \phi_{l_4(e_4,jj)} n_z^4 \phi_i + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4}} \gamma^4_{l_4^4(e_4,jj)} \int\limits_{\partial\Omega^4_{e_4}} t_z^4 \phi_{l_4(e_4,jj)} \phi_i \right];
\end{aligned}
\tag{5.71}
$$

where double-letter indices are used to reference local node numbers, n_v^e is the number of nodes where velocity is calculated in element e, n_p^e is the number of nodes where pressure is calculated in element e, n_v^{i,e_i} is the number of nodes where velocity is to be calculated on line element e_i of boundary i, $l(e,jj)$ maps the local number jj of a node in element e to its global number, $l^p(e,jj)$ maps the local node number jj of element e to its pressure-node number, and similarly $l_k(e_k,jj)$ maps the local node number jj of line-element e_k in boundary k to its global node number, and $l_k^k(e_k,jj)$ maps the local node number jj of line-element e_k on boundary k to its boundary node number. Moreover, we have introduced functions p_j^g, σ_j^1, σ_j^2, u_j^s, w_j^s, λ_j^2, λ_j^3, γ_j^3, λ_j^4 and γ_j^4 whose index j corresponds to a boundary numbering and are defined as the non-identically-null subset of the functions with the same name except for the tilde $(\tilde{\cdot})$ symbol. The same analogy applies to functions l_k^k. That is to say $\lambda^2_{l_2^2(e_2,jj)} = \tilde{\lambda}^2_{l_2(e_2,jj)}$.

A form of optimisation that is similar to what was done with j can be done in terms of index i. For a given index i, only the integrals on the elements that contain this node can have a non-zero contribution to the i-th residual. Hence, its is more convenient to loop over each element's nodes and find the contribution to $\mathcal{M}_i^z$ for each of the is that are indices of the nodes in the element at hand, and to sum this contribution to each $\mathcal{M}_i^z$. We thus define

$$\mathcal{M}_i^{z,0a} = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}} \left[-\frac{2\Delta_t St}{3} \int_{\Omega_e} \phi_{l(e,ii)} \hat{g}_z \right], \tag{5.72}$$

$$\begin{aligned}
\mathcal{M}_i^{z,0} = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}} \Bigg[& Re \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,jj)} - \frac{4Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)}(t_{n-1}) \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,jj)} \\
& + \frac{Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)}(t_{n-2}) \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,jj)} + \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \frac{2\Delta_t}{3} \int_{\Omega_e} \partial_r\phi_{l(e,jj)}\partial_r\phi_{l(e,ii)} \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^e} u_{l(e,jj)} \int_{\Omega_e} \partial_z\phi_{l(e,jj)}\partial_r\phi_{l(e,ii)} + \frac{4\Delta_t}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \int_{\Omega_e} \partial_z\phi_{l(e,jj)}\partial_z\phi_{l(e,ii)} \Bigg],
\end{aligned} \tag{5.73}$$

$$\begin{aligned}
\mathcal{M}_i^{z,0c} = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}} \Bigg[& \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} u_{l(e,kk)} \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} w_{l(e,kk)} \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
& - Re \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)} \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& + \frac{4Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& - \frac{Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
& - Re \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)} \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
& + \frac{4Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
& - \frac{Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) \int\limits_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \Bigg],
\end{aligned} \tag{5.74}$$

$$\mathcal{M}_i^{z,0d} = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}} \left[-\frac{2\Delta_t}{3} \sum_{jj=1}^{n_p^e} p_{l^p(e,jj)} \int\limits_{\Omega_e} \psi_{l^p(e,jj)} \partial_z \phi_{l(e,ii)} \right], \tag{5.75}$$

$$\begin{aligned}
\mathcal{M}_i^{z,1} = & \frac{2\Delta_t}{3Ca} \sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1}) + \frac{2\Delta_t}{3Ca} \sigma^1(r_a, z_a) \phi_i(r_a, z_a) m_z^1(r_a, z_a) \\
& + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n^1_{\mathrm{el}}} \Bigg[\frac{2\Delta_t}{3Ca} \sum_{jj=1}^{n_v^{1,e_1}} \sigma^1_{l_1^1(e_1,jj)} \int\limits_{\partial\Omega^1_{e_1}} \phi^1_{l_1(e_1,jj)} t_z^1 \partial_s \phi_{l_1(e_1,ii)} \\
& \qquad - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{e_1}} p^g_{l_1^1(e_1,jj)} \int\limits_{\partial\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n_z^1 \Bigg],
\end{aligned} \tag{5.76}$$

$$
\begin{aligned}
\mathcal{M}_i^{z,2} = \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \Bigg[& \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} \int\limits_{\partial\Omega_{e_2}^2} \phi_{l_2(e_2,ii)} t_r^2 t_z^2 \phi_{l_2(e_2,jj)} \\
& + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} \int\limits_{\partial\Omega_{e_2}^2} \phi_{l_2(e_2,ii)} t_z^2 t_z^2 \phi_{l_2(e_2,jj)} \\
& - \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} u^s_{l_2^2(e_2,jj)} \int\limits_{\partial\Omega_{e_2}^2} \phi_{l_2(e_2,ii)} \phi_{l_2(e_2,jj)} t_r^2 t_z^2 \\
& - \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} w^s_{l_2^2(e_2,jj)} \int\limits_{\partial\Omega_{e_2}^2} \phi_{l_2(e_2,ii)} \phi_{l_2(e_2,jj)} t_z^2 t_z^2 \\
& - \frac{\Delta_t}{3Ca} \sum_{jj=1}^{n_v^{e2}} \sigma^2_{l_2^2(e_2,jj)} \int\limits_{\partial\Omega_{e_2}^2} \phi^2_{l_2(e_2,ii)} \boldsymbol{t}_z^2 \partial_s \phi^2_{l_2(e_2,jj)} \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2}} \lambda^2_{l_2^2(e_2,jj)} \int\limits_{\partial\Omega_{e_2}^2} \phi_{l_2(e_2,jj)} \phi_{l_2(e_2,ii)} n_z^2 \Bigg],
\end{aligned} \tag{5.77}
$$

$$
\begin{aligned}
\mathcal{M}_i^{z,3} = \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{n_{\mathrm{el}}^3} \Bigg[& \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \lambda^3_{l_3^3(e_3,jj)} \int\limits_{\partial\Omega_{e_3}^3} \phi_{l_3(e_3,jj)} n_z^3 \phi_{l_3(e_3,ii)} \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \gamma^3_{l_3^3(e_3,jj)} \int\limits_{\partial\Omega_{e_3}^3} t_z^3 \phi_{l_3(e_3,jj)} \phi_{l_3(e_3,ii)} \Bigg].
\end{aligned} \tag{5.78}
$$

$$
\begin{aligned}
\mathcal{M}_i^{z,4} = \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{n_{\mathrm{el}}^4} \Bigg[& \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4}} \lambda^4_{l_4^4(e_4,jj)} \int\limits_{\partial\Omega_{e_4}^4} \phi_{l_4(e_4,jj)} n_z^4 \phi_{l_4(e_4,ii)} \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{4,e_4}} \gamma^4_{l_4^4(e_4,jj)} \int\limits_{\partial\Omega_{e_4}^4} t_z^4 \phi_{l_4(e_4,jj)} \phi_{l_4(e_4,ii)} \Bigg].
\end{aligned} \tag{5.79}
$$

We can now re-write the z-momentum residual as

$$
\begin{aligned}
\mathcal{M}_i^z \;=\; & \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \mathcal{M}_{e,ii}^{z,0}}_{\mathcal{M}_i^{z,0}} + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar{n}_{\text{el}}^1} \mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \underbrace{\sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\text{el}}^2} \mathcal{M}_{e,ii}^{z,2}}_{\mathcal{M}_i^{z,2}} \\
& + \underbrace{\sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\text{el}}^3} \mathcal{M}_{e_3,ii}^{z,3}}_{\mathcal{M}_i^{z,3}} + \underbrace{\sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\text{el}}^4} \mathcal{M}_{e_4,ii}^{z,4}}_{\mathcal{M}_i^{z,4}};
\end{aligned}
\tag{5.80}
$$

where

$$
\begin{aligned}
\mathcal{M}_i^{z,1} = & \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca},
\end{aligned}
\tag{5.81}
$$

and

$$
\mathcal{M}_{e,ii}^{z,0a} = -\frac{2\Delta_t St}{3}\underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\hat{g}_z}_{a_{ii,g_z}(e)}, \tag{5.82}
$$

$$
\begin{aligned}
\mathcal{M}_{e,ii}^{z,0b} = & Re\sum_{jj=1}^{n_v^e} w_{l(e,jj)}\underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)} - \frac{4Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)}(t_{n-1})\underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)} \\
& + \frac{Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)}(t_{n-2})\underbrace{\int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)} + \sum_{jj=1}^{n_v^e} w_{l(e,jj)}\frac{2\Delta_t}{3}\underbrace{\int_{\Omega_e} \partial_r\phi_{l(e,jj)}\partial_r\phi_{l(e,ii)}}_{a_{ii,jj}^{r,r}(e)} \\
& + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\underbrace{\int_{\Omega_e} \partial_r\phi_{l(e,ii)}\partial_z\phi_{l(e,jj)}}_{a_{ii,jj}^{r,z}(e)} + \frac{4\Delta_t}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)}\underbrace{\int_{\Omega_e} \partial_z\phi_{l(e,jj)}\partial_z\phi_{l(e,ii)}}_{a_{ii,jj}^{z,z}(e)},
\end{aligned}
\tag{5.83}
$$

$$
\begin{aligned}
\mathcal{M}^{z,0c}_{e,ii} = &\frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} u_{l(e,kk)} \underbrace{\int_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
&+ \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} w_{l(e,kk)} \underbrace{\int_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
&- Re \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)} \underbrace{\int_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
&+ \frac{4Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) \underbrace{\int_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
&- \frac{Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) \underbrace{\int_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
&- Re \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)} \underbrace{\int_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
&+ \frac{4Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) \underbrace{\int_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
&- \frac{Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) \underbrace{\int_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)},
\end{aligned}
\tag{5.84}
$$

$$
\mathcal{M}^{z,0d}_{e,ii} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{n_p^e} p_{l^p(e,jj)} \underbrace{\int_{\Omega_e} \psi_{l^p(e,jj)} \partial_z \phi_{l(e,ii)}}_{b^z_{jj,ii}(e)}, \tag{5.85}
$$

$$
\begin{aligned}
\mathcal{M}^{z,1}_{e_1,ii} = \frac{2\Delta_t}{3Ca} \sum_{jj=1}^{n_v^{1,e_1}} \sigma^1_{l_1^1(e_1,jj)} & \underbrace{\int_{\partial\Omega^1_{e_1}} \phi^1_{l_1(e_1,jj)} t^1_z \partial_s \phi_{l_1(e_1,ii)}}_{c^s jj,ii,t_z} \\
- \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{e_1}} p^g_{l_1^1(e_1,jj)} & \underbrace{\int_{\partial\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_z}_{c_{ii,jj,n_z}(e_1)},
\end{aligned}
\tag{5.86}
$$

$$
\begin{aligned}
\mathcal{M}^{z,2}_{e_2,ii} = \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} & \underbrace{\int_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,ii)} t^2_r t^2_z \phi_{l_2(e_2,jj)}}_{d_{ii,jj,t_r,t_z}(e_2)} \\
+ \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} & \underbrace{\int_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,ii)} t^2_z t^2_z \phi_{l_2(e_2,jj)}}_{d_{ii,jj,t_z,t_z}(e_2)} \\
- \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} u^s_{l_2^2(e_2,jj)} & \underbrace{\int_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,ii)} \phi_{l_2(e_2,jj)} t^2_r t^2_z}_{d_{ii,jj,t_r,t_z}(e_2)} \\
- \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} w^s_{l_2^2(e_2,jj)} & \underbrace{\int_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,ii)} \phi_{l_2(e_2,jj)} t^2_z t^2_z}_{d_{ii,jj,t_z,t_z}(e_2)} \\
- \frac{\Delta_t}{3Ca} \sum_{jj=1}^{n_v^{e_2}} \sigma^2_{l_2^2(e_2,jj)} & \underbrace{\int_{\partial\Omega^2_{e_2}} \phi^2_{l_2(e_2,ii)} t^2_z \partial_s \phi^2_{l_2(e_2,jj)}}_{d^s_{ii,jj,t_z}(e_2)} \\
+ \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2}} \lambda^2_{l_2^2(e_2,jj)} & \underbrace{\int_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,jj)} \phi_{l_2(e_2,ii)} n^2_z}_{d_{ii,jj,n_z}(e_2)},
\end{aligned}
\tag{5.87}
$$

$$\mathcal{M}^{z,3}_{e_3,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \lambda^3_{l^3_3(e_3,jj)} \underbrace{\int\limits_{\partial\Omega^3_{e_3}} \phi_{l_3(e_3,jj)} n^3_z \phi_{l_3(e_3,ii)}}_{f_{ii,jj,n_z}(e_3)} \tag{5.88}$$

$$+\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \gamma^3_{l^3_3(e_3,jj)} \underbrace{\int\limits_{\partial\Omega^3_{e_3}} t^3_z \phi_{l_3(e_3,jj)} \phi_{l_3(e_3,ii)}}_{f_{ii,jj,t_z}(e_3)} .$$

$$\mathcal{M}^{z,4}_{e_4,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \lambda^4_{l^4_4(e_4,jj)} \underbrace{\int\limits_{\partial\Omega^4_{e_4}} \phi_{l_4(e_4,jj)} n^4_z \phi_{l_4(e_4,ii)}}_{e_{ii,jj,n_z}(e_4)} \tag{5.89}$$

$$+\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \gamma^4_{l^4_4(e_4,jj)} \underbrace{\int\limits_{\partial\Omega^4_{e_4}} t^4_z \phi_{l_4(e_4,jj)} \phi_{l_4(e_4,ii)}}_{e_{ii,jj,t_z}(e_4)} .$$

Using the notation introduced above and re-arranging terms we have

$$\mathcal{M}^{z,0a}_{e,ii} = -\frac{2\Delta_t St}{3} a_{ii,g_z}(e), \tag{5.90}$$

$$\mathcal{M}^{z,0b}_{e,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)} a^{r,z}_{ii,jj}(e) + \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \frac{2\Delta_t}{3} a^{r,r}_{ii,jj}(e)$$

$$+\frac{4\Delta_t}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)} a^{z,z}_{ii,jj}(e) + Re\sum_{jj=1}^{n_v^e} w_{l(e,jj)} a_{ii,jj}(e) \tag{5.91}$$

$$-\frac{4Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)}(t_{n-1}) a_{ii,jj}(e) + \frac{Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)}(t_{n-2}) a_{ii,jj}(e),$$

$$
\begin{aligned}
\mathcal{M}^{z,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} u_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} w_{l(e,kk)} a^z_{ii,kk,jj}(e) \\
& - Re \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
& + \frac{4Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) a^r_{ii,kk,jj}(e) \\
& - \frac{Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) a^r_{ii,kk,jj}(e) \\
& - Re \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)} a^z_{ii,kk,jj}(e) \\
& + \frac{4Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) a^z_{ii,kk,jj}(e) \\
& - \frac{Re}{3} \sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{5.92}
$$

$$
\mathcal{M}^{z,0d}_{e,ii} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{n_p^e} p_{l^p(e,jj)} b^z_{jj,ii}(e), \tag{5.93}
$$

$$
\mathcal{M}^{z,1}_{e_1,ii} = \frac{2\Delta_t}{3Ca} \sum_{jj=1}^{n_v^{1,e_1}} \sigma^1_{l_1^1(e_1,jj)} c^s_{jj,ii,t_z} - \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{e_1}} p^g_{l_1^1(e_1,jj)} c_{ii,jj,n_z}(e_1), \tag{5.94}
$$

$$
\begin{aligned}
\mathcal{M}^{z,2}_{e_2,ii} = {} & \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2) + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} d_{ii,jj,t_z,t_z}(e_2) \\
& - \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} u^s_{l_2^2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2) - \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v^{2,e_2}} w^s_{l_2^2(e_2,jj)} d_{ii,jj,t_z,t_z}(e_2) \\
& - \frac{\Delta_t}{3Ca} \sum_{jj=1}^{n_v^{e_2}} \sigma^2_{l_2^2(e_2,jj)} d^s_{ii,jj,t_z}(e_2) + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2}} \lambda^2_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e_2),
\end{aligned}
\tag{5.95}
$$

$$
\mathcal{M}^{z,3}_{e_3,ii} = \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \lambda^3_{l_3^3(e_3,jj)} f_{ii,jj,n_z}(e_3) + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{3,e_3}} \gamma^3_{l_3^3(e_3,jj)} f_{ii,jj,t_z}(e_3), \tag{5.96}
$$

and

$$\mathcal{M}^{z,4}_{e_4,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \lambda^4_{l_4^4(e_4,jj)} e_{ii,jj,n_z}(e_4) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \gamma^4_{l_4^4(e_4,jj)} e_{ii,jj,t_z}(e_4). \quad (5.97)$$

Finally, we summarise the z-momentum residual as

$$\begin{aligned}
\mathcal{M}^z_i \; = \; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \mathcal{M}^{z,0a}_{e,ii} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \mathcal{M}^{z,0b}_{e,ii} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \mathcal{M}^{z,0c}_{e,ii} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \mathcal{M}^{z,0d}_{e,ii} + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar{n}^1_{\rm el}} \mathcal{M}^{z,1}_{e_1,ii} \\
& + \frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1}) m_z^{1,n}(r_{J^1}, z_{J^1})}{Ca} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_a, z_a)\phi_i(r_a, z_a) m_z^1(r_a, z_a)}{Ca} \\
& + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}^2_{\rm el}} \mathcal{M}^{z,2}_{e,ii} + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}^3_{\rm el}} \mathcal{M}^{z,3}_{e_3,ii} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}^4_{\rm el}} \mathcal{M}^{z,4}_{e_4,ii};
\end{aligned} \quad (5.98)$$

where

$$\mathcal{M}^{z,0a}_{e,ii} = -\frac{2\Delta_t St}{3} a_{ii,g_z}(e), \quad (5.99)$$

$$\begin{aligned}
\mathcal{M}^{z,0b}_{e,ii} = & \sum_{jj=1}^{n_v^e} \frac{2\Delta_t}{3}\left[u_{l(e,jj)} a^{r,z}_{ii,jj}(e) + w_{l(e,jj)}\left(a^{r,r}_{ii,jj}(e) + 2a^{z,z}_{ii,jj}(e)\right)\right] \\
& + \sum_{jj=1}^{n_v^e} Re\, a_{ii,jj}(e)\left[w_{l(e,jj)} - \frac{4}{3}w_{l(e,jj)}(t_{n-1}) + \frac{1}{3}w_{l(e,jj)}(t_{n-2})\right],
\end{aligned} \quad (5.100)$$

$$\begin{aligned}
\mathcal{M}^{z,0c}_{e,ii} = & \sum_{jj=1}^{n_v^e} Re\, w_{l(e,jj)} \left[\frac{2\Delta_t}{3} \underbrace{\sum_{kk=1}^{n_v^e}\left[u_{l(e,kk)} a^r_{ii,kk,jj}(e) + w_{l(e,kk)} a^z_{ii,kk,jj}(e)\right]}_{A_{ii,jj}(e)}\right. \\
& - \underbrace{\sum_{kk=1}^{n_v^e} a^r_{ii,kk,jj}(e)\left[r^c_{l(e,kk)} - \frac{4}{3}r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3}r^c_{l(e,kk)}(t_{n-2})\right]}_{B_{ii,jj}(e)} \\
& \left. - \underbrace{\sum_{kk=1}^{n_v^e} a^z_{ii,kk,jj}(e)\left[z^c_{l(e,kk)} - \frac{4}{3}z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3}z^c_{l(e,kk)}(t_{n-2})\right]}_{C_{ii,jj}(e)}\right],
\end{aligned} \quad (5.101)$$

$$\mathcal{M}^{z,0d}_{e,ii} = \sum_{jj=1}^{n_p^e} -\frac{2\Delta_t}{3} p_{l^p(e,jj)} b^z_{jj,ii}(e), \quad (5.102)$$

$$\mathcal{M}^{z,1}_{e_1,ii} = \sum_{jj=1}^{n_v^{1,e_1}} \left[\frac{2\Delta_t}{3}\left(\frac{1}{Ca}\sigma^1_{l_1^1(e_1,jj)} c^s jj, ii, t_z - p^g_{l_1^1(e_1,jj)} c_{ii,jj,n_z}(e_1)\right)\right], \tag{5.103}$$

$$\begin{aligned}\mathcal{M}^{z,2}_{e_2,ii} = \sum_{jj=1}^{n_v^{2,e_2}} \frac{2\Delta_t}{3}\Bigg[Be\,\Big(d_{ii,jj,t_r,t_z}(e_2)\left\{u_{l_2(e_2,jj)} - u^s_{l_2^2(e_2,jj)}\right\} \\ + d_{ii,jj,t_z,t_z}(e_2)\left\{w_{l_2(e_2,jj)} - w^s_{l_2^2(e_2,jj)}\right\}\Big) \\ - \frac{1}{2Ca}\sigma^2_{l_2^2(e_2,jj)} d^s_{ii,jj,t_z}(e_2) + \lambda^2_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e_2)\Bigg],\end{aligned} \tag{5.104}$$

$$\mathcal{M}^{z,3}_{e_3,ii} = \sum_{jj=1}^{n_v^{3,e_3}} \frac{2\Delta_t}{3}\left[\lambda^3_{l_3^3(e_3,jj)} f_{ii,jj,n_z}(e_3) + \gamma^3_{l_3^3(e_3,jj)} f_{ii,jj,t_z}(e_3)\right], \tag{5.105}$$

and

$$\mathcal{M}^{z,4}_{e_4,ii} = \sum_{jj=1}^{n_v^{4,e_4}} \frac{2\Delta_t}{3}\left[\lambda^4_{l_4^4(e_4,jj)} e_{ii,jj,n_z}(e_4) + \gamma^4_{l_4^4(e_4,jj)} e_{ii,jj,t_z}(e_4)\right]. \tag{5.106}$$

5.1. Jacobian terms

We now calculate the derivatives of $\mathcal{M}_i^z$ with respect to u_q, w_q, p_q, σ_q^1, σ_q^2, λ_q^2, λ_q^3, γ_q^3, λ_q^4, γ_q^4 and h_q.

5.1.1. Derivatives of $\mathcal{M}_i^z$ with respect to u_q

From equation (5.98) we have

$$\begin{aligned}
\partial_{u_q}\mathcal{M}_i^z \ = & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{u_q}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{u_q}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{u_q}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{u_q}\mathcal{M}_{e,ii}^{z,0d} \\
& + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{u_q}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\partial_{u_q}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{u_q}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{u_q}\mathcal{M}_{e,ii}^{z,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\mathrm{el}}^3} \partial_{u_q}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{u_q}\mathcal{M}_{e_4,ii}^{z,4};
\end{aligned} \tag{5.107}$$

which yields

$$\partial_{u_q}\mathcal{M}_i^z = \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{u_q}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{u_q}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{u_q}\mathcal{M}_{e,ii}^{z,2}. \tag{5.108}$$

Now, from equation (5.91) we have

$$\begin{aligned}
\partial_{u_q}\mathcal{M}_{e,ii}^{z,0b} = & \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}^{r,z}(e)\partial_{u_q}u_{l(e,jj)} + \sum_{jj=1}^{n_v^e}\frac{2\Delta_t}{3}a_{ii,jj}^{r,r}(e)\partial_{u_q}w_{l(e,jj)} \\
& + \frac{4\Delta_t}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}^{z,z}(e)\partial_{u_q}w_{l(e,jj)} + Re\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{u_q}w_{l(e,jj)} \\
& - \frac{4Re}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{u_q}w_{l(e,jj)}(t_{n-1}) + \frac{Re}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{u_q}w_{l(e,jj)}(t_{n-2}),
\end{aligned} \tag{5.109}$$

i.e.

$$\partial_{u_q}\mathcal{M}_{e,ii}^{z,0b} = \frac{2\Delta_t}{3}a_{ii,jj}^{r,z}(e)|_{q=l(e,jj)}. \tag{5.110}$$

From equation (5.92) we have

$$
\begin{aligned}
\partial_{u_q}\mathcal{M}^{z,0c}_{e,ii} = {}& \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} \partial_{u_q} u_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
&+ \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} \partial_{u_q} w_{l(e,kk)} a^z_{ii,kk,jj}(e) \\
&- Re\sum_{jj=1}^{n_v^e} \partial_{u_q} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
&+ \frac{4Re}{3}\sum_{jj=1}^{n_v^e} \partial_{u_q} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1}) a^r_{ii,kk,jj}(e) \\
&- \frac{Re}{3}\sum_{jj=1}^{n_v^e} \partial_{u_q} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2}) a^r_{ii,kk,jj}(e) \\
&- Re\sum_{jj=1}^{n_v^e} \partial_{u_q} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)} a^z_{ii,kk,jj}(e) \\
&+ \frac{4Re}{3}\sum_{jj=1}^{n_v^e} \partial_{u_q} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1}) a^z_{ii,kk,jj}(e) \\
&- \frac{Re}{3}\sum_{jj=1}^{n_v^e} \partial_{u_q} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2}) a^z_{ii,kk,jj}(e),
\end{aligned} \tag{5.111}
$$

i.e.

$$
\partial_{u_q}\mathcal{M}^{z,0c}_{e,ii} = \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)} a^r_{ii,kk,jj}(e)|_{q=l(e,kk)}. \tag{5.112}
$$

Finally from equation (5.95) we have

$$
\begin{aligned}
\partial_{u_q}\mathcal{M}^{z,2}_{e_2,ii} = {}& \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{u_q} u_{l_2(e_2,jj)} \\
&+ \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_z,t_z}(e_2)\partial_{u_q} w_{l_2(e_2,jj)} \\
&- \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{u_q} u^s_{l_2^2(e_2,jj)} \\
&- \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_z,t_z}(e_2)\partial_{u_q} w^s_{l_2^2(e_2,jj)} \\
&- \frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{e_2}} d^s_{ii,jj,t_z}(e_2)\partial_{u_q}\sigma^2_{l_2^2(e_2,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,n_z}(e_2)\partial_{u_q}\lambda^2_{l_2^2(e_2,jj)},
\end{aligned} \tag{5.113}
$$

i.e.

$$\partial_{u_q}\mathcal{M}^{z,2}_{e_2,ii} = \frac{2\Delta_t Be}{3} d_{ii,jj,t_r,t_z}(e_2)|_{q=l_2(e_2,jj)}. \tag{5.114}$$

5.1.2. Derivatives of $\mathcal{M}_i^z$ with respect to w_q

From equation (5.98) we have

$$\begin{aligned}\partial_{w_q}\mathcal{M}_i^z \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar n_{\rm el}} \partial_{w_q}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar n_{\rm el}} \partial_{w_q}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar n_{\rm el}} \partial_{w_q}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar n_{\rm el}} \partial_{w_q}\mathcal{M}_{e,ii}^{z,0d}\\ & + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar n_{\rm el}^1} \partial_{w_q}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\partial_{w_q}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca}\\ & + \frac{2\Delta_t}{3}\partial_{w_q}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar n_{\rm el}^2} \partial_{w_q}\mathcal{M}_{e,ii}^{z,2}\\ & + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar n_{\rm el}^3} \partial_{w_q}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar n_{\rm el}^4} \partial_{w_q}\mathcal{M}_{e_4,ii}^{z,4};\end{aligned} \tag{5.115}$$

which yields

$$\partial_{w_q}\mathcal{M}_i^z = \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar n_{\rm el}} \partial_{w_q}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar n_{\rm el}} \partial_{w_q}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar n_{\rm el}^2} \partial_{w_q}\mathcal{M}_{e,ii}^{z,2}. \tag{5.116}$$

Now, from equation (5.91) we have

$$\begin{aligned}\partial_{w_q}\mathcal{M}_{e,ii}^{z,0b} = & \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}^{r,z}(e)\partial_{w_q}u_{l(e,jj)} + \sum_{jj=1}^{n_v^e}\frac{2\Delta_t}{3}a_{ii,jj}^{r,r}(e)\partial_{w_q}w_{l(e,jj)}\\ & + \frac{4\Delta_t}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}^{z,z}(e)\partial_{w_q}w_{l(e,jj)} + Re\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{w_q}w_{l(e,jj)}\\ & - \frac{4Re}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{w_q}w_{l(e,jj)}(t_{n-1}) + \frac{Re}{3}\sum_{jj=1}^{n_v^e} a_{ii,jj}(e)\partial_{w_q}w_{l(e,jj)}(t_{n-2}),\end{aligned} \tag{5.117}$$

i.e.

$$\partial_{w_q}\mathcal{M}_{e,ii}^{z,0b} = \frac{2\Delta_t}{3}a_{ii,jj}^{r,r}(e)|_{q=l(e,jj)} + \frac{4\Delta_t}{3}a_{ii,jj}^{z,z}(e)|_{q=l(e,jj)} + Re\, a_{ii,jj}(e)|_{q=l(e,jj)}, \tag{5.118}$$

or, equivalently,

$$\partial_{w_q}\mathcal{M}_{e,ii}^{z,0b} = \frac{2\Delta_t}{3}\left[Re\, a_{ii,jj}(e) + a_{ii,jj}^{r,r}(e) + 2a_{ii,jj}^{z,z}(e)\right]_{q=l(e,jj)}, \tag{5.119}$$

From equation (5.92) we have

$$
\begin{aligned}
\partial_{w_q}\mathcal{M}^{z,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e}\partial_{w_q}w_{l(e,jj)}\sum_{kk=1}^{n_v^e}u_{l(e,kk)}a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e}\partial_{w_q}w_{l(e,jj)}\sum_{kk=1}^{n_v^e}w_{l(e,kk)}a^z_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e}w_{l(e,jj)}\sum_{kk=1}^{n_v^e}\partial_{w_q}w_{l(e,kk)}a^z_{ii,kk,jj}(e) \\
& - Re\sum_{jj=1}^{n_v^e}\partial_{w_q}w_{l(e,jj)}\sum_{kk=1}^{n_v^e}r^c_{l(e,kk)}a^r_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{jj=1}^{n_v^e}\partial_{w_q}w_{l(e,jj)}\sum_{kk=1}^{n_v^e}r^c_{l(e,kk)}(t_{n-1})a^r_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{jj=1}^{n_v^e}\partial_{w_q}w_{l(e,jj)}\sum_{kk=1}^{n_v^e}r^c_{l(e,kk)}(t_{n-2})a^r_{ii,kk,jj}(e) \\
& - Re\sum_{jj=1}^{n_v^e}\partial_{w_q}w_{l(e,jj)}\sum_{kk=1}^{n_v^e}z^c_{l(e,kk)}a^z_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{jj=1}^{n_v^e}\partial_{w_q}w_{l(e,jj)}\sum_{kk=1}^{n_v^e}z^c_{l(e,kk)}(t_{n-1})a^z_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{jj=1}^{n_v^e}\partial_{w_q}w_{l(e,jj)}\sum_{kk=1}^{n_v^e}z^c_{l(e,kk)}(t_{n-2})a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{5.120}
$$

i.e.

$$
\begin{aligned}
\partial_{w_q}\mathcal{M}^{z,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3}\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} u_{l(e,kk)}a^{r}_{ii,kk,jj}(e) + \frac{2\Delta_t Re}{3}\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} w_{l(e,kk)}a^{z}_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3}\sum_{\substack{jj=1\\ q=l(e,kk)}}^{n_v^e} w_{l(e,jj)}a^{z}_{ii,kk,jj}(e) - Re\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} r^{c}_{l(e,kk)}a^{r}_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} r^{c}_{l(e,kk)}(t_{n-1})a^{r}_{ii,kk,jj}(e) - \frac{Re}{3}\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} r^{c}_{l(e,kk)}(t_{n-2})a^{r}_{ii,kk,jj}(e) \\
& - Re\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} z^{c}_{l(e,kk)}a^{z}_{ii,kk,jj}(e) + \frac{4Re}{3}\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} z^{c}_{l(e,kk)}(t_{n-1})a^{z}_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} z^{c}_{l(e,kk)}(t_{n-2})a^{z}_{ii,kk,jj}(e),
\end{aligned}
\tag{5.121}
$$

or, equivalently,

$$
\begin{aligned}
\partial_{w_q}\mathcal{M}^{z,0c}_{e,ii} = {} & \sum_{\substack{jj=1\\ q=l(e,kk)}}^{n_v^e} \frac{2\Delta_t Re}{3} w_{l(e,jj)}a^{z}_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3}\underbrace{\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e}\left[u_{l(e,kk)}a^{r}_{ii,kk,jj}(e) + w_{l(e,kk)}a^{z}_{ii,kk,jj}(e)\right]}_{A_{ii,jj}(e)} \\
& -Re\underbrace{\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} a^{r}_{ii,kk,jj}(e)\left[r^{c}_{l(e,kk)} - \frac{4}{3}r^{c}_{l(e,kk)}(t_{n-1}) + \frac{1}{3}r^{c}_{l(e,kk)}(t_{n-2})\right]}_{B_{ii,jj}(e)} \\
& -Re\underbrace{\sum_{\substack{kk=1\\ q=l(e,jj)}}^{n_v^e} a^{z}_{ii,kk,jj}(e)\left[z^{c}_{l(e,kk)} - \frac{4}{3}z^{c}_{l(e,kk)}(t_{n-1}) - \frac{1}{3}z^{c}_{l(e,kk)}(t_{n-2})\right]}_{C_{ii,jj}(e)}.
\end{aligned}
\tag{5.122}
$$

Finally from equation (5.95) we have

$$\begin{aligned}
\partial_{w_q}\mathcal{M}^{z,2}_{e_2,ii} = & \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{w_q}u_{l_2(e_2,jj)} \\
& + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_z,t_z}(e_2)\partial_{w_q}w_{l_2(e_2,jj)} \\
& - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{w_q}u^s_{l_2^2(e_2,jj)} \\
& - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_z,t_z}(e_2)\partial_{w_q}w^s_{l_2^2(e_2,jj)} \\
& - \frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{e_2}} d^s_{ii,jj,t_z}(e_2)\partial_{w_q}\sigma^2_{l_2^2(e_2,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,n_z}(e_2)\partial_{w_q}\lambda^2_{l_2^2(e_2,jj)},
\end{aligned} \tag{5.123}$$

i.e.

$$\partial_{w_q}\mathcal{M}^{z,2}_{e_2,ii} = \frac{2\Delta_t Be}{3} d_{ii,jj,t_z,t_z}(e_2)|_{q=l(e,jj)}. \tag{5.124}$$

5.1.3. Derivatives of $\mathcal{M}_i^z$ with respect to p_q

From equation (5.98) we have

$$\begin{aligned}
\partial_{p_q}\mathcal{M}_i^z \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\mathcal{M}_{e,ii}^{z,0d} \\
& + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{p_q}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\partial_{p_q}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{p_q}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{p_q}\mathcal{M}_{e,ii}^{z,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\mathrm{el}}^3} \partial_{p_q}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{p_q}\mathcal{M}_{e_4,ii}^{z,4};
\end{aligned} \tag{5.125}$$

which yields

$$\partial_{p_q}\mathcal{M}_i^z = \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\mathcal{M}_{e,ii}^{z,0d}; \tag{5.126}$$

Now, from equation (5.93), we have

$$\partial_{p_q}\mathcal{M}_{e,ii}^{z,0d} = -\frac{2\Delta_t}{3}\sum_{jj=1}^{n_p^e} b_{jj,ii}^z(e)\partial_{p_q}p_{l^p(e,jj)}, \tag{5.127}$$

i.e.

$$\partial_{p_q}\mathcal{M}_{e,ii}^{z,0d} = -\frac{2\Delta_t}{3} b_{jj,ii}^z(e)|_{q=l(e,jj)}. \tag{5.128}$$

5.1.4. Derivatives of $\mathcal{M}_i^z$ with respect to σ_q^1

From equation (5.98) we have

$$\begin{aligned}
\partial_{\sigma_q^1}\mathcal{M}_i^z \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\sigma_q^1}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\sigma_q^1}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\sigma_q^1}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e_1,ii)}}^{\bar{n}_{\text{el}}} \partial_{\sigma_q^1}\mathcal{M}_{e,ii}^{z,0d} \\
& + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{\sigma_q^1}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\partial_{\sigma_q^1}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{\sigma_q^1}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{\sigma_q^1}\mathcal{M}_{e,ii}^{z,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\text{el}}^3} \partial_{\sigma_q^1}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\text{el}}^4} \partial_{\sigma_q^1}\mathcal{M}_{e_4,ii}^{z,4},
\end{aligned} \tag{5.129}$$

which yields

$$\begin{aligned}
\partial_{\sigma_q^1}\mathcal{M}_i^z = & \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{\sigma_q^1}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\frac{\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca}\partial_{\sigma_q^1}\sigma^1(r_{J^1},z_{J^1}) \\
& + \frac{2\Delta_t}{3}\frac{\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca}\partial_{\sigma_q^1}\sigma^1(r_a,z_a),
\end{aligned} \tag{5.130}$$

i.e.

$$\begin{aligned}
\partial_{\sigma_q^1}\mathcal{M}_i^z = & \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{\sigma_q^1}\mathcal{M}_{e_1,ii}^{z,1} \\
& + \frac{2\Delta_t}{3}\frac{\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca}|_{q=J^1} + \frac{2\Delta_t}{3}\frac{\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca}|_{q=a}.
\end{aligned} \tag{5.131}$$

Now from equation (5.94) we have

$$\partial_{\sigma_q^1}\mathcal{M}_{e_1,ii}^{z,1} = \frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{1,e_1}} c_{jj,ii,t_z}^s\partial_{\sigma_q^1}\sigma_{l_1^1(e_1,jj)}^1 - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{e_1}} c_{ii,jj,n_z}(e_1)\partial_{\sigma_q^1}p_{l_1^1(e_1,jj)}^g, \tag{5.132}$$

i.e.

$$\partial_{\sigma_q^1}\mathcal{M}_{e_1,ii}^{z,1} = \frac{2\Delta_t}{3Ca}c_{jj,ii,t_z}^s|_{q=l_1^1(e_1,jj)}. \tag{5.133}$$

5.1.5. Derivatives of $\mathcal{M}_i^z$ with respect to σ_q^2

From equation (5.98) we have

$$
\begin{aligned}
\partial_{\sigma_q^2}\mathcal{M}_i^z \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\sigma_q^2}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\sigma_q^2}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\sigma_q^2}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\sigma_q^2}\mathcal{M}_{e,ii}^{z,0d} \\
& + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\sigma_q^2}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\partial_{\sigma_q^2}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{\sigma_q^2}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\sigma_q^2}\mathcal{M}_{e,ii}^{z,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\mathrm{el}}^3} \partial_{\sigma_q^2}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{\sigma_q^2}\mathcal{M}_{e_4,ii}^{z,4},
\end{aligned}
\tag{5.134}
$$

which yields

$$
\partial_{\sigma_q^2}\mathcal{M}_i^z = \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\sigma_q^2}\mathcal{M}_{e,ii}^{z,2}. \tag{5.135}
$$

From equation (5.95) we have

$$
\begin{aligned}
\partial_{\sigma_q^2}\mathcal{M}_{e_2,ii}^{z,2} = & \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{\sigma_q^2}u_{l_2(e_2,jj)} \\
& + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_z,t_z}(e_2)\partial_{\sigma_q^2}w_{l_2(e_2,jj)} \\
& - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{\sigma_q^2}u^s_{l_2^2(e_2,jj)} \\
& - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_z,t_z}(e_2)\partial_{\sigma_q^2}w^s_{l_2^2(e_2,jj)} \\
& - \frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{e_2}} d^s_{ii,jj,t_z}(e_2)\partial_{\sigma_q^2}\sigma^2_{l_2^2(e_2,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,n_z}(e_2)\partial_{\sigma_q^2}\lambda^2_{l_2^2(e_2,jj)},
\end{aligned}
\tag{5.136}
$$

i.e.

$$
\partial_{\sigma_q^2}\mathcal{M}_{e_2,ii}^{z,2} = -\frac{\Delta_t}{3Ca} d^s_{ii,jj,t_z}(e_2)|_{q=l_2^2 e_2,jj}. \tag{5.137}
$$

5.1.6. Derivatives of $\mathcal{M}_i^z$ with respect to λ_q^2

From equation (5.98) we have

$$\begin{aligned}\partial_{\lambda_q^2}\mathcal{M}_i^z \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\lambda_q^2}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\lambda_q^2}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\lambda_q^2}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\lambda_q^2}\mathcal{M}_{e,ii}^{z,0d} \\ & + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar{n}_{\rm el}^1} \partial_{\lambda_q^2}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\partial_{\lambda_q^2}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca} \\ & + \frac{2\Delta_t}{3}\partial_{\lambda_q^2}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\rm el}^2} \partial_{\lambda_q^2}\mathcal{M}_{e,ii}^{z,2} \\ & + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\rm el}^3} \partial_{\lambda_q^2}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\rm el}^4} \partial_{\lambda_q^2}\mathcal{M}_{e_4,ii}^{z,4},\end{aligned} \tag{5.138}$$

which yields

$$\partial_{\lambda_q^2}\mathcal{M}_i^z = \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\rm el}^2} \partial_{\lambda_q^2}\mathcal{M}_{e,ii}^{z,2}. \tag{5.139}$$

From equation (5.95) we have

$$\begin{aligned}\partial_{\lambda_q^2}\mathcal{M}_{e_2,ii}^{z,2} = & \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{\lambda_q^2}u_{l_2(e_2,jj)} \\ & + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_z,t_z}(e_2)\partial_{\lambda_q^2}w_{l_2(e_2,jj)} \\ & - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_r,t_z}(e_2)\partial_{\lambda_q^2}u^s_{l_2^2(e_2,jj)} \\ & - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,t_z,t_z}(e_2)\partial_{\lambda_q^2}w^s_{l_2^2(e_2,jj)} \\ & - \frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{e_2}} d^s_{ii,jj,t_z}(e_2)\partial_{\lambda_q^2}\sigma^2_{l_2^2(e_2,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} d_{ii,jj,n_z}(e_2)\partial_{\lambda_q^2}\lambda^2_{l_2^2(e_2,jj)},\end{aligned} \tag{5.140}$$

i.e.

$$\partial_{\lambda_q^2}\mathcal{M}_{e_2,ii}^{z,2} = \frac{2\Delta_t}{3}d_{ii,jj,n_z}(e_2)|_{q=l_2^2(e_2,jj)}. \tag{5.141}$$

5.1.7. Derivatives of $\mathcal{M}_i^z$ with respect to λ_q^3

From equation (5.98) we have

$$\begin{aligned}
\partial_{\lambda_q^3}\mathcal{M}_i^z \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^3}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^3}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^3}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^3}\mathcal{M}_{e,ii}^{z,0d} \\
& + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\lambda_q^3}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\partial_{\lambda_q^3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{\lambda_q^3}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\lambda_q^3}\mathcal{M}_{e,ii}^{z,2} \\
& + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\mathrm{el}}^3} \partial_{\lambda_q^3}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{\lambda_q^3}\mathcal{M}_{e_4,ii}^{z,4},
\end{aligned} \tag{5.142}$$

which yields

$$\partial_{\lambda_q^3}\mathcal{M}_i^z = \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\mathrm{el}}^3} \partial_{\lambda_q^3}\mathcal{M}_{e_3,ii}^{z,3}. \tag{5.143}$$

From equation (5.96) we have

$$\partial_{\lambda_q^3}\mathcal{M}_{e_3,ii}^{z,3} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} f_{ii,jj,n_z}(e_3)\partial_{\lambda_q^3}\lambda^3_{l_3^3(e_3,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} f_{ii,jj,t_z}(e_3)\partial_{\lambda_q^3}\gamma^3_{l_3^3(e_3,jj)}, \tag{5.144}$$

i.e.

$$\partial_{\lambda_q^3}\mathcal{M}_{e_3,ii}^{z,3} = \frac{2\Delta_t}{3} f_{ii,jj,n_z}(e_3)|_{q=l_3^3(e_3,jj))}. \tag{5.145}$$

5.1.8. Derivatives of $\mathcal{M}_i^z$ with respect to γ_q^3

From equation (5.98) we have

$$\begin{aligned}\partial_{\gamma_q^3}\mathcal{M}_i^z \;=\; &\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\gamma_q^3}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\gamma_q^3}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\gamma_q^3}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\gamma_q^3}\mathcal{M}_{e,ii}^{z,0d}\\
&+ \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{\gamma_q^3}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\partial_{\gamma_q^3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca}\\
&+ \frac{2\Delta_t}{3}\partial_{\gamma_q^3}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{\gamma_q^3}\mathcal{M}_{e,ii}^{z,2}\\
&+ \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\text{el}}^3} \partial_{\gamma_q^3}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\text{el}}^4} \partial_{\gamma_q^3}\mathcal{M}_{e_4,ii}^{z,4},\end{aligned} \tag{5.146}$$

which yields

$$\partial_{\gamma_q^3}\mathcal{M}_i^z = \sum_{\substack{e_3=1\\ i=l_3(e,ii)}}^{\bar{n}_{\text{el}}^3} \partial_{\gamma_q^3}\mathcal{M}_{e_3,ii}^{z,3}, \tag{5.147}$$

From equation (5.96) we have

$$\partial_{\gamma_q^3}\mathcal{M}_{e_3,ii}^{z,3} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} f_{ii,jj,n_z}(e_3)\partial_{\gamma_q^3}\lambda^3_{l_3^3(e_3,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} f_{ii,jj,t_z}(e_3)\partial_{\gamma_q^3}\gamma^3_{l_3^3(e_3,jj)}, \tag{5.148}$$

i.e.

$$\partial_{\gamma_q^3}\mathcal{M}_{e_3,ii}^{z,3} = \frac{2\Delta_t}{3} f_{ii,jj,t_z}(e_3)|_{q=l_3^3(e_3,jj)}. \tag{5.149}$$

5.1.9. Derivatives of $\mathcal{M}_i^z$ with respect to λ_q^4

From equation (5.98) we have

$$\begin{aligned}\partial_{\lambda_q^4}\mathcal{M}_i^z \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^4}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^4}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^4}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^4}\mathcal{M}_{e,ii}^{z,0d} \\ & + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\lambda_q^4}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\partial_{\lambda_q^4}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca} \\ & + \frac{2\Delta_t}{3}\partial_{\lambda_q^4}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\lambda_q^4}\mathcal{M}_{e,ii}^{z,2} \\ & + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\mathrm{el}}^3} \partial_{\lambda_q^4}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{\lambda_q^4}\mathcal{M}_{e_4,ii}^{z,4},\end{aligned} \tag{5.150}$$

which yields

$$\partial_{\lambda_q^4}\mathcal{M}_i^z = \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{\lambda_q^4}\mathcal{M}_{e_4,ii}^{z,4}. \tag{5.151}$$

From equation (5.97) we have

$$\partial_{\lambda_q^4}\mathcal{M}_{e_4,ii}^{z,4} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} e_{ii,jj,n_z}(e_4)\partial_{\lambda_q^4}\lambda^4_{l_4^4(e_4,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} e_{ii,jj,t_z}(e_4)\partial_{\lambda_q^4}\gamma^4_{l_4^4(e_4,jj)}, \tag{5.152}$$

i.e.

$$\partial_{\lambda_q^4}\mathcal{M}_{e_4,ii}^{z,4} = \frac{2\Delta_t}{3} e_{ii,jj,n_z}(e_4)|_{q=l_4^4(e_4,jj))}. \tag{5.153}$$

5.1.10. Derivatives of $\mathcal{M}_i^z$ with respect to γ_q^4

From equation (5.98) we have

$$\begin{aligned}\partial_{\gamma_q^4}\mathcal{M}_i^z &= \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\gamma_q^4}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\gamma_q^4}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\gamma_q^4}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e_1,ii)}}^{\bar{n}_{\text{el}}} \partial_{\gamma_q^4}\mathcal{M}_{e,ii}^{z,0d}\\
&+ \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{\gamma_q^4}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\partial_{\gamma_q^4}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^{1,n}(r_{J^1},z_{J^1})}{Ca}\\
&+ \frac{2\Delta_t}{3}\partial_{\gamma_q^4}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)m_z^1(r_a,z_a)}{Ca} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{\gamma_q^4}\mathcal{M}_{e,ii}^{z,2}\\
&+ \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\text{el}}^3} \partial_{\gamma_q^4}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\text{el}}^4} \partial_{\gamma_q^4}\mathcal{M}_{e_4,ii}^{z,4},\end{aligned} \tag{5.154}$$

which yields

$$\partial_{\gamma_q^4}\mathcal{M}_i^z = \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\text{el}}^4} \partial_{\gamma_q^4}\mathcal{M}_{e_4,ii}^{z,4}, \tag{5.155}$$

From equation (5.97) we have

$$\partial_{\gamma_q^4}\mathcal{M}_{e_4,ii}^{z,4} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} e_{ii,jj,n_z}(e_4)\partial_{\gamma_q^4}\lambda^4_{l_4^4(e_4,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} e_{ii,jj,t_z}(e_4)\partial_{\gamma_q^4}\gamma^4_{l_4^4(e_4,jj)}, \tag{5.156}$$

i.e.

$$\partial_{\gamma_q^4}\mathcal{M}_{e_4,ii}^{z,4} = \frac{2\Delta_t}{3}e_{ii,jj,t_z}(e_4)|_{q=l_4^4(e_4,jj)}. \tag{5.157}$$

5.1.11. **Derivatives of $\mathcal{M}_i^z$ with respect to h_q**

From equation (5.98) we have

$$\begin{aligned}\partial_{h_q}\mathcal{M}_i^z = &\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{h_q}\mathcal{M}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{h_q}\mathcal{M}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{h_q}\mathcal{M}_{e,ii}^{z,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{h_q}\mathcal{M}_{e,ii}^{z,0d}\\
&+ \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\rm el}^1} \partial_{h_q}\mathcal{M}_{e_1,ii}^{z,1} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})}{Ca}\partial_{h_q}m_z^{1,n}(r_{J^1},z_{J^1})\\
&+ \frac{2\Delta_t}{3}\frac{\sigma^1(r_a,z_a)\phi_i(r_a,z_a)}{Ca}\partial_{h_q}m_z^1(r_a,z_a) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{\bar{n}_{\rm el}^2} \partial_{h_q}\mathcal{M}_{e,ii}^{z,2}\\
&+ \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{\bar{n}_{\rm el}^3} \partial_{h_q}\mathcal{M}_{e_3,ii}^{z,3} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{\bar{n}_{\rm el}^4} \partial_{h_q}\mathcal{M}_{e_4,ii}^{z,4}.\end{aligned} \tag{5.158}$$

From equation (5.90)

$$\partial_{h_q}\mathcal{M}_{e,ii}^{z,0a} = -\frac{2\Delta_t St}{3}\partial_{h_q}a_{ii,g_z}(e), \tag{5.159}$$

and passing to local spine numbers

$$\partial_{h_q}\mathcal{M}_{e,ii}^{z,0a} = -\frac{2\Delta_t St}{3}\partial_{h_{S(e,qq)}}a_{ii,g_z}(e)|_{q=S(e,qq)}. \tag{5.160}$$

Now, from equation (5.91) we have

$$\begin{aligned}\partial_{h_q}\mathcal{M}_{e,ii}^{z,0b} = &\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^e} u_{l(e,jj)}\partial_{h_q}a_{ii,jj}^{r,z}(e) + \sum_{jj=1}^{n_v^e}\frac{2\Delta_t}{3}w_{l(e,jj)}\partial_{h_q}a_{ii,jj}^{r,r}(e)\\
&+ \frac{4\Delta_t}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)}\partial_{h_q}a_{ii,jj}^{z,z}(e) + Re\sum_{jj=1}^{n_v^e} w_{l(e,jj)}\partial_{h_q}a_{ii,jj}(e)\\
&- \frac{4Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)}(t_{n-1})\partial_{h_q}a_{ii,jj}(e) + \frac{Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)}(t_{n-2})\partial_{h_q}a_{ii,jj}(e),\end{aligned} \tag{5.161}$$

i.e.

$$\begin{aligned}\partial_{h_q}\mathcal{M}_{e,ii}^{z,0b} = \sum_{jj=1}^{n_v^e}\Bigg\{&\frac{2\Delta_t}{3}\left[u_{l(e,jj)}\partial_{h_q}a_{ii,jj}^{r,z}(e) + w_{l(e,jj)}\left(\partial_{h_q}a_{ii,jj}^{r,r}(e) + 2\partial_{h_q}a_{ii,jj}^{z,z}(e)\right)\right]\\
&+ Re\,\partial_{h_q}a_{ii,jj}(e)\left[w_{l(e,jj)} - \frac{4}{3}w_{l(e,jj)}(t_{n-1}) + \frac{1}{3}w_{l(e,jj)}(t_{n-2})\right]\Bigg\},\end{aligned} \tag{5.162}$$

and passing to local spine numbers we have

$$\partial_{h_q}\mathcal{M}^{z,0b}_{e,ii} = \sum_{\substack{jj=1\\ q=S(e,qq)}}^{n_v^e} \left\{\frac{2\Delta_t}{3}\left[u_{l(e,jj)}\partial_{h_{S(e,qq)}} a^{r,z}_{ii,jj}(e)\right.\right.$$
$$\left. + w_{l(e,jj)}\left(\partial_{h_{S(e,qq)}} a^{r,r}_{ii,jj}(e) + 2\partial_{h_{S(e,qq)}} a^{z,z}_{ii,jj}(e)\right)\right]$$
$$\left. + Re\,\partial_{h_{S(e,qq)}} a_{ii,jj}(e)\left[w_{l(e,jj)} - \frac{4}{3}w_{l(e,jj)}(t_{n-1}) + \frac{1}{3}w_{l(e,jj)}(t_{n-2})\right]\right\}. \tag{5.163}$$

From equation (5.92) we have

$$\begin{aligned}
\partial_{h_q}\mathcal{M}^{z,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} u_{l(e,kk)}\partial_{h_q} a^{r}_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} w_{l(e,kk)}\partial_{h_q} a^{z}_{ii,kk,jj}(e) \\
& - Re\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} a^{r}_{ii,kk,jj}(e)\partial_{h_q} r^c_{l(e,kk)} \\
& - Re\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}\partial_{h_q} a^{r}_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-1})\partial_{h_q} a^{r}_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} r^c_{l(e,kk)}(t_{n-2})\partial_{h_q} a^{r}_{ii,kk,jj}(e) \\
& - Re\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} a^{z}_{ii,kk,jj}(e)\partial_{h_q} z^c_{l(e,kk)} \\
& - Re\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}\partial_{h_q} a^{z}_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-1})\partial_{h_q} a^{z}_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{jj=1}^{n_v^e} w_{l(e,jj)} \sum_{kk=1}^{n_v^e} z^c_{l(e,kk)}(t_{n-2})\partial_{h_q} a^{z}_{ii,kk,jj}(e),
\end{aligned} \tag{5.164}$$

i.e.

$$\partial_{h_q}\mathcal{M}^{z,0c}_{e,ii} = \sum_{jj=1}^{n_v^e} Re\, w_{l(e,jj)} \left\{ \frac{2\Delta_t}{3} \underbrace{\sum_{kk=1}^{n_v^e} \left[u_{l(e,kk)}\partial_{h_q} a^r_{ii,kk,jj}(e) + w_{l(e,kk)}\partial_{h_q} a^z_{ii,kk,jj}(e)\right]}_{\partial_{h_q} A_{ii,jj}(e)} \right. \tag{5.165}$$

$$\underbrace{-\sum_{kk=1}^{n_v^e}\left[a^r_{ii,kk,jj}(e)\partial_{h_q} r^c_{l(e,kk)} + \partial_{h_q} a^r_{ii,kk,jj}(e)\left(r^c_{l(e,kk)} - \frac{4}{3}r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3}r^c_{l(e,kk)}(t_{n-2})\right)\right]}_{\partial_{h_q} B_{ii,jj}(e)}$$

$$\left. \underbrace{-\sum_{kk=1}^{n_v^e}\left[a^z_{ii,kk,jj}(e)\partial_{h_q} z^c_{l(e,kk)} + \partial_{h_q} a^z_{ii,kk,jj}(e)\left(z^c_{l(e,kk)} - \frac{4}{3}z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3}z^c_{l(e,kk)}(t_{n-2})\right)\right]}_{\partial_{h_q} C_{ii,jj}(e)} \right\},$$

and passing to local spine numbers we have

$$\partial_{h_q}\mathcal{M}^{z,0c}_{e,ii} =$$

$$\sum_{\substack{jj=1 \\ q=S(e,qq)}}^{n_v^e} Re\, w_{l(e,jj)} \left\{ \frac{2\Delta_t}{3} \underbrace{\sum_{kk=1}^{n_v^e} \left[u_{l(e,kk)}\partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) + w_{l(e,kk)}\partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e)\right]}_{\partial_{h_{S(e,qq)}} A_{ii,jj}(e)} \right.$$

$$- \sum_{kk=1}^{n_v^e}\left[a^r_{ii,kk,jj}(e)\partial_{h_{S(e,qq)}} r^c_{l(e,kk)}\right.$$

$$\left. + \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e)\left(r^c_{l(e,kk)} - \frac{4}{3}r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3}r^c_{l(e,kk)}(t_{n-2})\right)\right]$$

$$- \sum_{kk=1}^{n_v^e}\left[a^z_{ii,kk,jj}(e)\partial_{h_{S(e,qq)}} z^c_{l(e,kk)}\right.$$

$$\left.\left. + \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e)\left(z^c_{l(e,kk)} - \frac{4}{3}z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3}z^c_{l(e,kk)}(t_{n-2})\right)\right] \right\}. \tag{5.166}$$

Now, from equation (5.93), we have

$$\partial_{h_q}\mathcal{M}^{z,0d}_{e,ii} = \sum_{jj=1}^{n_p^e} -\frac{2\Delta_t}{3} p_{l^p(e,jj)}\partial_{h_q} b^z_{jj,ii}(e), \tag{5.167}$$

and passing to local spine numbers we have

$$\partial_{h_q}\mathcal{M}^{z,0d}_{e,ii} = \sum_{\substack{jj=1\\ q=S(e,qq)}}^{n_p^e} -\frac{2\Delta_t}{3} p_{l^p(e,jj)} \partial_{h_{S(e,qq)}} b^z_{jj,ii}(e). \tag{5.168}$$

Now from equation (5.94) we have

$$\partial_{h_q}\mathcal{M}^{z,1}_{e_1,ii} = \frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{1,e_1}} \sigma^1_{l_1^1(e_1,jj)}\partial_{h_q} c^s_{jj,ii,t_z} - \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{e_1}} p^g_{l_1^1(e_1,jj)}\partial_{h_q} c_{ii,jj,n_z}(e_1), \tag{5.169}$$

i.e.

$$\partial_{h_q}\mathcal{M}^{z,1}_{e_1,ii} = \sum_{jj=1}^{n_v^{1,e_1}} \frac{2\Delta_t}{3}\left[\frac{1}{Ca}\sigma^1_{l_1^1(e_1,jj)}\partial_{h_q} c^s_{jj,ii,t_z} - p^g_{l_1^1(e_1,jj)}\partial_{h_q} c_{ii,jj,n_z}(e_1)\right], \tag{5.170}$$

and passing to local spine numbers

$$\partial_{h_q}\mathcal{M}^{z,1}_{e_1,ii} = \sum_{\substack{jj=1\\ q=S_1(e_1,qq)}}^{n_v^{1,e_1}} \frac{2\Delta_t}{3}\left[\frac{1}{Ca}\sigma^1_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}} c^s_{jj,ii,t_z} - p^g_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_z}(e_1)\right]. \tag{5.171}$$

From equation (5.95) we have

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{z,2}_{e_2,ii} = {} & \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)}\partial_{h_q} d_{ii,jj,t_r,t_z}(e_2)\\ & + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)}\partial_{h_q} d_{ii,jj,t_z,t_z}(e_2)\\ & - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} u^s_{l_2^2(e_2,jj)}\partial_{h_q} d_{ii,jj,t_r,t_z}(e_2)\\ & - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v^{2,e_2}} w^s_{l_2^2(e_2,jj)}\partial_{h_q} d_{ii,jj,t_z,t_z}(e_2)\\ & - \frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v^{e_2}} \sigma^2_{l_2^2(e_2,jj)}\partial_{h_q} d^s_{ii,jj,t_z}(e_2) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{2,e_2}} \lambda^2_{l_2^2(e_2,jj)}\partial_{h_q} d_{ii,jj,n_z}(e_2),\end{aligned} \tag{5.172}$$

i.e.

$$\begin{aligned}\partial_{h_q}\mathcal{M}^{z,2}_{e_2,ii} = \sum_{jj=1}^{n_v^{2,e_2}} \Bigg[\frac{2\Delta_t}{3}\Bigg(& Be\left\{\partial_{h_q} d_{ii,jj,t_r,t_z}(e_2)\left[u_{l_2(e_2,jj)} - u^s_{l_2^2(e_2,jj)}\right]\right.\\ & \left. + \partial_{h_q} d_{ii,jj,t_z,t_z}(e_2)\left[w_{l_2(e_2,jj)} - w^s_{l_2^2(e_2,jj)}\right]\right\}\\ & - \frac{1}{2Ca}\sigma^2_{l_2^2(e_2,jj)}\partial_{h_q} d^s_{ii,jj,t_z}(e_2) + \lambda^2_{l_2^2(e_2,jj)}\partial_{h_q} d_{ii,jj,n_z}(e_2)\Bigg)\Bigg],\end{aligned} \tag{5.173}$$

and passing to local spine numbers

$$\partial_{h_q}\mathcal{M}^{z,2}_{e_2,ii} = \sum_{\substack{jj=1\\ q=S_2(e_2,qq)}}^{n_v^{2,e_2}} \left[\frac{2\Delta_t}{3}\left(Be\left\{\partial_{h_{S_2(e_2,qq)}} d_{ii,jj,t_r,t_z}(e_2)\left[u_{l_2(e_2,jj)} - u^s_{l_2^2(e_2,jj)}\right]\right.\right.\right.$$
$$\left. + \partial_{h_{S_2(e_2,qq)}} d_{ii,jj,t_z,t_z}(e_2)\left[w_{l_2(e_2,jj)} - w^s_{l_2^2(e_2,jj)}\right]\right\}$$
$$\left.\left. - \frac{1}{2Ca}\sigma^2_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}} d^s_{ii,jj,t_z}(e_2) + \lambda^2_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}} d_{ii,jj,n_z}(e_2)\right)\right]. \tag{5.174}$$

From equation (5.96) we have

$$\partial_{h_q}\mathcal{M}^{z,3}_{e_3,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \lambda^3_{l_3^3(e_3,jj)}\partial_{h_q} f_{ii,jj,n_z}(e_3) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{3,e_3}} \gamma^3_{l_3^3(e_3,jj)}\partial_{h_q} f_{ii,jj,t_z}(e_3), \tag{5.175}$$

i.e.

$$\partial_{h_q}\mathcal{M}^{z,3}_{e_3,ii} = \sum_{jj=1}^{n_v^{3,e_3}} \frac{2\Delta_t}{3}\left[\lambda^3_{l_3^3(e_3,jj)}\partial_{h_q} f_{ii,jj,n_z}(e_3) + \gamma^3_{l_3^3(e_3,jj)}\partial_{h_q} f_{ii,jj,t_z}(e_3)\right], \tag{5.176}$$

and passing to local spine numbers

$$\partial_{h_q}\mathcal{M}^{z,3}_{e_3,ii}$$
$$= \sum_{\substack{jj=1\\ q=S_3(e_3,qq)}}^{n_v^{3,e_3}} \frac{2\Delta_t}{3}\left[\lambda^3_{l_3^3(e_3,jj)}\partial_{h_{S_3(e_3,qq)}} f_{ii,jj,n_z}(e_3) + \gamma^3_{l_3^3(e_3,jj)}\partial_{h_{S_3(e_3,qq)}} f_{ii,jj,t_z}(e_3)\right]. \tag{5.177}$$

From equation (5.97) we have

$$\partial_{h_q}\mathcal{M}^{z,4}_{e_4,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \lambda^4_{l_4^4(e_4,jj)}\partial_{h_q} e_{ii,jj,n_z}(e_4) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{4,e_4}} \gamma^4_{l_4^4(e_4,jj)}\partial_{h_q} e_{ii,jj,t_z}(e_4), \tag{5.178}$$

i.e.

$$\partial_{h_q}\mathcal{M}^{z,4}_{e_4,ii} = \sum_{jj=1}^{n_v^{4,e_4}} \frac{2\Delta_t}{3}\left[\lambda^4_{l_4^4(e_4,jj)}\partial_{h_q} e_{ii,jj,n_z}(e_4) + \gamma^4_{l_4^4(e_4,jj)}\partial_{h_q} e_{ii,jj,t_z}(e_4)\right], \tag{5.179}$$

and passing to local spine numbers

$$\partial_{h_q}\mathcal{M}^{z,4}_{e_4,ii}$$
$$= \sum_{\substack{jj=1\\ q=S_4(e_4,qq)}}^{n_v^{4,e_4}} \frac{2\Delta_t}{3}\left[\lambda^4_{l_4^4(e_4,jj)}\partial_{h_{S_4(e_4,qq)}} e_{ii,jj,n_z}(e_4) + \gamma^4_{l_4^4(e_4,jj)}\partial_{h_{S_4(e_4,qq)}} e_{ii,jj,t_z}(e_4)\right]. \tag{5.180}$$

6. The continuity equation

We consider

$$\partial_r u + \partial_z w = 0, \tag{6.1}$$

and we define

$$C_i = \int\limits_{\Omega^f} \psi_i \partial_r u + \int\limits_{\Omega^f} \psi_i \partial_z w, \tag{6.2}$$

where i is an index that runs through the pressure node numbering. Substituting approximations (4.66) and (4.67) we have

$$\mathcal{C}_i = \int\limits_{\Omega^f} \psi_i \partial_r \left(\sum_{j=1}^{n_v} u_j \phi_j \right) + \int\limits_{\Omega^f} \psi_i \partial_z \left(\sum_{j=1}^{n_v} w_j \phi_j \right), \tag{6.3}$$

where $\hat{C}_i$ results from the substitution of the approximation of u and w into C_i.

We can re-write this as

$$\mathcal{C}_i = \sum_{j=1}^{n_v} u_j \int\limits_{\Omega^f} \psi_i \partial_r \phi_j + \sum_{j=1}^{n_v} w_j \int\limits_{\Omega^f} \psi_i \partial_z \phi_j, \tag{6.4}$$

gathering the sums we have

$$\mathcal{C}_i = \sum_{j=1}^{n_v} \left[u_j \int\limits_{\Omega^f} \psi_i \partial_r \phi_j + w_j \int\limits_{\Omega^f} \psi_i \partial_z \phi_j \right]. \tag{6.5}$$

We now express the integrals as a sum over the integrals on each element

$$\mathcal{C}_i = \sum_{e=n^f_{\text{el},s}}^{n_{\text{el}}} \sum_{j=1}^{n_v} \left[u_j \int\limits_{\Omega_e} \psi_i \partial_r \phi_j + w_j \int\limits_{\Omega_e} \psi_i \partial_z \phi_j \right], \tag{6.6}$$

where $n^f_{\text{el},s}$ is the first element in the far field.

Moving to local numbering in variable j we have

$$\mathcal{C}_i = \sum_{e=n^f_{\text{el},s}}^{n_{\text{el}}} \sum_{jj=1}^{n^e_v} \left[u_{l(e,jj)} \int\limits_{\Omega_e} \psi_i \partial_r \phi_{l(e,jj)} + w_{l(e,jj)} \int\limits_{\Omega_e} \psi_i \partial_z \phi_{l(e,jj)} \right]. \tag{6.7}$$

Moving to local node numbers for index variable i we have

$$\mathcal{C}_i = \sum_{e=n^f_{\text{el},s}}^{n_{\text{el}}} \sum_{jj=1}^{n^e_v} \left[u_{l(e,jj)} \underbrace{\int\limits_{\Omega_e} \psi_{l^p(e,ii)} \partial_r \phi_{l(e,jj)}}_{b^r_{ii,jj}(e)} + w_{l(e,jj)} \underbrace{\int\limits_{\Omega_e} \psi_{l^p(e,ii)} \partial_z \phi_{l(e,jj)}}_{b^z_{ii,jj}(e)} \right], \tag{6.8}$$

i.e.

$$\mathcal{C}_i = \sum_{\substack{e=n^f_{\text{el},s} \\ i=l^p(e,ii)}}^{n_{\text{el}}} \mathcal{C}_{e,ii}. \tag{6.9}$$

where

$$\mathcal{C}_{e,ii} = \sum_{jj=1}^{n_v^e} \left[u_{l(e,jj)} b^r_{ii,jj}(e) + w_{l(e,jj)} b^z_{ii,jj}(e) \right]. \tag{6.10}$$

6.1. Jacobian terms

We now consider the derivatives of $\mathcal{C}_i$ with respect to u_q, w_q and h_q.

6.1.1. Derivatives of $\mathcal{C}_i$ with respect to u_q

From equation (6.9) we have

$$\partial_{u_q}\mathcal{C}_i = \sum_{\substack{e=n_{\mathrm{el},s}^f \\ i=l^p(e,ii)}}^{n_{\mathrm{el}}} \partial_{u_q}\mathcal{C}_{e,ii}, \tag{6.11}$$

and from equation (6.10) we have

$$\partial_{u_q}\mathcal{C}_{e,ii} = \sum_{jj=1}^{n_v^e} \left[b_{ii,jj}^r(e)\partial_{u_q}u_{l(e,jj)} + b_{ii,jj}^z(e)\partial_{u_q}w_{l(e,jj)}\right], \tag{6.12}$$

i.e.

$$\partial_{u_q}\mathcal{C}_{e,ii} = b_{ii,jj}^r(e)|_{q=l(e,jj)}. \tag{6.13}$$

6.1.2. Derivatives of $\mathcal{C}_i$ with respect to w_q

From equation (6.9) we have

$$\partial_{w_q}\mathcal{C}_i = \sum_{\substack{e=n_{\mathrm{el},s}^f \\ i=l^p(e,ii)}}^{n_{\mathrm{el}}} \partial_{w_q}\mathcal{C}_{e,ii}, \tag{6.14}$$

and from equation (6.10) we have

$$\partial_{w_q}\mathcal{C}_{e,ii} = \sum_{jj=1}^{n_v^e} \left[b_{ii,jj}^r(e)\partial_{w_q}u_{l(e,jj)} + b_{ii,jj}^z(e)\partial_{w_q}w_{l(e,jj)}\right], \tag{6.15}$$

i.e.

$$\partial_{w_q}\mathcal{C}_{e,ii} = b_{ii,jj}^z(e)|_{q=l(e,jj)}. \tag{6.16}$$

6.1.3. Derivatives of $\mathcal{C}_i$ with respect to h_q

From equation (6.9) we have

$$\partial_{h_q}\mathcal{C}_i = \sum_{\substack{e=n_{\mathrm{el},s}^f \\ i=l^p(e,ii)}}^{n_{\mathrm{el}}} \partial_{h_q}\mathcal{C}_{e,ii}, \tag{6.17}$$

and from equation (6.10) we have

$$\partial_{h_q}\mathcal{C}_{e,ii} = \sum_{jj=1}^{n_v^e} \left[u_{l(e,jj)}\partial_{h_q}b_{ii,jj}^r(e) + w_{l(e,jj)}\partial_{h_q}b_{ii,jj}^z(e)\right], \tag{6.18}$$

and passing to local spine numbers we have

$$\partial_{h_q}\mathcal{C}_{e,ii} = \sum_{\substack{jj=1 \\ q=S(e,qq)}}^{n_v^e} \left[u_{l(e,jj)}\partial_{h_{S(e,qq)}}b_{ii,jj}^r(e) + w_{l(e,jj)}\partial_{h_{S(e,qq)}}b_{ii,jj}^z(e)\right]. \tag{6.19}$$

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7. The slip condition on the liquid-solid interface (SC2)

We recall equation (2.52)

$$\left[\boldsymbol{v}^{s_2} - \frac{1}{2}\left(\boldsymbol{u} + \boldsymbol{u}^s\right)\right] \cdot \left(\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2\right) = Es\, \nabla^s \sigma^2, \tag{7.1}$$

and we define the i-th SC2 residual as

$$S_i^2 = \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[\boldsymbol{v}^{s_2} - \frac{1}{2}\left(\boldsymbol{u} + \boldsymbol{u}^s\right)\right] \cdot \boldsymbol{t}^2 - Es \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \boldsymbol{t}^2 \cdot \nabla^s \sigma^2, \tag{7.2}$$

which, of course we wish to make identically null.

We thus have

$$S_i^2 = \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[\boldsymbol{v}^{s_2} \cdot \boldsymbol{t}^2 - \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{t}^2 - \frac{1}{2}\boldsymbol{u}^s \cdot \boldsymbol{t}^2\right] - Es \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left(\partial_s \sigma^2\right) \boldsymbol{t}^2 \cdot \boldsymbol{t}^2, \tag{7.3}$$

i.e.

$$S_i^2 = \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[u^{s_2} t_r^2 + w^{s_2} t_z^2 - \frac{1}{2} u t_r^2 - \frac{1}{2} w t_z^2 - \frac{1}{2} u^s t_r^2 - \frac{1}{2} w^s t_z^2\right] - Es \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \partial_s \sigma^2, \tag{7.4}$$

equivalently

$$\begin{aligned} S_i^2 = &\int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[u^{s_2} t_r^2\right] + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[w^{s_2} t_z^2\right] + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[-\frac{1}{2} u t_r^2\right] + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[-\frac{1}{2} w t_z^2\right] \\ &+ \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[-\frac{1}{2} u^s t_r^2\right] + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[-\frac{1}{2} w^s t_z^2\right] - Es \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \partial_s \sigma^2, \end{aligned} \tag{7.5}$$

i.e.

$$\begin{aligned} S_i^2 = &\int\limits_{\partial\Omega^{2,n}} \phi_i^2 u^{s_2} t_r^2 + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 w^{s_2} t_z^2 - \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 u t_r^2 - \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 w t_z^2 \\ &- \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 u^s t_r^2 - \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 w^s t_z^2 - Es \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \partial_s \sigma^2, \end{aligned} \tag{7.6}$$

We consider the last integral on the right hand side above and we integrate by parts to obtain

$$\int\limits_{\partial\Omega^2} \phi_i^2 \partial_s \sigma^2 = \phi_i^2 \sigma^2 \Big|_{(r_c,z_c)}^{(r_o,z_o)} - \int\limits_{\partial\Omega^2} \sigma^2 \partial_s \phi_i^2. \tag{7.7}$$

This yields

$$\begin{aligned} S_i^2 = &\, Es\, \phi_i^2(r_c,z_c)\sigma^2(r_c,z_c) - Es\, \phi_i^2(r_o,z_o)\sigma^2(r_o,z_o) + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 u^{s_2} t_r^2 + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 w^{s_2} t_z^2 \\ &- \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 u t_r^2 - \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 w t_z^2 - \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 u^s t_r^2 - \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 w^s t_z^2 + Es \int\limits_{\partial\Omega^{2,n}} \sigma^2 \partial_s \phi_i^2, \end{aligned} \tag{7.8}$$

We recall the approximations

$$u \approx \sum_{j=1}^{n_v} u_j \phi_j, \tag{7.9}$$

$$w \approx \sum_{j=1}^{n_v} w_j \phi_j, \tag{7.10}$$

$$\sigma^2 \approx \sum_{j=1}^{n_v} \sigma_j^2 \phi_j^2, \tag{7.11}$$

$$u^s \approx \sum_{j=1}^{n_v} u_j^s \phi_j^2, \tag{7.12}$$

$$w^s \approx \sum_{j=1}^{n_v} w_j^s \phi_j^2, \tag{7.13}$$

$$u^{s_2} \approx \sum_{j=1}^{n_v} u_j^{s_2} \phi_j^2, \tag{7.14}$$

and

$$w^{s_2} \approx \sum_{j=1}^{n_v} w_j^{s_2} \phi_j^2. \tag{7.15}$$

We thus have

$$\begin{aligned}
\mathcal{S}_i^2 &= Es\, \phi_i^2(r_c, z_c)\sigma^2(r_c, z_c) - Es\, \phi_i^2(r_o, z_o)\sigma^2(r_o, z_o) \\
&\quad + \int_{\partial\Omega^{2,n}} \phi_i^2 \left(\sum_{j=1}^{n_v} u_j^{s_2} \phi_j^2 \right) t_r^2 + \int_{\partial\Omega^{2,n}} \phi_i^2 \left(\sum_{j=1}^{n_v} w_j^{s_2} \phi_j^2 \right) t_z^2 \\
&\quad - \frac{1}{2} \int_{\partial\Omega^{2,n}} \phi_i^2 \left(\sum_{j=1}^{n_v} u_j \phi_j^2 \right) t_r^2 - \frac{1}{2} \int_{\partial\Omega^{2,n}} \phi_i^2 \left(\sum_{j=1}^{n_v} w_j \phi_j^2 \right) t_z^2 \\
&\quad - \frac{1}{2} \int_{\partial\Omega^{2,n}} \phi_i^2 \left(\sum_{j=1}^{n_v} u_j^s \phi_j^2 \right) t_r^2 - \frac{1}{2} \int_{\partial\Omega^{2,n}} \phi_i^2 \left(\sum_{j=1}^{n_v} w_j^s \phi_j^2 \right) t_z^2 \\
&\quad + Es \int_{\partial\Omega^{2,n}} \left(\sum_{j=1}^{n_v} \sigma_j^2 \phi_j^2 \right) \partial_s \phi_i^2.
\end{aligned} \tag{7.16}$$

Moving the integrals into the sums, we have

$$\begin{aligned}
\mathcal{S}_i^2 &= Es\, \phi_i^2(r_c, z_c)\sigma^2(r_c, z_c) - Es\, \phi_i^2(r_o, z_o)\sigma^2(r_o, z_o) \\
&\quad + \sum_{j=1}^{n_v} u_j^{s_2} \int_{\partial\Omega^{2,n}} \phi_i^2 \phi_j^2 t_r^2 + \sum_{j=1}^{n_v} w_j^{s_2} \int_{\partial\Omega^{2,n}} \phi_i^2 \phi_j^2 t_z^2 \\
&\quad - \frac{1}{2} \sum_{j=1}^{n_v} u_j \int_{\partial\Omega^{2,n}} \phi_i^2 \phi_j^2 t_r^2 - \frac{1}{2} \sum_{j=1}^{n_v} w_j \int_{\partial\Omega^{2,n}} \phi_i^2 \phi_j^2 t_z^2 \\
&\quad - \frac{1}{2} \sum_{j=1}^{n_v} u_j^s \int_{\partial\Omega^{2,n}} \phi_i^2 \phi_j^2 t_r^2 - \frac{1}{2} \sum_{j=1}^{n_v} w_j^s \int_{\partial\Omega^{2,n}} \phi_i^2 \phi_j^2 t_z^2 + Es \sum_{j=1}^{n_v} \sigma_j^2 \int_{\partial\Omega^{2,n}} \phi_j^2 \partial_s \phi_i^2.
\end{aligned} \tag{7.17}$$

Decomposing the integrals into sums of integrals over each individual element and passing to local element node numbers we have

$$\mathcal{S}_i^2 = Es\,\phi_i^2(r_c,z_c)\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\text{el}}^2} \mathcal{S}_{e_2,ii}^2, \tag{7.18}$$

where

$$\begin{aligned}
\mathcal{S}_{e_2,ii}^2 = &\sum_{jj=1}^{n_v^{2,e_2}} u_{l_2^2(e_2,jj)}^{s_2} \underbrace{\int_{\partial\Omega^{2,n}} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2 t_r^2}_{d_{ii,jj,t_r}(e_2)} \\
&+ \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2^2(e_2,jj)}^{s_2} \underbrace{\int_{\partial\Omega^{2,n}} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2 t_z^2}_{d_{ii,jj,t_z}(e_2)} \\
&- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} \underbrace{\int_{\partial\Omega^{2,n}} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2 t_r^2}_{d_{ii,jj,t_r}(e_2)} \\
&- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} \underbrace{\int_{\partial\Omega^{2,n}} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2 t_z^2}_{d_{ii,jj,t_z}(e_2)} \\
&- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} u_{l_2^2(e_2,jj)}^{s} \underbrace{\int_{\partial\Omega^{2,n}} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2 t_r^2}_{d_{ii,jj,t_r}(e_2)} \\
&- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} w_{l_2^2(e_2,jj)}^{s} \underbrace{\int_{\partial\Omega^{2,n}} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2 t_z^2}_{d_{ii,jj,t_z}(e_2)} \\
&+ Es\sum_{jj=1}^{n_v^{2,e_2}} \sigma_{l_2^2(e_2,jj)}^2 \underbrace{\int_{\partial\Omega^{2,n}} \phi_{l_2(e_2,jj)}^2 \partial_s \phi_{l_2(e_2,ii)}^2}_{d_{jj,ii}^s(e_2)}.
\end{aligned} \tag{7.19}$$

i.e.

$$\mathcal{S}_i^2 = Es\,\phi_i^2(r_c,z_c)\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\text{el}}^2} \mathcal{S}_{e_2,ii}^2, \tag{7.20}$$

where

$$
\begin{aligned}
\mathcal{S}^2_{e_2,ii} = {} & \sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2) \\
& + \sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
& - \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
& - \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} u^{s}_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} w^{s}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
& + Es \sum_{jj=1}^{n_v^{2,e_2}} \sigma^2_{l_2^2(e_2,jj)} d^s_{jj,ii}(e_2).
\end{aligned}
\tag{7.21}
$$

Summarising and re-arranging terms we have

$$
\mathcal{S}^2_i = Es\,\phi^2_i(r_c,z_c)\sigma^2(r_c,z_c) - Es\,\phi^2_i(r_o,z_o)\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1 \\ i=l_2(e_2,ii)}}^{n^2_{\mathrm{el}}} \mathcal{S}^2_{e_2,ii}, \tag{7.22}
$$

where

$$
\begin{aligned}
\mathcal{S}^2_{e_2,ii} = \sum_{jj=1}^{n_v^{2,e_2}} \Bigg\{ & d_{ii,jj,t_r}(e_2) \left[u^{s_2}_{l_2^2(e_2,jj)} - \frac{1}{2} u_{l_2(e_2,jj)} - \frac{1}{2} u^s_{l_2^2(e_2,jj)} \right] \\
& + d_{ii,jj,t_z}(e_2) \left[w^{s_2}_{l_2^2(e_2,jj)} - \frac{1}{2} w_{l_2(e_2,jj)} - \frac{1}{2} w^s_{l_2^2(e_2,jj)} \right] \\
& + Es\,\sigma^2_{l_2^2(e_2,jj)} d^s_{jj,ii}(e_2) \Bigg\}.
\end{aligned}
\tag{7.23}
$$

7.1. Jacobian terms

Here we find the derivative of $S_i^{2,r}$ with respect to u_q, w_q, u^{s_2}, w^{s_2}, σ^2 and h_q.

7.1.1. Derivatives of $\mathcal{S}_i^2$ with respect to u_q

From equation (7.20)

$$\partial_{u_q}\mathcal{S}_i^2 = Es\,\phi_i^2(r_c,z_c)\partial_{u_q}\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\partial_{u_q}\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\rm el}^2} \partial_{u_q}\mathcal{S}_{e_2,ii}^2, \tag{7.24}$$

and from equation (7.21) we have

$$\begin{aligned}\partial_{u_q}\mathcal{S}_{e_2,ii}^2 = &\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2)\\ &+\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\ &-\frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q} u_{l_2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q} w_{l_2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\ &-\frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q} u^s_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q} w^s_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\ &+Es\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q}\sigma^2_{l_2^2(e_2,jj)} d^s_{jj,ii}(e_2),\end{aligned} \tag{7.25}$$

i.e.

$$\partial_{u_q}\mathcal{S}_{e_2,ii}^2 = -\frac{1}{2} d_{ii,jj,t_r}(e_2)|_{q=l_2(e_2,jj))}, \tag{7.26}$$

7.1.2. Derivatives of $\mathcal{S}_i^2$ with respect to w_q

From equation (7.20)

$$\partial_{w_q}\mathcal{S}_i^2 = Es\,\phi_i^2(r_c,z_c)\partial_{w_q}\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\partial_{w_q}\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{w_q}\mathcal{S}_{e_2,ii}^2, \tag{7.27}$$

and from equation (7.21) we have

$$\begin{aligned}\partial_{w_q}\mathcal{S}_{e_2,ii}^2 = &\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2)\\ &+ \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\ &- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q} u_{l_2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q} w_{l_2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\ &- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q} u^{s}_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q} w^{s}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\ &+ Es\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q} \sigma^2_{l_2^2(e_2,jj)} d^{s}_{jj,ii}(e_2),\end{aligned} \tag{7.28}$$

i.e.

$$\partial_{w_q}\mathcal{S}_{e_2,ii}^{2,r} = -\frac{1}{2} d_{ii,jj,t_z}(e_2)|_{q=l_2(e_2,jj)}. \tag{7.29}$$

7.1.3. Derivatives of $\mathcal{S}_i^2$ with respect to $u_q^{s_2}$

From equation (7.20)

$$\partial_{u_q^{s_2}} \mathcal{S}_i^2 = Es\,\phi_i^2(r_c, z_c)\partial_{u_q^{s_2}} \sigma^2(r_c, z_c) - Es\,\phi_i^2(r_o, z_o)\partial_{u_q^{s_2}} \sigma^2(r_o, z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\rm el}^2} \partial_{u_q^{s_2}} \mathcal{S}_{e_2,ii}^2, \tag{7.30}$$

and from equation (7.21) we have

$$\begin{aligned}
\partial_{u_q^{s_2}} \mathcal{S}_{e_2,ii}^2 = & \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2) \\
& + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
& - \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} u_{l_2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} w_{l_2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
& - \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} u^{s}_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} w^{s}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
& + Es \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \sigma^2_{l_2^2(e_2,jj)} d^s_{jj,ii}(e_2),
\end{aligned} \tag{7.31}$$

i.e.

$$\partial_{u_q^{s_2}} \mathcal{S}_{e_2,ii}^2 = d_{ii,jj,t_r}(e_2)|_{q=l_2^2(e_2,jj)}. \tag{7.32}$$

7.1.4. Derivatives of $\mathcal{S}_i^2$ with respect to $w_q^{s_2}$

From equation (7.20)

$$\partial_{w_q^{s_2}} \mathcal{S}_i^2 = Es\,\phi_i^2(r_c, z_c)\partial_{w_q^{s_2}}\sigma^2(r_c, z_c) - Es\,\phi_i^2(r_o, z_o)\partial_{w_q^{s_2}}\sigma^2(r_o, z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{w_q^{s_2}} \mathcal{S}_{e_2,ii}^2, \tag{7.33}$$

and from equation (7.21) we have

$$\begin{aligned}
\partial_{w_q^{s_2}} \mathcal{S}_{e_2,ii}^2 = & \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} u_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,t_r}(e_2) \\
& + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} w_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,t_z}(e_2) \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} u_{l_2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} w_{l_2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} u_{l_2^2(e_2,jj)}^{s} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} w_{l_2^2(e_2,jj)}^{s} d_{ii,jj,t_z}(e_2) \\
& + Es\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \sigma_{l_2^2(e_2,jj)}^2 d_{jj,ii}^s(e_2),
\end{aligned} \tag{7.34}$$

i.e.

$$\partial_{w_q^{s_2}} \mathcal{S}_{e_2,ii}^2 = d_{ii,jj,t_z}(e_2)\big|_{q=l_2^2(e_2,jj)}. \tag{7.35}$$

7.1.5. Derivatives of $\mathcal{S}_i^2$ with respect to σ_q^2

From equation (7.20)

$$\partial_{\sigma_q^2}\mathcal{S}_i^2 = Es\,\phi_i^2(r_c,z_c)\partial_{\sigma_q^2}\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\partial_{\sigma_q^2}\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\rm el}^2} \partial_{\sigma_q^2}\mathcal{S}_{e_2,ii}^2, \tag{7.36}$$

and from equation (7.21) we have

$$\begin{aligned}\partial_{\sigma_q^2}\mathcal{S}_{e_2,ii}^2 = &\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\sigma_q^2} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2)\\ &+ \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\sigma_q^2} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\ &- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\sigma_q^2} u_{l_2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\sigma_q^2} w_{l_2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\ &- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\sigma_q^2} u^{s}_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\sigma_q^2} w^{s}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\ &+ Es\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\sigma_q^2} \sigma^2_{l_2^2(e_2,jj)} d^s_{jj,ii}(e_2),\end{aligned} \tag{7.37}$$

i.e.

$$\partial_{\sigma_q^2}\mathcal{S}_{e_2,ii}^2 = Es\, d^s_{jj,ii}(e_2)|_{q=l_2^2(e_2,jj)}. \tag{7.38}$$

7.1.6. Derivatives of $\mathcal{S}_i^2$ with respect to h_q

From equation (7.20)

$$\partial_{h_q}\mathcal{S}_i^2 = Es\,\phi_i^2(r_c,z_c)\partial_{h_q}\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\partial_{h_q}\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\mathrm{el}}^2} \partial_{h_{S_2(e_2,qq)}}\mathcal{S}_{e_2,ii}^2, \tag{7.39}$$

and from equation (7.21) we have

$$\begin{aligned}
\partial_{h_{S_2(e_2,qq)}}\mathcal{S}_{e_2,ii}^2 = {} & \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2^2(e_2,jj)}^{s_2}\partial_{h_{S_2(e_2,qq)}} d_{ii,jj,t_r}(e_2) \\
& + \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2^2(e_2,jj)}^{s_2}\partial_{h_{S_2(e_2,qq)}} d_{ii,jj,t_z}(e_2) \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}} d_{ii,jj,t_r}(e_2) \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}} d_{ii,jj,t_z}(e_2) \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} u_{l_2^2(e_2,jj)}^{s}\partial_{h_{S_2(e_2,qq)}} d_{ii,jj,t_r}(e_2) \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} w_{l_2^2(e_2,jj)}^{s}\partial_{h_{S_2(e_2,qq)}} d_{ii,jj,t_z}(e_2) \\
& + Es\sum_{jj=1}^{n_v^{2,e_2}} \sigma_{l_2^2(e_2,jj)}^2\partial_{h_{S_2(e_2,qq)}} d_{jj,ii}^{s}(e_2),
\end{aligned} \tag{7.40}$$

i.e.

$$\begin{aligned}
\partial_{h_{S_2(e_2,qq)}}\mathcal{S}_{e_2,ii}^2 = \sum_{jj=1}^{n_v^{2,e_2}} \Bigg\{ & \partial_{h_{S_2(e_2,qq)}} d_{ii,jj,t_r}(e_2)\left[u_{l_2^2(e_2,jj)}^{s_2} - \frac{1}{2}u_{l_2(e_2,jj)} - \frac{1}{2}u_{l_2^2(e_2,jj)}^{s}\right] \\
& + \partial_{h_{S_2(e_2,qq)}} d_{ii,jj,t_z}(e_2)\left[w_{l_2^2(e_2,jj)}^{s_2} - \frac{1}{2}w_{l_2(e_2,jj)} - \frac{1}{2}w_{l_2^2(e_2,jj)}^{s}\right] \\
& + Es\,\sigma_{l_2^2(e_2,jj)}^2 \underbrace{\partial_{h_{S_2(e_2,qq)}} d_{jj,ii}^{s}(e_2)}_{=0}\Bigg\},
\end{aligned} \tag{7.41}$$

8. Impermeability condition (I2)

we recall equation (2.51) which states

$$\left(\boldsymbol{v}^2 - \boldsymbol{u}^s\right) \cdot \boldsymbol{n}^2 = 0, \tag{8.1}$$

i.e.

$$(u^{s_2} - u^s)n_r^2 + (w^{s_2} - w^s)n_z^2 = 0, \tag{8.2}$$

where $\boldsymbol{v}^{s_2} = (u^{s_2}, w^{s_2})$, and we define the i-th residual of the impermeability equation as

$$I_i = \int\limits_{\partial\Omega^{2,f}} \phi_i^2 u^{s_2} n_r^2 + \int\limits_{\partial\Omega^{2,f}} \phi_i^2 w^{s_2} n_z^2 - \int\limits_{\partial\Omega^{2,f}} \phi_i^2 u^s n_r^2 - \int\limits_{\partial\Omega^{2,f}} \phi_i^2 w^s n_z^2, \tag{8.3}$$

where i is an index that runs through the boundary 2 node numbering.

We recall the approximations given by

$$u^s \approx \sum_{j=1}^{n_v} u_j^2 \phi_j^2 \tag{8.4}$$

and

$$w^s \approx \sum_{j=1}^{n_v} w_j^s \phi_j^2, \tag{8.5}$$

and we introduce

$$u^{s_2} \approx \sum_{j=1}^{n_v} u_j^{s_2} \phi_j^2, \tag{8.6}$$

and

$$w^{s_2} \approx \sum_{j=1}^{n_v} w_j^{s_2} \phi_j^2. \tag{8.7}$$

Using these, we have

$$\begin{aligned} \mathcal{I}_i = & \int\limits_{\partial\Omega^{2,f}} \phi_i^2 \left(\sum_{j=1}^{n_v^{1,e_2}} u_j^{s_2} \phi_j^2 \right) n_r^2 + \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v^{1,e_2}} w_j^{s_2} \phi_j^2 \right) n_z^2 \\ & - \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v^{1,e_2}} \tilde{u}_j^s \phi_j^2 \right) n_r^2 - \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v^{1,e_2}} \tilde{w}_j^s \phi_j^2 \right) n_z^2, \end{aligned} \tag{8.8}$$

i.e.

$$\mathcal{I}_i = \sum_{j=1}^{n_v^{1,e_2}} u_j^{s_2} \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_r^2 + \sum_{j=1}^{n_v^{1,e_2}} w_j^{s_2} \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_z^2 - \sum_{j=1}^{n_v^{1,e_2}} \tilde{u}_j^s \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_r^2 - \sum_{j=1}^{n_v^{1,e_2}} \tilde{w}_j^s \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_z^2, \tag{8.9}$$

gathering the sums we obtain

$$\mathcal{I}_i = \sum_{j=1}^{n_v^{1,e_2}} \left(u_j^{s_2} \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_r^2 + w_j^{s_2} \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_z^2 - \tilde{u}_j^s \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_r^2 - \tilde{w}_j^s \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_z^2 \right). \tag{8.10}$$

Now, we decompose the integrals into the sum of the integrals over each line-element on boundary 2

$$\mathcal{I}_i = \sum_{e_2=1}^{n_{\text{el}}^2} \mathcal{I}_{e_2,ii}, \tag{8.11}$$

where

$$\mathcal{I}_{e_2,ii} = \sum_{j=1}^{n_v^{1,e_2}} \left(u_j^{s_2} \int\limits_{\partial\Omega_{e_2}} \phi_i^2 \phi_j^2 n_r^2 + w_j^{s_2} \int\limits_{\partial\Omega_{e_2}} \phi_i^2 \phi_j^2 n_z^2 - u_j^s \int\limits_{\partial\Omega_{e_2}} \phi_i^2 \phi_j^2 n_r^2 - w_j^s \int\limits_{\partial\Omega_{e_2}} \phi_i^2 \phi_j^2 n_z^2 \right), \tag{8.12}$$

and expressing sums over index j in terms of local node numbers we have

$$\begin{aligned} \mathcal{I}_{e_2,ii} = \sum_{jj=1}^{n_v^{e_2}} \Bigg(& u^{s_2}_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega_{e_2}} \phi_i^2 \phi_{l_2(e_2,jj)}^2 n_r^2}_{d_{ii,jj,n_r}(e_2)} + w^{s_2}_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega_{e_2}} \phi_i^2 \phi_{l_2(e_2,jj)}^2 n_z^2}_{d_{ii,jj,n_z}(e_2)} \\ & - u^{s}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega_{e_2}} \phi_i^2 \phi_{l_2(e_2,jj)}^2 n_r^2}_{d_{ii,jj,n_r}(e_2)} - w^{s}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega_{e_2}} \phi_i^2 \phi_{l_2(e_2,jj)}^2 n_z^2}_{d_{ii,jj,n_z}(e_2)} \Bigg), \end{aligned} \tag{8.13}$$

i.e.

$$\begin{aligned} \mathcal{I}_{e_2,ii} = \sum_{jj=1}^{n_v^{e_2}} \Big(& u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) + w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e_2) \\ & - u^{s}_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) - w^{s}_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e_2) \Big). \end{aligned} \tag{8.14}$$

Summarising and re-arranging we have

$$\mathcal{I}_i = \sum_{e_2=1}^{n_{\text{el}}^2} \mathcal{I}_{e_2,ii}, \tag{8.15}$$

where

$$\boxed{\mathcal{I}_{e_2,ii} = \sum_{jj=1}^{n_v^{e_2}} \left(d_{ii,jj,n_r}(e_2) \left[u^{s_2}_{l_2^2(e_2,jj)} - u^{s}_{l_2^2(e_2,jj)} \right] + d_{ii,jj,n_z}(e_2) \left[w^{s_2}_{l_2^2(e_2,jj)} - w^{s}_{l_2^2(e_2,jj)} \right] \right)}. \tag{8.16}$$

8.1. Jacobian terms

We now calculate the derivatives of $\mathcal{I}_i$ with respect to $u_q^{s_2}$, $w_q^{s_2}$ and h_q.

8.1.1. Derivatives of $\mathcal{I}_i$ with respect to u_q

We consider

$$\begin{aligned}\partial_{u_q^{s_2}}\mathcal{I}_i = \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{u_q^{s_2}} \sum_{jj=1}^{n_v^{e_2}} \Big[& u^{s_2}_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) + w^{s_2}_{l_2(e,jj)} d_{ii,jj,n_z}(e_2) \\ & - u^{s}_{l_2^s(e_2,jj)} d_{ii,jj,n_r}(e_2) - w^{s}_{l_2^s(e,jj)} d_{ii,jj,n_z}(e_2)\Big],\end{aligned} \tag{8.17}$$

passing the derivative into the sum we have

$$\partial_{u_q^{s_2}}\mathcal{I}_i = \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \sum_{jj=1}^{n_v^{e_2}} \underbrace{\partial_{u_q^{s_2}} u_{l_2(e_2,jj)}}_{\delta_{q,l_2(e_2,jj)}} d_{ii,jj,n_r}(e_2), \tag{8.18}$$

i.e.

$$\boxed{\partial_{u_q^{s_2}}\mathcal{I}_i = \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)\\ q=l_2(e_2,jj)}}^{n_{\text{el}}^2} d_{ii,jj,n_r}(e_2)}. \tag{8.19}$$

8.1.2. Derivatives of $\mathcal{I}_i$ with respect to $w_q^{s_2}$

We consider

$$\partial_{w_q^{s_2}} \mathcal{I}_i = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{w_q^{s_2}} \sum_{jj=1}^{n_v^{e_2}} \Big[u^{s_2}_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) + w^{s_2}_{l_2(e,jj)} d_{ii,jj,n_z}(e_2) \\ - u^{s}_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) - w^{s}_{l_2^2(e,jj)} d_{ii,jj,n_z}(e_2) \Big], \tag{8.20}$$

passing the derivative into the sum we have

$$\partial_{w_q^{s_2}} \mathcal{I}_i = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \sum_{jj=1}^{n_v^{e_2}} \underbrace{\partial_{w_q^{s_2}} w^{s_2}_{l_2(e_2,jj)}}_{\delta_{q,l_2(e_2,jj)}} d_{ii,jj,n_z}(e_2), \tag{8.21}$$

i.e.

$$\boxed{\partial_{w_q^{s_2}} \mathcal{I}_i = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii) \\ q=l_2(e_2,jj)}}^{n_{\mathrm{el}}^2} d_{ii,jj,n_z}(e_2)}. \tag{8.22}$$

8.1.3. **Derivatives of $\mathcal{I}_i$ with respect to h_q**

From equation (8.14) we have

$$\begin{aligned}\partial_{h_q}\mathcal{I}_i = \sum_{\substack{e_2=1\\ i=l_2^2(e,ii)}}^{n_{\text{el}}^2}\sum_{jj=1}^{n_v^{e_2}}\partial_{h_q}\Big(u^{s_2}_{l_2(e_2,jj)}d_{ii,jj,n_r}(e_2)+w^{s_2}_{l_2(e_2,jj)}d_{ii,jj,n_z}(e_2)\\ - u^{s}_{l_2^2(e_2,jj)}d_{ii,jj,n_r}(e_2)-w^{s}_{l_2^2(e_2,jj)}d_{ii,jj,n_z}(e_2)\Big).\end{aligned} \tag{8.23}$$

passing to local spine numbers we have

$$\begin{aligned}\partial_{h_q}\mathcal{I}_i = \sum_{\substack{e_2=1\\ i=l_2^2(e,ii)\\ q=S_2(e_2,qq)}}^{n_{\text{el}}^2}\sum_{jj=1}^{n_v^{e_2}}\Big(u^{s_2}_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2)+w^{s_2}_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_z}(e_2)\\ - u^{s}_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2)-w^{s}_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_z}(e_2)\Big).\end{aligned} \tag{8.24}$$

i.e.

$$\begin{aligned}\partial_{h_q}\mathcal{I}_i = \sum_{\substack{e_2=1\\ i=l_2^2(e,ii)\\ q=S_2(e_2,qq)}}^{n_{\text{el}}^2}\sum_{jj=1}^{n_v^{e_2}}\Big(\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2)\left[u^{s_2}_{l_2(e_2,jj)}-u^{s}_{l_2^2(e_2,jj)}\right]\\ +\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_z}(e_2)\left[w^{s_2}_{l_2(e_2,jj)}-w^{s}_{l_2^2(e_2,jj)}\right]\Big).\end{aligned} \tag{8.25}$$

9. The mass exchange equation on boundary 2 (MEC2)

We recall equation (2.55), which states

$$(\boldsymbol{u}-\boldsymbol{v}^{s_2})\cdot\boldsymbol{n}^2 = Fs\,(\rho^{s_2}-Ds\,), \tag{9.1}$$

and we combine it with the transport equation for surface 2, (2.56), we have

$$(\boldsymbol{u}-\boldsymbol{v}^{s_2})\cdot\boldsymbol{n}^2 + Ls\,\{\partial_t\rho^{s_2}+\rho^{s_2}\nabla^s\cdot\boldsymbol{c}+\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]\}=0, \tag{9.2}$$

where $Ls\,=Fs\,Ts\,=\rho/(\rho^s_{(0)}L)$. Treatment of terms that involve Ls is the same as the one given in condition DTC2 with terms involving Ts.

We thus have

$$\begin{aligned}(u-u^{s_2})n_r^2+(w-w^{s_2})n_z^2 Ls\,\partial_t\rho^{s_2}+Ls\,\rho^{s_2}t_r^2\partial_s\partial_t r^c\\ +Ls\,\rho^{s_2}t_z^2\partial_s\partial_t z^c+Ls\,\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=0,\end{aligned} \tag{9.3}$$

and define the i-th MEC2 residual as

$$\begin{aligned}E_i^2 = Ls\int\limits_{\partial\Omega^2}\phi_i^2\partial_t\rho^{s_2}+Ls\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s\partial_t r^c+Ls\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s\partial_t z^c\\ +Ls\int\limits_{\partial\Omega^2}\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]+\int\limits_{\partial\Omega^2}\phi_i^2 u n_r^2+\int\limits_{\partial\Omega^2}\phi_i^2 w n_z^2-\int\limits_{\partial\Omega^2}\phi_i^2 u^{s_2}n_r^2-\int\limits_{\partial\Omega^2}\phi_i^2 w^{s_2}n_z^2,\end{aligned} \tag{9.4}$$

where i is an index that runs through the boundary 2 node numbering.

We consider now the term

$$Ls\int\limits_{\partial\Omega^2}\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})], \tag{9.5}$$

and we recall the vector calculus identity

$$\nabla^s\cdot(\phi\boldsymbol{A})=\boldsymbol{A}\cdot\nabla^s\phi+\phi\nabla^s\cdot\boldsymbol{A} \tag{9.6}$$

Using this identity with $\phi=\phi_i^2$ and $\boldsymbol{A}=\rho^{s_2}\left(\boldsymbol{v}^{s-2}-\boldsymbol{c}\right)$, we have

$$\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\right]=\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2+\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]\,, \tag{9.7}$$

i.e.

$$\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2+\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\right], \tag{9.8}$$

equivalently

$$\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\right]-\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2, \tag{9.9}$$

i.e.

$$\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\right]-\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2. \tag{9.10}$$

We now separate the normal and tangential components of $\boldsymbol{v}^{s_2}$ and $\boldsymbol{c}$, obtaining

$$\begin{aligned}\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\left(\boldsymbol{v}_\parallel^{s_2}-\boldsymbol{c}_\parallel\right)\right]+\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\,\underbrace{(v_\perp^{s_2}-c_\perp)}_{=0}\,\boldsymbol{n}^2\right]\\ -\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2,\end{aligned} \tag{9.11}$$

where the underbraced factor is equal to zero by the impermeability condition.

Taking this into the integral above, we have

$$Ls \int\limits_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = -Ls \int\limits_{C^2} \boldsymbol{m}^2 \cdot \left[\phi_i^2 \rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] - Ls \int\limits_{\partial\Omega^2} \rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right) \cdot \nabla^s \phi_i^2, \tag{9.12}$$

where we have applied the surface divergence theorem to the first term on the right-hand side above.

Here we notice that in the 2D case which we are considering, the boundary of $\partial\Omega^2$, given by C^2 is simply the end points of boundary 2, where the appropriate conditions are to be applied.

$$\begin{aligned} Ls \int\limits_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = &-Ls \int\limits_{\partial\Omega^2} \rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right) \cdot \nabla^s \phi_i^2 \\ &- Ls\, \phi_i^2(c) \rho_c^{s_2}\left(\boldsymbol{v}_c^{s_2} \cdot \boldsymbol{m}_c^2 - \boldsymbol{c}_c \cdot \boldsymbol{m}_c^2\right) \\ &\underbrace{-Ls\, \phi_i^2(o) \rho_o^{s_2} \boldsymbol{v}_o^{s_2} \cdot \boldsymbol{m}_o^2 + Ls\, \phi_i^2(o) \rho_o^{s_2} \boldsymbol{c}_o \cdot \boldsymbol{m}_o^2}_{=0}, \end{aligned} \tag{9.13}$$

where the o sub-index stands for the origin, where there velocity of the coordinates and the surface are both zero. This yields

$$\begin{aligned} Ls \int\limits_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = &-Ls \int\limits_{\partial\Omega^2} \rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right) \cdot \nabla^s \phi_i^2 \\ &- Ls\, \phi_i^2(c) \rho_c^{s_2} \boldsymbol{v}_c^{s_2} \cdot \boldsymbol{m}_c^2 + Ls\, \phi_i^2(c) \rho_c^{s_2} \boldsymbol{c}_c \cdot \boldsymbol{m}_c^2, \end{aligned} \tag{9.14}$$

We notice here that we have not decomposed this equation into two parts (near-field and far-field), as it does not involve the bulk velocity variables, which are the only ones that require a separate treatment.

Re-writing the expression above we have

$$\begin{aligned} Ls \int\limits_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = &-Ls \int\limits_{\partial\Omega^2} \rho^{s_2}\left(\partial_s \phi_i^2\right)\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right) \cdot \boldsymbol{t}^2 \\ &- Ls\, \delta_{i,c} \rho_c^{s_2} u_c^{s_2} m_r^2(c) - Ls\, \delta_{i,c} \rho_c^{s_2} w_c^{s_2} m_z^2(c) \\ &+ Ls\, \delta_{i,c} \rho_c^{s_2} m_r^2(c) \partial_t r_c^c + Ls\, \delta_{i,c} \rho_c^{s_2} m_z^2(c) \partial_t z_c^c, \end{aligned} \tag{9.15}$$

i.e.

$$\begin{aligned} Ls \int\limits_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = &-Ls \int\limits_{\partial\Omega^2} \rho^{s_2}\left(\partial_s \phi_i^2\right) \boldsymbol{v}^{s_2} \cdot \boldsymbol{t}^2 + Ls \int\limits_{\partial\Omega^2} \rho^{s_2}\left(\partial_s \phi_i^2\right) \boldsymbol{c} \cdot \boldsymbol{t}^2 \\ &- Ls\, \delta_{i,c} \rho_c^{s_2} u_c^{s_2} m_r^2(c) - Ls\, \delta_{i,c} \rho_c^{s_2} w_c^{s_2} m_z^2(c) \\ &+ Ls\, \delta_{i,c} \rho_c^{s_2} m_r^2(c) \partial_t r_c^c + Ls\, \delta_{i,c} \rho_c^{s_2} m_z^2(c) \partial_t z_c^c, \end{aligned} \tag{9.16}$$

which is

$$
\begin{aligned}
Ls \int\limits_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2} \left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = & -Ls \int\limits_{\partial\Omega^2} \rho^{s_2} u^{s_2} t_r^2 \partial_s \phi_i^2 - Ls \int\limits_{\partial\Omega^2} \rho^{s_2} w^{s_2} t_z^2 \partial_s \phi_i^2 \\
& + Ls \int\limits_{\partial\Omega^2} \rho^{s_2} t_r^2 \left(\partial_s \phi_i^2\right) \partial_t r^c + Ls \int\limits_{\partial\Omega^2} \rho^{s_2} t_z^2 \left(\partial_s \phi_i^2\right) \partial_t z^c \\
& - Ls\, \delta_{i,c} \rho_c^{s_2} u_c^{s_2} m_r^2(c) - Ls\, \delta_{i,c} \rho_c^{s_2} w_c^{s_2} m_z^2(c) \\
& + Ls\, \delta_{i,c} \rho_c^{s_2} m_r^2(c) \partial_t r_c^c + Ls\, \delta_{i,c} \rho_c^{s_2} m_z^2(c) \partial_t z_c^c.
\end{aligned}
\tag{9.17}
$$

We substitute this into the residual equation obtaining

$$
\begin{aligned}
E_i^2 = & Ls \int\limits_{\partial\Omega^2} \phi_i^2 \partial_t \rho^{s_2} + Ls \int\limits_{\partial\Omega^2} \phi_i^2 \rho^{s_2} t_r^2 \partial_s \partial_t r^c + Ls \int\limits_{\partial\Omega^2} \phi_i^2 \rho^{s_2} t_z^2 \partial_s \partial_t z^c - Ls \int\limits_{\partial\Omega^2} \rho^{s_2} u^{s_2} t_r^2 \partial_s \phi_i^2 \\
& - Ls \int\limits_{\partial\Omega^2} \rho^{s_2} w^{s_2} t_z^2 \partial_s \phi_i^2 + Ls \int\limits_{\partial\Omega^2} \rho^{s_2} t_r^2 \left(\partial_s \phi_i^2\right) \partial_t r^c + Ls \int\limits_{\partial\Omega^2} \rho^{s_2} t_z^2 \left(\partial_s \phi_i^2\right) \partial_t z^c \\
& - Ls\, \delta_{i,c} \rho_c^{s_2} u_c^{s_2} m_r^2(c) - Ls\, \delta_{i,c} \rho_c^{s_2} w_c^{s_2} m_z^2(c) + Ls\, \delta_{i,c} \rho_c^{s_2} m_r^2(c) \partial_t r_c^c \\
& + Ls\, \delta_{i,c} \rho_c^{s_2} m_z^2(c) \partial_t z_c^c + \int\limits_{\partial\Omega^2} \phi_i^2 u n_r^2 + \int\limits_{\partial\Omega^2} \phi_i^2 w n_z^2 - \int\limits_{\partial\Omega^2} \phi_i^2 u^{s_2} n_r^2 - \int\limits_{\partial\Omega^2} \phi_i^2 w^{s_2} n_z^2,
\end{aligned}
\tag{9.18}
$$

We now recall the approximation

$$
\partial_t r^c \approx \frac{3r^c - 4r^c(t_{n-1}) + r^c(t_{n-2})}{2\Delta_t} \tag{9.19}
$$

and

$$
\partial_t z^c \approx \frac{3z^c - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t}; \tag{9.20}
$$

and we introduce

$$
\partial_t \rho^{s_2} \approx \frac{3\rho^{s_2} - 4\rho^{s_2}(t_{n-1}) + \rho^{s_2}(t_{n-2})}{2\Delta_t}. \tag{9.21}
$$

Substituting these approximations in the residual equation we have

$$
\begin{aligned}
E_i^2 = & -Ls\,\delta_{i,c}\rho_c^{s_2}u_c^{s_2}m_r^2(c) - Ls\,\delta_{i,c}\rho_c^{s_2}w_c^{s_2}m_z^2(c) \\
& + Ls\,\delta_{i,c}\rho_c^{s_2}m_r^2(c)\frac{3r_c^c - 4r_c^c(t_{n-1}) + r_c^c(t_{n-2})}{2\Delta_t} \\
& + Ls\,\delta_{i,c}\rho_c^{s_2}m_z^2(c)\frac{3z^c - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t} \\
& - Ls\int\limits_{\partial\Omega^2}\rho^{s_2}u^{s_2}t_r^2\partial_s\phi_i^2 - Ls\int\limits_{\partial\Omega^2}\rho^{s_2}w^{s_2}t_z^2\partial_s\phi_i^2 \\
& + Ls\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2\frac{3r^c - 4r^c(t_{n-1}) + r^c(t_{n-2})}{2\Delta_t}\partial_s\phi_i^2 \\
& + Ls\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2\frac{3z^c - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t}\partial_s\phi_i^2 \\
& + Ls\int\limits_{\partial\Omega^2}\phi_i^2\frac{3\rho^{s_2} - 4\rho^{s_2}(t_{n-1}) + \rho^{s_2}(t_{n-2})}{2\Delta_t} \\
& + Ls\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\frac{3\partial_s r^c - 4\partial_s r^c(t_{n-1}) + \partial_s r^c(t_{n-2})}{2\Delta_t} \\
& + Ls\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\frac{3\partial_s z^c - 4\partial_s z^c(t_{n-1}) + \partial_s z^c(t_{n-2})}{2\Delta_t} \\
& + \int\limits_{\partial\Omega^2}\phi_i^2 u n_r^2 + \int\limits_{\partial\Omega^2}\phi_i^2 w n_z^2 - \int\limits_{\partial\Omega^2}\phi_i^2 u^{s_2}n_r^2 - \int\limits_{\partial\Omega^2}\phi_i^2 w^{s_2}n_z^2.
\end{aligned}
\tag{9.22}
$$

Multiplying the residual equation by $2\Delta_t/3$ we have

$$
\begin{aligned}
\mathscr{E}_i^2 = &-\frac{2\Delta_t Ls}{3}\delta_{i,c}\rho_c^{s_2}u_c^{s_2}m_r^2(c) - \frac{2\Delta_t Ls}{3}\delta_{i,c}\rho_c^{s_2}w_c^{s_2}m_z^2(c) \\
&+ Ls\,\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c - \frac{4Ls}{3}\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c(t_{n-1}) + \frac{Ls}{3}\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c(t_{n-2}) \\
&+ Ls\,\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c - \frac{4Ls}{3}\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c(t_{n-1}) + \frac{Ls}{3}\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c(t_{n-2}) \\
&+ Ls\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2} - \frac{4Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}(t_{n-1}) + \frac{Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}(t_{n-2}) \\
&- \frac{2\Delta_t Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}u^{s_2}t_r^2\partial_s\phi_i^2 - \frac{2\Delta_t Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}w^{s_2}t_z^2\partial_s\phi_i^2 \\
&+ Ls\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2r^c\partial_s\phi_i^2 - \frac{4Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2r^c(t_{n-1})\partial_s\phi_i^2 + \frac{Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2r^c(t_{n-2})\partial_s\phi_i^2 \\
&+ Ls\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2z^c\partial_s\phi_i^2 - \frac{4Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2z^c(t_{n-1})\partial_s\phi_i^2 + \frac{Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2z^c(t_{n-2})\partial_s\phi_i^2 \\
&+ Ls\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c - \frac{4Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c(t_{n-1}) + \frac{Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c(t_{n-2}) \\
&+ Ls\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c - \frac{4Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c(t_{n-1}) + \frac{Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c(t_{n-2}) \\
&+ \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^2}\phi_i^2un_r^2 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^2}\phi_i^2wn_z^2 - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^2}\phi_i^2u^{s_2}n_r^2 - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^2}\phi_i^2w^{s_2}n_z^2.
\end{aligned}
\tag{9.23}
$$

We now introduce the decomposition

$$
\begin{aligned}
\mathscr{E}_i^2 = &-\frac{2\Delta_t Ls}{3}\delta_{i,c}\rho_c^{s_2}u_c^{s_2}m_r^2(c) - \frac{2\Delta_t Ls}{3}\delta_{i,c}\rho_c^{s_2}w_c^{s_2}m_z^2(c) \\
&+ Ls\,\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c - \frac{4Ls}{3}\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c(t_{n-1}) + \frac{Ls}{3}\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c(t_{n-2}) \\
&+ Ls\,\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c - \frac{4Ls}{3}\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c(t_{n-1}) + \frac{Ls}{3}\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c(t_{n-2}) \\
&+ \mathscr{E}_i^{2,b} + \mathscr{E}_i^{2,c},
\end{aligned}
\tag{9.24}
$$

where

$$
\begin{aligned}
\mathscr{E}_i^{2,b} = &\,Ls\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2} - \frac{4Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}(t_{n-1}) + \frac{Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}(t_{n-2}) \\
&+ \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^2}\phi_i^2un_r^2 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^2}\phi_i^2wn_z^2 - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^2}\phi_i^2u^{s_2}n_r^2 - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^2}\phi_i^2w^{s_2}n_z^2,
\end{aligned}
\tag{9.25}
$$

and

$$\begin{aligned}
\mathscr{E}_i^{2,c} = &-\frac{2\Delta_t Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}u^{s_2}t_r^2\partial_s\phi_i^2 - \frac{2\Delta_t Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}w^{s_2}t_z^2\partial_s\phi_i^2\\
&+ Ls\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2 r^c\partial_s\phi_i^2 - \frac{4Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2 r^c(t_{n-1})\partial_s\phi_i^2 + \frac{Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2 r^c(t_{n-2})\partial_s\phi_i^2\\
&+ Ls\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2 z^c\partial_s\phi_i^2 - \frac{4Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2 z^c(t_{n-1})\partial_s\phi_i^2 + \frac{Ls}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2 z^c(t_{n-2})\partial_s\phi_i^2\\
&+ Ls\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c - \frac{4Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c(t_{n-1}) + \frac{Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c(t_{n-2})\\
&+ Ls\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c - \frac{4Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c(t_{n-1}) + \frac{Ls}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c(t_{n-2}).
\end{aligned} \tag{9.26}$$

We recall approximations

$$u \approx \sum_{j=1}^{n_v} u_j\phi_j, \tag{9.27}$$

$$w \approx \sum_{j=1}^{n_v} w_j\phi_j, \tag{9.28}$$

$$u^{s_2} \approx \sum_{j=1}^{n_v} u_j^{s_2}\phi_j^2, \tag{9.29}$$

$$w^{s_2} \approx \sum_{j=1}^{n_v} w_j^{s_2}\phi_j \tag{9.30}$$

and

$$\rho^{s_2} \approx \sum_{j=1}^{n_v} \rho_j^{s_2}\phi_j. \tag{9.31}$$

Furthermore, we recall

$$\rho^{s_2} \approx \sum_{j=1}^{n_v} \rho_j^{s_2}\phi_j^2, \tag{9.32}$$

$$\rho^{s_2}(t_{n-1}) \approx \sum_{j=1}^{n_v} \rho_j^{s_2}(t_{n-1})\phi_j^2, \tag{9.33}$$

$$\rho^{s_2}(t_{n-2}) \approx \sum_{j=1}^{n_v} \rho_j^{s_2}(t_{n-2})\phi_j^2, \tag{9.34}$$

$$r^c \approx \sum_{j=1}^{n_v} r_j^c\phi_j, \tag{9.35}$$

$$r^c(t_{n-1}) \approx \sum_{j=1}^{n_v} r_j^c(t_{n-1})\phi_j, \tag{9.36}$$

$$r^c(t_{n-2}) \approx \sum_{j=1}^{n_v} r_j^c(t_{n-2})\phi_j, \tag{9.37}$$

$$z^c \approx \sum_{j=1}^{n_v} z_j^c\phi_j \tag{9.38}$$

$$z^c(t_{n-1}) \approx \sum_{j=1}^{n_v} z_j^c(t_{n-1})\phi_j \tag{9.39}$$

$$z^c(t_{n-2}) \approx \sum_{j=1}^{n_v} z_j^c(t_{n-2})\phi_j. \tag{9.40}$$

We take these approximation into the residual equation above and obtain

$$\begin{aligned}\mathcal{E}_i^2 = &\int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} u_j\phi_j\right) n_r^2 + \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} w_j\phi_j\right) n_z^2 - \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} u_j^{s_2}\phi_j^2\right) n_r^2 \\ &- \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} w_j^{s_2}\phi_j\right) n_z^2 - Fs \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} \rho_j^{s_2}\phi_j\right) + Fs\,Ds \int\limits_{\partial\Omega^2} \phi_i^2.\end{aligned} \tag{9.41}$$

Moving the integrals into the sums and re-arranging terms we have

$$\begin{aligned}\mathcal{E}_i^2 = &Fs\,Ds \int\limits_{\partial\Omega^2} \phi_i^2 + \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^2} \phi_i^2\phi_j^2 n_r^2 + \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^2} \phi_i^2\phi_j^2 n_z^2 \\ &- \sum_{j=1}^{n_v} u_j^{s_2} \int\limits_{\partial\Omega^2} \phi_i^2\phi_j^2 n_r^2 - \sum_{j=1}^{n_v} w_j^{s_2} \int\limits_{\partial\Omega^2} \phi_i^2\phi_j^2 n_z^2 - Fs \sum_{j=1}^{n_v} \rho_j^{s_2} \int\limits_{\partial\Omega^2} \phi_i^2\phi_j^2.\end{aligned} \tag{9.42}$$

Decomposing the integrals into sums of integrals over each line-element and using local node-numbers we have

$$\mathcal{E}_i^2 = \sum_{e_2=1}^{n_{\mathrm{el}}^2} \mathcal{E}_{e_2,ii}^2, \tag{9.43}$$

where

$$
\begin{aligned}
\mathcal{E}^2_{e_2,ii} = Fs\,Ds \underbrace{\int\limits_{\partial\Omega^2} \phi^2_{l_2(e_2,ii)}}_{d_{ii}(e_2)} + \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^2} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)} n_r^2}_{d_{ii,jj,n_r}(e_2)} \\
+ \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^2} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)} n_z^2}_{d_{ii,jj,n_z}(e_2)} - \sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^2} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)} n_r^2}_{d_{ii,jj,n_r}(e_2)} \\
- \sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^2} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)} n_z^2}_{d_{ii,jj,n_z}(e_2)} - Fs \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^2} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)}}_{d_{ii,jj}(e_2)},
\end{aligned}
\tag{9.44}
$$

i.e.

$$
\begin{aligned}
\mathcal{E}^2_{e_2,ii} = Fs\,Ds\,d_{ii}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) \\
+ \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)} d_{ii,jj,n_z}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) \\
- \sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e_2) - Fs \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj}(e_2).
\end{aligned}
\tag{9.45}
$$

Summarising and re-arranging terms we have

$$
\mathcal{E}^2_i = \sum_{e_2=1}^{n^2_{\mathrm{el}}} \mathcal{E}^2_{e_2,ii},
\tag{9.46}
$$

where

$$
\begin{aligned}
\mathcal{E}^2_{e_2,ii} = Fs\,Ds\,d_{ii}(e_2) \\
+ \sum_{jj=1}^{n_v^{2,e_2}} \Big\{ d_{ii,jj,n_r}(e_2) \left[u_{l_2(e_2,jj)} - u^{s_2}_{l_2^2(e_2,jj)} \right] \\
+ d_{ii,jj,n_z}(e_2) \left[w_{l_2(e_2,jj)} - w^{s_2}_{l_2^2(e_2,jj)} \right] \\
- Fs\,\rho^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj}(e_2) \Big\}.
\end{aligned}
\tag{9.47}
$$

9.1. **Jacobian terms**

We now calculate the derivatives of $\mathcal{E}_i^2$ with respect to u_q, w_q, $u_q^{s_2}$, $w_q^{s_2}$, $\rho_q^{s_2}$ and h_q.

9.1.1. **Derivatives of $\mathcal{E}_i^2$ with respect to u_q**

Using equation (9.43) we have

$$\partial_{u_q}\mathcal{E}_i^2 = \sum_{e_2=1}^{n_{\text{el}}^2} \partial_{u_q}\mathcal{E}_{e_2,ii}^2, \tag{9.48}$$

and from equation (9.45) we have

$$\begin{aligned}\partial_{u_q}\mathcal{E}_{e_2,ii}^2 = {} & Fs\,Ds\,\partial_{u_q}d_{ii}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q}u_{l_2(e_2,jj)}d_{ii,jj,n_r}(e_2) \\ & + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q}w_{l_2(e_2,jj)}d_{ii,jj,n_z}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q}u^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,n_r}(e_2) \\ & - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q}w^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,n_z}(e_2) - Fs\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q}\rho^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj}(e_2),\end{aligned} \tag{9.49}$$

i.e.

$$\partial_{u_q}\mathcal{E}_{e_2,ii}^2 = d_{ii,jj,n_r}(e_2)|_{q=l_2(e_2,jj)}. \tag{9.50}$$

9.1.2. **Derivatives of $\mathcal{E}_i^2$ with respect to w_q**

Using equation (9.43) we have

$$\partial_{w_q}\mathcal{E}_i^2 = \sum_{e_2=1}^{n_{\mathrm{el}}^2} \partial_{w_q}\mathcal{E}_{e_2,ii}^2, \tag{9.51}$$

and from equation (9.45) we have

$$\begin{aligned}\partial_{w_q}\mathcal{E}_{e_2,ii}^2 = Fs\,Ds\,\partial_{w_q}d_{ii}(e_2) &+ \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q}u_{l_2(e_2,jj)}d_{ii,jj,n_r}(e_2)\\ &+ \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q}w_{l_2(e_2,jj)}d_{ii,jj,n_z}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q}u^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,n_r}(e_2)\\ &- \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q}w^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,n_z}(e_2) - Fs\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q}\rho^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj}(e_2),\end{aligned} \tag{9.52}$$

i.e.

$$\partial_{w_q}\mathcal{E}_{e_2,ii}^2 = d_{ii,jj,n_z}(e_2)|_{q=l_2(e_2,jj)}. \tag{9.53}$$

9.1.3. Derivatives of $\mathcal{E}_i^2$ with respect to $u_q^{s_2}$

Using equation (9.43) we have

$$\partial_{u_q^{s_2}} \mathcal{E}_i^2 = \sum_{e_2=1}^{n_{\mathrm{el}}^2} \partial_{u_q^{s_2}} \mathcal{E}_{e_2,ii}^2, \tag{9.54}$$

and from equation (9.45) we have

$$\begin{aligned}\partial_{u_q^{s_2}} \mathcal{E}_{e_2,ii}^2 = Fs\, Ds\, \partial_{u_q^{s_2}} d_{ii}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} u_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) \\ + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} w_{l_2(e_2,jj)} d_{ii,jj,n_z}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) \\ - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e_2) - Fs \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj}(e_2),\end{aligned} \tag{9.55}$$

i.e.

$$\partial_{u_q^{s_2}} \mathcal{E}_{e_2,ii}^2 = -d_{ii,jj,n_r}(e_2)|_{q=l_2(e_2,jj)}. \tag{9.56}$$

9.1.4. Derivatives of $\mathcal{E}_i^2$ with respect to $w_q^{s_2}$

Using equation (9.43) we have

$$\partial_{w_q^{s_2}} \mathcal{E}_i^2 = \sum_{e_2=1}^{n_{\mathrm{el}}^2} \partial_{w_q^{s_2}} \mathcal{E}_{e_2,ii}^2, \tag{9.57}$$

and from equation (9.45) we have

$$\begin{aligned}
\partial_{w_q^{s_2}} \mathcal{E}_{e_2,ii}^2 = Fs\, Ds\, \partial_{w_q^{s_2}} d_{ii}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} u_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) \\
+ \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} w_{l_2(e_2,jj)} d_{ii,jj,n_z}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) \\
- \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e_2) - Fs \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj}(e_2),
\end{aligned} \tag{9.58}$$

i.e.

$$\partial_{w_q^{s_2}} \mathcal{E}_{e_2,ii}^2 = -d_{ii,jj,n_z}(e_2)|_{q=l_2(e_2,jj)}. \tag{9.59}$$

9.1.5. Derivatives of $\mathcal{E}_i^2$ with respect to $\rho_q^{s_2}$

Using equation (9.43) we have

$$\partial_{\rho_q^{s_2}} \mathcal{E}_i^2 = \sum_{e_2=1}^{n_{\mathrm{el}}^2} \partial_{\rho_q^{s_2}} \mathcal{E}_{e_2,ii}^2, \tag{9.60}$$

and from equation (9.45) we have

$$\begin{aligned}\partial_{\rho_q^{s_2}} \mathcal{E}_{e_2,ii}^2 = {} & Fs\, Ds\, \partial_{\rho_q^{s_2}} d_{ii}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} u_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) \\ & + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} w_{l_2(e_2,jj)} d_{ii,jj,n_z}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) \\ & - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e_2) - Fs \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj}(e_2),\end{aligned} \tag{9.61}$$

i.e.

$$\partial_{\rho_q^{s_2}} \mathcal{E}_{e_2,ii}^2 = -Fs\, d_{ii,jj}(e_2)|_{q=l_2(e_2,jj)}. \tag{9.62}$$

9.1.6. **Derivatives of $\mathcal{E}_i^2$ with respect to h_q**

Using equation (9.43) we have

$$\partial_{h_q}\mathcal{E}_i^2 = \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\rm el}^2} \partial_{h_{S_2(e_2,qq)}}\mathcal{E}_{e_2,ii}^2, \tag{9.63}$$

and from equation (9.45) we have

$$\begin{aligned}
\partial_{h_{S_2(e_2,qq)}}\mathcal{E}_{e_2,ii}^2 = Fs\,Ds\,\partial_{h_{S_2(e_2,qq)}}d_{ii}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2)\\
+\sum_{jj=1}^{n_v^{2,e_2}} w_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_z}(e_2)\\
-\sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2)\\
-\sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_z}(e_2)\\
-Fs\sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj}(e_2),
\end{aligned} \tag{9.64}$$

i.e.

$$\begin{aligned}
\partial_{h_{S_2(e_2,qq)}}\mathcal{E}_{e_2,ii}^2 = Fs\,Ds\,\partial_{h_{S_2(e_2,qq)}}d_{ii}(e_2)\\
+\sum_{jj=1}^{n_v^{2,e_2}}\Big[\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2)\left\{u_{l_2(e_2,jj)} - u^{s_2}_{l_2^2(e_2,jj)}\right\}\\
+\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_z}(e_2)\left\{w_{l_2(e_2,jj)} - w^{s_2}_{l_2^2(e_2,jj)}\right\}\\
-Fs\,\rho^{s_2}_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj}(e_2)\Big].
\end{aligned} \tag{9.65}$$

10. The density transport equation on boundary 2 (DTC2)

Derivations for this equation in the far field are identical to those in the near field, so we refer the reader to section 31

11. The $\sigma - \rho$ state equation on boundary 2 (TDC2)

We recall equation (2.54), which states the dependence of surface tension on density TDC2, given by

$$\sigma^2 = Cs\,(1 - \rho^{s_2})\,. \tag{11.1}$$

The i-th residual for TDC2 is given by

$$T_i^2 = \sigma_i^2 + Cs\,\rho_i^{s_2} - Cs\,. \tag{11.2}$$

11.1. Jacobian terms

Here we find the derivatives of T_i^2 with respect to σ_q^2 and ρ_q^2.

11.1.1. Derivatives of T_i^2 with respect to σ_q^2

$$\partial_{\sigma_q^2} T_i^2 = \delta_{q,i}. \tag{11.3}$$

11.1.2. Derivatives of T_i^2 with respect to ρ_q^2

$$\partial_{\rho_q^{s_2}} T_i^2 = Cs\,\delta_{q,i}. \tag{11.4}$$

12. The slip condition equation on boundary 1 (SC1)

We recall equation (2.47), which states

$$(\boldsymbol{v}^{s_1} - \boldsymbol{u}) \cdot \left(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1\right) = \frac{1 + 4Eg\,Bg}{4Bg} \nabla^s \sigma^1. \tag{12.1}$$

We define the i-th SC1 residual as

$$S_i^1 = \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 \left(\boldsymbol{v}^{s_1} - \boldsymbol{u}\right) \cdot \boldsymbol{t}^1 - \frac{1 + 4Eg\,Bg}{4Bg} \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 \boldsymbol{t}^1 \cdot \nabla^s \sigma^1, \tag{12.2}$$

i.e.

$$S_i^1 = \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 \boldsymbol{v}^{s_1} \cdot \boldsymbol{t}^1 - \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 \boldsymbol{u} \cdot \boldsymbol{t}^1 - \frac{1 + 4Eg\,Bg}{4Bg} \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 \left(\partial_s \sigma^1\right) \boldsymbol{t}^1 \cdot \boldsymbol{t}^1, \tag{12.3}$$

equivalently

$$S_i^1 = \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 u^{s_1} t_r^1 + \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 w^{s_1} t_z^1 - \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 u t_r^1 - \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 w t_z^1 - \frac{1 + 4Eg\,Bg}{4Bg} \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 \partial_s \sigma^1. \tag{12.4}$$

We consider the last integral on the right hand side above and we integrate by parts to obtain

$$- \int\limits_{\partial\Omega^{1,\mathrm{f}}} \phi_i^1 \partial_s \sigma^1 = -\phi_i^1 \sigma^1 \Big|_{(r_J, z_J)}^{(r_a, z_a)} + \int\limits_{\partial\Omega^1} \sigma^1 \partial_s \phi_i^1. \tag{12.5}$$

This yields

$$\begin{aligned} S_i^1 = &\int\limits_{\partial\Omega^1} \phi_i^1 u^{s_1} t_r^1 + \int\limits_{\partial\Omega^1} \phi_i^1 w^{s_1} t_z^1 - \int\limits_{\partial\Omega^1} \phi_i^1 u t_r^1 - \int\limits_{\partial\Omega^1} \phi_i^1 w t_z^1 + \frac{1 + 4Eg\,Bg}{4Bg} \int\limits_{\partial\Omega^1} \sigma^1 \partial_s \phi_i^1 \\ &+ \frac{1 + 4Eg\,Bg}{4Bg} \phi_i^1(r_J, z_J) \sigma^1(r_J, z_J) - \frac{1 + 4Eg\,Bg}{4Bg} \phi_i^1(r_a, z_a) \sigma^1(r_a, z_a). \end{aligned} \tag{12.6}$$

We now recall the approximations

$$u \approx \sum_{j=1}^{n_v} u_j \phi_j, \tag{12.7}$$

$$w \approx \sum_{j=1}^{n_v} w_j \phi_j \tag{12.8}$$

and

$$\sigma^1 \approx \sum_{j=1}^{n_v} \sigma_j^1 \phi_j \tag{12.9}$$

and we introduce

$$u^{s_1} \approx \sum_{j=1}^{n_v} u_j^{s_1} \phi_j \tag{12.10}$$

and

$$w^{s_1} \approx \sum_{j=1}^{n_v} w_j^{s_1} \phi_j. \tag{12.11}$$

Substituting these approximations into the residual equation we have

$$\begin{aligned}
\mathcal{S}_i^1 = & \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} u_j^{s_1}\phi_j\right) t_r^1 + \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} w_j^{s_1}\phi_j\right) t_z^1 \\
& - \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} u_j\phi_j\right) t_r^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} w_j\phi_j\right) t_z^1 \\
& + \frac{1+4Eg\,Bg}{4Bg} \int\limits_{\partial\Omega^1} \left(\sum_{j=1}^{n_v} \sigma_j^1\phi_j\right) \partial_s\phi_i^1 \\
& + \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\sigma^1(r_c,z_c) - \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\sigma^1(r_a,z_a).
\end{aligned} \tag{12.12}$$

Moving the integrals into the sum we have

$$\begin{aligned}
\mathcal{S}_i^1 = & \sum_{j=1}^{n_v} u_j^{s_1} \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 t_r^1 + \sum_{j=1}^{n_v} w_j^{s_1} \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 t_z^1 \\
& - \sum_{j=1}^{n_v} u_j \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 t_r^1 - \sum_{j=1}^{n_v} w_j \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 t_z^1 \\
& + \frac{1+4Eg\,Bg}{4Bg} \sum_{j=1}^{n_v} \sigma_j^1 \int\limits_{\partial\Omega^1} \phi_j^1\partial_s\phi_i^1.
\end{aligned} \tag{12.13}$$

Decomposing the integral into sums over line-elements and passing to local node numbers we have

$$\mathcal{S}_i^1 = \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\sigma^1(r_c,z_c) - \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\sigma^1(r_a,z_a) + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1} \mathcal{S}_{e_1,ii}^1, \tag{12.14}$$

where

$$\begin{aligned}
\mathcal{S}_{e_1,ii}^1 = & \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1^1(e_1,jj)}^{s_1} \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1\phi_{l_1(e_1,jj)}^1 t_r^1}_{c_{ii,jj,t_r}(e_1)} + \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1^1(e_1,jj)}^{s_1} \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1\phi_{l_1(e_1,jj)}^1 t_z^1}_{c_{ii,jj,t_z}(e_1)} \\
& - \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1(e_1,jj)} \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1\phi_{l_1(e_1,jj)}^1 t_r^1}_{c_{ii,jj,t_r}(e_1)} - \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1(e_1,jj)} \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1\phi_{l_1(e_1,jj)}^1 t_z^1}_{c_{ii,jj,t_z}(e_1)} \\
& + \frac{1+4Eg\,Bg}{4Bg} \sum_{jj=1}^{n_v^{1,e_1}} \sigma_{l_1^1(e_1,jj)}^1 \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \partial_s\phi_{l_1(e_1,jj)}^1}_{c_{jj,ii}^s(e_1)},
\end{aligned} \tag{12.15}$$

i.e.

$$\begin{aligned}
\mathcal{S}^1_{e_1,ii} = {} & \sum_{jj=1}^{n_v^{1,e_1}} u^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} w^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj,t_z}(e_1) \\
& - \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1(e_1,jj)} c_{ii,jj,t_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1(e_1,jj)} c_{ii,jj,t_z}(e_1) \\
& + \frac{1+4Eg\,Bg}{4Bg} \sum_{jj=1}^{n_v^{1,e_1}} \sigma^1_{l_1^1(e_1,jj)} c^s_{jj,ii}(e_1).
\end{aligned} \tag{12.16}$$

Summarising and re-arraging terms we have

$$\mathcal{S}^1_i = \frac{1+4Eg\,Bg}{4Bg} \phi^1_i(r_c,z_c)\sigma^1(r_c,z_c) - \frac{1+4Eg\,Bg}{4Bg} \phi^1_i(r_a,z_a)\sigma^1(r_a,z_a) + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n^1_{\mathrm{el}}} \mathcal{S}^1_{e_1,ii}, \tag{12.17}$$

where

$$\begin{aligned}
\mathcal{S}^1_{e_1,ii} = \sum_{jj=1}^{n_v^{1,e_1}} \Bigg\{ & c_{ii,jj,t_r}(e_1) \left[u^{s_1}_{l_1^1(e_1,jj)} - u_{l_1(e_1,jj)} \right] \\
& + c_{ii,jj,t_z}(e_1) \left[w^{s_1}_{l_1^1(e_1,jj)} - w_{l_1(e_1,jj)} \right] \\
& + \frac{1+4Eg\,Bg}{4Bg} \sigma^1_{l_1^1(e_1,jj)} c^s_{jj,ii}(e_1) \Bigg\}.
\end{aligned} \tag{12.18}$$

12.1. Jacobian terms

Here we find the derivatives of S_i^1 with respect to u_q, w_q, $u_q^{s_1}$, $w_q^{s_1}$, σ_q^1 and h_q.

12.1.1. Derivatives of S_i^1 with respect to u_q

Using equation (12.14) we have

$$\begin{aligned}\partial_{u_q}\mathcal{S}_i^1 =& \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{u_q}\sigma^1(r_c,z_c)\\ &-\frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{u_q}\sigma^1(r_a,z_a)+\sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{u_q}\mathcal{S}_{e_1,ii}^1.\end{aligned}\tag{12.19}$$

Form equation (12.16) we have

$$\begin{aligned}\partial_{u_q}\mathcal{S}_{e_1,ii}^1 =& \sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q}u^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,t_r}(e_1)+\sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q}w^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,t_z}(e_1)\\ &-\sum_{jj=1}^{n_v^{1,e_1}}c_{ii,jj,t_r}(e_1)\partial_{u_q}u_{l_1(e_1,jj)}-\sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q}w_{l_1(e_1,jj)}c_{ii,jj,t_z}(e_1)\\ &+\frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q}\sigma^1_{l_1^1(e_1,jj)}c^s_{jj,ii}(e_1),\end{aligned}\tag{12.20}$$

i.e.

$$\partial_{u_q}\mathcal{S}_{e_1,ii}^1 = -c_{ii,jj,t_r}(e_1)|_{q=l_1(e_1,jj)}.\tag{12.21}$$

12.1.2. Derivatives of S_i^1 with respect to w_q

Using equation (12.14) we have

$$\begin{aligned}\partial_{w_q}\mathcal{S}_i^1 = {}& \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{w_q}\sigma^1(r_c,z_c)\\ &-\frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{w_q}\sigma^1(r_a,z_a)+\sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{w_q}\mathcal{S}_{e_1,ii}^1.\end{aligned} \tag{12.22}$$

Form equation (12.16) we have

$$\begin{aligned}\partial_{w_q}\mathcal{S}_{e_1,ii}^1 = {}& \sum_{jj=1}^{n_v^{1,e_1}}\partial_{w_q}u_{l_1^1(e_1,jj)}^{s_1}c_{ii,jj,t_r}(e_1)+\sum_{jj=1}^{n_v^{1,e_1}}\partial_{w_q}w_{l_1^1(e_1,jj)}^{s_1}c_{ii,jj,t_z}(e_1)\\ &-\sum_{jj=1}^{n_v^{1,e_1}}c_{ii,jj,t_r}(e_1)\partial_{w_q}u_{l_1(e_1,jj)}-\sum_{jj=1}^{n_v^{1,e_1}}\partial_{w_q}w_{l_1(e_1,jj)}c_{ii,jj,t_z}(e_1)\\ &+\frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{w_q}\sigma_{l_1^1(e_1,jj)}^1c_{jj,ii}^s(e_1),\end{aligned} \tag{12.23}$$

i.e.

$$\partial_{w_q}\mathcal{S}_{e_1,ii}^1 = -c_{ii,jj,t_z}(e_1)|_{q=l_1(e_1,jj)}. \tag{12.24}$$

12.1.3. Derivatives of S_i^1 with respect to $u_q^{s_1}$

Using equation (12.14) we have

$$\begin{aligned} \partial_{u_q^{s_1}} \mathcal{S}_i^1 = & \frac{1+4Eg\,Bg}{4Bg} \phi_i^1(r_c,z_c) \partial_{u_q^{s_1}} \sigma^1(r_c,z_c) \\ & - \frac{1+4Eg\,Bg}{4Bg} \phi_i^1(r_a,z_a) \partial_{u_q^{s_1}} \sigma^1(r_a,z_a) + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_{u_q^{s_1}} \mathcal{S}_{e_1,ii}^1 . \end{aligned} \tag{12.25}$$

Form equation (12.16) we have

$$\begin{aligned} \partial_{u_q^{s_1}} \mathcal{S}_{e_1,ii}^1 = & \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} u_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} w_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,t_z}(e_1) \\ & - \sum_{jj=1}^{n_v^{1,e_1}} c_{ii,jj,t_r}(e_1) \partial_{u_q^{s_1}} u_{l_1(e_1,jj)} - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} w_{l_1(e_1,jj)} c_{ii,jj,t_z}(e_1) \\ & + \frac{1+4Eg\,Bg}{4Bg} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \sigma_{l_1^1(e_1,jj)}^1 c_{jj,ii}^s(e_1), \end{aligned} \tag{12.26}$$

i.e.

$$\partial_{u_q^{s_1}} \mathcal{S}_{e_1,ii}^1 = c_{ii,jj,t_r}(e_1)|_{q=l_1^1(e_1,jj)} . \tag{12.27}$$

12.1.4. Derivatives of S_i^1 with respect to $w_q^{s_1}$

Using equation (12.14) we have

$$\partial_{w_q^{s_1}} \mathcal{S}_i^1 = \frac{1+4Eg\,Bg}{4Bg} \phi_i^1(r_c, z_c) \partial_{w_q^{s_1}} \sigma^1(r_c, z_c) \\ - \frac{1+4Eg\,Bg}{4Bg} \phi_i^1(r_a, z_a) \partial_{w_q^{s_1}} \sigma^1(r_a, z_a) + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_{w_q^{s_1}} \mathcal{S}_{e_1,ii}^1. \tag{12.28}$$

Form equation (12.16) we have

$$\partial_{w_q^{s_1}} \mathcal{S}_{e_1,ii}^1 = \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} u_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} w_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,t_z}(e_1) \\ - \sum_{jj=1}^{n_v^{1,e_1}} c_{ii,jj,t_r}(e_1) \partial_{w_q^{s_1}} u_{l_1(e_1,jj)} - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} w_{l_1(e_1,jj)} c_{ii,jj,t_z}(e_1) \\ + \frac{1+4Eg\,Bg}{4Bg} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \sigma_{l_1^1(e_1,jj)}^1 c_{jj,ii}^s(e_1), \tag{12.29}$$

i.e.

$$\partial_{w_q^{s_1}} \mathcal{S}_{e_1,ii}^1 = c_{ii,jj,t_z}(e_1)|_{q=l_1^1(e_1,jj)}. \tag{12.30}$$

12.1.5. **Derivatives of S_i^1 with respect to σ_q^1**

Using equation (12.14) and local spine numbers we have

$$\begin{aligned}\partial_{\sigma_q^1}\mathcal{S}_i^1 &= \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{\sigma_q^1}\sigma^1(r_c,z_c)\\ &\quad-\frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{\sigma_q^1}\sigma^1(r_a,z_a)+\sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\text{el}}^1}\partial_{\sigma_q^1}\mathcal{S}_{e_1,ii}^1.\end{aligned}\tag{12.31}$$

Form equation (12.16) we have

$$\begin{aligned}\partial_{\sigma_q^1}\mathcal{S}_{e_1,ii}^1 &= \sum_{jj=1}^{n_v^{1,e_1}}\partial_{\sigma_q^1}u_{l_1^1(e_1,jj)}^{s_1}c_{ii,jj,t_r}(e_1)+\sum_{jj=1}^{n_v^{1,e_1}}\partial_{\sigma_q^1}w_{l_1^1(e_1,jj)}^{s_1}c_{ii,jj,t_z}(e_1)\\ &\quad-\sum_{jj=1}^{n_v^{1,e_1}}\partial_{\sigma_q^1}u_{l_1(e_1,jj)}c_{ii,jj,t_r}(e_1)-\sum_{jj=1}^{n_v^{1,e_1}}\partial_{\sigma_q^1}w_{l_1(e_1,jj)}c_{ii,jj,t_z}(e_1)\\ &\quad+\frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}}c_{jj,ii}^s(e_1)\partial_{\sigma_q^1}\sigma_{l_1^1(e_1,jj)}^1,\end{aligned}\tag{12.32}$$

i.e.

$$\sigma_{l_1^1(e_1,jj)}^1\mathcal{S}_{e_1,ii}^1=\frac{1+4Eg\,Bg}{4Bg}c_{jj,ii}^s(e_1)|_{q=L_1^1(e_1,jj)}.\tag{12.33}$$

12.1.6. Derivatives of S_i^1 with respect to h_q

Using equation (12.14) and local spine numbers we have

$$\begin{aligned}\partial_{h_q}\mathcal{S}_i^1 &= \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{h_q}\sigma^1(r_c,z_c)\\ &\quad - \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{h_q}\sigma^1(r_a,z_a) + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\mathrm{el}}^1} \partial_{h_{S_1(e_1,qq)}}\mathcal{S}_{e_1,ii}^1. \end{aligned} \tag{12.34}$$

Form equation (12.16) we have

$$\begin{aligned}\partial_{h_{S_1(e_1,qq)}}\mathcal{S}_{e_1,ii}^1 = &\sum_{jj=1}^{n_v^{1,e_1}} u_{l_1^1(e_1,jj)}^{s_1}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1^1(e_1,jj)}^{s_1}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_z}(e_1)\\ &- \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_z}(e_1)\\ &+ \frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}} \sigma_{l_1^1(e_1,jj)}^1\partial_{h_{S_1(e_1,qq)}}c_{jj,ii}^s(e_1),\end{aligned} \tag{12.35}$$

i.e.

$$\begin{aligned}\partial_{h_{S_1(e_1,qq)}}\mathcal{S}_{e_1,ii}^1 = \sum_{jj=1}^{n_v^{1,e_1}} \Bigg[&\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_r}(e_1)\left\{u_{l_1^1(e_1,jj)}^{s_1} - u_{l_1(e_1,jj)}\right\}\\ &+ \partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_z}(e_1)\left\{w_{l_1^1(e_1,jj)}^{s_1} - w_{l_1(e_1,jj)}\right\}\\ &- \frac{1+4Eg\,Bg}{4Bg}\sigma_{l_1^1(e_1,jj)}^1 \underbrace{\partial_{h_{S_1(e_1,qq)}}c_{jj,ii}^s(e_1)}_{=0}\Bigg].\end{aligned} \tag{12.36}$$

13. Kinematic boundary condition (KBC)

We consider equation (2.44) which states

$$(\boldsymbol{v}^{s_1} - \boldsymbol{c}) \cdot \boldsymbol{n}^1 = 0, \tag{13.1}$$

i.e.

$$u^{s_1} n_r^1 - u^c n_r^1 + w^{s_1} n_z^1 - w^c n_z^1 = 0, \tag{13.2}$$

and define

$$K_i = \int\limits_{\partial\Omega^{1,f}} \phi_i^1 u^{s_1} n_r^1 - \int\limits_{\partial\Omega^{1,f}} \phi_i^1 u^c n_r^1 + \int\limits_{\partial\Omega^{1,f}} \phi_i^1 w^{s_1} n_z^1 - \int\limits_{\partial\Omega^{1,f}} \phi_i^1 w^c n_z^1, \tag{13.3}$$

where i is an index that runs on the boundary 1 numbering of the free surface nodes.

we substitute approximations (4.54) and (4.55) and obtain

$$\begin{aligned} \mathfrak{K}_i = &\int\limits_{\partial\Omega^1} \phi_i^1 u^{s_1} n_r^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \frac{3r^c(t_n) - 4r^c(t_{n-1}) + r^c(t_{n-2})}{2\Delta_t} n_r^1 \\ &+ \int\limits_{\partial\Omega^1} \phi_i^1 w^{s_1} n_z^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \frac{3z^c(t_n) - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t} n_z^1, \end{aligned} \tag{13.4}$$

multiplying by $2\Delta_t/3$ we have

$$\begin{aligned} \mathscr{K}_i = &\frac{2\Delta_t}{3} \int\limits_{\partial\Omega^1} \phi_i^1 u^{s_1} n_r^1 - \int\limits_{\partial\Omega^1} \phi_i^1 r^c n_r^1 + \frac{4}{3} \int\limits_{\partial\Omega^1} \phi_i^1 r^c(t_{n-1}) n_r^1 - \frac{1}{3} \int\limits_{\partial\Omega^1} \phi_i^1 r^c(t_{n-2}) n_r^1 \\ &+ \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^1} \phi_i^1 w^{s_1} n_z^1 - \int\limits_{\partial\Omega^1} \phi_i^1 z^c n_z^1 + \frac{4}{3} \int\limits_{\partial\Omega^1} \phi_i^1 z^c(t_{n-1}) n_z^1 - \frac{1}{3} \int\limits_{\partial\Omega^1} \phi_i^1 z^c(t_{n-2}) n_z^1, \end{aligned} \tag{13.5}$$

We substitute approximations (4.66)-(4.69) into the equation above and obtain

$$\begin{aligned} \mathcal{K}_i = &\frac{2\Delta_t}{3} \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} u_j^{s_1} \phi_j^1 \right) n_r^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} r_j^c \phi_j^1 \right) n_r^1 \\ &+ \frac{4}{3} \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} r_j^c(t_{n-1}) \phi_j^1 \right) n_r^1 - \frac{1}{3} \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} r_j^c(t_{n-2}) \phi_j^1 \right) n_r^1 \\ &+ \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} w_j^{s_1} \phi_j^1 \right) n_z^1 - \int\limits_{\partial\Omega^1} \phi_i^1 z^c \left(\sum_{j=1}^{n_v} z_j^c \phi_j^1 \right) n_z^1 \\ &+ \frac{4}{3} \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} z_j^c(t_{n-1}) \phi_j^1 \right) n_z^1 - \frac{1}{3} \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} z_j^c(t_{n-2}) \phi_j^1 \right) n_z^1. \end{aligned} \tag{13.6}$$

Moving the integrals into the sums we have

$$
\begin{aligned}
\mathcal{K}_i = {} & \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} u_j^{s_1} \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 n_r^1 - \sum_{j=1}^{n_v} r_j^c \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 n_r^1 \\
& + \frac{4}{3}\sum_{j=1}^{n_v} r_j^c(t_{n-1}) \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 n_r^1 - \frac{1}{3}\sum_{j=1}^{n_v} r_j^c(t_{n-2}) \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 n_r^1 \\
& + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} w_j^{s_1} \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 n_z^1 - \sum_{j=1}^{n_v} z_j^c \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 n_z^1 \\
& + \frac{4}{3}\sum_{j=1}^{n_v} z_j^c(t_{n-1}) \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 n_z^1 - \frac{1}{3}\sum_{j=1}^{n_v} z_j^c(t_{n-2}) \int\limits_{\partial\Omega^1} \phi_i^1\phi_j^1 n_z^1.
\end{aligned}
\tag{13.7}
$$

Now, we decompose the integrals into the sum of the integrals over each line-element on boundary 1

$$
\begin{aligned}
\mathcal{K}_i = \sum_{e_1=1}^{n_{\mathrm{el}}^1} \Bigg[& \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} u_j^{s_1} \int\limits_{\Omega_{e_1}^1} \phi_i^1\phi_j^1 n_r^1 - \sum_{j=1}^{n_v} r_j^c \int\limits_{\Omega_{e_1}^1} \phi_i^1\phi_j^1 n_r^1 + \frac{4}{3}\sum_{j=1}^{n_v} r_j^c(t_{n-1}) \int\limits_{\Omega_{e_1}^1} \phi_i^1\phi_j^1 n_r^1 \\
& - \frac{1}{3}\sum_{j=1}^{n_v} r_j^c(t_{n-2}) \int\limits_{\Omega_{e_1}^1} \phi_i^1\phi_j^1 n_r^1 \\
& + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v} w_j^{s_1} \int\limits_{\Omega_{e_1}^1} \phi_i^1\phi_j^1 n_z^1 - \sum_{j=1}^{n_v} z_j^c \int\limits_{\Omega_{e_1}^1} \phi_i^1\phi_j^1 n_z^1 + \frac{4}{3}\sum_{j=1}^{n_v} z_j^c(t_{n-1}) \int\limits_{\Omega_{e_1}^1} \phi_i^1\phi_j^1 n_z^1 \\
& - \frac{1}{3}\sum_{j=1}^{n_v} z_j^c(t_{n-2}) \int\limits_{\Omega_{e_1}^1} \phi_i^1\phi_j^1 n_z^1 \Bigg],
\end{aligned}
\tag{13.8}
$$

and expressing sums over index j in terms of local node numbers we have

$$
\begin{aligned}
\mathcal{K}_i = \sum_{e_1=1}^{n_{\text{el}}^1} \Bigg[& \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1}} u^{s_1}_{l_1(e_1,jj)} \int\limits_{\Omega^1_{e_1}} \phi^1_i \phi^1_{l_1(e_1,jj)} n^1_r - \sum_{jj=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,jj)} \int\limits_{\Omega^1_{e_1}} \phi^1_i \phi^1_{l_1(e_1,jj)} n^1_r \\
& + \frac{4}{3} \sum_{jj=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,jj)}(t_{n-1}) \int\limits_{\Omega^1_{e_1}} \phi^1_i \phi^1_{l_1(e_1,jj)} n^1_r - \frac{1}{3} \sum_{jj=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,jj)}(t_{n-2}) \int\limits_{\Omega^1_{e_1}} \phi^1_i \phi^1_{l_1(e_1,jj)} n^1_r \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1}} w^{s_1}_{l_1(e_1,jj)} \int\limits_{\Omega^1_{e_1}} \phi^1_i \phi^1_{l_1(e_1,jj)} n^1_z - \sum_{jj=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,jj)} \int\limits_{\Omega^1_{e_1}} \phi^1_i \phi^1_{l_1(e_1,jj)} n^1_z \\
& + \frac{4}{3} \sum_{jj=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,jj)}(t_{n-1}) \int\limits_{\Omega^1_{e_1}} \phi^1_i \phi^1_{l_1(e_1,jj)} n^1_z \\
& - \frac{1}{3} \sum_{jj=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,jj)}(t_{n-2}) \int\limits_{\Omega^1_{e_1}} \phi^1_i \phi^1_{l_1(e_1,jj)} n^1_z \Bigg],
\end{aligned}
\tag{13.9}
$$

and passing to local node numbers for index i, we have

$$
\begin{aligned}
\mathcal{K}_i = \sum_{\substack{e_1=1 \\ i=l_1(e_1,ii)}}^{n_{\text{el}}^1} \Bigg[& \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1}} u^{s_1}_{l_1(e_1,jj)} \int\limits_{\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_r \\
& - \sum_{jj=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,jj)} \int\limits_{\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_r \\
& + \frac{4}{3} \sum_{jj=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,jj)}(t_{n-1}) \int\limits_{\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_r \\
& - \frac{1}{3} \sum_{jj=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,jj)}(t_{n-2}) \int\limits_{\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_r \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1}} w^{s_1}_{l_1(e_1,jj)} \int\limits_{\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_z - \sum_{jj=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,jj)} \int\limits_{\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_z \\
& + \frac{4}{3} \sum_{jj=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,jj)}(t_{n-1}) \int\limits_{\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_z \\
& - \frac{1}{3} \sum_{jj=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,jj)}(t_{n-2}) \int\limits_{\Omega^1_{e_1}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_z \Bigg].
\end{aligned}
\tag{13.10}
$$

Re-writing we have

$$\begin{aligned}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n^1_{\mathrm{el}}} \Bigg[&\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1}} u^{s_1}_{l_1(e_1,jj)} c_{ii,jj,n_r} - \sum_{jj=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,jj)} c_{ii,jj,n_r} \\
&+ \frac{4}{3}\sum_{jj=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,jj)}(t_{n-1}) c_{ii,jj,n_r} - \frac{1}{3}\sum_{jj=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,jj)}(t_{n-2}) c_{ii,jj,n_r} \\
&+ \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1}} w^{s_1}_{l_1(e_1,jj)} c_{ii,jj,n_z} - \sum_{jj=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,jj)} c_{ii,jj,n_z} + \frac{4}{3}\sum_{jj=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,jj)}(t_{n-1}) c_{ii,jj,n_z} \\
&- \frac{1}{3}\sum_{jj=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,jj)}(t_{n-2}) c_{ii,jj,n_z} \Bigg]. \end{aligned} \tag{13.11}$$

Re-arranging we have

$$\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n^1_{\mathrm{el}}} \mathcal{K}_{e_1,ii} \tag{13.12}$$

where

$$\begin{aligned}\mathcal{K}_{e_1,ii} = \sum_{jj=1}^{n_v^{1,e_1}} \Bigg[&c_{ii,jj,n_r}\left\{ \frac{2\Delta_t}{3} u^{s_1}_{l_1(e_1,jj)} - r^c_{l_1(e_1,jj)} + \frac{4}{3} r^c_{l_1(e_1,jj)}(t_{n-1}) - \frac{1}{3} r^c_{l_1(e_1,jj)}(t_{n-2}) \right\} \\
&+ c_{ii,jj,n_z}\left\{ \frac{2\Delta_t}{3} w^{s_1}_{l_1(e_1,jj)} - z^c_{l_1(e_1,jj)} + \frac{4}{3} z^c_{l_1(e_1,jj)}(t_{n-1}) - \frac{1}{3} z^c_{l_1(e_1,jj)}(t_{n-2}) \right\} \Bigg]. \end{aligned} \tag{13.13}$$

13.1. **Jacobian terms**

We now calculate the derivatives of $\mathcal{K}_i$ with respect to $u_q^{s_1}$, $w_q^{s_1}$ and h_q.

13.1.1. **Derivatives of $\mathcal{K}_i$ with respect to $u_q^{s_1}$**

We consider (13.11), from where we have

$$\begin{aligned}\partial_{u_q^{s_1}}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\text{el}}^1} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \Bigg[\frac{2\Delta_t}{3} \left(u_{l_1(e_1,jj)} c_{ii,jj,n_r} + w_{l_1(e_1,jj)} c_{ii,jj,n_z}\right) \\ - \left(r^c_{l_1(e_1,jj)} c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)} c_{ii,jj,n_z}\right) \\ + \frac{4}{3}\left(r^c_{l_1(e_1,jj)}(t_{n-1}) c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}(t_{n-1}) c_{ii,jj,n_z}\right) \\ - \frac{1}{3}\left(r^c_{l_1(e_1,jj)}(t_{n-2}) c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}(t_{n-2}) c_{ii,jj,n_z}\right)\Bigg],\end{aligned} \tag{13.14}$$

i.e.

$$\partial_{u_q^{s_1}}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\text{el}}^1} \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{1,e_1}} c_{ii,jj,n_r} \partial_{u_q^{s_1}} u^{s_1}_{l_1(e_1,jj)}, \tag{13.15}$$

i.e.

$$\boxed{\partial_{u_q^{s_1}}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)\\ q=l_1(e_1,jj)}}^{n_{\text{el}}^1} \frac{2\Delta_t}{3} c_{ii,jj,n_r}}. \tag{13.16}$$

13.1.2. Derivatives of $\mathcal{K}_i$ with respect to $w_q^{s_1}$

We consider (13.13), from where we have

$$\begin{aligned}\partial_{w_q^{s_1}}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\text{el}}^1}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{w_q^{s_1}}\Bigg[\frac{2\Delta_t}{3}\left(u_{l_1(e_1,jj)}c_{ii,jj,n_r}+w_{l_1(e_1,jj)}c_{ii,jj,n_z}\right)\\ -\left(r^c_{l_1(e_1,jj)}c_{ii,jj,n_r}+z^c_{l_1(e_1,jj)}c_{ii,jj,n_z}\right)\\ +\frac{4}{3}\left(r^c_{l_1(e_1,jj)}(t_{n-1})c_{ii,jj,n_r}+z^c_{l_1(e_1,jj)}(t_{n-1})c_{ii,jj,n_z}\right)\\ -\frac{1}{3}\left(r^c_{l_1(e_1,jj)}(t_{n-2})c_{ii,jj,n_r}+z^c_{l_1(e_1,jj)}(t_{n-2})c_{ii,jj,n_z}\right)\Bigg],\end{aligned} \tag{13.17}$$

i.e.

$$\partial_{w_q^{s_1}}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\text{el}}^1}\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v^{1,e_1}}c_{ii,jj,n_z}\partial_{w_q^{s_1}}w^{s_1}_{l_1(e_1,jj)}, \tag{13.18}$$

i.e.

$$\boxed{\partial_{w_q^{s_1}}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)\\ q=l_1(e_1,jj)}}^{n_{\text{el}}^1}\frac{2\Delta_t}{3}c_{ii,jj,n_z}}. \tag{13.19}$$

13.1.3. Derivatives of $\mathcal{K}_i$ with respect to h_q

We consider (13.13), from where we have

$$\begin{aligned}\partial_{h_q}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\rm el}^1} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{h_q}\Bigg[\frac{2\Delta_t}{3}\left(u^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,n_r} + w_{l_1(e_1,jj)}c_{ii,jj,n_z}\right)\\ -\left(r^c_{l_1(e_1,jj)}c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}c_{ii,jj,n_z}\right)\\ +\frac{4}{3}\left(r^c_{l_1(e_1,jj)}(t_{n-1})c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}(t_{n-1})c_{ii,jj,n_z}\right)\\ -\frac{1}{3}\left(r^c_{l_1(e_1,jj)}(t_{n-2})c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}(t_{n-2})c_{ii,jj,n_z}\right)\Bigg],\end{aligned} \tag{13.20}$$

i.e.

$$\begin{aligned}\partial_{h_q}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\rm el}^1} \sum_{jj=1}^{n_v^{1,e_1}} \Bigg[\frac{2\Delta_t}{3}\left(u^{s_1}_{l_1^1(e_1,jj)}\partial_{h_q}c_{ii,jj,n_r} + w^{s_1}_{l_1^1(e_1,jj)}\partial_{h_q}c_{ii,jj,n_z}\right)\\ -\Big(r^c_{l_1(e_1,jj)}\partial_{h_q}c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}\partial_{h_q}c_{ii,jj,n_z} + c_{ii,jj,n_r}\partial_{h_q}r^c_{l_1(e_1,jj)}\\ + c_{ii,jj,n_z}\partial_{h_q}z^c_{l_1(e_1,jj)}\Big)\\ +\frac{4}{3}\left(r^c_{l_1(e_1,jj)}(t_{n-1})\partial_{h_q}c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}(t_{n-1})\partial_{h_q}c_{ii,jj,n_z}\right)\\ -\frac{1}{3}\left(r^c_{l_1(e_1,jj)}(t_{n-2})\partial_{h_q}c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}(t_{n-2})\partial_{h_q}c_{ii,jj,n_z}\right)\Bigg],\end{aligned} \tag{13.21}$$

and using local spine numbers we have

$$\begin{aligned}\partial_{h_q}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\rm el}^1} \sum_{\substack{jj=1\\ q=S_1(e_1,qq)}}^{n_v^{1,e_1}} \Bigg[\frac{2\Delta_t}{3}\Big(u^{s_1}_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_r}\\ + w^{s_1}_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_z}\Big)\\ -\Big(r^c_{l_1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_z}\\ + c_{ii,jj,n_r}\partial_{h_{S_1(e_1,qq)}}r^c_{l_1(e_1,jj)} + c_{ii,jj,n_z}\partial_{h_{S_1(e_1,qq)}}z^c_{l_1(e_1,jj)}\Big)\\ +\frac{4}{3}\left(r^c_{l_1(e_1,jj)}(t_{n-1})\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}(t_{n-1})\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_z}\right)\\ -\frac{1}{3}\left(r^c_{l_1(e_1,jj)}(t_{n-2})\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_r} + z^c_{l_1(e_1,jj)}(t_{n-2})\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_z}\right)\Bigg],\end{aligned} \tag{13.22}$$

i.e.

$$\partial_{h_q}\mathcal{K}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\rm el}^1} \partial_{h_{S_1(e_1,qq)}}\mathcal{K}_{e_1,ii}, \tag{13.23}$$

where

$$\begin{aligned}\partial_{h_{S_1(e_1,qq)}}\mathcal{K}_{e_1,ii} = \sum_{jj=1}^{n_v^{1,e_1}} \Bigg[&\frac{2\Delta_t}{3} u^{s_1}_{l_1^1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_r} - c_{ii,jj,n_r} \partial_{h_{S_1(e_1,qq)}} r^c_{l_1(e_1,jj)} \\ &- \partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_r} \left\{ r^c_{l_1(e_1,jj)} - \frac{4}{3} r^c_{l_1(e_1,jj)}(t_{n-1}) + \frac{1}{3} r^c_{l_1(e_1,jj)}(t_{n-2}) \right\} \\ &+ \frac{2\Delta_t}{3} w^{s_1}_{l_1^1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_z} - c_{ii,jj,n_z} \partial_{h_{S_1(e_1,qq)}} z^c_{l_1(e_1,jj)} \\ &- z^c_{l_1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} \left\{ c_{ii,jj,n_z} - \frac{4}{3} z^c_{l_1(e_1,jj)}(t_{n-1}) + \frac{1}{3} z^c_{l_1(e_1,jj)}(t_{n-2}) \right\} \Bigg].\end{aligned} \tag{13.24}$$

14. The mass exchange equation on boundary 1 (MEC1)

We recall equation (2.49), which states

$$(\boldsymbol{u}-\boldsymbol{v}^{s_1})\cdot\boldsymbol{n}^1 = Fg\,(\rho^{s_1}-Dg\,), \tag{14.1}$$

i.e.

$$(u-u^{s_1})n_r^1+(w-w^{s_1})n_z^1-Fg\,\rho^{s_1}+Fg\,Dg\,=0, \tag{14.2}$$

and define the i-th MEC1 residual as

$$E_i^1=\int\limits_{\partial\Omega^1}\phi_i^1 u n_r^1+\int\limits_{\partial\Omega^1}\phi_i^1 w n_z^1-\int\limits_{\partial\Omega^1}\phi_i^1 u^{s_1}n_r^1-\int\limits_{\partial\Omega^1}\phi_i^1 w^{s_1}n_z^1-Fg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}+Fg\,Dg\int\limits_{\partial\Omega^1}\phi_i^1, \tag{14.3}$$

where i is an index that runs through the boundary 1 node numbering.

We substitute approximations

$$u\approx\sum_{j=1}^{n_v}u_j\phi_j, \tag{14.4}$$

$$w\approx\sum_{j=1}^{n_v}w_j\phi_j, \tag{14.5}$$

$$u^{s_1}\approx\sum_{j=1}^{n_v}u_j^{s_1}\phi_j^1, \tag{14.6}$$

$$w^{s_1}\approx\sum_{j=1}^{n_v}w_j^{s_1}\phi_j^1 \tag{14.7}$$

and

$$\rho^{s_1}\approx\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j. \tag{14.8}$$

into the residual equation above and obtain

$$\begin{aligned}\mathcal{E}_i^1=&\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}u_j\phi_j\right)n_r^1+\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}w_j\phi_j\right)n_z^1-\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}u_j^{s_1}\phi_j^1\right)n_r^1\\&-\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}w_j^{s_1}\phi_j\right)n_z^1-Fg\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j\right)+Fg\,Dg\int\limits_{\partial\Omega^1}\phi_i^1.\end{aligned} \tag{14.9}$$

Moving the integrals into the sums and re-arranging terms we have

$$\begin{aligned}\mathcal{E}_i^1=&\,Fg\,Dg\int\limits_{\partial\Omega^1}\phi_i^1+\sum_{j=1}^{n_v}u_j\int\limits_{\partial\Omega^1}\phi_i^1\phi_j^1 n_r^1+\sum_{j=1}^{n_v}w_j\int\limits_{\partial\Omega^1}\phi_i^1\phi_j^1 n_z^1\\&-\sum_{j=1}^{n_v}u_j^{s_1}\int\limits_{\partial\Omega^1}\phi_i^1\phi_j^1 n_r^1-\sum_{j=1}^{n_v}w_j^{s_1}\int\limits_{\partial\Omega^1}\phi_i^1\phi_j^1 n_z^1-Fg\sum_{j=1}^{n_v}\rho_j^{s_1}\int\limits_{\partial\Omega^1}\phi_i^1\phi_j^1.\end{aligned} \tag{14.10}$$

Decomposing the integrals into sums of integrals over each line-element and using local node-numbers we have

$$\mathcal{E}_i^1 = \sum_{\substack{e_1=1 \\ i=l_1(e_1,ii)}}^{n_{\text{el}}^1} \mathcal{E}_{e_1,ii}^1, \tag{14.11}$$

where

$$\begin{aligned}\mathcal{E}_{e_1,ii}^1 = Fg\,Dg \underbrace{\int_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1}_{c_{ii}(e_1)} + \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1(e_1,jj)} \underbrace{\int_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \phi_{l_1(e_1,jj)}^1 n_r^1}_{c_{ii,jj,n_r}(e_1)} \\ + \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1(e_1,jj)} \underbrace{\int_{\partial\Omega^2} \phi_{l_1(e_1,ii)}^1 \phi_{l_1(e_1,jj)}^1 n_z^1}_{c_{ii,jj,n_z}(e_1)} - \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1^1(e_1,jj)}^{s_1} \underbrace{\int_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \phi_{l_1(e_1,jj)}^1 n_r^1}_{c_{ii,jj,n_r}(e_1)} \\ - \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1^1(e_1,jj)}^{s_1} \underbrace{\int_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \phi_{l_1(e_1,jj)}^1 n_z^1}_{c_{ii,jj,n_z}(e_1)} - Fg \sum_{jj=1}^{n_v^{1,e_1}} \rho_{l_1^1(e_1,jj)}^{s_1} \underbrace{\int_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \phi_{l_1(e_1,jj)}^1}_{c_{ii,jj}(e_1)},\end{aligned} \tag{14.12}$$

i.e.

$$\begin{aligned}\mathcal{E}_{e_1,ii}^1 = Fg\,Dg\,c_{ii}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1) \\ - \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,n_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,n_z}(e_1) - Fg \sum_{jj=1}^{n_v^{1,e_1}} \rho_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj}(e_1).\end{aligned} \tag{14.13}$$

Summarising and re-arranging terms we have

$$\mathcal{E}_i^1 = \sum_{\substack{e_1=1 \\ i=l_1(e_1,ii)}}^{n_{\text{el}}^1} \mathcal{E}_{e_1,ii}^1, \tag{14.14}$$

where

$$\begin{aligned}\mathcal{E}_{e_1,ii}^1 = Fg\,Dg\,c_{ii}(e_1) \\ + \sum_{jj=1}^{n_v^{1,e_1}} \Big\{ c_{ii,jj,n_r}(e_1) \left[u_{l_1(e_1,jj)} - u_{l_1^1(e_1,jj)}^{s_1} \right] \\ + c_{ii,jj,n_z}(e_1) \left[w_{l_1(e_1,jj)} - w_{l_1^1(e_1,jj)}^{s_1} \right] \\ - Fg\,\rho_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj}(e_1) \Big\}.\end{aligned} \tag{14.15}$$

14.1. Jacobian terms

We now calculate the derivatives of $\mathcal{E}_i^1$ with respect to u_q, w_q, $u_q^{s_1}$, $w_q^{s_1}$, $\rho_q^{s_1}$ and h_q.

14.1.1. Derivatives of $\mathcal{E}_i^1$ with respect to u_q

Using equation (14.11) we have

$$\partial_{u_q}\mathcal{E}_i^1 = \sum_{e_1=1}^{n_{\rm el}^1} \partial_{u_q}\mathcal{E}_{e_1,ii}^1, \tag{14.16}$$

and from equation (14.13) we have

$$\begin{aligned}\partial_{u_q}\mathcal{E}_{e_1,ii}^1 = Fg\,Dg\,\partial_{u_q}c_{ii}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q}u_{l_1(e_1,jj)}c_{ii,jj,n_r}(e_1)\\ + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q}w_{l_1(e_1,jj)}c_{ii,jj,n_z}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q}u^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,n_r}(e_1)\\ - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q}w^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,n_z}(e_1) - Fg\sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q}\rho^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj}(e_1),\end{aligned} \tag{14.17}$$

i.e.

$$\partial_{u_q}\mathcal{E}_{e_2,ii}^2 = c_{ii,jj,n_r}(e_2)|_{q=l_2(e_2,jj)}. \tag{14.18}$$

14.1.2. Derivatives of $\mathcal{E}_i^1$ with respect to w_q

Using equation (14.11) we have

$$\partial_{w_q}\mathcal{E}_i^1 = \sum_{e_1=1}^{n_{\mathrm{el}}^1} \partial_{w_q}\mathcal{E}_{e_1,ii}^1, \tag{14.19}$$

and from equation (14.13) we have

$$\begin{aligned}
\partial_{w_q}\mathcal{E}_{e_1,ii}^1 = Fg\,Dg\,\partial_{w_q}c_{ii}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q}u_{l_1(e_1,jj)}c_{ii,jj,n_r}(e_1)& \\
+ \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q}w_{l_1(e_1,jj)}c_{ii,jj,n_z}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q}u^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,n_r}(e_1)& \\
- \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q}w^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,n_z}(e_1) - Fg\sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q}\rho^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj}(e_1)&,
\end{aligned} \tag{14.20}$$

i.e.

$$\partial_{w_q}\mathcal{E}_{e_1,ii}^1 = c_{ii,jj,n_z}(e_1)|_{q=l_1(e_1,jj)}. \tag{14.21}$$

14.1.3. Derivatives of $\mathcal{E}_i^1$ with respect to $u_q^{s_1}$

Using equation (14.11) we have

$$\partial_{u_q^{s_1}} \mathcal{E}_i^1 = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\rm el}^1} \partial_{u_q^{s_1}} \mathcal{E}_{e_1,ii}^1, \tag{14.22}$$

and from equation (14.13) we have

$$\begin{aligned} \partial_{u_q^{s_1}} \mathcal{E}_{e_1,ii}^1 = Fg\, Dg\, \partial_{u_q^{s_1}} c_{ii}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} u_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1) \\ + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} w_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} u^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj,n_r}(e_1) \\ - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} w^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj,n_z}(e_1) - Fg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj}(e_1), \end{aligned} \tag{14.23}$$

i.e.

$$\partial_{u_q^{s_1}} \mathcal{E}_{e_1,ii}^2 = -c_{ii,jj,n_r}(e_1)|_{q=l_1(e_1,jj)}. \tag{14.24}$$

14.1.4. Derivatives of $\mathcal{E}_i^1$ with respect to $w_q^{s_1}$

Using equation (14.11) we have

$$\partial_{w_q^{s_1}} \mathcal{E}_i^1 = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^2} \partial_{w_q^{s_1}} \mathcal{E}_{e_1,ii}^1, \tag{14.25}$$

and from equation (14.13) we have

$$\begin{aligned} \partial_{w_q^{s_1}} \mathcal{E}_{e_1,ii}^1 = Fg\, Dg\, \partial_{w_q^{s_1}} c_{ii}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} u_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1) \\ + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} w_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} u^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj,n_r}(e_1) \\ - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} w^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj,n_z}(e_1) - Fg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj}(e_1), \end{aligned} \tag{14.26}$$

i.e.

$$\partial_{w_q^{s_1}} \mathcal{E}_{e_1,ii}^1 = -c_{ii,jj,n_z}(e_1)|_{q=l_2(e_1,jj)}. \tag{14.27}$$

14.1.5. Derivatives of $\mathcal{E}_i^1$ with respect to $\rho_q^{s_1}$

Using equation (14.11) we have

$$\partial_{\rho_q^{s_1}} \mathcal{E}_i^1 = \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_{\rho_q^{s_1}} \mathcal{E}_{e_1,ii}^1, \tag{14.28}$$

and from equation (14.13) we have

$$\begin{aligned}\partial_{\rho_q^{s_1}} \mathcal{E}_{e_1,ii}^1 = Fg\,Dg\,\partial_{\rho_q^{s_1}} c_{ii}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} u_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1)\\ + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} w_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} u^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj,n_r}(e_1)\\ - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} w^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj,n_z}(e_1) - Fg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj}(e_1),\end{aligned} \tag{14.29}$$

i.e.

$$\partial_{\rho_q^{s_1}} \mathcal{E}_{e_1,ii}^1 = -Fg\, c_{ii,jj}(e_1)|_{q=l_1(e_1,jj)}. \tag{14.30}$$

14.1.6. **Derivatives of $\mathcal{E}_i^1$ with respect to h_q**

Using equation (14.11) we have

$$\partial_{h_q}\mathcal{E}_i^1 = \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\rm el}^1} \partial_{h_{S_1(e_1,qq)}}\mathcal{E}_{e_1,ii}^1, \tag{14.31}$$

and from equation (14.13) we have

$$\begin{aligned}\partial_{h_{S_1(e_1,qq)}}\mathcal{E}_{e_1,ii}^1 = Fg\,Dg\,\partial_{h_{S_1(e_1,qq)}}c_{ii}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_r}(e_1)\\ + \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_z}(e_1)\\ - \sum_{jj=1}^{n_v^{1,e_1}} u^{s_1}_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_r}(e_1)\\ - \sum_{jj=1}^{n_v^{1,e_1}} w^{s_1}_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_z}(e_1)\\ - Fg\sum_{jj=1}^{n_v^{1,e_1}} \rho^{s_1}_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj}(e_1),\end{aligned} \tag{14.32}$$

i.e.

$$\begin{aligned}\partial_{h_{S_1(e_1,qq)}}\mathcal{E}_{e_1,ii}^1 = Fg\,Dg\,\partial_{h_{S_1(e_1,qq)}}c_{ii}(e_1)\\ + \sum_{jj=1}^{n_v^{1,e_1}}\Big[\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_r}(e_1)\left\{u_{l_1(e_1,jj)} - u^{s_1}_{l_1^1(e_1,jj)}\right\}\\ + \partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n_z}(e_1)\left\{w_{l_1(e_1,jj)} - w^{s_1}_{l_1^1(e_1,jj)}\right\}\\ - Fg\,\rho^{s_1}_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj}(e_1)\Big].\end{aligned} \tag{14.33}$$

15. The density transport equation on the free surface (DTC1)

Derivations for this equation in the far field are identical to those in the near field, so we refer the reader to section 36

16. The $\sigma - \rho$ state equation on boundary 1 (TDC1)

We recall equation (2.48), which states the dependence of surface tension on density TDC1, given by

$$\sigma^1 = Cg\,(1 - \rho^{s_1})\,. \tag{16.1}$$

The i-th residual for TDC2 is given by

$$T_i^1 = \sigma_i^2 + Cg\,\rho_i^{s_1} - Cg\,, \tag{16.2}$$

i.e.

$$T_i^1 = \sigma_i^1 + Cg\,(\rho_i^{s_1} - 1)\,, \tag{16.3}$$

16.1. Jacobian terms

Here we find the derivatives of T_i^1 with respect to σ_q^1 and $\rho_q^{s_1}$.

16.1.1. Derivatives of T_i^2 with respect to σ_q^1

$$\partial_{\sigma_q^1} T_i^1 = \delta_{q,i}. \tag{16.4}$$

16.1.2. Derivatives of T_i^1 with respect to $\rho_q^{s_1}$

$$\partial_{\rho_q^{s_1}} T_i^1 = Cg\,\delta_{q,i}. \tag{16.5}$$

17. Summary of residual equations

17.1. Bulk terms

17.2. r-momentum residuals

$$
\begin{aligned}
\hat{M}_i^r = &\sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}} \hat{M}_{e,ii}^r + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\mathrm{el}}^1} \hat{M}_{e_1,ii}^{r,1} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \hat{M}_{e_2,ii}^{r,2} \\
&+ \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{n_{\mathrm{el}}^3} \hat{M}_{e_3,ii}^{r,3} + \frac{\sigma_d^1 \phi_i(r_d, z_d) - \sigma_c^1 \phi_i(r_c, 0)\cos(\theta)}{Ca},
\end{aligned} \tag{17.1}
$$

where

$$
\begin{aligned}
\hat{M}_{e,ii}^r = &\sum_{jj=1}^{n_v} \Bigg(u_{l(e,jj)} \Bigg\{ Re \underbrace{\sum_{k=1}^{n_v} \left[u_{l(e,kk)} a_{ii,kk,jj}^r(e) + w_{l(e,kk)} a_{ii,kk,jj}^z(e) \right]}_{A_{ii,jj}(u,w,e)} \\
&+ 2a_{ii,jj}^{r,r}(e) + a_{ii,jj}^{z,z}(e) \Bigg\} + w_{l(e,jj)} a_{ii,jj}^{z,r}(e) \Bigg) - \sum_{j=1}^{n_p^e} p_{l^p(e,jj)} b_{jj,ii}^r(e),
\end{aligned} \tag{17.2}
$$

$$
\hat{M}_{e_1,ii}^{r,1} = \frac{1}{Ca} \sum_{jj=1}^{n_v} \sigma_{l_1^1(e_1,jj)}^1 c_{t_r,jj,ii}^s(e_1), \tag{17.3}
$$

$$
\hat{M}_{e_2,ii}^{r,2} = Be \left\{ \sum_{jj=1}^{n_v} \left[u_{l_2(e_2,jj)} d_{t_r,t_r,ii,jj}(e_2) + w_{l_2(e_2,jj)} d_{t_r,t_z,ii,jj}(e_2) \right] + d_{t_r,t_r,ii}(e_2) \right\}, \tag{17.4}
$$

$$
\hat{M}_{e_3,ii}^{r,3} = \sum_{jj=1}^{n_v^{e_3}} \left[\lambda_{l_3^3(e_3,jj)}^3 f_{ii,jj,n_r}(e_3) + \gamma_{l_3^3(e_3,jj)}^3 f_{t_r,ii,jj}(e_3) \right]. \tag{17.5}
$$

17.3. z-momentum residuals

$$\hat{M}_i^z = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\rm el}} \hat{M}_{e,ii}^z + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\rm el}^1} \hat{M}_{e_1,ii}^{z,1} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\rm el}^2} \hat{M}_{e_2,ii}^{z,2} + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{n_{\rm el}^3} \hat{M}_{e_3,ii}^{z,3} + \frac{\sigma_c^1 \phi_i(r_c,0)\sin(\theta)}{Ca}, \tag{17.6}$$

where

$$\hat{M}_{e,ii}^z = \sum_{jj=1}^{n_v^e} \left[w_{l(e,jj)} \left\{ \underbrace{Re \sum_{kk=1}^{n_v^e} \left[u_{l(e,kk)} a_{ii,kk,jj}^r(e) + w_{l(e,kk)} a_{ii,kk,jj}^z(e) \right]}_{A_{ii,jj}(u,w,e)} + 2a_{ii,jj}^{z,z}(e) + a_{ii,jj}^{r,r}(e) \right\} + u_{l(e,jj)} a_{ii,jj}^{r,z}(e) \right] - \sum_{jj=1}^{n_p^e} p_{l^p(e,jj)} b_{jj,ii}^z(e) + St\, a_{ii}(e), \tag{17.7}$$

$$\hat{M}_{e_1,ii}^{z,1} = \frac{1}{Ca} \sum_{jj=1}^{n_v^{e1}} \sigma_{l_1^1(e_1,jj)}^1 c_{t_z,jj,ii}^s(e_1), \tag{17.8}$$

$$\hat{M}_{e_2,ii}^{z,2} = Be \left\{ \sum_{jj=1}^{n_v^{2,e_2}} \left[u_{l_2(e_2,jj)} d_{t_r,t_r,ii,jj}(e_2) + w_{l_2(e_2,jj)} d_{t_r,t_z,ii,jj}(e_2) \right] + d_{t_r,t_r,ii}(e_2) \right\} + \sum_{jj=1}^{n_v^{e2}} \lambda_{l_2^2(e_2,jj)}^2 d_{ii,jj,n_r}(e_2), \tag{17.9}$$

$$\hat{M}_{e_3,ii}^{z,3} = \sum_{jj=1}^{n_v^{e3}} \gamma_{l_3(e_3,jj)}^3 f_{t_r,ii,jj}(e). \tag{17.10}$$

17.4. Continuity residuals

$$\hat{C}_i = \sum_{\substack{e=1 \\ i=l^p(e,jj)}}^{n_{\mathrm{el}}} \sum_{ii=1}^{n_v^e} \left[u_{l(e,ii)} b^r_{jj,ii}(e) + w_{l(e,ii)} b^z_{jj,ii}(e_3) \right]. \tag{17.11}$$

17.5. Kinematic boundary condition residuals

$$\hat{K}_i = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \sum_{jj=1}^{n_v^{e1}} \left[u_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1) + w_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1) \right]. \tag{17.12}$$

17.6. Impermeability residuals

$$\hat{I}_i = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \sum_{jj=1}^{n_v^{e2}} \left[u_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) + w_{l_2(e,jj)} d_{ii,jj,n_z}(e_2) \right]. \tag{17.13}$$

18. System Jacobian

The system of equations given by the Residuals has the following Jabcobian

$$J_R = \begin{bmatrix} \partial_u \hat{M}^r & \partial_w \hat{M}^r & \partial^p \hat{M}^r & \partial_{\lambda^2} \hat{M}^r & \partial_{\lambda^3} \hat{M}^r & \partial_{\gamma^3} \hat{M}^r & \partial_h \hat{M}^r \\ \partial_u \hat{M}^z & \partial_w \hat{M}^z & \partial^p \hat{M}^z & \partial_{\lambda^2} \hat{M}^z & \partial_{\lambda^3} \hat{M}^z & \partial_{\gamma^3} \hat{M}^z & \partial_h \hat{M}^z \\ \partial_u \hat{C} & \partial_w \hat{C} & \partial^p \hat{C} & \partial_{\lambda^2} \hat{C} & \partial_{\lambda^3} \hat{C} & \partial_{\gamma^3} \hat{C} & \partial_h \hat{C} \\ \partial_u \hat{K} & \partial_w \hat{K} & \partial^p \hat{K} & \partial_{\lambda^2} \hat{K} & \partial_{\lambda^3} \hat{K} & \partial_{\gamma^3} \hat{K} & \partial_h \hat{K} \\ \partial_u \hat{I} & \partial_w \hat{I} & \partial^p \hat{I} & \partial_{\lambda^2} \hat{I} & \partial_{\lambda^3} \hat{I} & \partial_{\gamma^3} \hat{I} & \partial_h \hat{I} \end{bmatrix}. \tag{18.1}$$

Every entry in the Jacobian matrix can be calculated analytically, with the only exception of the elements on the first column on each block of the last block-column. These elements are calculated quasi-analytically, since the derivatives of the nodal positions with respect to the focal length is approximated numerically.

18.1. Entries on the first block-row

Blocks in the first row are given by

$$\begin{aligned} \left(\partial_u \hat{M}^r\right)_{i,q} = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\text{el}}} \Bigg\{ & Re \sum_{\substack{jj=1\\ q=l(e,kk)}}^{n_v^e} u_{l(e,jj)} a^r_{ii,kk,jj}(e) \\ & + \sum_{\substack{jj=1\\ q=l(e,jj)}}^{n_v^e} \left[Re\, A_{ii,jj}(u,w,e) + 2a^{r,r}_{ii,jj}(e) + a^{z,z}_{ii,jj}(e)\right] \Bigg\} \\ & + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=l_2(e_2,jj)}}^{n_{\text{el}}^2} Be\, d_{t_r,t_r,ii,jj}(e_2), \end{aligned} \tag{18.2}$$

$$\begin{aligned} \left(\partial_w \hat{M}^r\right)_{i,q} = & \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\text{el}}} \left\{ \sum_{\substack{jj=1\\ q=l(e,kk)}}^{n_v^e} Re\ u_{l(e,jj)} a^z_{ii,kk,jj}(e) \right\} \\ & + \sum_{\substack{e=1\\ i=l(e,ii)\\ q=l(e,jj)}}^{n_{\text{el}}} a^{z,r}_{ii,jj}(e) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=l_2(e_2,jj)}}^{n_{\text{el}}^2} Be\, d_{t_r,t_z,ii,jj}(e_2), \end{aligned} \tag{18.3}$$

$$\left(\partial^p \hat{M}^r\right)_{i,q} = \sum_{\substack{e=1\\ i=l(e,ii)\\ q=l^p(e,jj)}}^{n_{\text{el}}} -b^r_{jj,ii}(e), \tag{18.4}$$

$$\left(\partial_{\lambda^2}\hat{M}^r\right)_{i,q} = \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=l_2^2(e_2,jj)}}^{n_{\mathrm{el}}^2} d_{ii,jj,n_r}(e_2), \tag{18.5}$$

$$\left(\partial_{\lambda^3}\hat{M}^r\right)_{i,q} = \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)\\ q=l_3^3(e_3,jj)}}^{n_{\mathrm{el}}^3} f_{ii,jj,n_r}(e_3), \tag{18.6}$$

$$\left(\partial_{\gamma^3}\hat{M}^r\right)_{i,q} = \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)\\ q=l_3^3(e_3,jj)}}^{n_{\mathrm{el}}^3} f_{t_r,ii,jj}(e_3), \tag{18.7}$$

$$
\begin{aligned}
\left(\partial_h \hat{M}^r\right)_{i,q} = & \sum_{\substack{e=1 \\ i=l(e,ii)}}^{n_{\mathrm{el}}} \sum_{\substack{jj=1 \\ q=S(e,qq)}}^{n_v^e} \Bigg(u_{l(e,jj)} \Bigg\{ \\
& Re \underbrace{\sum_{kk=1}^{n_v^e} \left[u_{l(e,kk)} \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) + w_{l(e,kk)} \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \right]}_{\partial_{h_{S(e,qq)}} A_{ii,jj}(u,w,e)} \\
& \left. +2\partial_{h_{S(e,qq)}} a^{r,r}_{ii,jj}(e) + \partial_{h_{S(e,qq)}} a^{z,z}_{ii,jj}(e) \Bigg\} + w_{l(e,jj)} \partial_{h_{S(e,qq)}} a^{z,r}_{ii,jj}(e) \right) \\
& - \sum_{\substack{e=1 \\ i=l(e,ii)}}^{n_{\mathrm{el}}} \sum_{\substack{jj=1 \\ q=S(e,qq)}}^{n_p^e} p_{l^p(e,jj)} \partial_{h_{S(e,qq)}} b^r_{jj,ii}(e) \\
& + \sum_{\substack{e_1=1 \\ i=l_1(e_1,ii)}}^{n^1_{\mathrm{el}}} \frac{1}{Ca} \sum_{\substack{jj=1 \\ q=S_1(e_1,qq)}}^{n_v} \sigma^1_{l^1_1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c^s_{t_r,jj,ii}(e_1) \\
& + \sum_{\substack{e_2=1 \\ i=l_2(e_2,ii) \\ q=S_2(e_2,qq)}}^{n^2_{\mathrm{el}}} Be\, \partial_{h_{S_2(e_2,qq)}} d_{t_r,t_r,ii}(e_2) \\
& + \sum_{\substack{e_2=1 \\ i=l_2(e_2,ii)}}^{n^2_{\mathrm{el}}} Be \Bigg\{ \sum_{\substack{jj=1 \\ q=S_2(e_2,qq)}}^{n_v^{2,e_2}} \Big[u_{l_2(e_2,jj)} \partial_{h_{S_2(e_2,qq)}} d_{t_r,t_r,ii,jj}(e_2) \\
& \qquad + w_{l_2(e_2,jj)} \partial_{h_{S_2(e_2,qq)}} d_{t_r,t_z,ii,jj}(e_2) \Big] \Bigg\} \\
& + \sum_{\substack{e_2=1 \\ i=l_2(e_2,ii)}}^{n^2_{\mathrm{el}}} \sum_{\substack{j=1 \\ q=S_2(e_2,qq)}}^{n_v^{e2}} \lambda^2_{l^2_2(e_2,jj)} \partial_{h_{S_2(e_2,qq)}} d_{ii,jj,n_r}(e_2) \\
& + \sum_{\substack{e_3=1 \\ i=l_3(e_3,ii)}}^{n^3_{\mathrm{el}}} \sum_{\substack{j=1 \\ q=S_3(e_3,qq)}}^{n_v^{e3}} \Big[\lambda^3_{l^3_3(e_3,jj)} \partial_{h_{S_3(e_3,qq)}} f_{ii,jj,n_r}(e_3) \\
& \qquad + \gamma^3_{l^3_3(e_3,jj)} \partial_{h_{S_3(e_3,qq)}} f_{t_r,ii,jj}(e_3) \Big].
\end{aligned} \tag{18.8}
$$

18.2. Entries on the second block-row

Blocks in the second row are given by

$$\left(\partial_u \hat{M}^z\right)_{i,q} = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\text{el}}} \sum_{\substack{jj=1\\ q=l(e,kk)}}^{n_v^e} Re\; w_{l(e,jj)} a^r_{ii,kk,jj}(e)$$
$$+ \sum_{\substack{e=1\\ i=l(e,ii)\\ q=l(e,jj)}}^{n_{\text{el}}} a^{r,z}_{ii,jj}(e) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=l_2(e_2,jj)}}^{n^2_{\text{el}}} Be\, d_{t_r,t_z,ii,jj}, \tag{18.9}$$

$$\left(\partial_w \hat{M}^z\right)_{i,q} = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\text{el}}} \left\{ \sum_{\substack{jj=1\\ q=l(e,jj)}}^{n_v^e} \left[Re\, A_{ii,jj}(u,w,e) + 2a^{z,z}_{ii,jj}(e) + a^{r,r}_{ii,jj}(e)\right]\right.$$
$$\left. + \sum_{\substack{jj=1\\ q=l(e,kk)}}^{n_v^e} Re\; w_{l(e,jj)} a^z_{ii,kk,jj}(e)\right\} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=l_2(e_2,jj)}}^{n^2_{\text{el}}} Be\, d_{t_z,t_z,ii,jj}, \tag{18.10}$$

$$\left(\partial^p \hat{M}^z\right)_{i,q} = \sum_{\substack{e=1\\ i=l(e,ii)\\ q=l^p(e,jj)}}^{n_{\text{el}}} -b^z_{jj,ii}(e), \tag{18.11}$$

$$\left(\partial_{\lambda^2} \hat{M}^z\right)_{i,q} = \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=l^2_2(e_2,jj)}}^{n^2_{\text{el}}} d_{ii,jj,n_z}(e_2), \tag{18.12}$$

$$\left(\partial_{\lambda^3} \hat{M}^z\right)_{i,q} = \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)\\ q=l^3_3(e_3,jj)}}^{n^3_{\text{el}}} f_{ii,jj,n_z}(e_3), \tag{18.13}$$

$$\left(\partial_{\gamma^3} \hat{M}^z\right)_{i,q} = \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)\\ q=l^3_3(e_3,jj)}}^{n^3_{\text{el}}} f_{t_z,ii,jj}(e_3), \tag{18.14}$$

$$
\begin{aligned}
\left(\partial_h \hat{M}^z\right)_{i,q} = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}} \sum_{\substack{jj=1\\ q=S(e,qq)}}^{n_v^e} \Bigg(& w_{l(e,jj)} \Bigg\{ \\
& Re \underbrace{\sum_{kk=1}^{n_v^e} \left[u_{l(e,kk)} \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) + w_{l(e,kk)} \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e)\right]}_{\partial_{h_{S(e,qq)}} A_{ii,jj}(u,w,e)} \\
& +2\partial_{h_{S(e,qq)}} a^{z,z}_{ii,jj}(e) + \partial_{h_{S(e,qq)}} a^{r,r}_{ii,jj}(e) \Bigg\} + u_{l(e,jj)} \partial_{h_{S(e,qq)}} a^{r,z}_{ii,jj}(e) \Bigg) \\
& + \sum_{\substack{e=1\\ i=l(e,ii)\\ q=S(e,qq)}}^{n_{\mathrm{el}}} St\, \partial_{h_{S(e,qq)}} a_{ii}(e) - \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}} \sum_{\substack{jj=1\\ q=S(e,qq)}}^{n_p^e} p_{l^p(e,jj)} \partial_{h_{S(e,qq)}} b^z_{jj,ii}(e) \\
& + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n^1_{\mathrm{el}}} \frac{1}{Ca} \sum_{\substack{jj=1\\ q=S_1(e_1,qq)}}^{n_v^{e1}} \sigma^1_{l^1_1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c^s_{t_z,jj,ii}(e_1) \\
& + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n^2_{\mathrm{el}}} Be\, \partial_{h_{S_2(e_2,qq)}} d_{t_r,t_z,ii} \\
& + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n^2_{\mathrm{el}}} Be \sum_{\substack{jj=1\\ q=S_2(e_2,qq)}}^{n_v^{2,e_2}} \Big[u_{l_2(e_2,jj)} \partial_{h_{S_2(e_2,qq)}} d_{t_r,t_z,ii,jj} \\
& \qquad + w_{l_2(e_2,jj)} \partial_{h_{S_2(e_2,qq)}} d_{t_z,t_z,ii,jj} \Big] \\
& + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n^2_{\mathrm{el}}} \sum_{\substack{jj=1\\ q=S_2(e_2,qq)}}^{n_v^{e_2}} \lambda^2_{l^2_2(e_2,jj)} \partial_{h_{S_2(e_2,qq)}} d_{ii,jj,n_z}(e_2) \\
& + \sum_{\substack{e_3=1\\ i=l_3(e_3,ii)}}^{n^3_{\mathrm{el}}} \sum_{\substack{jj=1\\ q=S_3(e_3,qq)}}^{n_v^{3,e_3}} \Big[\lambda^3_{l^3_3(e_3,jj)} \partial_{h_{S_3(e_3,qq)}} f_{ii,jj,n_z} \\
& \qquad + \gamma^3_{l_3(e_3,jj)} \partial_{h_{S_3(e_3,qq)}} f_{t_z,ii,jj}(e_3) \Big].
\end{aligned}
\tag{18.15}
$$

18.3. Entries on the third block-row

Blocks in the third row are given by

$$\left(\partial_u \hat{C}\right)_{k,q} = \sum_{\substack{e=1\\k=l^p(e,jj)\\q=l(e,ii)}}^{n_{\mathrm{el}}} b^r_{jj,ii}(e), \tag{18.16}$$

$$\left(\partial_w \hat{C}\right)_{k,q} = \sum_{\substack{e=1\\k=l^p(e,jj)\\q=l(e,ii)}}^{n_{\mathrm{el}}} b^z_{jj,ii}(e), \tag{18.17}$$

$$\left(\partial^p \hat{C}\right)_{i,q} = 0, \tag{18.18}$$

$$\left(\partial_{\lambda^2} \hat{C}\right)_{i,q} = 0, \tag{18.19}$$

$$\left(\partial_{\lambda^3} \hat{C}\right)_{i,q} = 0, \tag{18.20}$$

$$\left(\partial_{\gamma^3} \hat{C}\right)_{i,q} = 0, \tag{18.21}$$

$$\left(\partial_h \hat{C}\right)_{i,q} = \sum_{\substack{e=1\\i=l^p(e,ii)}}^{n_{\mathrm{el}}} \sum_{\substack{jj=1\\q=S(e,qq)}}^{n_v^e} \left[u_{l(e,jj)} \partial_{h_{S(e,qq)}} b^r_{ii,jj}(e) + w_{l(e,jj)} \partial_{h_{S(e,qq)}} b^z_{ii,jj}(e)\right]. \tag{18.22}$$

18.4. Entries on the fourth block-row

Blocks in the fourth row are given by

$$\left(\partial_u \hat{K}\right)_{i,q} = \sum_{\substack{e_1=1\\i=l_1^1(e_1,ii)\\q=l_1(e_1,jj)}}^{n^1_{\mathrm{el}}} c_{ii,jj,n_r}(e_1), \tag{18.23}$$

$$\left(\partial_w \hat{K}\right)_{i,q} = \sum_{\substack{e_1=1\\i=l_1^1(e_1,ii)\\q=l_1(e_1,jj)}}^{n^1_{\mathrm{el}}} c_{ii,jj,n_z}(e_1), \tag{18.24}$$

$$\left(\partial^p \hat{K}\right)_{i,q} = 0, \tag{18.25}$$

$$\left(\partial_{\lambda^2} \hat{K}\right)_{i,q} = 0, \tag{18.26}$$

$$\left(\partial_{\lambda^3} \hat{K}\right)_{i,q} = 0, \tag{18.27}$$

$$\left(\partial_{\gamma^3} \hat{K}\right)_{i,q} = 0, \tag{18.28}$$

$$\left(\partial_h \hat{K}\right)_{i,q} = \sum_{\substack{e_1=1 \\ i=l_1(e_1,ii)}}^{n_{\mathrm{el}}^1} \sum_{\substack{jj=1 \\ q=S(e,qq)}}^{n_v^{e_1}} \left[u_{l_1(e_1,jj)} \partial_{h_{S(e,qq)}} c_{ii,jj,n_r}(e_1) + w_{l_1(e_1,jj)} \partial_{h_{S(e,qq)}} c_{ii,jj,n_z}(e_1)\right]. \tag{18.29}$$

18.5. **Entries on the fifth block-row**

Blocks in the fifth row are given by

$$\left(\partial_u \hat{I}\right)_{i,q} = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii) \\ q=l_2(e_2,jj)}}^{n_{\mathrm{el}}^2} d_{ii,jj,n_r}(e_2), \tag{18.30}$$

$$\left(\partial_w \hat{I}\right)_{i,q} = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii) \\ q=l_2(e_2,jj)}}^{n_{\mathrm{el}}^2} d_{ii,jj,n_z}(e_2), \tag{18.31}$$

$$\left(\partial^p \hat{I}\right)_{i,q} = 0, \tag{18.32}$$

$$\left(\partial_{\lambda^2} \hat{I}\right)_{i,q} = 0, \tag{18.33}$$

$$\left(\partial_{\lambda^3} \hat{I}\right)_{i,q} = 0, \tag{18.34}$$

$$\left(\partial_{\gamma^3} \hat{I}\right)_{i,q} = 0, \tag{18.35}$$

$$\left(\partial_h \hat{I}\right)_{i,q} = \sum_{\substack{e_2=1 \\ i=l_2^2(e,ii)}}^{n_{\mathrm{el}}^2} \sum_{\substack{jj=1 \\ q=S_2(e_2,qq)}}^{n_v^{e_2}} \left[u_{l_2(e_2,jj)} \partial_{h_q} d_{ii,jj,n_r}(e_2) + w_{l_2(e,jj)} \partial_{h_q} d_{ii,jj,n_z}(e_2)\right]. \tag{18.36}$$

18.6. **Jacobian with explicit zero-blocks**

Re-writing the Jacobian indicating null blocks we have

$$J_R = \begin{bmatrix} \partial_u \hat{M}^r & \partial_w \hat{M}^r & \partial^p \hat{M}^r & \partial_{\lambda_2} \hat{M}^r & \partial_{\lambda^3} \hat{M}^r & \partial_{\gamma^3} \hat{M}^r & \partial_h \hat{M}^r \\ \partial_u \hat{M}^z & \partial_w \hat{M}^z & \partial^p \hat{M}^z & \partial_{\lambda^2} \hat{M}^z & \partial_{\lambda^3} \hat{M}^z & \partial_{\gamma^3} \hat{M}^z & \partial_h \hat{M}^z \\ \partial_u \hat{C} & \partial_w \hat{C} & 0 & 0 & 0 & 0 & \partial_h \hat{C} \\ \partial_u \hat{K} & \partial_w \hat{K} & 0 & 0 & 0 & 0 & \partial_h \hat{K} \\ \partial_u \hat{I} & \partial_w \hat{I} & 0 & 0 & 0 & 0 & \partial_h \hat{I} \end{bmatrix}, \quad (18.37)$$

We highlight that when boundary 2 is a straight line parallel to the r axis, we also have

$$\partial_{\lambda^2} \hat{M}^r = 0, \quad (18.38)$$

and when boundary 3 is a straight line parallel to the z axis, we also have

$$\partial_{\gamma^3} \hat{M}^r = 0, \quad (18.39)$$

$$\partial_{\lambda^3} \hat{M}^z = 0. \quad (18.40)$$

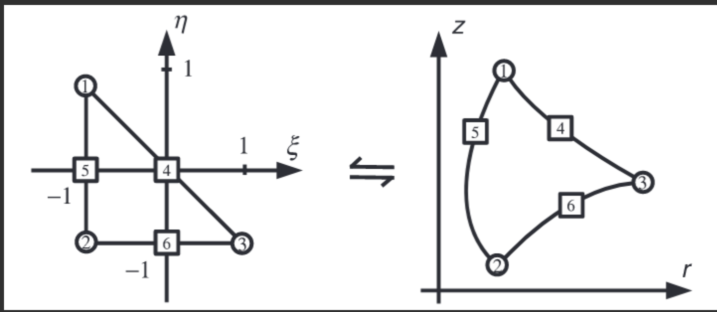

FIGURE 10. V6P3 Taylor-Hood master element. Reproduction of figure 13 in Sprittles & Shikhmurzaev (2012*b*)
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19. Basis functions

We follow Sprittles & Shikhmurzaev (2012*b*) and use the V6P3 Taylor-Hood master element, see figure 10, to which we map every element in our domain isoparametrically (see below for details).

Pressure interpolating functions ψ are given by:

$$\psi_1 = \frac{1+\eta}{2}, \tag{19.1}$$

$$\psi_2 = -\frac{\xi+\eta}{2}, \tag{19.2}$$

$$\psi_3 = \frac{1+\xi}{2}. \tag{19.3}$$

Nodes 1, 2 and 3 are pressure-velocity nodes, while 4, 5 and 6 are velocity only nodes. The sub-index in functions ψ indicate on which of the pressure-velocity nodes the functions attain the value 1, being zero in the other two, and linear over each element.

Velocity interpolating functions ϕ are defined as follows

$$\phi_1 = \psi_1(2\psi_1 - 1) = \frac{1+\eta}{2}\,(1+\eta-1) = \frac{\eta(\eta+1)}{2} = \frac{\eta^2+\eta}{2}, \tag{19.4}$$

$$\phi_2 = \psi_2(2\psi_2 - 1) = -\frac{\xi+\eta}{2}\,(-\xi-\eta-1) = \frac{(\xi+\eta)(\xi+\eta+1)}{2} = \frac{\xi^2+\eta^2+2\xi\eta+\xi+\eta}{2}, \tag{19.5}$$

$$\phi_3 = \psi_3(2\psi_3 - 1) = \frac{1+\xi}{2}\,(1+\xi-1) = \frac{\xi(\xi+1)}{2} = \frac{\xi^2+\xi}{2}, \tag{19.6}$$

$$\phi_4 = 4\psi_1\psi_3 = 4\frac{1+\eta}{2}\frac{1+\xi}{2} = (\xi+1)(\eta+1) = \xi\eta+\xi+\eta+1, \tag{19.7}$$

$$\phi_5 = 4\psi_2\psi_1 = -4\frac{\xi+\eta}{2}\frac{1+\eta}{2} = -(\xi+\eta)(\eta+1) = -\eta^2-\xi\eta-\xi-\eta, \tag{19.8}$$

$$\phi_6 = 4\psi_3\psi_2 = -4\frac{1+\xi}{2}\frac{\xi+\eta}{2} = -(\xi+\eta)(\xi+1) = -\xi^2-\xi\eta-\xi-\eta. \tag{19.9}$$

Function $S_e(\xi,\eta)$ maps the coordinates of the master element to the coordinates of element number e. More specifically

$$(r^e, z^e) = S_e(\xi,\eta). \tag{19.10}$$

The fact that our mapping is isoparametric means that, much like the pressure and velocity functions, the r_e and z_e coordinates of the points in a given element are described as

$$r_e = \sum_{jj=1}^{6} r_{e,jj}\phi_{jj}, \tag{19.11}$$

and

$$z_e = \sum_{jj=1}^{6} z_{e,jj}\phi_{jj}, \tag{19.12}$$

with $r_{e,jj}$ being the r coordinate of local node jj in element e. The obvious analogy applies to z.

19.1. **Derivatives of the basis functions**

The derivatives of the velocity interpolating functions with respect to ξ are given by:

$$\partial_\xi \phi_1 = \partial_\xi \left[\frac{\eta^2+\eta}{2}\right] = 0, \tag{19.13}$$

$$\partial_\xi \phi_2 = \partial_\xi \left[\frac{\xi^2+\eta^2+2\xi\eta+\xi+\eta}{2}\right] = \frac{2\xi+2\eta+1}{2} = \xi+\eta+\frac{1}{2}, \tag{19.14}$$

$$\partial_\xi \phi_3 = \partial_\xi \left[\frac{\xi^2+\xi}{2}\right] = \xi+\frac{1}{2}, \tag{19.15}$$

$$\partial_\xi \phi_4 = \partial_\xi \left[\xi\eta+\xi+\eta+1\right] = \eta+1, \tag{19.16}$$

$$\partial_\xi \phi_5 = \partial_\xi \left[-\eta^2-\xi\eta-\xi-\eta\right] = -\eta-1, \tag{19.17}$$

$$\partial_\xi \phi_6 = \partial_\xi \left[-\xi^2-\xi\eta-\xi-\eta\right] = -2\xi-\eta-1. \tag{19.18}$$

The derivatives of the velocity interpolating functions with respect to η are given by:

$$\partial_\eta \phi_1 = \partial_\eta \left[\frac{\eta^2+\eta}{2}\right] = \eta+\frac{1}{2}, \tag{19.19}$$

$$\partial_\eta \phi_2 = \partial_\eta \left[\frac{\xi^2+\eta^2+2\xi\eta+\xi+\eta}{2}\right] = \frac{2\eta+2\xi+1}{2} = \xi+\eta+\frac{1}{2}, \tag{19.20}$$

$$\partial_\eta \phi_3 = \partial_\eta \left[\frac{\xi^2+\xi}{2}\right] = 0, \tag{19.21}$$

$$\partial_\eta \phi_4 = \partial_\eta \left[\xi\eta+\xi+\eta+1\right] = \xi+1, \tag{19.22}$$

$$\partial_\eta \phi_5 = \partial_\eta \left[-\eta^2-\xi\eta-\xi-\eta\right] = -2\eta-\xi-1, \tag{19.23}$$

$$\partial_\eta \phi_6 = \partial_\eta \left[-\xi^2-\xi\eta-\xi-\eta\right] = -\xi-1. \tag{19.24}$$

20. Integrals over triangular elements

In order to simplify our calculations, it is convenient to express all integrals over an element as integrals over the master element. This requires changing the r and z variables of the element coordinates to the ξ and η variables of the master element. That is to say, we need to consider

$$\int_{\Omega_e} f(r,z)d\Omega_e = \int_{\eta=-1}^{\eta=1}\int_{\xi=-1}^{\xi=-\eta} f(\xi,\eta)|\det J_e|\mathrm{d}\xi\mathrm{d}\eta, \tag{20.1}$$

where J_e is the Jacobian of the isoparametric map S_e for that element, given by

$$J_e = \begin{bmatrix} \dfrac{\partial r_e}{\partial \xi} & \dfrac{\partial r_e}{\partial \eta} \\ \\ \dfrac{\partial z_e}{\partial \xi} & \dfrac{\partial z_e}{\partial \eta} \end{bmatrix}, \tag{20.2}$$

and therefore

$$\det J_e = \frac{\partial r_e}{\partial \xi}\frac{\partial z_e}{\partial \eta} - \frac{\partial r_e}{\partial \eta}\frac{\partial z_e}{\partial \xi}, \tag{20.3}$$

and from (19.11) and (19.12) we have

$$\det J_e = \left(\sum_{ii=1}^{6} r_{e,ii}\frac{\partial \phi_{ii}}{\partial \xi}\right)\left(\sum_{jj=1}^{6} z_{e,jj}\frac{\partial \phi_{jj}}{\partial \eta}\right) - \left(\sum_{ii=1}^{6} r_{e,ii}\frac{\partial \phi_{ii}}{\partial \eta}\right)\left(\sum_{jj=1}^{6} z_{e,jj}\frac{\partial \phi_{jj}}{\partial \xi}\right), \tag{20.4}$$

i.e.

$$\det J_e = \sum_{ii=1}^{6}\sum_{jj=1}^{6} r_{e,ii}\frac{\partial \phi_{ii}}{\partial \xi}\frac{\partial \phi_{jj}}{\partial \eta} z_{e,jj} - \sum_{jj=1}^{6}\sum_{ii=1}^{6} r_{e,ii}\frac{\partial \phi_{jj}}{\partial \xi}\frac{\partial \phi_{ii}}{\partial \eta} z_{e,jj}, \tag{20.5}$$

which yields

$$\det J_e = \sum_{ii=1}^{6}\sum_{jj=1}^{6} r_{e,ii}\underbrace{\left(\frac{\partial \phi_{ii}}{\partial \xi}\frac{\partial \phi_{jj}}{\partial \eta} - \frac{\partial \phi_{jj}}{\partial \xi}\frac{\partial \phi_{ii}}{\partial \eta}\right)}_{T_{ii,jj}} z_{e,jj}. \tag{20.6}$$

Furthermore, if we make sure that the local numbering in all our elements is done in the same direction as our master element (counter-clockwise), and we do not have degenerate elements, we can ensure that the determinant of the Jacobian is always positive (which we assume from here on).

Many of the integrals over elements that are needed to create our system of equations and its Jacobian require that we calculate derivatives of the interpolating functions with respect to r and z. Following Sprittles & Shikhmurzaev (2012b), we will calculate these using the chain rule, first taking derivatives with respect to ξ and η and then differentiating ξ and η with respect to r_e and z_e. We do this by noticing that

$$J_e^{-1} = \begin{bmatrix} \dfrac{\partial \xi}{\partial r_e} & \dfrac{\partial \xi}{\partial z_e} \\ \\ \dfrac{\partial \eta}{\partial r_e} & \dfrac{\partial \eta}{\partial z_e} \end{bmatrix} = \frac{1}{\det J_e}\begin{bmatrix} \dfrac{\partial z_e}{\partial \eta} & -\dfrac{\partial r_e}{\partial \eta} \\ \\ -\dfrac{\partial r_e}{\partial \xi} & \dfrac{\partial z_e}{\partial \xi} \end{bmatrix}, \tag{20.7}$$

where the left equality follows by definition and the right one is the computation of the inverse from the expression in (20.2). We thus have

$$\frac{\partial \phi_{jj}}{\partial r_e} = \frac{\partial \phi_{jj}}{\partial \xi}\frac{\partial \xi}{\partial r_e} + \frac{\partial \phi_{jj}}{\partial \eta}\frac{\partial \eta}{\partial r_e} = \frac{1}{\det J_e}\left(\frac{\partial \phi}{\partial \xi}\frac{\partial z_e}{\partial \eta} - \frac{\partial \phi}{\partial \eta}\frac{\partial z_e}{\partial \xi}\right), \tag{20.8}$$

and hence

$$\frac{\partial \phi_{jj}}{\partial r_e} = \frac{1}{\det J_e}\left[\frac{\partial \phi_{jj}}{\partial \xi}\left(\sum_{kk=1}^{6} z_{e,kk}\frac{\partial \phi_{kk}}{\partial \eta}\right) - \frac{\partial \phi_{jj}}{\partial \eta}\left(\sum_{kk=1}^{6} z_{e,kk}\frac{\partial \phi_{kk}}{\partial \xi}\right)\right], \tag{20.9}$$

i.e.

$$\frac{\partial \phi_{jj}}{\partial r_e} = \frac{1}{\det J_e}\left[\left(\sum_{kk=1}^{6} z_{e,kk}\frac{\partial \phi_{jj}}{\partial \xi}\frac{\partial \phi_{kk}}{\partial \eta}\right) - \left(\sum_{kk=1}^{6} z_{e,kk}\frac{\partial \phi_{jj}}{\partial \eta}\frac{\partial \phi_{kk}}{\partial \xi}\right)\right], \tag{20.10}$$

yielding

$$\boxed{\frac{\partial \phi_{jj}}{\partial r_e} = \frac{1}{\det J_e}\sum_{kk=1}^{6}\underbrace{\left(\frac{\partial \phi_{jj}}{\partial \xi}\frac{\partial \phi_{kk}}{\partial \eta} - \frac{\partial \phi_{jj}}{\partial \eta}\frac{\partial \phi_{kk}}{\partial \xi}\right)}_{T_{jj,kk}} z_{e,kk}.} \tag{20.11}$$

Similarly

$$\frac{\partial \phi_{jj}}{\partial z^e} = \frac{\partial \phi_{jj}}{\partial \xi}\frac{\partial \xi}{\partial z^e} + \frac{\partial \phi_{jj}}{\partial \eta}\frac{\partial \eta}{\partial z^e} = \frac{1}{\det J_e}\left(-\frac{\partial \phi}{\partial \xi}\frac{\partial r^e}{\partial \eta} + \frac{\partial \phi}{\partial \eta}\frac{\partial r^e}{\partial \xi}\right), \tag{20.12}$$

and hence

$$\frac{\partial \phi_{jj}}{\partial z^e} = \frac{1}{\det J_e}\left[-\frac{\partial \phi_{jj}}{\partial \xi}\left(\sum_{kk=1}^{6} r^e_{kk}\frac{\partial \phi_{kk}}{\partial \eta}\right) + \frac{\partial \phi_{jj}}{\partial \eta}\left(\sum_{kk=1}^{6} r^e_{kk}\frac{\partial \phi_{kk}}{\partial \xi}\right)\right], \tag{20.13}$$

i.e.

$$\frac{\partial \phi_{jj}}{\partial z^e} = \frac{1}{\det J_e}\left[-\left(\sum_{kk=1}^{6} r^e_{kk}\frac{\partial \phi_{jj}}{\partial \xi}\frac{\partial \phi_{kk}}{\partial \eta}\right) + \left(\sum_{kk=1}^{6} r^e_{kk}\frac{\partial \phi_{jj}}{\partial \eta}\frac{\partial \phi_{kk}}{\partial \xi}\right)\right], \tag{20.14}$$

yielding

$$\boxed{\frac{\partial \phi_{jj}}{\partial z^e} = \frac{1}{\det J_e}\sum_{kk=1}^{6}\underbrace{\left(-\frac{\partial \phi_{jj}}{\partial \xi}\frac{\partial \phi_{kk}}{\partial \eta} + \frac{\partial \phi_{jj}}{\partial \eta}\frac{\partial \phi_{kk}}{\partial \xi}\right)}_{-T_{jj,kk}} r^e_{kk}.} \tag{20.15}$$

Hence, we have now all the elements needed to compute all integrals over elements using the map to the master element.

20.1. **Integrals over elements mapped to master element**

Integration over the master element will be carried out using Gaussian quadrature, following Zhang *et al.* (2009). Therefore, all integrals will be transformed into sums over sampling points. Here we give the expression for each integral over a triangular element that was used above. Boxed equations correspond to the most convenient way of expressing the terms when coding them.

20.1.1. a **terms**

From (**??**), we have

$$a_{g_r,ii}(e) = \int_{\Omega_e} g_r \phi_{l(e,ii)}, \tag{20.16}$$

which we can re-write as

$$a_{g_z,ii}(e) = \int_{E} g_r \phi_{l(e,ii)} \det J_e, \tag{20.17}$$

and using Gaussian quadrature we have

$$\boxed{a_{g_z,ii} \approx \sum_{p=1}^{n_G} W(p) g_r \phi_{ii}(p) \det J_e(p)}. \tag{20.18}$$

From (**??**), we have

$$a_{ii,jj}(e) = \int_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,jj)}, \tag{20.19}$$

which we can re-write as

$$a_{ii,jj}(e) = \int_{E} \phi_{l(e,ii)} \phi_{l(e,jj)} \det J_e, \tag{20.20}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,jj}(e) \approx \sum_{p=1}^{n_G} W(p) \phi_{ii}(p) \phi_{jj}(p) \det J_e(p)}. \tag{20.21}$$

From (**??**) we also have

$$a^r_{ii,kk,jj}(e) = \int_{\Omega_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)}, \tag{20.22}$$

which we can re-write as

$$a^r_{ii,kk,jj}(e) = \int_{E} \phi_{ii} \phi_{kk} \left(\sum_{mm=1}^{6} T_{jj,mm} z_{e,mm} \right), \tag{20.23}$$

where we have cancelled the $\det J_e$ in the denominator of the expression for the derivative of $\phi_{l(e,jj)}$ with the one that corresponds to the Jacobian of the change of coordinates. Moreover,

$$\phi_{ii}(\cdot) = \phi_{l(e,ii)}(S_e(\cdot)), \tag{20.24}$$

where, as mentioned before, S_e maps points in the master element onto points in the element being considered. We highlight that ϕ_{ii} can be named in this way (with no reference to the original element, i.e. element number e) because once the integral is mapped to the master element, all information about the original element is stored in J_e (i.e. the Jacobian of S_e). That is to say, ϕ_{ii} no longer depends on the specific element.

Now, using Gaussian quadrature we have

$$\boxed{a^r_{ii,kk,jj}(e) \approx \sum_{p=1}^{n_G} W(p)\phi_{ii}(p)\phi_{kk}(p)\left(\sum_{mm=1}^{6} T_{jj,mm}(p)z_{e,mm}\right)}, \tag{20.25}$$

where we are using the notation $f(p)$ as a short version for $f(\xi_p,\eta_p)$, with (ξ_p,η_p) being the p-th Gaussian quadrature point, and $W(p)$ is the weight associated to the p-th Gaussian quadrature point (out of n_G points); and, as usual, double letter indexes are used to indicate local numbering.

Also from (**??**) we have

$$a^z_{ii,kk,jj}(e) = \int_{\Omega_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)}, \tag{20.26}$$

which we can re-write as

$$a^z_{ii,kk,jj}(e) = -\int_E \phi_{ii}\phi_{kk}\left(\sum_{mm=1}^{6} T_{jj,mm}r_{e,mm}\right), \tag{20.27}$$

and using Gaussian quadrature we have

$$\boxed{a^z_{ii,kk,jj}(e) \approx -\sum_{p=1}^{n_G} W(p)\phi_{ii}(p)\phi_{kk}(p)\left(\sum_{mm=1}^{6} T_{jj,mm}(p)r_{e,mm}\right)}. \tag{20.28}$$

From (**??**) we also have

$$a^{r,r}_{ii,jj}(e) = \int_{\Omega_e} \partial_r\phi_{l(e,ii)}\partial_r\phi_{l(e,jj)}, \tag{20.29}$$

which we can re-write as

$$a^{r,r}_{ii,jj}(e) = \int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm}z_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn}z_{e,nn}\right)}{\det J_e}, \tag{20.30}$$

where we have cancelled one of the $\det Je$ terms in the denominators of the derivatives with the Jacobian of the change of coordinates.

Now, using Gaussian quadrature we have

$$\boxed{a^{r,r}_{ii,jj}(e) \approx \sum_{p=1}^{n_G} W(p)\frac{\left(\sum_{mm=1}^{6} T_{ii,mm}(p)z_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn}(p)z_{e,nn}\right)}{\det J_e(p)}}. \tag{20.31}$$

From (**??**) we also have

$$a^{z,r}_{ii,jj}(e) = \int_{\Omega_e} \partial_z\phi_{l(e,ii)}\partial_r\phi_{l(e,jj)} \tag{20.32}$$

which we can re-write as

$$a^{z,r}_{ii,jj}(e) = -\int\limits_{E} \frac{\left(\sum\limits_{mm=1}^{6} T_{ii,mm} r_{e,mm}\right)\left(\sum\limits_{nn=1}^{6} T_{jj,nn} z_{e,nn}\right)}{\det J_e}. \tag{20.33}$$

Using Gaussian quadrature we have

$$\boxed{a^{z,r}_{ii,jj}(e) \approx -\sum_{p=1}^{n_G} W(p) \frac{\left(\sum\limits_{mm=1}^{6} T_{ii,mm}(p) r^{e}_{mm}\right)\left(\sum\limits_{nn=1}^{6} T_{jj,nn}(p) z^{e}_{nn}\right)}{\det J_e(p)}}. \tag{20.34}$$

From (**??**) we also have

$$a^{z,z}_{ii,jj}(e) = \int\limits_{\Omega_e} \partial_z \phi_{l(e,ii)} \partial_z \phi_{l(e,jj)}, \tag{20.35}$$

which we can re-write as

$$a^{z,z}_{ii,jj}(e) = \int\limits_{E} \frac{\left(\sum\limits_{mm=1}^{6} T_{ii,mm} r_{e,mm}\right)\left(\sum\limits_{nn=1}^{6} T_{jj,nn} r_{e,nn}\right)}{\det J_e}, \tag{20.36}$$

and using Gaussian quadrature we have

$$\boxed{a^{z,z}_{ii,jj}(e) \approx \sum_{p=1}^{n_G} W(p) \frac{\left(\sum\limits_{mm=1}^{6} T_{ii,mm}(p) r_{e,mm}\right)\left(\sum\limits_{nn=1}^{6} T_{jj,nn}(p) r_{e,nn}\right)}{\det J_e(p)}}. \tag{20.37}$$

From (**??**) we have

$$a^{r,z}_{ii,jj}(e) = \int\limits_{\Omega_e} \partial_r \phi_{l(e,ii)} \partial_z \phi_{l(e,jj)}, \tag{20.38}$$

which we can re-write as

$$a^{r,z}_{ii,jj}(e) = -\int\limits_{E} \frac{\left(\sum\limits_{mm=1}^{6} T_{ii,mm} z_{e,mm}\right)\left(\sum\limits_{nn=1}^{6} T_{jj,nn} r_{e,nn}\right)}{\det J_e}, \tag{20.39}$$

where we have once again cancelled one of the $\det Je$ terms in the denominators of the derivatives with the Jacobian of the change of coordinates.

Using Gaussian quadrature we have

$$\boxed{a^{r,z}_{ii,jj}(e) \approx -\sum_{p=1}^{n_G} W(p) \frac{\left(\sum\limits_{mm=1}^{6} T_{ii,mm}(p) z_{e,mm}\right)\left(\sum\limits_{nn=1}^{6} T_{jj,nn}(p) r_{e,nn}\right)}{\det J_e(p)}}. \tag{20.40}$$

Finally, from (**??**), we have

$$a_{g_z,ii}(e) = \int\limits_{\Omega_e} g_z \phi_{l(e,ii)}, \tag{20.41}$$

which we can re-write as

$$a_{g_z,ii}(e) = \int\limits_E g_z \phi_{l(e,ii)} \det J_e, \tag{20.42}$$

and using Gaussian quadrature we have

$$\boxed{a_{g_z,ii} \approx \sum_{p=1}^{n_G} W(p) g_z \phi_{ii}(p) \det J_e(p)}. \tag{20.43}$$

20.1.2. b terms

From equation (**??**) we have

$$b^r_{jj,ii}(e) = \int\limits_{\Omega_e} \psi_{l^p(e,jj)} \partial_r \phi_{l(e,ii)}, \tag{20.44}$$

which we can re-write as

$$b^r_{jj,ii}(e) = \int\limits_E \psi_{jj} \left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm} \right), \tag{20.45}$$

where we have cancelled the terms $\det J_e$ in the denominator of the expression for the derivative of ϕ with the one in the Jacobian of the change of coordinates. Here we have once again used the notation

$$\psi_{jj}(\cdot) = \psi_{l^p(e,jj)}(S_e(\cdot)), \tag{20.46}$$

where S_e maps the master element onto the specific element being considered. We highlight again that once the maps to the master element is done there is no need to preserve a reference to the specific element in the notation (which is why the is no mention of the element index e in the notation introduced). Once again, all the information regarding the specific element is stored in S_e and, consequently, in its Jacobian J_e. However, there is one exception to this rule that is worth noting. An element that contains the contact line is likely to contain a pressure singularity, and therefore it can be convenient to use a different (specifically a singular) interpolating function for pressure at this node. For our meshing this corresponds to element 1 only, which might therefore need to be treated separately.

Now, using Gaussian quadrature we have

$$\boxed{b^r_{jj,ii} \approx \sum_{p=1}^{n_G} W(p) \psi_{jj}(p) \sum_{mm=1}^{6} T_{ii,mm}(p) z_{e,mm}}. \tag{20.47}$$

Similarly, from equation (**??**), we have

$$b^z_{jj,ii}(e) = \int\limits_{\Omega_e} \psi_{l^p(e,jj)} \partial_z \phi_{l(e,ii)} \tag{20.48}$$

which we can re-write as

$$b^z_{jj,ii}(e) = -\int\limits_E \psi_{jj} \left(\sum_{mm=1}^{6} T_{ii,mm} r_{e,mm} \right), \tag{20.49}$$

and using Gaussian quadrature we have

$$\boxed{b^z_{jj,ii}(e) \approx -\sum_{p=1}^{n_G} W(p)\psi_{jj}(p) \sum_{mm=1}^{6} T_{ii,mm}(p) r_{e,mm}}. \tag{20.50}$$

20.2. Derivatives of integrals over triangle elements

The expressions above contain all terms that depend of the coordinates of each element. That is to say, the residuals are given by the product of these expressions by variables that (for the purpose of the resulting non-linear system of equations) do not depend of the variables (h) that determine the shape of our domain. Therefore, in order to calculate the derivatives of the residuals with respect to the lengths of the spines, we need to calculate the derivatives of these expressions.

Furthermore, in the above expressions, all functions are independent of the location of the nodes, except for $r_{e,mm}$, $z_{e,mm}$ and $\det J_e$. We consider now the derivative of $\det J_e$

$$\partial_{h_q} \det J_e = \sum_{ii=1}^{6} \sum_{jj=1}^{6} \partial_{h_q} \left(r_{e,ii} T_{ii,jj} z_{e,jj} \right), \tag{20.51}$$

which yields

$$\boxed{\partial_{h_q} \det J_e = \sum_{ii=1}^{6} \sum_{jj=1}^{6} \left[\left(\partial_{h_q} r^e_{ii}\right) T_{ii,jj} z_{e,jj} + r_{e,ii} T_{ii,jj} \left(\partial_{h_q} z_{e,jj}\right) \right]}, \tag{20.52}$$

which reduces the problem to finding the derivatives of $r_{e,ii}$ and $z_{e,jj}$ with respect to each h_q. The way to calculate the latter two derivatives will be explained in later sections.

20.2.1. Derivatives of a terms

From (20.17) we have

$$\partial_{h_q} a_{g_r,ii}(e) = \partial_{h_q} \int_E g_r \phi_{l(e,ii)} \det J_e, \tag{20.53}$$

i.e.

$$\partial_{h_q} a_{g_r,ii}(e) = \int_E g_r \phi_{l(e,ii)} \left(\partial_{h_q} \det J_e\right), \tag{20.54}$$

and using Gaussian quadrature

$$\boxed{\partial_{h_q} a_{g_r,ii} \approx \sum_{p=1}^{n_G} W(p) g_r \phi_{ii}(p) \left(\partial_{h_q} \det J_e(p)\right)}. \tag{20.55}$$

From (20.20) we have

$$\partial_{h_q} a_{ii,jj}(e) = \partial_{h_q} \int_E \phi_{l(e,ii)} \phi_{l(e,jj)} \det J_e, \tag{20.56}$$

i.e.

$$\partial_{h_q} a_{ii,jj}(e) = \int_E \phi_{l(e,ii)} \phi_{l(e,jj)} \left(\partial_{h_q} \det J_e\right), \tag{20.57}$$

and using Gaussian quadrature

$$\boxed{\partial_{h_q} a_{ii,jj} \approx \sum_{p=1}^{n_G} W(p)\phi_{l(e,ii)}\phi_{l(e,jj)}\left(\partial_{h_q} \det J_e(p)\right)}. \tag{20.58}$$

From (20.23), we have

$$\partial_{h_q} a^r_{ii,kk,jj}(e) = \partial_{h_q} \int_E \phi_{ii}\phi_{kk}\left(\sum_{mm=1}^{6} T_{jj,mm} z^e_{mm}\right), \tag{20.59}$$

i.e.

$$\partial_{h_q} a^r_{ii,kk,jj}(e) = \int_E \phi_{ii}\phi_{kk}\left(\sum_{mm=1}^{6} T_{jj,mm}\left(\partial_{h_q} z_{e,mm}\right)\right), \tag{20.60}$$

and using Gaussian quadrature we have

$$\boxed{\partial_{h_q} a^r_{ii,kk,jj}(e) \approx \sum_{p=1}^{n_G} W(p)\phi_{ii}(p)\phi_{kk}(p)\left[\sum_{mm=1}^{6} T_{jj,mm}(p)\left(\partial_{h_q} z_{e,mm}\right)\right]}. \tag{20.61}$$

Similarly, from (20.27), we have

$$\partial_{h_q} a^z_{ii,kk,jj}(e) = -\partial_{h_q} \int_E \phi_{ii}\phi_{kk}\left(\sum_{mm=1}^{6} T_{jj,mm} r_{e,mm}\right), \tag{20.62}$$

i.e.

$$\partial_{h_q} a^z_{ii,kk,jj}(e) = -\int_E \phi_{ii}\phi_{kk}\left(\sum_{mm=1}^{6} T_{jj,mm}\left(\partial_{h_q} r_{e,mm}\right)\right), \tag{20.63}$$

and using Gaussian quadrature we have

$$\boxed{\partial_{h_q} a^z_{ii,kk,jj}(e) \approx -\sum_{p=1}^{n_G} W(p)\phi_{ii}(p)\phi_{kk}(p)\left[\sum_{mm=1}^{6} T_{jj,mm}(p)\left(\partial_{h_q} r_{e,mm}\right)\right]}. \tag{20.64}$$

From (20.30) we have

$$\partial_{h_q} a^{r,r}_{ii,jj}(e) = \partial_{h_q} \int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn} z_{e,nn}\right)}{\det J_e}, \tag{20.65}$$

i.e.

$$\partial_{h_q} a^{r,r}_{ii,jj}(e) = \int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm}\left(\partial_{h_q} z_{e,mm}\right)\right)\left(\sum_{nn=1}^{6} T_{jj,nn} z_{e,nn}\right)}{\det J_e} + \int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn}\left(\partial_{h_q} z_{e,nn}\right)\right)}{\det J_e} - \int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn} z_{e,nn}\right)}{(\det J_e)^2} \partial_{h_q} \det J_e, \tag{20.66}$$

and using Gaussian quadrature we have

$$\begin{aligned}
&\partial_{h_q} a^{r,r}_{ii,jj}(e) \\
&\quad\approx \sum_{p=1}^{n_G} W(p) \frac{\left[\sum_{mm=1}^{6} T_{ii,mm}(p)\left(\partial_{h_q} z_{e,mm}\right)\right]\left[\sum_{nn=1}^{6} T_{jj,nn}(p) z_{e,nn}\right]}{\det J_e(p)} \\
&\qquad+ \sum_{p=1}^{n_G} W(p) \frac{\left[\sum_{mm=1}^{6} T_{ii,mm}(p) z_{e,mm}\right]\left[\sum_{nn=1}^{6} T_{jj,nn}(p)\left(\partial_{h_q} z_{e,nn}\right)\right]}{\det J_e(p)} \\
&\qquad- \sum_{p=1}^{n_G} W(p) \frac{\left[\sum_{mm=1}^{6} T_{ii,mm}(p) z_{e,mm}\right]\left[\sum_{nn=1}^{6} T_{jj,nn}(p) z_{e,nn}\right]}{\left[\det J_e(p)\right]^2} \partial_{h_q} \det J_e(p),
\end{aligned} \tag{20.67}$$

Similarly, from (20.39), we have

$$\partial_{h_q} a^{r,z}_{ii,jj}(e) = -\partial_{h_q} \int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn} r_{e,nn}\right)}{\det J_e}, \tag{20.68}$$

i.e.

$$\partial_{h_q} a^{r,z}_{ii,jj}(e) = -\int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm}\left(\partial_{h_q} z_{e,mm}\right)\right)\left(\sum_{nn=1}^{6} T_{jj,nn} r_{e,nn}\right)}{\det J_e} - \int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn}\left(\partial_{h_q} r_{e,nn}\right)\right)}{\det J_e} + \int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn} r_{e,nn}\right)}{(\det J_e)^2} \partial_{h_q} \det J_e, \tag{20.69}$$

and using Gaussian quadrature we have

$$
\begin{aligned}
&\partial_{h_q} a^{r,z}_{ii,jj}(e) \\
&\quad \approx -\sum_{p=1}^{n_G} W(p) \frac{\left[\sum\limits_{mm=1}^{6} T_{ii,mm}(p)\left(\partial_{h_q} z_{e,mm}\right)\right]\left[\sum\limits_{nn=1}^{6} T_{jj,nn}(p) r_{e,nn}\right]}{\det J_e(p)} \\
&\quad -\sum_{p=1}^{n_G} W(p) \frac{\left[\sum\limits_{mm=1}^{6} T_{ii,mm}(p) z_{e,mm}\right]\left[\sum\limits_{nn=1}^{6} T_{jj,nn}(p)\left(\partial_{h_q} r_{e,nn}\right)\right]}{\det J_e(p)} \\
&\quad +\sum_{p=1}^{n_G} W(p) \frac{\left[\sum\limits_{mm=1}^{6} T_{ii,mm}(p) z_{e,mm}\right]\left[\sum\limits_{nn=1}^{6} T_{jj,nn}(p) r_{e,nn}\right]}{\left[\det J_e(p)\right]^2} \partial_{h_q} \det J_e(p).
\end{aligned}
\tag{20.70}
$$

From (20.33) we have

$$
\partial_{h_q} a^{z,r}_{ii,jj}(e) = -\partial_{h_q} \int_E \frac{\left(\sum\limits_{mm=1}^{6} T_{ii,mm} r_{e,mm}\right)\left(\sum\limits_{nn=1}^{6} T_{jj,nn} z_{e,nn}\right)}{\det J_e}, \tag{20.71}
$$

i.e.

$$
\begin{aligned}
\partial_{h_q} a^{z,r}_{ii,jj}(e) = &-\int_E \frac{\left(\sum\limits_{mm=1}^{6} T_{ii,mm}\left(\partial_{h_q} r_{e,mm}\right)\right)\left(\sum\limits_{nn=1}^{6} T_{jj,nn} z_{e,nn}\right)}{\det J_e} \\
&-\int_E \frac{\left(\sum\limits_{mm=1}^{6} T_{ii,mm} r_{e,mm}\right)\left(\sum\limits_{nn=1}^{6} T_{jj,nn}\left(\partial_{h_q} z_{e,nn}\right)\right)}{\det J_e} \\
&+\int_E \frac{\left(\sum\limits_{mm=1}^{6} T_{ii,mm} r_{e,mm}\right)\left(\sum\limits_{nn=1}^{6} T_{jj,nn} z_{e,nn}\right)}{\left(\det J_e\right)^2} \partial_{h_q} \det J_e,
\end{aligned}
\tag{20.72}
$$

which, using Gaussian quadrature yields,

$$\begin{aligned}
&\partial_{h_q} a^{z,r}_{ii,jj}(e) \\
&\quad \approx -\sum_{p=1}^{n_G} W(p) \frac{\left[\sum_{mm=1}^{6} T_{ii,mm}(p)\left(\partial_{h_q} r_{e,mm}\right)\right]\left[\sum_{nn=1}^{6} T_{jj,nn}(p) z_{e,nn}\right]}{\det J_e(p)} \\
&\qquad -\sum_{p=1}^{n_G} W(p) \frac{\left[\sum_{mm=1}^{6} T_{ii,mm}(p) r_{e,mm}\right]\left[\sum_{nn=1}^{6} T_{jj,nn}(p)\left(\partial_{h_q} z_{e,nn}\right)\right]}{\det J_e(p)} \\
&\qquad +\sum_{p=1}^{n_G} W(p) \frac{\left[\sum_{mm=1}^{6} T_{ii,mm}(p) r_{e,mm}\right]\left[\sum_{nn=1}^{6} T_{jj,nn}(p) z_{e,nn}\right]}{\left[\det J_e(p)\right]^2} \partial_{h_q} \det J_e(p).
\end{aligned} \tag{20.73}$$

From (20.36) we have

$$\partial_{h_q} a^{z,z}_{ii,jj}(e) = \partial_{h_q} \int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm} r_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn} r_{e,nn}\right)}{\det J_e}, \tag{20.74}$$

i.e.

$$\begin{aligned}
\partial_{h_q} a^{z,z}_{ii,jj}(e) = &\int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm}\left(\partial_{h_q} r_{e,mm}\right)\right)\left(\sum_{nn=1}^{6} T_{jj,nn} r_{e,nn}\right)}{\det J_e} \\
&+\int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm} r_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn}\left(\partial_{h_q} r_{e,nn}\right)\right)}{\det J_e} \\
&-\int_E \frac{\left(\sum_{mm=1}^{6} T_{ii,mm} r_{e,mm}\right)\left(\sum_{nn=1}^{6} T_{jj,nn} r_{e,nn}\right)}{\left(\det J_e\right)^2} \partial_{h_q} \det J_e,
\end{aligned} \tag{20.75}$$

and using Gaussian quadrature we have

$$
\begin{aligned}
&\partial_{h_q} a^{z,z}_{ii,jj}(e) \\
&\quad \approx \sum_{p=1}^{n_G} W(p) \frac{\left[\sum\limits_{mm=1}^{6} T_{ii,mm}(p)\partial_{h_q}\left(r_{e,mm}\right)\right]\left[\sum\limits_{nn=1}^{6} T_{jj,nn}(p) r_{e,nn}\right]}{\det J_e(p)} \\
&\qquad + \sum_{p=1}^{n_G} W(p) \frac{\left[\sum\limits_{mm=1}^{6} T_{ii,mm}(p) r_{e,mm}\right]\left[\sum\limits_{nn=1}^{6} T_{jj,nn}(p)\partial_{h_q}\left(r_{e,nn}\right)\right]}{\det J_e(p)} \\
&\qquad - \sum_{p=1}^{n_G} W(p) \frac{\left[\sum\limits_{mm=1}^{6} T_{ii,mm}(p) r_{e,mm}\right]\left[\sum\limits_{nn=1}^{6} T_{jj,nn}(p) r_{e,nn}\right]}{\left[\det J_e(p)\right]^2}\partial_{h_q}\left(\det J_e(p)\right).
\end{aligned}
\tag{20.76}
$$

Finally, from (20.42) we have

$$\partial_{h_q} a_{g_z,ii}(e) = \partial_{h_q} \int_E g_z \phi_{l(e,ii)} \det J_e, \tag{20.77}$$

i.e.

$$\partial_{h_q} a_{g_z,ii}(e) = \int_E g_z \phi_{l(e,ii)} \left(\partial_{h_q} \det J_e\right), \tag{20.78}$$

and using Gaussian quadrature

$$\boxed{\partial_{h_q} a_{g_z,ii} \approx \sum_{p=1}^{n_G} W(p) g_z \phi_{ii}(p) \left(\partial_{h_q} \det J_e(p)\right)}. \tag{20.79}$$

20.2.2. **Derivatives of b terms**

From equation (20.44) we have

$$\partial_{h_q} b^r_{jj,ii}(e) = \partial_{h_q} \int_E \psi_{jj} \left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm}\right), \tag{20.80}$$

i.e.

$$\partial_{h_q} b^r_{jj,ii}(e) = \int_E \psi_{jj} \left(\sum_{mm=1}^{6} T_{ii,mm} \left(\partial_{h_q} z_{e,mm}\right)\right), \tag{20.81}$$

which using Gaussian quadrature yields

$$\boxed{\partial_{h_q} b^r_{jj,ii} \approx \sum_{p=1}^{n_G} W(p) \psi_{jj}(p) \left[\sum_{mm=1}^{6} T_{ii,mm}(p) \left(\partial_{h_q} z_{e,mm}\right)\right]}. \tag{20.82}$$

Similarly, from (20.48) we have

$$\partial_{h_q} b^z_{jj,ii}(e) = -\partial_{h_q} \int_E \psi_{jj} \left(\sum_{mm=1}^{6} T_{ii,mm} r_{e,mm}\right), \tag{20.83}$$

i.e.

$$\partial_{h_q} b^z_{jj,ii}(e) = -\int_E \psi_{jj} \left(\sum_{mm=1}^{6} T_{ii,mm} \left(\partial_{h_q} r_{e,mm} \right) \right), \tag{20.84}$$

and using Gaussian quadrature we have

$$\boxed{\partial_{h_q} b^r_{jj,ii} \approx -\sum_{p=1}^{n_G} W(p)\psi_{jj}(p) \left[\sum_{mm=1}^{6} T_{ii,mm}(p) \left(\partial_{h_q} r_{e,mm} \right) \right]}. \tag{20.85}$$

21. Integrals over line elements

21.1. The free-surface line elements

From (**??**) we have

$$c^s_{t_r,jj,ii}(e_1) = \int_{\partial\Omega_{e_1}} t^1_r \phi^1_{l_1(e_1,jj)} \partial_s \phi^1_{l_1(e_1,ii)}. \tag{21.1}$$

In order to simplify our calculations when we consider the integral above and others on the same boundary, we will ensure that our line elements that lay on the free-surface always correspond to the side of the element containing the nodes of local number 2, 6 and 3 (see figure 10). Hence, line-elements on boundary 1 are easily parameterised by the variable ξ. More specifically we have

$$\phi^1_1(\xi) = \phi_2(\xi, \eta = -1), \tag{21.2}$$

$$\phi^1_2(\xi) = \phi_6(\xi, \eta = -1) \tag{21.3}$$

and

$$\phi^1_3(\xi) = \phi_3(\xi, \eta = -1); \tag{21.4}$$

since the line element is given by the equation $\eta = -1$ in the master element.

From equation (19.5) we have

$$\phi^1_1(\xi) = \frac{\xi^2 + \eta^2 + 2\xi\eta + \xi + \eta}{2}\Big|_{\eta=-1} = \frac{\xi^2 + 1 - 2\xi + \xi - 1}{2}, \tag{21.5}$$

i.e.

$$\boxed{\phi^1_1(\xi) = \frac{\xi^2 - \xi}{2}}; \tag{21.6}$$

from (19.9) we have

$$\phi^1_2(\xi) = -\xi^2 - \xi\eta - \xi - \eta\big|_{\eta=-1} = -\xi^2 + \xi - \xi + 1, \tag{21.7}$$

i.e.

$$\boxed{\phi^1_2(\xi) = -\xi^2 + 1}; \tag{21.8}$$

and from (19.6) we have

$$\boxed{\phi^1_3(\xi) = \frac{\xi^2 + \xi}{2}}. \tag{21.9}$$

Consequently,

$$\boxed{\partial_\xi \phi^1_1(\xi) = \xi - \frac{1}{2}}, \tag{21.10}$$

$$\boxed{\partial_\xi \phi_2^1(\xi) = -2\xi}, \tag{21.11}$$

$$\boxed{\partial_\xi \phi_3^1(\xi) = \xi + \frac{1}{2}}. \tag{21.12}$$

We can calculate the tangent to the line element by

$$\boldsymbol{t}^1 = \frac{(\partial_\xi r_{e_1}^1, \partial_\xi z_{e_1}^1)}{\sqrt{\left(\partial_\xi r_{e_1}^1\right)^2 + \left(\partial_\xi z_{e_1}^1\right)^2}}, \tag{21.13}$$

where the tangent points in the direction of increasing ξ and $r_{e_1}^1$ is the r coordinate along on element e_1 on boundary 1. $r_{e_1}^1$ and its analogue for z are defined by the map $S_{e_1}^1$ which takes the interval $[-1, 1]$ to the line element in boundary one, i.e.

$$(r_{e_1}^1, z_{e_1}^1) = S_{e_1}^1(\xi). \tag{21.14}$$

Moreover, we have

$$\boxed{\partial_\xi r_{e_1}^1 = \sum_{jj=1}^{3} r_{e_1,jj}^1 \partial_\xi \phi_{jj}^1}, \tag{21.15}$$

and

$$\boxed{\partial_\xi z_{e_1}^1 = \sum_{jj=1}^{3} z_{e_1,jj}^1 \partial_\xi \phi_{jj}^1}, \tag{21.16}$$

where we have once again used the fact that once we have mapped to the master element (in this case the interval $[-1, 1]$) the interpolating functions ϕ no longer depend on the coordinate of the specific element to introduce the notation

$$\phi_{ii}^1(\cdot) = \phi_{l_1(e_1,jj)}^1(S_{e_1}^1(.)). \tag{21.17}$$

Furthermore, the derivatives with respect to the arc-length s, can be calculated using

$$\partial_s f = \partial_\xi f \partial_s \xi, \tag{21.18}$$

and we introduce

$$\boxed{J_{e_1}^1 := \partial_\xi s = \sqrt{\left(\partial_\xi r_{e_1}^1\right)^2 + \left(\partial_\xi z_{e_1}^1\right)^2}}, \tag{21.19}$$

which is the determinant of the Jacobian of $S_{e_1}^1$.

We also highlight the the integral we are considering is a line integral and therefore when parameterising by ξ to actually perform the calculation we need to multiply the integrand by the derivative of the arc-length, yielding

$$c_{t_r,jj,ii}^s(e_1) = \int_{\xi=-1}^{\xi=1} \frac{\partial_\xi r_{e_1}^1(\xi)}{J_{e_1}^1(\xi)} \phi_{jj}^1(\xi) \partial_s \phi_{ii}^1(\xi) \partial_\xi s. \tag{21.20}$$

We now use the chain rule on the last two terms in the integrand, which yields

$$c_{t_r,jj,ii}^s(e_1) = \int_{\xi=-1}^{\xi=1} \frac{\partial_\xi r_{e_1}^1(\xi)}{J_{e_1}^1(\xi)} \phi_{jj}^1(\xi) \partial_\xi \phi_{ii}^1(\xi), \tag{21.21}$$

Hence, using Gaussian quadrature we have

$$\boxed{c^s_{t_r,jj,ii}(e_1) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{\partial_\xi r^1_{e_1}(p)}{J^1_{e_1}(p)} \phi^1_{jj}(p) \partial_\xi \phi^1_{ii}(p)}, \tag{21.22}$$

where we are using the notation $f(p)$ as a short version of $f(\xi_p)$, with ξ_p is the p-th Gaussian quadrature point and $W_l(p)$ is the p-th Gaussian quadrature weights (out of n_{lG} total points).

Similarly, from (5.86) we have

$$c^s_{t_z,jj,ii}(e_1) = \int_{\partial\Omega_{e_1}} t^1_z \phi^1_{l_1(e_1,jj)} \partial_s \phi^1_{l_1(e_1,ii)}, \tag{21.23}$$

which can be re-written as

$$c^s_{t_z,jj,ii}(e_1) = \int_{\xi=-1}^{\xi=1} \frac{\partial_\xi z^1_{e_1}(\xi)}{J^1_{e_1}(\xi)} \phi^1_{jj}(\xi) \partial_\xi \phi^1_{ii}(\xi), \tag{21.24}$$

which, using Gaussian quadrature, yields

$$\boxed{c^s_{t_z,jj,ii}(e_1) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{\partial_\xi z(p)}{J^1_{e_1}(p)} \phi^1_{jj}(p) \partial_\xi \phi^1_{ii}(p)}. \tag{21.25}$$

From equation (13.13) we have

$$c_{ii,jj,n_r}(e_1) = \int_{\Omega_{e_1}} n^1_r \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)}. \tag{21.26}$$

Here, we recall that

$$\boldsymbol{n}^1 = \frac{\alpha(-\partial_\xi z^1_{e_1}, \partial_\xi r^1_{e_1})}{\sqrt{\left(\partial_\xi r^1_{e_1}\right)^2 + \left(\partial_\xi z^1_{e_1}\right)^2}}, \tag{21.27}$$

where $\alpha = 1$ is the rotation is counter-clockwise and $\alpha = -1$ if the rotation is clockwise. On boundary 1 we decided to have the local line-element numbering so as to have $\alpha = 1$.

We can now re-write the integral above as

$$c_{ii,jj,n_r}(e_1) = -\int_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \phi^1_{jj}(\xi), \tag{21.28}$$

where we have cancelled the denominator of the expression for the normal to surface 1 with the Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{c_{ii,jj,n_r}(e_1) \approx -\sum_{p=1}^{n_{lG}} W_l(p) \partial_\xi z^1_{e_1}(p) \phi^1_{ii}(p) \phi^1_{jj}(p)}. \tag{21.29}$$

From equation (13.13) we also have

$$c_{ii,jj,n_z}(e_1) = \int_{\Omega_{e_1}} n^1_z \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)}, \tag{21.30}$$

which can be re-written as

$$c_{ii,jj,n_z}(e_1) = \int_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\phi^1_{jj}(\xi), \tag{21.31}$$

and using Gaussian quadrature we have

$$\boxed{c_{ii,jj,n_z}(e_1) \approx \sum_{p=1}^{n_{lG}} W_l(p)\partial_\xi r^1_{e_1}(p)\phi^1_{ii}(p)\phi^1_{jj}(p)}. \tag{21.32}$$

21.2. **The liquid-solid surface line elements**

We can also arrange all local node numbering to guarantee that every side of a triangular element that falls on boundary 2 is the side containing local nodes 1, 5 and 2 (see figure 10). This allows us to have a natural parametrisation of these line elements using variable η. More specifically we have

$$\phi^2_1(\xi) = \phi_2(\xi = -1, \eta), \tag{21.33}$$

$$\phi^2_2(\xi) = \phi_5(\xi = -1, \eta) \tag{21.34}$$

and

$$\phi^2_3(\xi) = \phi_1(\xi = -1, \eta); \tag{21.35}$$

since the line element is given by the equation $\xi = -1$ in the master element.

From equation (19.5) we have

$$\phi^2_1(\eta) = \frac{\xi^2 + \eta^2 + 2\xi\eta + \xi + \eta}{2}|_{\xi=-1} = \frac{1 + \eta^2 - 2\eta - 1 + \eta}{2}, \tag{21.36}$$

i.e.

$$\boxed{\phi^2_1(\eta) = \frac{\eta^2 - \eta}{2}}; \tag{21.37}$$

from (19.8) we have

$$\phi^2_2(\eta) = -\eta^2 - \xi\eta - \xi - \eta|_{\xi=-1} = -\eta^2 + \eta + 1 - \eta, \tag{21.38}$$

i.e.

$$\boxed{\phi^2_2(\eta) = -\eta^2 + 1}; \tag{21.39}$$

and from (19.4) we have

$$\boxed{\phi^2_3(\eta = \frac{\eta^2 + \eta}{2}}. \tag{21.40}$$

Consequently

$$\boxed{\partial_\eta \phi^2_1(\eta) = \eta - \frac{1}{2}}, \tag{21.41}$$

$$\boxed{\partial_\eta \phi^2_2(\eta) = -2\eta}, \tag{21.42}$$

and

$$\boxed{\partial_\eta \phi^2_3(\eta) = \eta + \frac{1}{2}}. \tag{21.43}$$

Now, from (**??**) we have

$$d_{t_r,t_r,ii,jj}(e_2) = \int_{\partial\Omega_{e_2}} t_r^2 t_r^2 \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)}, \tag{21.44}$$

where we now have

$$\boldsymbol{t}^2 = \frac{(\partial_\eta r^2_{e_2}, \partial_\eta z^2_{e_2})}{\sqrt{\left(\partial_\eta r^2_{e_2}\right)^2 + \left(\partial_\eta z^2_{e_2}\right)^2}}, \tag{21.45}$$

which yields a tangent vector that points in the direction of increasing η.

Naturally, we also have

$$\boxed{\partial_\eta r^2_{e_2} = \sum_{jj=1}^{3} r^2_{e_2,jj} \partial_\eta \phi^1_{jj}}, \tag{21.46}$$

$$\boxed{\partial_\eta z^2_{e_2} = \sum_{jj=1}^{3} z^2_{e_2,jj} \partial_\eta \phi^1_{jj}} \tag{21.47}$$

and

$$\boxed{J^2_{e_2} \coloneqq \partial_\eta s = \sqrt{\left(\partial_\eta r\right)^2 + \left(\partial_\eta z\right)^2}}. \tag{21.48}$$

Hence, we can re-write (21.44) as

$$d_{t_r,t_r,ii,jj}(e_2) = \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2}{J^2_{e_2}} \phi^2_{ii}\phi^2_{jj}, \tag{21.49}$$

where we have cancelled one of the square roots from the denominator of the tangent vector components with the $\partial_\eta s$.

Now, using Gaussian quadrature we have

$$\boxed{d_{t_r,t_r,ii,jj}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r(p))^2}{J^2_{e_2}(p)} \phi^2_{ii}(p)\phi^2_{jj}(p)}, \tag{21.50}$$

where we are using the notation $f(p)$ as a short version of $f(\eta_p)$, with η_p is the p-th Gaussian quadrature point and $W_l(p)$ is the p-th Gaussian quadrature weight (out of n_{lG} total points).

Similarly, from (**??**) we also have

$$d_{t_r,t_z,ii,jj} = \int_{\partial\Omega^2} t_r^2 t_z^2 \phi_{l_2(e_2,ii)} \phi_{l_2(e_2,jj)}, \tag{21.51}$$

which can be re-written as

$$d_{t_r,t_z,ii,jj}(e_2) = \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{J^2_{e_2}} \phi^2_{ii}\phi^2_{jj}, \tag{21.52}$$

and using Gaussian quadrature we have

$$\boxed{d_{t_r,t_z,ii,jj}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))(\partial_\eta z^2_{e_2}(p))}{J^2_{e_2}(p)} \phi^2_{ii}(p)\phi^2_{jj}(p)}. \tag{21.53}$$

Now from (5.87) we have

$$d_{t_z,t_z,ii,jj} = \int\limits_{\partial\Omega^2} t_z^2 t_z^2 \phi_{l_2(e_2,ii)}\phi_{l_2(e_2,jj)}, \tag{21.54}$$

which can be re-written as

$$d_{t_z,t_z,ii,jj}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2}{J_{e_2}^2}\phi_{ii}^2\phi_{jj}^2, \tag{21.55}$$

and using Gaussian quadrature we have

$$\boxed{d_{t_z,t_z,ii,jj}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p)\frac{(\partial_\eta z_{e_2}^2(p))^2}{J_{e_2}^2(p)}\phi_{ii}^2(p)\phi_{jj}^2(p)}. \tag{21.56}$$

From (5.87) we also have

$$d_{ii,jj,n_r}(e_2) = \int\limits_{\partial\Omega_{e_2}} n_r^2 \phi_{l_2(e_2,ii)}\phi_{l_2(e_2,jj)}. \tag{21.57}$$

Here, we recall that

$$\boldsymbol{n}^2 = \frac{\alpha(-\partial_\eta z_{e_2}^2, \partial_\eta r_{e_2}^2)}{\sqrt{\left(\partial_\eta r_{e_2}^2\right)^2 + \left(\partial_\eta z_{e_2}^2\right)^2}}, \tag{21.58}$$

where $\alpha = 1$ is the rotation is counter-clockwise and $\alpha = -1$ if the rotation is clockwise. In boundary 2 we decided to have the local line-element numbering so as to have $\alpha = -1$. Hence, we have

$$d_{ii,jj,n_r}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{\partial_\eta z_{e_2}^2}{J_{e_2}^2}\phi_{ii}\phi_{jj}, \tag{21.59}$$

and, using Gaussian quadrature, we have

$$\boxed{d_{ii,jj,n_r}(e_2) \approx \sum_{p=1}^{n_{lG}} \frac{\partial_\eta z_{e_2}^2(p)}{J_{e_2}^2(p)}\phi_{ii}(p)\phi_{jj}(p)}. \tag{21.60}$$

Similarly, from (5.87) we have

$$d_{ii,jj,n_z}(e_2) = \int\limits_{\partial\Omega_{e_2}} n_z^2 \phi_{l_2(e_2,ii)}\phi_{l_2(e_2,jj)}, \tag{21.61}$$

which can be re-written as

$$d_{ii,jj,n_z}(e_2) = -\int\limits_{\eta=-1}^{\eta=1} \frac{\partial_\eta r_{e_2}^2}{J_{e_2}^2}\phi_{ii}\phi_{jj}, \tag{21.62}$$

and, using Gaussian quadrature, we have

$$\boxed{d_{ii,jj,n_z}(e_2) \approx -\sum_{p=1}^{n_{lG}} \frac{\partial_\eta r_{e_2}^2(p)}{J_{e_2}^2(p)}\phi_{ii}(p)\phi_{jj}(p)}. \tag{21.63}$$

21.3. The inflow boundary line elements

As a consequence of our prior choice of local numbering, the line elements along boundary 3 must correspond to the side of the master element that contains nodes 3, 4 and 1 (see figure 10). We then choose to parameterise these line elements using variable ξ. This implies that we have

$$\phi_1^3(\xi) = \phi_1(\xi, \eta = -\xi), \tag{21.64}$$

$$\phi_2^3(\xi) = \phi_4(\xi, \eta = -\xi) \tag{21.65}$$

and

$$\phi_3^3(\xi) = \phi_3(\xi, \eta = -\xi); \tag{21.66}$$

since the line element is given by the equation $\eta = -\xi$ in the master element.

From equation (19.4) we have

$$\phi_1^3(\xi) = \frac{\eta^2 + \eta}{2}|_{\eta=-\xi}, \tag{21.67}$$

i.e.

$$\boxed{\phi_1^3(\xi) = \frac{\xi^2 - \xi}{2}}; \tag{21.68}$$

from (19.7) we have

$$\phi_2^3(\xi) = \xi\eta + \xi + \eta + 1|_{\eta=-\xi} = -\xi^2 + \xi - \xi + 1, \tag{21.69}$$

i.e.

$$\boxed{\phi_2^3(\xi) = -\xi^2 + 1}; \tag{21.70}$$

and from (19.6) we have

$$\boxed{\phi_3^3(\xi) = \frac{\xi^2 + \xi}{2}}. \tag{21.71}$$

Consequently,

$$\boxed{\partial_\xi \phi_1^3(\xi) = \xi - \frac{1}{2}}, \tag{21.72}$$

$$\boxed{\partial_\xi \phi_2^3(\xi) = -2\xi} \tag{21.73}$$

and

$$\boxed{\partial_\xi \phi_3^3(\xi) = \xi + \frac{1}{2}}. \tag{21.74}$$

Now, from equation (**??**) we have

$$f_{t_r,ii,jj}(e_3) = \int\limits_{\partial\Omega_{e_3}} t_r^3 \phi_{l_3(e_3,ii)} \phi_{l_3(e_3,jj)}. \tag{21.75}$$

where we now have

$$\boldsymbol{t}^3 = \frac{(\partial_\xi r_{e_3}^3, \partial_\xi z_{e_3}^3)}{\sqrt{\left(\partial_\xi r_{e_3}^3\right)^2 + \left(\partial_\xi z_{e_3}^3\right)^2}}, \tag{21.76}$$

which yields a tangent vector that points in the direction of increasing η.

Naturally, we also have

$$\boxed{\partial_\eta r^3_{e_3} = \sum_{jj=1}^{3} r^3_{e_3,jj}\partial_\eta \phi^1_{jj}}, \tag{21.77}$$

$$\boxed{\partial_\eta r^3_{e_3} = \sum_{jj=1}^{3} r^3_{e_3,jj}\partial_\eta \phi^1_{jj}}, \tag{21.78}$$

and

$$J^3_{e_3} := \partial_\xi s = \sqrt{\left(\partial_\xi r^3_{e_3}\right)^2 + \left(\partial_\eta z^3_{e_3}\right)^2}. \tag{21.79}$$

Hence, we can re-write (21.75) as

$$f_{t_r,ii,jj}(e_3) = \int_{\xi=-1}^{\xi=1} \partial_\xi r^3_{e_3}\phi^3_{ii}\phi^3_{jj}, \tag{21.80}$$

where we have cancelled the denominator of the expression for the tangent with the Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{f_{t_r,ii,jj}(e_3) \approx \sum_{p=1}^{n_{lG}} \partial_\eta r^3_{e_3}(p)\phi_{ii}(p)\phi_{jj}(p)}. \tag{21.81}$$

From equation (5.88) we have

$$f_{t_z,ii,jj}(e_3) = \int_{\partial\Omega_{e_3}} t^3_z\phi^3_{l_3(e_3,ii)}\phi^3_{l_3(e_3,jj)}, \tag{21.82}$$

which can be re-written as

$$f_{t_z,ii,jj}(e_3) = \int_{\xi=-1}^{\xi=1} \partial_\xi z^3_{e_3}\phi^3_{ii}\phi^3_{jj}, \tag{21.83}$$

and using Gaussian quadrature we have

$$\boxed{f_{t_z,ii,jj}(e_3) \approx \sum_{p=1}^{n_{lG}} \partial_\eta z^3_{e_3}(p)\phi_{ii}(p)\phi_{jj}(p)}. \tag{21.84}$$

From equation (**??**) we have

$$f_{ii,jj,n_r}(e_3) = \int_{\partial\Omega_{e_3}} n^3_r\phi_{l_3(e_3,ii)}\phi_{l_3(e_3,jj)}. \tag{21.85}$$

Here, we recall that

$$\boldsymbol{n}^3 = \frac{\alpha(-\partial_\xi z^3_{e_3}, \partial_\xi r^3_{e_3})}{\sqrt{\left(\partial_\xi r^3_{e_3}\right)^2 + \left(\partial_\xi z^3_{e_3}\right)^2}}, \tag{21.86}$$

where $\alpha = 1$ is the rotation is counter-clockwise and $\alpha = -1$ if the rotation is clockwise. For boundary 3 we decided to have the local line-element numbering so as to have $\alpha = -1$.

Hence, we have

$$f_{ii,jj,n_r}(e_3) = \int_{\eta=-1}^{\eta=1} \partial_\xi z^3_{e_3} \phi_{ii}\phi_{jj}, \tag{21.87}$$

where we have cancelled the Jacobian of the change of coordinates with the denominator of the expression for the normal. Now, using Gaussian quadrature we have

$$\boxed{f_{ii,jj,n_r}(e_3) \approx \sum_{p=1}^{n_{lG}} \partial_\xi z^3_{e_3}(p)\phi_{ii}(p)\phi_{jj}(p)}. \tag{21.88}$$

From equation (5.88) we have

$$f_{ii,jj,n_z}(e_3) = \int_{\partial\Omega_{e_3}} n^3_z \phi^3_{l3(e_3,ii)}\phi^3_{l3(e_3,jj)}, \tag{21.89}$$

which can be re-written as

$$f_{ii,jj,n_z}(e_3) = -\int_{\xi=-1}^{\xi=1} \partial_\xi r^3_{e_3}\phi^3_{ii}\phi^3_{jj}, \tag{21.90}$$

and using Gaussian quadrature we have

$$\boxed{f_{ii,jj,n_z}(e_3) \approx -\sum_{p=1}^{n_{lG}} \partial_\eta r^3_{e_3}(p)\phi_{ii}(p)\phi_{jj}(p)}. \tag{21.91}$$

21.4. Derivatives of line-element integrals

We recall

$$J^i_{e_i}(p) = \sqrt{\left(\partial_{\xi_i} r^i_{e_i}(p)\right)^2 + \left(\partial_{\xi_i} z^i_{e_i}(p)\right)^2}, \tag{21.92}$$

where $\xi_i = \xi$ for $i = 1, 3$ and $\xi_i = \eta$ for $i = 2$; and we notice that

$$\partial_{h_q} J^i_{e_i}(p) = \frac{1}{2}\frac{1}{J^i_{e_i}(p)}\left[2\partial_{\xi_i} r^i_{e_i}(p)\partial_{h_q}\left(\partial_{\xi_i} r^i_{e_i}(p)\right) + 2\partial_{\xi_i} z^i_{e_i}(p)\partial_{h_q}\left(\partial_{\xi_i} z^i_{e_i}(p)\right)\right], \tag{21.93}$$

i.e.

$$\boxed{\partial_{h_q} J^i_{e_i}(p) = \frac{\left[\partial_{\xi_i} r^i_{e_i}(p)\partial_{h_q}\left(\partial_{\xi_i} r^i_{e_i}(p)\right) + \partial_{\xi_i} z^i_{e_i}(p)\partial_{h_q}\left(\partial_{\xi_i} z^i_{e_i}(p)\right)\right]}{J^i_{e_i}(p)}}, \tag{21.94}$$

which reduces the problem to finding

$$\partial_{h_q}\left(\partial_{\xi_i} r^i_{e_i}\right) = \partial_{h_q}\partial_{\xi_i}\left(\sum_{mm=1}^{3} r^i_{e_i,mm}\phi^i_{mm}\right), \tag{21.95}$$

i.e.

$$\partial_{h_q}\left(\partial_{\xi_i} r^i_{e_i}\right) = \partial_{h_q}\left(\sum_{mm=1}^{3} r^i_{e_i,mm}\partial_{\xi_i}\phi^i_{mm}\right), \tag{21.96}$$

which yields

$$\boxed{\partial_{h_q}\left(\partial_{\xi_i} r^i_{e_i}\right) = \sum_{mm=1}^{3}\left(\partial_{\xi_i}\phi^i_{mm}\right)\left(\partial_{h_q} r^i_{e_i,mm}\right)}; \tag{21.97}$$

and, similarly,

$$\boxed{\partial_{h_q}\left(\partial_{\xi_i} z^i_{e_i}\right) = \sum_{mm=1}^{3}\left(\partial_{\xi_i}\phi^i_{mm}\right)\left(\partial_{h_q} z^i_{e_i,mm}\right)}. \tag{21.98}$$

21.4.1. Derivatives of c terms

From equation (21.21) we have

$$\partial_{h_q} c^s_{t_r,jj,ii}(e_1) = \partial_{h_q}\int_{\xi=-1}^{\xi=1}\frac{\partial_\xi r^1_{e_1}(\xi)}{J^1_{e_1}(\xi)}\phi^1_{jj}(\xi)\partial_\xi\phi^1_{ii}(\xi), \tag{21.99}$$

i.e.

$$\begin{aligned}
&\partial_{h_q} c^s_{t_r,jj,ii}(e_1)\\
&= \int_{\xi=-1}^{\xi=1}\frac{\partial_{h_q}\partial_\xi r^1_{e_1}(\xi)}{J^1_{e_1}(\xi)}\phi^1_{jj}(\xi)\partial_\xi\phi^1_{ii}(\xi) - \int_{\xi=-1}^{\xi=1}\frac{\left[\partial_\xi r^1_{e_1}(\xi)\right]\partial_{h_q} J^1_{e_1}(\xi)}{\left(J^1_{e_1}(\xi)\right)^2}\phi^1_{jj}(\xi)\partial_\xi\phi^1_{ii}(\xi),
\end{aligned} \tag{21.100}$$

and, using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}
\partial_{h_q} c^s_{t_r,jj,ii}(e_1) \approx &\sum_{p=1}^{n_{lG}}\frac{\phi^1_{jj}(p)\left[\partial_\xi\phi^1_{ii}(p)\right]\partial_{h_q}\partial_\xi r^1_{e_1}(p)}{J^1_{e_1}(p)}\\
&-\sum_{p=1}^{n_{lG}}\frac{\phi^1_{jj}(\xi)\left[\partial_\xi\phi^1_{ii}(p)\right]\left[\partial_\xi r^1_{e_1}(p)\right]\partial_{h_q} J^1_{e_1}(p)}{\left(J^1_{e_1}(p)\right)^2}.
\end{aligned}} \tag{21.101}$$

From equation (21.24) we have

$$\partial_{h_q} c^s_{t_z,jj,ii}(e_1) = \partial_{h_q}\int_{\xi=-1}^{\xi=1}\frac{\partial_\xi z^1_{e_1}(\xi)}{J^1_{e_1}(\xi)}\phi^1_{jj}(\xi)\partial_\xi\phi^1_{ii}(\xi), \tag{21.102}$$

i.e.

$$\begin{aligned}
&\partial_{h_q} c^s_{t_z,jj,ii}(e_1)\\
&= \int_{\xi=-1}^{\xi=1}\frac{\partial_{h_q}\partial_\xi z^1_{e_1}(\xi)}{J^1_{e_1}(\xi)}\phi^1_{jj}(\xi)\partial_\xi\phi^1_{ii}(\xi) - \int_{\xi=-1}^{\xi=1}\frac{\left[\partial_\xi z^1_{e_1}(\xi)\right]\partial_{h_q} J^1_{e_1}(\xi)}{\left(J^1_{e_1}(\xi)\right)^2}\phi^1_{jj}(\xi)\partial_\xi\phi^1_{ii}(\xi),
\end{aligned} \tag{21.103}$$

and, using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}
\partial_{h_q} c^s_{t_z,jj,ii}(e_1) \approx &\sum_{p=1}^{n_{lG}}\frac{\phi^1_{jj}(p)\left[\partial_\xi\phi^1_{ii}(p)\right]\partial_{h_q}\partial_\xi z^1_{e_1}(p)}{J^1_{e_1}(p)}\\
&-\sum_{p=1}^{n_{lG}}\frac{\phi^1_{jj}(\xi)\left[\partial_\xi\phi^1_{ii}(p)\right]\left[\partial_\xi z^1_{e_1}(p)\right]\partial_{h_q} J^1_{e_1}(p)}{\left(J^1_{e_1}(p)\right)^2}.
\end{aligned}} \tag{21.104}$$

From (21.28) we have

$$\partial_{h_q} c_{ii,jj,n_r}(e_1) = -\partial_{h_q} \int_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi)\phi^1_{ii}(\xi)\phi^1_{jj}(\xi), \tag{21.105}$$

i.e.

$$\begin{aligned}\partial_{h_q} c_{n_r,jj,ii}(e_1) = &- \int_{\xi=-1}^{\xi=1} \frac{\partial_{h_q}\partial_\xi z^1_{e_1}(\xi)}{J^1_{e_1}(\xi)} \phi^1_{jj}(\xi)\partial_\xi\phi^1_{ii}(\xi) \\ &+ \int_{\xi=-1}^{\xi=1} \frac{\left[\partial_\xi z^1_{e_1}(\xi)\right]\partial_{h_q} J^1_{e_1}(\xi)}{\left(J^1_{e_1}(\xi)\right)^2} \phi^1_{jj}(\xi)\partial_\xi\phi^1_{ii}(\xi),\end{aligned} \tag{21.106}$$

and, using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}\partial_{h_q} c_{n_r,jj,ii}(e_1) \approx &- \sum_{p=1}^{n_{IG}} \frac{\phi^1_{jj}(p)\left[\partial_\xi\phi^1_{ii}(p)\right]\partial_{h_q}\partial_\xi z^1_{e_1}(p)}{J^1_{e_1}(p)} \\ &+ \sum_{p=1}^{n_{IG}} \frac{\phi^1_{jj}(\xi)\left[\partial_\xi\phi^1_{ii}(p)\right]\left[\partial_\xi z^1_{e_1}(p)\right]\partial_{h_q} J^1_{e_1}(p)}{\left(J^1_{e_1}(p)\right)^2}.\end{aligned}} \tag{21.107}$$

From (21.31) we have

$$\partial_{h_q} c_{ii,jj,n_z}(e_1) = \partial_{h_q} \int_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\phi^1_{jj}(\xi), \tag{21.108}$$

i.e.

$$\begin{aligned}&\partial_{h_q} c_{n_z,jj,ii}(e_1) \\ &= \int_{\xi=-1}^{\xi=1} \frac{\partial_{h_q}\partial_\xi r^1_{e_1}(\xi)}{J^1_{e_1}(\xi)} \phi^1_{jj}(\xi)\partial_\xi\phi^1_{ii}(\xi) - \int_{\xi=-1}^{\xi=1} \frac{\left[\partial_\xi r^1_{e_1}(\xi)\right]\partial_{h_q} J^1_{e_1}(\xi)}{\left(J^1_{e_1}(\xi)\right)^2} \phi^1_{jj}(\xi)\partial_\xi\phi^1_{ii}(\xi),\end{aligned} \tag{21.109}$$

and, using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}\partial_{h_q} c^s_{n_z,jj,ii}(e_1) \approx &\sum_{p=1}^{n_{IG}} \frac{\phi^1_{jj}(p)\left[\partial_\xi\phi^1_{ii}(p)\right]\partial_{h_q}\partial_\xi r^1_{e_1}(p)}{J^1_{e_1}(p)} \\ &- \sum_{p=1}^{n_{IG}} \frac{\phi^1_{jj}(\xi)\left[\partial_\xi\phi^1_{ii}(p)\right]\left[\partial_\xi r^1_{e_1}(p)\right]\partial_{h_q} J^1_{e_1}(p)}{\left(J^1_{e_1}(p)\right)^2}.\end{aligned}} \tag{21.110}$$

21.4.2. **Derivatives of d terms**

From equation (21.49) we have

$$\partial_{h_q} d_{t_r,t_r,ii,jj}(e_2) = \partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{\left[\partial_\eta r^2_{e_2}(\eta)\right]^2}{J^2_{e_2}(\eta)} \phi^2_{ii}(\eta)\phi^2_{jj}(\eta), \tag{21.111}$$

i.e.

$$\partial_{h_q} d_{t_r,t_r,ii,jj}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{2\left[\partial_\eta r^2_{e_2}(\eta)\right]\partial_{h_q}\partial_\eta r^2_{e_2}(\eta)}{J^2_{e_2}(\eta)}\phi^2_{ii}(\eta)\phi^2_{jj}(\eta) - \int\limits_{\eta=-1}^{\eta=1} \frac{\left[\partial_\eta r^2_{e_2}(\eta)\right]^2\partial_{h_q} J^2_{e_2}(\eta)}{\left(J^2_{e_2}(\eta)\right)^2}\phi^2_{ii}(\eta)\phi^2_{jj}(\eta), \tag{21.112}$$

and, using Gaussian quadrature, we have

$$\boxed{\partial_{h_q} d_{t_r,t_r,ii,jj}(e_2) \approx 2\sum_{p=1}^{n_{lG}} \frac{\phi^2_{ii}(p)\phi^2_{jj}(p)\left[\partial_\eta r^2_{e_2}(p)\right]\partial_{h_q}\partial_\eta r^2_{e_2}(p)}{J^2_{e_2}(p)} - \sum_{p=1}^{n_{lG}} \frac{\phi^2_{ii}(p)\phi^2_{jj}(p)\left[\partial_\eta r^2_{e_2}(p)\right]^2\partial_{h_q} J^2_{e_2}(p)}{\left(J^2_{e_2}(p)\right)^2}.} \tag{21.113}$$

From equation (21.52)

$$\partial_{h_q} d_{t_r,t_z,ii,jj}(e_2) = \partial_{h_q}\int\limits_{\eta=-1}^{\eta=1} \frac{\left[\partial_\eta r^2_{e_2}(\eta)\right]\partial_\eta z^2_{e_2}(\eta)}{J^2_{e_2}(\eta)}\phi^2_{ii}(\eta)\phi^2_{jj}(\eta), \tag{21.114}$$

i.e.

$$\partial_{h_q} d_{t_r,t_z,ii,jj}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{\left[\partial_{h_q}\partial_\eta r^2_{e_2}(\eta)\right]\partial_\eta z^2_{e_2}(\eta)}{J^2_{e_2}(\eta)}\phi^2_{ii}(\eta)\phi^2_{jj}(\eta) + \int\limits_{\eta=-1}^{\eta=1} \frac{\left[\partial_\eta r^2_{e_2}(\eta)\right]\partial_{h_q}\partial_\eta z^2_{e_2}(\eta)}{J^2_{e_2}(\eta)}\phi^2_{ii}(\eta)\phi^2_{jj}(\eta) - \int\limits_{\eta=-1}^{\eta=1} \frac{\left[\partial_\eta r^2_{e_2}(\eta)\right]\left[\partial_\eta z^2_{e_2}(\eta)\right]\partial_{h_q} J^2_{e_2}(\eta)}{\left(J^2_{e_2}(\eta)\right)^2}\phi^2_{ii}(\eta)\phi^2_{jj}(\eta), \tag{21.115}$$

and, using Gaussian quadrature, we have

$$\boxed{\partial_{h_q} d_{t_r,t_z,ii,jj}(e_2) \approx \sum_{p=1}^{n_{lG}} \frac{\phi^2_{ii}(p)\phi^2_{jj}(p)\left[\partial_\eta r^2_{e_2}(p)\right]\partial_{h_q}\partial_\eta z^2_{e_2}(p)}{J^2_{e_2}(p)} + \sum_{p=1}^{n_{lG}} \frac{\phi^2_{ii}(p)\phi^2_{jj}(p)\left[\partial_\eta z^2_{e_2}(p)\right]\partial_{h_q}\partial_\eta r^2_{e_2}(p)}{J^2_{e_2}(p)} - \sum_{p=1}^{n_{lG}} \frac{\phi^2_{ii}(p)\phi^2_{jj}(p)\left[\partial_\eta r^2_{e_2}(p)\right]\left[\partial_\eta z^2_{e_2}(p)\right]\partial_{h_q} J^2_{e_2}(p)}{\left(J^2_{e_2}(p)\right)^2}.} \tag{21.116}$$

From (21.55) we have

$$\partial_{h_q} d_{t_z,t_z,ii,jj}(e_2) = \partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{\left(\partial_\eta z_{e_2}^2\right)^2}{J_{e_2}^2} \phi_{ii}^2 \phi_{jj}^2, \tag{21.117}$$

i.e.

$$\begin{aligned} \partial_{h_q} d_{t_z,t_z,ii,jj}(e_2) = & \int_{\eta=-1}^{\eta=1} \frac{2\left[\partial_\eta z_{e_2}^2(\eta)\right] \partial_{h_q} \partial_\eta z_{e_2}^2(\eta)}{J_{e_2}^2(\eta)} \phi_{ii}^2(\eta) \phi_{jj}^2(\eta) \\ & - \int_{\eta=-1}^{\eta=1} \frac{\left[\partial_\eta z_{e_2}^2(\eta)\right]^2 \partial_{h_q} J_{e_2}^2(\eta)}{\left(J_{e_2}^2(\eta)\right)^2} \phi_{ii}^2(\eta) \phi_{jj}^2(\eta), \end{aligned} \tag{21.118}$$

and, using Gaussian quadrature, we have

$$\boxed{\begin{aligned} \partial_{h_q} d_{t_z,t_z,ii,jj}(e_2) \approx 2 & \sum_{p=1}^{n_{lG}} \frac{\phi_{ii}^2(p) \phi_{jj}^2(p) \left[\partial_\eta z_{e_2}^2(p)\right] \partial_{h_q} \partial_\eta z_{e_2}^2(p)}{J_{e_2}^2(p)} \\ & - \sum_{p=1}^{n_{lG}} \frac{\phi_{ii}^2(p) \phi_{jj}^2(p) \left[\partial_\eta z_{e_2}^2(p)\right]^2 \partial_{h_q} J_{e_2}^2(p)}{\left(J_{e_2}^2(p)\right)^2}. \end{aligned} \quad (21.119)}$$

From (21.59)

$$\partial_{h_q} d_{ii,jj,n_r}(e_2) = \partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{\partial_\eta z_{e_2}^2(\eta)}{J_{e_2}^2(\eta)} \phi_{ii}(\eta) \phi_{jj}(\eta), \tag{21.120}$$

i.e.

$$\partial_{h_q} d_{ii,jj,n_r}(e_2) = \int_{\eta=-1}^{\eta=1} \frac{\partial_{h_q} \partial_\eta z_{e_2}^2(\eta)}{J_{e_2}^2(\eta)} \phi_{ii}^2(\eta) \phi_{jj}^2(\eta) - \int_{\eta=-1}^{\eta=1} \frac{\left[\partial_\eta z_{e_2}^2(\eta)\right] \partial_{h_q} J_{e_2}^2(\eta)}{\left(J_{e_2}^2(\eta)\right)^2} \phi_{ii}^2(\eta) \phi_{jj}^2(\eta), \tag{21.121}$$

and, using Gaussian quadrature, we have

$$\boxed{\begin{aligned} & \partial_{h_q} d_{ii,jj,n_r}(e_2) \\ & \quad \approx \sum_{p=1}^{n_{lG}} \frac{\phi_{ii}^2(p) \phi_{jj}^2(p) \partial_{h_q} \partial_\eta z_{e_2}^2(p)}{J_{e_2}^2(p)} - \sum_{p=1}^{n_{lG}} \frac{\phi_{ii}^2(p) \phi_{jj}^2(p) \left[\partial_\eta z_{e_2}^2(p)\right] \partial_{h_q} J_{e_2}^2(p)}{\left(J_{e_2}^2(p)\right)^2}. \end{aligned} \quad (21.122)}$$

From (21.62) we have

$$\partial_{h_q} d_{ii,jj,n_z}(e_2) = -\partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{\partial_\eta r_{e_2}^2}{J_{e_2}^2} \phi_{ii} \phi_{jj}, \tag{21.123}$$

i.e.

$$\partial_{h_q} d_{ii,jj,n_z}(e_2) = -\int_{\eta=-1}^{\eta=1} \frac{\partial_{h_q}\partial_\eta r^2_{e_2}(\eta)}{J^2_{e_2}(\eta)} \phi^2_{ii}(\eta)\phi^2_{jj}(\eta) + \int_{\eta=-1}^{\eta=1} \frac{\left[\partial_\eta r^2_{e_2}(\eta)\right]\partial_{h_q} J^2_{e_2}(\eta)}{\left(J^2_{e_2}(\eta)\right)^2} \phi^2_{ii}(\eta)\phi^2_{jj}(\eta), \tag{21.124}$$

and, using Gaussian quadrature, we have

$$\boxed{\partial_{h_q} d_{ii,jj,n_z}(e_2) \approx \sum_{p=1}^{n_{lG}} \frac{\phi^2_{ii}(p)\phi^2_{jj}(p)\partial_{h_q}\partial_\eta z^2_{e_2}(p)}{J^2_{e_2}(p)} - \sum_{p=1}^{n_{lG}} \frac{\phi^2_{ii}(p)\phi^2_{jj}(p)\left[\partial_\eta z^2_{e_2}(p)\right]\partial_{h_q} J^2_{e_2}(p)}{\left(J^2_{e_2}(p)\right)^2}.} \tag{21.125}$$

21.4.3. Derivatives of f terms

From equation (21.75) we have

$$\partial_{h_q} f_{t_r,ii,jj}(e_3) = \partial_{h_q} \int_{\xi=-1}^{\xi=1} \partial_\xi r^3_{e_3}(\xi)\phi^3_{ii}(\xi)\phi^3_{jj}(\xi), \tag{21.126}$$

i.e.

$$\partial_{h_q} f_{t_r,ii,jj}(e_3) = \int_{\xi=-1}^{\xi=1} \phi^3_{ii}(\xi)\phi^3_{jj}(\xi)\partial_{h_q}\partial_\xi r^3_{e_3}(\xi), \tag{21.127}$$

and, using Gaussian quadrature, we have

$$\boxed{\partial_{h_q} f_{t_r,ii,jj}(e_3) \approx \sum_{p=1}^{n_{lG}} \phi^3_{ii}(p)\phi^3_{jj}(p)\partial_{h_q}\partial_\xi r^3_{e_3}(p)}. \tag{21.128}$$

From equation (21.83) we have

$$\partial_{h_q} f_{t_z,ii,jj}(e_3) = \partial_{h_q} \int_{\xi=-1}^{\xi=1} \partial_\xi z^3_{e_3}\phi^3_{ii}\phi^3_{jj}, \tag{21.129}$$

i.e.

$$\partial_{h_q} f_{t_z,ii,jj}(e_3) = \int_{\xi=-1}^{\xi=1} \phi^3_{ii}(\xi)\phi^3_{jj}(\xi)\partial_{h_q}\partial_\xi z^3_{e_3}(\xi), \tag{21.130}$$

and, using Gaussian quadrature, we have

$$\boxed{\partial_{h_q} f_{t_z,ii,jj}(e_3) \approx \sum_{p=1}^{n_{lG}} \phi^3_{ii}(p)\phi^3_{jj}(p)\partial_{h_q}\partial_\xi z^3_{e_3}(p)}. \tag{21.131}$$

From equation (21.87) we have

$$\partial_{h_q} f_{ii,jj,n_r}(e_3) = \partial_{h_q} \int_{\eta=-1}^{\eta=1} \partial_\xi z^3_{e_3}\phi_{ii}\phi_{jj}, \tag{21.132}$$

i.e.

$$\partial_{h_q} f_{ii,jj,n_r}(e_3) = \int_{\xi=-1}^{\xi=1} \phi_{ii}^3(\xi)\phi_{jj}^3(\xi)\partial_{h_q}\partial_\xi z_{e_3}^3(\xi), \tag{21.133}$$

and, using Gaussian quadrature, we have

$$\boxed{\partial_{h_q} f_{ii,jj,n_r}(e_3) \approx \sum_{p=1}^{n_{lG}} \phi_{ii}^3(p)\phi_{jj}^3(p)\partial_{h_q}\partial_\xi z_{e_3}^3(p)}. \tag{21.134}$$

From equation (21.90)

$$\partial_{h_q} f_{ii,jj,n_z}(e_3) = -\partial_{h_q} \int_{\xi=-1}^{\xi=1} \partial_\xi r_{e_3}^3 \phi_{ii}^3 \phi_{jj}^3, \tag{21.135}$$

i.e.

$$\partial_{h_q} f_{ii,jj,n_z}(e_3) = -\int_{\xi=-1}^{\xi=1} \phi_{ii}^3(\xi)\phi_{jj}^3(\xi)\partial_{h_q}\partial_\xi r_{e_3}^3(\xi), \tag{21.136}$$

and, using Gaussian quadrature, we have

$$\boxed{\partial_{h_q} f_{ii,jj,n_z}(e_3) \approx -\sum_{p=1}^{n_{lG}} \phi_{ii}^3(p)\phi_{jj}^3(p)\partial_{h_q}\partial_\xi r_{e_3}^3(p)}. \tag{21.137}$$

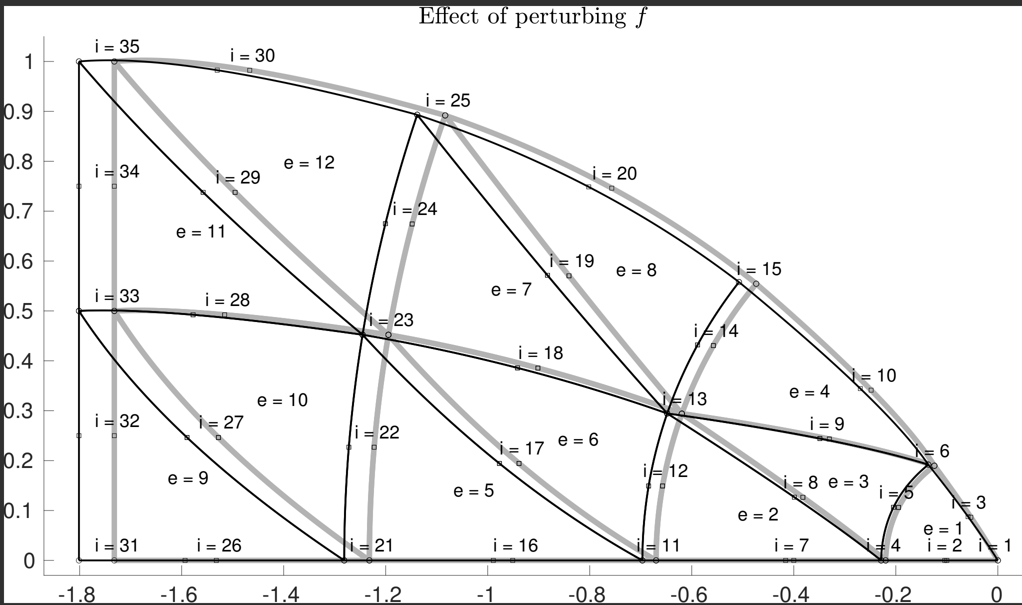

FIGURE 11. Effect of the forward perturbation of f. Grey lines correspond to the unperturbed mesh, black lines to the perturbed mesh. Element numbers and global node numbers are indicated.

22. Derivatives of r and z with respect to h

We will calculate the derivatives of the nodal positions with respect to all spine lengths exactly, and the derivatives of nodal positions with respect to the focal length f (which we consider to be spine number 1) numerically.

The n_q nodes placed along spine q are equally spaced along it, with the first node on the solid surface and the last node on the free surface. The spines are arcs of circumference; except for the first and last ones, which are a horizontal and a vertical segment, respectively. Hence, the i-th node along a spine, that is not the first or last, is located along the q-th spine at a fraction $\alpha_i^q = (i-1)/(n_q-1)$ of its total length, measuring from the solid towards the free surface; and, therefore, its coordinates are

$$r_{i,q} = r_q - R_q \cos\left(\frac{\alpha_i^q h_q}{R_q}\right) \tag{22.1}$$

and

$$z_{i,q} = R_q \sin\left(\frac{\alpha_i^q h_q}{R_q}\right), \tag{22.2}$$

where r_q and R_q are the r coordinate of the centre, and the radius of the q-th spine; respectively. Consequently, we have

$$\partial_{h_q} r_{i,q} = \alpha_i^q \sin\left(\frac{\alpha_i^q h_q}{R_q}\right) \tag{22.3}$$

and

$$\partial_{\alpha_q} z_{i,q} = \alpha_i^q \cos\left(\frac{\alpha_i^q h_q}{R_q}\right). \tag{22.4}$$

The coordinates of the i-th node along the last spine are given by

$$r_{i,q} = -f \tag{22.5}$$

and

$$z_{i,q} = \alpha_i^q h_q, \tag{22.6}$$

and, therefore, derivatives with respect to the length of the last spine are simply given by

$$\partial_{h_q} r_{i,q} = 0 \tag{22.7}$$

and

$$\partial_{h_q} z_{i,q} = \alpha_i^q. \tag{22.8}$$

Finally, to calculate derivatives with respect to f (the focal length, i.e. spine number 1), we perturb the length of f forward (see figure 11) and backward and we assess the differential quotient given by

$$\partial_f r \approx \frac{r^+ - r^-}{f^+ - f^-}, \tag{22.9}$$

where the $+$ and $-$ signs indicate the result of the forward and the backward perturbation, respectively. A completely analogue formula applies to the z coordinate of the nodes.

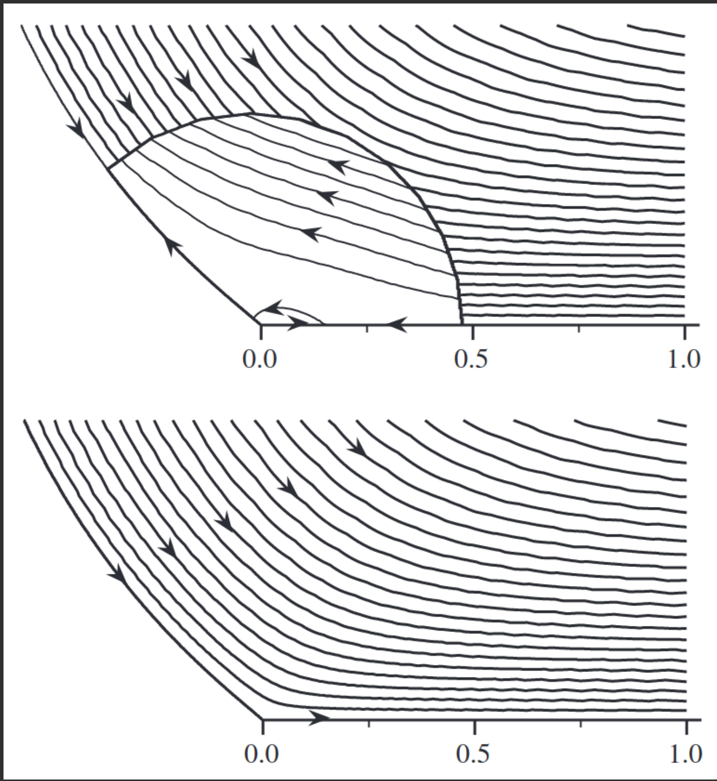

FIGURE 12. Figure from Sprittles & Shikhmurzaev (2011a). Top: Domain split into two region, with streamlines for $\boldsymbol{u}$ shown in the outer region and streamlines for $\tilde{\boldsymbol{u}}$ shown in the inner region. Bottom: Result of the application of the method, once the eigen-solution is added.

23. Formulation for obtuse contact angles

In Sprittles & Shikhmurzaev (2011a) it was shown that, when the contact angle is greater than $\pi/2$, a different treatment is necessary in the vicinity of the contact line. As it happens, the solution that we would obtain form directly applying the numerical scheme above described, even including the singular element at the contact line, fails at capturing the correct fluid behaviour. Instead, we need a combination of finite elements and asymptotic methods to obtain the physically relevant solution. The method developed in Sprittles & Shikhmurzaev (2011a), essentially requires that we write the solution of the Navier-Stokes equations for incompressible flow in the vicinity of the contact line (see figure 12) as the sum of two velocity fields, namely

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + A\check{\boldsymbol{u}}, \tag{23.1}$$

where $\bar{\boldsymbol{u}}$ and A are to be determined and $\check{\boldsymbol{u}}$ is an eigen-solution; that is to say, it is a non-zero element of the 1-dimensional vector space of solutions to the Stokes equation for incompressible flow in a wedge of angle θ_c, with no tangential stress conditions on the bounding surfaces; i.e.

$$\nabla \cdot \check{\boldsymbol{P}} = 0, \tag{23.2}$$

and

$$\nabla \cdot \check{\boldsymbol{u}} = 0, \tag{23.3}$$

where

$$\check{\boldsymbol{P}} = -\check{p}\boldsymbol{I} + \nabla\check{\boldsymbol{u}} + (\nabla\check{\boldsymbol{u}})^T \tag{23.4}$$

which hold on† $0 < r$ and $0 < \theta < \theta_c$, and are subject to

$$\boldsymbol{n}^i \cdot \check{\boldsymbol{P}} \cdot \left(\boldsymbol{I} - \boldsymbol{n}^i\boldsymbol{n}^i\right) = 0, \tag{23.5}$$

for both bounding surfaces ($i = 1$ and $i = 2$). Equations above are presented in dimensionless form.

Solution $\check{\boldsymbol{u}}$ is found using a stream function, i.e.

$$\check{\boldsymbol{u}} = \nabla^{\perp}\check{\psi}, \tag{23.6}$$

where

$$\check{\psi} = \left(\sqrt{r^2 + z^2}\right)^{\lambda} \sin\left(\lambda\left(\pi - \theta\right)\right), \tag{23.7}$$

with $\lambda = \pi/\theta_c$, $\theta = \arctan(z/r)$, and function arctan is assumed to take values between 0 and π. See Appendix A for a derivation of the solution.

We notice that the pressure for the this solution is globally constant and only determined up to an additive constant, which means we can impose

$$\check{p} = 0 \tag{23.8}$$

in (23.4).

23.1. **Bulk equations**

We thus substitute this decomposition into the Navier-Stokes equation for incompressible flow of uniform density with an ALE system of reference, obtaining in the conservation of mass

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \bar{\boldsymbol{u}} + A\underbrace{\nabla \cdot \check{\boldsymbol{u}}}_{=0} = 0, \tag{23.9}$$

i.e.

$$\nabla \cdot \bar{\boldsymbol{u}} = 0; \tag{23.10}$$

in the conservation of momentum

$$Re\left[\partial_t(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}}) + (\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{c}) \cdot \nabla\left(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}}\right)\right] = \nabla \cdot \bar{\boldsymbol{P}} + A\underbrace{\nabla \cdot \check{\boldsymbol{P}}}_{=0} + St\,\hat{\boldsymbol{g}}, \tag{23.11}$$

where

$$\bar{\boldsymbol{P}} = -p\boldsymbol{I} + \nabla\bar{\boldsymbol{u}} + (\nabla\bar{\boldsymbol{u}})^T, \tag{23.12}$$

and $\boldsymbol{c}$ is the velocity of the ALE coordinate. Where we highlight that the pressure of the numerical part of the solution is the pressure of the full problem as a consequence of our choice of $\check{p}$.

We thus have

$$\begin{aligned} Re\Big[&\partial_t\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}} \cdot \nabla\bar{\boldsymbol{u}} - \boldsymbol{c} \cdot \nabla\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} \cdot \nabla\bar{\boldsymbol{u}} + A\bar{\boldsymbol{u}} \cdot \nabla\check{\boldsymbol{u}} \\ &+ A\partial_t\check{\boldsymbol{u}} - A\boldsymbol{c} \cdot \nabla\check{\boldsymbol{u}} + (A)^2\,\check{\boldsymbol{u}} \cdot \nabla\check{\boldsymbol{u}}\Big] - \nabla \cdot \bar{\boldsymbol{P}} - St\,\hat{\boldsymbol{g}} = 0, \end{aligned} \tag{23.13}$$

which must hold in $\bar{\Omega}$, i.e. the domain where the modified equation is solved.

† Actually, the solution we will find holds on the entire plane and this is important in cases when the fluid in the vicinity of the contact line does not form a convex wedge. However, the description given above is enough to specify the problem and to identify its solution (as is shown in Appendix A)

23.2. Interface formation equations

23.2.1. Free surface

On the free surface, the kinematic boundary condition (2.44) retains its form as

$$(\boldsymbol{v}^{s_1} - \boldsymbol{c}) \cdot \boldsymbol{n}^1 = 0, \tag{23.14}$$

the dynamic boundary condition given by (2.46) becomes

$$\left(p^g + \bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) \cdot \boldsymbol{n}^1 = -\frac{\nabla^s \cdot \left[\sigma^1(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)\right]}{Ca}. \tag{23.15}$$

The slip condition on boundary 1, SC1, given in equation (2.47) becomes

$$(\boldsymbol{v}^{s_1} - \bar{\boldsymbol{u}} - A\check{\boldsymbol{u}}) \cdot \left(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1\right) = \frac{1 + 4Eg\,Bg}{4Bg} \nabla^s \sigma^1. \tag{23.16}$$

The dependence of surface tension on surface density TDC1 (2.48) remains in the same form

$$\sigma^1 = Cg\,\left(1 - \rho^{s_1}\right). \tag{23.17}$$

The mass exchange between the free surface and the bulk (2.49), MEC1, now reads

$$\left(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{v}^{s_1}\right) \boldsymbol{n}^1 = Fg\,\left(\rho^{s_1} - Dg\right). \tag{23.18}$$

The density transport condition (2.50), DTC1, maintains the form

$$Tg\,\left\{\partial_t \rho^{s_1} + \rho^{s_1} \nabla^s \cdot \boldsymbol{c} + \nabla^s \cdot \left[\rho^{s_1}\left(\boldsymbol{v}^{s_1} - \boldsymbol{c}\right)\right]\right\} = Dg\, - \rho^{s_1}. \tag{23.19}$$

In this split-domain formulation, boundary conditions for SC1 and the DBC1 are required at the contact line and at the junction point of the near-field and the far-field sections of the free surface. The conditions for the near field are given by the angle at which the surface is pulled by the far-field half of boundary 1. Similarly, the boundary conditions for the same equations in the far field are defined by the surface tension pull from the near field. This interchange of tensions ensures that the surface will not have a corner at the junction point.

Regarding the boundary conditions for the surface-density transport DTC1, the mass flux coming in from the far-field portion of surface 1 is simply equated to the flux going into the near-field portion of surface 1.

23.2.2. Liquid-solid interface

On the liquid-solid interface, the IC (2.51) maintains its form

$$\left(\boldsymbol{v}^2 - \boldsymbol{u}^s\right) \cdot \boldsymbol{n}^2 = 0; \tag{23.20}$$

where $\boldsymbol{u}^s$ is the velocity of the solid.

The GNSC (2.53) becomes

$$\boldsymbol{n}^2 \cdot \left(\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2) + \frac{1}{2Ca}\nabla^s\sigma^2 = Be\,(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{u}^s) \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2), \tag{23.21}$$

We highlight that the free surface and the liquid-solid surface for the modified obtuse-angle formulation are the same free as those for the original formulation; whereas the eigen-solution was derived for the case when the free surface is planar (though the eigen-solution is defined on the entire plane). In particular, the boundaries of the problem to be solved are not planar, and thus one must be sure not to apply the no-tangential-stress condition for the eigen-solution when evaluating it on the full problem's free surface or liquid-solid surface. If the liquid-solid interface happens to be planar, some terms could be dropped; however, for greater generality here we considered the case where it could either be planar or not.

Slip condition SC2 (2.52) becomes

$$\left[\boldsymbol{v}^{s_2} - \frac{1}{2}\left(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} + \boldsymbol{u}^s\right)\right] \cdot \left(\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2\right) = Es\,\nabla^s\sigma^2. \tag{23.22}$$

The dependence of surface tension on density TDC2 (2.54) retains its form

$$\sigma^2 = Cs\,\left(1 - \rho^{s_2}\right). \tag{23.23}$$

The mass exchange between the bulk and the surface MEC2 (2.55) is given by

$$\left(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{v}^{s_2}\right) \cdot \boldsymbol{n}^2 = Fs\,\left(\rho^{s_2} - Ds\right). \tag{23.24}$$

The density transport equation DTC2 (2.56) retains its form

$$Ts\,\left\{\partial_t\rho^{s_2} + \rho^{s_2}\nabla^s \cdot \boldsymbol{c} + \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right]\right\} = Ds\, - \rho^{s_2}. \tag{23.25}$$

The required boundary conditions for SC2, GNSC and DTC2 at the junction point between the near-field and the far-field halves of boundary 2 is provided by imposing continuity of surface tension and mass flux at the junction point.

23.2.3. **Contact line**

The contact angle condition CAC (2.57) is given by

$$\sigma_c^1\cos\theta_c + \sigma_c^2 = So\,. \tag{23.26}$$

The mass balance condition at the contact line MBCL is given by

$$\rho^{s_1}\left(\boldsymbol{v}_{\parallel}^{s_1} - \boldsymbol{c}_c\right) \cdot \boldsymbol{m}^1 + \rho^{s_2}\left(\boldsymbol{v}_{\parallel}^{s_2} - \boldsymbol{c}_c\right) \cdot \boldsymbol{m}^2 = 0, \tag{23.27}$$

i.e.

$$\rho^{s_1}\boldsymbol{v}^{s_1} \cdot \boldsymbol{m}^1 - \rho^{s_1}\boldsymbol{c}_c \cdot \boldsymbol{m}^1 + \rho^{s_2}\boldsymbol{v}^{s_2} \cdot \boldsymbol{m}^2 - \rho^{s_2}\boldsymbol{c}_c \cdot \boldsymbol{m}^2 = 0. \tag{23.28}$$

For this 2D problem, we can always choose $\boldsymbol{t}^1 = \boldsymbol{m}^1$ and $\boldsymbol{t}^2 = \boldsymbol{m}^2$. We thus have

$$\begin{aligned}&\rho^{s_1}u_c^{s_1}t_r^1 + \rho^{s_1}w_c^{s_1}t_z^1 - \rho^{s_1}t_r^1\partial_t r_c^c - \rho^{s_1}t_z^1\partial_t z_c^c\\ &\quad + \rho^{s_2}u^{s_2}t_r^2 + \rho^{s_2}w^{s_2}t_z^2 - \rho^{s_2}t_r^2\partial_t r_c^c - \rho^{s_2}t_z^2\partial_t z_c^c = 0.\end{aligned} \tag{23.29}$$

23.3. **Separatrix**

Finally, we impose continuity conditions for the velocity and stress at the boundary that separates the regions where the standard and the modified Navier-Stokes equations are solved. That is,

$$\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} = \boldsymbol{u}_{\text{out}} \tag{23.30}$$

and

$$\boldsymbol{n}^5 \cdot \left(\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) = -\boldsymbol{n}^4 \cdot \boldsymbol{P}_{\text{out}}, \tag{23.31}$$

where $\boldsymbol{n}^5 = -\boldsymbol{n}^4$ is the normal to the surface that separates the two regions, and points into the modified equation region. The subscript $_{\text{out}}$ indicates the velocity in the region where we solve the standard Navier-Stokes equation. It is important to notice that the equations for both regions are coupled through these conditions, and they must be solved as a joint system.

24. Component equations near obtuse contact angle

Defining $\bar{\boldsymbol{u}} = (\bar{u}, \bar{w})$, $\check{\boldsymbol{u}} = (\check{u}, v_e)$ and re-writing the governing equations by components we have the r-momentum equation

$$\begin{aligned}
&Re\,\partial_t\bar{u} + Re\,\bar{u}\partial_r\bar{u} + Re\,\bar{w}\partial_z\bar{u} - Re\,u_c\partial_r\bar{u} - Re\,w_c\partial_z\bar{u} \\
&\quad + ARe\,\check{u}\partial_r\bar{u} + ARe\,\check{w}\partial_z\bar{u} + ARe\,\bar{u}\partial_r\check{u} + ARe\,\bar{w}\partial_z\check{u} \\
&\quad + ARe\,\partial_t\check{u} - ARe\,u_c\partial_r\check{u} - ARe\,w_c\partial_z\check{u} \\
&\quad + (A)^2\,Re\,\check{u}\partial_r\check{u} + (A)^2\,Re\,\check{w}\partial_z\check{u} \\
&\quad - \boldsymbol{e}_r\cdot\nabla\cdot\bar{\boldsymbol{P}} - St\,\underbrace{\boldsymbol{e}_r\cdot\hat{\boldsymbol{g}}}_{\hat{\boldsymbol{g}}_r} = 0,
\end{aligned} \tag{24.1}$$

the z-momentum equation

$$\begin{aligned}
&Re\,\partial_t\bar{w} + Re\,\bar{u}\partial_r\bar{w} + Re\,\bar{w}\partial_z\bar{w} - Re\,u_c\partial_r\bar{w} - Re\,w_c\partial_z\bar{w} \\
&\quad + ARe\,\check{u}\partial_r\bar{w} + ARe\,\check{w}\partial_z\bar{w} + ARe\,\bar{u}\partial_r\check{w} + ARe\,\bar{w}\partial_z\check{w} \\
&\quad + ARe\,\partial_t\check{w} - ARe\,u_c\partial_r\check{w} - ARe\,w_c\partial_z\check{w} \\
&\quad + (A)^2\,Re\,\check{u}\partial_r\check{w} + (A)^2\,Re\,\check{w}\partial_z\check{w} \\
&\quad - \boldsymbol{e}_z\cdot\nabla\cdot\bar{\boldsymbol{P}} - St\,\underbrace{\boldsymbol{e}_z\cdot\hat{\boldsymbol{g}}}_{\hat{\boldsymbol{g}}_z} = 0,
\end{aligned} \tag{24.2}$$

and the continuity equation

$$\partial_r\bar{u} + \partial_z\bar{w} = 0; \tag{24.3}$$

which, on the free surface, are subject to

$$\bar{u}n_r^1 + \bar{w}n_z^1 + A\check{u}n_r^1 + A\check{w}n_z^1 - u_c n_r^1 - w_c n_z^1 = 0, \tag{24.4}$$

on the solid surface, must satisfy

$$\bar{u}n_r^2 + \bar{w}n_z^2 + A\check{u}n_r^2 + A\check{w}n_z^2 = 0, \tag{24.5}$$

and on the separatrix of the domains we have

$$\bar{u} + A\check{u} = u_{\mathrm{out}}. \tag{24.6}$$

For the time being we will not write equations (23.15), (**??**) and (23.31) in components.

25. The r-momentum residuals in the near-field

We define the i-th residuals of the r-momentum equation as

$$
\begin{aligned}
\bar{M}_i^r = {} & Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{u} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{u} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{u} \\
& + Re\, A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \breve{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \breve{u} \\
& + ARe \int\limits_{\Omega^n} \phi_i \partial_t \breve{u} - ARe \int\limits_{\Omega^n} \phi_i u_c \partial_r \breve{u} - ARe \int\limits_{\Omega^n} \phi_i w_c \partial_z \breve{u} \\
& + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{u} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{u} \\
& - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_r - \int\limits_{\Omega^n} \phi_i \boldsymbol{e}_r \cdot \nabla \cdot \bar{\boldsymbol{P}},
\end{aligned}
\tag{25.1}
$$

where Ω^n in the near-field (with respect to the contact line) sub-domain, in which we solve the modified Navier-Stokes equation.

We recall the tensor identity†

$$
\nabla \cdot (\boldsymbol{x} \cdot \boldsymbol{Q}) = \boldsymbol{x} \cdot \nabla \cdot \boldsymbol{Q} + \nabla \boldsymbol{x} : \boldsymbol{Q}, \tag{25.2}
$$

taking $\boldsymbol{x} = \phi_i \boldsymbol{e}_r$ and $\boldsymbol{Q} = \bar{\boldsymbol{P}}$ we have

$$
-\phi_i \boldsymbol{e}_r \cdot \nabla \cdot \bar{\boldsymbol{P}} = -\nabla \cdot \left(\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}}\right) + \nabla \left(\phi_i \boldsymbol{e}_r\right) : \bar{\boldsymbol{P}}, \tag{25.3}
$$

which reduces $\bar{M}_i^r$ to

$$
\begin{aligned}
\bar{M}_i^r = {} & Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{u} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{u} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{u} \\
& + Re\, A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \breve{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \breve{u} \\
& + ARe \int\limits_{\Omega^n} \phi_i \partial_t \breve{u} - ARe \int\limits_{\Omega^n} \phi_i u_c \partial_r \breve{u} - ARe \int\limits_{\Omega^n} \phi_i w_c \partial_z \breve{u} \\
& + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{u} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{u} \\
& - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_r + \int\limits_{\Omega^n} \nabla \left(\phi_i \boldsymbol{e}_r\right) : \bar{\boldsymbol{P}} - \int\limits_{\Omega^n} \nabla \cdot \left(\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}}\right),
\end{aligned}
\tag{25.4}
$$

we can now apply the divergence theorem to the last term on the right hand side above

† In the case of Cartesian coordinate, the : symbol can be thought of just as the canonical inner product of matrices when used between two tensors of second order.

to obtain

$$\begin{aligned}\bar{M}_i^r = {} & Re\int_{\Omega^n}\phi_i\partial_t\bar{u} + Re\int_{\Omega^n}\phi_i\bar{u}\partial_r\bar{u} + Re\int_{\Omega^n}\phi_i\bar{w}\partial_z\bar{u} - Re\int_{\Omega^n}\phi_i u_c\partial_r\bar{u} - Re\int_{\Omega^n}\phi_i w_c\partial_z\bar{u}\\ & + Re\,A\int_{\Omega^n}\phi_i\breve{u}\partial_r\bar{u} + Re\,A\int_{\Omega^n}\phi_i\breve{w}\partial_z\bar{u} + Re\,A\int_{\Omega^n}\phi_i\bar{u}\partial_r\breve{u} + Re\,A\int_{\Omega^n}\phi_i\bar{w}\partial_z\breve{u}\\ & + ARe\int_{\Omega^n}\phi_i\partial_t\breve{u} - ARe\int_{\Omega^n}\phi_i u_c\partial_r\breve{u} - ARe\int_{\Omega^n}\phi_i w_c\partial_z\breve{u}\\ & + Re\,(A)^2\int_{\Omega^n}\phi_i\breve{u}\partial_r\breve{u} + Re\,(A)^2\int_{\Omega^n}\phi_i\breve{w}\partial_z\breve{u}\\ & - St\int_{\Omega^n}\phi_i\hat{\boldsymbol{g}}_r + \int_{\Omega^n}\nabla(\phi_i\boldsymbol{e}_r):\bar{\boldsymbol{P}} + \int_{\partial\bar{\Omega}}\phi_i\boldsymbol{e}_r\cdot\bar{\boldsymbol{P}}\cdot\boldsymbol{n},\end{aligned} \tag{25.5}$$

where $\partial\bar{\Omega}$ is the boundary of $\bar{\Omega}$ and $\boldsymbol{n}$ is its normal, that points into $\bar{\Omega}$.

We notice that

$$\nabla(\phi_i\boldsymbol{e}_r):\bar{\boldsymbol{P}} = \begin{bmatrix}\partial_r\phi_i & \partial_z\phi_i\\ 0 & 0\end{bmatrix} : \begin{bmatrix}\bar{P}_{rr} & \bar{P}_{rz}\\ \bar{P}_{zr} & \bar{P}_{zz}\end{bmatrix} \tag{25.6}$$

i.e.

$$\nabla(\phi_i\boldsymbol{e}_r):\bar{\boldsymbol{P}} = \begin{bmatrix}\partial_r\phi_i & \partial_z\phi_i\\ 0 & 0\end{bmatrix} : \begin{bmatrix}-p+2\partial_r\bar{u} & \partial_z\bar{u}+\partial_r\bar{w}\\ \partial_r\bar{w}+\partial_z\bar{u} & -p+2\partial_z\bar{w}\end{bmatrix}, \tag{25.7}$$

which is

$$\nabla(\phi_i\boldsymbol{e}_r):\bar{\boldsymbol{P}} = \partial_r\phi_i\bar{P}_{rr} + \partial_z\phi_i\bar{P}_{rz} = -p\partial_r\phi_i + 2\partial_r\bar{u}\partial_r\phi_i + \partial_z\bar{u}\partial_z\phi_i + \partial_r\bar{w}\partial_z\phi_i. \tag{25.8}$$

Therefore we have

$$\begin{aligned}\bar{M}_i^r = {} & Re\int_{\Omega^n}\phi_i\partial_t\bar{u} + Re\int_{\Omega^n}\phi_i\bar{u}\partial_r\bar{u} + Re\int_{\Omega^n}\phi_i\bar{w}\partial_z\bar{u} - Re\int_{\Omega^n}\phi_i u_c\partial_r\bar{u} - Re\int_{\Omega^n}\phi_i w_c\partial_z\bar{u}\\ & + Re\,A\int_{\Omega^n}\phi_i\breve{u}\partial_r\bar{u} + Re\,A\int_{\Omega^n}\phi_i\breve{w}\partial_z\bar{u} + Re\,A\int_{\Omega^n}\phi_i\bar{u}\partial_r\breve{u} + Re\,A\int_{\Omega^n}\phi_i\bar{w}\partial_z\breve{u}\\ & + ARe\int_{\Omega^n}\phi_i\partial_t\breve{u} - ARe\int_{\Omega^n}\phi_i u_c\partial_r\breve{u} - ARe\int_{\Omega^n}\phi_i w_c\partial_z\breve{u}\\ & + Re\,(A)^2\int_{\Omega^n}\phi_i\breve{u}\partial_r\breve{u} + Re\,(A)^2\int_{\Omega^n}\phi_i\breve{w}\partial_z\breve{u}\\ & - St\int_{\Omega^n}\phi_i\hat{\boldsymbol{g}}_r - \int_{\Omega^n}p\partial_r\phi_i + 2\int_{\Omega^n}\partial_r\bar{u}\partial_r\phi_i + \int_{\Omega^n}\partial_z\bar{u}\partial_z\phi_i + \int_{\Omega^n}\partial_r\bar{w}\partial_z\phi_i + \int_{\partial\bar{\Omega}}\phi_i\boldsymbol{e}_r\cdot\bar{\boldsymbol{P}}\cdot\boldsymbol{n}.\end{aligned} \tag{25.9}$$

We now consider the penultimate term in the RHS of the equation above, i.e.

$$\int_{\Omega^n}\partial_r\bar{w}\partial_z\phi_i. \tag{25.10}$$

We recall the multi-variable integration by parts formula given by

$$\int_{\Omega^n} f\partial_{x_i} g = -\int_{\Omega^n} g\partial_{x_i} f - \int_{\partial\Omega^n} fgn^i, \tag{25.11}$$

where n^i is the i-th Cartesian component of the inward-pointing unit normal to Ω. †

Taking $f = \partial_r \bar{w}$ and $g = \phi_i$, we have

$$\int_{\Omega^f} \partial_r \bar{w} \partial_z \phi_i = -\int_{\Omega^f} \phi_i \partial_z \partial_r \bar{w} - \int_{\partial\Omega^f} \phi_i n_z \partial_r \bar{w}. \tag{25.12}$$

We can then exchange the order of the derivatives of $\bar{w}$ in the first integral on the RHS above to obtain

$$\int_{\Omega^n} \partial_r \bar{w} \partial_z \phi_i = -\int_{\Omega^n} \underbrace{\phi_i}_{f} \partial_r \underbrace{\partial_z \bar{w}}_{g} - \int_{\partial\Omega^n} \phi_i n_z \partial_r \bar{w}, \tag{25.13}$$

and taking $f = \phi_i$ and $g = \partial_z \bar{w}$ above, we can apply integration by parts once more, obtaining

$$\int_{\Omega^n} \partial_r \bar{w} \partial_z \phi_i = \int_{\Omega^n} \partial_r \phi_i \partial_z \bar{w} + \int_{\partial\Omega^n} \phi_i n_r \partial_z \bar{w} - \int_{\partial\Omega^n} \phi_i n_z \partial_r \bar{w}, \tag{25.14}$$

i.e.

$$\int_{\Omega^n} \partial_r \bar{w} \partial_z \phi_i = \int_{\Omega^n} \partial_r \phi_i \partial_z \bar{w} + \int_{\partial\Omega^n} \phi_i n_r \partial_z \bar{w} - \int_{\partial\Omega^n} \phi_i n_z \partial_r \bar{w}, \tag{25.15}$$

We now recall equation 2.41, which implies that $\partial_z \bar{w} = -\partial_r \bar{u}$, and we substitute this expression into the second integral on the RHS above, obtaining

$$\int_{\Omega^n} \partial_r \bar{w} \partial_z \phi_i = \int_{\Omega^n} \partial_r \phi_i \partial_z \bar{w} - \int_{\partial\Omega^n} \phi_i n_r \partial_r \bar{u} - \int_{\partial\Omega^n} \phi_i n_z \partial_r \bar{w}, \tag{25.16}$$

i.e.

$$\begin{aligned} \int_{\Omega^n} \partial_r \bar{w} \partial_z \phi_i = \int_{\Omega^n} \partial_r \phi_i \partial_z \bar{w} - \int_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \bar{u} - \int_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \bar{w} - \int_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \bar{u} \\ - \int_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \bar{w} - \int_{\partial\Omega^5} \phi_i n_r^5 \partial_r \bar{u} - \int_{\partial\Omega^5} \phi_i n_z^5 \partial_r \bar{w}, \end{aligned} \tag{25.17}$$

† This expression can be derived from the Gauss-Green theorem, which is a scalar version of the Gauss divergence theorem that can, in turn, be derived from the standard vector version of the Gauss divergence theorem.

We substitute this into equation 25.9 and we have

$$\begin{aligned}
\bar{M}_i^r &= Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{u} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{u} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{u} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \breve{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \breve{u} \\
&+ ARe \int\limits_{\Omega^n} \phi_i \partial_t \breve{u} - ARe \int\limits_{\Omega^n} \phi_i u_c \partial_r \breve{u} - ARe \int\limits_{\Omega^n} \phi_i w_c \partial_z \breve{u} \\
&+ Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{u} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{u} \\
&- St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_r - \int\limits_{\Omega^n} p \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r \bar{u} \partial_r \phi_i + \int\limits_{\Omega^n} \partial_z \bar{u} \partial_z \phi_i \\
&+ \int\limits_{\Omega^n} \partial_r \phi_i \underbrace{(\partial_r \bar{u} + \partial_z \bar{w})}_{=0} - \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \bar{u} - \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \bar{w} - \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \bar{u} \\
&- \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \bar{w} - \int\limits_{\partial\Omega^5} \phi_i n_r^5 \partial_r \bar{u} - \int\limits_{\partial\Omega^5} \phi_i n_z^5 \partial_r \bar{w} + \int\limits_{\partial\bar{\Omega}} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}.
\end{aligned} \tag{25.18}$$

Here, we will introduce a couple of terms that cancel each other out, but which will be useful as we proceed with the derivations. From the divergence theorem we have

$$-A \int\limits_{\partial\Omega^n} \phi_i \left(\nabla \breve{u}\right) \cdot \boldsymbol{n} = A \int\limits_{\Omega^n} \nabla \cdot (\phi_i \nabla \breve{u}), \tag{25.19}$$

i.e.

$$-A \int\limits_{\partial\Omega^n} \phi_i \left(\nabla \breve{u}\right) \cdot \boldsymbol{n} = A \int\limits_{\Omega^n} (\nabla \phi_i) \cdot (\nabla \breve{u}) + A \int\limits_{\Omega^n} \phi_i \underbrace{\nabla \cdot \nabla \breve{u}}_{=0}, \tag{25.20}$$

where the last term is zero because the eigensolution velocity field satisfies the Stokes equation.

Equivalently, we have

$$A \int\limits_{\Omega^n} \partial_r \phi_i \partial_r \breve{u} + A \int\limits_{\Omega^n} \partial_z \phi_i \partial_z \breve{u} + A \int\limits_{\partial\Omega^n} \phi_i \left(\nabla \breve{u}\right) \cdot \boldsymbol{n} = 0, \tag{25.21}$$

i.e.

$$\begin{aligned}
&A \int\limits_{\Omega^n} \partial_r \phi_i \partial_r \breve{u} + A \int\limits_{\Omega^n} \partial_z \phi_i \partial_z \breve{u} + A \int\limits_{\partial\Omega^{1,n}} \phi_i \left(\nabla \breve{u}\right) \cdot \boldsymbol{n}^1 \\
&\quad + A \int\limits_{\partial\Omega^{2,n}} \phi_i \left(\nabla \breve{u}\right) \cdot \boldsymbol{n}^2 + A \int\limits_{\partial\Omega^5} \phi_i \left(\nabla \breve{u}\right) \cdot \boldsymbol{n}^5 = 0.
\end{aligned} \tag{25.22}$$

We now consider the last integral on the right hand side of equation (25.18)

$$\int\limits_{\partial\Omega^n} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n} = \int\limits_{\partial\Omega^{1,n}} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^1 + \int\limits_{\partial\Omega^{2,n}} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 + \int\limits_{\partial\Omega^5} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5, \tag{25.23}$$

where $\partial\Omega^{1,n}$ is the near-field portion of the free surface, $\partial\Omega^{2,n}$ is the near-field portion of the liquid-solid interface, and $\partial\Omega^5$ is the separatrix surface (whose normal $\boldsymbol{n}^5$ points into the near-field sub-domain).

Substituting equations (25.22) and (25.18) into 25.9, we have

$$\bar{M}_i^r = \bar{M}_i^{r,0} + \bar{M}_i^{r,1} + \bar{M}_i^{r,2} + \bar{M}_i^{r,3} + \bar{M}_i^{r,5} \tag{25.24}$$

where

$$\begin{aligned}\bar{M}_i^{r,0} = {} & Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{u} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{u} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{u} \\ & + Re\, A \int\limits_{\Omega^n} \phi_i \check{u} \partial_r \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \check{w} \partial_z \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \check{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \check{u} \\ & + A Re \int\limits_{\Omega^n} \phi_i \partial_t \check{u} - A Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \check{u} - A Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \check{u} \\ & + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \check{u} \partial_r \check{u} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \check{w} \partial_z \check{u} \\ & - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_r - \int\limits_{\Omega^n} p \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r u \partial_r \phi_i + \int\limits_{\Omega^n} \partial_z u \partial_z \phi_i + A \int\limits_{\Omega} \partial_r \phi_i \partial_r \check{u} + A \int\limits_{\Omega} \partial_z \phi_i \partial_z \check{u},\end{aligned} \tag{25.25}$$

$$\bar{M}_i^{r,1} = A \int\limits_{\partial\Omega^1} \phi_i \left(\nabla \check{u}\right) \cdot \boldsymbol{n}^1 - \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \bar{u} - \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \bar{w} + \int\limits_{\partial\Omega^{1,n}} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^1, \tag{25.26}$$

$$\bar{M}_i^{r,2} = A \int\limits_{\partial\Omega^2} \phi_i \left(\nabla \check{u}\right) \cdot \boldsymbol{n}^2 - \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \bar{u} - \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \bar{w} + \int\limits_{\partial\Omega^{2,n}} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2, \tag{25.27}$$

and

$$\bar{M}_i^{r,5} = A \int\limits_{\partial\Omega^5} \phi_i \left(\nabla \check{u}\right) \cdot \boldsymbol{n}^5 - \int\limits_{\partial\Omega^5} \phi_i n_r^5 \partial_r \bar{u} - \int\limits_{\partial\Omega^5} \phi_i n_z^5 \partial_r \bar{w} + \int\limits_{\partial\Omega^5} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5. \tag{25.28}$$

For the free-surface, equation (25.26), we have equation (23.15), which states

$$\left(\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) \cdot \boldsymbol{n}^1 = -p^g \boldsymbol{n}^1 - \frac{\nabla^s \cdot \left[\sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)\right]}{Ca}. \tag{25.29}$$

and therefore

$$\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^1 = -\phi_i p^g \boldsymbol{e}_r \cdot \boldsymbol{n}^1 - \frac{1}{Ca} \phi_i \boldsymbol{e}_r \cdot \nabla^s \cdot \left[\sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)\right] - A \phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^1. \tag{25.30}$$

Now, we have the following surface vector calculus identity

$$\nabla^s \cdot (\boldsymbol{x} \cdot \boldsymbol{Q}) = \boldsymbol{Q} : \nabla^s \boldsymbol{x} + \boldsymbol{x} \cdot \nabla^s \cdot \boldsymbol{Q}, \tag{25.31}$$

and taking $\boldsymbol{x} = \phi_i \boldsymbol{e}_r$ and $\boldsymbol{Q} = \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)$, we have

$$\nabla^s \cdot (\phi_i \boldsymbol{e}_r \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)) = \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) : \nabla^s (\phi_i \boldsymbol{e}_r) + \phi_i \boldsymbol{e}_r \cdot \nabla^s \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) \tag{25.32}$$

which yields

$$\phi_i \boldsymbol{e}_r \cdot \nabla^s \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) = \nabla^s \cdot (\phi_i \boldsymbol{e}_r \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)) - \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) : \nabla^s \phi_i \boldsymbol{e}_r. \tag{25.33}$$

In this $1D$-surface case, we have

$$\nabla^s \phi_i \boldsymbol{e}_r = \begin{bmatrix} t_r^1 \partial_s \phi_i & 0 \\ t_z^1 \partial_s \phi_i & 0 \end{bmatrix}, \tag{25.34}$$

where $\boldsymbol{t}^1 = (t_r^1, t_z^1)$, and the tangent vector must be pointing in the direction of increasing arc-length s, therefore

$$(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) : \nabla^s \phi_i \boldsymbol{e}_r = \begin{bmatrix} 1 - n_r^1 n_r^1 & -n_r^1 n_z^1 \\ -n_z^1 n_r^1 & 1 - n_z^1 n_z^1 \end{bmatrix} : \begin{bmatrix} t_r^1 \partial_s \phi_i & 0 \\ t_z^1 \partial_s \phi_i & 0 \end{bmatrix}, \tag{25.35}$$

where $\boldsymbol{n}^1 = (n_r^1, n_z^1)$, i.e.

$$(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) : \nabla^s \phi_i \boldsymbol{e}_r = t_r^1 \partial_s \phi_i - (\boldsymbol{t}^1 \cdot \boldsymbol{n}^1) n_r^1 \partial_s \phi_i = t_r^1 \partial_s \phi_i. \tag{25.36}$$

We therefore have in equation (25.33)

$$\phi_i \boldsymbol{e}_r \cdot \nabla^s \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) = \nabla^s \cdot (\phi_i \boldsymbol{e}_r \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)) - \sigma^1 t_r^1 \partial_s \phi_i. \tag{25.37}$$

Taking this result into (25.30), the result of that into (25.23) and the subsequent result into equation (25.26) we have

$$\begin{aligned} \bar{M}_i^{r,1} = {} & A \int_{\partial\Omega^1} \phi_i \, (\nabla \check{u}) \cdot \boldsymbol{n}^1 - \int_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \bar{u} - \int_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \bar{w} - \int_{\partial\Omega^{1,n}} \phi_i p^g n_r^1 \\ & - \frac{1}{Ca} \int_{\partial\Omega^{1,n}} \nabla^s \cdot (\sigma^1 \phi_i \boldsymbol{e}_r \cdot (\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)) + \frac{1}{Ca} \int_{\partial\Omega^{1,n}} t_r^1 \sigma^1 \partial_s \phi_i \\ & - A \int_{\partial\Omega^{1,n}} \phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^1. \end{aligned} \tag{25.38}$$

Using the surface divergence theorem and the definition of the surface divergence for a $1D$ surface, we have

$$\begin{aligned} \bar{M}_i^{r,1} = {} & A \int_{\partial\Omega^1} \phi_i \, (\nabla \check{u}) \cdot \boldsymbol{n}^1 - \int_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \bar{u} - \int_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \bar{w} \\ & + \frac{1}{Ca} \int_{C_1^n} \sigma^1 \phi_i \boldsymbol{e}_r \cdot \boldsymbol{m}^1 + \frac{1}{Ca} \int_{\partial\Omega^{1,n}} t_r^1 \sigma^1 \partial_s \phi_i \\ & + A \int_{\partial\Omega^1} \phi_i \, (\nabla \check{u}) \cdot \boldsymbol{n}^1 - A \int_{\partial\Omega^{1,n}} \phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^1, \end{aligned} \tag{25.39}$$

where C_1^n is actually the two points bounding the free surface, and $\boldsymbol{m}^1$ is the vector that is tangent to the free surface, normal to the contact line and points into the free surface.

We can thus reduce the expression above to

$$\begin{aligned}\bar{M}_i^{r,1} = A \int\limits_{\partial\Omega^1} \phi_i \left(\nabla \breve{u}\right) \cdot \boldsymbol{n}^1 - \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \bar{u} - \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \bar{w} - \int\limits_{\partial\Omega^{1,n}} \phi_i p^g n_r^1 \\ + \frac{\sigma^1(r_c, z_c)\phi_i(r_c, z_c) m_r^1(r_c, z_c)}{Ca} + \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1}) m_r^{1,f}(r_{J^1}, z_{J^1})}{Ca} \\ + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_r^1 \sigma^1 \partial_s \phi_i + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_r^1 \sigma^1 \partial_s \phi_i \\ - A \int\limits_{\partial\Omega^{1,n}} \phi_i \boldsymbol{e}_r \cdot \breve{\boldsymbol{P}} \cdot \boldsymbol{n}^1,\end{aligned} \tag{25.40}$$

where (r_c, z_c) is the location of the contact line and (r_{J^1}, z_{J^1}) are the coordinates of the junction between near-field and far-field on the free surface. Moreover, $\boldsymbol{m}_r^{1,f}$ is the r-component of the unit vector that is tangent to the far field half of the free surface at the junction point and points into the near-field sub-domain.

We consider now the integrand in the last term of equation (25.40), i.e.

$$\phi_i \boldsymbol{e}_r \cdot \breve{\boldsymbol{P}} \cdot \boldsymbol{n}^1 = \begin{bmatrix} \phi_i \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2\partial_r \breve{u} & \partial_z \breve{u} + \partial_r \breve{w} \\ \partial_r \breve{w} + \partial_z \breve{u} & 2\partial_z \breve{w} \end{bmatrix} \cdot \begin{bmatrix} n_r^1 \\ n_z^1 \end{bmatrix}, \tag{25.41}$$

i.e.

$$\phi_i \boldsymbol{e}_r \cdot \breve{\boldsymbol{P}} \cdot \boldsymbol{n}^1 = \begin{bmatrix} \phi_i \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2n_r^1 \partial_r \breve{u} + n_z^1 \left(\partial_z \breve{u} + \partial_r \breve{w}\right) \\ n_r^1 \left(\partial_r \breve{w} + \partial_z \breve{u}\right) + 2n_z^1 \partial_z \breve{w} \end{bmatrix}. \tag{25.42}$$

Hence

$$\phi_i \boldsymbol{e}_r \cdot \breve{\boldsymbol{P}} \cdot \boldsymbol{n}^1 = 2\phi_i n_r^1 \partial_r \breve{u} + \phi_i n_z^1 \partial_z \breve{u} + \phi_i n_z^1 \partial_r \breve{w}. \tag{25.43}$$

Integrating over the boundary of the domain, we have

$$\int\limits_{\partial\Omega^n} \phi_i \boldsymbol{e}_r \cdot \breve{\boldsymbol{P}} \cdot \boldsymbol{n}^1 = 2 \int\limits_{\partial\Omega^n} \phi_i n_r^1 \partial_r \breve{u} + \int\limits_{\partial\Omega^n} \phi_i n_z^1 \partial_z \breve{u} + \int\limits_{\partial\Omega^n} \phi_i n_z^1 \partial_r \breve{w}, \tag{25.44}$$

i.e.

$$\int\limits_{\partial\Omega^{1,n}} \phi_i \boldsymbol{e}_r \cdot \breve{\boldsymbol{P}} \cdot \boldsymbol{n}^1 = \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \breve{u} + \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \breve{w} + \underbrace{\int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \breve{u} + \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_z \breve{u}}_{\int\limits_{\partial\Omega^{1,n}} \phi_i (\nabla \breve{u}) \cdot \boldsymbol{n}^1}. \tag{25.45}$$

For an explicit formula for each of the velocities of the eigen-solution and its derivatives that are required throughout this text, see Appendix B.

we can now take (25.45) into (25.40), yielding

$$\bar{M}_i^{r,1} = -\int_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \bar{u} - \int_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \bar{w} - \int_{\partial\Omega^{1,n}} \phi_i p^g n_r^1 + \frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca}$$
$$+ \frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca} + \frac{1}{Ca}\int_{\partial\Omega^{1,n}} t_r^1\sigma^1\partial_s\phi_i + \frac{1}{Ca}\int_{\partial\Omega^{1,n}} t_r^1\sigma^1\partial_s\phi_i$$
$$- A\int_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \check{u} - A\int_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \check{w}. \tag{25.46}$$

On equation (25.27), we consider now the term

$$\int_{\partial\Omega^2} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2, \tag{25.47}$$

where we have

$$\phi_i \boldsymbol{e}_r \cdot \left(\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) \cdot \boldsymbol{n}^2 = \phi_i \boldsymbol{e}_r \cdot \underbrace{\boldsymbol{n}^2 \cdot \left(\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2)}_{Be\,(\bar{\boldsymbol{u}}+A\check{\boldsymbol{u}}-\boldsymbol{u}^s)\cdot(\boldsymbol{I}-\boldsymbol{n}^2\boldsymbol{n}^2) - \frac{1}{2Ca\,Es}\left[\boldsymbol{v}^{s_2} - \frac{1}{2}(\bar{\boldsymbol{u}}+A\check{\boldsymbol{u}}+\boldsymbol{u}^s)\right]\cdot(\boldsymbol{I}-\boldsymbol{n}^2\boldsymbol{n}^2)}$$
$$+\phi_i \boldsymbol{e}_r \cdot \underbrace{\left(\boldsymbol{n}^2 \cdot \left(\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) \cdot \boldsymbol{n}^2\right)}_{\lambda^2} \boldsymbol{n}^2, \tag{25.48}$$

where we have used equations (23.21) and (23.22). Moreover, we recall variable λ^2, i.e. the normal stress on boundary 2.

Hence, we have

$$\begin{aligned}\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = \phi_i \boldsymbol{e}_r \cdot Be\,(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{u}^s) \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2) \\ -\frac{1}{2Ca\,Es}\phi_i \boldsymbol{e}_r \cdot \left[\boldsymbol{v}^{s_2} - \frac{1}{2}(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} + \boldsymbol{u}^s)\right] \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2) \\ + \phi_i \boldsymbol{e}_r \cdot \lambda^2 \boldsymbol{n}^2 - A\phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2,\end{aligned} \tag{25.49}$$

i.e.

$$\begin{aligned}\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = Be\,\phi_i \boldsymbol{e}_r \cdot \left[(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{u}^s) \cdot \boldsymbol{t}^2\right] \boldsymbol{t}^2 \\ -\frac{1}{2Ca\,Es}\phi_i \boldsymbol{e}_r \cdot \left[\boldsymbol{v}^{s_2} \cdot \boldsymbol{t}^2 - \frac{1}{2}(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} + \boldsymbol{u}^s) \cdot \boldsymbol{t}^2\right] \boldsymbol{t}^2 \\ + \lambda^2 \phi_i n_r^2 - A\phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2,\end{aligned} \tag{25.50}$$

where we have used that, in 2D, $\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2 = \boldsymbol{t}^2\boldsymbol{t}^2$, with $\boldsymbol{t}^2$ being the unit tangent to the boundary 2 which points in the direction of increasing arc-length s. We highlight that the term $\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2)$ is zero when the solid surface is flat, which follows from the no-tangential-stress condition for the eigen-solution; however, for a generic curved solid surface it is not zero.

Re-writing we have

$$\begin{aligned}\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = {} & Be\,\phi_i \boldsymbol{e}_r \cdot \left(\bar{\boldsymbol{u}} \cdot \boldsymbol{t}^2\right) \boldsymbol{t}^2 + Be\,A\phi_i \boldsymbol{e}_r \cdot \left(\check{\boldsymbol{u}} \cdot \boldsymbol{t}^2\right) \boldsymbol{t}^2 - Be\,\phi_i \boldsymbol{e}_r \cdot \left(\boldsymbol{u}^s \cdot \boldsymbol{t}^2\right) \boldsymbol{t}^2 \\ & - \frac{1}{2Ca\,Es} \phi_i \boldsymbol{e}_r \cdot \left[\boldsymbol{v}^{s_2} \cdot \boldsymbol{t}^2\right] \boldsymbol{t}^2 \\ & - \frac{1}{2Ca\,Es} \phi_i \boldsymbol{e}_r \cdot \left[-\frac{1}{2}\left(\bar{\boldsymbol{u}}\right) \cdot \boldsymbol{t}^2\right] \boldsymbol{t}^2 \\ & - \frac{1}{2Ca\,Es} \phi_i \boldsymbol{e}_r \cdot \left[-\frac{1}{2}\left(A\check{\boldsymbol{u}}\right) \cdot \boldsymbol{t}^2\right] \boldsymbol{t}^2 \\ & - \frac{1}{2Ca\,Es} \phi_i \boldsymbol{e}_r \cdot \left[-\frac{1}{2}\left(\boldsymbol{u}^s\right) \cdot \boldsymbol{t}^2\right] \boldsymbol{t}^2 \\ & + \lambda^2 \phi_i n_r^2 - A\phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2, \end{aligned} \tag{25.51}$$

i.e.

$$\begin{aligned}\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = {} & Be\,\phi_i t_r^2 \left(\bar{\boldsymbol{u}} \cdot \boldsymbol{t}^2\right) + Be\,A\phi_i t_r^2 \left(\check{\boldsymbol{u}} \cdot \boldsymbol{t}^2\right) - Be\,\phi_i t_r^2 \left(\boldsymbol{u}^s \cdot \boldsymbol{t}^2\right) \\ & - \frac{1}{2Ca\,Es} \phi_i t_r^2 \left[\boldsymbol{v}^{s_2} \cdot \boldsymbol{t}^2\right] \\ & + \frac{1}{4Ca\,Es} \phi_i t_r^2 \left[\bar{\boldsymbol{u}} \cdot \boldsymbol{t}^2\right] \\ & + \frac{1}{4Ca\,Es} A\phi_i t_r^2 \left[\check{\boldsymbol{u}} \cdot \boldsymbol{t}^2\right] \\ & + \frac{1}{4Ca\,Es} \phi_i t_r^2 \left[\boldsymbol{u}^s \cdot \boldsymbol{t}^2\right] \\ & + \lambda^2 \phi_i n_r^2 - A\phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2. \end{aligned} \tag{25.52}$$

Expanding the innermost products in the equations above we have

$$\begin{aligned}\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = {} & Be\,\phi_i t_r^2 \left(\bar{u} t_r^2 + \bar{w} t_z^2\right) + Be\,A\phi_i t_r^2 \left(\check{u} t_r^2 + \check{w} t_z^2\right) - Be\,\phi_i t_r^2 \left(u^s t_r^2 + w^s t_z^2\right) \\ & - \frac{1}{2Ca\,Es} \phi_i t_r^2 \left(u^{s_2} t_r^2 + w^{s_2} t_z^2\right) \\ & + \frac{1}{4Ca\,Es} \phi_i t_r^2 \left(\bar{u} t_r^2 + \bar{w} t_z^2\right) \\ & + \frac{1}{4Ca\,Es} A\phi_i t_r^2 \left(\check{u} t_r^2 + \check{w} t_z^2\right) \\ & + \frac{1}{4Ca\,Es} \phi_i t_r^2 \left(u^s t_r^2 + w^s t_z^2\right) \\ & + \lambda^2 \phi_i n_r^2 - A\phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2. \end{aligned} \tag{25.53}$$

Grouping terms we have

$$
\begin{aligned}
\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = \lambda^2 \phi_i n_r^2 & \\
& + \left(\frac{1}{4Ca\,Es} + Be\right)\phi_i \bar{u} t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} + Be\right)\phi_i \bar{w} t_r^2 t_z^2 \\
& + \left(\frac{1}{4Ca\,Es} + Be\right) A\phi_i t_r^2 \check{u} t_r^2 + \left(\frac{1}{4Ca\,Es} + Be\right) A\phi_i t_r^2 \check{w} t_z^2 \\
& + \left(\frac{1}{4Ca\,Es} - Be\right)\phi_i t_r^2 t_r^2 u^s + \left(\frac{1}{4Ca\,Es} - Be\right)\phi_i t_r^2 t_z^2 w^s \\
& - \frac{1}{2Ca\,Es}\phi_i t_r^2 t_r^2 u^{s_2} - \frac{1}{2Ca\,Es}\phi_i t_r^2 t_z^2 w^{s_2} \\
& - A\phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2 .
\end{aligned}
\tag{25.54}
$$

We consider now the term

$$
-A\phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = -A\phi_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2\partial_r \check{u} & \partial_z \check{u} + \partial_r \check{w} \\ \partial_r \check{w} + \partial_z \check{u} & 2\partial_z \check{w} \end{bmatrix} \cdot \begin{bmatrix} n_r^2 \\ n_z^2 \end{bmatrix}, \tag{25.55}
$$

i.e.

$$
-A\phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = -2A\phi_i n_r^2 \partial_r \check{u} - A\phi_i n_z^2 \partial_r \check{w} - A\phi_i n_z^2 \partial_z \check{u}. \tag{25.56}
$$

Taking this result into 25.54, we have

$$
\begin{aligned}
\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = \lambda^2 \phi_i n_r^2 & \\
& + \left(\frac{1}{4Ca\,Es} + Be\right)\phi_i \bar{u} t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} + Be\right)\phi_i \bar{w} t_r^2 t_z^2 \\
& + \left(\frac{1}{4Ca\,Es} + Be\right) A\phi_i t_r^2 \check{u} t_r^2 + \left(\frac{1}{4Ca\,Es} + Be\right) A\phi_i t_r^2 \check{w} t_z^2 \\
& + \left(\frac{1}{4Ca\,Es} - Be\right)\phi_i t_r^2 t_r^2 u^s + \left(\frac{1}{4Ca\,Es} - Be\right)\phi_i t_r^2 t_z^2 w^s \\
& - \frac{1}{2Ca\,Es}\phi_i t_r^2 t_r^2 u^{s_2} - \frac{1}{2Ca\,Es}\phi_i t_r^2 t_z^2 w^{s_2} \\
& - 2A\phi_i n_r^2 \partial_r \check{u} - A\phi_i n_r^2 \partial_r \check{w} - A\phi_i n_z^2 \partial_z \check{u}.
\end{aligned}
\tag{25.57}
$$

Integrating we have

$$
\begin{aligned}
\int\limits_{\partial\Omega^{2,n}} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = &\left(\frac{1}{4Ca\,Es} + Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{u} t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} + Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{w} t_z^2 t_r^2 \\
&+\left(\frac{1}{4Ca\,Es} + Be\right) \int\limits_{\partial\Omega^{2,n}} A\phi_i \check{u} t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} + Be\right) A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{w} t_z^2 t_r^2 \\
&+ \left(\frac{1}{4Ca\,Es} - Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i u^s t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} - Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i w^s t_z^2 t_r^2 \\
&- \frac{1}{2Ca\,Es} \int\limits_{\partial\Omega^{2,n}} \phi_i u^{s_2} t_r^2 t_r^2 - \frac{1}{2Ca\,Es} \int\limits_{\partial\Omega^{2,n}} \phi_i w^{s_2} t_z^2 t_r^2 \\
&+ \int\limits_{\partial\Omega^{2,n}} \lambda^2 \phi_i n_r^2 \\
&- A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \check{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \check{w} \\
&\underbrace{-A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \check{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_z \check{u}\,.}_{-A \int\limits_{\partial\Omega^{2,n}} \phi_i (\nabla \check{u}) \cdot \boldsymbol{n}^2}
\end{aligned}
\tag{25.58}
$$

Taking the result above into equation (25.27) we have

$$
\begin{aligned}
\bar{M}_i^{r,2} = &- \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \bar{u} - \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \bar{w} \\
&+ \left(\frac{1}{4Ca\,Es} + Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{u} t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} + Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{w} t_z^2 t_r^2 \\
&+\left(\frac{1}{4Ca\,Es} + Be\right) \int\limits_{\partial\Omega^{2,n}} A\phi_i \check{u} t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} + Be\right) A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{w} t_z^2 t_r^2 \\
&+ \left(\frac{1}{4Ca\,Es} - Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i u^s t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} - Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i w^s t_z^2 t_r^2 \\
&- \frac{1}{2Ca\,Es} \int\limits_{\partial\Omega^{2,n}} \phi_i u^{s_2} t_r^2 t_r^2 - \frac{1}{2Ca\,Es} \int\limits_{\partial\Omega^{2,n}} \phi_i w^{s_2} t_z^2 t_r^2 \\
&+ \int\limits_{\partial\Omega^{2,n}} \lambda^2 \phi_i n_r^2 - A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \check{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \check{w},
\end{aligned}
\tag{25.59}
$$

We consider now the term

$$\int\limits_{\partial\Omega^5} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5, \tag{25.60}$$

where we have

$$\phi_i \boldsymbol{e}_r \cdot \left(\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) \cdot \boldsymbol{n}^5 = \phi_i \boldsymbol{e}_r \cdot \boldsymbol{P} \cdot \boldsymbol{n}^5, \tag{25.61}$$

i.e.

$$\phi_i \boldsymbol{e}_r \cdot \left(\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) \cdot \boldsymbol{n}^5 = \phi_i \boldsymbol{e}_r \cdot \left[\underbrace{(\boldsymbol{n}^5 \cdot \boldsymbol{P} \cdot \boldsymbol{n}^5)}_{\lambda^5} \boldsymbol{n}^5 + \underbrace{\boldsymbol{n}^5 \cdot \boldsymbol{P} \cdot (\boldsymbol{I} - \boldsymbol{n}^5 \boldsymbol{n}^5)}_{\gamma^5 \boldsymbol{t}^5}\right], \tag{25.62}$$

hence we have

$$\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5 = \phi_i \lambda^5 \boldsymbol{e}_r \cdot \boldsymbol{n}^5 + \phi_i \gamma^5 \boldsymbol{e}_r \cdot \boldsymbol{t}^5 - A\phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^5, \tag{25.63}$$

i.e.

$$\begin{aligned} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5 \; &= \\ \phi_i \lambda^5 n_r^5 + \phi_i \gamma^5 t_r^5 &+ \begin{bmatrix} \phi_i \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2A\partial_r \check{u} & -A\partial_r \check{w} - A\partial_z \check{u} \\ -A\partial_z \check{u} - A\partial_r \check{w} & -2A\partial_z \check{w} \end{bmatrix} \cdot \begin{bmatrix} n_r^5 \\ n_z^5 \end{bmatrix}, \end{aligned} \tag{25.64}$$

or equivalently

$$\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5 \; = \phi_i \lambda^4 n_r^4 + \phi_i \gamma^4 t_r^4 + \begin{bmatrix} \phi_i \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2An_r^5 \partial_r \check{u} - An_z^5 \partial_r \check{w} - An_z^5 \partial_z \check{u} \\ -An_r^5 \partial_z \check{u} - An_r^5 \partial_r \check{w} - 2An_z^5 \partial_z \check{w} \end{bmatrix}, \tag{25.65}$$

which yields

$$\phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5 = \phi_i \lambda^5 n_r^5 + \phi_i \gamma^5 t_r^5 + -2An_r^5 \phi_i \partial_r \check{u} - An_z^5 \phi_i \partial_r \check{w} - An_z^5 \phi_i \partial_z \check{u}. \tag{25.66}$$

Integrating we have

$$\begin{aligned} \int_{\partial\Omega^5} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5 = \int_{\partial\Omega^5} \phi_i \lambda^5 n_r^5 &+ \int_{\partial\Omega^5} \phi_i \gamma^5 t_r^5 - A \int_{\partial\Omega^5} n_r^5 \phi_i \partial_r \check{u} \\ &- A \int_{\partial\Omega^5} n_z^5 \phi_i \partial_r \check{w} \underbrace{- A \int_{\partial\Omega^5} n_r^5 \phi_i \partial_r \check{u} - A \int_{\partial\Omega^5} n_z^5 \phi_i \partial_z \check{u}}_{-A \int_{\partial\Omega^5} \phi_i (\nabla \check{u}) \cdot \boldsymbol{n}^5} . \end{aligned} \tag{25.67}$$

Taking this result into equation (25.28) we have

$$\begin{aligned} \bar{M}_i^{r,5} = - \int_{\partial\Omega^5} \phi_i n_r^5 \partial_r \bar{u} - \int_{\partial\Omega^5} \phi_i n_z^5 \partial_r \bar{w} + \int_{\partial\Omega^5} \phi_i \lambda^5 n_r^5 \\ + \int_{\partial\Omega^5} \phi_i \gamma^5 t_r^5 - A \int_{\partial\Omega^5} n_r^5 \phi_i \partial_r \check{u} - A \int_{\partial\Omega^5} n_z^5 \phi_i \partial_r \check{w}. \end{aligned} \tag{25.68}$$

We thus have

$$\bar{M}_i^r = \bar{M}_i^{r,0} + \bar{M}_i^{r,1} + \bar{M}_i^{r,2} + \bar{M}_i^{r,5}, \tag{25.69}$$

where

$$
\begin{aligned}
\bar{M}_i^{r,0} = & -St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_r + A \int\limits_{\Omega^n} \partial_r \phi_i \partial_r \check{u} + A \int\limits_{\Omega^n} \partial_z \phi_i \partial_z \check{u} \\
& + Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{u} \\
& - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{u} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{u} \\
& + Re\, A \int\limits_{\Omega^n} \phi_i \check{u} \partial_r \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \check{w} \partial_z \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \check{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \check{u} \\
& - ARe \int\limits_{\Omega^n} \phi_i u_c \partial_r \check{u} - Re\, A \int\limits_{\Omega^n} \phi_i w_c \partial_z \check{u} \\
& - \int\limits_{\Omega^n} p \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r u \partial_r \phi_i + \int\limits_{\Omega^n} \partial_z u \partial_z \phi_i \\
& + ARe \int\limits_{\Omega^n} \phi_i \partial_t \check{u} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \check{u} \partial_r \check{u} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \check{w} \partial_z \check{u},
\end{aligned}
\tag{25.70}
$$

$$
\begin{aligned}
\bar{M}_i^{r,1} = & \frac{\sigma^1(r_c, z_c) \phi_i(r_c, z_c) m_r^1(r_c, z_c)}{Ca} + \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_r^{1,f}(r_{J^1}, z_{J^1})}{Ca} \\
& - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \check{u} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \check{w} \\
& - \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \bar{u} - \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \bar{w} - \int\limits_{\partial\Omega^{1,n}} \phi_i p^g n_r^1 + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_r^1 \sigma^1 \partial_s \phi_i,
\end{aligned}
\tag{25.71}
$$

$$
\begin{aligned}
\bar{M}_i^{r,2} = & -A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \check{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \check{w} \\
& + \left(\frac{1}{4Ca\,Es} + Be\right) \int\limits_{\partial\Omega^{2,n}} A\phi_i \check{u} t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} + Be\right) A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{w} t_z^2 t_r^2 \\
& - \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \bar{u} - \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \bar{w} \\
& + \left(\frac{1}{4Ca\,Es} + Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{u} t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} + Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{w} t_z^2 t_r^2 \\
& + \left(\frac{1}{4Ca\,Es} - Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i u^s t_r^2 t_r^2 + \left(\frac{1}{4Ca\,Es} - Be\right) \int\limits_{\partial\Omega^{2,n}} \phi_i w^s t_z^2 t_r^2 \\
& - \frac{1}{2Ca\,Es} \int\limits_{\partial\Omega^{2,n}} \phi_i u^{s_2} t_r^2 t_r^2 - \frac{1}{2Ca\,Es} \int\limits_{\partial\Omega^{2,n}} \phi_i w^{s_2} t_z^2 t_r^2 \\
& + \int\limits_{\partial\Omega^{2,n}} \lambda^2 \phi_i n_r^2,
\end{aligned}
\tag{25.72}
$$

and

$$
\begin{aligned}
\bar{M}_i^{r,5} = & -A \int\limits_{\partial\Omega^5} n_r^5 \phi_i \partial_r \check{u} - A \int\limits_{\partial\Omega^5} n_z^5 \phi_i \partial_r \check{w} - \int\limits_{\partial\Omega^5} \phi_i n_r^5 \partial_r \bar{u} \\
& - \int\limits_{\partial\Omega^5} \phi_i n_z^5 \partial_r \bar{w} + \int\limits_{\partial\Omega^5} \phi_i \lambda^5 n_r^5 + \int\limits_{\partial\Omega^5} \phi_i \gamma^5 t_r^5.
\end{aligned}
\tag{25.73}
$$

We now recall the approximations made in (4.54) and (4.55), namely

$$
u^c(t_n) = \partial_t r^c(t_n) \approx \frac{(1+2q_n) r^c(t_n) - (1+q_n)^2 r^c(t_{n-1}) + q_n^2 r^c(t_{n-2})}{(1+q_n)\Delta_t^n}, \tag{25.74}
$$

$$
w^c(t_n) = \partial_t z^c(t_n) \approx \frac{(1+2q_n) z^c(t_n) - (1+q_n)^2 z^c(t_{n-1}) + q_n^2 z^c(t_{n-2})}{(1+q_n)\Delta_t^n}, \tag{25.75}
$$

and we introduce the following approximations

$$
\partial_t \bar{u}(t_n) \approx \frac{(1+2q_n)\bar{u}(t_n) - (1+q_n)^2 \bar{u}(t_{n-1}) + q_n^2 \bar{u}(t_{n-2})}{(1+q_n)\Delta_t^n}, \tag{25.76}
$$

$$
\partial_t \bar{w}(t_n) \approx \frac{(1+2q_n)\bar{w}(t_n) - (1+q_n)^2 \bar{w}(t_{n-1}) + q_n^2 \bar{w}(t_{n-2})}{(1+q_n)\Delta_t^n}, \tag{25.77}
$$

$$
\partial_t \check{u}(t_n) \approx \frac{(1+2q_n)\check{u}(t_n) - (1+q_n)^2 \check{u}(t_{n-1}) + q_n^2 \check{u}(t_{n-2})}{(1+q_n)\Delta_t^n}, \tag{25.78}
$$

$$
\partial_t \check{w}(t_n) \approx \frac{(1+2q_n)\check{w}(t_n) - (1+q_n)^2 \check{w}(t_{n-1}) + q_n^2 \check{w}(t_{n-2})}{(1+q_n)\Delta_t^n}, \tag{25.79}
$$

Substituting these into (25.69) and (**??**)-(**??**), though the last three remain unaltered, we obtain

$$\bar{\mathfrak{M}}_i^r = \bar{\mathfrak{M}}_i^{r,0} + \bar{\mathfrak{M}}_i^{r,1} + \bar{\mathfrak{M}}_i^{r,2} + \bar{\mathfrak{M}}_i^{r,4}, \tag{25.80}$$

$$\begin{aligned}
\bar{\mathfrak{M}}_i^{r,0} = &-St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_r + A \int\limits_{\Omega^n} \partial_r \phi_i \partial_r \breve{u} + A \int\limits_{\Omega^n} \partial_z \phi_i \partial_z \breve{u} \\
&+ Re \int\limits_{\Omega^n} \phi_i \frac{(1+2q_n)\bar{u}(t_n) - (1+q_n)^2 \bar{u}(t_{n-1}) + q_n^2 \bar{u}(t_{n-2})}{(1+q_n)\Delta_t^n} \\
&+ Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{u} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{u} \\
&- Re \int\limits_{\Omega^n} \phi_i \frac{(1+2q_n)r^c(t_n) - (1+q_n)^2 r^c(t_{n-1}) + q_n^2 r^c(t_{n-2})}{(1+q_n)\Delta_t^n} \partial_r \bar{u} \\
&- Re \int\limits_{\Omega^n} \phi_i \frac{(1+2q_n)z^c(t_n) - (1+q_n)^2 z^c(t_{n-1}) + q_n^2 z^c(t_{n-2})}{(1+q_n)\Delta_t^n} \partial_z \bar{u} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \bar{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \breve{u} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \breve{u} \\
&- Re\, A \int\limits_{\Omega^n} \phi_i \frac{(1+2q_n)r^c(t_n) - (1+q_n)^2 r^c(t_{n-1}) + q_n^2 r^c(t_{n-2})}{(1+q_n)\Delta_t^n} \partial_r \breve{u} \\
&- Re\, A \int\limits_{\Omega^n} \phi_i \frac{(1+2q_n)z^c(t_n) - (1+q_n)^2 z^c(t_{n-1}) + q_n^2 z^c(t_{n-2})}{(1+q_n)\Delta_t^n} \partial_z \breve{u} \\
&- \int\limits_{\Omega^n} p \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r \bar{u} \partial_r \phi_i + \int\limits_{\Omega^n} \partial_z \bar{u} \partial_z \phi_i \\
&+ A Re \int\limits_{\Omega^n} \phi_i \frac{(1+2q_n)\breve{u}(t_n) - (1+q_n)^2 \breve{u}(t_{n-1}) + q_n^2 \breve{u}(t_{n-2})}{(1+q_n)\Delta_t^n} \\
&+ Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{u} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{u}.
\end{aligned} \tag{25.81}$$

$$\begin{aligned}
\bar{\mathfrak{M}}_i^{r,1} = &\frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_r^1 \sigma^1 \partial_s \phi_i - \int\limits_{\partial\Omega^{1,n}} \phi_i p^g n_r^1 - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \breve{u} \\
&- A \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \breve{w} - \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \bar{u} - \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_r \bar{w} \\
&+ \frac{\sigma^1(r_c, z_c)\phi_i(r_c, z_c) m_r^1(r_c, z_c)}{Ca} + \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1}) m_r^{1,f}(r_{J^1}, z_{J^1})}{Ca},
\end{aligned} \tag{25.82}$$

$$\bar{\mathfrak{M}}_i^{r,2} = Be \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{u} t_r^2 t_r^2 + Be \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{w} t_r^2 t_z^2 + Be\, A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{u} t_r^2 t_r^2 + Be\, A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{w} t_r^2 t_z^2$$

$$- Be \int\limits_{\partial\Omega^{2,n}} \phi_i u^s t_r^2 t_r^2 - Be \int\limits_{\partial\Omega^{2,n}} \phi_i w^s t_r^2 t_z^2 - \frac{1}{2Ca} \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 \partial_s \sigma^2 + \int\limits_{\partial\Omega^{2,n}} \lambda^2 \phi_i n_r^2$$

$$- A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \check{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \check{w} - \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \bar{u} - \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \bar{w}, \tag{25.83}$$

and

$$\bar{\mathfrak{M}}_i^{r,5} = \int\limits_{\partial\Omega^5} \lambda^5 n_r^5 \phi_i + \int\limits_{\partial\Omega^5} \gamma^5 t_r^5 \phi_i - A \int\limits_{\partial\Omega^5} n_r^5 \phi_i \partial_r \check{u} \tag{25.84}$$

$$- A \int\limits_{\partial\Omega^5} n_z^5 \phi_i \partial_r \check{w} - \int\limits_{\partial\Omega^5} \phi_i n_r^5 \partial_r \bar{u} - \int\limits_{\partial\Omega^5} \phi_i n_z^5 \partial_r \bar{w}.$$

We now multiply equations (25.81)-(25.84) by $2\Delta_t/3$ and re-arrange terms to obtain

$$\bar{\mathscr{M}}_i^r = \bar{\mathscr{M}}_i^{r,0} + \bar{\mathscr{M}}_i^{r,1} + \bar{\mathscr{M}}_i^{r,2} + \bar{\mathscr{M}}_i^{r,4}, \tag{25.85}$$

with

$$
\begin{aligned}
\bar{\mathscr{M}}_i^{r,0} = &-\frac{2\Delta_t St}{3}\int\limits_{\Omega^n}\phi_i\hat{\boldsymbol{g}}_r + \frac{2\Delta_t A}{3}\int\limits_{\Omega^n}\partial_r\phi_i\partial_r\check{u} + \frac{2\Delta_t A}{3}\int\limits_{\Omega^n}\partial_z\phi_i\partial_z\check{u} + \underbrace{\frac{2(1+2q_n)}{3(1+q_n)}}_{a_n}Re\int\limits_{\Omega^n}\phi_i\bar{u}\\
&-\underbrace{\frac{2(1+q_n)}{3}}_{a_{n-1}}Re\int\limits_{\Omega^n}\phi_i u(t_{n-1}) + \underbrace{\frac{2q_n^2}{3(1+q_n)}}_{a_{n-2}}Re\int\limits_{\Omega^n}\phi_i u(t_{n-2})\\
&+\frac{2\Delta_t Re}{3}\int\limits_{\Omega^n}\phi_i\bar{u}\partial_r\bar{u} + \frac{2\Delta_t Re}{3}\int\limits_{\Omega^n}\phi_i\bar{w}\partial_z\bar{u}\\
&-a_n Re\int\limits_{\Omega^n}\phi_i r^c\partial_r\bar{u} + a_{n-1}Re\int\limits_{\Omega^n}\phi_i r^c(t_{n-1})\partial_r\bar{u} - a_{n-2}Re\int\limits_{\Omega^n}\phi_i r^c(t_{n-2})\partial_r\bar{u}\\
&-a_n Re\int\limits_{\Omega^n}\phi_i z^c\partial_z\bar{u} + a_{n-1}Re\int\limits_{\Omega^n}\phi_i z^c(t_{n-1})\partial_z\bar{u} - a_{n-2}Re\int\limits_{\Omega^n}\phi_i z^c(t_{n-2})\partial_z\bar{u}\\
&+\frac{2\Delta_t Re}{3}A\int\limits_{\Omega^n}\phi_i\check{u}\partial_r\bar{u} + \frac{2\Delta_t Re}{3}A\int\limits_{\Omega^n}\phi_i\check{w}\partial_z\bar{u}\\
&+\frac{2\Delta_t Re}{3}A\int\limits_{\Omega^n}\phi_i\bar{u}\partial_r\check{u} + \frac{2\Delta_t Re}{3}A\int\limits_{\Omega^n}\phi_i\bar{w}\partial_z\check{u}\\
&-a_n Re\,A\int\limits_{\Omega^n}\phi_i r^c\partial_r\check{u} + a_{n-1}Re\,A\int\limits_{\Omega^n}\phi_i r^c(t_{n-1})\partial_r\check{u} - a_{n-2}Re\,A\int\limits_{\Omega^n}\phi_i r^c(t_{n-2})\partial_r\check{u}\\
&-a_n Re\,A\int\limits_{\Omega^n}\phi_i z^c\partial_z\check{u} + a_{n-1}Re\,A\int\limits_{\Omega^n}\phi_i z^c(t_{n-1})\partial_z\check{u} - a_{n-2}Re\,A\int\limits_{\Omega^n}\phi_i z^c(t_{n-2})\partial_z\check{u}\\
&-\frac{2\Delta_t}{3}\int\limits_{\Omega^n}p\partial_r\phi_i + \frac{2\Delta_t}{3}\int\limits_{\Omega^n}\partial_r\bar{u}\partial_r\phi_i + \frac{2\Delta_t}{3}\int\limits_{\Omega^n}\partial_z\bar{u}\partial_z\phi_i\\
&+Aa_n Re\int\limits_{\Omega^n}\phi_i\check{u}\\
&+\frac{2\Delta_t Re}{3}(A)^2\int\limits_{\Omega^n}\phi_i\check{u}\partial_r\check{u} + \frac{2\Delta_t Re}{3}(A)^2\int\limits_{\Omega^n}\phi_i\check{w}\partial_z\check{u}.
\end{aligned}
\tag{25.86}
$$

$$
\begin{aligned}
\bar{\mathscr{M}}_i^{r,1} = &\frac{2\Delta_t}{3Ca}\int\limits_{\partial\Omega^{1,n}}t_r^1\sigma^1\partial_s\phi_i - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}}\phi_i p^g n_r^1 - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{1,n}}\phi_i n_r^1\partial_r\check{u}\\
&-\frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{1,n}}\phi_i n_z^1\partial_r\check{w} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}}\phi_i n_r^1\partial_r\bar{u} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}}\phi_i n_z^1\partial_r\bar{w}\\
&+\frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca},
\end{aligned}
\tag{25.87}
$$

$$\begin{aligned}\bar{\mathscr{M}}_i^{r,2} &= \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}} \phi_i \bar{u} t_r^2 t_r^2 + \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}} \phi_i \bar{w} t_r^2 t_z^2 + \frac{2\Delta_t Be}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{u} t_r^2 t_r^2 \\ &+ \frac{2\Delta_t Be}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{w} t_r^2 t_z^2 - \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}} \phi_i u^s t_r^2 t_r^2 - \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}} \phi_i w^s t_r^2 t_z^2 \\ &- \frac{\Delta_t}{3Ca}\int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 \partial_s \sigma^2 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,n}} \lambda^2 \phi_i n_r^2 - \frac{2\Delta_t A}{3}\int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \check{u} \\ &- \frac{2\Delta_t A}{3}\int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \check{w} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \bar{u} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \bar{w},\end{aligned} \tag{25.88}$$

and

$$\begin{aligned}\bar{\mathscr{M}}_i^{r,5} &= \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5} \lambda^5 n_r^5 \phi_i + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5} \gamma^5 t_r^5 \phi_i - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_r^5 \phi_i \partial_r \check{u} \\ &- \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_z^5 \phi_i \partial_r \check{w} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5} \phi_i n_r^5 \partial_r \bar{u} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5} \phi_i n_z^5 \partial_r \bar{w},\end{aligned} \tag{25.89}$$

where time-dependent functions for which the time argument is omitted are evaluated at time t_n and we have used $u(t_{n-1}) = \bar{u}(t_{n-1}) + A\check{u}(t_{n-1})$ and $u(t_{n-2}) = \bar{u}(t_{n-2}) + A\check{u}(t_{n-2})$.

We now recall the approximations made in (4.68)-(4.80), given by

$$u(r,z,t) \approx \sum_{j=1}^{n_v} u_j(t)\phi_j(r,z), \tag{25.90}$$

$$w(r,z,t) \approx \sum_{j=1}^{n_v} w_j(t)\phi_j(r,z), \tag{25.91}$$

$$r^c(r,z,t) \approx \sum_{j=1}^{n_v} r_j^c(t)\phi_j(r,z), \tag{25.92}$$

$$z^c(r,z,t) \approx \sum_{j=1}^{n_v} z_j^c(t)\phi_j(r,z), \tag{25.93}$$

$$p(r,z,t) \approx \sum_{j=1}^{n_p} p_j(t)\psi_j(r,z), \tag{25.94}$$

$$\sigma^1(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\sigma}_j^1(t)\phi_j^1(r,z), \tag{25.95}$$

$$\lambda^2(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\lambda}_j^2(t)\phi_j^2(r,z), \tag{25.96}$$

$$\lambda^3(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\lambda}_j^3(t)\phi_j^3(r,z), \tag{25.97}$$

$$\gamma^3(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\gamma}_j^3(t)\phi_j^3(r,z), \tag{25.98}$$

$$\lambda^4(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\lambda}_j^4(t)\phi_j^4(r,z), \tag{25.99}$$

$$\gamma^4(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\gamma}_j^4(t)\phi_j^4(r,z), \tag{25.100}$$

$$u^s(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{u}_j^s(t)\phi_j(r,z), \tag{25.101}$$

$$w^s(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{w}_j^s(t)\phi_j(r,z), \tag{25.102}$$

$$p^g(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{p}_j^g(t)\phi_j^1(r,z), \tag{25.103}$$

$$\sigma^2(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\sigma}_j^2(t)\phi_j^2(r,z); \tag{25.104}$$

where n_v is the total number of velocity nodes, n_p is the number of nodes where pressure is calculated, the j index indicates global node numbers that we will use in the Galerkin method (that is to say, ϕ_j is the hat function centred at the j-th node), and ϕ_j^k coincides on the k-th boundary with ϕ_j, and is identically null elsewhere. Moreover, we can assume functions $\tilde{\sigma}_j^1$ and $\tilde{\lambda}_j^2$ are identically null (as all functions these multiply will be null everywhere by our construction of the basis functions) for all j such that $\phi_j = 0$ on boundary 1 and 2, respectively. Furthermore, functions p_j are numbered following the pressure-node numbering.

In a similar spirit, we now introduce the following approximations

$$\bar{u}(r,z,t) \approx \sum_{j=1}^{\bar{n}_v} \bar{u}_j(t)\phi_j(r,z) \tag{25.105}$$

$$\bar{w}(r,z,t) \approx \sum_{j=1}^{\bar{n}_v} \bar{w}_j(t)\phi_j(r,z), \tag{25.106}$$

$$\lambda^5(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\lambda}_j^5(t)\phi_j^5(r,z) \tag{25.107}$$

and

$$\gamma^5(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\gamma}_j^5(t)\phi_j^5(r,z) \tag{25.108}$$

where n_v^n is the number of velocity nodes in Ω^n. We highlight that the numbering convention is chosen so as to have the first $\bar{n}_v$ nodes correspond to $\bar{\Omega}$.

Substituting these approximations into equations (25.86)-(25.89) we define

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{r,0} =& -\frac{2\Delta_t St}{3}\int\limits_{\Omega^n}\phi_i\hat{\boldsymbol{g}}_r + \frac{2\Delta_t A}{3}\int\limits_{\Omega^n}\partial_r\phi_i\partial_r\check{u} + \frac{2\Delta_t A}{3}\int\limits_{\Omega^n}\partial_z\phi_i\partial_z\check{u} + a_n Re\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right)\\
&- a_{n-1}Re\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{\bar{n}_v}u_j(t_{n-1})\phi_j\right) + a_{n-2}Re\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{\bar{n}_v}u_j(t_{n-2})\phi_j\right)\\
&+ \frac{2\Delta_t Re}{3}\int\limits_{\Omega^n}\phi_i\left(\sum_{k=1}^{\bar{n}_v}\bar{u}_k\phi_k\right)\partial_r\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right) + \frac{2\Delta_t Re}{3}\int\limits_{\Omega^n}\phi_i\left(\sum_{k=1}^{\bar{n}_v}\bar{w}_k\phi_k\right)\partial_z\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right)\\
&- a_n Re\int\limits_{\Omega^n}\phi_i\left(\sum_{k=1}^{n_v}r_k^c\phi_k\right)\partial_r\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right) + a_{n-1}Re\int\limits_{\Omega^n}\phi_i\left(\sum_{k=1}^{n_v}r_k^c(t_{n-1})\phi_k\right)\partial_r\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right)\\
&- a_{n-2}Re\int\limits_{\Omega^n}\phi_i\left(\sum_{k=1}^{n_v}r_k^c(t_{n-2})\phi_k\right)\partial_r\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right)\\
&- a_n Re\int\limits_{\Omega^n}\phi_i\left(\sum_{k=1}^{n_v}z_k^c\phi_k\right)\partial_z\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right) + a_{n-1}Re\int\limits_{\Omega^n}\phi_i\left(\sum_{k=1}^{n_v}z_k^c(t_{n-1})\phi_k\right)\partial_z\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right)\\
&- a_{n-2}Re\int\limits_{\Omega^n}\phi_i\left(\sum_{k=1}^{n_v}z_k^c(t_{n-2})\phi_k\right)\partial_z\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right) + \frac{2\Delta_t Re}{3}A\int\limits_{\Omega^n}\phi_i\check{u}\partial_r\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right)\\
&+ \frac{2\Delta_t Re}{3}A\int\limits_{\Omega^n}\phi_i\check{w}\partial_z\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right) + \frac{2\Delta_t Re}{3}A\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right)\partial_r\check{u}\\
&+ \frac{2\Delta_t Re}{3}A\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{\bar{n}_v}\bar{w}_j\phi_j\right)\partial_z\check{u} - a_n Re\,A\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{n_v}r_j^c\phi_j\right)\partial_r\check{u}\\
&+ a_{n-1}Re\,A\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{n_v}r_j^c(t_{n-1})\phi_j\right)\partial_r\check{u} - a_{n-2}Re\,A\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{n_v}r_j^c(t_{n-2})\phi_j\right)\partial_r\check{u}\\
&- a_n Re\,A\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{n_v}z_j^c\phi_j\right)\partial_z\check{u} + a_{n-1}Re\,A\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{n_v}z_j^c(t_{n-1})\phi_j\right)\partial_z\check{u}\\
&- a_{n-2}Re\,A\int\limits_{\Omega^n}\phi_i\left(\sum_{j=1}^{n_v}z_j^c(t_{n-2})\phi_j\right)\partial_z\check{u} - \frac{2\Delta_t}{3}\int\limits_{\Omega^n}\left(\sum_{j=1}^{\bar{n}_p}p_j\psi_j\right)\partial_r\phi_i\\
&+ \frac{2\Delta_t}{3}\int\limits_{\Omega^n}\partial_r\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right)\partial_r\phi_i + \frac{2\Delta_t}{3}\int\limits_{\Omega^n}\partial_z\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right)\partial_z\phi_i\\
&+ a_n Re\,A\int\limits_{\Omega^n}\phi_i\check{u} + \frac{2\Delta_t Re}{3}(A)^2\int\limits_{\Omega^n}\phi_i\check{u}\partial_r\check{u} + \frac{2\Delta_t Re}{3}(A)^2\int\limits_{\Omega^n}\phi_i\check{w}\partial_z\check{u},
\end{aligned}
\tag{25.109}
$$

where n_p^n is the number of pressure nodes in Ω^n,

$$\begin{aligned}\bar{\mathcal{M}}_i^{r,1} = {}& \frac{2\Delta_t}{3Ca}\int\limits_{\partial\Omega^{1,n}} t_r^1\left(\sum_{j=1}^{\bar{n}^v}\tilde{\sigma}^1\phi_j^1\right)\partial_s\phi_i - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}}\phi_i\left(\sum_{j=1}^{n_v^n}\tilde{p}_j^g\phi_j^1\right)n_r^1\\ &- \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{1,n}}\phi_i n_r^1\partial_r\check{u} - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{1,n}}\phi_i n_z^1\partial_r\check{w} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}}\phi_i n_r^1\partial_r\bar{u}\\ &- \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}}\phi_i n_z^1\partial_r\bar{w} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca}\\ &+ \frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^1(r_{J^1},z_{J^1})}{Ca},\end{aligned} \tag{25.110}$$

$$\begin{aligned}\bar{\mathcal{M}}_i^{r,2} = {}& \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i^2\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j^2\right)t_r^2t_r^2 + \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i^2\left(\sum_{j=1}^{\bar{n}_v}\bar{w}_j\phi_j^2\right)t_r^2t_z^2\\ &+ \frac{2\Delta_t Be}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i\check{u}t_r^2t_r^2 + \frac{2\Delta_t Be}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i\check{w}t_r^2t_z^2\\ &- \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i\left(\sum_{j=1}^{n_v}\tilde{u}_j^s\phi_j\right)t_r^2t_r^2 - \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i\left(\sum_{j=1}^{n_v}\tilde{w}_j^s\phi_j\right)t_r^2t_z^2\\ &- \frac{\Delta_t}{3Ca}\int\limits_{\partial\Omega^{2,n}}\phi_i t_r^2\partial_s\left(\sum_{j=1}^{n_v}\tilde{\sigma}_j^2\phi_j^2\right)\sigma^2 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,n}}\left(\sum_{j=1}^{\bar{n}_v}\tilde{\lambda}_j^2\phi_j^2\right)\phi_i^2n_r^2\\ &- \frac{2\Delta_t A}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i n_r^2\partial_r\check{u} - \frac{2\Delta_t A}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i n_z^2\partial_r\check{w}\\ &- \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i n_r^2\partial_r\bar{u} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i n_z^2\partial_r\bar{w},\end{aligned} \tag{25.111}$$

and

$$\begin{aligned}\bar{\mathcal{M}}_i^{r,5} = {}& \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\left(\sum_{j=1}^{n_v}\tilde{\lambda}_j^5\phi_j^5\right)n_r^5\phi_i + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\left(\sum_{j=1}^{n_v}\tilde{\gamma}_j^5\phi_j^5\right)t_r^5\phi_i - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^5}n_r^5\phi_i\partial_r\check{u}\\ &- \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^5}n_z^5\phi_i\partial_r\check{w} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\phi_i n_r^5\partial_r\bar{u} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\phi_i n_z^5\partial_r\bar{w},\end{aligned} \tag{25.112}$$

where we have replaced each instance of ϕ_i by ϕ_i^k, whenever the integral takes place on the k-th boundary (recalling that ϕ_i^k is equal to ϕ_i on the k-th boundary and zero everywhere else). The same was done for basis functions ψ_i where applicable.

Moving the integrals into the sums, we can re-write the expressions above as

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{r,0} = &-\frac{2\Delta_t St}{3}\int\limits_{\Omega^n}\phi_i\hat{\boldsymbol{g}}_r+\frac{2\Delta_t A}{3}\int\limits_{\Omega^n}\partial_r\phi_i\partial_r\check{u}+\frac{2\Delta_t A}{3}\int\limits_{\Omega^n}\partial_z\phi_i\partial_z\check{u}+a_n Re\,A\int\limits_{\Omega^n}\phi_i\check{u}\\
&+\frac{2\Delta_t Re}{3}(A)^2\int\limits_{\Omega^n}\phi_i\check{u}\partial_r\check{u}+\frac{2\Delta_t Re}{3}(A)^2\int\limits_{\Omega^n}\phi_i\check{w}\partial_z\check{u}+a_n Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\Omega^n}\phi_i\phi_j\\
&-a_{n-1}Re\sum_{j=1}^{\bar{n}_v}u_j(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_j+a_{n-2}Re\sum_{j=1}^{\bar{n}_v}u_j(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_j\\
&+\frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\Omega^n}\phi_i\check{u}\partial_r\phi_j+\frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\Omega^n}\phi_i\check{w}\partial_z\phi_j\\
&+\frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{u}+\frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{u}\\
&-a_n Re\,A\sum_{j=1}^{n_v}r_j^c\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{u}+a_{n-1}Re\,A\sum_{j=1}^{n_v}r_j^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{u}\\
&-a_{n-2}Re\,A\sum_{j=1}^{n_v}r_j^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{u}-a_n Re\,A\sum_{j=1}^{n_v}z_j^c\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{u}\\
&+a_{n-1}Re\,A\sum_{j=1}^{n_v}z_j^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{u}-a_{n-2}Re\,A\sum_{j=1}^{n_v}z_j^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{u}\\
&-\frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_p}p_j\int\limits_{\Omega^n}\psi_j\partial_r\phi_i+\frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\Omega^n}\partial_r\phi_j\partial_r\phi_i+\frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\Omega^n}\partial_z\phi_j\partial_z\phi_i\\
&+\frac{2\Delta_t Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{\bar{n}_v}\bar{u}_k\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j+\frac{2\Delta_t Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{\bar{n}_v}\bar{w}_k\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j\\
&-a_n Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}r_k^c\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j+a_{n-1}Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}r_k^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j\\
&-a_{n-2}Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}r_k^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j-a_n Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}z_k^c\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j\\
&+a_{n-1}Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}z_k^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j-a_{n-2}Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}z_k^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j,
\end{aligned}
\tag{25.113}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{r,1} = &-\frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{1,n}}\phi_i n_r^1\partial_r\check{u} - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{1,n}}\phi_i n_z^1\partial_r\check{w} + \frac{2\Delta_t}{3Ca}\sum_{j=1}^{\bar{n}^v}\tilde{\sigma}^1\int\limits_{\partial\Omega^{1,n}} t_r^1\phi_j^1\partial_s\phi_i \\
&-\frac{2\Delta_t}{3}\sum_{j=1}^{n_v^n}\tilde{p}_j^g\int\limits_{\partial\Omega^{1,n}}\phi_i^1\phi_j^1 n_r^1 - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}}\phi_i n_r^1\partial_r\bar{u} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}}\phi_i n_z^1\partial_r\bar{w} \\
&+\frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca},
\end{aligned}
\tag{25.114}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{r,2} = &\frac{2\Delta_t Be}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i\check{u}t_r^2t_r^2 + \frac{2\Delta_t Be}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i\check{w}t_r^2t_z^2 - \frac{2\Delta_t A}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i n_r^2\partial_r\check{u} \\
&-\frac{2\Delta_t A}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i n_z^2\partial_r\check{w} - \frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v}\tilde{u}_j^s\int\limits_{\partial\Omega^{2,n}}\phi_i\phi_j t_r^2t_r^2 \\
&-\frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v}\tilde{w}_j^s\int\limits_{\partial\Omega^{2,n}}\phi_i\phi_j t_r^2t_z^2 + \frac{2\Delta_t Be}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\partial\Omega^{2,n}}\phi_i^2\phi_j^2t_r^2t_r^2 \\
&+\frac{2\Delta_t Be}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\partial\Omega^{2,n}}\phi_i^2\phi_j^2t_r^2t_z^2 - \frac{\Delta_t}{3Ca}\sum_{j=1}^{n_v}\tilde{\sigma}_j^2\int\limits_{\partial\Omega^{2,n}}\phi_i t_r^2\partial_s\phi_j^2 \\
&+\frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\tilde{\lambda}_j^2\int\limits_{\partial\Omega^{2,n}}\phi_j^2\phi_i^2n_r^2 - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i n_r^2\partial_r\bar{u} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i n_z^2\partial_r\bar{w},
\end{aligned}
\tag{25.115}
$$

and

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{r,5} = &-\frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^5}n_r^5\phi_i\partial_r\check{u} - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^5}n_z^5\phi_i\partial_r\check{w} + \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\lambda}_j^5\int\limits_{\partial\Omega^5}\phi_j^5n_r^5\phi_i \\
&+\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{\gamma}_j^5\int\limits_{\partial\Omega^5}\phi_j^5t_r^5\phi_i - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\phi_i n_r^5\partial_r\bar{u} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\phi_i n_z^5\partial_r\bar{w}.
\end{aligned}
\tag{25.116}
$$

We now partition the domain into closed curve-sided triangular *elements* (see figure 3), whose interiors are disjoint, and proceed to decompose the integrals above in a sum of integrals over each element. The boundary integrals, are in turn converted into a sum of integrals over line elements in the boundary, i.e. those portions of the boundary of the triangular elements that lie on the domain boundary $\partial\Omega$. Figure 4 shows that we have chosen only corner nodes of the elements to be pressure-and-velocity nodes, and illustrates the pressure-node numbering convention used.

This yields (recall equation 25.69)

$$
\begin{aligned}
\bar{\mathcal{M}}_i^r = & \underbrace{\sum_{\substack{e=1 \\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \bar{\mathcal{M}}_{e,ii}^{r,0}}_{\bar{\mathcal{M}}_i^{r,0}} \\
& + \underbrace{\sum_{\substack{e_1=1 \\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_{J1},z_{J1})\phi_i(r_{J1},z_{J1})m_r^{1,f}(r_{J1},z_{J1})}{Ca}}_{\bar{\mathcal{M}}_i^{r,1}} \\
& + \underbrace{\sum_{\substack{e_2=1 \\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \bar{\mathcal{M}}_{e,ii}^{r,2}}_{\bar{\mathcal{M}}_i^{r,2}} + \underbrace{\sum_{\substack{e_5=1 \\ i=l_5(e,ii)}}^{\bar{n}_{\text{el}}^5} \bar{\mathcal{M}}_{e_5,ii}^{r,5}}_{\bar{\mathcal{M}}_i^{r,5}},
\end{aligned}
\tag{25.117}
$$

where $\bar{n}_{\text{el}}$ is the number of triangular elements in $\bar{\Omega}$, $\bar{n}_{\text{el}}^k$ is the number of line elements on the k-th boundary of $\bar{\Omega}$. We now introduce *local node numbering*, i.e. give each node another number for each element in which the node is contained, we represent local node numbers using double letter indices. Moreover, function $l(e, ii)$ maps the local number ii of a node in element e to its global number i, i.e. $l(e, ii) = i$ (see figure 5), and similarly $l_k(e_k, ii)$ maps the local node number ii of line-element e_k in boundary k to its global node number i, i.e. $l_k(e_k, ii) = 1$ (see figures 7 and 8).

Moreover, we have

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,0}_{e,ii} = & -\frac{2\Delta_t St}{3}\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\hat{\boldsymbol{g}}_r + \frac{2\Delta_t A}{3}\int\limits_{\Omega^n}\partial_r\phi_i\partial_r\breve{u} + \frac{2\Delta_t A}{3}\int\limits_{\Omega^n}\partial_z\phi_i\partial_z\breve{u} + a_n Re\, A\int\limits_{\Omega^n}\phi_{l(e,ii)}\breve{u} \\
& + \frac{2\Delta_t Re}{3}\left(A\right)^2\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\breve{u}\partial_r\breve{u} + \frac{2\Delta_t Re}{3}\left(A\right)^2\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\breve{w}\partial_z\breve{u} + a_n Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j \\
& - a_{n-1}Re\sum_{j=1}^{\bar{n}_v}u_j(t_{n-1})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j + a_{n-2}Re\sum_{j=1}^{\bar{n}_v}u_j(t_{n-2})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j \\
& + \frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\breve{u}\partial_r\phi_j + \frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\breve{w}\partial_z\phi_j \\
& + \frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j\partial_r\breve{u} + \frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j\partial_z\breve{u} \\
& - a_n Re\, A\sum_{j=1}^{n_v}r^c_j\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j\partial_r\breve{u} + a_{n-1}Re\, A\sum_{j=1}^{n_v}r^c_j(t_{n-1})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j\partial_r\breve{u} \\
& - a_{n-2}Re\, A\sum_{j=1}^{n_v}r^c_j(t_{n-2})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j\partial_r\breve{u} - a_n Re\, A\sum_{j=1}^{n_v}z^c_j\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j\partial_z\breve{u} \\
& + a_{n-1}Re\, A\sum_{j=1}^{n_v}z^c_j(t_{n-1})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j\partial_z\breve{u} - a_{n-2}Re\, A\sum_{j=1}^{n_v}z^c_j(t_{n-2})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_j\partial_z\breve{u} \\
& - \frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_p}p_j\int\limits_{\bar{\Omega}_e}\psi_j\partial_r\phi_{l(e,ii)} + \frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\bar{\Omega}_e}\partial_r\phi_j\partial_r\phi_{l(e,ii)} + \frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\bar{\Omega}_e}\partial_z\phi_j\partial_z\phi_{l(e,ii)} \\
& + \frac{2\Delta_t Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{\bar{n}_v}\bar{u}_k\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_r\phi_j + \frac{2\Delta_t Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{\bar{n}_v}\bar{w}_k\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_z\phi_j \\
& - a_n Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}r^c_k\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_r\phi_j + a_{n-1}Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}r^c_k(t_{n-1})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_r\phi_j \\
& - a_{n-2}Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}r^c_k(t_{n-2})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_r\phi_j - a_n Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}z^c_k\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_z\phi_j \\
& + a_{n-1}Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}z^c_k(t_{n-1})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_z\phi_j \\
& - a_{n-2}Re\sum_{j=1}^{\bar{n}_v}\bar{u}_j\sum_{k=1}^{n_v}z^c_k(t_{n-2})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_z\phi_j,
\end{aligned}
\tag{25.118}
$$

$$\bar{\mathcal{M}}^{r,1}_{e_1,ii} = -\frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^1_{e_1}} \phi^1_{l_1(e_1,ii)} n^1_r \partial_r \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^1_{e_1}} \phi^1_{l_1(e_1,ii)} n^1_z \partial_r \check{w}$$
$$+ \frac{2\Delta_t}{3Ca} \sum_{j=1}^{\bar{n}_v} \tilde{\sigma}^1_j \int\limits_{\partial\Omega^{\bar{1},n}{}^1_{e_1}} t^1_r \phi^1_j \partial_s \phi_{l_1(e_1,ii)} - \frac{2\Delta_t}{3} \sum_{j=1}^{n^n_v} \tilde{p}^g_j \int\limits_{\partial\Omega^{1,n}} \phi^1_{l_1(e_1,ii)} \phi^1_j n^1_r \quad (25.119)$$
$$- \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{1,n}} \phi_i n^1_r \partial_r \bar{u} - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{1,n}} \phi_i n^1_z \partial_r \bar{w},$$

$$\bar{\mathcal{M}}^{r,2}_{e_2,ii} = \frac{2\Delta_t Be}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \check{u} t^2_r t^2_r + \frac{2\Delta_t Be}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \check{w} t^2_r t^2_z - \frac{2\Delta_t A}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n^2_r \partial_r \check{u}$$
$$- \frac{2\Delta_t A}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n^2_z \partial_r \check{w} - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{u}^s_j \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi_j t^2_r t^2_r$$
$$- \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{w}^s_j \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi_j t^2_r t^2_z + \frac{2\Delta_t Be}{3} \sum_{j=1}^{\bar{n}_v} \bar{u}_j \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_j t^2_r t^2_r$$
$$+ \frac{2\Delta_t Be}{3} \sum_{j=1}^{\bar{n}_v} \bar{w}_j \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_j t^2_r t^2_z - \frac{\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}^2_j \int\limits_{\partial\Omega^{2,n}} \phi_{l_2(e_2,ii)} t^2_r \partial_s \phi^2_j$$
$$+ \frac{2\Delta_t}{3} \sum_{j=1}^{\bar{n}_v} \tilde{\lambda}^2_j \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_j n^2_r - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n^2_r \partial_r \bar{u} - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n^2_z \partial_r \bar{w},$$
$$(25.120)$$

and

$$\bar{\mathcal{M}}^{r,5}_{e,ii} = -\frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^5_{e_5}} \phi^5_{l_5(e_5,ii)} n^5_r \partial_r \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^5_{e_5}} \phi^5_{l_5(e_5,ii)} n^5_z \partial_r \check{w}$$
$$+ \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}^5_j \int\limits_{\partial\bar{\Omega}^5_{e_5}} \phi^5_j n^5_r \phi_{l_5(e_5,ii)} + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma}^5_j \int\limits_{\partial\bar{\Omega}^5_{e_5}} \phi^5_j t^5_r \phi_{l_5(e_5,ii)} \quad (25.121)$$
$$- \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i n^5_r \partial_r \bar{u} - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i n^5_z \partial_r \bar{w},$$

with $\partial\Omega_{e_k}$ is the part of $\partial\Omega_k$ that is contained in e_k.

Now, we impose that each function ϕ_j, ψ_j will only be supported on the elements that contain node j. Upon imposing this, we notice that the vast majority of the j and k indexed terms that are added in the sum on each element is identically null. This is, of course, because the integral of the product of these functions will be summing zero unless all functions involved are associated to (i.e. attain the value 1 in) some node on the element. Therefore, a more efficient way to express this sums is to resort to *local node numbering*. That is to say, when we are calculating the integral on each element, we know that non-zero contributions can only come from a basis function whose index corresponds to one of the node indices of the element at hand and it is therefore better to have the sums over k and j above to only go over the nodes contained in that element. Hence, it is more convenient to re-write the expressions above as

$$\begin{aligned}\bar{\mathcal{M}}_{e,ii}^{r,0a} &= -\frac{2\Delta_t St}{3}\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}g_r + \frac{2\Delta_t A}{3}\int\limits_{\Omega^n}\partial_r\phi_i\partial_r\check{u} + \frac{2\Delta_t A}{3}\int\limits_{\Omega^n}\partial_z\phi_i\partial_z\check{u} \\ &+ a_n Re\, A\int\limits_{\Omega^n}\phi_{l(e,ii)}\check{u} + \frac{2\Delta_t Re}{3}(A)^2\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\check{u}\partial_r\check{u} + \frac{2\Delta_t Re}{3}(A)^2\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\check{w}\partial_z\check{u},\end{aligned} \tag{25.122}$$

$$\begin{aligned}\bar{\mathcal{M}}_{e,ii}^{r,0b} &= a_n Re\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)} - a_{n-1}Re\sum_{jj=1}^{\bar{n}_v^e}u_{l(e,jj)}(t_{n-1})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)} \\ &+ a_{n-2}Re\sum_{jj=1}^{\bar{n}_v^e}u_{l(e,jj)}(t_{n-2})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)} \\ &+ \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\check{u}\partial_r\phi_{l(e,jj)} \\ &+ \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\check{w}\partial_z\phi_{l(e,jj)} + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)}\partial_r\check{u} \\ &+ \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l(e,jj)}\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)}\partial_z\check{u} - a_n Re\, A\sum_{jj=1}^{\bar{n}_v^e}r^c_{l(e,jj)}\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)}\partial_r\check{u} \\ &+ a_{n-1}Re\, A\sum_{j=1}^{\bar{n}_v^e}r^c_{l(e,jj)}(t_{n-1})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)}\partial_r\check{u} \\ &- a_{n-2}Re\, A\sum_{jj=1}^{\bar{n}_v^e}r^c_{l(e,jj)}(t_{n-2})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)}\partial_r\check{u} \\ &- a_n Re\, A\sum_{jj=1}^{\bar{n}_v^e}z^c_{l(e,jj)}\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)}\partial_z\check{u} \\ &+ a_{n-1}Re\, A\sum_{jj=1}^{\bar{n}_v^e}z^c_{l(e,jj)}(t_{n-1})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)}\partial_z\check{u} \\ &- a_{n-2}Re\, A\sum_{jj=1}^{\bar{n}_v^e}z^c_{l(e,jj)}(t_{n-2})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_{l(e,jj)}\partial_z\check{u} \\ &+ \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\int\limits_{\bar{\Omega}_e}\partial_r\phi_{l(e,jj)}\partial_r\phi_{l(e,ii)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\int\limits_{\bar{\Omega}_e}\partial_z\phi_{l(e,jj)}\partial_z\phi_{l(e,ii)},\end{aligned} \tag{25.123}$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,0c}_{e,ii} &= \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}^e_v} \bar{u}_{l(e,kk)} \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
&+ \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}^e_v} \bar{w}_{l(e,kk)} \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
&- a_n Re \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}^e_v} r^c_{l(e,kk)} \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
&+ a_{n-1} Re \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}^e_v} r^c_{l(e,kk)}(t_{n-1}) \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
&- a_{n-2} Re \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}^e_v} r^c_{l(e,kk)}(t_{n-2}) \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)} \\
&- a_n Re \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}^e_v} z^c_{l(e,kk)} \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
&+ a_{n-1} Re \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}^e_v} z^c_{l(e,kk)}(t_{n-1}) \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)} \\
&- a_{n-2} Re \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}^e_v} z^c_{l(e,kk)}(t_{n-2}) \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)},
\end{aligned} \tag{25.124}
$$

$$
\bar{\mathcal{M}}^{r,0d}_{e,ii} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}^e_p} p_{l^p(e,jj)} \int_{\bar{\Omega}_e} \psi_{l^p(e,jj)} \partial_r \phi_{l(e,ii)} \tag{25.125}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,1}_{e_1,ii} &= -\frac{2\Delta_t}{3} A \int_{\partial\bar{\Omega}^1_{e_1}} \phi^1_{l_1(e_1,ii)} n^1_r \partial_r \breve{u} - \frac{2\Delta_t}{3} A \int_{\partial\bar{\Omega}^1_{e_1}} \phi^1_{l_1(e_1,ii)} n^1_z \partial_r \breve{w} \\
&+ \frac{2\Delta_t}{3Ca} \sum_{jj=1}^{\bar{n}^{e1}_v} \tilde{\sigma}^1_{l_1(e_1,jj)} \int_{\partial\bar{\Omega}^1_{e_1}} t^1_r \phi^1_{l_1(e_1,jj)} \partial_s \phi_{l_1(e_1,ii)} \\
&- \frac{2\Delta_t}{3} \sum_{j=1}^{n^n_v} \tilde{p}^g_j \int_{\partial\Omega^{1,n}} \phi^1_{l_1(e_1,ii)} \phi^1_j n^1_r - \frac{2\Delta_t}{3} \int_{\partial\Omega^{1,n}} \phi_i n^1_r \partial_r \bar{u} - \frac{2\Delta_t}{3} \int_{\partial\Omega^{1,n}} \phi_i n^1_z \partial_r \bar{w},
\end{aligned} \tag{25.126}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,2}_{e_2,ii} = {} & \frac{2\Delta_t Be}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \check{u} t^2_r t^2_r + \frac{2\Delta_t Be}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \check{w} t^2_r t^2_z \\
& - \frac{2\Delta_t A}{3} \int\limits_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,ii)} n^2_r \partial_r \check{u} - \frac{2\Delta_t A}{3} \int\limits_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,ii)} n^2_z \partial_r \check{w} \\
& + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l_2(e_2,jj)} \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_r \\
& + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{\bar{n}^e_v} \bar{w}_{l_2(e_2,jj)} \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_z \\
& - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{u}^s_{l_2(e_2,jj)} \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_r \\
& - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{w}^s_{l_2(e_2,jj)} \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_z \\
& - \frac{\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}^2_{l_2(e_2,jj)} \int\limits_{\partial\Omega^{2,n}} \phi_{l_2(e_2,ii)} t^2_r \partial_s \phi^2_{l_2(e_2,jj)} \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}^{e_2}_v} \tilde{\lambda}^2_{l_2(e_2,jj)} \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} n^2_r \\
& - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n^2_r \partial_r \bar{u} - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n^2_z \partial_r \bar{w},
\end{aligned}
\tag{25.127}
$$

and

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,5}_{e,ii} = {} & -\frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^5_{e_5}} \phi^5_{l_5(e_5,ii)} n^5_r \partial_r \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^5_{e_5}} \phi^5_{l_5(e_5,ii)} n^5_z \partial_r \check{w} \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}^e_v} \tilde{\lambda}^5_{l_5(e_5,jj)} \int\limits_{\partial\bar{\Omega}^5_{e_5}} \phi^5_{l_5(e_5,jj)} n^5_r \phi_{l_5(e_5,ii)} \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v} \tilde{\gamma}^5_{l_5(e_5,jj)} \int\limits_{\partial\bar{\Omega}^5_{e_5}} \phi^5_{l_5(e_5,jj)} t^5_r \phi_{l_5(e_5,ii)} \\
& - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i n^5_r \partial_r \bar{u} - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i n^5_z \partial_r \bar{w};
\end{aligned}
\tag{25.128}
$$

where double-letter indices are used to reference local node numbers, n^e_v is the number of velocity nodes in element e, n^e_p is the number of pressure nodes e, $n^{e_i}_v$ is the number of velocity nodes on line element e_i of boundary i, $n^{e_i}_p$ is the number of pressure nodes on line element e_i of boundary i, and $l(e,jj) = j$, i.e. $l(e,jj)$ maps the local number jj of a node in element e to its global number j (see figure 5), $l^p(e,jj) = j$ maps the local node number jj of element e onto its pressure-node number j (see figure 6), and similarly $l_k(e_k, jj) = j$ maps the local node number jj of line-element e_k in boundary k to its global node number j (see figures 7 and 8. Naturally, $l_k^p(e_k, jj) = j$, maps the local pressure node number jj of element e_k on boundary k, to its global pressure-node number j.

Moreover, we have introduced $\mathcal{M}_{e,ii}^{\bar{r},0a}$, $\mathcal{M}_{e,ii}^{\bar{r},0b}$ and $\mathcal{M}_{e,ii}^{\bar{r},0c}$, where

$$\mathcal{M}_{e,ii}^{\bar{r},0} = \mathcal{M}_{e,ii}^{\bar{r},0a} + \mathcal{M}_{e,ii}^{\bar{r},0b} + \mathcal{M}_{e,ii}^{\bar{r},0c}. \tag{25.129}$$

We now consider functions $\tilde{\sigma}_j^1$, $\tilde{\lambda}_j^2$, $\tilde{\lambda}_j^4$ and $\tilde{\gamma}_j^4$. We recall that for all j indices that correspond to nodes outside their respective boundaries these functions are identically zero. It is therefore more convenient to introduce functions σ_j^1, λ_j^2, λ_j^4 and γ_j^4 where j is a numbering of the nodes that lie on the corresponding boundary. We also introduce functions $l_1^1(e_1, jj)$ which maps local node number jj on element e_1 on boundary 1 to its corresponding boundary-node number, and the analogue functions $l_2^2(e_2, jj)$ and $l_4^4(e_4, jj)$ (see figures 9 and **??**; and compare them to 9 and **??**, respectively). The only difference between functions $l_k(e_k, jj)$ and $l_k^k(e_k, jj)$ is that the image of the latter is the node number in the boundary numbering and in the former it is the node number in the global numbering. Re-writing the equations above under this new convention we have

$$\begin{aligned}\bar{\mathcal{M}}_{e,ii}^{r,0a} = &-\frac{2\Delta_t St}{3}\underbrace{\int_{\bar{\Omega}_e}\phi_{l(e,ii)}\hat{\boldsymbol{g}}_r}_{a_{ii,g_r}(e)}+\frac{2\Delta_t A}{3}\underbrace{\int_{\Omega^n}\partial_r\phi_i\partial_r\check{u}}_{a^r_{ii,\partial_r\check{u}}(e)}+\frac{2\Delta_t A}{3}\underbrace{\int_{\Omega^n}\partial_z\phi_i\partial_z\check{u}}_{a^z_{ii,\partial_z\check{u}}(e)}\\ &+a_n Re\, A\underbrace{\int_{\Omega^n}\phi_{l(e,ii)}\check{u}}_{a_{ii,\check{u}}(e)}+\frac{2\Delta_t Re}{3}(A)^2\underbrace{\int_{\bar{\Omega}_e}\phi_{l(e,ii)}\check{u}\partial_r\check{u}}_{a_{ii,\check{u},\partial_r\check{u}}(e)}+\frac{2\Delta_t Re}{3}(A)^2\underbrace{\int_{\bar{\Omega}_e}\phi_{l(e,ii)}\check{w}\partial_z\check{u}}_{a_{ii,\check{w},\partial_z\check{u}}(e)},\end{aligned} \tag{25.130}$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,0b}_{e,ii} &= a_n Re \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)} - a_{n-1} Re \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)}(t_{n-1}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)} \\
&+ a_{n-2} Re \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)}(t_{n-2}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)} + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{u} \partial_r \phi_{l(e,jj)}}_{a^r_{ii,jj,\check{u}}(e)} \\
&+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{w} \partial_z \phi_{l(e,jj)}}_{a^z_{ii,jj,\check{w}}(e)} + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)} \partial_r \check{u}}_{a_{ii,jj,\partial_r \check{u}}(e)} \\
&+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}^e_v} \bar{w}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)} \partial_z \check{u}}_{a_{ii,jj,\partial_z \check{u}}(e)} - a_n Re\, A \sum_{jj=1}^{\bar{n}^e_v} r^c_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)} \partial_r \check{u}}_{a_{ii,jj,\partial_r \check{u}}(e)} \\
&+ a_{n-1} Re\, A \sum_{j=1}^{\bar{n}^e_v} r^c_{l(e,jj)}(t_{n-1}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)} \partial_r \check{u}}_{a_{ii,jj,\partial_r \check{u}}(e)} \\
&- a_{n-2} Re\, A \sum_{jj=1}^{\bar{n}^e_v} r^c_{l(e,jj)}(t_{n-2}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)} \partial_r \check{u}}_{a_{ii,jj,\partial_r \check{u}}(e)} \\
&- a_n Re\, A \sum_{jj=1}^{\bar{n}^e_v} z^c_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)} \partial_z \check{u}}_{a_{ii,jj,\partial_z \check{u}}(e)} + a_{n-1} Re\, A \sum_{jj=1}^{\bar{n}^e_v} z^c_{l(e,jj)}(t_{n-1}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)} \partial_z \check{u}}_{a_{ii,jj,\partial_z \check{u}}(e)} \\
&- a_{n-2} Re\, A \sum_{jj=1}^{\bar{n}^e_v} z^c_{l(e,jj)}(t_{n-2}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)} \partial_z \check{u}}_{a_{ii,jj,\partial_z \check{u}}(e)} \\
&+ \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \partial_r \phi_{l(e,jj)} \partial_r \phi_{l(e,ii)}}_{a^{r,r}_{ii,jj}(e)} + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \partial_z \phi_{l(e,jj)} \partial_z \phi_{l(e,ii)}}_{a^{z,z}_{ii,jj}(e)},
\end{aligned}
\tag{25.131}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,0c}_{e,ii} = \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{u}_{l(e,kk)} & \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
+\frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{w}_{l(e,kk)} & \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
-a_n Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)} & \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
+a_{n-1} Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-1}) & \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
-a_{n-2} Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-2}) & \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_r\phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
-a_n Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)} & \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
+a_{n-1} Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-1}) & \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
-a_{n-2} Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-2}) & \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,kk)}\partial_z\phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)},
\end{aligned}
\tag{25.132}
$$

$$
\bar{\mathcal{M}}^{r,0d}_{e,ii} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_p^e} p_{l^p(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \psi_{l^p(e,jj)}\partial_r\phi_{l(e,ii)}}_{b^r_{jj,ii}(e)}, \tag{25.133}
$$

$$\bar{\mathcal{M}}^{r,1}_{e_1,ii} = -\frac{2\Delta_t}{3}A \underbrace{\int\limits_{\partial\bar{\Omega}^1_{e_1}} \phi^1_{l_1(e_1,ii)} n^1_r \partial_r \breve{u}}_{c_{ii,n_r,\partial_r\breve{u}}(e_1)} - \frac{2\Delta_t}{3}A \underbrace{\int\limits_{\partial\bar{\Omega}^1_{e_1}} \phi^1_{l_1(e_1,ii)} n^1_z \partial_r \breve{w}}_{c_{ii,n_z,\partial_r\breve{w}}(e_1)} \tag{25.134}$$

$$+\frac{2\Delta_t}{3Ca}\sum_{jj=1}^{\bar{n}^{e_1}_v} \sigma^1_{l^1_1(e_1,jj)} \underbrace{\int\limits_{\partial\bar{\Omega}^1_{e_1}} t^1_r \phi^1_{l_1(e_1,jj)} \partial_s \phi_{l_1(e_1,ii)}}_{c^s_{jj,ii,t_r}(e_1)}$$

$$-\frac{2\Delta_t}{3}\sum_{j=1}^{n^n_v} \tilde{p}^g_{l_1(e_1,jj)} \underbrace{\int\limits_{\partial\Omega^{1,n}} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} n^1_r}_{c_{ii,jj,n^r}(e_1)}$$

$$-\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}} \phi_i n^1_r \partial_r \bar{u} - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}} \phi_i n^1_z \partial_r \bar{w},$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,2}_{e_2,ii} =& \frac{2\Delta_t Be}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \check{u} t^2_r t^2_r}_{d_{ii,t_r,t_r,\check{u}}(e_2)} + \frac{2\Delta_t Be}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \check{w} t^2_r t^2_z}_{d_{ii,t_r,t_z,\check{w}}(e_2)} \\
& - \frac{2\Delta_t A}{3} \underbrace{\int\limits_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,ii)} n^2_r \partial_r \check{u}}_{d_{ii,n_r,\partial_r\check{u}}(e_2)} - \frac{2\Delta_t A}{3} \underbrace{\int\limits_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,ii)} n^2_z \partial_r \check{w}}_{d_{ii,n_z,\partial_r\check{w}}(e_2)} \\
& + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_r}_{d_{ii,jj,t_r,t_r}(e_2)} \\
& + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{\bar{n}^e_v} \bar{w}_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_z}_{d_{ii,jj,t_r,t_z}(e_2)} \\
& - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} u^s_{l^2_2(e_2,jj)} \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_r}_{d_{ii,jj,t_r,t_r}(e_2)} \\
& - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} w^s_{l^2_2(e_2,jj)} \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t^2_r t^2_z}_{d_{ii,jj,t_r,t_z}(e_2)} \\
& - \frac{\Delta_t}{3Ca} \sum_{j=1}^{n_v} \sigma^2_{l^2_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,n}} \phi_{l_2(e_2,ii)} t^2_r \partial_s \phi^2_j}_{d^s_{ii,jj,t_r}(e_2)} + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}^{e_2}_v} \lambda^2_{l^2_2(e_2,jj)} \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} n^2_r}_{d_{ii,jj,n_r}(e_2)} \\
& - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n^2_r \partial_r \bar{u} - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n^2_z \partial_r \bar{w},
\end{aligned}
\tag{25.135}
$$

and

$$\begin{aligned}\bar{\mathcal{M}}^{r,5}_{e,ii} = &-\frac{2\Delta_t}{3}A\underbrace{\int\limits_{\partial\bar{\Omega}^5_{e_5}}\phi^5_{l_5(e_5,ii)}n^5_r\partial_r\breve{u}}_{g_{ii,n_r,\partial_r\breve{u}}(e_2)}-\frac{2\Delta_t}{3}A\underbrace{\int\limits_{\partial\bar{\Omega}^5_{e_5}}\phi^5_{l_5(e_5,ii)}n^5_z\partial_r\breve{w}}_{g_{ii,n_z,\partial_r\breve{w}}(e_5)}\\ &+\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}^e_v}\lambda^5_{l_5(e_5,jj)}\underbrace{\int\limits_{\partial\bar{\Omega}^5_{e_5}}\phi^5_{l_5(e_5,jj)}n^5_r\phi_{l_5(e_5,ii)}}_{g_{ii,jj,n_r}(e_5)}\\ &+\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v}\gamma^5_{l_5(e_5,jj)}\underbrace{\int\limits_{\partial\bar{\Omega}^5_{e_5}}\phi^5_{l_5(e_5,jj)}t^5_r\phi_{l_5(e_5,ii)}}_{g_{ii,jj,t_r}(e_5)}\\ &-\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\phi_i n^5_r\partial_r\bar{u}-\frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\phi_i n^5_z\partial_r\bar{w}.\end{aligned} \tag{25.136}$$

The names of the integral quantities above, are chosen so that a and b always stand for integrals in a triangular element, with a been integral of the velocity-interpolating functions ϕ and their spatial derivatives, and b been integrals of the pressure-interpolating functions ψ potentially multiplied by some ϕ functions and/or their derivatives. Similarly, integrals identified as c, d and g have a domain on the free, solid and separatrix boundary, respectively. Moreover, sub-indices indicate the quantities been integrated and super-indices indicate which derivatives are taken of these quantities. Hence, the number of super-indices is always lower than the number of sub-indices. Furthermore, k super-indices indicate that the last k sub-indices correspond to differentiated variables and each one of these last k variables (last k sub-indices) is differentiated with respect to its matching number of k super-indices.

In practice, we loop over the element nodes once again for index ii (i.e. the local index of the i-th residual component) defining and calculating $\hat{M}^r_{e,ii}$ for each local node ii on each element and then adding the contribution to the $\hat{M}^r$ vector at entry $i=(e,ii)$.

Re-writing equations (25.117) and (25.130)-(25.136) we have

$$\begin{aligned}\bar{\mathcal{M}}^r_i \;=& \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}}\left[\bar{\mathcal{M}}^{r,0a}_{e,ii}+\bar{\mathcal{M}}^{r,0b}_{e,ii}+\bar{\mathcal{M}}^{r,0c}_{e,ii}+\bar{\mathcal{M}}^{r,0d}_{e,ii}\right]\\ &+\sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}^1_{\text{el}}}\bar{\mathcal{M}}^{r,1}_{e_1,ii}+\frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m^1_r(r_c,z_c)}{Ca}\\ &+\frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m^{1,f}_r(r_{J^1},z_{J^1})}{Ca}+\sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}^2_{\text{el}}}\bar{\mathcal{M}}^{r,2}_{e,ii}+\sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}^5_{\text{el}}}\bar{\mathcal{M}}^{r,5}_{e_5,ii},\end{aligned} \tag{25.137}$$

where

$$\bar{\mathcal{M}}^{r,0a}_{e,ii} = -\frac{2\Delta_t St}{3} a_{ii,g_r}(e) + \frac{2\Delta_t A}{3} a^{r}_{ii,\partial_r \check{u}}(e) + \frac{2\Delta_t A}{3} a^{z}_{ii,\partial_z \check{u}}(e)$$

$$+ a_n Re\, A a_{ii,\check{u}}(e) + \frac{2\Delta_t Re}{3} (A)^2 a_{ii,\check{u},\partial_r \check{u}}(e) + \frac{2\Delta_t Re}{3} (A)^2 a_{ii,\check{w},\partial_z \check{u}}(e), \tag{25.138}$$

$$\bar{\mathcal{M}}^{r,0b}_{e,ii} = \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} a^{r,r}_{ii,jj}(e) + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} a^{z,z}_{ii,jj}(e)$$

$$+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} a^{r}_{ii,jj,\check{u}}(e) + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} a^{z}_{ii,jj,\check{w}}(e)$$

$$+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} a_{ii,jj,\partial_r \check{u}}(e) + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}^e_v} \bar{w}_{l(e,jj)} a_{ii,jj,\partial_z \check{u}}(e)$$

$$+ a_n Re \sum_{jj=1}^{\bar{n}^e_v} \bar{u}_{l(e,jj)} a_{ii,jj}(e) - a_{n-1} Re \sum_{jj=1}^{\bar{n}^e_v} u_{l(e,jj)}(t_{n-1}) a_{ii,jj}(e)$$

$$+ a_{n-2} Re \sum_{jj=1}^{\bar{n}^e_v} u_{l(e,jj)}(t_{n-2}) a_{ii,jj}(e) - a_n Re\, A \sum_{jj=1}^{\bar{n}^e_v} r^c_{l(e,jj)} a_{ii,jj,\partial_r \check{u}}(e)$$

$$+ a_{n-1} Re\, A \sum_{j=1}^{\bar{n}^e_v} r^c_{l(e,jj)}(t_{n-1}) a_{ii,jj,\partial_r \check{u}}(e) - a_{n-2} Re\, A \sum_{jj=1}^{\bar{n}^e_v} r^c_{l(e,jj)}(t_{n-2}) a_{ii,jj,\partial_r \check{u}}(e)$$

$$- a_n Re\, A \sum_{jj=1}^{\bar{n}^e_v} z^c_{l(e,jj)} a_{ii,jj,\partial_z \check{u}}(e) + a_{n-1} Re\, A \sum_{jj=1}^{\bar{n}^e_v} z^c_{l(e,jj)}(t_{n-1}) a_{ii,jj,\partial_z \check{u}}(e)$$

$$- a_{n-2} Re\, A \sum_{jj=1}^{\bar{n}^e_v} z^c_{l(e,jj)}(t_{n-2}) a_{ii,jj,\partial_z \check{u}}(e), \tag{25.139}$$

$$\begin{aligned}
\bar{\mathcal{M}}^{r,0c}_{e,ii} = &\frac{2\Delta_t Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}\bar{u}_{l(e,kk)}a^r_{ii,kk,jj}(e)\\
&+\frac{2\Delta_t Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}\bar{w}_{l(e,kk)}a^z_{ii,kk,jj}(e)\\
&-a_n Re\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}a^r_{ii,kk,jj}(e)\\
&+a_{n-1} Re\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}(t_{n-1})a^r_{ii,kk,jj}(e)\\
&-a_{n-2} Re\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}(t_{n-2})a^r_{ii,kk,jj}(e)\\
&-a_n Re\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}a^z_{ii,kk,jj}(e)\\
&+a_{n-1} Re\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}(t_{n-1})a^z_{ii,kk,jj}(e)\\
&-a_{n-2} Re\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}(t_{n-2})a^z_{ii,kk,jj}(e),
\end{aligned}\tag{25.140}$$

$$\bar{\mathcal{M}}^{r,0d}_{e,ii} = -\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_p^e}p_{l^p(e,jj)}b^r_{jj,ii}(e),\tag{25.141}$$

$$\begin{aligned}
\bar{\mathcal{M}}^{r,1}_{e_1,ii} = &-\frac{2\Delta_t}{3}Ac_{ii,n_r,\partial_r\check{u}}(e_1)-\frac{2\Delta_t}{3}Ac_{ii,n_z,\partial_r\check{w}}(e_1)+\frac{2\Delta_t}{3Ca}\sum_{jj=1}^{\bar{n}_v^{e_1}}\sigma^1_{l_1^1(e_1,jj)}c^s_{jj,ii,t_r}(e_1)\\
&-\frac{2\Delta_t}{3}\sum_{j=1}^{n_v^n}p^g_{l_1^1(e_1,jj)}c_{ii,jj,n^r}(e_1)-\frac{2\Delta_t}{3}\int_{\partial\Omega^{1,n}}\phi_i n^1_r\partial_r\bar{u}-\frac{2\Delta_t}{3}\int_{\partial\Omega^{1,n}}\phi_i n^1_z\partial_r\bar{w},
\end{aligned}\tag{25.142}$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,2}_{e_2,ii} =& \frac{2\Delta_t Be}{3} A d_{ii,t_r,t_r,\breve{u}}(e_2) + \frac{2\Delta_t Be}{3} A d_{ii,t_r,t_z,\breve{w}}(e_2) - \frac{2\Delta_t A}{3} d_{ii,n_r,\partial_r\breve{u}}(e_2) \\
& - \frac{2\Delta_t A}{3} d_{ii,n_z,\partial_r\breve{w}}(e_2) + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l_2(e_2,jj)} d_{ii,jj,t_r,t_r}(e_2) \\
& + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l_2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2) - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} u^s_{l_2^2(e_2,jj)} d_{ii,jj,t_r,t_r}(e_2) \\
& - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} w^s_{l_2^2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2) - \frac{\Delta_t}{3Ca} \sum_{j=1}^{n_v} \sigma^2_{l_2^2(e_2,jj)} d^s_{ii,jj,t_r}(e_2) \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e2}} \lambda^2_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \bar{u} - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \bar{w},
\end{aligned}
\tag{25.143}
$$

and

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,5}_{e,ii} =& -\frac{2\Delta_t}{3} A g_{ii,n_r,\partial_r\breve{u}}(e_5) - \frac{2\Delta_t}{3} A g_{ii,n_z,\partial_r\breve{w}}(e_5) + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \lambda^5_{l_5(e_5,jj)} g_{ii,jj,n_r}(e_5) \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v} \gamma^5_{l_5(e_5,jj)} g_{ii,jj,t_r}(e_5) - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i n_r^5 \partial_r \bar{u} - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i n_z^5 \partial_r \bar{w}.
\end{aligned}
\tag{25.144}
$$

Summarising and grouping terms we have

$$
\begin{aligned}
\bar{\mathcal{M}}^{r}_{i} =& \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}^{r,0a}_{e,ii} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}^{r,0b}_{e,ii} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}^{r,0c}_{e,ii} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}^{r,0d}_{e,ii} \\
& + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}^1_{\mathrm{el}}} \bar{\mathcal{M}}^{r,1}_{e_1,ii} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}^2_{\mathrm{el}}} \bar{\mathcal{M}}^{r,2}_{e,ii} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}^5_{\mathrm{el}}} \bar{\mathcal{M}}^{r,5}_{e_5,ii} \\
& + \frac{2\Delta_t}{3} \frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca} + \frac{2\Delta_t}{3} \frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca},
\end{aligned}
\tag{25.145}
$$

where

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,0a}_{e,ii} =& a_n Re\, A a_{ii,\breve{u}}(e) + \frac{2\Delta_t}{3} \left[-St\, a_{ii,g_r}(e) + A \left(a^r_{ii,\partial_r\breve{u}}(e) + a^z_{ii,\partial_z\breve{u}}(e) \right) \right. \\
& \left. + Re\, A^2 \left(a_{ii,\breve{u},\partial_r\breve{u}}(e) + a_{ii,\breve{w},\partial_z\breve{u}}(e) \right) \right],
\end{aligned}
\tag{25.146}
$$

$$\bar{\mathcal{M}}^{r,0b}_{e,ii} = \sum_{jj=1}^{\bar{n}_v^e} \left(\frac{2\Delta_t}{3} \left\{ \bar{u}_{l(e,jj)} \left[a^{r,r}_{ii,jj}(e) + a^{z,z}_{ii,jj}(e) \right] \right. \right.$$
$$+ Re\, A \left[\bar{w}_{l(e,jj)} a_{ii,jj,\partial_z \check{u}}(e) + \bar{u}_{l(e,jj)} \left(a^{r}_{ii,jj,\check{u}}(e) + a^{z}_{ii,jj,\check{w}}(e) + a_{ii,jj,\partial_r \check{u}}(e) \right) \right] \Big\}$$
$$+ Re \Big\{ a_{ii,jj}(e) \left[a_n \bar{u}_{l(e,jj)} - a_{n-1} u_{l(e,jj)}(t_{n-1}) + a_{n-2} u_{l(e,jj)}(t_{n-2}) \right]$$
$$- A a_{ii,jj,\partial_r \check{u}}(e) \left[a_n r^c_{l(e,jj)} - a_{n-1} r^c_{l(e,jj)}(t_{n-1}) + a_{n-2} r^c_{l(e,jj)}(t_{n-2}) \right]$$
$$\left. \left. - A a_{ii,jj,\partial_z \check{u}}(e) \left[a_n z^c_{l(e,jj)} - a_{n-1} z^c_{l(e,jj)}(t_{n-1}) + a_{n-2} z^c_{l(e,jj)}(t_{n-2}) \right] \right\} \right), \tag{25.147}$$

$$\bar{\mathcal{M}}^{r,0c}_{e,ii} = Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \left[\frac{2\Delta_t}{3} \underbrace{\sum_{kk=1}^{\bar{n}_v^e} \left\{ \bar{u}_{l(e,kk)} a^{r}_{ii,kk,jj}(e) + \bar{w}_{l(e,kk)} a^{z}_{ii,kk,jj}(e) \right\}}_{\bar{A}_{ii,jj}(e)} \right.$$
$$- \underbrace{\sum_{kk=1}^{\bar{n}_v^e} a^{r}_{ii,kk,jj}(e) \left\{ a_n r^c_{l(e,kk)} - a_{n-1} r^c_{l(e,kk)}(t_{n-1}) + a_{n-2} r^c_{l(e,kk)}(t_{n-2}) \right\}}_{B_{ii,jj}(e)}$$
$$\left. - \underbrace{\sum_{kk=1}^{\bar{n}_v^e} a^{z}_{ii,kk,jj}(e) \left\{ a_n z^c_{l(e,kk)} - a_{n-1} z^c_{l(e,kk)}(t_{n-1}) + a_{n-2} z^c_{l(e,kk)}(t_{n-2}) \right\}}_{C_{ii,jj}(e)} \right], \tag{25.148}$$

$$\bar{\mathcal{M}}^{r,0d}_{e,ii} = \sum_{jj=1}^{\bar{n}_p^e} -\frac{2\Delta_t}{3} p_{l^p(e,jj)} b^{r}_{jj,ii}(e), \tag{25.149}$$

$$\bar{\mathcal{M}}^{r,1}_{e_1,ii} = -\frac{2\Delta_t}{3} A \left[c_{ii,n_r,\partial_r \check{u}}(e_1) + c_{ii,n_z,\partial_z \check{u}}(e_1) \right]$$
$$+ \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e_1}} \left[\frac{1}{Ca} \sigma^1_{l^1_1(e_1,jj)} c^{s}_{jj,ii,t_r}(e_1) - p^{g}_{l^1_1(e_1,jj)} c_{ii,jj,n_r}(e_1) \right] \tag{25.150}$$
$$- \frac{2\Delta_t}{3} \int_{\partial\Omega^{1,n}} \phi_i n^1_r \partial_r \bar{u} - \frac{2\Delta_t}{3} \int_{\partial\Omega^{1,n}} \phi_i n^1_z \partial_r \bar{w},$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,2}_{e_2,ii} =& \frac{2\Delta_t}{3} A \left[Be \left(d_{ii,t_r,t_r,\breve{u}}(e_2) + d_{ii,t_r,t_z,\breve{w}}(e_2) \right) - d_{ii,n_r,\partial_r \breve{u}}(e_2) - d_{ii,n_z,\partial_r \breve{w}}(e_2) \right] \\
&+ \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \left[Be \left(d_{ii,jj,t_r,t_r}(e_2) \left\{ \bar{u}_{l_2(e_2,jj)} - u^s_{l_2^2(e_2,jj)} \right\} \right. \right. \\
&\qquad \left. + d_{ii,jj,t_r,t_z}(e_2) \left\{ \bar{w}_{l_2(e_2,jj)} - w^s_{l_2^2(e_2,jj)} \right\} \right) \\
&\qquad \left. - \frac{1}{2Ca} \sigma^2_{l_2^2(e_2,jj)} d^s_{ii,jj,t_r}(e_2) + \lambda^2_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) \right] \\
&- \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 \partial_r \bar{u} - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 \partial_r \bar{w},
\end{aligned}
\tag{25.151}
$$

and

$$
\begin{aligned}
\bar{\mathcal{M}}^{r,5}_{e,ii} =& -\frac{2\Delta_t}{3} A g_{ii,n_r,\partial_r \breve{u}}(e_5) - \frac{2\Delta_t}{3} A g_{ii,n_z,\partial_r \breve{w}}(e_5) \\
&+ \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \left[\lambda^5_{l_5(e_5,jj)} g_{ii,jj,n_r}(e_5) + \gamma^5_{l_5(e_5,jj)} g_{ii,jj,t_r}(e_5) \right] \\
&- \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i n_r^5 \partial_r \bar{u} - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i n_z^5 \partial_r \bar{w}.
\end{aligned}
\tag{25.152}
$$

25.1. Jacobian terms

We now calculate the derivatives of $\bar{\mathcal{M}}_i^r$ with respect to $\bar{u}_q$, $\bar{w}_q$, p_q, σ_q^1, θ_c, σ_q^2, λ_q^2, λ_q^5, γ_q^5, A and h_q.

25.1.1. Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to $\bar{u}_q$

Using equation (25.137) we have

$$\begin{aligned}
\partial_{\bar{u}_q}\bar{\mathcal{M}}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,ii}^{r,0c} \\
& + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,ii}^{r,0d} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\partial_{\bar{u}_q}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{\bar{u}_q}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca} \\
& + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\mathrm{el}}^5} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e_5,ii}^{r,5},
\end{aligned} \tag{25.153}$$

i.e.

$$\partial_{\bar{u}_q}\bar{\mathcal{M}}_i^r \;=\; \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,ii}^{r,0c} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,ii}^{r,2}, \tag{25.154}$$

Now, from equation (25.139)

$$
\begin{aligned}
\partial_{\bar u_q}\bar{\mathcal{M}}^{r,0b}_{e,ii} =& \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar n_v^e} a^{z,r}_{ii,jj}(e)\partial_{\bar u_q}\bar w_{l(e,jj)} + \frac{4\Delta_t}{3}\sum_{jj=1}^{\bar n_v^e} a^{r,r}_{ii,jj}(e)\partial_{\bar u_q}\bar u_{l(e,jj)}\\
&+ \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar n_v^e} a^{z,z}_{ii,jj}(e)\partial_{\bar u_q}\bar u_{l(e,jj)} + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar n_v^e} a^{r}_{ii,jj,\check u}(e)\partial_{\bar u_q}\bar u_{l(e,jj)}\\
&+ \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar n_v^e} a^{z}_{ii,jj,\check w}(e)\partial_{\bar u_q}\bar u_{l(e,jj)} + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar n_v^e} a_{ii,jj,\partial_r\check u}(e)\partial_{\bar u_q}\bar u_{l(e,jj)}\\
&+ \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar n_v^e} a_{ii,jj,\partial_z\check u}(e)\partial_{\bar u_q}\bar w_{l(e,jj)} + Re\sum_{jj=1}^{\bar n_v^e} a_{ii,jj}(e)\partial_{\bar u_q}\bar u_{l(e,jj)}\\
&- \frac{4Re}{3}\sum_{jj=1}^{\bar n_v^e} a_{ii,jj}(e)\partial_{\bar u_q}u_{l(e,jj)}(t_{n-1}) + \frac{Re}{3}\sum_{jj=1}^{\bar n_v^e} a_{ii,jj}(e)\partial_{\bar u_q}u_{l(e,jj)}(t_{n-2})\\
&- Re\,A\sum_{jj=1}^{\bar n_v^e} a_{ii,jj,\partial_r\check u}(e)\partial_{\bar u_q}r^c_{l(e,jj)} + \frac{4Re}{3}A\sum_{j=1}^{\bar n_v^e} a_{ii,jj,\partial_r\check u}(e)\partial_{\bar u_q}r^c_{l(e,jj)}(t_{n-1})\\
&- \frac{Re}{3}A\sum_{jj=1}^{\bar n_v^e} a_{ii,jj,\partial_r\check u}(e)\partial_{\bar u_q}r^c_{l(e,jj)}(t_{n-2}) - Re\,A\sum_{jj=1}^{\bar n_v^e} a_{ii,jj,\partial_z\check u}(e)\partial_{\bar u_q}z^c_{l(e,jj)}\\
&+ \frac{4Re}{3}A\sum_{jj=1}^{\bar n_v^e} a_{ii,jj,\partial_z\check u}(e)\partial_{\bar u_q}z^c_{l(e,jj)}(t_{n-1}) - \frac{Re}{3}A\sum_{jj=1}^{\bar n_v^e} a_{ii,jj,\partial_z\check u}(e)\partial_{\bar u_q}z^c_{l(e,jj)}(t_{n-2}),
\end{aligned}
\tag{25.155}
$$

i.e.

$$
\begin{aligned}
\partial_{\bar u_q}\bar{\mathcal{M}}^{r,0b}_{e,ii} =& \frac{4\Delta_t}{3}a^{r,r}_{ii,jj}(e)|_{q=l(e,jj)} + \frac{2\Delta_t}{3}a^{z,z}_{ii,jj}(e)|_{q=l(e,jj)}\\
&+ \frac{2\Delta_t Re}{3}Aa^{r}_{ii,jj,\check u}(e)|_{q=l(e,jj)} + \frac{2\Delta_t Re}{3}Aa^{z}_{ii,jj,\check w}(e)|_{q=l(e,jj)}\\
&+ \frac{2\Delta_t Re}{3}Aa_{ii,jj,\partial_r\check u}(e)|_{q=l(e,jj)} + Re\,a_{ii,jj}(e)|_{q=l(e,jj)}.
\end{aligned}
\tag{25.156}
$$

equivalently

$$
\begin{aligned}
\partial_{\bar u_q}\bar{\mathcal{M}}^{r,0b}_{e,ii} =& Re\,a_{ii,jj}(e)|_{q=l(e,jj)}\\
&+ \frac{2\Delta_t}{3}\left[2a^{r,r}_{ii,jj}(e) + a^{z,z}_{ii,jj}(e)\right.\\
&\left.+ Re\,A\left(a^{r}_{ii,jj,\check u}(e) + a^{z}_{ii,jj,\check w}(e) + a_{ii,jj,\partial_r\check u}(e)\right)\right]|_{q=l(e,jj)}.
\end{aligned}
\tag{25.157}
$$

From equation (25.140)

$$
\begin{aligned}
\partial_{\bar{u}_q} \bar{\mathcal{M}}^{r,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \partial_{\bar{u}_q} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{u}_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \partial_{\bar{u}_q} \bar{u}_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \partial_{\bar{u}_q} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{w}_{l(e,kk)} a^z_{ii,kk,jj}(e) \\
& - Re \sum_{jj=1}^{\bar{n}_v^e} \partial_{\bar{u}_q} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
& + \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \partial_{\bar{u}_q} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-1}) a^r_{ii,kk,jj}(e) \\
& - \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \partial_{\bar{u}_q} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-2}) a^r_{ii,kk,jj}(e) \\
& - Re \sum_{jj=1}^{\bar{n}_v^e} \partial_{\bar{u}_q} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)} a^z_{ii,kk,jj}(e) \\
& + \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \partial_{\bar{u}_q} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-1}) a^z_{ii,kk,jj}(e) \\
& - \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \partial_{\bar{u}_q} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-2}) a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{25.158}
$$

i.e.

$$\begin{aligned}
\partial_{\bar{u}_q}\bar{\mathcal{M}}^{r,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3} \sum_{\substack{jj=1\\ q=l(e,kk)}}^{\bar{n}^e_v} \partial_{\bar{u}_q}\bar{u}_{l(e,jj)} a^r_{ii,kk,jj}(e) + \frac{2\Delta_t Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{\bar{n}^e_v} \bar{u}_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{\bar{n}^e_v} \bar{w}_{l(e,kk)} a^z_{ii,kk,jj}(e) - Re \sum_{\substack{kk=1\\ q=l(e,jj)}} r^c_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
& + \frac{4Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{\bar{n}^e_v} r^c_{l(e,kk)}(t_{n-1}) a^r_{ii,kk,jj}(e) - \frac{Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{\bar{n}^e_v} r^c_{l(e,kk)}(t_{n-2}) a^r_{ii,kk,jj}(e) \\
& - Re \sum_{\substack{kk=1\\ q=l(e,jj)}}^{\bar{n}^e_v} z^c_{l(e,kk)} a^z_{ii,kk,jj}(e) + \frac{4Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{\bar{n}^e_v} z^c_{l(e,kk)}(t_{n-1}) a^z_{ii,kk,jj}(e) \\
& - \frac{Re}{3} \sum_{\substack{kk=1\\ q=l(e,jj)}}^{\bar{n}^e_v} z^c_{l(e,kk)}(t_{n-2}) a^z_{ii,kk,jj}(e),
\end{aligned} \tag{25.159}$$

equivalently

$$\begin{aligned}
\partial_{\bar{u}_q}\bar{\mathcal{M}}^{r,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3} \sum_{\substack{jj=1\\ q=l(e,kk)}}^{\bar{n}^e_v} \partial_{\bar{u}_q}\bar{u}_{l(e,jj)} a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3} \underbrace{\sum_{\substack{kk=1\\ q=l(e,jj)}}^{\bar{n}^e_v} \left[\bar{u}_{l(e,kk)} a^r_{ii,kk,jj}(e) + \bar{w}_{l(e,kk)} a^z_{ii,kk,jj}(e)\right]}_{\bar{A}_{ii,jj}(e)} \\
& - Re \underbrace{\sum_{\substack{kk=1\\ q=l(e,jj)}}^{\bar{n}^e_v} a^r_{ii,kk,jj}(e) \left[r^c_{l(e,kk)} - \frac{4}{3} r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,kk)}(t_{n-2})\right]}_{B_{ii,jj}(e)} \\
& - Re \underbrace{\sum_{\substack{kk=1\\ q=l(e,jj)}}^{\bar{n}^e_v} a^z_{ii,kk,jj}(e) \left[z^c_{l(e,kk)} - \frac{4}{3} z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,kk)}(t_{n-2})\right]}_{C_{ii,jj}(e)}.
\end{aligned} \tag{25.160}$$

Finally from equation (25.143) we have

$$\begin{aligned}
\partial_{\bar{u}_q}\bar{\mathcal{M}}^{r,2}_{e_2,ii} =& \frac{2\Delta_t Be}{3} A\partial_{\bar{u}_q} d_{ii,t_r,t_r,\breve{u}}(e_2) + \frac{2\Delta_t Be}{3} A\partial_{\bar{u}_q} d_{ii,t_r,t_z,\breve{w}}(e_2) \\
&- \frac{2\Delta_t A}{3}\partial_{\bar{u}_q} d_{ii,n_r,\partial_r\breve{u}}(e_2) - \frac{2\Delta_t A}{3}\partial_{\bar{u}_q} d_{ii,n_z,\partial_r\breve{w}}(e_2) \\
&+ \frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^e} d_{ii,jj,t_r,t_r}(e_2)\partial_{\bar{u}_q}\bar{u}_{l_2(e_2,jj)} + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^e} d_{ii,jj,t_r,t_z}(e_2)\partial_{\bar{u}_q}\bar{w}_{l_2(e_2,jj)} \\
&- \frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v} d_{ii,jj,t_r,t_r}(e_2)\partial_{\bar{u}_q} u^s_{l_2^2(e_2,jj)} - \frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v} d_{ii,jj,t_r,t_z}(e_2)\partial_{\bar{u}_q} w^s_{l_2^2(e_2,jj)} \\
&- \frac{\Delta_t}{3Ca}\sum_{j=1}^{n_v} d^s_{ii,jj,t_r}(e_2)\partial_{\bar{u}_q}\sigma^2_{l_2^2(e_2,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^{e2}} d_{ii,jj,n_r}(e_2)\partial_{\bar{u}_q}\lambda^2_{l_2^2(e_2,jj)},
\end{aligned} \tag{25.161}$$

i.e.

$$\partial_{\bar{u}_q}\bar{\mathcal{M}}^{r,2}_{e_2,ii} = \frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^e} d_{ii,jj,t_r,t_r}(e_2)|_{q=l_2(e_2,jj)}. \tag{25.162}$$

25.1.2. Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to $\bar{w}_q$

Using equation (25.137) we have

$$\begin{aligned}
\partial_{\bar{w}_q}\bar{\mathcal{M}}_i^r = & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{r,0c} \\
& + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{r,0d} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\rm el}^1} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\partial_{\bar{w}_q}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{\bar{w}_q}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca} \\
& + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\rm el}^2} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\rm el}^5} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e_5,ii}^{r,5},
\end{aligned} \tag{25.163}$$

i.e.

$$\partial_{\bar{w}_q}\bar{\mathcal{M}}_i^r = \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{r,0c} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\rm el}^2} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{r,2}, \tag{25.164}$$

Now, from equation (25.139)

$$\begin{aligned}
\partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{r,0b} = & \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}^{z,r}(e)\partial_{\bar{w}_q}\bar{w}_{l(e,jj)} + \frac{4\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}^{r,r}(e)\partial_{\bar{w}_q}\bar{u}_{l(e,jj)} \\
& + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}^{z,z}(e)\partial_{\bar{w}_q}\bar{u}_{l(e,jj)} + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\breve{u}}^{r}(e)\partial_{\bar{w}_q}\bar{u}_{l(e,jj)} \\
& + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\breve{w}}^{z}(e)\partial_{\bar{w}_q}\bar{u}_{l(e,jj)} + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\breve{u}}(e)\partial_{\bar{w}_q}\bar{u}_{l(e,jj)} \\
& + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\breve{u}}(e)\partial_{\bar{w}_q}\bar{w}_{l(e,jj)} + Re\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}(e)\partial_{\bar{w}_q}\bar{u}_{l(e,jj)} \\
& - \frac{4Re}{3}\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}(e)\partial_{\bar{w}_q}u_{l(e,jj)}(t_{n-1}) + \frac{Re}{3}\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}(e)\partial_{\bar{w}_q}u_{l(e,jj)}(t_{n-2}) \\
& - Re\,A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\breve{u}}(e)\partial_{\bar{w}_q}r_{l(e,jj)}^c + \frac{4Re}{3}A\sum_{j=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\breve{u}}(e)\partial_{\bar{w}_q}r_{l(e,jj)}^c(t_{n-1}) \\
& - \frac{Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\breve{u}}(e)\partial_{\bar{w}_q}r_{l(e,jj)}^c(t_{n-2}) - Re\,A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\breve{u}}(e)\partial_{\bar{w}_q}z_{l(e,jj)}^c \\
& + \frac{4Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\breve{u}}(e)\partial_{\bar{w}_q}z_{l(e,jj)}^c(t_{n-1}) - \frac{Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\breve{u}}(e)\partial_{\bar{w}_q}z_{l(e,jj)}^c(t_{n-2}),
\end{aligned} \tag{25.165}$$

i.e.

$$\partial_{\bar{w}_q}\bar{\mathcal{M}}^{r,0b}_{e,ii} = \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e}\left[a^{z,r}_{ii,jj}(e) + Re\, Aa_{ii,jj,\partial_z\check{u}}(e)\right]_{q=l(e,jj)}. \tag{25.166}$$

From equation (25.140)

$$\begin{aligned}
\partial_{\bar{w}_q}\bar{\mathcal{M}}^{r,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\partial_{\bar{w}_q}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}\bar{u}_{l(e,kk)}a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}\partial_{\bar{w}_q}\bar{u}_{l(e,kk)}a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}a^z_{ii,kk,jj}(e)\partial_{\bar{w}_q}\bar{w}_{l(e,kk)} \\
& - Re\sum_{jj=1}^{\bar{n}_v^e}\partial_{\bar{w}_q}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}a^r_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\partial_{\bar{w}_q}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}(t_{n-1})a^r_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\partial_{\bar{w}_q}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}(t_{n-2})a^r_{ii,kk,jj}(e) \\
& - Re\sum_{jj=1}^{\bar{n}_v^e}\partial_{\bar{w}_q}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}a^z_{ii,kk,jj}(e) \\
& + \frac{4Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\partial_{\bar{w}_q}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}(t_{n-1})a^z_{ii,kk,jj}(e) \\
& - \frac{Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\partial_{\bar{w}_q}\bar{u}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}(t_{n-2})a^z_{ii,kk,jj}(e),
\end{aligned} \tag{25.167}$$

i.e.

$$\partial_{\bar{w}_q}\bar{\mathcal{M}}^{r,0c}_{e,ii} = \frac{2\Delta_t Re}{3}\sum_{\substack{jj=1\\ q=l(e,kk)}}^{\bar{n}_v^e}\bar{u}_{l(e,jj)}a^z_{ii,kk,jj}(e). \tag{25.168}$$

Finally from equation (25.143) we have

$$\begin{aligned}\partial_{\bar{w}_q}\bar{\mathcal{M}}^{r,2}_{e_2,ii} =&\frac{2\Delta_t Be}{3}A\partial_{\bar{w}_q}d_{ii,t_r,t_r,\check{u}}(e_2)+\frac{2\Delta_t Be}{3}A\partial_{\bar{w}_q}d_{ii,t_r,t_z,\check{w}}(e_2)\\ &-\frac{2\Delta_t A}{3}\partial_{\bar{w}_q}d_{ii,n_r,\partial_r\check{u}}(e_2)-\frac{2\Delta_t A}{3}\partial_{\bar{w}_q}d_{ii,n_z,\partial_r\check{w}}(e_2)\\ &+\frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^e}d_{ii,jj,t_r,t_r}(e_2)\partial_{\bar{w}_q}\bar{u}_{l_2(e_2,jj)}+\frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^e}d_{ii,jj,t_r,t_z}(e_2)\partial_{\bar{w}_q}\bar{w}_{l_2(e_2,jj)}\\ &-\frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v}d_{ii,jj,t_r,t_r}(e_2)\partial_{\bar{w}_q}u^s_{l_2^2(e_2,jj)}-\frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v}d_{ii,jj,t_r,t_z}(e_2)\partial_{\bar{w}_q}w^s_{l_2^2(e_2,jj)}\\ &-\frac{\Delta_t}{3Ca}\sum_{j=1}^{n_v}d^s_{ii,jj,t_r}(e_2)\partial_{\bar{w}_q}\sigma^2_{l_2^2(e_2,jj)}+\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^{e2}}d_{ii,jj,n_r}(e_2)\partial_{\bar{w}_q}\lambda^2_{l_2^2(e_2,jj)},\end{aligned}\tag{25.169}$$

i.e.

$$\partial_{\bar{w}_q}\bar{\mathcal{M}}^{r,2}_{e_2,ii} =\frac{2\Delta_t Be}{3}d_{ii,jj,t_r,t_z}(e_2)|_{q=l_2(e_2,jj)}.\tag{25.170}$$

25.1.3. Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to p_q

Using equation (25.137) we have

$$\begin{aligned}
\partial_{p_q}\bar{\mathcal{M}}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\bar{\mathcal{M}}_{e,ii}^{r,0c} \\
& + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{p_q}\bar{\mathcal{M}}_{e,ii}^{r,0d} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{p_q}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\partial_{p_q}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca} \\
& + \frac{2\Delta_t}{3}\partial_{p_q}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca} \\
& + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{p_q}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\mathrm{el}}^5} \partial_{p_q}\bar{\mathcal{M}}_{e_5,ii}^{r,5},
\end{aligned} \tag{25.171}$$

i.e.

$$\partial_{p_q}\bar{\mathcal{M}}_i^r \;=\; \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{p_q}\bar{\mathcal{M}}_{e_1,ii}^{r,1}. \tag{25.172}$$

From equation (25.141) we have

$$\partial_{p_q}\bar{\mathcal{M}}_{e,ii}^{r,0d} = -\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_p^e} b_{jj,ii}^r(e)\partial_{p_q}p_{l^p(e,jj)}, \tag{25.173}$$

i.e.

$$\partial_{p_q}\bar{\mathcal{M}}_{e,ii}^{r,0d} = -\frac{2\Delta_t}{3}b_{jj,ii}^r(e)|_{q=l(e,jj)}, \tag{25.174}$$

25.1.4. **Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to σ_q^1**

Using equation (25.137) we have

$$\begin{aligned}
\partial_{\sigma_q^1}\bar{\mathcal{M}}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\sigma_q^1}\bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\sigma_q^1}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\sigma_q^1}\bar{\mathcal{M}}_{e,ii}^{r,0c} \\
& + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\sigma_q^1}\bar{\mathcal{M}}_{e,ii}^{r,0d} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\sigma_q^1}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca}\partial_{\sigma_q^1}\sigma^1(r_c,z_c) \\
& + \frac{2\Delta_t}{3}\frac{\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca}\partial_{\sigma_q^1}\sigma^1(r_{J^1},z_{J^1}) \\
& + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\sigma_q^1}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\mathrm{el}}^5} \partial_{\sigma_q^1}\bar{\mathcal{M}}_{e_5,ii}^{r,5},
\end{aligned} \tag{25.175}$$

i.e.

$$\partial_{\sigma_q^1}\bar{\mathcal{M}}_i^r \;=\; \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\sigma_q^1}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{m_r^1(r_c,z_c)}{Ca}\delta_{c,i}\delta_{c,q} + \frac{2\Delta_t}{3}\frac{m_r^{1,f}(r_{J^1},z_{J^1})}{Ca}\delta_{J^1,i}\delta_{J^1,q}. \tag{25.176}$$

From equation (25.142) we have

$$\begin{aligned}
\partial_{\sigma_q^1}\bar{\mathcal{M}}_{e_1,ii}^{r,1} = & -\frac{2\Delta_t}{3}\partial_{\sigma_q^1}Ac_{ii,n_r,\partial_r\check{u}}(e_1) - \frac{2\Delta_t}{3}\partial_{\sigma_q^1}Ac_{ii,n_z,\partial_r\check{w}}(e_1) \\
& + \frac{2\Delta_t}{3Ca}\sum_{jj=1}^{\bar{n}_v^{e_1}} c_{jj,ii,t_r}^s(e_1)\partial_{\sigma_q^1}\sigma_{l_1^1(e_1,jj)}^1 - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v^n} c_{ii,jj,n^r}(e_1)\partial_{\sigma_q^1}p_{l_1^1(e_1,jj)}^g,
\end{aligned} \tag{25.177}$$

i.e.

$$\partial_{\sigma_q^1}\bar{\mathcal{M}}_{e_1,ii}^{r,1} = \frac{2\Delta_t}{3Ca}c_{jj,ii,t_r}^s(e_1)|_{q=l_1^1(e_1,jj)}. \tag{25.178}$$

25.1.5. Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to θ_c

Using equation (25.137) we have

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}_i^r \; &= \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,0c} \\
&+ \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,0d} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{\theta_c}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)}{Ca}\underbrace{\partial_{\theta_c} m_r^1(r_c,z_c)}_{\partial_{\theta_c}(-\cos(\theta_c))} \\
&+ \frac{2\Delta_t}{3}\partial_{\theta_c}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca} \\
&+ \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\text{el}}^5} \partial_{\theta_c}\bar{\mathcal{M}}_{e_5,ii}^{r,5},
\end{aligned}
\tag{25.179}
$$

i.e.

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}_i^r \; &= \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,0c} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,0d} \\
&+ \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \partial_{\theta_c}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)}{Ca}\delta_{i,c}\sin(\theta_c) + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\text{el}}^5} \partial_{\theta_c}\bar{\mathcal{M}}_{e_5,ii}^{r,5}.
\end{aligned}
\tag{25.180}
$$

Now, from equation (25.138) we have

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,0a} \; &= -\frac{2\Delta_t St}{3}\partial_{\theta_c} a_{ii,g_r}(e) + \frac{2\Delta_t A}{3}\partial_{\theta_c} a_{ii,\partial_r\check{u}}^r(e) + \frac{2\Delta_t A}{3}\partial_{\theta_c} a_{ii,\partial_z\check{u}}^z(e) \\
&+ Re\, A\partial_{\theta_c} a_{ii,\check{u}}(e) + \frac{2\Delta_t Re}{3}(A)^2\,\partial_{\theta_c} a_{ii,\check{u},\partial_r\check{u}}(e) + \frac{2\Delta_t Re}{3}(A)^2\,\partial_{\theta_c} a_{ii,\check{w},\partial_z\check{u}}(e),
\end{aligned}
\tag{25.181}
$$

i.e.

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{r,0a} \; =& A\left[Re\,\partial_{\theta_c} a_{ii,\check{u}}(e) + \frac{2\Delta_t}{3}\partial_{\theta_c} a_{ii,\partial_r\check{u}}^r(e) + \frac{2\Delta_t}{3}\partial_{\theta_c} a_{ii,\partial_z\check{u}}^z(e)\right. \\
&\left. + \frac{2\Delta_t}{3} Re\, A\left(\partial_{\theta_c} a_{ii,\check{u},\partial_r\check{u}}(e) + \partial_{\theta_c} a_{ii,\check{w},\partial_z\check{u}}(e)\right)\right].
\end{aligned}
\tag{25.182}
$$

From equation (25.139)

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}^{r,0b}_{e,ii} \;=& \frac{2\Delta_t}{3}\sum_{jj\,=1}^{\bar{n}^e_v}\bar{w}_{l(e,jj)}\partial_{\theta_c}a^{z,r}_{ii,jj}(e)+\frac{4\Delta_t}{3}\sum_{jj\,=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{\theta_c}a^{r,r}_{ii,jj}(e)\\
&+\frac{2\Delta_t}{3}\sum_{jj\,=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{\theta_c}a^{z,z}_{ii,jj}(e)+\frac{2\Delta_t Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{\theta_c}a^{r}_{ii,jj,\breve{u}}(e)\\
&+\frac{2\Delta_t Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{\theta_c}a^{z}_{ii,jj,\breve{w}}(e)+\frac{2\Delta_t Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_r\breve{u}}(e)\\
&+\frac{2\Delta_t Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}\bar{w}_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_z\breve{u}}(e)+Re\sum_{jj\,=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{\theta_c}a_{ii,jj}(e)\\
&-\frac{4Re}{3}\sum_{jj\,=1}^{\bar{n}^e_v}u_{l(e,jj)}(t_{n-1})\partial_{\theta_c}a_{ii,jj}(e)+\frac{Re}{3}\sum_{jj\,=1}^{\bar{n}^e_v}u_{l(e,jj)}(t_{n-2})\partial_{\theta_c}a_{ii,jj}(e)\\
&-Re\,A\sum_{jj\,=1}^{\bar{n}^e_v}r^c_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_r\breve{u}}(e)+\frac{4Re}{3}A\sum_{j\,=1}^{\bar{n}^e_v}r^c_{l(e,jj)}(t_{n-1})\partial_{\theta_c}a_{ii,jj,\partial_r\breve{u}}(e)\\
&-\frac{Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}r^c_{l(e,jj)}(t_{n-2})\partial_{\theta_c}a_{ii,jj,\partial_r\breve{u}}(e)-Re\,A\sum_{jj\,=1}^{\bar{n}^e_v}z^c_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_z\breve{u}}(e)\\
&+\frac{4Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}z^c_{l(e,jj)}(t_{n-1})\partial_{\theta_c}a_{ii,jj,\partial_z\breve{u}}(e)-\frac{Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}z^c_{l(e,jj)}(t_{n-2})\partial_{\theta_c}a_{ii,jj,\partial_z\breve{u}}(e),
\end{aligned}
\tag{25.183}
$$

i.e.

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}^{r,0b}_{e,ii} \;=& \frac{2\Delta_t Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{\theta_c}a^{r}_{ii,jj,\breve{u}}(e)+\frac{2\Delta_t Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{\theta_c}a^{z}_{ii,jj,\breve{w}}(e)\\
&+\frac{2\Delta_t Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_r\breve{u}}(e)+\frac{2\Delta_t Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}\bar{w}_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_z\breve{u}}(e)\\
&-Re\,A\sum_{jj\,=1}^{\bar{n}^e_v}r^c_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_r\breve{u}}(e)+\frac{4Re}{3}A\sum_{j\,=1}^{\bar{n}^e_v}r^c_{l(e,jj)}(t_{n-1})\partial_{\theta_c}a_{ii,jj,\partial_r\breve{u}}(e)\\
&-\frac{Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}r^c_{l(e,jj)}(t_{n-2})\partial_{\theta_c}a_{ii,jj,\partial_r\breve{u}}(e)-Re\,A\sum_{jj\,=1}^{\bar{n}^e_v}z^c_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_z\breve{u}}(e)\\
&+\frac{4Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}z^c_{l(e,jj)}(t_{n-1})\partial_{\theta_c}a_{ii,jj,\partial_z\breve{u}}(e)-\frac{Re}{3}A\sum_{jj\,=1}^{\bar{n}^e_v}z^c_{l(e,jj)}(t_{n-2})\partial_{\theta_c}a_{ii,jj,\partial_z\breve{u}}(e),
\end{aligned}
\tag{25.184}
$$

equivalently

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}^{r,0b}_{e,ii} =& Re\, A \sum_{jj=1}^{\bar{n}_v^e} \Bigg\{ \frac{2\Delta_t}{3} \big[\bar{w}_{l(e,jj)} \partial_{\theta_c} a_{ii,jj,\partial_z \breve{u}}(e) \\
&+ \bar{u}_{l(e,jj)} \left(\partial_{\theta_c} a_{ii,jj,\partial_r \breve{u}}(e) + \partial_{\theta_c} a^r_{ii,jj,\breve{u}}(e) + \partial_{\theta_c} a^z_{ii,jj,\breve{w}}(e) \right) \big] \\
&- \partial_{\theta_c} a_{ii,jj,\partial_r \breve{u}}(e) \left[r^c_{l(e,jj)} - \frac{4}{3} r^c_{l(e,jj)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,jj)}(t_{n-2}) \right] \\
&- \partial_{\theta_c} a_{ii,jj,\partial_z \breve{u}}(e) \left[z^c_{l(e,jj)} - \frac{4}{3} z^c_{l(e,jj)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,jj)}(t_{n-2}) \right] \Bigg\},
\end{aligned}
\tag{25.185}
$$

From equation (25.140)

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}^{r,0c}_{e,ii} =& \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{u}_{l(e,kk)} \partial_{\theta_c} a^r_{ii,kk,jj}(e) \\
&+ \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{w}_{l(e,kk)} \partial_{\theta_c} a^z_{ii,kk,jj}(e) \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)} \partial_{\theta_c} a^r_{ii,kk,jj}(e) \\
&+ \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-1}) \partial_{\theta_c} a^r_{ii,kk,jj}(e) \\
&- \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-2}) \partial_{\theta_c} a^r_{ii,kk,jj}(e) \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)} \partial_{\theta_c} a^z_{ii,kk,jj}(e) \\
&+ \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-1}) \partial_{\theta_c} a^z_{ii,kk,jj}(e) \\
&- \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-2}) \partial_{\theta_c} a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{25.186}
$$

i.e.

$$
\partial_{\theta_c}\bar{\mathcal{M}}^{r,0c}_{e,ii} = 0. \tag{25.187}
$$

From equation (25.141)

$$
\partial_{\theta_c}\bar{\mathcal{M}}^{r,0d}_{e,ii} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_p^e} p_{l^p(e,jj)} \partial_{\theta_c} b^r_{jj,ii}(e), \tag{25.188}
$$

i.e.

$$
\partial_{\theta_c}\bar{\mathcal{M}}^{r,0d}_{e,ii} = 0. \tag{25.189}
$$

From equation (25.142)

$$\partial_{\theta_c}\bar{\mathcal{M}}^{r,1}_{e_1,ii} = -\frac{2\Delta_t}{3}A\partial_{\theta_c}c_{ii,n_r,\partial_r\breve{u}}(e_1) - \frac{2\Delta_t}{3}A\partial_{\theta_c}c_{ii,n_z,\partial_r\breve{w}}(e_1) + \frac{2\Delta_t}{3Ca}\sum_{jj=1}^{\bar{n}_v^{e1}}\sigma^1_{l_1^1(e_1,jj)}\partial_{\theta_c}c^s_{jj,ii,t_r}(e_1) - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v^n}p^g_{l_1^1(e_1,jj)}\partial_{\theta_c}c_{ii,jj,n^r}(e_1), \tag{25.190}$$

i.e.

$$\partial_{\theta_c}\bar{\mathcal{M}}^{r,1}_{e_1,ii} = -\frac{2\Delta_t}{3}A\partial_{\theta_c}c_{ii,n_r,\partial_r\breve{u}}(e_1) - \frac{2\Delta_t}{3}A\partial_{\theta_c}c_{ii,n_z,\partial_r\breve{w}}(e_1), \tag{25.191}$$

equivalently

$$\partial_{\theta_c}\bar{\mathcal{M}}^{r,1}_{e_1,ii} = -\frac{2\Delta_t}{3}A\left[\partial_{\theta_c}c_{ii,n_r,\partial_r\breve{u}}(e_1) + \partial_{\theta_c}c_{ii,n_z,\partial_r\breve{w}}(e_1)\right]. \tag{25.192}$$

From equation (25.143)

$$\begin{aligned}\partial_{\theta_c}\bar{\mathcal{M}}^{r,2}_{e_2,ii} =& \frac{2\Delta_t Be}{3}A\partial_{\theta_c}d_{ii,t_r,t_r,\breve{u}}(e_2) + \frac{2\Delta_t Be}{3}A\partial_{\theta_c}d_{ii,t_r,t_z,\breve{w}}(e_2)\\ &- \frac{2\Delta_t A}{3}\partial_{\theta_c}d_{ii,n_r,\partial_r\breve{u}}(e_2) - \frac{2\Delta_t A}{3}\partial_{\theta_c}d_{ii,n_z,\partial_r\breve{w}}(e_2)\\ &+ \frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{u}_{l_2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,t_r,t_r}(e_2) + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l_2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,t_r,t_z}(e_2)\\ &- \frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v}u^s_{l_2^2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,t_r,t_r}(e_2) - \frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v}w^s_{l_2^2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,t_r,t_z}(e_2)\\ &- \frac{\Delta_t}{3Ca}\sum_{j=1}^{n_v}\sigma^2_{l_2^2(e_2,jj)}\partial_{\theta_c}d^s_{ii,jj,t_r}(e_2) + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^{e2}}\lambda^2_{l_2^2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,n_r}(e_2),\end{aligned} \tag{25.193}$$

i.e.

$$\begin{aligned}\partial_{\theta_c}\bar{\mathcal{M}}^{r,2}_{e_2,ii} =& \frac{2\Delta_t Be}{3}A\partial_{\theta_c}d_{ii,t_r,t_r,\breve{u}}(e_2) + \frac{2\Delta_t Be}{3}A\partial_{\theta_c}d_{ii,t_r,t_z,\breve{w}}(e_2)\\ &- \frac{2\Delta_t A}{3}\partial_{\theta_c}d_{ii,n_r,\partial_r\breve{u}}(e_2) - \frac{2\Delta_t A}{3}\partial_{\theta_c}d_{ii,n_z,\partial_r\breve{w}}(e_2),\end{aligned} \tag{25.194}$$

equivalently

$$\begin{aligned}\partial_{\theta_c}\bar{\mathcal{M}}^{r,2}_{e_2,ii} =& \frac{2\Delta_t}{3}A\left[Be\left(\partial_{\theta_c}d_{ii,t_r,t_r,\breve{u}}(e_2) + \partial_{\theta_c}d_{ii,t_r,t_z,\breve{w}}(e_2)\right) - \partial_{\theta_c}d_{ii,n_r,\partial_r\breve{u}}(e_2)\right.\\ &\left. - \partial_{\theta_c}d_{ii,n_z,\partial_r\breve{w}}(e_2)\right]\end{aligned} \tag{25.195}$$

From (25.144)

$$\partial_{\theta_c}\bar{\mathcal{M}}^{r,5}_{e,ii} = -\frac{2\Delta_t}{3}A\partial_{\theta_c}g_{ii,n_r,\partial_r\breve{u}}(e_5) - \frac{2\Delta_t}{3}A\partial_{\theta_c}g_{ii,n_z,\partial_r\breve{w}}(e_5) + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e}\lambda^5_{l_5(e_5,jj)}\partial_{\theta_c}g_{ii,jj,n_r}(e_5) + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v}\gamma^5_{l_5(e_5,jj)}\partial_{\theta_c}g_{ii,jj,t_r}(e_5), \tag{25.196}$$

i.e.

$$\partial_{\theta_c}\bar{\mathcal{M}}^{r,5}_{e,ii} = -\frac{2\Delta_t}{3}A\partial_{\theta_c}g_{ii,n_r,\partial_r\check{u}}(e_5) - \frac{2\Delta_t}{3}A\partial_{\theta_c}g_{ii,n_z,\partial_r\check{w}}(e_5), \tag{25.197}$$

equivalently

$$\partial_{\theta_c}\bar{\mathcal{M}}^{r,5}_{e,ii} = -\frac{2\Delta_t}{3}A\left[\partial_{\theta_c}g_{ii,n_r,\partial_r\check{u}}(e_5) + \partial_{\theta_c}g_{ii,n_z,\partial_r\check{w}}(e_5)\right]. \tag{25.198}$$

25.1.6. **Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to σ_q^2**

Using equation (25.137) we have

$$\begin{aligned}\partial_{\sigma_q^2}\bar{\mathcal{M}}_i^r \;=& \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\sigma_q^2}\bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\sigma_q^2}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\sigma_q^2}\bar{\mathcal{M}}_{e,ii}^{r,0c}\\ &+ \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\sigma_q^2}\bar{\mathcal{M}}_{e,ii}^{r,0d} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\rm el}^1} \partial_{\sigma_q^2}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca}\partial_{\sigma_q^2}\sigma^1(r_c,z_c)\\ &+ \frac{2\Delta_t}{3}\frac{\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca}\partial_{\sigma_q^2}\sigma^1(r_{J^1},z_{J^1})\\ &+ \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\rm el}^2} \partial_{\sigma_q^2}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\rm el}^5} \partial_{\sigma_q^2}\bar{\mathcal{M}}_{e_5,ii}^{r,5},\end{aligned} \tag{25.199}$$

i.e.

$$\partial_{\sigma_q^2}\bar{\mathcal{M}}_i^r \;=\; \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\rm el}^2} \partial_{\sigma_q^2}\bar{\mathcal{M}}_{e,ii}^{r,2}. \tag{25.200}$$

From equation (25.143) we have

$$\begin{aligned}\partial_{\sigma_q^2}\bar{\mathcal{M}}_{e_2,ii}^{r,2} \;=& \frac{2\Delta_t Be}{3}A\partial_{\sigma_q^2}d_{ii,t_r,t_r,\breve{u}}(e_2) + \frac{2\Delta_t Be}{3}A\partial_{\sigma_q^2}d_{ii,t_r,t_z,\breve{w}}(e_2)\\ &- \frac{2\Delta_t A}{3}\partial_{\sigma_q^2}d_{ii,n_r,\partial_r\breve{u}}(e_2) - \frac{2\Delta_t A}{3}\partial_{\sigma_q^2}d_{ii,n_z,\partial_r\breve{w}}(e_2)\\ &+ \frac{2\Delta_t Be}{3}\sum_{jj\;=1}^{\bar{n}_v^e} d_{ii,jj,t_r,t_r}(e_2)\partial_{\sigma_q^2}\bar{u}_{l_2(e_2,jj)} + \frac{2\Delta_t Be}{3}\sum_{jj\;=1}^{\bar{n}_v^e} d_{ii,jj,t_r,t_z}(e_2)\partial_{\sigma_q^2}\bar{w}_{l_2(e_2,jj)}\\ &- \frac{2\Delta_t Be}{3}\sum_{j\;=1}^{n_v} d_{ii,jj,t_r,t_r}(e_2)\partial_{\sigma_q^2}u^s_{l_2^2(e_2,jj)} - \frac{2\Delta_t Be}{3}\sum_{j\;=1}^{n_v} d_{ii,jj,t_r,t_z}(e_2)\partial_{\sigma_q^2}w^s_{l_2^2(e_2,jj)}\\ &- \frac{\Delta_t}{3Ca}\sum_{j\;=1}^{n_v} d^s_{ii,jj,t_r}(e_2)\partial_{\sigma_q^2}\sigma^2_{l_2^2(e_2,jj)} + \frac{2\Delta_t}{3}\sum_{jj\;=1}^{\bar{n}_v^{e2}} d_{ii,jj,n_r}(e_2)\partial_{\sigma_q^2}\lambda^2_{l_2^2(e_2,jj)},\end{aligned} \tag{25.201}$$

i.e.

$$\partial_{\sigma_q^2}\bar{\mathcal{M}}_{e_2,ii}^{r,2} \;= -\frac{\Delta_t}{3Ca}d^s_{ii,jj,t_r}(e_2)|_{q=l_2^2(e_2,jj)}. \tag{25.202}$$

25.1.7. **Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to λ_q^2**

Using equation (25.137) we have

$$
\begin{aligned}
\partial_{\lambda_q^2}\bar{\mathcal{M}}_i^r \ =& \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^2}\bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^2}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^2}\bar{\mathcal{M}}_{e,ii}^{r,0c} \\
&+ \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^2}\bar{\mathcal{M}}_{e,ii}^{r,0d} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\lambda_q^2}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca}\partial_{\lambda_q^2}\sigma^1(r_c,z_c) \\
&+ \frac{2\Delta_t}{3}\frac{\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca}\partial_{\lambda_q^2}\sigma^1(r_{J^1},z_{J^1}) \\
&+ \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\lambda_q^2}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\mathrm{el}}^5} \partial_{\lambda_q^2}\bar{\mathcal{M}}_{e_5,ii}^{r,5},
\end{aligned}
\tag{25.203}
$$

i.e.

$$
\partial_{\lambda_q^2}\bar{\mathcal{M}}_i^r \ = \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\lambda_q^2}\bar{\mathcal{M}}_{e,ii}^{r,2}. \tag{25.204}
$$

From equation (25.143) we have

$$
\begin{aligned}
\partial_{\lambda_q^2}\bar{\mathcal{M}}_{e_2,ii}^{r,2} \ =& \frac{2\Delta_t Be}{3}A\partial_{\lambda_q^2}d_{ii,t_r,t_r,\breve{u}}(e_2) + \frac{2\Delta_t Be}{3}A\partial_{\lambda_q^2}d_{ii,t_r,t_z,\breve{w}}(e_2) \\
&- \frac{2\Delta_t A}{3}\partial_{\lambda_q^2}d_{ii,n_r,\partial_r\breve{u}}(e_2) - \frac{2\Delta_t A}{3}\partial_{\lambda_q^2}d_{ii,n_z,\partial_r\breve{w}}(e_2) \\
&+ \frac{2\Delta_t Be}{3}\sum_{jj\ =1}^{\bar{n}_v^e} d_{ii,jj,t_r,t_r}(e_2)\partial_{\lambda_q^2}\bar{u}_{l_2(e_2,jj)} + \frac{2\Delta_t Be}{3}\sum_{jj\ =1}^{\bar{n}_v^e} d_{ii,jj,t_r,t_z}(e_2)\partial_{\lambda_q^2}\bar{w}_{l_2(e_2,jj)} \\
&- \frac{2\Delta_t Be}{3}\sum_{j\ =1}^{n_v} d_{ii,jj,t_r,t_r}(e_2)\partial_{\lambda_q^2}u^s_{l_2^2(e_2,jj)} - \frac{2\Delta_t Be}{3}\sum_{j\ =1}^{n_v} d_{ii,jj,t_r,t_z}(e_2)\partial_{\lambda_q^2}w^s_{l_2^2(e_2,jj)} \\
&- \frac{\Delta_t}{3Ca}\sum_{j\ =1}^{n_v} d^s_{ii,jj,t_r}(e_2)\partial_{\lambda_q^2}\sigma^2_{l_2^2(e_2,jj)} + \frac{2\Delta_t}{3}\sum_{jj\ =1}^{\bar{n}_v^{e2}} d_{ii,jj,n_r}(e_2)\partial_{\lambda_q^2}\lambda^2_{l_2^2(e_2,jj)},
\end{aligned}
\tag{25.205}
$$

i.e.

$$
\partial_{\lambda_q^2}\bar{\mathcal{M}}_{e_2,ii}^{r,2} \ = \frac{2\Delta_t}{3}d_{ii,jj,n_r}(e_2)|_{q=l_2^2(e_2,jj)}. \tag{25.206}
$$

25.1.8. **Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to λ_q^5**

Using equation (25.137) we have

$$
\begin{aligned}
\partial_{\lambda_q^5}\bar{\mathcal{M}}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^5}\bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^5}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^5}\bar{\mathcal{M}}_{e,ii}^{r,0c} \\
& + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\lambda_q^5}\bar{\mathcal{M}}_{e,ii}^{r,0d} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\lambda_q^5}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca}\partial_{\lambda_q^5}\sigma^1(r_c,z_c) \\
& + \frac{2\Delta_t}{3}\frac{\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca}\partial_{\lambda_q^5}\sigma^1(r_{J^1},z_{J^1}) \\
& + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\lambda_q^5}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\mathrm{el}}^5} \partial_{\lambda_q^5}\bar{\mathcal{M}}_{e_5,ii}^{r,5},
\end{aligned}
\tag{25.207}
$$

i.e.

$$
\partial_{\lambda_q^5}\bar{\mathcal{M}}_i^r \;=\; \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\mathrm{el}}^5} \partial_{\lambda_q^5}\bar{\mathcal{M}}_{e_5,ii}^{r,5}. \tag{25.208}
$$

From equation (25.144)

$$
\begin{aligned}
\partial_{\lambda_q^5}\bar{\mathcal{M}}_{e,ii}^{r,5} = & -\frac{2\Delta_t}{3}\partial_{\lambda_q^5}Ag_{ii,n_r,\partial_r\breve{u}}(e_5) - \frac{2\Delta_t}{3}\partial_{\lambda_q^5}Ag_{ii,n_z,\partial_r\breve{w}}(e_5) \\
& + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} g_{ii,jj,n_r}(e_5)\partial_{\lambda_q^5}\lambda_{l_5(e_5,jj)}^5 + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v} g_{ii,jj,t_r}(e_5)\partial_{\lambda_q^5}\gamma_{l_5(e_5,jj)}^5,
\end{aligned}
\tag{25.209}
$$

i.e.

$$
\partial_{\lambda_q^5}\bar{\mathcal{M}}_{e,ii}^{r,5} = \frac{2\Delta_t}{3}g_{ii,jj,n_r}(e_5)|_{q=l_5^5 e_5,jj}. \tag{25.210}
$$

25.1.9. **Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to γ_q^5**

Using equation (25.137) we have

$$\begin{aligned}
\partial_{\gamma_q^5}\bar{\mathcal{M}}_i^r \; = & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\gamma_q^5}\bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\gamma_q^5}\bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\gamma_q^5}\bar{\mathcal{M}}_{e,ii}^{r,0c} \\
& + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_{\gamma_q^5}\bar{\mathcal{M}}_{e,ii}^{r,0d} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{\gamma_q^5}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca}\partial_{\gamma_q^5}\sigma^1(r_c,z_c) \\
& + \frac{2\Delta_t}{3}\frac{\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca}\partial_{\gamma_q^5}\sigma^1(r_{J^1},z_{J^1}) \\
& + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{\gamma_q^5}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\mathrm{el}}^5} \partial_{\gamma_q^5}\bar{\mathcal{M}}_{e_5,ii}^{r,5},
\end{aligned} \tag{25.211}$$

i.e.

$$\partial_{\gamma_q^5}\bar{\mathcal{M}}_i^r \; = \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\mathrm{el}}^5} \partial_{\gamma_q^5}\bar{\mathcal{M}}_{e_5,ii}^{r,5}. \tag{25.212}$$

From equation (25.144)

$$\begin{aligned}
\partial_{\gamma_q^5}\bar{\mathcal{M}}_{e,ii}^{r,5} = & -\frac{2\Delta_t}{3}\partial_{\gamma_q^5}Ag_{ii,n_r,\partial_r\breve{u}}(e_5) - \frac{2\Delta_t}{3}\partial_{\gamma_q^5}Ag_{ii,n_z,\partial_r\breve{w}}(e_5) \\
& + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} g_{ii,jj,n_r}(e_5)\partial_{\gamma_q^5}\lambda_{l_5(e_5,jj)}^5 + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v} g_{ii,jj,t_r}(e_5)\partial_{\gamma_q^5}\gamma_{l_5(e_5,jj)}^5,
\end{aligned} \tag{25.213}$$

i.e.

$$\partial_{\gamma_q^5}\bar{\mathcal{M}}_{e,ii}^{r,5} = \frac{2\Delta_t}{3}g_{ii,jj,t_r}(e_5)|_{q=l_5^5 e_5,jj}. \tag{25.214}$$

25.1.10. **Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to A**

Using equation (25.137) we have

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_A \bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_A \bar{\mathcal{M}}_{e,ii}^{r,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_A \bar{\mathcal{M}}_{e,ii}^{r,0c} \\
& + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_A \bar{\mathcal{M}}_{e,ii}^{r,0d} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_A \bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca}\partial_A\sigma^1(r_c,z_c) \\
& + \frac{2\Delta_t}{3}\frac{\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca}\partial_A\sigma^1(r_{J^1},z_{J^1}) + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_A \bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\mathrm{el}}^5} \partial_A \bar{\mathcal{M}}_{e_5,ii}^{r,5},
\end{aligned}
\tag{25.215}
$$

i.e.

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}_i^r \;=\; & \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_A \bar{\mathcal{M}}_{e,ii}^{r,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \partial_A \bar{\mathcal{M}}_{e,ii}^{r,0b} \\
& + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \partial_A \bar{\mathcal{M}}_{e_1,ii}^{r,1} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \partial_A \bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\mathrm{el}}^5} \partial_A \bar{\mathcal{M}}_{e_5,ii}^{r,5}.
\end{aligned}
\tag{25.216}
$$

From equation (25.138)

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}_{e,ii}^{r,0a} = & -\frac{2\Delta_t St}{3}\partial_A a_{ii,g_r}(e) + \frac{2\Delta_t}{3} a^r_{ii,\partial_r \breve{u}}(e)\partial_A A + \frac{2\Delta_t}{3} a^z_{ii,\partial_z \breve{u}}(e)\partial_A A \\
& + Re\, a_{ii,\breve{u}}(e)\partial_A A + \frac{2\Delta_t Re}{3} a_{ii,\breve{u},\partial_r\breve{u}}(e)\partial_A\left(A\right)^2 + \frac{2\Delta_t Re}{3} a_{ii,\breve{w},\partial_z\breve{u}}(e)\partial_A\left(A\right)^2,
\end{aligned}
\tag{25.217}
$$

i.e.

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}_{e,ii}^{r,0a} = & Re\, a_{ii,\breve{u}}(e) + \frac{2\Delta_t}{3} a^r_{ii,\partial_r \breve{u}}(e) + \frac{2\Delta_t}{3} a^z_{ii,\partial_z \breve{u}}(e) \\
& + \frac{2\Delta_t Re}{3} a_{ii,\breve{u},\partial_r\breve{u}}(e)2A + \frac{2\Delta_t Re}{3} a_{ii,\breve{w},\partial_z\breve{u}}(e)2A,
\end{aligned}
\tag{25.218}
$$

equivalently

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}_{e,ii}^{r,0a} = Re \Bigg[& a_{ii,\breve{u}}(e) + \frac{2\Delta_t}{3}\left(a^r_{ii,\partial_r \breve{u}}(e) + a^z_{ii,\partial_z \breve{u}}(e)\right) \\
& + \frac{4\Delta_t A}{3}\left(a_{ii,\breve{u},\partial_r\breve{u}}(e) + a_{ii,\breve{w},\partial_z\breve{u}}(e)\right)\Bigg].
\end{aligned}
\tag{25.219}
$$

Now, from equation (25.139)

$$\begin{aligned}
\partial_A \bar{\mathcal{M}}^{r,0b}_{e,ii} \; =& \frac{2\Delta_t}{3}\sum_{jj\;=1}^{\bar{n}_v^e} a^{z,r}_{ii,jj}(e)\partial_A \bar{w}_{l(e,jj)} + \frac{4\Delta_t}{3}\sum_{jj\;=1}^{\bar{n}_v^e} a^{r,r}_{ii,jj}(e)\partial_A \bar{u}_{l(e,jj)} \\
&+ \frac{2\Delta_t}{3}\sum_{jj\;=1}^{\bar{n}_v^e} a^{z,z}_{ii,jj}(e)\partial_A \bar{u}_{l(e,jj)} + \frac{2\Delta_t Re}{3}\partial_A A\sum_{jj\;=1}^{\bar{n}_v^e} a^{r}_{ii,jj,\breve{u}}(e)\bar{u}_{l(e,jj)} \\
&+ \frac{2\Delta_t Re}{3}\partial_A A\sum_{jj\;=1}^{\bar{n}_v^e} a^{z}_{ii,jj,\breve{w}}(e)\bar{u}_{l(e,jj)} + \frac{2\Delta_t Re}{3}\partial_A A\sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\breve{u}}(e)\bar{u}_{l(e,jj)} \\
&+ \frac{2\Delta_t Re}{3}\partial_A A\sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\breve{u}}(e)\bar{w}_{l(e,jj)} + Re\sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj}(e)\partial_A \bar{u}_{l(e,jj)} \\
&- \frac{4Re}{3}\sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj}(e)\partial_A u_{l(e,jj)}(t_{n-1}) + \frac{Re}{3}\sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj}(e)\partial_A u_{l(e,jj)}(t_{n-2}) \\
&- Re\,\partial_A A\sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\breve{u}}(e)r^c_{l(e,jj)} + \frac{4Re}{3}\partial_A A\sum_{j\;=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\breve{u}}(e)r^c_{l(e,jj)}(t_{n-1}) \\
&- \frac{Re}{3}\partial_A A\sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\breve{u}}(e)r^c_{l(e,jj)}(t_{n-2}) - Re\,\partial_A A\sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\breve{u}}(e)z^c_{l(e,jj)} \\
&+ \frac{4Re}{3}\partial_A A\sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\breve{u}}(e)z^c_{l(e,jj)}(t_{n-1}) - \frac{Re}{3}\partial_A A\sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\breve{u}}(e)z^c_{l(e,jj)}(t_{n-2}),
\end{aligned} \tag{25.220}$$

i.e.

$$\begin{aligned}
\partial_A \bar{\mathcal{M}}^{r,0b}_{e,ii} \; =& Re\sum_{jj\;=1}^{\bar{n}_v^e}\left[\frac{2\Delta_t}{3}\left\{a_{ii,jj,\partial_z\breve{u}}(e)\bar{w}_{l(e,jj)}\right.\right. \\
&\left. + \bar{u}_{l(e,jj)}\left[a^{r}_{ii,jj,\breve{u}}(e) + a^{z}_{ii,jj,\breve{w}}(e) + a_{ii,jj,\partial_r\breve{u}}(e)\right]\right\} \\
&- a_{ii,jj,\partial_r\breve{u}}(e)\left\{r^c_{l(e,jj)} - \frac{4}{3}r^c_{l(e,jj)}(t_{n-1}) + \frac{1}{3}r^c_{l(e,jj)}(t_{n-2})\right\} \\
&\left. - a_{ii,jj,\partial_z\breve{u}}(e)\left\{z^c_{l(e,jj)} - \frac{4}{3}z^c_{l(e,jj)}(t_{n-1}) + \frac{1}{3}z^c_{l(e,jj)}(t_{n-2})\right\}\right].
\end{aligned} \tag{25.221}$$

From equation (25.142) we have

$$\begin{aligned}
\partial_A \bar{\mathcal{M}}^{r,1}_{e_1,ii} =& -\frac{2\Delta_t}{3}\partial_A A c_{ii,n_r,\partial_r\breve{u}}(e_1) - \frac{2\Delta_t}{3}\partial_A A c_{ii,n_z,\partial_r\breve{w}}(e_1) \\
&+ \frac{2\Delta_t}{3Ca}\sum_{jj=1}^{\bar{n}_v^{e1}} c^s_{jj,ii,t_r}(e_1)\partial_A \sigma^1_{l_1^1(e_1,jj)} - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v^n} c_{ii,jj,n^r}(e_1)\partial_A p^g_{l_1^1(e_1,jj)},
\end{aligned} \tag{25.222}$$

i.e.

$$\partial_A \bar{\mathcal{M}}^{r,1}_{e_1,ii} = -\frac{2\Delta_t}{3}c_{ii,n_r,\partial_r\breve{u}}(e_1) - \frac{2\Delta_t}{3}c_{ii,n_z,\partial_r\breve{w}}(e_1), \tag{25.223}$$

equivalently

$$\partial_A \bar{\mathcal{M}}^{r,1}_{e_1,ii} = -\frac{2\Delta_t}{3}\left[c_{ii,n_r,\partial_r\check{u}}(e_1) + c_{ii,n_z,\partial_r\check{w}}(e_1)\right]. \tag{25.224}$$

From equation (25.143) we have

$$\begin{aligned}\partial_A \bar{\mathcal{M}}^{r,2}_{e_2,ii} =& \frac{2\Delta_t Be}{3}\partial_A A d_{ii,t_r,t_r,\check{u}}(e_2) + \frac{2\Delta_t Be}{3} d_{ii,t_r,t_z,\check{w}}(e_2)\partial_A A \\ &- \frac{2\Delta_t}{3} d_{ii,n_r,\partial_r\check{u}}(e_2)\partial_A A - \frac{2\Delta_t}{3} d_{ii,n_z,\partial_r\check{w}}(e_2)\partial_A A \\ &+ \frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^e} d_{ii,jj,t_r,t_r}(e_2)\partial_A \bar{u}_{l_2(e_2,jj)} + \frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^e} d_{ii,jj,t_r,t_z}(e_2)\partial_A \bar{w}_{l_2(e_2,jj)} \\ &- \frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v} d_{ii,jj,t_r,t_r}(e_2)\partial_A u^s_{l_2^2(e_2,jj)} - \frac{2\Delta_t Be}{3}\sum_{j=1}^{n_v} d_{ii,jj,t_r,t_z}(e_2)\partial_A w^s_{l_2^2(e_2,jj)} \\ &- \frac{\Delta_t}{3Ca}\sum_{j=1}^{n_v} d^s_{ii,jj,t_r}(e_2)\partial_A \sigma^2_{l_2^2(e_2,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^{e2}} d_{ii,jj,n_r}(e_2)\partial_A \lambda^2_{l_2^2(e_2,jj)},\end{aligned} \tag{25.225}$$

i.e.

$$\begin{aligned}\partial_A \bar{\mathcal{M}}^{r,2}_{e_2,ii} =& \frac{2\Delta_t Be}{3} d_{ii,t_r,t_r,\check{u}}(e_2) + \frac{2\Delta_t Be}{3} d_{ii,t_r,t_z,\check{w}}(e_2) \\ &- \frac{2\Delta_t}{3} d_{ii,n_r,\partial_r\check{u}}(e_2) - \frac{2\Delta_t}{3} d_{ii,n_z,\partial_r\check{w}}(e_2),\end{aligned} \tag{25.226}$$

equivalently

$$\partial_A \bar{\mathcal{M}}^{r,2}_{e_2,ii} = \frac{2\Delta_t}{3}\left[Be\left(d_{ii,t_r,t_r,\check{u}}(e_2) + d_{ii,t_r,t_z,\check{w}}(e_2)\right) - d_{ii,n_r,\partial_r\check{u}}(e_2) - d_{ii,n_z,\partial_r\check{w}}(e_2)\right]. \tag{25.227}$$

From equation (25.144)

$$\begin{aligned}\partial_A \bar{\mathcal{M}}^{r,5}_{e,ii} =& -\frac{2\Delta_t}{3} g_{ii,n_r,\partial_r\check{u}}(e_5)\partial_A A - \frac{2\Delta_t}{3} g_{ii,n_z,\partial_r\check{w}}(e_5)\partial_A A \\ &+ \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} g_{ii,jj,n_r}(e_5)\partial_A \lambda^5_{l_5(e_5,jj)} + \frac{2\Delta_t}{3}\sum_{jj=1}^{n_v} g_{ii,jj,t_r}(e_5)\partial_A \gamma^5_{l_5(e_5,jj)},\end{aligned} \tag{25.228}$$

i.e.

$$\partial_A \bar{\mathcal{M}}^{r,5}_{e,ii} = -\frac{2\Delta_t}{3} g_{ii,n_r,\partial_r\check{u}}(e_5) - \frac{2\Delta_t}{3} g_{ii,n_z,\partial_r\check{w}}(e_5), \tag{25.229}$$

equivalently

$$\partial_A \bar{\mathcal{M}}^{r,5}_{e,ii} = -\frac{2\Delta_t}{3}\left[g_{ii,n_r,\partial_r\check{u}}(e_5) + g_{ii,n_z,\partial_r\check{w}}(e_5)\right]. \tag{25.230}$$

25.1.11. Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to h_q

We denote the spine lengths by h, and we consider the derivatives of the residuals with respect to each spine.

From equation (25.117) we have

$$\bar{\mathcal{M}}_i^r = \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}_{e,ii}^{r,0}}_{\bar{\mathcal{M}}_i^{r,0}} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca}$$
$$+\frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_r^{1,f}(r_{J^1},z_{J^1})}{Ca} + \underbrace{\sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \bar{\mathcal{M}}_{e,ii}^{r,2}}_{\bar{\mathcal{M}}_i^{r,2}} + \underbrace{\sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\mathrm{el}}^4} \bar{\mathcal{M}}_{e_4,ii}^{r,4}}_{\bar{\mathcal{M}}_i^{r,4}}, \tag{25.231}$$

We notice that in the sum by elements above, it is only those spines that contain nodes in these elements that are going to have an effect on each of the derivatives shown above. Put differently, the vast majority of the derivatives above will be identically null. Hence, we once again resort to a function that maps objects in the element to the global number of these elements. Here we define as the "local spines" of an element those spines that contain nodes that are part of the element being considered, and we number those spines with a local spine number (from 1 to the number of spines that contain nodes of the element). We then introduce the local-spine-number to global-spine-number map $S(e,qq) = q$, which maps the qq-th local spine number on element e to its global spines number (previously referred to as simply *the spine number*) q. Similarly, we define $S_i(e_i,qq) = q$, which maps the local spine number qq of element e_i on boundary i to its global spine number q.

Thus using local spine numbers we have

$$\partial_{h_q}\bar{\mathcal{M}}_i^r = \sum_{\substack{e=1\\ i=l(e,ii)\\ q=S(e,qq)}}^{\bar{n}_{\mathrm{el}}} \partial_{h_{S(e,qq)}}\bar{\mathcal{M}}_{e,ii}^{r,0} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)\\ q=S(e,qq)}}^{\bar{n}_{\mathrm{el}}^1} \partial_{h_{S(e,qq)}}\bar{\mathcal{M}}_{e_1,ii}^{r,1}$$
$$+\frac{2\Delta_t}{3}\underbrace{\partial_{h_q}\frac{\sigma_c^1\delta_{i,c}m_r^1(r_c,z_c)}{Ca}}_{=0} + \frac{2\Delta_t}{3}\frac{\sigma_{J^1}^1\delta_{i,J^1}}{Ca}\partial_{h_q}m_r^{1,f}(r_{J^1},z_{J^1}) \tag{25.232}$$
$$+\sum_{\substack{e_2=1\\ i=l_2(e,ii)\\ q=S(e,qq)}}^{\bar{n}_{\mathrm{el}}^2} \partial_{h_{S(e,qq)}}\bar{\mathcal{M}}_{e,ii}^{r,2} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)\\ q=S(e,qq)}}^{\bar{n}_{\mathrm{el}}^4} \partial_{h_{S(e,qq)}}\bar{\mathcal{M}}_{e_4,ii}^{r,4}.$$

Then; we have, from equation (25.138),

$$\partial_{h_{S(e,qq)}}\bar{\mathcal{M}}^{r,0a}_{e,ii} = -\frac{2\Delta_t St}{3}\partial_{h_{S(e,qq)}}a_{ii,g_r}(e) + \frac{2\Delta_t A}{3}\partial_{h_{S(e,qq)}}a^{r}_{ii,\partial_r\breve{u}}(e) + \frac{2\Delta_t A}{3}\partial_{h_{S(e,qq)}}a^{z}_{ii,\partial_z\breve{u}}(e) + Re\,A\partial_{h_{S(e,qq)}}a_{ii,\breve{u}}(e) + \frac{2\Delta_t Re}{3}\left(A\right)^2\partial_{h_{S(e,qq)}}a_{ii,\breve{u},\partial_r\breve{u}}(e) + \frac{2\Delta_t Re}{3}\left(A\right)^2\partial_{h_{S(e,qq)}}a_{ii,\breve{w},\partial_z\breve{u}}(e), \tag{25.233}$$

i.e.

$$\partial_{h_{S(e,qq)}}\bar{\mathcal{M}}^{r,0a}_{e,ii} = Re\,A\partial_{h_{S(e,qq)}}a_{ii,\breve{u}}(e) + \frac{2\Delta_t}{3}\Big\{-St\,\partial_{h_{S(e,qq)}}a_{ii,g_r}(e) + A\left[\partial_{h_{S(e,qq)}}a^{r}_{ii,\partial_r\breve{u}}(e) + \partial_{h_{S(e,qq)}}a^{z}_{ii,\partial_z\breve{u}}(e)\right] + Re\,\left(A\right)^2\left[\partial_{h_{S(e,qq)}}a_{ii,\breve{u},\partial_r\breve{u}}(e) + \partial_{h_{S(e,qq)}}a_{ii,\breve{w},\partial_z\breve{u}}(e)\right]\Big\}; \tag{25.234}$$

from equation (25.131),

$$\begin{aligned}
\partial_{h_{S(e,qq)}}\bar{\mathcal{M}}^{r,0b}_{e,ii} &= Re\sum_{jj=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{h_{S(e,qq)}}a_{ii,jj}(e) - \frac{4Re}{3}\sum_{jj=1}^{\bar{n}^e_v}u_{l(e,jj)}(t_{n-1})\partial_{h_{S(e,qq)}}a_{ii,jj}(e)\\
&+ \frac{Re}{3}\sum_{jj=1}^{\bar{n}^e_v}u_{l(e,jj)}(t_{n-2})\partial_{h_{S(e,qq)}}a_{ii,jj}(e) + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{h_{S(e,qq)}}a^{r}_{ii,jj,\breve{u}}(e)\\
&+ \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{h_{S(e,qq)}}a^{z}_{ii,jj,\breve{w}}(e) + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{h_{S(e,qq)}}a_{ii,jj,\partial_r\breve{u}}(e)\\
&+ \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}^e_v}\bar{w}_{l(e,jj)}\partial_{h_{S(e,qq)}}a_{ii,jj,\partial_z\breve{u}}(e) - Re\,A\sum_{jj=1}^{\bar{n}^e_v}a_{ii,jj,\partial_r\breve{u}}(e)\partial_{h_{S(e,qq)}}r^c_{l(e,jj)}\\
&- Re\,A\sum_{jj=1}^{\bar{n}^e_v}r^c_{l(e,jj)}\partial_{h_{S(e,qq)}}a_{ii,jj,\partial_r\breve{u}}(e) + \frac{4Re}{3}A\sum_{j=1}^{\bar{n}^e_v}r^c_{l(e,jj)}(t_{n-1})\partial_{h_{S(e,qq)}}a_{ii,jj,\partial_r\breve{u}}(e)\\
&- \frac{Re}{3}A\sum_{jj=1}^{\bar{n}^e_v}r^c_{l(e,jj)}(t_{n-2})\partial_{h_{S(e,qq)}}a_{ii,jj,\partial_r\breve{u}}(e) - Re\,A\sum_{jj=1}^{\bar{n}^e_v}a_{ii,jj,\partial_z\breve{u}}(e)\partial_{h_{S(e,qq)}}z^c_{l(e,jj)}\\
&- Re\,A\sum_{jj=1}^{\bar{n}^e_v}z^c_{l(e,jj)}\partial_{h_{S(e,qq)}}a_{ii,jj,\partial_z\breve{u}}(e) + \frac{4Re}{3}A\sum_{jj=1}^{\bar{n}^e_v}z^c_{l(e,jj)}(t_{n-1})\partial_{h_{S(e,qq)}}a_{ii,jj,\partial_z\breve{u}}(e)\\
&- \frac{Re}{3}A\sum_{jj=1}^{\bar{n}^e_v}z^c_{l(e,jj)}(t_{n-2})\partial_{h_{S(e,qq)}}a_{ii,jj,\partial_z\breve{u}}(e) + \frac{4\Delta_t}{3}\sum_{jj=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{h_{S(e,qq)}}a^{r,r}_{ii,jj}(e)\\
&+ \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}^e_v}\bar{u}_{l(e,jj)}\partial_{h_{S(e,qq)}}a^{z,z}_{ii,jj}(e) + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}^e_v}\bar{w}_{l(e,jj)}\partial_{h_{S(e,qq)}}a^{z,r}_{ii,jj}(e),
\end{aligned} \tag{25.235}$$

i.e.

$$\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{r,0b}_{e,ii} = \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \left[\partial_{h_{S(e,qq)}} a^{z,r}_{ii,jj}(e) + Re\, A \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_z \breve{u}}(e)\right]$$

$$+ \frac{2\Delta_t}{3} \sum_{jj\ =1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \left\{2\partial_{h_{S(e,qq)}} a^{r,r}_{ii,jj}(e) + \partial_{h_{S(e,qq)}} a^{z,z}_{ii,jj}(e)\right.$$

$$\left. + Re\, A \left[\partial_{h_{S(e,qq)}} a^{r}_{ii,jj,\breve{u}}(e) + \partial_{h_{S(e,qq)}} a^{z}_{ii,jj,\breve{w}}(e) + \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_r \breve{u}}(e)\right]\right\}$$

$$+ Re \sum_{jj\ =1}^{\bar{n}_v^e} \partial_{h_{S(e,qq)}} a_{ii,jj}(e) \left[\bar{u}_{l(e,jj)} - \frac{4}{3} u_{l(e,jj)}(t_{n-1}) + \frac{1}{3} u_{l(e,jj)}(t_{n-2})\right]$$

$$- Re\, A \sum_{jj\ =1}^{\bar{n}_v^e} a_{ii,jj,\partial_r \breve{u}}(e) \partial_{h_{S(e,qq)}} r^c_{l(e,jj)}$$

$$- Re\, A \sum_{jj\ =1}^{\bar{n}_v^e} \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_r \breve{u}}(e) \left[r^c_{l(e,jj)} - \frac{4}{3} r^c_{l(e,jj)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,jj)}(t_{n-2})\right]$$

$$- Re\, A \sum_{jj\ =1}^{\bar{n}_v^e} a_{ii,jj,\partial_z \breve{u}}(e) \partial_{h_{S(e,qq)}} z^c_{l(e,jj)}$$

$$- Re\, A \sum_{jj\ =1}^{\bar{n}_v^e} \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_z \breve{u}}(e) \left[z^c_{l(e,jj)} - \frac{4}{3} z^c_{l(e,jj)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,jj)}(t_{n-2})\right]$$

$$- \frac{2\Delta_t}{3} \sum_{jj\ =1}^{\bar{n}_p^e} p_{l^p(e,jj)} \partial_{h_{S(e,qq)}} b^r_{jj,ii}(e); \tag{25.236}$$

from equation (25.132),

$$
\begin{aligned}
\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{r,0c}_{e,ii} = {} & \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{u}_{l(e,kk)} \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{w}_{l(e,kk)} \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \\
& - Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} a^r_{ii,kk,jj}(e) \partial_{h_{S(e,qq)}} r^c_{l(e,kk)} \\
& - Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)} \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \\
& + \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-1}) \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \\
& - \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-2}) \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \\
& - Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(ee,jj)} \sum_{kk=1}^{\bar{n}_v^e} a^z_{ii,kk,jj}(e) \partial_{h_{S(e,qq)}} z^c_{l(e,kk)} \\
& - Re \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(ee,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)} \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \\
& + \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-1}) \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \\
& - \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-2}) \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{25.237}
$$

i.e.

$$\begin{aligned}
&\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{r,0c}_{e,ii} = \\
&\sum_{jj=1}^{\bar{n}_v^e} Re\, \bar{u}_{l(e,jj)} \left\{ \frac{2\Delta_t}{3} \underbrace{\sum_{kk=1}^{\bar{n}_v^e} \left[\bar{u}_{l(e,kk)} \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) + \bar{w}_{l(e,kk)} \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \right]}_{\partial_{h_{S(e,qq)}} \bar{A}_{ij}} \right. \\
&\qquad - \sum_{kk=1}^{\bar{n}_v^e} \left(a^r_{ii,kk,jj}(e) \partial_{h_{S(e,qq)}} r^c_{l(e,kk)} \right. \\
&\qquad \left. + \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \left[r^c_{l(e,kk)} - \frac{4}{3} r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,kk)}(t_{n-2}) \right] \right) \\
&\qquad - \sum_{kk=1}^{\bar{n}_v^e} \left(a^z_{ii,kk,jj}(e) \partial_{h_{S(e,qq)}} z^c_{l(e,kk)} \right. \\
&\qquad \left. \left. + \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \left[z^c_{l(e,kk)} - \frac{4}{3} z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,kk)}(t_{n-2}) \right] \right) \right\}
\end{aligned} \tag{25.238}$$

or, equivalently,

$$\begin{aligned}
&\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{r,0c}_{e,ii} = \\
&\sum_{jj=1}^{\bar{n}_v^e} Re\, \bar{u}_{l(e,jj)} \left\{ \frac{2\Delta_t}{3} \partial_{h_{S(e,qq)}} \bar{A}_{ij} - \partial_{h_{S(e,qq)}} B_{ii,jj} - \partial_{h_{S(e,qq)}} C_{ii,jj} \right\},
\end{aligned} \tag{25.239}$$

where

$$\begin{aligned}
\partial_{h_{S(e,qq)}} B_{ii,jj} = \sum_{kk=1}^{\bar{n}_v^e} \left(a^r_{ii,kk,jj}(e) \partial_{h_{S(e,qq)}} r^c_{l(e,kk)} + \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \left[r^c_{l(e,kk)} \right.\right. \\
\left.\left. - \frac{4}{3} r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,kk)}(t_{n-2}) \right] \right)
\end{aligned} \tag{25.240}$$

and

$$\begin{aligned}
\partial_{h_{S(e,qq)}} C_{ii,jj} = \left(\sum_{kk=1}^{\bar{n}_v^e} a^z_{ii,kk,jj}(e) \partial_{h_{S(e,qq)}} z^c_{l(e,kk)} + \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \left[z^c_{l(e,kk)} \right.\right. \\
\left.\left. - \frac{4}{3} z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,kk)}(t_{n-2}) \right] \right);
\end{aligned} \tag{25.241}$$

moreover

$$\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{r,0d}_{e,ii} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_p^e} p_{l^p(e,jj)} \partial_{h_{S(e,qq)}} b^r_{jj,ii}(e). \tag{25.242}$$

From equation (25.142),

$$\begin{aligned}\partial_{h_{S_1(e_1,qq)}}\bar{\mathcal{M}}^{r,1}_{e_1,ii} = &-\frac{2\Delta_t}{3}A\partial_{h_{S_1(e_1,qq)}}c_{ii,n_r,\partial_r\breve{u}}(e_1) - \frac{2\Delta_t}{3}A\partial_{h_{S_1(e_1,qq)}}c_{ii,n_z,\partial_r\breve{w}}(e_1)\\ &+\frac{2\Delta_t}{3Ca}\sum_{jj=1}^{\bar{n}_v^{e1}}\sigma^1_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c^s_{jj,ii,t_r}(e_1)\\ &-\frac{2\Delta_t}{3}\sum_{j=1}^{n_v^n}p^g_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,n^r}(e_1);\end{aligned} \tag{25.243}$$

from equation (25.143),

$$\begin{aligned}\partial_{h_{S_2(e_2,qq)}}\bar{\mathcal{M}}^{r,2}_{e_2,ii} \;=&\frac{2\Delta_t Be}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,t_r,t_r,\breve{u}}(e_2) + \frac{2\Delta_t Be}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,t_r,t_z,\breve{w}}(e_2)\\ &-\frac{2\Delta_t A}{3}\partial_{h_{S_2(e_2,qq)}}d_{ii,n_r,\partial_r\breve{u}}(e_2) - \frac{2\Delta_t A}{3}\partial_{h_{S_2(e_2,qq)}}d_{ii,n_z,\partial_r\breve{w}}(e_2)\\ &-\frac{2\Delta_t Be}{3}\sum_{jj\,=1}^{\bar{n}_v^e}u^s_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_r,t_r}(e_2)\\ &-\frac{2\Delta_t Be}{3}\sum_{jj\,=1}^{\bar{n}_v^e}w^s_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_r,t_z}(e_2)\\ &+\frac{2\Delta_t Be}{3}\sum_{jj\,=1}^{\bar{n}_v^e}\bar{u}_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_r,t_r}(e_2)\\ &+\frac{2\Delta_t Be}{3}\sum_{jj\,=1}^{\bar{n}_v^e}\bar{w}_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_r,t_z}(e_2)\\ &-\frac{\Delta_t}{3Ca}\sum_{j\,=1}^{n_v}\sigma^2_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d^s_{ii,jj,t_r}(e_2) + \frac{2\Delta_t}{3}\sum_{jj\,=1}^{\bar{n}_v^{e2}}\lambda^2_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2),\end{aligned} \tag{25.244}$$

and from equation (25.144),

$$\begin{aligned}\partial_{h_{S_5(e_5,qq)}}\bar{\mathcal{M}}^{r,5}_{e_5,ii} = &-\frac{2\Delta_t}{3}A\partial_{h_{S_5(e_5,qq)}}g_{ii,n_r,\partial_r\breve{u}}(e_5) - \frac{2\Delta_t}{3}A\partial_{h_{S_5(e_5,qq)}}g_{ii,n_z,\partial_r\breve{w}}(e_5)\\ &+\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e}\lambda^5_{l_5(e_5,jj)}\partial_{h_{S_5(e_5,qq)}}g_{ii,jj,n_r}(e_5)\\ &+\frac{2\Delta_t}{3}\sum_{jj=1}^{n_v}\gamma^5_{l_5(e_5,jj)}\partial_{h_{S_5(e_5,qq)}}g_{ii,jj,t_r,t_z}(e_5).\end{aligned} \tag{25.245}$$

26. The z-momentum residuals near an obtuse contact angle

We recall equation (24.2)

$$\begin{aligned} &Re\,\partial_t \bar{w} + Re\,\bar{u}\partial_r \bar{w} + Re\,\bar{w}\partial_z \bar{w} - Re\,u_c\partial_r \bar{w} - Re\,w_c\partial_z \bar{w} \\ &+ ARe\,\breve{u}\partial_r \bar{w} + ARe\,\breve{w}\partial_z \bar{w} + ARe\,\bar{u}\partial_r \breve{w} + ARe\,\bar{w}\partial_z \breve{w} \\ &+ ARe\,\partial_t \breve{w} - ARe\,u_c\partial_r \breve{w} - ARe\,w_c\partial_z \breve{w} \\ &+ (A)^2\,Re\,\breve{u}\partial_r \breve{w} + (A)^2\,Re\,\breve{w}\partial_z \breve{w} \\ &- \boldsymbol{e}_z \cdot \nabla \cdot \bar{\boldsymbol{P}} - St\,\underbrace{\boldsymbol{e}_z \cdot \hat{\boldsymbol{g}}}_{\hat{\boldsymbol{g}}_z} = 0, \end{aligned} \tag{26.1}$$

and we define the i-th residuals of the z-momentum equation as

$$\begin{aligned} \bar{M}_i^z \;=& Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{u}\partial_r \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{w}\partial_z \bar{w} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{w} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{w} \\ &+Re\,A \int\limits_{\Omega^n} \phi_i \breve{u}\partial_r \bar{w} + Re\,A \int\limits_{\Omega^n} \phi_i \breve{w}\partial_z \bar{w} + Re\,A \int\limits_{\Omega^n} \phi_i \bar{u}\partial_r \breve{w} + Re\,A \int\limits_{\Omega^n} \phi_i \bar{w}\partial_z \breve{w} \\ &+Re\,A \int\limits_{\Omega^n} \phi_i \partial_t \breve{w} - Re\,A \int\limits_{\Omega^n} u_c \partial_r \breve{w} - Re\,A \int\limits_{\Omega^n} \phi_i w_c \partial_z \breve{w} \\ &+ Re\,(A)^2 \int\limits_{\Omega^n} \phi_i \breve{u}\partial_r \breve{w} + Re\,(A)^2 \int\limits_{\Omega^n} \phi_i \breve{w}\partial_z \breve{w} \\ &- St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z - \int\limits_{\Omega^n} \phi_i \boldsymbol{e}_z \cdot \nabla \cdot \bar{\boldsymbol{P}}, \end{aligned} \tag{26.2}$$

where $\bar{\Omega}$ in the domain in which we solve the modified Navier-Stokes equation.

We recall the tensor identity†

$$\nabla \cdot (\boldsymbol{x} \cdot \boldsymbol{Q}) = \boldsymbol{x} \cdot \nabla \cdot \boldsymbol{Q} + \nabla \boldsymbol{x} : \boldsymbol{Q}, \tag{26.3}$$

taking $\boldsymbol{x} = \phi_i \boldsymbol{e}_z$ and $\boldsymbol{Q} = \bar{\boldsymbol{P}}$ we have

$$-\phi_i \boldsymbol{e}_z \cdot \nabla \cdot \bar{\boldsymbol{P}} = -\nabla \cdot \left(\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}}\right) + \nabla\left(\phi_i \boldsymbol{e}_z\right) : \bar{\boldsymbol{P}}, \tag{26.4}$$

† In the case of Cartesian coordinate, the : symbol can be thought of just as the canonical inner product of matrices when used between two tensors of second order.

which reduces $\bar{M}_i^z$ to

$$\begin{aligned}\bar{M}_i^z &= Re\int_{\Omega^n}\phi_i\partial_t\bar{w} + Re\int_{\Omega^n}\phi_i\bar{u}\partial_r\bar{w} + Re\int_{\Omega^n}\phi_i\bar{w}\partial_z\bar{w} - Re\int_{\Omega^n}\phi_i u_c\partial_r\bar{w} - Re\int_{\Omega^n}\phi_i w_c\partial_z\bar{w}\\ &+Re\,A\int_{\Omega^n}\phi_i\breve{u}\partial_r\bar{w} + Re\,A\int_{\Omega^n}\phi_i\breve{w}\partial_z\bar{w} + Re\,A\int_{\Omega^n}\phi_i\bar{u}\partial_r\breve{w} + Re\,A\int_{\Omega^n}\phi_i\bar{w}\partial_z\breve{w}\\ &-Re\,A\int_{\Omega^n}\phi_i u_c\partial_r\breve{w} - Re\,A\int_{\Omega^n}\phi_i w_c\partial_z\breve{w} + Re\,A\int_{\Omega^n}\phi_i\partial_t\breve{w}\\ &+ Re\,(A)^2\int_{\Omega^n}\phi_i\breve{u}\partial_r\breve{w} + Re\,(A)^2\int_{\Omega^n}\phi_i\breve{w}\partial_z\breve{w}\\ &- St\int_{\Omega^n}\phi_i\hat{\boldsymbol{g}}_z + \int_{\Omega^n}\nabla\left(\phi_i\boldsymbol{e}_z\right)\,:\bar{\boldsymbol{P}} - \int_{\Omega^n}\nabla\cdot\left(\phi_i\boldsymbol{e}_z\cdot\bar{\boldsymbol{P}}\right),\end{aligned} \tag{26.5}$$

we can now apply the divergence theorem to the last term on the right hand side above to obtain

$$\begin{aligned}\bar{M}_i^z &= Re\int_{\Omega^n}\phi_i\partial_t\bar{w} + Re\int_{\Omega^n}\phi_i\bar{u}\partial_r\bar{w} + Re\int_{\Omega^n}\phi_i\bar{w}\partial_z\bar{w} - Re\int_{\Omega^n}\phi_i u_c\partial_r\bar{w} - Re\int_{\Omega^n}\phi_i w_c\partial_z\bar{w}\\ &+Re\,A\int_{\Omega^n}\phi_i\breve{u}\partial_r\bar{w} + Re\,A\int_{\Omega^n}\phi_i\breve{w}\partial_z\bar{w} + Re\,A\int_{\Omega^n}\phi_i\bar{u}\partial_r\breve{w} + Re\,A\int_{\Omega^n}\phi_i\bar{w}\partial_z\breve{w}\\ &-Re\,A\int_{\Omega^n}\phi_i u_c\partial_r\breve{w} - Re\,A\int_{\Omega^n}\phi_i w_c\partial_z\breve{w} + Re\,A\int_{\Omega^n}\phi_i\partial_t\breve{w}\\ &+ Re\,(A)^2\int_{\Omega^n}\phi_i\breve{u}\partial_r\breve{w} + Re\,(A)^2\int_{\Omega^n}\phi_i\breve{w}\partial_z\breve{w}\\ &- St\int_{\Omega^n}\phi_i\hat{\boldsymbol{g}}_z + \int_{\Omega^n}\nabla\left(\phi_i\boldsymbol{e}_z\right)\,:\bar{\boldsymbol{P}} + \int_{\partial\Omega^n}\phi_i\boldsymbol{e}_z\cdot\bar{\boldsymbol{P}}\cdot\boldsymbol{n},\end{aligned} \tag{26.6}$$

where $\partial\Omega^n$ is the boundary of Ω^n and $\boldsymbol{n}$ is its unit normal that points into $\bar{\Omega}$.

We notice that

$$\nabla(\phi_i\boldsymbol{e}_z):\bar{\boldsymbol{P}} = \begin{bmatrix} 0 & 0 \\ \partial_r\phi_i & \partial_z\phi_i \end{bmatrix} : \begin{bmatrix} \bar{P}_{rr} & \bar{P}_{rz} \\ \bar{P}_{zr} & \bar{P}_{zz} \end{bmatrix} \tag{26.7}$$

i.e.

$$\nabla(\phi_i\boldsymbol{e}_z):\bar{\boldsymbol{P}} = \begin{bmatrix} 0 & 0 \\ \partial_r\phi_i & \partial_z\phi_i \end{bmatrix} : \begin{bmatrix} -p+2\partial_r\bar{u} & \partial_z\bar{u}+\partial_r\bar{w} \\ \partial_r\bar{w}+\partial_z\bar{u} & -p+2\partial_z\bar{w} \end{bmatrix}, \tag{26.8}$$

which is

$$\nabla(\phi_i\boldsymbol{e}_r):\bar{\boldsymbol{P}} = \partial_r\phi_i\bar{P}_{zr} + \partial_z\phi_i\bar{P}_{zz} = \partial_r\bar{w}\partial_r\phi_i + \partial_z\bar{u}\partial_r\phi_i - p\partial_z\phi_i + 2\partial_z\bar{w}\partial_z\phi_i. \tag{26.9}$$

Therefore we have

$$
\begin{aligned}
\bar{M}_i^z = & \; Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{w} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{w} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{w} \\
& + Re\, A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \breve{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \breve{w} \\
& - Re\, A \int\limits_{\Omega^n} \phi_i u_c \partial_r \breve{w} - Re\, A \int\limits_{\Omega^n} \phi_i w_c \partial_z \breve{w} \\
& + Re\, A \int\limits_{\Omega^n} \phi_i \partial_t \breve{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{w} \\
& - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z - \int\limits_{\Omega^n} p \partial_z \phi_i + 2 \int\limits_{\Omega^n} \partial_z \bar{w} \partial_z \phi_i + \int\limits_{\Omega^n} \partial_z \bar{u} \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r \bar{w} \partial_r \phi_i + \int\limits_{\partial\Omega^n} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}.
\end{aligned}
\tag{26.10}
$$

We now consider the last integral on the right hand side of the equation above

$$
\int\limits_{\partial\bar{\Omega}} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n} = \int\limits_{\partial\Omega^{1,n}} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^1 + \int\limits_{\partial\Omega^{2,n}} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 + \int\limits_{\partial\Omega^5} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5, \tag{26.11}
$$

where $\partial\bar{\Omega}_1$ is the free surface, $\partial\bar{\Omega}_2$ is the solid surface, and $\partial\bar{\Omega}_4$ is the surface that separates the domain where the two different PDEs are solved numerically.

For the free-surface we have equation (23.15), which states

$$
\left(\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) \cdot \boldsymbol{n}^1 = -p^g \boldsymbol{n}^1 - \frac{\nabla^s \cdot \left[\sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)\right]}{Ca}. \tag{26.12}
$$

and therefore

$$
\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^1 = -p^g \phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^1 - \frac{1}{Ca} \phi_i \boldsymbol{e}_z \cdot \nabla^s \cdot \left[\sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)\right] - A \phi_i \boldsymbol{e}_z \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^1. \tag{26.13}
$$

Now, we have the following surface vector calculus identity

$$
\nabla^s \cdot (\boldsymbol{x} \cdot \boldsymbol{Q}) = \boldsymbol{Q} : \nabla^s \boldsymbol{x} + \boldsymbol{x} \cdot \nabla^s \cdot \boldsymbol{Q}, \tag{26.14}
$$

and taking $\boldsymbol{x} = \phi_i \boldsymbol{e}_z$ and $\boldsymbol{Q} = \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)$, we have

$$
\nabla^s \cdot (\phi_i \boldsymbol{e}_z \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)) = \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) : \nabla^s (\phi_i \boldsymbol{e}_r) + \phi_i \boldsymbol{e}_z \cdot \nabla^s \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) \tag{26.15}
$$

which yields

$$
\phi_i \boldsymbol{e}_z \cdot \nabla^s \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) = \nabla^s \cdot (\phi_i \boldsymbol{e}_z \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1)) - \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) : \nabla^s \phi_i \boldsymbol{e}_z. \tag{26.16}
$$

In this $1D$-surface case, we have

$$
\nabla^s \phi_i \boldsymbol{e}_z = \begin{bmatrix} 0 & t_r^1 \partial_s \phi_i \\ 0 & t_z^1 \partial_s \phi_i \end{bmatrix}, \tag{26.17}
$$

where $\boldsymbol{t}^1 = (t_r^1, t_z^1)$, and the tangent vector must be pointing in the direction of increasing arclength s, therefore

$$
(\boldsymbol{I} - \boldsymbol{n}^1 \boldsymbol{n}^1) : \nabla^s \phi_i \boldsymbol{e}_z = \begin{bmatrix} 1 - n_r^1 n_r^1 & -n_r^1 n_z^1 \\ -n_z^1 n_r^1 & 1 - n_z^1 n_z^1 \end{bmatrix} : \begin{bmatrix} 0 & t_r^1 \partial_s \phi_i \\ 0 & t_z^1 \partial_s \phi_i \end{bmatrix}, \tag{26.18}
$$

where $\boldsymbol{n}^1 = (n_r^1, n_z^1)$, i.e.

$$(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) : \nabla^s \phi_i \boldsymbol{e}_z = t_z^1 \partial_s \phi_i - (\boldsymbol{t}^1 \cdot \boldsymbol{n}^1) n_z^1 \partial_s \phi_i = t_z^1 \partial_s \phi_i. \tag{26.19}$$

We therefore have in equation (26.16)

$$\phi_i \boldsymbol{e}_z \cdot \nabla^s \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1) = \nabla^s \cdot (\phi_i \boldsymbol{e}_z \cdot \sigma^1 (\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)) - \sigma^1 t_z^1 \partial_s \phi_i. \tag{26.20}$$

Taking this result into (26.11) and the result of that into the i-th z-momentum residual equation (26.10) we have

$$\begin{aligned}
\bar{M}_i^z &= Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{w} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{w} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \check{u} \partial_r \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \check{w} \partial_z \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \check{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \check{w} \\
&- \int\limits_{\Omega^n} p \partial_z \phi_i + 2 \int\limits_{\Omega^n} \partial_z \bar{w} \partial_z \phi_i + \int\limits_{\Omega^n} \partial_z \bar{u} \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r \bar{w} \partial_r \phi_i \\
&- Re\, A \int\limits_{\Omega^n} \phi_i u_c \partial_r \check{w} - Re\, A \int\limits_{\Omega^n} \phi_i w_c \partial_z \check{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \partial_t \check{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \check{u} \partial_r \check{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \check{w} \partial_z \check{w} - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z \\
&- \int\limits_{\partial\Omega^{1,n}} p^g \phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^1 - \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} \nabla^s \cdot (\sigma^1 \phi_i \boldsymbol{e}_z \cdot (\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1)) + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_z^1 \sigma^1 \partial_s \phi_i \\
&- A \int\limits_{\partial\Omega^{1,n}} \phi_i \boldsymbol{e}_z \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^1 + \int\limits_{\partial\Omega^{2,n}} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 + \int\limits_{\partial\Omega^5} \phi_i \boldsymbol{e}_r \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5.
\end{aligned} \tag{26.21}$$

Using the surface divergence theorem and the definition of the surface divergence for a

$1D$ surface, we have

$$
\begin{aligned}
\bar{M}_i^z &= Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{w} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{w} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \breve{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \breve{w} \\
&- \int\limits_{\Omega^n} p \partial_z \phi_i + 2 \int\limits_{\Omega^n} \partial_z \bar{w} \partial_z \phi_i + \int\limits_{\Omega^n} \partial_z \bar{u} \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r \bar{w} \partial_r \phi_i \\
&- Re\, A \int\limits_{\Omega^n} \phi_i u_c \partial_r \breve{w} - Re\, A \int\limits_{\Omega^n} \phi_i w_c \partial_z \breve{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \partial_t \breve{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{w} - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z \\
&+ \frac{1}{Ca} \int\limits_{\bar{C}_1} \sigma^1 \phi_i \boldsymbol{e}_z \cdot \boldsymbol{m}^1 + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_z^1 \sigma^1 \partial_s \phi_i \\
&\quad - A \int\limits_{\partial\Omega^{1,n}} \phi_i \boldsymbol{e}_z \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^1 + \int\limits_{\partial\Omega^{2,n}} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 + \int\limits_{\partial\Omega^5} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^4,
\end{aligned}
\tag{26.22}
$$

where $\bar{C}_1$ is actually the two points bounding the free surface, and $\boldsymbol{m}^1$ is the vector that is tangent to the free surface, normal to the contact line and points into the free surface. We can thus reduce the expression above to

$$
\begin{aligned}
\bar{M}_i^z &= Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{w} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{w} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \breve{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \breve{w} \\
&- \int\limits_{\Omega^n} p \partial_z \phi_i + 2 \int\limits_{\Omega^n} \partial_z \bar{w} \partial_z \phi_i + \int\limits_{\Omega^n} \partial_z \bar{u} \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r \bar{w} \partial_r \phi_i \\
&- Re\, A \int\limits_{\Omega^n} \phi_i u_c \partial_r \breve{w} - Re\, A \int\limits_{\Omega^n} \phi_i w_c \partial_z \breve{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \partial_t \breve{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{w} - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z \\
&\quad - \int\limits_{\partial\Omega^{1,n}} p^g \phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^1 + \frac{\sigma^1(r_c, z_c)\phi_i(r_c, z_c) m_z^1(r_c, z_c)}{Ca} + \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1})}{Ca} \\
&\quad + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_z^1 \sigma^1 \partial_s \phi_i - A \int\limits_{\partial\Omega^{1,n}} \phi_i \boldsymbol{e}_z \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^1 \\
&\quad + \int\limits_{\partial\Omega^{2,n}} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 + \int\limits_{\partial\Omega^5} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^4,
\end{aligned}
\tag{26.23}
$$

where (r_c, z_c) is the location of the contact line and (r_d, z_d) are the coordinates of the inflow end of the free surface.

We consider now the integrand in the third to last term above

$$\phi_i \boldsymbol{e}_z \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^1 = \begin{bmatrix} 0 \\ \phi_i \end{bmatrix} \cdot \begin{bmatrix} 2\partial_r \check{u} & \partial_z \check{u} + \partial_r \check{w} \\ \partial_r \check{w} + \partial_z \check{u} & 2\partial_z \check{w} \end{bmatrix} \cdot \begin{bmatrix} n_r^1 \\ n_z^1 \end{bmatrix}, \tag{26.24}$$

i.e.

$$\phi_i \boldsymbol{e}_z \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^1 = \begin{bmatrix} 0 \\ \phi_i \end{bmatrix} \cdot \begin{bmatrix} 2n_r^1 \partial_r \check{u} + n_z^1 (\partial_z \check{u} + \partial_r \check{w}) \\ n_r^1 (\partial_r \check{w} + \partial_z \check{u}) + 2n_z^1 \partial_z \check{w} \end{bmatrix}. \tag{26.25}$$

Hence

$$\phi_i \boldsymbol{e}_r \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^1 = \phi_i n_r^1 \partial_r \check{w} + \phi_i n_r^1 \partial_z \check{u} + 2\phi_i n_z^1 \partial_z \check{w}. \tag{26.26}$$

The expressions for $\check{u}$, $\check{w}$ and its derivatives above, are to be obtained form the expressions in (B 1), (B 2), (**??**), (**??**), (**??**) and (**??**). Having these explicit expressions, we can take (26.26) into (26.23), yielding

$$\begin{aligned}
\bar{M}_i^z &= Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{w} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{w} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \check{u} \partial_r \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \check{w} \partial_z \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \check{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \check{w} \\
&- \int\limits_{\Omega^n} p \partial_z \phi_i + 2 \int\limits_{\Omega^n} \partial_z \bar{w} \partial_z \phi_i + \int\limits_{\Omega^n} \partial_z \bar{u} \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r \bar{w} \partial_r \phi_i \\
&- Re\, A \int\limits_{\Omega^n} \phi_i u_c \partial_r \check{w} - Re\, A \int\limits_{\Omega^n} \phi_i w_c \partial_z \check{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \partial_t \check{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \check{u} \partial_r \check{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \check{w} \partial_z \check{w} - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z \\
&- \int\limits_{\partial\Omega^{1,n}} p^g \phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^1 + \frac{\sigma^1(r_c, z_c) \phi_i(r_c, z_c) m_z^1(r_c, z_c)}{Ca} + \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1})}{Ca} \\
&+ \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_z^1 \sigma^1 \partial_s \phi_i - 2A \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_z \check{w} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_z \check{u} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \check{w} \\
&+ \int\limits_{\partial\Omega^{2,n}} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 + \int\limits_{\partial\Omega^5} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5,
\end{aligned} \tag{26.27}$$

We consider now the term

$$\int\limits_{\partial\Omega^2} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2, \tag{26.28}$$

where we have

$$\phi_i \boldsymbol{e}_z \cdot (\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}) \cdot \boldsymbol{n}^2 = \phi_i \boldsymbol{e}_z \cdot \underbrace{\boldsymbol{n}^2 \cdot (\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}) \cdot (\boldsymbol{I} - \boldsymbol{n}^2 \boldsymbol{n}^2)}_{Be\, (\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{u}^s) \cdot (\boldsymbol{I} - \boldsymbol{n}^2 \boldsymbol{n}^2) - \frac{1}{2Ca} \nabla^s \sigma^2} + \phi_i \boldsymbol{e}_z \cdot \underbrace{\left(\boldsymbol{n}^2 \cdot (\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}) \cdot \boldsymbol{n}^2\right)}_{\lambda^2} \boldsymbol{n}^2, \tag{26.29}$$

where we have used equations (23.21) and variable λ^2, i.e. the normal stress on boundary 2.

Hence, we have

$$\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = \phi_i \boldsymbol{e}_z \cdot Be\left(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{u}^s\right) \cdot \left(\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2\right) - \frac{1}{2Ca}\phi_i \boldsymbol{e}_z \cdot \nabla^s \sigma^2 + \phi_i \boldsymbol{e}_z \cdot \lambda^2 \boldsymbol{n}^2 - A\phi_i \boldsymbol{e}_z \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2, \tag{26.30}$$

i.e.

$$\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = Be\,\phi_i \boldsymbol{e}_z \cdot \left(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{u}^s\right) \cdot \left(\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2\right) - \frac{1}{2Ca}\phi_i(\partial_s\sigma^2)\boldsymbol{e}_z \cdot \boldsymbol{t}^2 + \lambda^2\phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^2 - A\phi_i \boldsymbol{e}_z \cdot \left[\left(\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2\right)\boldsymbol{n}^2 + \boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \left(\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2\right)\right], \tag{26.31}$$

where we have used that $\nabla^s\sigma^2 = \partial_s\sigma^2\boldsymbol{t}^2$, with s being the arc-length variable and $\boldsymbol{t}^2$ the unit tangent to the surface which points in the direction of increasing s. We notice that the term $\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot (\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2)$ is zero when the solid surface is flat, which follows from the no-tangential-stress condition for the eigen-solution, i.e. equation (**??**); however, for a generic curved solid surface it is not zero.

Re-writing we have

$$\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = Be\,\phi_i \boldsymbol{e}_z \cdot \left(\left(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{u}^s\right) \cdot \boldsymbol{t}^2\right)\boldsymbol{t}^2 - \frac{1}{2Ca}\phi_i t_z^2 \partial_s\sigma^2 + \lambda^2\phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^2 - A\phi_i \boldsymbol{e}_z \cdot \left(\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2\right)\boldsymbol{n}^2 - A\phi_i \boldsymbol{e}_z \cdot \left(\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{t}^2\right)\boldsymbol{t}^2, \tag{26.32}$$

i.e.

$$\begin{aligned}\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = {} & Be\,\phi_i \boldsymbol{e}_z \cdot \left(\bar{\boldsymbol{u}} \cdot \boldsymbol{t}^2\right)\boldsymbol{t}^2 + Be\,A\phi_i \boldsymbol{e}_z \cdot \left(\check{\boldsymbol{u}} \cdot \boldsymbol{t}^2\right)\boldsymbol{t}^2 \\ & - Be\,\phi_i \boldsymbol{e}_z \cdot \left(\boldsymbol{u}^s \cdot \boldsymbol{t}^2\right)\boldsymbol{t}^2 - \frac{1}{2Ca}\phi_i t_z^2 \partial_s\sigma^2 + \lambda^2\phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^2 \\ & - A\phi_i \boldsymbol{e}_z \cdot \left(\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2\right)\boldsymbol{n}^2 - A\phi_i \left(\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{t}^2\right)\boldsymbol{e}_z \cdot \boldsymbol{t}^2.\end{aligned} \tag{26.33}$$

Expanding the innermost products in the equations above we have

$$\begin{aligned}\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = {} & Be\,\phi_i \left(\bar{u}t_r^2 + \bar{w}t_z^2\right)\boldsymbol{e}_z \cdot \boldsymbol{t}^2 + Be\,A\phi_i \left(\check{u}t_r^2 + \check{w}t_z^2\right)\boldsymbol{e}_z \cdot \boldsymbol{t}^2 \\ & - Be\,\phi_i \left(u^s t_r^2 + w^s t_z^2\right)\boldsymbol{e}_z \cdot \boldsymbol{t}^2 - \frac{1}{2Ca}\phi_i t_z^2 \partial_s\sigma^2 + \lambda^2\phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^2 \\ & - A\phi_i \left(\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2\right)\boldsymbol{e}_z \cdot \boldsymbol{n}^2 - A\phi_i \left(\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{t}^2\right)t_z^2,\end{aligned} \tag{26.34}$$

and performing one further product we have

$$\begin{aligned}\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = {} & Be\,\phi_i \bar{u}t_r^2 t_z^2 + Be\,\phi_i \bar{w}t_z^2 t_z^2 + Be\,A\phi_i \check{u}t_r^2 t_z^2 + Be\,A\phi_i \check{w}t_z^2 t_z^2 \\ & - Be\,\phi_i u^s t_r^2 t_z^2 - Be\,\phi_i w^s t_z^2 t_z^2 - \frac{1}{2Ca}\phi_i t_z^2 \partial_s\sigma^2 + \lambda^2\phi_i n_z^2 \\ & - A\phi_i \left(\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2\right)n_z^2 - A\phi_i \left(\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{t}^2\right)t_z^2.\end{aligned} \tag{26.35}$$

We consider now the term

$$\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = \begin{bmatrix} n_r^2 \\ n_z^2 \end{bmatrix} \cdot \begin{bmatrix} 2\partial_r\check{u} & \partial_z\check{u} + \partial_r\check{w} \\ \partial_r\check{w} + \partial_z\check{u} & 2\partial_z\check{w} \end{bmatrix} \cdot \begin{bmatrix} n_r^2 \\ n_z^2 \end{bmatrix}, \tag{26.36}$$

in the equation above. Expanding we have

$$\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = 2n_r^2 n_r^2 \partial_r\check{u} + 2n_r^2 n_z^2 \partial_z\check{u} + 2n_r^2 n_z^2 \partial_r\check{w} + 2n_z^2 n_z^2 \partial_z\check{w}. \tag{26.37}$$

Taking this result back into equation (26.35) we have

$$\begin{aligned}\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 = & Be\, \phi_i \bar{u} t_r^2 t_z^2 + Be\, \phi_i \bar{w} t_z^2 t_z^2 + Be\, A\phi_i \check{u} t_r^2 t_z^2 + Be\, A\phi_i \check{w} t_z^2 t_z^2 \\ & - Be\, \phi_i u^s t_r^2 t_z^2 - Be\, \phi_i w^s t_z^2 t_z^2 - \frac{1}{2Ca} \phi_i t_z^2 \partial_s \sigma^2 + \lambda^2 \phi_i n_z^2 \\ & - 2A\phi_i n_r^2 n_r^2 n_z^2 \partial_r \check{u} - 2A\phi_i n_r^2 n_z^2 n_z^2 \partial_z \check{u} - 2A\phi_i n_r^2 n_z^2 n_z^2 \partial_r \check{w} \\ & - 2A\phi_i n_z^2 n_z^2 n_z^2 \partial_z \check{w} - A\phi_i \left(\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{t}^2 \right) t_z^2. \end{aligned} \tag{26.38}$$

We consider now the term

$$\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{t}^2 = \begin{bmatrix} n_r^2 \\ n_z^2 \end{bmatrix} \cdot \begin{bmatrix} 2\partial_r \check{u} & \partial_z \check{u} + \partial_r \check{w} \\ \partial_r \check{w} + \partial_z \check{u} & 2\partial_z \check{w} \end{bmatrix} \cdot \begin{bmatrix} t_r^2 \\ t_z^2 \end{bmatrix}, \tag{26.39}$$

in the equation above. Expanding we have

$$\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{t}^2 = \begin{bmatrix} n_r^2 \\ n_z^2 \end{bmatrix} \cdot \begin{bmatrix} 2t_r^2 \partial_r \check{u} + t_z^2 \partial_z \check{u} + t_z^2 \partial_r \check{w} \\ t_r^2 \partial_r \check{w} + t_r^2 \partial_z \check{u} + 2t_z^2 \partial_z \check{w} \end{bmatrix}, \tag{26.40}$$

i.e.

$$\boldsymbol{n}^2 \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{t}^2 = 2t_r^2 n_r^2 \partial_r \check{u} + t_r^2 n_z^2 \partial_z \check{u} + t_z^2 n_r^2 \partial_z \check{u} + t_z^2 n_r^2 \partial_r \check{w} + t_r^2 n_z^2 \partial_r \check{w} + 2t_z^2 n_z^2 \partial_z \check{w}. \tag{26.41}$$

Taking this result back into equation (26.38) we have

$$\begin{aligned}\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^2 \; = & Be\, \phi_i \bar{u} t_r^2 t_z^2 + Be\, \phi_i \bar{w} t_z^2 t_z^2 - Be\, \phi_i u^s t_r^2 t_z^2 - Be\, \phi_i w^s t_z^2 t_z^2 - \frac{1}{2Ca} \phi_i t_z^2 \partial_s \sigma^2 \\ & + \lambda^2 \phi_i n_z^2 \\ & + Be\, A\phi_i \check{u} t_r^2 t_z^2 + Be\, A\phi_i \check{w} t_z^2 t_z^2 \\ & - 2A\phi_i n_r^2 n_r^2 n_z^2 \partial_r \check{u} - 2A\phi_i n_r^2 n_z^2 n_z^2 \partial_z \check{u} - 2A\phi_i n_r^2 n_z^2 n_z^2 \partial_r \check{w} - 2A\phi_i n_z^2 n_z^2 n_z^2 \partial_z \check{w} \\ & - 2A\phi_i t_r^2 t_z^2 n_r^2 \partial_r \check{u} - A\phi_i t_r^2 t_z^2 n_z^2 \partial_z \check{u} - A\phi_i t_z^2 t_z^2 n_r^2 \partial_z \check{u} \\ & - A\phi_i t_z^2 t_z^2 n_r^2 \partial_r \check{w} - A\phi_i t_r^2 t_z^2 n_z^2 \partial_r \check{w} - 2A\phi_i t_z^2 t_z^2 n_z^2 \partial_z \check{w}. \end{aligned} \tag{26.42}$$

Taking the result above into equation (26.23) we have

$$
\begin{aligned}
\bar{M}_i^z \; =& \; Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{w} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{w} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \breve{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \breve{w} \\
&- \int\limits_{\Omega^n} p \partial_z \phi_i + 2 \int\limits_{\Omega^n} \partial_z \bar{w} \partial_z \phi_i + \int\limits_{\Omega^n} \partial_z \bar{u} \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r \bar{w} \partial_r \phi_i \\
&- Re\, A \int\limits_{\Omega^n} \phi_i u_c \partial_r \breve{w} - Re\, A \int\limits_{\Omega^n} \phi_i w_c \partial_z \breve{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \partial_t \breve{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{w} - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z \\
&- \int\limits_{\partial\Omega^{1,n}} p^g \phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^1 + \frac{\sigma^1(r_c, z_c) \phi_i(r_c, z_c) m_z^1(r_c, z_c)}{Ca} \\
&+ \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1})}{Ca} \\
&+ \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_z^1 \sigma^1 \partial_s \phi_i - 2A \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_z \breve{w} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_z \breve{u} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \breve{w} \\
&+ Be \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{u} t_r^2 t_z^2 + Be \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{w} t_z^2 t_z^2 - Be \int\limits_{\partial\Omega^{2,n}} \phi_i u^s t_r^2 t_z^2 \\
&- Be \int\limits_{\partial\Omega^{2,n}} \phi_i w^s t_z^2 t_z^2 - \frac{1}{2Ca} \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 \partial_s \sigma^2 + \int\limits_{\partial\Omega^{2,n}} \lambda^2 \phi_i n_z^2 \\
&+ Be\, A \int\limits_{\partial\Omega^{2,n}} \phi_i \breve{u} t_r^2 t_z^2 + Be\, A \int\limits_{\partial\Omega^{2,n}} \phi_i \breve{w} t_z^2 t_z^2 - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_r^2 n_z^2 \partial_r \breve{u} \\
&- 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_z \breve{u} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_r \breve{w} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 n_z^2 n_z^2 \partial_z \breve{w} \\
&- 2A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_r^2 \partial_r \breve{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_z \breve{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_z \breve{u} \\
&- A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_r \breve{w} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_r \breve{w} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_z^2 \partial_z \breve{w} + \int\limits_{\partial\Omega^5} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5.
\end{aligned}
\tag{26.43}
$$

We consider now the term

$$
\int\limits_{\partial\Omega^5} \phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5, \tag{26.44}
$$

where we have

$$
\phi_i \boldsymbol{e}_z \cdot \left(\bar{\boldsymbol{P}} + A \breve{\boldsymbol{P}} \right) \cdot \boldsymbol{n}^5 = \phi_i \boldsymbol{e}_z \cdot \boldsymbol{P} \cdot \boldsymbol{n}^5, \tag{26.45}
$$

i.e.

$$\phi_i \boldsymbol{e}_z \cdot \left(\bar{\boldsymbol{P}} + A\check{\boldsymbol{P}}\right) \cdot \boldsymbol{n}^5 = \phi_i \boldsymbol{e}_z \cdot \underbrace{\left(\boldsymbol{n}^5 \cdot \boldsymbol{P} \cdot \boldsymbol{n}^5\right)}_{\lambda^5} \boldsymbol{n}^5 + \phi_i \boldsymbol{e}_z \cdot \underbrace{\left(\boldsymbol{n}^5 \cdot \boldsymbol{P} \cdot (\boldsymbol{I} - \boldsymbol{n}^5\boldsymbol{n}^5)\right)}_{\gamma^5 \boldsymbol{t}^5}, \quad (26.46)$$

hence we have

$$\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5 = \phi_i \lambda^5 \boldsymbol{e}_z \cdot \boldsymbol{n}^5 + \phi_i \gamma^5 \boldsymbol{e}_z \cdot \boldsymbol{t}^5 - A\phi_i \boldsymbol{e}_z \cdot \check{\boldsymbol{P}} \cdot \boldsymbol{n}^5, \quad (26.47)$$

i.e.

$$\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5 \;= \phi_i \lambda^5 n_z^4 + \phi_i \gamma^5 t_z^5 \quad (26.48)$$

$$+A \begin{bmatrix} 0 \\ \phi_i \end{bmatrix} \cdot \begin{bmatrix} -2\partial_r \check{u} & -\partial_r \check{w} - \partial_z \check{u} \\ -\partial_z \check{u} - \partial_r \check{w} & -2\partial_z \check{w} \end{bmatrix} \cdot \begin{bmatrix} n_r^5 \\ n_z^5 \end{bmatrix},$$

or equivalently

$$\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5 = \phi_i \lambda^5 n_z^5 + \phi_i \gamma^5 t_z^5 + A \begin{bmatrix} 0 \\ \phi_i \end{bmatrix} \cdot \begin{bmatrix} -2n_r^5 \partial_r \check{u} - n_z^5 \partial_r \check{w} - n_z^5 \partial_z \check{u} \\ -n_r^5 \partial_z \check{u} - n_r^5 \partial_r \check{w} - 2n_z^5 \partial_z \check{w} \end{bmatrix}, \quad (26.49)$$

which yields

$$\phi_i \boldsymbol{e}_z \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{n}^5 = \phi_i \lambda^5 n_z^5 + \phi_i \gamma^5 t_z^5 - A n_r^5 \phi_i \partial_z \check{u} - A n_r^5 \phi_i \partial_r \check{w} - 2A n_z^5 \phi_i \partial_z \check{w}. \quad (26.50)$$

Taking this result into equation (26.43) we have

$$
\begin{aligned}
\bar{M}_i^z \;&= Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{w} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{w} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \breve{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \breve{w} \\
&- \int\limits_{\Omega^n} p \partial_z \phi_i + 2 \int\limits_{\Omega^n} \partial_z \bar{w} \partial_z \phi_i + \int\limits_{\Omega^n} \partial_z \bar{u} \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r \bar{w} \partial_r \phi_i \\
&- Re\, A \int\limits_{\Omega^n} \phi_i u_c \partial_r \breve{w} - Re\, A \int\limits_{\Omega^n} \phi_i w_c \partial_z \breve{w} \\
&+ Re\, A \int\limits_{\Omega^n} \phi_i \partial_t \breve{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{w} - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z \\
&- \int\limits_{\partial\Omega^{1,n}} p^g \phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^1 + \frac{\sigma^1(r_c, z_c) \phi_i(r_c, z_c) m_z^1(r_c, z_c)}{Ca} \\
&+ \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1})}{Ca} + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_z^1 \sigma^1 \partial_s \phi_i \\
&- 2A \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_z \breve{w} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_z \breve{u} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \breve{w} \\
&+ Be \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{u} t_r^2 t_z^2 + Be \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{w} t_z^2 t_z^2 - Be \int\limits_{\partial\Omega^{2,n}} \phi_i u^s t_r^2 t_z^2 \\
&- Be \int\limits_{\partial\Omega^{2,n}} \phi_i w^s t_z^2 t_z^2 - \frac{1}{2Ca} \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 \partial_s \sigma^2 + \int\limits_{\partial\Omega^{2,n}} \lambda^2 \phi_i n_z^2 \\
&+ Be\, A \int\limits_{\partial\Omega^{2,n}} \phi_i \breve{u} t_r^2 t_z^2 + Be\, A \int\limits_{\partial\Omega^{2,n}} \phi_i \breve{w} t_z^2 t_z^2 - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_r^2 n_z^2 \partial_r \breve{u} \\
&- 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_z \breve{u} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_r \breve{w} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 n_z^2 n_z^2 \partial_z \breve{w} \\
&- 2A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_r^2 \partial_r \breve{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_z \breve{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_z \breve{u} \\
&- A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_r \breve{w} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_r \breve{w} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_z^2 \partial_z \breve{w} \\
&+ \int\limits_{\partial\Omega^5} \phi_i \lambda^4 n_z^4 + \int\limits_{\partial\Omega^5} \phi_i \gamma^4 t_z^4 - 2A \int\limits_{\partial\Omega^5} n_z^4 \phi_i \partial_z \breve{w} - A \int\limits_{\partial\Omega^5} n_r^4 \phi_i \partial_z \breve{u} - A \int\limits_{\partial\Omega^5} n_r^4 \phi_i \partial_r \breve{w}.
\end{aligned}
\tag{26.51}
$$

Given the size of the equation above, it is more convenient to introduce the following equality

$$\bar{M}_i^z = \bar{M}_i^{z,0} + \bar{M}_i^{z,1} + \bar{M}_i^{z,2} + \bar{M}_i^{z,4}, \tag{26.52}$$

where

$$
\begin{aligned}
\bar{M}_i^{z,0} &= Re \int\limits_{\Omega^n} \phi_i \partial_t \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{w} + Re \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{w} - Re \int\limits_{\Omega^n} \phi_i u_c \partial_r \bar{w} - Re \int\limits_{\Omega^n} \phi_i w_c \partial_z \bar{w} \\
&+Re\, A \int\limits_{\Omega^n} \phi_i \check{u} \partial_r \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \check{w} \partial_z \bar{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \check{w} + Re\, A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \check{w} \\
&- \int\limits_{\Omega^n} p \partial_z \phi_i + 2 \int\limits_{\Omega^n} \partial_z \bar{w} \partial_z \phi_i + \int\limits_{\Omega^n} \partial_z \bar{u} \partial_r \phi_i + \int\limits_{\Omega^n} \partial_r \bar{w} \partial_r \phi_i \\
&-Re\, A \int\limits_{\Omega^n} \phi_i u_c \partial_r \check{w} - Re\, A \int\limits_{\Omega^n} \phi_i w_c \partial_z \check{w} \\
&+Re\, A \int\limits_{\Omega^n} \phi_i \partial_t \check{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \check{u} \partial_r \check{w} + Re\, (A)^2 \int\limits_{\Omega^n} \phi_i \check{w} \partial_z \check{w} - St \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z,
\end{aligned}
\tag{26.53}
$$

$$
\begin{aligned}
\bar{M}_i^{z,1} &= \frac{\sigma^1(r_c, z_c)\phi_i(r_c, z_c) m_z^1(r_c, z_c)}{Ca} + \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1})}{Ca} \\
&- \int\limits_{\partial\Omega^{1,n}} p^g \phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^1 + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_z^1 \sigma^1 \partial_s \phi_i \\
&-2A \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_z \check{w} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_z \check{u} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \check{w},
\end{aligned}
\tag{26.54}
$$

$$
\begin{aligned}
\bar{M}_i^{z,2} &= Be \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{u} t_r^2 t_z^2 + Be \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{w} t_z^2 t_z^2 - Be \int\limits_{\partial\Omega^{2,n}} \phi_i u^s t_r^2 t_z^2 - Be \int\limits_{\partial\Omega^{2,n}} \phi_i w^s t_z^2 t_z^2 \\
&- \frac{1}{2Ca} \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 \partial_s \sigma^2 + \int\limits_{\partial\Omega^{2,n}} \lambda^2 \phi_i n_z^2 \\
&+ Be\, A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{u} t_r^2 t_z^2 + Be\, A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{w} t_z^2 t_z^2 \\
&- 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_r^2 n_z^2 \partial_r \check{u} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_z \check{u} \\
&- 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_r \check{w} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 n_z^2 n_z^2 \partial_z \check{w} \\
&- 2A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_r^2 \partial_r \check{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_z \check{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_z \check{u} \\
&- A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_r \check{w} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_r \check{w} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_z^2 \partial_z \check{w},
\end{aligned}
\tag{26.55}
$$

and

$$
\bar{M}_i^{z,5} = \int\limits_{\partial\Omega^5} \phi_i \lambda^5 n_z^5 + \int\limits_{\partial\Omega^5} \phi_i \gamma^5 t_z^5 - 2A \int\limits_{\partial\Omega^5} n_z^5 \phi_i \partial_z \check{w} - A \int\limits_{\partial\Omega^5} n_r^5 \phi_i \partial_z \check{u} - A \int\limits_{\partial\Omega^5} n_r^5 \phi_i \partial_r \check{w}.
\tag{26.56}
$$

We now recall approximations (25.74), (??), (25.76), (25.77), (25.78), (25.79),

$$u^c(t_n) = \partial_t r^c(t_n) \approx \frac{3r^c(t_n) - 4r^c(t_{n-1}) + r^c(t_{n-2})}{2\Delta_t}, \tag{26.57}$$

$$w^c(t_n) = \partial_t z^c(t_n) \approx \frac{3z^c(t_n) - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t}, \tag{26.58}$$

$$\partial_t \bar{u}(t_n) \approx \frac{3\bar{u}(t_n) - 4\bar{u}(t_{n-1}) + \bar{u}(t_{n-2})}{2\Delta_t}, \tag{26.59}$$

$$\partial_t \bar{w}(t_n) \approx \frac{3\bar{w}(t_n) - 4\bar{w}(t_{n-1}) + \bar{w}(t_{n-2})}{2\Delta_t}, \tag{26.60}$$

$$\partial_t \check{u}(t_n) \approx \frac{3\check{u}(t_n) - 4\check{u}(t_{n-1}) + \check{u}(t_{n-2})}{2\Delta_t}, \tag{26.61}$$

$$\partial_t \check{w}(t_n) \approx \frac{3\check{w}(t_n) - 4\check{w}(t_{n-1}) + \check{w}(t_{n-2})}{2\Delta_t}. \tag{26.62}$$

Substituting these into (26.52) and (26.53)-(26.56), though the last three remain unaltered, we obtain

$$\bar{M}_i^z \approx \bar{\mathfrak{M}}_i^z := \bar{\mathfrak{M}}_i^{z,0} + \bar{\mathfrak{M}}_i^{z,1} + \bar{\mathfrak{M}}_i^{z,2} + \bar{\mathfrak{M}}_i^{z,4}, \tag{26.63}$$

where

$$\begin{aligned}
\bar{\mathfrak{M}}_i^{z,0} :=\ & Re \int_{\Omega^n} \phi_i \frac{3\bar{w}(t_n) - 4\bar{w}(t_{n-1}) + \bar{w}(t_{n-2})}{2\Delta_t} + Re \int_{\Omega^n} \phi_i \bar{u} \partial_r \bar{w} + Re \int_{\Omega^n} \phi_i \bar{w} \partial_z \bar{w} \qquad (26.64) \\
& - Re \int_{\Omega^n} \phi_i \frac{3r^c(t_n) - 4r^c(t_{n-1}) + r^c(t_{n-2})}{2\Delta_t} \partial_r \bar{w} - Re \int_{\Omega^n} \phi_i \frac{3z^c(t_n) - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t} \partial_z \bar{w} \\
& + Re\, A \int_{\Omega^n} \phi_i \check{u} \partial_r \bar{w} + Re\, A \int_{\Omega^n} \phi_i \check{w} \partial_z \bar{w} + Re\, A \int_{\Omega^n} \phi_i \bar{u} \partial_r \check{w} + Re\, A \int_{\Omega^n} \phi_i \bar{w} \partial_z \check{w} \\
& - \int_{\Omega^n} p \partial_z \phi_i + 2 \int_{\Omega^n} \partial_z \bar{w} \partial_z \phi_i + \int_{\Omega^n} \partial_z \bar{u} \partial_r \phi_i + \int_{\Omega^n} \partial_r \bar{w} \partial_r \phi_i \\
& - Re\, A \int_{\Omega^n} \phi_i \frac{3r^c(t_n) - 4r^c(t_{n-1}) + r^c(t_{n-2})}{2\Delta_t} \partial_r \check{w} - Re\, A \int_{\Omega^n} \phi_i \frac{3z^c(t_n) - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t} \partial_z \check{w} \\
& + Re\, A \int_{\Omega^n} \phi_i \frac{3\check{w}(t_n) - 4\check{w}(t_{n-1}) + \check{w}(t_{n-2})}{2\Delta_t} + Re\, (A)^2 \int_{\Omega^n} \phi_i \check{u} \partial_r \check{w} + Re\, (A)^2 \int_{\Omega^n} \phi_i \check{w} \partial_z \check{w} \\
& - St \int_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z,
\end{aligned}$$

$$\bar{\mathfrak{M}}_i^{z,1} := \frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_z^1(r_c,z_c)}{Ca} + \frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^1(r_{J^1},z_{J^1})}{Ca}$$
$$- \int\limits_{\partial\Omega^{1,n}} p^g \phi_i \boldsymbol{e}_z \cdot \boldsymbol{n}^1 + \frac{1}{Ca} \int\limits_{\partial\Omega^{1,n}} t_z^1 \sigma^1 \partial_s \phi_i \tag{26.65}$$
$$-2A \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_z \breve{w} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_z \breve{u} - A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \breve{w},$$

$$\bar{\mathfrak{M}}_i^{z,2} := Be \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{u} t_r^2 t_z^2 + Be \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{w} t_z^2 t_z^2 - Be \int\limits_{\partial\Omega^{2,n}} \phi_i u^s t_r^2 t_z^2 - Be \int\limits_{\partial\Omega^{2,n}} \phi_i w^s t_z^2 t_z^2$$
$$- \frac{1}{2Ca} \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 \partial_s \sigma^2 + \int\limits_{\partial\Omega^{2,n}} \lambda^2 \phi_i n_z^2$$
$$+ Be\, A \int\limits_{\partial\Omega^{2,n}} \phi_i \breve{u} t_r^2 t_z^2 + Be\, A \int\limits_{\partial\Omega^{2,n}} \phi_i \breve{w} t_z^2 t_z^2$$
$$- 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_r^2 n_z^2 \partial_r \breve{u} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_z \breve{u}$$
$$- 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_r \breve{w} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 n_z^2 n_z^2 \partial_z \breve{w}$$
$$- 2A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_r^2 \partial_r \breve{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_z \breve{u} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_z \breve{u}$$
$$- A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_r \breve{w} - A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_r \breve{w} - 2A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_z^2 \partial_z \breve{w}, \tag{26.66}$$

and

$$\bar{\mathfrak{M}}_i^{z,4} := \int\limits_{\partial\Omega^5} \phi_i \lambda^4 n_z^4 + \int\limits_{\partial\Omega^5} \phi_i \gamma^4 t_z^4 - 2A \int\limits_{\partial\Omega^5} n_z^4 \phi_i \partial_z \breve{w} - A \int\limits_{\partial\Omega^5} n_r^4 \phi_i \partial_z \breve{u} - A \int\limits_{\partial\Omega^5} n_r^4 \phi_i \partial_r \breve{w}. \tag{26.67}$$

We now multiply equations (26.63), and therefore (26.64)-(26.67), by $2\Delta_t/3$ and rearrange terms to obtain

$$\bar{\mathscr{M}}_i^z := \frac{2\Delta_t}{3} \bar{\mathfrak{M}}_i^z, \tag{26.68}$$

where

$$\bar{\mathscr{M}}_i^z = \bar{\mathscr{M}}_i^{z,0} + \bar{\mathscr{M}}_i^{z,1} + \bar{\mathscr{M}}_i^{z,2} + \bar{\mathscr{M}}_i^{z,4}, \tag{26.69}$$

with

$$
\begin{aligned}
\bar{\mathscr{M}}_i^{z,0} :=\ & Re \int\limits_{\Omega^n} \phi_i \bar{w} - \frac{4Re}{3} \int\limits_{\Omega^n} \phi_i w(t_{n-1}) + \frac{Re}{3} \int\limits_{\Omega^n} \phi_i w(t_{n-2}) \qquad (26.70)\\
& + \frac{2\Delta_t Re}{3} \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \bar{w} + \frac{2\Delta_t Re}{3} \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \bar{w}\\
& - Re \int\limits_{\Omega^n} \phi_i r^c \partial_r \bar{w} + \frac{4Re}{3} \int\limits_{\Omega^n} \phi_i r^c(t_{n-1}) \partial_r \bar{w} - \frac{Re}{3} \int\limits_{\Omega^n} \phi_i r^c(t_{n-2}) \partial_r \bar{w}\\
& - Re \int\limits_{\Omega^n} \phi_i z^c \partial_z \bar{w} + \frac{4Re}{3} \int\limits_{\Omega^n} \phi_i z^c(t_{n-1}) \partial_z \bar{w} - \frac{Re}{3} \int\limits_{\Omega^n} \phi_i z^c(t_{n-2}) \partial_z \bar{w}\\
& + \frac{2\Delta_t Re}{3} A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \bar{w} + \frac{2\Delta_t Re}{3} A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \bar{w} + \frac{2\Delta_t Re}{3} A \int\limits_{\Omega^n} \phi_i \bar{u} \partial_r \breve{w} + \frac{2\Delta_t Re}{3} A \int\limits_{\Omega^n} \phi_i \bar{w} \partial_z \breve{w}\\
& - \frac{2\Delta_t}{3} \int\limits_{\Omega^n} p \partial_z \phi_i + \frac{4\Delta_t}{3} \int\limits_{\Omega^n} \partial_z \bar{w} \partial_z \phi_i + \frac{2\Delta_t}{3} \int\limits_{\Omega^n} \partial_z \bar{u} \partial_r \phi_i + \frac{2\Delta_t}{3} \int\limits_{\Omega^n} \partial_r \bar{w} \partial_r \phi_i\\
& - Re\, A \int\limits_{\Omega^n} \phi_i r^c \partial_r \breve{w} + \frac{4Re}{3} A \int\limits_{\Omega^n} \phi_i r^c(t_{n-1}) \partial_r \breve{w} - \frac{Re}{3} A \int\limits_{\Omega^n} \phi_i r^c(t_{n-2}) \partial_r \breve{w}\\
& - Re\, A \int\limits_{\Omega^n} \phi_i z^c \partial_z \breve{w} + \frac{4Re}{3} A \int\limits_{\Omega^n} \phi_i z^c(t_{n-1}) \partial_z \breve{w} - \frac{Re}{3} A \int\limits_{\Omega^n} \phi_i z^c(t_{n-2}) \partial_z \breve{w}\\
& + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} + \frac{2\Delta_t Re}{3} (A)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{w} + \frac{2\Delta_t Re}{3} (A)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{w}\\
& - \frac{2\Delta_t St}{3} \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z,
\end{aligned}
$$

$$
\begin{aligned}
\bar{\mathscr{M}}_i^{z,1} :=\ & \frac{2\Delta_t}{3} \frac{\sigma^1(r_c, z_c) \phi_i(r_c, z_c) m_z^1(r_c, z_c)}{Ca} + \frac{2\Delta_t}{3} \frac{\sigma^1(r_{J^1}, z_{J^1}) \phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1})}{Ca}\\
& - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{1,n}} p^g \phi_i \boldsymbol{n}_z^1 + \frac{2\Delta_t}{3Ca} \int\limits_{\partial\Omega^{1,n}} t_z^1 \sigma^1 \partial_s \phi_i\\
& - \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_z \breve{w} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_z \breve{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \breve{w},
\end{aligned}
$$

(26.71)

$$
\begin{aligned}
\bar{\mathscr{M}}_i^{z,2} := & \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{u} t_r^2 t_z^2 + \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i \bar{w} t_z^2 t_z^2 \\
& -\frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i u^s t_r^2 t_z^2 - \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i w^s t_z^2 t_z^2 \\
& -\frac{\Delta_t}{3Ca} \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 \partial_s \sigma^2 + \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \lambda^2 \phi_i n_z^2 \\
& +\frac{2\Delta_t Be}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{u} t_r^2 t_z^2 + \frac{2\Delta_t Be}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{w} t_z^2 t_z^2 \\
& -\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_r^2 n_z^2 \partial_r \check{u} - \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_z \check{u} \\
& -\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_r \check{w} - \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 n_z^2 n_z^2 \partial_z \check{w} \\
& -\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_r^2 \partial_r \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_z \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_z \check{u} \\
& -\frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_r \check{w} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_r \check{w} - \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_z^2 \partial_z \check{w},
\end{aligned}
\tag{26.72}
$$

and

$$
\begin{aligned}
\bar{\mathscr{M}}_i^{z,5} := & \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i \lambda^5 n_z^5 + \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i \gamma^5 t_z^5 \\
& -\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_z^5 \phi_i \partial_z \check{w} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_r^5 \phi_i \partial_z \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_r^5 \phi_i \partial_r \check{w}.
\end{aligned}
\tag{26.73}
$$

where all time-dependent function whose time argument is not indicated are evaluated at time $t = t_n$, and we have used $\bar{w}(t_{n-1}) + \check{w}(t_{n-1}) = w(t_{n-1})$ and $\bar{w}(t_{n-2}) + \check{w}(t_{n-2}) = w(t_{n-2})$, in $\mathscr{M}_i^0$.

We now recall the approximations made in (25.92)-(25.108); which are given by

$$r^c(r, z, t) \approx \sum_{j=1}^{n_v} r_j^c(t) \phi_j(r, z) \tag{26.74}$$

$$z^c(r, z, t) \approx \sum_{j=1}^{n_v} z_j^c(t) \phi_j(r, z) \tag{26.75}$$

$$p(r, z, t) \approx \sum_{j=1}^{n_p} p_j(t) \psi_j(r, z) \tag{26.76}$$

$$\sigma^1(r, z, t) \approx \sum_{j=1}^{n_v} \tilde{\sigma}_j^1(t) \phi_j^1(r, z) \tag{26.77}$$

$$p^g(r, z, t) \approx \sum_{j=1}^{n_v} \tilde{p}_j^g(t) \phi_j^1(r, z) \tag{26.78}$$

$$\sigma^2(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\sigma}_j^2(t)\phi_j^1(r,z) \tag{26.79}$$

$$\lambda^2(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\lambda}_j^2(t)\phi_j^2(r,z) \tag{26.80}$$

$$u^s(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{u}_j^s(t)\phi_j^2(r,z) \tag{26.81}$$

$$w^s(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{w}_j^s(t)\phi_j^2(r,z) \tag{26.82}$$

$$\bar{u}(r,z,t) \approx \sum_{j=1}^{\bar{n}_v} \bar{u}_j(t)\phi_j(r,z) \tag{26.83}$$

$$\bar{w}(r,z,t) \approx \sum_{j=1}^{\bar{n}_v} \bar{w}_j(t)\phi_j(r,z), \tag{26.84}$$

$$\lambda^5(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\lambda}_j^5(t)\phi_j^5(r,z) \tag{26.85}$$

and

$$\gamma^5(r,z,t) \approx \sum_{j=1}^{n_v} \tilde{\gamma}_j^5(t)\phi_j^5(r,z) \tag{26.86}$$

where n_v is the total number of velocity nodes, n_p is the number of nodes where pressure is calculated, the j index indicates global node numbers that we will use in the Galerkin method (that is to say, ϕ_j is the hat function centred at the j-th node), and ϕ_j^k coincides on the k-th boundary with ϕ_j, and is identically null elsewhere. Moreover, we can assume functions $\tilde{\sigma}_j^1$ and $\tilde{\lambda}_j^2$ are identically null (as all functions these multiply will be null everywhere by our construction of the basis functions) for all j such that $\phi_j = 0$ on boundary 1 and 2, respectively. Furthermore, functions p_j are numbered following the pressure-node numbering. where $\bar{n}_v$ is the number of velocity nodes in $\bar{\Omega}$. We highlight that the numbering convention is chosen so as to have the first $\bar{n}_v$ nodes correspond to $\bar{\Omega}$.

Substituting these approximations into (26.69), and consequently into(26.70)-(26.73) we define

$$\bar{\mathscr{M}}_i^z \approx \bar{\mathcal{M}}_i^z := \bar{\mathcal{M}}_i^{z,0} + \bar{\mathcal{M}}_i^{z,1} + \bar{\mathcal{M}}_i^{z,2} + \bar{\mathcal{M}}_i^{z,5}, \tag{26.87}$$

with

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{z,0} := {} & Re \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) - \frac{4Re}{3} \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} w_j(t_{n-1}) \phi_j \right) \\
& + \frac{Re}{3} \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} w_j(t_{n-2}) \phi_j \right) + \frac{2\Delta_t Re}{3} \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} \bar{u}_j \phi_j \right) \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& + \frac{2\Delta_t Re}{3} \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& - Re \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} r_j^c \phi_j \right) \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) + \frac{4Re}{3} \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} r_j^c(t_{n-1}) \phi_j \right) \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& - \frac{Re}{3} \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} r_j^c(t_{n-2}) \phi_j \right) \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& - Re \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} z_j^c \phi_j \right) \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) + \frac{4Re}{3} \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} z_j^c(t_{n-1}) \phi_j \right) \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& - \frac{Re}{3} \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} z_j^c(t_{n-2}) \phi_j \right) \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& + \frac{2\Delta_t Re}{3} A \int_{\Omega^n} \phi_i \check{u} \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) + \frac{2\Delta_t Re}{3} A \int_{\Omega^n} \phi_i \check{w} \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& + \frac{2\Delta_t Re}{3} A \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} \bar{u}_j \phi_j \right) \partial_r \check{w} + \frac{2\Delta_t Re}{3} A \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \partial_z \check{w} \\
& - \frac{2\Delta_t}{3} \int_{\Omega^n} \left(\sum_{j=1}^{\bar{n}_p} p_j \psi_j \right) \partial_z \phi_i + \frac{4\Delta_t}{3} \int_{\Omega^n} \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \partial_z \phi_i \\
& + \frac{2\Delta_t}{3} \int_{\Omega^n} \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{u}_j \phi_j \right) \partial_r \phi_i + \frac{2\Delta_t}{3} \int_{\Omega^n} \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \partial_r \phi_i \\
& - Re\, A \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} r_j^c \phi_j \right) \partial_r \check{w} + \frac{4Re}{3} A \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} r_j^c(t_{n-1}) \phi_j \right) \partial_r \check{w} \\
& - \frac{Re}{3} A \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} r_j^c(t_{n-2}) \phi_j \right) \partial_r \check{w} - Re\, A \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} z_j^c \phi_j \right) \partial_z \check{w} \\
& + \frac{4Re}{3} A \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} z_j^c(t_{n-1}) \phi_j \right) \partial_z \check{w} - \frac{Re}{3} A \int_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} z_j^c(t_{n-2}) \phi_j \right) \partial_z \check{w} \\
& + Re\, A \int_{\Omega^n} \phi_i \check{w} + \frac{2\Delta_t Re}{3} (A)^2 \int_{\Omega^n} \phi_i \check{u} \partial_r \check{w} + \frac{2\Delta_t Re}{3} (A)^2 \int_{\Omega^n} \phi_i \check{w} \partial_z \check{w} - \frac{2\Delta_t St}{3} \int_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z,
\end{aligned}
\tag{26.88}
$$

$$\begin{aligned}\bar{\mathcal{M}}_i^{z,1} := & -\frac{4\Delta_t}{3}A\int\limits_{\partial\Omega^{1,n}}\phi_i n_z^1\partial_z\breve{w} - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{1,n}}\phi_i n_r^1\partial_z\breve{u} - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{1,n}}\phi_i n_r^1\partial_r\breve{w}\\ & + \frac{2\Delta_t}{3Ca}\int\limits_{\partial\Omega^{1,n}} t_z^1\left(\sum_{j=1}^{n_v}\tilde{\sigma}_j^1\phi_j^1\right)\partial_s\phi_i - \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{1,n}}\left(\sum_{j=1}^{n_v}\tilde{p}_j^g\phi_j^1\right)\phi_i\boldsymbol{n}_z^1\\ & + \frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_z^1(r_c,z_c)}{Ca} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^1(r_{J^1},z_{J^1})}{Ca},\end{aligned} \tag{26.89}$$

$$\begin{aligned}\bar{\mathcal{M}}_i^{z,2} := & \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i t_r^2 t_z^2\left(\sum_{j=1}^{\bar{n}_v}\bar{u}_j\phi_j\right) + \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i t_z^2 t_z^2\left(\sum_{j=1}^{\bar{n}_v}\bar{w}_j\phi_j\right)\\ & -\frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i\left(\sum_{j=1}^{n_v}\tilde{u}_j^s(t)\phi_j^2\right)t_r^2 t_z^2 - \frac{2\Delta_t Be}{3}\int\limits_{\partial\Omega^{2,n}}\phi_i\left(\sum_{j=1}^{n_v}\tilde{w}_j^s(t)\phi_j^2\right)t_z^2 t_z^2\\ & -\frac{\Delta_t}{3Ca}\int\limits_{\partial\Omega^{2,n}}\phi_i t_z^2\partial_s\left(\sum_{j=1}^{n_v}\tilde{\sigma}_j^2(t)\phi_j^1\right) + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^{2,n}}\left(\sum_{j=1}^{n_v}\tilde{\lambda}_j^2\phi_j^2\right)\phi_i n_z^2\\ & +\frac{2\Delta_t Be}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i\breve{u}t_r^2 t_z^2 + \frac{2\Delta_t Be}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i\breve{w}t_z^2 t_z^2\\ & -\frac{4\Delta_t}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i n_r^2 n_r^2 n_z^2\partial_r\breve{u} - \frac{4\Delta_t}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i n_r^2 n_z^2 n_z^2\partial_z\breve{u}\\ & -\frac{4\Delta_t}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i n_r^2 n_z^2 n_z^2\partial_r\breve{w} - \frac{4\Delta_t}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i n_z^2 n_z^2 n_z^2\partial_z\breve{w}\\ & -\frac{4\Delta_t}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i t_r^2 t_z^2 n_r^2\partial_r\breve{u} - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i t_r^2 t_z^2 n_z^2\partial_z\breve{u} - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i t_z^2 t_z^2 n_r^2\partial_z\breve{u}\\ & -\frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i t_z^2 t_z^2 n_r^2\partial_r\breve{w} - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i t_r^2 t_z^2 n_z^2\partial_r\breve{w} - \frac{4\Delta_t}{3}A\int\limits_{\partial\Omega^{2,n}}\phi_i t_z^2 t_z^2 n_z^2\partial_z\breve{w},\end{aligned} \tag{26.90}$$

and

$$\begin{aligned}\bar{\mathcal{M}}_i^{z,5} := & \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\phi_i\left(\sum_{j=1}^{n_v}\tilde{\lambda}_j^5\phi_j^5\right)n_z^5 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^5}\phi_i\left(\sum_{j=1}^{n_v}\tilde{\gamma}_j^5\phi_j^5\right)t_z^5\\ & -\frac{4\Delta_t}{3}A\int\limits_{\partial\Omega^5}n_z^5\phi_i\partial_z\breve{w} - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^5}n_r^5\phi_i\partial_z\breve{u} - \frac{2\Delta_t}{3}A\int\limits_{\partial\Omega^5}n_r^5\phi_i\partial_r\breve{w}.\end{aligned} \tag{26.91}$$

Re-arranging terms we have

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{z,0} \; = & -\frac{2\Delta_t St}{3} \int\limits_{\Omega^n} \phi_i \hat{\boldsymbol{g}}_z + Re\, A \int\limits_{\Omega^n} \phi_i \breve{w} + \frac{2\Delta_t Re}{3} \left(A\right)^2 \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \breve{w} \\
& + \frac{2\Delta_t Re}{3} \left(A\right)^2 \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \breve{w} \\
& + Re \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) - \frac{4Re}{3} \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} w_j(t_{n-1}) \phi_j \right) + \frac{Re}{3} \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} w_j(t_{n-2}) \phi_j \right) \\
& + \frac{2\Delta_t Re}{3} A \int\limits_{\Omega^n} \phi_i \breve{u} \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) + \frac{2\Delta_t Re}{3} A \int\limits_{\Omega^n} \phi_i \breve{w} \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& + \frac{2\Delta_t Re}{3} A \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} \bar{u}_j \phi_j \right) \partial_r \breve{w} + \frac{2\Delta_t Re}{3} A \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \partial_z \breve{w} \\
& - \frac{2\Delta_t}{3} \int\limits_{\Omega^n} \left(\sum_{j=1}^{\bar{n}_p} p_j \psi_j \right) \partial_z \phi_i + \frac{4\Delta_t}{3} \int\limits_{\Omega^n} \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \partial_z \phi_i \\
& + \frac{2\Delta_t}{3} \int\limits_{\Omega^n} \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{u}_j \phi_j \right) \partial_r \phi_i + \frac{2\Delta_t}{3} \int\limits_{\Omega^n} \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \partial_r \phi_i \\
& - Re\, A \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} r_j^c \phi_j \right) \partial_r \breve{w} + \frac{4Re}{3} A \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} r_j^c(t_{n-1}) \phi_j \right) \partial_r \breve{w} - \frac{Re}{3} A \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} r_j^c(t_{n-2}) \phi_j \right) \partial \\
& - Re\, A \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} z_j^c \phi_j \right) \partial_z \breve{w} + \frac{4Re}{3} A \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} z_j^c(t_{n-1}) \phi_j \right) \partial_z \breve{w} - \frac{Re}{3} A \int\limits_{\Omega^n} \phi_i \left(\sum_{j=1}^{n_v} z_j^c(t_{n-2}) \phi_j \right) \partial \\
& + \frac{2\Delta_t Re}{3} \int\limits_{\Omega^n} \phi_i \left(\sum_{k=1}^{\bar{n}_v} \bar{u}_k \phi_k \right) \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) + \frac{2\Delta_t Re}{3} \int\limits_{\Omega^n} \phi_i \left(\sum_{k=1}^{\bar{n}_v} \bar{w}_k \phi_k \right) \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& - Re \int\limits_{\Omega^n} \phi_i \left(\sum_{k=1}^{n_v} r_k^c \phi_k \right) \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) + \frac{4Re}{3} \int\limits_{\Omega^n} \phi_i \left(\sum_{k=1}^{n_v} r_k^c(t_{n-1}) \phi_k \right) \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& - \frac{Re}{3} \int\limits_{\Omega^n} \phi_i \left(\sum_{k=1}^{n_v} r_k^c(t_{n-2}) \phi_k \right) \partial_r \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& - Re \int\limits_{\Omega^n} \phi_i \left(\sum_{k=1}^{n_v} z_k^c \phi_k \right) \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) + \frac{4Re}{3} \int\limits_{\Omega^n} \phi_i \left(\sum_{k=1}^{n_v} z_k^c(t_{n-1}) \phi_k \right) \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
& - \frac{Re}{3} \int\limits_{\Omega^n} \phi_i \left(\sum_{k=1}^{n_v} z_k^c(t_{n-2}) \phi_k \right) \partial_z \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right),
\end{aligned}
\tag{26.92}
$$

$$\begin{aligned}
\bar{\mathcal{M}}_i^{z,1} &= -\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_z \breve{w} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_z \breve{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \breve{w} \\
&+ \frac{2\Delta_t}{3Ca} \int\limits_{\partial\Omega^{1,n}} t_z^1 \left(\sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \phi_j^1 \right) \partial_s \phi_i - \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{1,n}} \left(\sum_{j=1}^{n_v} \tilde{p}_j^g \phi_j^1 \right) \phi_i \boldsymbol{n}_z^1 \\
&+ \frac{2\Delta_t}{3} \frac{\sigma^1(r_c, z_c)\phi_i(r_c, z_c) m_z^1(r_c, z_c)}{Ca} + \frac{2\Delta_t}{3} \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1})}{Ca},
\end{aligned} \tag{26.93}$$

$$\begin{aligned}
\bar{\mathcal{M}}_i^{z,2} &= \frac{2\Delta_t Be}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i \breve{u} t_r^2 t_z^2 + \frac{2\Delta_t Be}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i \breve{w} t_z^2 t_z^2 \\
&- \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_r^2 n_z^2 \partial_r \breve{u} - \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_z \breve{u} \\
&- \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_r \breve{w} - \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 n_z^2 n_z^2 \partial_z \breve{w} \\
&- \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_r^2 \partial_r \breve{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_z \breve{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_z \breve{u} \\
&- \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_r \breve{w} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_r \breve{w} - \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_z^2 \partial_z \breve{w} \\
&- \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i \left(\sum_{j=1}^{n_v} \tilde{u}_j^s \phi_j^2 \right) t_r^2 t_z^2 - \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i \left(\sum_{j=1}^{n_v} \tilde{w}_j^s \phi_j^2 \right) t_z^2 t_z^2 \\
&+ \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 \left(\sum_{j=1}^{\bar{n}_v} \bar{u}_j \phi_j \right) + \frac{2\Delta_t Be}{3} \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 \left(\sum_{j=1}^{\bar{n}_v} \bar{w}_j \phi_j \right) \\
&- \frac{\Delta_t}{3Ca} \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 \partial_s \left(\sum_{j=1}^{n_v} \tilde{\sigma}_j^2 \phi_j^1 \right) + \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^{2,n}} \left(\sum_{j=1}^{n_v} \tilde{\lambda}_j^2 \phi_j^2 \right) \phi_i n_z^2,
\end{aligned} \tag{26.94}$$

and

$$\begin{aligned}
\bar{\mathcal{M}}_i^{z,5} &= -\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_z^5 \phi_i \partial_z \breve{w} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_r^5 \phi_i \partial_z \breve{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_r^5 \phi_i \partial_r \breve{w} \\
& \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i \left(\sum_{j=1}^{n_v} \tilde{\lambda}_j^5 \phi_j^5 \right) n_z^5 + \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^5} \phi_i \left(\sum_{j=1}^{n_v} \tilde{\gamma}_j^5 \phi_j^5 \right) t_z^5.
\end{aligned} \tag{26.95}$$

Moving the integrals into the sums

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{z,0} &= -\frac{2\Delta_t St}{3}\int\limits_{\Omega^n}\phi_i\hat{\boldsymbol{g}}_z + Re\,A\int\limits_{\Omega^n}\phi_i\check{w} \qquad (26.96)\\
&+\frac{2\Delta_t Re}{3}(A)^2\int\limits_{\Omega^n}\phi_i\check{u}\partial_r\check{w} + \frac{2\Delta_t Re}{3}(A)^2\int\limits_{\Omega^n}\phi_i\check{w}\partial_z\check{w}\\
&+Re\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\phi_i\phi_j - \frac{4Re}{3}\sum_{j=1}^{\bar{n}_v}w_j(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_j + \frac{Re}{3}\sum_{j=1}^{\bar{n}_v}w_j(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_j\\
&+\frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\phi_i\check{u}\partial_r\phi_j + \frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\phi_i\check{w}\partial_z\phi_j\\
&+\frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{w} + \frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{w}\\
&-\frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_p}p_j\int\limits_{\Omega^n}\psi_j\partial_z\phi_i + \frac{4\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\partial_z\phi_j\partial_z\phi_i\\
&+\frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\Omega^n}\partial_z\phi_j\partial_r\phi_i + \frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\partial_r\phi_j\partial_r\phi_i\\
&-Re\,A\sum_{j=1}^{n_v}r_j^c\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{w} + \frac{4Re}{3}A\sum_{j=1}^{n_v}r_j^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{w} - \frac{Re}{3}A\sum_{j=1}^{n_v}r_j^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{w}\\
&-Re\,A\sum_{j=1}^{n_v}z_j^c\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{w} + \frac{4Re}{3}A\sum_{j=1}^{n_v}z_j^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{w} - \frac{Re}{3}A\sum_{j=1}^{n_v}z_j^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{w}\\
&+\frac{2\Delta_t Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{\bar{n}_v}\bar{u}_k\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j + \frac{2\Delta_t Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{\bar{n}_v}\bar{w}_k\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j\\
&-Re\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}r_k^c\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j + \frac{4Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}r_k^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j\\
&-\frac{Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}r_k^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j\\
&-Re\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}z_k^c\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j + \frac{4Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}z_k^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j\\
&-\frac{Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}z_k^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j,
\end{aligned}
$$

$$\begin{aligned}\bar{\mathcal{M}}_i^{z,1} &= -\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{1,n}} \phi_i n_z^1 \partial_z \check{w} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_z \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{1,n}} \phi_i n_r^1 \partial_r \check{w} \\ &+ \frac{2\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^1 \int\limits_{\partial\Omega^{1,n}} t_z^1 \phi_j^1 \partial_s \phi_i - \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{p}_j^g \int\limits_{\partial\Omega^{1,n}} \phi_j^1 \phi_i n_z^1 \\ &+ \frac{2\Delta_t}{3} \frac{\sigma^1(r_c, z_c)\phi_i(r_c, z_c) m_z^1(r_c, z_c)}{Ca} + \frac{2\Delta_t}{3} \frac{\sigma^1(r_{J^1}, z_{J^1})\phi_i(r_{J^1}, z_{J^1}) m_z^1(r_{J^1}, z_{J^1})}{Ca},\end{aligned} \tag{26.97}$$

$$\begin{aligned}\bar{\mathcal{M}}_i^{z,2} &= \frac{2\Delta_t Be}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{u} t_r^2 t_z^2 + \frac{2\Delta_t Be}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i \check{w} t_z^2 t_z^2 \\ &-\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_r^2 n_z^2 \partial_r \check{u} - \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_z \check{u} \\ &-\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_r^2 n_z^2 n_z^2 \partial_r \check{w} - \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i n_z^2 n_z^2 n_z^2 \partial_z \check{w} \\ &-\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_r^2 \partial_r \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_z \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_z \check{u} \\ &-\frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_r^2 \partial_r \check{w} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_r^2 t_z^2 n_z^2 \partial_r \check{w} - \frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 t_z^2 n_z^2 \partial_z \check{w} \\ &-\frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{u}_j^s \int\limits_{\partial\Omega^{2,n}} \phi_i \phi_j^2 t_r^2 t_z^2 - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{w}_j^s \int\limits_{\partial\Omega^{2,n}} \phi_i \phi_j^2 t_z^2 t_z^2 \\ &+\frac{2\Delta_t Be}{3} \sum_{j=1}^{\bar{n}_v} \bar{u}_j \int\limits_{\partial\Omega^{2,n}} \phi_i \phi_j t_r^2 t_z^2 + \frac{2\Delta_t Be}{3} \sum_{j=1}^{\bar{n}_v} \bar{w}_j \int\limits_{\partial\Omega^{2,n}} \phi_i \phi_j t_z^2 t_z^2 \\ &-\frac{\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}_j^2 \int\limits_{\partial\Omega^{2,n}} \phi_i t_z^2 \partial_s \phi_j^1 + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}_j^2 \int\limits_{\partial\Omega^{2,n}} \phi_j^2 \phi_i n_z^2,\end{aligned} \tag{26.98}$$

and

$$\begin{aligned}\bar{\mathcal{M}}_i^{z,4} &= -\frac{4\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_z^4 \phi_i \partial_z \check{w} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_r^4 \phi_i \partial_z \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\Omega^5} n_r^4 \phi_i \partial_r \check{w} \\ &\frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}_j^4 \int\limits_{\partial\Omega^5} \phi_i \phi_j^2 n_z^4 + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma}_j^4 \int\limits_{\partial\Omega^5} \phi_i \phi_j^2 t_z^4.\end{aligned} \tag{26.99}$$

We now decompose $\bar{\mathcal{M}}_i^{z,0}$ as follows

$$\bar{\mathcal{M}}_i^{z,0} = \bar{\mathcal{M}}_i^{z,0a} + \bar{\mathcal{M}}_i^{z,0b} + \bar{\mathcal{M}}_i^{z,0c} + \bar{\mathcal{M}}_i^{z,0d}, \tag{26.100}$$

where

$$\begin{aligned}\bar{\mathcal{M}}_i^{z,0a} &= -\frac{2\Delta_t St}{3}\int\limits_{\Omega^n}\phi_i\hat{\boldsymbol{g}}_z + Re\,A\int\limits_{\Omega^n}\phi_i\check{w}\\ &+\frac{2\Delta_t Re}{3}\left(A\right)^2\int\limits_{\Omega^n}\phi_i\check{u}\partial_r\check{w} + \frac{2\Delta_t Re}{3}\left(A\right)^2\int\limits_{\Omega^n}\phi_i\check{w}\partial_z\check{w},\end{aligned} \tag{26.101}$$

$$\begin{aligned}\bar{\mathcal{M}}_i^{z,0b} &= Re\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\phi_i\phi_j - \frac{4Re}{3}\sum_{j=1}^{\bar{n}_v}w_j(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_j + \frac{Re}{3}\sum_{j=1}^{\bar{n}_v}w_j(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_j\\ &+\frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\phi_i\check{u}\partial_r\phi_j + \frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\phi_i\check{w}\partial_z\phi_j\\ &+\frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{w} + \frac{2\Delta_t Re}{3}A\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{w}\\ &+\frac{4\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\partial_z\phi_j\partial_z\phi_i\\ &+\frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\bar{u}_j\int\limits_{\Omega^n}\partial_z\phi_j\partial_r\phi_i + \frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\int\limits_{\Omega^n}\partial_r\phi_j\partial_r\phi_i\\ &-Re\,A\sum_{j=1}^{n_v}r_j^c\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{w} + \frac{4Re}{3}A\sum_{j=1}^{n_v}r_j^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{w} - \frac{Re}{3}A\sum_{j=1}^{n_v}r_j^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_j\partial_r\check{w}\\ &-Re\,A\sum_{j=1}^{n_v}z_j^c\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{w} + \frac{4Re}{3}A\sum_{j=1}^{n_v}z_j^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{w} - \frac{Re}{3}A\sum_{j=1}^{n_v}z_j^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_j\partial_z\check{w},\end{aligned} \tag{26.102}$$

$$\begin{aligned}\bar{\mathcal{M}}_i^{z,0c} &= \frac{2\Delta_t Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{\bar{n}_v}\bar{u}_k\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j + \frac{2\Delta_t Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{\bar{n}_v}\bar{w}_k\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j\\ &-Re\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}r_k^c\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j + \frac{4Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}r_k^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j\\ &-\frac{Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}r_k^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_k\partial_r\phi_j\\ &-Re\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}z_k^c\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j + \frac{4Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}z_k^c(t_{n-1})\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j\\ &-\frac{Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}z_k^c(t_{n-2})\int\limits_{\Omega^n}\phi_i\phi_k\partial_z\phi_j,\end{aligned} \tag{26.103}$$

$$\bar{\mathcal{M}}_i^{z,0d} = -\frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_p}p_j\int\limits_{\Omega^n}\psi_j\partial_z\phi_i. \tag{26.104}$$

Using the partition into elements that we described above and the local element-node numbers for variable i we have

$$\bar{\mathcal{M}}_i^z = \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \bar{\mathcal{M}}_{e,ii}^{z,0a}}_{\bar{\mathcal{M}}_i^{z,0a}} + \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \bar{\mathcal{M}}_{e,ii}^{z,0b}}_{\bar{\mathcal{M}}_i^{z,0b}} + \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \bar{\mathcal{M}}_{e,ii}^{z,0c}}_{\bar{\mathcal{M}}_i^{z,0c}} + \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \bar{\mathcal{M}}_{e,ii}^{z,0d}}_{\bar{\mathcal{M}}_i^{z,0d}}$$

$$+\frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_z^1(r_c,z_c)}{Ca} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^1(r_{J^1},z_{J^1})}{Ca}$$

$$+\sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\rm el}^1} \bar{\mathcal{M}}_{e_1,ii}^{z,1} + \underbrace{\sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\rm el}^2} \bar{\mathcal{M}}_{e,ii}^{z,2}}_{\bar{\mathcal{M}}_i^{z,2}} + \underbrace{\sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\rm el}^4} \bar{\mathcal{M}}_{e_4,ii}^{z,4}}_{\bar{\mathcal{M}}_i^{z,4}}, \tag{26.105}$$

and we have

$$\bar{\mathcal{M}}_{e,ii}^{z,0a} = -\frac{2\Delta_t St}{3}\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\hat{\boldsymbol{g}}_z + Re\, A\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\check{w} \tag{26.106}$$

$$+\frac{2\Delta_t Re}{3}(A)^2\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\check{u}\partial_r\check{w} + \frac{2\Delta_t Re}{3}(A)^2\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\check{w}\partial_z\check{w},$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{z,0b}_{e,ii} &= Re \sum_{j=1}^{\bar{n}_v} \bar{w}_j \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j - \frac{4Re}{3} \sum_{j=1}^{\bar{n}_v} w_j(t_{n-1}) \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j \\
&+ \frac{Re}{3} \sum_{j=1}^{\bar{n}_v} w_j(t_{n-2}) \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j \\
&+ \frac{2\Delta_t Re}{3} A \sum_{j=1}^{\bar{n}_v} \bar{w}_j \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{u}\partial_r\phi_j + \frac{2\Delta_t Re}{3} A \sum_{j=1}^{\bar{n}_v} \bar{w}_j \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{w}\partial_z\phi_j \\
&+ \frac{2\Delta_t Re}{3} A \sum_{j=1}^{\bar{n}_v} \bar{u}_j \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j\partial_r\check{w} + \frac{2\Delta_t Re}{3} A \sum_{j=1}^{\bar{n}_v} \bar{w}_j \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j\partial_z\check{w} \\
&+ \frac{4\Delta_t}{3} \sum_{j=1}^{\bar{n}_v} \bar{w}_j \int\limits_{\bar{\Omega}_e} \partial_z\phi_j\partial_z\phi_{l(e,ii)} \\
&+ \frac{2\Delta_t}{3} \sum_{j=1}^{\bar{n}_v} \bar{u}_j \int\limits_{\bar{\Omega}_e} \partial_z\phi_j\partial_r\phi_{l(e,ii)} + \frac{2\Delta_t}{3} \sum_{j=1}^{\bar{n}_v} \bar{w}_j \int\limits_{\bar{\Omega}_e} \partial_r\phi_j\partial_r\phi_{l(e,ii)} \\
&- Re\, A \sum_{j=1}^{n_v} r^c_j \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j\partial_r\check{w} + \frac{4Re}{3} A \sum_{j=1}^{n_v} r^c_j(t_{n-1}) \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j\partial_r\check{w} \\
&- \frac{Re}{3} A \sum_{j=1}^{n_v} r^c_j(t_{n-2}) \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j\partial_r\check{w} \\
&- Re\, A \sum_{j=1}^{n_v} z^c_j \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j\partial_z\check{w} + \frac{4Re}{3} A \sum_{j=1}^{n_v} z^c_j(t_{n-1}) \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j\partial_z\check{w} \\
&- \frac{Re}{3} A \sum_{j=1}^{n_v} z^c_j(t_{n-2}) \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_j\partial_z\check{w},
\end{aligned}
\tag{26.107}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{z,0c} &= \frac{2\Delta_t Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{\bar{n}_v}\bar{u}_k\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_r\phi_j + \frac{2\Delta_t Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{\bar{n}_v}\bar{w}_k\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_z\phi_j \\
&-Re\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}r_k^c\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_r\phi_j + \frac{4Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}r_k^c(t_{n-1})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_r\phi_j \\
&-\frac{Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}r_k^c(t_{n-2})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_r\phi_j \\
&-Re\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}z_k^c\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_z\phi_j + \frac{4Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}z_k^c(t_{n-1})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_z\phi_j \\
&-\frac{Re}{3}\sum_{j=1}^{\bar{n}_v}\bar{w}_j\sum_{k=1}^{n_v}z_k^c(t_{n-2})\int\limits_{\bar{\Omega}_e}\phi_{l(e,ii)}\phi_k\partial_z\phi_j,
\end{aligned}
\tag{26.108}
$$

$$
\bar{\mathcal{M}}_i^{z,0d} = -\frac{2\Delta_t}{3}\sum_{j=1}^{\bar{n}_p}p_j\int\limits_{\bar{\Omega}_e}\psi_j\partial_z\phi_{l(e,ii)},
\tag{26.109}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}_{e_1,ii}^{z,1} &= -\frac{4\Delta_t}{3}A\int\limits_{\partial\bar{\Omega}_{e_1}^1}\phi_{l_1(e_1,ii)}n_z^1\partial_z\check{w} - \frac{2\Delta_t}{3}A\int\limits_{\partial\bar{\Omega}_{e_1}^1}\phi_{l_1(e_1,ii)}n_r^1\partial_z\check{u} \\
&-\frac{2\Delta_t}{3}A\int\limits_{\partial\bar{\Omega}_{e_1}^1}\phi_{l_1(e_1,ii)}n_r^1\partial_r\check{w} \\
&+\frac{2\Delta_t}{3Ca}\sum_{j=1}^{n_v}\tilde{\sigma}_j^1\int\limits_{\partial\bar{\Omega}_{e_1}^1}t_z^1\phi_j^1\partial_s\phi_{l_1(e_1,ii)} - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\tilde{p}_j^g\int\limits_{\partial\Omega_{e_1}^1}\phi_j^1\phi_{l_1(e_1,ii)}n_z^1,
\end{aligned}
\tag{26.110}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{z,2}_{e_2,ii} &= \frac{2\Delta_t Be}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} \check{u} t_r^2 t_z^2 + \frac{2\Delta_t Be}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} \check{w} t_z^2 t_z^2 \\
&- \frac{4\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} n_r^2 n_r^2 n_z^2 \partial_r \check{u} - \frac{4\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} n_r^2 n_z^2 n_z^2 \partial_z \check{u} \\
&- \frac{4\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} n_r^2 n_z^2 n_z^2 \partial_r \check{w} - \frac{4\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} n_z^2 n_z^2 n_z^2 \partial_z \check{w} \\
&- \frac{4\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_r^2 t_z^2 n_r^2 \partial_r \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_r^2 t_z^2 n_z^2 \partial_z \check{u} \\
&- \frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_z^2 t_z^2 n_r^2 \partial_z \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_z^2 t_z^2 n_r^2 \partial_r \check{w} \\
&- \frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_r^2 t_z^2 n_z^2 \partial_r \check{w} - \frac{4\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_z^2 t_z^2 n_z^2 \partial_z \check{w} \\
&- \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{u}^s_j \int\limits_{\partial\Omega^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_j t_r^2 t_z^2 - \frac{2\Delta_t Be}{3} \sum_{j=1}^{n_v} \tilde{w}^s_j \int\limits_{\partial\Omega^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_j t_z^2 t_z^2 \\
&+ \frac{2\Delta_t Be}{3} \sum_{j=1}^{\bar{n}_v} \bar{u}_j \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} \phi_j t_r^2 t_z^2 + \frac{2\Delta_t Be}{3} \sum_{j=1}^{\bar{n}_v} \bar{w}_j \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} \phi_j t_z^2 t_z^2 \\
&- \frac{\Delta_t}{3Ca} \sum_{j=1}^{n_v} \tilde{\sigma}^2_j \int\limits_{\partial\Omega^2_{e_2}} \phi_i t_z^2 \partial_s \phi^1_j + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}^2_j \int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_j \phi_{l_2(e_2,ii)} n_z^2,
\end{aligned}
\tag{26.111}
$$

and

$$
\begin{aligned}
\bar{\mathcal{M}}^{z,5}_{e_5,ii} &= -\frac{4\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^5_{e_5}} n_z^5 \phi_{l_5(e_5,ii)} \partial_z \check{w} - \frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^5_{e_5}} n_r^5 \phi_{l_5(e_5,ii)} \partial_z \check{u} - \frac{2\Delta_t}{3} A \int\limits_{\partial\bar{\Omega}^5_{e_5}} n_r^5 \phi_i \partial_r \check{w} \\
&\frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\lambda}^5_j \int\limits_{\partial\bar{\Omega}^5_{e_5}} \phi_{l_5(e_4,ii)} \phi^5_j n_z^5 + \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \tilde{\gamma}^5_j \int\limits_{\partial\bar{\Omega}^5_{e_5}} \phi_{l_5(e_5,ii)} \phi^5_j t_z^5.
\end{aligned}
\tag{26.112}
$$

We now switch to local node numbers for variables j and k, obtaining

$$
\begin{aligned}
\bar{\mathcal{M}}^{z,0a}_{e,ii} = &-\frac{2\Delta_t St}{3} \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \hat{\boldsymbol{g}}_z}_{a_{ii,g_z}(e)} + Re\, A \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{w}}_{a_{ii,\check{w}}(e)} \\
&+ \frac{2\Delta_t Re}{3} (A)^2 \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{u} \partial_r \check{w}}_{a_{ii,\check{u},\partial_r \check{w}}(e)} + \frac{2\Delta_t Re}{3} (A)^2 \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{w} \partial_z \check{w}}_{a_{ii,\check{w},\partial_z \check{w}}(e)},
\end{aligned}
\tag{26.113}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{z,0b}_{e,ii} &= Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)} - \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} w_{l(e,jj)}(t_{n-1}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)} \\
&+ \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} w_{l(e,jj)}(t_{n-2}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}}_{a_{ii,jj}(e)} \\
&+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\check{u}\partial_r\phi_{l(e,jj)}}_{a^r_{ii,jj,\check{u}}(e)} + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\check{w}\partial_z\phi_{l(e,jj)}}_{a^z_{ii,jj,\check{w}}(e)} \\
&+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}\partial_r\check{w}}_{a_{ii,jj,\partial_r\check{w}}(e)} + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}\partial_z\check{w}}_{a_{ii,jj,\partial_z\check{w}}(e)} \\
&+ \frac{4\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \partial_z\phi_{l(e,ii)}\partial_z\phi_{l(e,jj)}}_{a^{z,z}_{ii,jj}(e)} \\
&+ \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \partial_r\phi_{l(e,ii)}\partial_z\phi_{l(e,jj)}}_{a^{r,z}_{ii,jj}(e)} + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \partial_r\phi_{l(e,ii)}\partial_r\phi_{l(e,jj)}}_{a^{r,r}_{ii,jj}(e)} \\
&- Re\, A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}\partial_r\check{w}}_{a_{ii,jj,\partial_r\check{w}}(e)} + \frac{4Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-1}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}\partial_r\check{w}}_{a_{ii,jj,\partial_r\check{w}}(e)} \\
&- \frac{Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-2}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}\partial_r\check{w}}_{a_{ii,jj,\partial_r\check{w}}(e)} \\
&- Re\, A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)} \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}\partial_z\check{w}}_{a_{ii,jj,\partial_z\check{w}}(e)} + \frac{4Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-1}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}\partial_z\check{w}}_{a_{ii,jj,\partial_z\check{w}}(e)} \\
&- \frac{Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-2}) \underbrace{\int_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}\partial_z\check{w}}_{a_{ii,jj,\partial_z\check{w}}(e)},
\end{aligned}
\tag{26.114}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{z,0c} \;&= \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{u}_{l(e,kk)} \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
&+ \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{w}_{l(e,kk)} \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)} \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
&+ \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-1}) \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
&- \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-2}) \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_r \phi_{l(e,jj)}}_{a^r_{ii,kk,jj}(e)} \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)} \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
&+ \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-1}) \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)} \\
&- \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-2}) \underbrace{\int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,kk)} \partial_z \phi_{l(e,jj)}}_{a^z_{ii,kk,jj}(e)},
\end{aligned}
\tag{26.115}
$$

$$
\bar{\mathcal{M}}_i^{z,0d} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_p^e} p_{l(e,jj)} \underbrace{\int\limits_{\bar{\Omega}_e} \psi_{l(e,jj)} \partial_z \phi_{l(e,ii)}}_{b^z_{jj,ii}(e)} \tag{26.116}
$$

$$\bar{\mathcal{M}}^{z,1}_{e_1,ii} = -\frac{4\Delta_t}{3}A\underbrace{\int\limits_{\partial\bar{\Omega}^1_{e_1}} \phi_{l_1(e_1,ii)}n^1_z\partial_z\check{w}}_{c_{ii,n_z,\partial_z\check{w}}(e)} - \frac{2\Delta_t}{3}A\underbrace{\int\limits_{\partial\bar{\Omega}^1_{e_1}} \phi_{l_1(e_1,ii)}n^1_r\partial_z\check{u}}_{c_{ii,n_r,\partial_z\check{u}}(e)} \tag{26.117}$$

$$-\frac{2\Delta_t}{3}A\underbrace{\int\limits_{\partial\bar{\Omega}^1_{e_1}} \phi_{l_1(e_1,ii)}n^1_r\partial_r\check{w}}_{c_{ii,n_r,\partial_r\check{w}}(e)}$$

$$+\frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^e}\sigma^1_{l_1^1(e_1,jj)}\underbrace{\int\limits_{\partial\bar{\Omega}^1_{e_1}} t^1_z\phi^1_{l_1(e_1,jj)}\partial_s\phi_{l_1(e_1,ii)}}_{c^s_{jj,ii,t_z}(e)}$$

$$-\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}p^g_{l_1^1(e_1,jj)}\underbrace{\int\limits_{\partial\Omega^1_{e_1}} \phi^1_{l_1(e_1,jj)}\phi_{l_1(e_1,ii)}n^1_z,}_{c_{ii,jj,n_z}(e_1)}$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{z,2}_{e_2,ii} = & \frac{2\Delta_t Be}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} \breve{u} t_r^2 t_z^2}_{d_{ii,t_r,t_z,\breve{u}}(e)} + \frac{2\Delta_t Be}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} \breve{w} t_z^2 t_z^2}_{d_{ii,t_z,t_z,\breve{w}}(e)} \qquad (26.118) \\
& - \frac{4\Delta_t}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} n_r^2 n_r^2 n_z^2 \partial_r \breve{u}}_{d_{ii,n_r,n_r,n_z,\partial_r\breve{u}}(e)} - \frac{4\Delta_t}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} n_r^2 n_z^2 n_z^2 \partial_z \breve{u}}_{d_{ii,n_r,n_z,n_z,\partial_z\breve{u}}(e)} \\
& - \frac{4\Delta_t}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} n_r^2 n_z^2 n_z^2 \partial_r \breve{w}}_{d_{ii,n_r,n_z,n_z,\partial_r\breve{w}}(e)} - \frac{4\Delta_t}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} n_z^2 n_z^2 n_z^2 \partial_z \breve{w}}_{d_{ii,n_z,n_z,n_z,\partial_z\breve{w}}(e)} \\
& - \frac{4\Delta_t}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_r^2 t_z^2 n_r^2 \partial_r \breve{u}}_{d_{ii,t_r,t_z,n_r,\partial_r\breve{u}}(e)} - \frac{2\Delta_t}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_r^2 t_z^2 n_z^2 \partial_z \breve{u}}_{d_{ii,t_r,t_z,n_z,\partial_z\breve{u}}(e)} \\
& - \frac{2\Delta_t}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_z^2 t_z^2 n_r^2 \partial_z \breve{u}}_{d_{ii,t_z,t_z,n_r,\partial_z\breve{u}}(e)} - \frac{2\Delta_t}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_z^2 t_z^2 n_r^2 \partial_r \breve{w}}_{d_{ii,t_z,t_z,n_r,\partial_r\breve{w}}(e)} \\
& - \frac{2\Delta_t}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_r^2 t_z^2 n_z^2 \partial_r \breve{w}}_{d_{ii,t_r,t_z,n_z,\partial_r\breve{w}}(e)} - \frac{4\Delta_t}{3} A \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} t_z^2 t_z^2 n_z^2 \partial_z \breve{w}}_{d_{ii,t_z,t_z,n_z,\partial_z\breve{w}}(e)} \\
& - \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v} \tilde{u}^s_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t_r^2 t_z^2}_{d_{ii,jj,t_r,t_z}(e_2)} - \frac{2\Delta_t Be}{3} \sum_{jj=1}^{n_v} \tilde{w}^s_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} t_z^2 t_z^2}_{d_{ii,jj,t_z,t_z}(e_2)} \\
& + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{\bar{n}_v^{e2}} \bar{u}_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} \phi_{l_2(e_2,jj)} t_r^2 t_z^2}_{d_{ii,jj,t_r,t_z}(e)} \\
& + \frac{2\Delta_t Be}{3} \sum_{jj=1}^{\bar{n}_v^{e2}} \bar{w}_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi_{l_2(e_2,ii)} \phi_{l_2(e_2,jj)} t_z^2 t_z^2}_{d_{ii,jj,t_z,t_z}(e)} \\
& - \frac{\Delta_t}{3Ca} \sum_{jj=1}^{n_v} \sigma^2_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^2_{e_2}} \phi_{l_2(e_2,ii)} t_z^2 \partial_s \phi^1_{l_2(e_2,jj)}}_{d^s_{ii,jj,t_z}(e_2)} + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e2}} \lambda^2_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\bar{\Omega}^2_{e_2}} \phi^2_{l_2(e_2,ii)} \phi^2_{l_2(e_2,jj)} n_z^2}_{d_{ii,jj,n_z}(e)},
\end{aligned}
$$

and

$$\bar{\mathcal{M}}^{z,5}_{e_5,ii} = -\frac{4\Delta_t}{3} A \underbrace{\int_{\partial\bar{\Omega}^5_{e_5}} n^5_z \phi_{l_5(e_5,ii)} \partial_z \check{w}}_{g_{ii,n_z,\partial_z\check{w}}} - \frac{2\Delta_t}{3} A \underbrace{\int_{\partial\bar{\Omega}^5_{e_5}} n^5_r \phi_{l_5(e_5,ii)} \partial_z \check{u}}_{g_{ii,n_r,\partial_z\check{u}}}$$

$$-\frac{2\Delta_t}{3} A \underbrace{\int_{\partial\bar{\Omega}^5_{e_5}} n^5_r \phi_{l_5(e_5,ii)} \partial_r \check{w}}_{g_{ii,n_r,\partial_r\check{w}}} \tag{26.119}$$

$$\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}^{e5}_v} \lambda^5_{l^5_5(e_5,jj)} \underbrace{\int_{\partial\bar{\Omega}^5_{e_5}} \phi^5_{l_5(e_5,ii)} \phi^5_{l_5(e_5,jj)} n^5_z}_{g_{ii,jj,n_z}}$$

$$+\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}^e_v} \gamma^5_{l^5_5(e_5,jj)} \underbrace{\int_{\partial\bar{\Omega}^5_{e_5}} \phi^5_{l_5(e_5,ii)} \phi^5_{l_5(e_5,jj)} t^5_z}_{g_{ii,j,t_z}}.$$

Summarising and re-writing we have

$$\bar{\mathcal{M}}^z_i = \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}^{z,0a}_{e,ii} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}^{z,0b}_{e,ii} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}^{z,0c}_{e,ii} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}^{z,0d}_{e,ii}$$

$$+ \frac{2\Delta_t}{3} \frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m^1_z(r_c,z_c)}{Ca} + \frac{2\Delta_t}{3} \frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m^1_z(r_{J^1},z_{J^1})}{Ca}$$

$$+ \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}^1_{\mathrm{el}}} \bar{\mathcal{M}}^{z,1}_{e_1,ii} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}^2_{\mathrm{el}}} \bar{\mathcal{M}}^{z,2}_{e,ii} + \sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}^4_{\mathrm{el}}} \bar{\mathcal{M}}^{z,4}_{e_4,ii}, \tag{26.120}$$

where

$$\bar{\mathcal{M}}^{z,0a}_{e,ii} = -\frac{2\Delta_t St}{3} a_{ii,g_z}(e) + Re\, A a_{ii,\check{w}}(e) \tag{26.121}$$

$$+\frac{2\Delta_t Re}{3} (A)^2 a_{ii,\check{u},\partial_r\check{w}}(e) + \frac{2\Delta_t Re}{3} (A)^2 a_{ii,\check{w},\partial_z\check{w}}(e),$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{z,0b}_{e,ii} &= Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a_{ii,jj}(e) - \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} w_{l(e,jj)}(t_{n-1}) a_{ii,jj}(e) \\
&+ \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} w_{l(e,jj)}(t_{n-2}) a_{ii,jj}(e) \\
&+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a^{r}_{ii,jj,\breve{u}}(e) + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a^{z}_{ii,jj,\breve{w}}(e) \\
&+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} a_{ii,jj,\partial_r \breve{w}}(e) + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a_{ii,jj,\partial_z \breve{w}}(e) \\
&+ \frac{4\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a^{z,z}_{ii,jj}(e) \\
&+ \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} a^{r,z}_{ii,jj}(e) + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a^{r,r}_{ii,jj}(e) \\
&- Re\, A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)} a_{ii,jj,\partial_r \breve{w}}(e) + \frac{4Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-1}) a_{ii,jj,\partial_r \breve{w}}(e) \\
&- \frac{Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-2}) a_{ii,jj,\partial_r \breve{w}}(e) \\
&- Re\, A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)} a_{ii,jj,\partial_z \breve{w}}(e) + \frac{4Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-1}) a_{ii,jj,\partial_z \breve{w}}(e) \\
&- \frac{Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-2}) a_{ii,jj,\partial_z \breve{w}}(e),
\end{aligned}
\tag{26.122}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}_i^{z,0c} = {} & \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{u}_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
& + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{w}_{l(e,kk)} a^z_{ii,kk,jj}(e) \\
& - Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)} a^r_{ii,kk,jj}(e) \\
& + \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-1}) a^r_{ii,kk,jj}(e) \\
& - \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-2}) a^r_{ii,kk,jj}(e) \\
& - Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)} a^z_{ii,kk,jj}(e) \\
& + \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-1}) a^z_{ii,kk,jj}(e) \\
& - \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-2}) a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{26.123}
$$

$$
\bar{\mathcal{M}}_i^{z,0d} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_p^e} p_{l^p(e,jj)} b^z_{jj,ii}(e) \tag{26.124}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}_{e_1,ii}^{z,1} & = -\frac{4\Delta_t}{3} Ac_{ii,n_z,\partial_z \breve{w}}(e) - \frac{2\Delta_t}{3} Ac_{ii,n_r,\partial_z \breve{u}}(e) \\
& -\frac{2\Delta_t}{3} Ac_{ii,n_r,\partial_r \breve{w}}(e) \\
& + \frac{2\Delta_t}{3Ca} \sum_{jj\ =1}^{n_v^e} \sigma^1_{l^1_1(e_1,jj)} c^s_{jj,ii,t_z}(e) - \frac{2\Delta_t}{3} \sum_{j\ =1}^{n_v} p^g_{l^1_1(e_1,jj)} c_{ii,jj,n_z}(e_1),
\end{aligned}
\tag{26.125}
$$

$$
\begin{aligned}
\bar{\mathcal{M}}^{z,2}_{e_2,ii} \; &= \frac{2\Delta_t Be}{3} Ad_{ii,t_r,t_z,\breve{u}}(e) + \frac{2\Delta_t Be}{3} Ad_{ii,t_z,t_z,\breve{w}}(e) \\
&-\frac{4\Delta_t}{3} Ad_{ii,n_r,n_r,n_z,\partial_r \breve{u}}(e) - \frac{4\Delta_t}{3} Ad_{ii,n_r,n_z,n_z,\partial_z \breve{u}}(e) \\
&-\frac{4\Delta_t}{3} Ad_{ii,n_r,n_z,n_z,\partial_r \breve{w}}(e) - \frac{4\Delta_t}{3} Ad_{ii,n_z,n_z,n_z,\partial_z \breve{w}}(e) \\
&-\frac{4\Delta_t}{3} Ad_{ii,t_r,t_z,n_r,\partial_r \breve{u}}(e) - \frac{2\Delta_t}{3} Ad_{ii,t_r,t_z,n_z,\partial_z \breve{u}}(e) \\
&-\frac{2\Delta_t}{3} Ad_{ii,t_z,t_z,n_r,\partial_z \breve{u}}(e) - \frac{2\Delta_t}{3} Ad_{ii,t_z,t_z,n_r,\partial_r \breve{w}}(e) \\
&-\frac{2\Delta_t}{3} Ad_{ii,t_r,t_z,n_z,\partial_r \breve{w}}(e) - \frac{4\Delta_t}{3} Ad_{ii,t_z,t_z,n_z,\partial_z \breve{w}}(e) \\
&-\frac{2\Delta_t Be}{3} \sum_{jj\;=1}^{n_v} \tilde{u}^s_{l_2^2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2) - \frac{2\Delta_t Be}{3} \sum_{jj\;=1}^{n_v} \tilde{w}^s_{l_2^2(e_2,jj)} d_{ii,jj,t_z,t_z}(e_2) \\
&+\frac{2\Delta_t Be}{3} \sum_{jj\;=1}^{\bar{n}_v^{e2}} \bar{u}_{l_2(e_2,jj)} d_{ii,jj,t_r,t_z}(e) \\
&+\frac{2\Delta_t Be}{3} \sum_{jj\;=1}^{\bar{n}_v^{e2}} \bar{w}_{l_2(e_2,jj)} d_{ii,jj,t_z,t_z}(e) \\
&-\frac{\Delta_t}{3Ca} \sum_{jj\;=1}^{n_v} \sigma^2_{l_2^2(e_2,jj)} d^s_{ii,jj,t_z}(e_2) + \frac{2\Delta_t}{3} \sum_{jj\;=1}^{\bar{n}_v^{e2}} \lambda^2_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e),
\end{aligned}
\tag{26.126}
$$

and

$$
\begin{aligned}
\bar{\mathcal{M}}^{z,5}_{e_5,ii} &= -\frac{4\Delta_t}{3} Ag_{ii,n_z,\partial_z \breve{w}} - \frac{2\Delta_t}{3} Ag_{ii,n_r,\partial_z \breve{u}} - \frac{2\Delta_t}{3} Ag_{ii,n_r,\partial_r \breve{w}} \\
&\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e5}} \lambda^5_{l_5^5(e_5,jj)} g_{ii,jj,n_z} \\
&+\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e}} \gamma^5_{l_5^5(e_5,jj)} g_{ii,j,t_z}.
\end{aligned}
\tag{26.127}
$$

26.1. Jacobian terms

We now calculate the derivatives of $\bar{\mathcal{M}}_i^z$ with respect to $\bar{u}_q$, $\bar{w}_q$, p_q, σ_q^1, θ_c, σ_q^2, λ_q^2, λ_q^5, γ_q^5, A and h_q.

26.1.1. Derivatives of $\bar{\mathcal{M}}_i^z$ with respect to $\bar{u}_q$

Using equation (26.87) and equations (26.121)-(**??**) we have

$$\partial_{\bar{w}_q}\bar{\mathcal{M}}_i^z = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\mathrm{el}}} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,ii}^{z,0c} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{\bar{u}_q}\bar{\mathcal{M}}_{e_2,ii}^{z,2}, \tag{26.128}$$

where the terms that do not depend on $\bar{u}_q$ have been removed. Expanding each term we have

$$\partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,i}^{z,0b} = \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\check{w}}(e)\, \underbrace{\partial_{\bar{u}_q}\bar{u}_{l(e,jj)}}_{\delta_{q,l(e,jj)}} + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}^{r,z}(e)\, \underbrace{\partial_{\bar{u}_q}\bar{u}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}, \tag{26.129}$$

$$\partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,i}^{z,0c} = \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} a_{ii,kk,jj}^{r}(e)\, \underbrace{\partial_{\bar{u}_q}\bar{u}_{l(e,kk)}}_{\delta_{q,l(e,kk)}}, \tag{26.130}$$

and

$$\partial_{\bar{u}_q}\bar{\mathcal{M}}_{e_2,ii}^{z,2} = \frac{2\Delta_t Be}{3} \sum_{j=1}^{\bar{n}_v^{e2}} d_{ii,jj,t_r,t_z}(e)\, \underbrace{\partial_{\bar{u}_q}\bar{u}_{l_2(e_2,jj)}}_{\delta_{q,l_2(e_2,jj)}}. \tag{26.131}$$

This yields

$$\partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,i}^{z,0b} = \frac{2\Delta_t Re}{3} A a_{ii,jj,\partial_r\check{w}}(e)|_{q=l(e,jj)} + \frac{2\Delta_t}{3} a_{ii,jj}^{r,z}(e)|_{q=l(e,jj)}, \tag{26.132}$$

i.e.

$$\partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,i}^{z,0b} = \frac{2\Delta_t}{3}\left[a_{ii,jj}^{r,z}(e) + Re\, A a_{ii,jj,\partial_r\check{w}}(e)\right]_{q=l(e,jj)}, \tag{26.133}$$

$$\partial_{\bar{u}_q}\bar{\mathcal{M}}_{e,i}^{z,0c} = \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a_{ii,kk,jj}^{r}(e)|_{q,l(e,kk)}, \tag{26.134}$$

and

$$\partial_{\bar{u}_q}\bar{\mathcal{M}}_{e_2,ii}^{z,2} = \frac{2\Delta_t Be}{3} d_{ii,jj,t_r,t_z}(e)|_{q=l_2(e_2,jj)}. \tag{26.135}$$

26.1.2. **Derivatives of $\bar{\mathcal{M}}_i^z$ with respect to $\bar{w}_q$**

Using equation (26.87) and equations (26.121)-(**??**) we have

$$\partial_{\bar{w}_q}\bar{\mathcal{M}}_i^z = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\text{el}}} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\text{el}}} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{z,0c} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{\bar{w}_q}\bar{\mathcal{M}}_{e_2,ii}^{z,2}, \quad (26.136)$$

where the terms that do not depend on $\bar{w}_q$ have been removed.

Expanding each term we have

$$\begin{aligned}
\partial_{\bar{w}_q}\bar{\mathcal{M}}_{e,ii}^{z,0b} &= Re\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}(e)\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}} + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\check{u}}^r(e)\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}} \\
&+\frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\check{w}}^z(e)\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}} + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\check{w}}(e)\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}} \\
&+\frac{4\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}^{z,z}(e)\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}} \\
&+\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}^{r,r}(e)\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}},
\end{aligned} \quad (26.137)$$

$$\partial_{\bar{w}_q}\bar{\mathcal{M}}^{z,0c}_{e,ii} = \frac{2\Delta_t Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}\sum_{kk=1}^{\bar{n}_v^e}\bar{u}_{l(e,kk)}a^r_{ii,kk,jj}(e) \tag{26.138}$$

$$+\frac{2\Delta_t Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}\sum_{kk=1}^{\bar{n}_v^e}\bar{w}_{l(e,kk)}a^z_{ii,kk,jj}(e)+\frac{2\Delta_t Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}a^z_{ii,kk,jj}(e)\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,kk)}}_{\delta_{q,l(e,kk)}}$$

$$-Re\sum_{jj=1}^{\bar{n}_v^e}\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}a^r_{ii,kk,jj}(e)$$

$$+\frac{4Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}(t_{n-1})a^r_{ii,kk,jj}(e)$$

$$-\frac{Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}(t_{n-2})a^r_{ii,kk,jj}(e)$$

$$-Re\sum_{jj=1}^{\bar{n}_v^e}\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}a^z_{ii,kk,jj}(e)$$

$$+\frac{4Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}(t_{n-1})a^z_{ii,kk,jj}(e)$$

$$-\frac{Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}(t_{n-2})a^z_{ii,kk,jj}(e),$$

and

$$\partial_{\bar{w}_q}\bar{\mathcal{M}}^{z,2}_{e_2,ii} = \frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^{e2}}d_{ii,jj,t_z,t_z}(e)\underbrace{\partial_{\bar{w}_q}\bar{w}_{l_2(e_2,jj)}}_{\delta_{q,l_2(e_2,jj)}}, \tag{26.139}$$

where we have ignored all terms that do not involve $\bar{w}$. This yields

$$\begin{aligned}\partial_{\bar{w}_q}\bar{\mathcal{M}}^{z,0b}_{e,ii} &= Re\,a_{ii,jj}(e)|_{q=l(e,jj)}+\frac{2\Delta_t Re}{3}Aa^r_{ii,jj,\check{u}}(e)|_{q=l(e,jj)}\\ &+\frac{2\Delta_t Re}{3}Aa^z_{ii,jj,\check{w}}(e)|_{q=l(e,jj)}+\frac{2\Delta_t Re}{3}Aa_{ii,jj,\partial_z\check{w}}(e)|_{q=l(e,jj)}\\ &+\frac{4\Delta_t}{3}a^{z,z}_{ii,jj}(e)|_{q=l(e,jj)}\\ &+\frac{2\Delta_t}{3}a^{r,r}_{ii,jj}(e)|_{q=l(e,jj)},\end{aligned} \tag{26.140}$$

i.e.

$$\begin{aligned}\partial_{\bar{w}_q}\bar{\mathcal{M}}^{z,0b}_{e,ii} = \frac{2\Delta_t}{3}&\left[a^{r,r}_{ii,jj}(e)+2a^{z,z}_{ii,jj}(e)\right]_{q=l(e,jj)}\\ &+Re\left\{a_{ii,jj}(e)+\frac{2\Delta_t}{3}A\left[a^r_{ii,jj,\check{u}}(e)+a^z_{ii,jj,\check{w}}(e)+a_{ii,jj,\partial_z\check{w}}(e)\right]\right\}_{q=l(e,jj)},\end{aligned} \tag{26.141}$$

$$
\begin{aligned}
\partial_{\bar{w}_q}\bar{\mathcal{M}}^{z,0c}_{e,ii} &= \frac{2\Delta_t Re}{3}\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v}\bar{u}_{l(e,kk)}a^r_{ii,kk,jj}(e) + \frac{2\Delta_t Re}{3}\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v}\bar{w}_{l(e,kk)}a^z_{ii,kk,jj}(e)\\
&+\frac{2\Delta_t Re}{3}\sum_{\substack{jj=1\\q=l(e,kk)}}^{\bar{n}^e_v}\bar{w}_{l(e,jj)}a^z_{ii,kk,jj}(e) - Re\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v} r^c_{l(e,kk)}a^r_{ii,kk,jj}(e)\\
&+\frac{4Re}{3}\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v} r^c_{l(e,kk)}(t_{n-1})a^r_{ii,kk,jj}(e) - \frac{Re}{3}\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v} r^c_{l(e,kk)}(t_{n-2})a^r_{ii,kk,jj}(e)\\
&-Re\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v} z^c_{l(e,kk)}a^z_{ii,kk,jj}(e) + \frac{4Re}{3}\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v} z^c_{l(e,kk)}(t_{n-1})a^z_{ii,kk,jj}(e)\\
&-\frac{Re}{3}\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v} z^c_{l(e,kk)}(t_{n-2})a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{26.142}
$$

i.e.

$$
\begin{aligned}
\partial_{\bar{w}_q}\bar{\mathcal{M}}^{z,0c}_{e,ii} &= \frac{2\Delta_t Re}{3}\sum_{\substack{jj=1\\q=l(e,kk)}}^{\bar{n}^e_v}\bar{w}_{l(e,jj)}a^z_{ii,kk,jj}(e)\\
&+\frac{2\Delta_t Re}{3}\underbrace{\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v}\left[\bar{u}_{l(e,kk)}a^r_{ii,kk,jj}(e)+\bar{w}_{l(e,kk)}a^z_{ii,kk,jj}(e)\right]}_{\bar{A}_{ii,jj}(e)}\\
&-Re\underbrace{\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v} a^r_{ii,kk,jj}(e)\left[r^c_{l(e,kk)}-\frac{4}{3}r^c_{l(e,kk)}(t_{n-1})+\frac{1}{3}r^c_{l(e,kk)}(t_{n-2})\right]}_{B_{ii,jj}(e)}\\
&-Re\underbrace{\sum_{\substack{kk=1\\q=l(e,jj)}}^{\bar{n}^e_v} a^z_{ii,kk,jj}(e)\left[z^c_{l(e,kk)}-\frac{4}{3}z^c_{l(e,kk)}(t_{n-1})+\frac{1}{3}z^c_{l(e,kk)}(t_{n-2})\right]}_{C_{ii,jj}(e)},
\end{aligned}
\tag{26.143}
$$

and

$$
\partial_{\bar{w}_q}\bar{\mathcal{M}}^{z,2}_{e_2,ii} = \frac{2\Delta_t Be}{3}d_{ii,jj,t_z,t_z}(e)|_{q=l_2(e_2,jj)}. \tag{26.144}
$$

26.1.3. Derivatives of $\bar{\mathcal{M}}_i^z$ with respect to p_q

Using equation (26.87) and equations (26.121)-(**??**) we have

$$\partial_{p_q} \bar{\mathcal{M}}_i^z = \sum_{\substack{e=1 \\ i=l(e,ii)}}^{n_{\text{el}}} \partial_{p_q} \bar{\mathcal{M}}_{e,ii}^{z,0d}, \tag{26.145}$$

ignoring terms that do not depend on p_q and expanding we have

$$\partial_{p_q} \bar{\mathcal{M}}_{e,ii}^{z,0b} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_p^e} b_{jj,ii}^z(e) \underbrace{\partial_{p_q} p_{l^p(e,jj)}}_{\delta_{q,l^p(e,jj)}}, \tag{26.146}$$

i.e.

$$\partial_{p_q} \bar{\mathcal{M}}_{e,ii}^{z,0b} = -\frac{2\Delta_t}{3} b_{jj,ii}^z(e)_{q=l^p(e,jj)}. \tag{26.147}$$

26.1.4. **Derivatives of $\bar{\mathcal{M}}_i^z$ with respect to σ_q^1**

Using equation (26.87) and equations (26.121)-(**??**) we have

$$\partial_{\sigma_q^1}\bar{\mathcal{M}}_i^z = \frac{2\Delta_t}{3}\partial_{\sigma_q^1}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_z^1(r_c,z_c)}{Ca} + \frac{2\Delta_t}{3}\partial_{\sigma_q^1}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^1(r_{J^1},z_{J^1})}{Ca} + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\sigma_q^1}\bar{\mathcal{M}}_{e_1,ii}^{z,1}, \tag{26.148}$$

ignoring terms that do not depend on σ^1 and expanding we have

$$\partial_{\sigma_q^1}\bar{\mathcal{M}}_i^z = \frac{2\Delta_t}{3}\frac{\phi_i(r_c,z_c)m_z^1(r_c,z_c)}{Ca}\underbrace{\partial_{\sigma_q^1}\sigma^1(r_c,z_c)}_{\delta_{q,c}} + \frac{2\Delta_t}{3}\frac{\phi_i(r_{J^1},z_{J^1})m_z^1(r_{J^1},z_{J^1})}{Ca}\underbrace{\partial_{\sigma_q^1}\sigma^1(r_{J^1},z_{J^1})}_{\delta_{q,J^1}} + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\mathrm{el}}^1}\frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^e}c_{jj,ii,t_z}^s(e)\underbrace{\partial_{\sigma_q^1}\sigma_{l_1^1(e_1,jj)}^1}_{\delta_{q,l_1^1(e_1,jj)}}, \tag{26.149}$$

where the sub-indices c and d indicate the boundary-1-node numbers that correspond to the contact line and the apex, respectively. This yields

$$\partial_{\sigma_q^1}\bar{\mathcal{M}}_i^z = \frac{2\Delta_t}{3}\frac{m_z^1(r_c,z_c)}{Ca}\delta_{i,c}\delta_{q,c} + \frac{2\Delta_t}{3}\frac{m_z^1(r_d,z_d)}{Ca}\delta_{i,d}\delta_{q,d} + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)\\ q=l_1^1(e_1,jj)}}^{n_{\mathrm{el}}^1}\frac{2\Delta_t}{3Ca}c_{jj,ii,t_z}^s(e), \tag{26.150}$$

where used that every basis function equals one on its node and zero at all other nodes.

26.1.5. Derivatives of $\bar{\mathcal{M}}_i^z$ with respect to θ_c

Using equation (26.87)

$$\begin{aligned}\partial_{\theta_c}\bar{\mathcal{M}}_i^z = &\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,0c}\\
&+ \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,0d} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)}{Ca}\underbrace{\partial_{\theta_c} m_z^1(r_c,z_c)}_{\partial_{\theta_c}(\sin(\theta_c))}\\
&+\frac{2\Delta_t}{3}\partial_{\theta_c}\frac{\sigma^1(r_{J^1},z_{J^1})\phi_i(r_{J^1},z_{J^1})m_z^1(r_{J^1},z_{J^1})}{Ca}\\
&+ \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\rm el}^1} \partial_{\theta_c}\bar{\mathcal{M}}_{e_1,ii}^{z,1} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\rm el}^2} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\rm el}^5} \partial_{\theta_c}\bar{\mathcal{M}}_{e_5,ii}^{z,5},\end{aligned} \tag{26.151}$$

i.e.

$$\begin{aligned}\partial_{\theta_c}\bar{\mathcal{M}}_i^z = &\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,0c}\\
&+ \sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\rm el}} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,0d} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)}{Ca}\delta_{i,c}\cos(\theta_c)\\
&+ \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\rm el}^1} \partial_{\theta_c}\bar{\mathcal{M}}_{e_1,ii}^{z,1} + \sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\rm el}^2} \partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,2} + \sum_{\substack{e_5=1\\ i=l_5(e,ii)}}^{\bar{n}_{\rm el}^5} \partial_{\theta_c}\bar{\mathcal{M}}_{e_5,ii}^{z,5}.\end{aligned} \tag{26.152}$$

Now, from equation (26.121)

$$\begin{aligned}\partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,0a} = &-\frac{2\Delta_t St}{3}\partial_{\theta_c} a_{ii,g_z}(e) + Re\, A\partial_{\theta_c} a_{ii,\check{w}}(e)\\
&+\frac{2\Delta_t Re}{3}\left(A\right)^2\partial_{\theta_c} a_{ii,\check{u},\partial_r\check{w}}(e) + \frac{2\Delta_t Re}{3}\left(A\right)^2\partial_{\theta_c} a_{ii,\check{w},\partial_z\check{w}}(e),\end{aligned} \tag{26.153}$$

i.e.

$$\partial_{\theta_c}\bar{\mathcal{M}}_{e,ii}^{z,0a} = Re\, A\left[\partial_{\theta_c} a_{ii,\check{w}}(e) + \frac{2\Delta_t}{3}A\left(\partial_{\theta_c} a_{ii,\check{u},\partial_r\check{w}}(e) + \partial_{\theta_c} a_{ii,\check{w},\partial_z\check{w}}(e)\right)\right]. \tag{26.154}$$

From equation (26.122) we have

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}^{z,0b}_{e,ii} &= Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}\partial_{\theta_c}a_{ii,jj}(e) - \frac{4Re}{3}\sum_{jj=1}^{\bar{n}_v^e} w_{l(e,jj)}(t_{n-1})\partial_{\theta_c}a_{ii,jj}(e) \\
&+\frac{Re}{3}\sum_{jj=1}^{\bar{n}_v^e} w_{l(e,jj)}(t_{n-2})\partial_{\theta_c}a_{ii,jj}(e) \\
&+\frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}\partial_{\theta_c}a^{r}_{ii,jj,\breve{u}}(e) + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}\partial_{\theta_c}a^{z}_{ii,jj,\breve{w}}(e) \\
&+\frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_r\breve{w}}(e) + \frac{2\Delta_t Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_z\breve{w}}(e) \\
&+\frac{4\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}\partial_{\theta_c}a^{z,z}_{ii,jj}(e) \\
&+\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)}\partial_{\theta_c}a^{r,z}_{ii,jj}(e) + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}\partial_{\theta_c}a^{r,r}_{ii,jj}(e) \\
&-Re\,A\sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_r\breve{w}}(e) + \frac{4Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-1})\partial_{\theta_c}a_{ii,jj,\partial_r\breve{w}}(e) \\
&-\frac{Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-2})\partial_{\theta_c}a_{ii,jj,\partial_r\breve{w}}(e) \\
&-Re\,A\sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}\partial_{\theta_c}a_{ii,jj,\partial_z\breve{w}}(e) + \frac{4Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-1})\partial_{\theta_c}a_{ii,jj,\partial_z\breve{w}}(e) \\
&-\frac{Re}{3}A\sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-2})\partial_{\theta_c}a_{ii,jj,\partial_z\breve{w}}(e),
\end{aligned}
\tag{26.155}
$$

i.e.

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}^{z,0b}_{e,ii} = {} & \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \partial_{\theta_c} a^r_{ii,jj,\breve{u}}(e) + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \partial_{\theta_c} a^z_{ii,jj,\breve{w}}(e) \\
& + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \partial_{\theta_c} a_{ii,jj,\partial_r \breve{w}}(e) + \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \partial_{\theta_c} a_{ii,jj,\partial_z \breve{w}}(e) \\
& - Re\, A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)} \partial_{\theta_c} a_{ii,jj,\partial_r \breve{w}}(e) + \frac{4Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-1}) \partial_{\theta_c} a_{ii,jj,\partial_r \breve{w}}(e) \\
& - \frac{Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-2}) \partial_{\theta_c} a_{ii,jj,\partial_r \breve{w}}(e) \\
& - Re\, A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)} \partial_{\theta_c} a_{ii,jj,\partial_z \breve{w}}(e) + \frac{4Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-1}) \partial_{\theta_c} a_{ii,jj,\partial_z \breve{w}}(e) \\
& - \frac{Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-2}) \partial_{\theta_c} a_{ii,jj,\partial_z \breve{w}}(e),
\end{aligned}
\tag{26.156}
$$

equivalently

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}^{z,0b}_{e,ii} = \sum_{jj=1}^{\bar{n}_v^e} Re\, A \Bigg\{ & \frac{2\Delta_t}{3} \big[\bar{u}_{l(e,jj)} \partial_{\theta_c} a_{ii,jj,\partial_r \breve{w}}(e) \\
& + \bar{w}_{l(e,jj)} \left(\partial_{\theta_c} a_{ii,jj,\partial_z \breve{w}}(e) + \partial_{\theta_c} a^r_{ii,jj,\breve{u}}(e) + \partial_{\theta_c} a^z_{ii,jj,\breve{w}}(e) \right) \big] \\
& - \partial_{\theta_c} a_{ii,jj,\partial_r \breve{w}}(e) \left[r^c_{l(e,jj)} - \frac{4}{3} r^c_{l(e,jj)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,jj)}(t_{n-2}) \right] \\
& - \partial_{\theta_c} a_{ii,jj,\partial_z \breve{w}}(e) \left[z^c_{l(e,jj)} - \frac{4}{3} z^c_{l(e,jj)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,jj)}(t_{n-2}) \right] \Bigg\}.
\end{aligned}
\tag{26.157}
$$

From equation (26.123) we have

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}_i^{z,0c} = \frac{2\Delta_t Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}\bar{u}_{l(e,kk)}\partial_{\theta_c}a^r_{ii,kk,jj}(e) \\
+\frac{2\Delta_t Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}\bar{w}_{l(e,kk)}\partial_{\theta_c}a^z_{ii,kk,jj}(e) \\
-Re\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}\partial_{\theta_c}a^r_{ii,kk,jj}(e) \\
+\frac{4Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}(t_{n-1})\partial_{\theta_c}a^r_{ii,kk,jj}(e) \\
-\frac{Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}r^c_{l(e,kk)}(t_{n-2})\partial_{\theta_c}a^r_{ii,kk,jj}(e) \\
-Re\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}\partial_{\theta_c}a^z_{ii,kk,jj}(e) \\
+\frac{4Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}(t_{n-1})\partial_{\theta_c}a^z_{ii,kk,jj}(e) \\
-\frac{Re}{3}\sum_{jj=1}^{\bar{n}_v^e}\bar{w}_{l(e,jj)}\sum_{kk=1}^{\bar{n}_v^e}z^c_{l(e,kk)}(t_{n-2})\partial_{\theta_c}a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{26.158}
$$

i.e.

$$\partial_{\theta_c}\bar{\mathcal{M}}_i^{z,0c} = 0. \tag{26.159}$$

From equation (26.124) we have

$$\partial_{\theta_c}\bar{\mathcal{M}}_i^{z,0d} = -\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_p^e} p_{l^p(e,jj)}\partial_{\theta_c}b^z_{jj,ii}(e), \tag{26.160}$$

i.e.

$$\partial_{\theta_c}\bar{\mathcal{M}}_i^{z,0d} = 0. \tag{26.161}$$

From equation (26.125) we have

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}_{e_1,ii}^{z,1} &= -\frac{4\Delta_t}{3}A\partial_{\theta_c}c_{ii,n_z,\partial_z\check{w}}(e) - \frac{2\Delta_t}{3}A\partial_{\theta_c}c_{ii,n_r,\partial_z\check{u}}(e) \\
-\frac{2\Delta_t}{3}A\partial_{\theta_c}c_{ii,n_r,\partial_r\check{w}}(e) \\
&+\frac{2\Delta_t}{3Ca}\sum_{jj=1}^{n_v^e}\sigma^1_{l_1^1(e_1,jj)}\partial_{\theta_c}c^s_{jj,ii,t_z}(e) - \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}p^g_{l_1^1(e_1,jj)}\partial_{\theta_c}c_{ii,jj,n_z}(e_1),
\end{aligned}
\tag{26.162}
$$

i.e.

$$
\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}_{e_1,ii}^{z,1} = -\frac{4\Delta_t}{3}A\partial_{\theta_c}c_{ii,n_z,\partial_z\check{w}}(e) - \frac{2\Delta_t}{3}A\partial_{\theta_c}c_{ii,n_r,\partial_z\check{u}}(e) \\
-\frac{2\Delta_t}{3}A\partial_{\theta_c}c_{ii,n_r,\partial_r\check{w}}(e),
\end{aligned}
\tag{26.163}
$$

equivalently

$$\partial_{\theta_c}\bar{\mathcal{M}}^{z,1}_{e_1,ii} = -\frac{2\Delta_t}{3}A\left[2\partial_{\theta_c}c_{ii,n_z,\partial_z\check{w}}(e) + \partial_{\theta_c}c_{ii,n_r,\partial_z\check{u}}(e) + \partial_{\theta_c}c_{ii,n_r,\partial_r\check{w}}(e)\right]. \quad (26.164)$$

From equation (26.126) we have

$$\begin{aligned}
&\partial_{\theta_c}\bar{\mathcal{M}}^{z,2}_{e_2,ii}\\
&= \frac{2\Delta_t Be}{3}A\partial_{\theta_c}d_{ii,t_r,t_z,\check{u}}(e) + \frac{2\Delta_t Be}{3}A\partial_{\theta_c}d_{ii,t_z,t_z,\check{w}}(e)\\
&-\frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,n_r,n_r,n_z,\partial_r\check{u}}(e) - \frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,n_r,n_z,n_z,\partial_z\check{u}}(e)\\
&-\frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,n_r,n_z,n_z,\partial_r\check{w}}(e) - \frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,n_z,n_z,n_z,\partial_z\check{w}}(e)\\
&-\frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_r,t_z,n_r,\partial_r\check{u}}(e) - \frac{2\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_r,t_z,n_z,\partial_z\check{u}}(e)\\
&-\frac{2\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_z,t_z,n_r,\partial_z\check{u}}(e) - \frac{2\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_z,t_z,n_r,\partial_r\check{w}}(e)\\
&-\frac{2\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_r,t_z,n_z,\partial_r\check{w}}(e) - \frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_z,t_z,n_z,\partial_z\check{w}}(e)\\
&-\frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v}\tilde{u}^s_{l_2^2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,t_r,t_z}(e_2) - \frac{2\Delta_t Be}{3}\sum_{jj=1}^{n_v}\tilde{w}^s_{l_2^2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,t_z,t_z}(e_2)\\
&+\frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^{e_2}}\bar{u}_{l_2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,t_r,t_z}(e)\\
&+\frac{2\Delta_t Be}{3}\sum_{jj=1}^{\bar{n}_v^{e_2}}\bar{w}_{l_2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,t_z,t_z}(e)\\
&-\frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v}\sigma^2_{l_2^2(e_2,jj)}\partial_{\theta_c}d^s_{ii,jj,t_z}(e_2) + \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^{e_2}}\lambda^2_{l_2^2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,n_z}(e),
\end{aligned} \quad (26.165)$$

i.e.

$$\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{M}}^{z,2}_{e_2,ii} = &\frac{2\Delta_t Be}{3}A\partial_{\theta_c}d_{ii,t_r,t_z,\check{u}}(e) + \frac{2\Delta_t Be}{3}A\partial_{\theta_c}d_{ii,t_z,t_z,\check{w}}(e)\\
&-\frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,n_r,n_r,n_z,\partial_r\check{u}}(e) - \frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,n_r,n_z,n_z,\partial_z\check{u}}(e)\\
&-\frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,n_r,n_z,n_z,\partial_r\check{w}}(e) - \frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,n_z,n_z,n_z,\partial_z\check{w}}(e)\\
&-\frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_r,t_z,n_r,\partial_r\check{u}}(e) - \frac{2\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_r,t_z,n_z,\partial_z\check{u}}(e)\\
&-\frac{2\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_z,t_z,n_r,\partial_z\check{u}}(e) - \frac{2\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_z,t_z,n_r,\partial_r\check{w}}(e)\\
&-\frac{2\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_r,t_z,n_z,\partial_r\check{w}}(e) - \frac{4\Delta_t}{3}A\partial_{\theta_c}d_{ii,t_z,t_z,n_z,\partial_z\check{w}}(e),
\end{aligned} \quad (26.166)$$

equivalently

$$
\begin{aligned}
\partial_{\theta_c} \bar{\mathcal{M}}^{z,2}_{e_2,ii} = \frac{2\Delta_t}{3} A \{ & Be \left[\partial_{\theta_c} d_{ii,t_r,t_z,\check{u}}(e) + \partial_{\theta_c} d_{ii,t_z,t_z,\check{w}}(e)\right] \\
& -2\partial_{\theta_c} d_{ii,n_r,n_r,n_z,\partial_r\check{u}}(e) - 2\partial_{\theta_c} d_{ii,n_r,n_z,n_z,\partial_z\check{u}}(e) \\
& -2\partial_{\theta_c} d_{ii,n_r,n_z,n_z,\partial_r\check{w}}(e) - 2\partial_{\theta_c} d_{ii,n_z,n_z,n_z,\partial_z\check{w}}(e) \\
& -2\partial_{\theta_c} d_{ii,t_r,t_z,n_r,\partial_r\check{u}}(e) - \partial_{\theta_c} d_{ii,t_r,t_z,n_z,\partial_z\check{u}}(e) \\
& -\partial_{\theta_c} d_{ii,t_z,t_z,n_r,\partial_z\check{u}}(e) - \partial_{\theta_c} d_{ii,t_z,t_z,n_r,\partial_r\check{w}}(e) \\
& -\partial_{\theta_c} d_{ii,t_r,t_z,n_z,\partial_r\check{w}}(e) - 2\partial_{\theta_c} d_{ii,t_z,t_z,n_z,\partial_z\check{w}}(e)\} \,.
\end{aligned} \tag{26.167}
$$

From equation (26.127) we have

$$
\begin{aligned}
\partial_{\theta_c} \bar{\mathcal{M}}^{z,5}_{e_5,ii} = & -\frac{4\Delta_t}{3} A \partial_{\theta_c} g_{ii,n_z,\partial_z\check{w}} - \frac{2\Delta_t}{3} A \partial_{\theta_c} g_{ii,n_r,\partial_z\check{u}} - \frac{2\Delta_t}{3} A \partial_{\theta_c} g_{ii,n_r,\partial_r\check{w}} \\
& \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e_5}} \lambda^5_{l_5^5(e_5,jj)} \partial_{\theta_c} g_{ii,jj,n_z} \\
& + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e}} \gamma^5_{l_5^5(e_5,jj)} \partial_{\theta_c} g_{ii,j,t_z} ,
\end{aligned} \tag{26.168}
$$

i.e.

$$
\partial_{\theta_c} \bar{\mathcal{M}}^{z,5}_{e_5,ii} = -\frac{4\Delta_t}{3} A \partial_{\theta_c} g_{ii,n_z,\partial_z\check{w}} - \frac{2\Delta_t}{3} A \partial_{\theta_c} g_{ii,n_r,\partial_z\check{u}} - \frac{2\Delta_t}{3} A \partial_{\theta_c} g_{ii,n_r,\partial_r\check{w}} , \tag{26.169}
$$

26.1.6. Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to σ_q^2

Using equation (26.87) and equations (26.121)-(**??**) we have

$$\partial_{\sigma_q^2}\bar{\mathcal{M}}_i^z = \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{\sigma_q^2}\bar{\mathcal{M}}_{e_2,ii}^{z,2}; \tag{26.170}$$

ignoring terms that do not depend on σ^2 and expanding we have

$$\partial_{\sigma_q^2}\bar{\mathcal{M}}_i^z = \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} -\frac{\Delta_t}{3Ca}\sum_{jj=1}^{n_v} d_{ii,jj,t_z}^s(e_2)\,\underbrace{\partial_{\sigma_q^2}\sigma_{l_2^2(e_2,jj)}^2}_{\delta_{q,l_2^2(e_2,jj)}}, \tag{26.171}$$

i.e.

$$\partial_{\lambda_q^2}\bar{\mathcal{M}}_i^z = \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=l_2^2(e_2,jj)}}^{n_{\mathrm{el}}^2} -\frac{\Delta_t}{3Ca} d_{ii,jj,t_z}^s(e_2). \tag{26.172}$$

26.1.7. Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to λ_q^2

Using equation (26.87) and equations (26.121)-(**??**) we have

$$\partial_{\lambda_q^2} \bar{\mathcal{M}}_i^z = \sum_{\substack{e_2=1 \\ i=l_2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{\lambda_q^2} \bar{\mathcal{M}}_{e_2,ii}^{z,2}; \tag{26.173}$$

ignoring terms that do not depend on λ^2 and expanding we have

$$\partial_{\lambda_q^2} \bar{\mathcal{M}}_i^z = \sum_{\substack{e_2=1 \\ i=l_2(e_2,ii)}}^{n_{\text{el}}^2} \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e_2}} d_{ii,jj,n_z}(e) \underbrace{\partial_{\lambda_q^2} \lambda_{l_2^2(e_2,jj)}^2}_{\delta_{q,l_2^2(e_2,jj)}}. \tag{26.174}$$

i.e.

$$\partial_{\lambda_q^2} \bar{\mathcal{M}}_i^z = \sum_{\substack{e_2=1 \\ i=l_2(e_2,ii) \\ q=l_2^2(e_2,jj)}}^{n_{\text{el}}^2} \frac{2\Delta_t}{3} d_{ii,jj,n_z}(e). \tag{26.175}$$

26.1.8. Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to λ_q^5

Using equation (26.87) and equations (26.121)-(**??**) we have

$$\partial_{\lambda_q^5} \bar{\mathcal{M}}_i^z = \sum_{\substack{e_2=1 \\ i=l_5(e_5,ii)}}^{n_{\text{el}}^5} \partial_{\lambda_q^5} \bar{\mathcal{M}}_{e_5,ii}^{z,5}; \tag{26.176}$$

ignoring terms that do not depend on λ^5 and expanding we have

$$\partial_{\lambda_q^5} \bar{\mathcal{M}}_i^z = \sum_{\substack{e_5=1 \\ i=l_5(e_5,ii)}}^{n_{\text{el}}^5} \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e_5}} g_{ii,jj,n_z}(e) \underbrace{\partial_{\lambda_q^5} \lambda_{l_5^5(e_5,jj)}^5}_{\delta_{q,l_5^5(e_5,jj)}}. \tag{26.177}$$

i.e.

$$\partial_{\lambda_q^5} \bar{\mathcal{M}}_i^z = \sum_{\substack{e_5=1 \\ i=l_5(e_5,ii) \\ q=l_5^5(e_5,jj)}}^{n_{\text{el}}^5} \frac{2\Delta_t}{3} g_{ii,jj,n_z}(e). \tag{26.178}$$

26.1.9. **Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to γ_q^5**

Using equation (26.87) and equations (26.121)-(**??**) we have

$$\partial_{\gamma_q^5} \bar{\mathcal{M}}_i^z = \sum_{\substack{e_2=1 \\ i=l_5(e_5,ii)}}^{n_{\mathrm{el}}^5} \partial_{\gamma_q^5} \bar{\mathcal{M}}_{e_5,ii}^{z,5}; \tag{26.179}$$

ignoring terms that do not depend on λ^5 and expanding we have

$$\partial_{\gamma_q^5} \bar{\mathcal{M}}_i^z = \sum_{\substack{e_5=1 \\ i=l_5(e_5,ii)}}^{n_{\mathrm{el}}^5} \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e_5}} g_{ii,jj,t_z}(e) \underbrace{\partial_{\gamma_q^5} \gamma_{l_5^5(e_5,jj)}^5}_{\delta_{q,l_5^5(e_5,jj)}}. \tag{26.180}$$

i.e.

$$\partial_{\gamma_q^5} \bar{\mathcal{M}}_i^z = \sum_{\substack{e_5=1 \\ i=l_5(e_5,ii) \\ q=l_5^5(e_5,jj)}}^{n_{\mathrm{el}}^5} \frac{2\Delta_t}{3} g_{ii,jj,t_z}(e). \tag{26.181}$$

26.1.10. Derivatives of $\bar{\mathcal{M}}_i^r$ with respect to A

Using equation (26.87) and equations (26.121)-(**??**) we have

$$\partial_A \bar{\mathcal{M}}_i^z = \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\rm el}} \partial_A \bar{\mathcal{M}}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\rm el}} \partial_A \bar{\mathcal{M}}_{e,ii}^{z,0b} + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\rm el}^1} \partial_A \bar{\mathcal{M}}_{e_1,ii}^{z,1} \tag{26.182}$$

$$+ \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\rm el}^2} \partial_A \bar{\mathcal{M}}_{e_2,ii}^{z,2} + \sum_{\substack{e_5=1\\ i=l_5(e_5,ii)}}^{n_{\rm el}^5} \partial_A \bar{\mathcal{M}}_{e_5,ii}^{z,5},$$

where

$$\partial_A \bar{\mathcal{M}}_{e,ii}^{z,0a} = -\frac{2\Delta_t St}{3} \partial_A a_{ii,g_z}(e) \tag{26.183}$$

$$+Re\, \partial_A (A)\, a_{ii,\breve{w}}(e) - \frac{4Re}{3} \partial_A (A)\, a_{ii,\breve{w}_{n-1}}(e) + \frac{Re}{3} \partial_A (A)\, a_{ii,\breve{w}_{n-2}}(e)$$

$$+\frac{2\Delta_t Re}{3} a_{ii,\breve{u},\partial_r \breve{w}}(e) \partial_A (A)^2 + \frac{2\Delta_t Re}{3} a_{ii,\breve{w},\partial_z \breve{w}}(e) \partial_A (A)^2,$$

i.e.

$$\boxed{\begin{aligned} \partial_A \bar{\mathcal{M}}_{e,ii}^{z,0a} &= Re\, a_{ii,\breve{w}}(e) \\ &+\frac{4\Delta_t Re}{3} A a_{ii,\breve{u},\partial_r \breve{w}}(e) + \frac{4\Delta_t Re}{3} A a_{ii,\breve{w},\partial_z \breve{w}}(e); \end{aligned} \qquad (26.184)}$$

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}^{z,0b}_{e,ii} &= Re\,\partial_A\left(\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a_{ii,jj}(e)\right) - \frac{4Re}{3}\partial_A\left(\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}(t_{n-1}) a_{ii,jj}(e)\right)\\
&+\frac{Re}{3}\partial_A\left(\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}(t_{n-2}) a_{ii,jj}(e)\right)\\
&+\frac{2\Delta_t Re}{3}\partial_A\left(A\right)\sum_{jj\,=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a^{r}_{ii,jj,\breve{u}}(e) + \frac{2\Delta_t Re}{3}\partial_A\left(A\right)\sum_{jj\,=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a^{z}_{ii,jj,\breve{w}}(e)\\
&+\frac{2\Delta_t Re}{3}\partial_A\left(A\right)\sum_{jj\,=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} a_{ii,jj,\partial_r\breve{w}}(e) + \frac{2\Delta_t Re}{3}\partial_A\left(A\right)\sum_{jj\,=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a_{ii,jj,\partial_z\breve{w}}(e)\\
&+\frac{4\Delta_t}{3}\partial_A\left(\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a^{z,z}_{ii,jj}(e)\right)\\
&+\frac{2\Delta_t}{3}\partial_A\left(\sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} a^{r,z}_{ii,jj}(e)\right) + \frac{2\Delta_t}{3}\partial_A\left(\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a^{r,r}_{ii,jj}(e)\right)\\
&-Re\,\partial_A\left(A\right)\sum_{jj\,=1}^{\bar{n}_v^e} r^c_{l(e,jj)} a_{ii,jj,\partial_r\breve{w}}(e) + \frac{4Re}{3}\partial_A\left(A\right)\sum_{jj\,=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-1}) a_{ii,jj,\partial_r\breve{w}}(e)\\
&-\frac{Re}{3}\partial_A\left(A\right)\sum_{jj\,=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-2}) a_{ii,jj,\partial_r\breve{w}}(e)\\
&-Re\,\partial_A\left(A\right)\sum_{jj\,=1}^{\bar{n}_v^e} z^c_{l(e,jj)} a_{ii,jj,\partial_z\breve{w}}(e) + \frac{4Re}{3}\partial_A\left(A\right)\sum_{jj\,=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-1}) a_{ii,jj,\partial_z\breve{w}}(e)\\
&-\frac{Re}{3}\partial_A\left(A\right)\sum_{jj\,=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-2}) a_{ii,jj,\partial_z\breve{w}}(e),
\end{aligned}
\tag{26.185}
$$

i.e.

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}_{e,ii}^{z,0b} &= \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a_{ii,jj,\breve{u}}^{r}(e) + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a_{ii,jj,\breve{w}}^{z}(e) \\
&+ \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} a_{ii,jj,\partial_r \breve{w}}(e) + \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a_{ii,jj,\partial_z \breve{w}}(e) \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} r_{l(e,jj)}^{c} a_{ii,jj,\partial_r \breve{w}}(e) + \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} r_{l(e,jj)}^{c}(t_{n-1}) a_{ii,jj,\partial_r \breve{w}}(e) \\
&- \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} r_{l(e,jj)}^{c}(t_{n-2}) a_{ii,jj,\partial_r \breve{w}}(e) \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} z_{l(e,jj)}^{c} a_{ii,jj,\partial_z \breve{w}}(e) + \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} z_{l(e,jj)}^{c}(t_{n-1}) a_{ii,jj,\partial_z \breve{w}}(e) \\
&- \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} z_{l(e,jj)}^{c}(t_{n-2}) a_{ii,jj,\partial_z \breve{w}}(e),
\end{aligned}
\tag{26.186}
$$

i.e.

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}_{e,ii}^{z,0b} &= \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} a_{ii,jj,\partial_r \breve{w}}(e) \\
&+ \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \left[a_{ii,jj,\breve{u}}^{r}(e) + a_{ii,jj,\breve{w}}^{z}(e) + a_{ii,jj,\partial_z \breve{w}}(e) \right] \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r \breve{w}}(e) \left[r_{l(e,jj)}^{c} - \frac{4}{3} r_{l(e,jj)}^{c}(t_{n-1}) + \frac{1}{3} r_{l(e,jj)}^{c}(t_{n-2}) \right] \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z \breve{w}}(e) \left[z_{l(e,jj)}^{c} - \frac{4}{3} z_{l(e,jj)}^{c}(t_{n-1}) + \frac{1}{3} z_{l(e,jj)}^{c}(t_{n-2}) \right],
\end{aligned}
\tag{26.187}
$$

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}_{e_1,ii}^{z,1} &= -\frac{4\Delta_t}{3} \partial_A (A)\, c_{ii,n_z,\partial_z \breve{w}}(e) - \frac{2\Delta_t}{3} \partial_A (A)\, c_{ii,n_r,\partial_z \breve{u}}(e) \\
&- \frac{2\Delta_t}{3} \partial_A (A)\, c_{ii,n_r,\partial_r \breve{w}}(e) \\
&+ \partial_A \left(\frac{2\Delta_t}{3Ca} \sum_{jj=1}^{n_v^e} \sigma_{l_1^1(e_1,jj)}^{1} c_{jj,ii,t_z}^{s}(e) \right) - \partial_A \left(\frac{2\Delta_t}{3} \sum_{j=1}^{n_v} p_{l_1^1(e_1,jj)}^{g} c_{ii,jj,n_z}(e_1) \right),
\end{aligned}
\tag{26.188}
$$

i.e.

$$
\boxed{
\begin{aligned}
\partial_A \bar{\mathcal{M}}_{e_1,ii}^{z,1} &= -\frac{4\Delta_t}{3} c_{ii,n_z,\partial_z \breve{w}}(e) - \frac{2\Delta_t}{3} c_{ii,n_r,\partial_z \breve{u}}(e) \\
&- \frac{2\Delta_t}{3} c_{ii,n_r,\partial_r \breve{w}}(e);
\end{aligned}
}
\tag{26.189}
$$

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}^{z,2}_{e_2,ii} &= \frac{2\Delta_t Be}{3}\partial_A\left(A\right) d_{ii,t_r,t_z,\breve{u}}(e) + \frac{2\Delta_t Be}{3}\partial_A\left(A\right) d_{ii,t_z,t_z,\breve{w}}(e) \\
&-\frac{4\Delta_t}{3}\partial_A\left(A\right) d_{ii,n_r,n_r,n_z,\partial_r\breve{u}}(e) - \frac{4\Delta_t}{3}\partial_A\left(A\right) d_{ii,n_r,n_z,n_z,\partial_z\breve{u}}(e) \\
&-\frac{4\Delta_t}{3}\partial_A\left(A\right) d_{ii,n_r,n_z,n_z,\partial_r\breve{w}}(e) - \frac{4\Delta_t}{3}\partial_A\left(A\right) d_{ii,n_z,n_z,n_z,\partial_z\breve{w}}(e) \\
&-\frac{4\Delta_t}{3}\partial_A\left(A\right) d_{ii,t_r,t_z,n_r,\partial_r\breve{u}}(e) - \frac{2\Delta_t}{3}\partial_A\left(A\right) d_{ii,t_r,t_z,n_z,\partial_z\breve{u}}(e) \\
&-\frac{2\Delta_t}{3}\partial_A\left(A\right) d_{ii,t_z,t_z,n_r,\partial_z\breve{u}}(e) - \frac{2\Delta_t}{3}\partial_A\left(A\right) d_{ii,t_z,t_z,n_r,\partial_r\breve{w}}(e) \\
&-\frac{2\Delta_t}{3}\partial_A\left(A\right) d_{ii,t_r,t_z,n_z,\partial_r\breve{w}}(e) - \frac{4\Delta_t}{3}\partial_A\left(A\right) d_{ii,t_z,t_z,n_z,\partial_z\breve{w}}(e) \\
&-\frac{2\Delta_t Be}{3}\partial_A \sum_{jj=1}^{n_v} \tilde{u}^s_{l_2^2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2) - \frac{2\Delta_t Be}{3}\partial_A \sum_{jj=1}^{n_v} \tilde{w}^s_{l_2^2(e_2,jj)} d_{ii,jj,t_z,t_z}(e_2) \\
&+\frac{2\Delta_t Be}{3}\partial_A \sum_{jj=1}^{\bar{n}_v^{e2}} \bar{u}_{l_2(e_2,jj)} d_{ii,jj,t_r,t_z}(e) \\
&+\frac{2\Delta_t Be}{3}\partial_A \sum_{jj=1}^{\bar{n}_v^{e2}} \bar{w}_{l_2(e_2,jj)} d_{ii,jj,t_z,t_z}(e) \\
&-\frac{\Delta_t}{3Ca}\partial_A \sum_{jj=1}^{n_v} \sigma^2_{l_2^2(e_2,jj)} d^s_{ii,jj,t_z}(e_2) + \frac{2\Delta_t}{3}\partial_A \sum_{jj=1}^{\bar{n}_v^{e2}} \lambda^2_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e),
\end{aligned}
\tag{26.190}
$$

i.e.

$$
\boxed{
\begin{aligned}
\partial_A \bar{\mathcal{M}}^{z,2}_{e_2,ii} &= \frac{2\Delta_t Be}{3} d_{ii,t_r,t_z,\breve{u}}(e) + \frac{2\Delta_t Be}{3} d_{ii,t_z,t_z,\breve{w}}(e) \\
&-\frac{4\Delta_t}{3} d_{ii,n_r,n_r,n_z,\partial_r\breve{u}}(e) - \frac{4\Delta_t}{3} d_{ii,n_r,n_z,n_z,\partial_z\breve{u}}(e) \\
&-\frac{4\Delta_t}{3} d_{ii,n_r,n_z,n_z,\partial_r\breve{w}}(e) - \frac{4\Delta_t}{3} d_{ii,n_z,n_z,n_z,\partial_z\breve{w}}(e) \\
&-\frac{4\Delta_t}{3} d_{ii,t_r,t_z,n_r,\partial_r\breve{u}}(e) - \frac{2\Delta_t}{3} d_{ii,t_r,t_z,n_z,\partial_z\breve{u}}(e) \\
&-\frac{2\Delta_t}{3} d_{ii,t_z,t_z,n_r,\partial_z\breve{u}}(e) - \frac{2\Delta_t}{3} d_{ii,t_z,t_z,n_r,\partial_r\breve{w}}(e) \\
&-\frac{2\Delta_t}{3} d_{ii,t_r,t_z,n_z,\partial_r\breve{w}}(e) - \frac{4\Delta_t}{3} d_{ii,t_z,t_z,n_z,\partial_z\breve{w}}(e),
\end{aligned}
\qquad (26.191)}
$$

and

$$
\begin{aligned}
\partial_A \bar{\mathcal{M}}^{z,5}_{e_5,ii} = &-\frac{4\Delta_t}{3}\partial_A\left(A\right) g_{ii,n_z,\partial_z\breve{w}} - \frac{2\Delta_t}{3}\partial_A\left(A\right) g_{ii,n_r,\partial_z\breve{u}} - \frac{2\Delta_t}{3}\partial_A\left(A\right) g_{ii,n_r,\partial_r\breve{w}} \\
&\frac{2\Delta_t}{3}\partial_A \sum_{jj=1}^{\bar{n}_v^{e5}} \lambda^5_{l_5^5(e_5,jj)} g_{ii,jj,n_z} \\
&+\frac{2\Delta_t}{3}\partial_A \sum_{jj=1}^{\bar{n}_v^{e}} \gamma^5_{l_5^5(e_5,jj)} g_{ii,j,t_z},
\end{aligned}
\tag{26.192}
$$

i.e.

$$\boxed{\partial_A \bar{\mathcal{M}}^{z,5}_{e_5,ii} = -\frac{4\Delta_t}{3} g_{ii,n_z,\partial_z \breve{w}} - \frac{2\Delta_t}{3} g_{ii,n_r,\partial_z \breve{u}} - \frac{2\Delta_t}{3} g_{ii,n_r,\partial_r \breve{w}}}. \tag{26.193}$$

26.1.11. Derivatives of $\bar{\mathcal{M}}_i^z$ with respect to h_q

We denote the spine lengths by h, and we consider the derivatives of the residuals with respect to the length of each spine.

From equation (**??**) we have

$$\partial_{h_q}\bar{\mathcal{M}}_i^z = \frac{2\Delta_t}{3}\frac{\sigma_c^1}{Ca}\delta_{i,c}\partial_{h_q}m_z^1(r_c,z_c) + \frac{2\Delta_t}{3}\frac{\sigma_d^1}{Ca}\delta_{i,d}\partial_{h_q}m_z^1(r_d,z_d) + \sum_{\substack{e=1\\ i=l(e,ii)}}^{n_{\text{el}}} \partial_{h_q}\bar{\mathcal{M}}_{e,ii}^{z,0}$$
$$+ \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n_{\text{el}}^1} \partial_{h_q}\bar{\mathcal{M}}_{e_1,ii}^{r,1} + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{h_q}\bar{\mathcal{M}}_{e_2,ii}^{r,2} + \sum_{\substack{e_4=1\\ i=l_4(e_4,ii)}}^{n_{\text{el}}^4} \partial_{h_q}\bar{\mathcal{M}}_{e_4,ii}^{r,4}. \tag{26.194}$$

We notice that in the sum by elements above, it is only those spines that contain nodes in these elements that are going to have an effect on each of the derivatives shown above. Put differently, the vast majority of the derivatives above will be identically null. Hence, we once again resort to a function that maps objects in the element to the global number of these elements. Here we define as the "local spines" of an element a those spines that contain nodes that are part of the element being considered, and we number those spines with a local spine number (from 1 to the number of spines that contain nodes of the element). We then introduce the local-spine-number to global-spine-number map $S(e,qq)=q$, which maps the qq-th local spine number on element e to its global spines number (previously referred to as simply *the spine number*) q. Similarly, we define $S_i(e_i,qq)=q$, which maps the local spine number qq of element e_i on boundary i to its global spine number q.

Thus using local spine numbers we have

$$\partial_{h_q}\bar{\mathcal{M}}_i^r = \frac{2\Delta_t}{3}\frac{\sigma_c^1}{Ca}\delta_{i,c}\partial_{h_q}m_z^1(r_c,z_c) + \frac{2\Delta_t}{3}\frac{\sigma_d^1}{Ca}\delta_{i,d}\partial_{h_q}m_z^1(r_d,z_d)$$
$$+ \sum_{\substack{e=1\\ i=l(e,ii)\\ q=S(e,qq)}}^{n_{\text{el}}} \partial_{h_{S(e,qq)}}\bar{\mathcal{M}}_{e,ii}^{z,0a} + \sum_{\substack{e=1\\ i=l(e,ii)\\ q=S(e,qq)}}^{n_{\text{el}}} \partial_{h_{S(e,qq)}}\bar{\mathcal{M}}_{e,ii}^{z,0b} + \sum_{\substack{e=1\\ i=l(e,ii)\\ q=S(e,qq)}}^{n_{\text{el}}} \partial_{h_{S(e,qq)}}\bar{\mathcal{M}}_{e,ii}^{z,0c}$$
$$+ \sum_{\substack{e=1\\ i=l(e,ii)\\ q=S(e,qq)}}^{n_{\text{el}}} \partial_{h_{S(e,qq)}}\bar{\mathcal{M}}_{e,ii}^{z,0d} + \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\text{el}}^1} \partial_{h_{S_1(e_1,qq)}}\bar{\mathcal{M}}_{e_1,ii}^{z,1}$$
$$+ \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\text{el}}^2} \partial_{h_{S_2(e_2,qq)}}\bar{\mathcal{M}}_{e_2,ii}^{z,2} + \sum_{\substack{e_5=1\\ i=l_5(e_5,ii)\\ q=S_5(e_5,qq)}}^{n_{\text{el}}^5} \partial_{h_{S_4(e_5,qq)}}\bar{\mathcal{M}}_{e_5,ii}^{z,5}. \tag{26.195}$$

Then; we have, from equation (26.121),

$$\begin{aligned}\partial_{h_{S(e,qq)}}\bar{\mathcal{M}}_{e,ii}^{z,0a} &= -\frac{2\Delta_t St}{3}\partial_{h_{S(e,qq)}}a_{ii,g_z}(e) + Re\,A\partial_{h_{S(e,qq)}}a_{ii,\breve{w}}(e)\\ +\frac{2\Delta_t Re}{3}(A)^2\,\partial_{h_{S(e,qq)}}a_{ii,\breve{u},\partial_r\breve{w}}(e) &+ \frac{2\Delta_t Re}{3}(A)^2\,\partial_{h_{S(e,qq)}}a_{ii,\breve{w},\partial_z\breve{w}}(e),\end{aligned} \tag{26.196}$$

i.e.

$$\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{z,0a}_{e,ii} = Re\, A \partial_{h_{S(e,qq)}} a_{ii,\check{w}}(e)$$
$$\frac{2\Delta_t}{3} \left\{ -St\, \partial_{h_{S(e,qq)}} a_{ii,g_z}(e) \right. \tag{26.197}$$
$$\left. + Re\, (A)^2 \left[\partial_{h_{S(e,qq)}} a_{ii,\check{u},\partial_r \check{w}}(e) + \partial_{h_{S(e,qq)}} a_{ii,\check{w},\partial_z \check{w}}(e) \right] \right\},$$

from equation (26.122),

$$\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{z,0b}_{e,ii} = Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \partial_{h_{S(e,qq)}} a_{ii,jj}(e) \tag{26.198}$$
$$- \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}(t_{n-1}) \partial_{h_{S(e,qq)}} a_{ii,jj}(e)$$
$$+ \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}(t_{n-2}) \partial_{h_{S(e,qq)}} a_{ii,jj}(e)$$
$$+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \partial_{h_{S(e,qq)}} a^{r}_{ii,jj,\check{u}}(e)$$
$$+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \partial_{h_{S(e,qq)}} a^{z}_{ii,jj,\check{w}}(e)$$
$$+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_r \check{w}}(e)$$
$$+ \frac{2\Delta_t Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_z \check{w}}(e)$$
$$+ \frac{4\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \partial_{h_{S(e,qq)}} a^{z,z}_{ii,jj}(e)$$
$$+ \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \partial_{h_{S(e,qq)}} a^{r,z}_{ii,jj}(e) + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \partial_{h_{S(e,qq)}} a^{r,r}_{ii,jj}(e)$$
$$- Re\, A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)} \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_r \check{w}}(e) - Re\, A \sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r \check{w}}(e) \partial_{h_{S(e,qq)}} r^c_{l(e,jj)}$$
$$+ \frac{4Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-1}) \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_r \check{w}}(e) - \frac{Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} r^c_{l(e,jj)}(t_{n-2}) \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_r \check{w}}(e)$$
$$- Re\, A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)} \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_z \check{w}}(e) - Re\, A \sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z \check{w}}(e) \partial_{h_{S(e,qq)}} z^c_{l(e,jj)}$$
$$+ \frac{4Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-1}) \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_z \check{w}}(e) - \frac{Re}{3} A \sum_{jj=1}^{\bar{n}_v^e} z^c_{l(e,jj)}(t_{n-2}) \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_z \check{w}}(e),$$

i.e.

$$\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{z,0b}_{e,ii} = \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)} \left[\partial_{h_{S(e,qq)}} a^{r,z}_{ii,jj}(e) + Re\, A \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_r \check{w}}(e)\right]$$

$$+ \frac{2\Delta_t}{3} \bar{w}_{l(e,jj)} \sum_{jj\;=1}^{\bar{n}_v^e} \left\{\partial_{h_{S(e,qq)}} a^{r,r}_{ii,jj}(e) + 2\partial_{h_{S(e,qq)}} a^{z,z}_{ii,jj}(e)\right.$$

$$\left. + Re\, A \left[\partial_{h_{S(e,qq)}} a^{r}_{ii,jj,\check{u}}(e) + \partial_{h_{S(e,qq)}} a^{z}_{ii,jj,\check{w}}(e) + \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_z \check{w}}(e)\right]\right\}$$

$$+ Re \sum_{jj\;=1}^{\bar{n}_v^e} \partial_{h_{S(e,qq)}} a_{ii,jj}(e) \left[\bar{w}_{l(e,jj)} - \frac{4}{3}\bar{w}_{l(e,jj)}(t_{n-1}) + \frac{1}{3}\bar{w}_{l(e,jj)}(t_{n-2})\right]$$

$$- Re\, A \sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r \check{w}}(e) \partial_{h_{S(e,qq)}} r^c_{l(e,jj)}$$

$$- Re\, A \sum_{jj\;=1}^{\bar{n}_v^e} \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_r \check{w}}(e) \left[r^c_{l(e,jj)} - \frac{4}{3} r^c_{l(e,jj)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,jj)}(t_{n-2})\right]$$

$$- Re\, A \sum_{jj\;=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z \check{w}}(e) \partial_{h_{S(e,qq)}} z^c_{l(e,jj)}$$

$$- Re\, A \sum_{jj\;=1}^{\bar{n}_v^e} \partial_{h_{S(e,qq)}} a_{ii,jj,\partial_z \check{w}}(e) \left[z^c_{l(e,jj)} - \frac{4}{3} z^c_{l(e,jj)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,jj)}(t_{n-2})\right]$$

$$- \frac{2\Delta_t}{3} \sum_{jj\;=1}^{\bar{n}_p^e} p_{l^p(e,jj)} \partial_{h_{S(e,qq)}} b^{z}_{jj,ii}(e), \tag{26.199}$$

from equation (26.123),

$$
\begin{aligned}
\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{z,0c}_{e,ii} &= \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{u}_{l(e,kk)} \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \\
&+ \frac{2\Delta_t Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} \bar{w}_{l(e,kk)} \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)} \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} a^r_{ii,kk,jj}(e) \partial_{h_{S(e,qq)}} r^c_{l(e,kk)} \\
&+ \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-1}) \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \\
&- \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} r^c_{l(e,kk)}(t_{n-2}) \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)} \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \\
&- Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} a^z_{ii,kk,jj}(e) \partial_{h_{S(e,qq)}} z^c_{l(e,kk)} \\
&+ \frac{4Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-1}) \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \\
&- \frac{Re}{3} \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \sum_{kk=1}^{\bar{n}_v^e} z^c_{l(e,kk)}(t_{n-2}) \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e),
\end{aligned}
\tag{26.200}
$$

i.e.

$$\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{z,0c}_{e,ii} = Re \sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} \left\{ \frac{2\Delta_t}{3} \sum_{kk=1}^{\bar{n}_v^e} \left[\bar{u}_{l(e,kk)} \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \right.\right.$$

$$\left. + \bar{w}_{l(e,kk)} \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \right]$$

$$- \sum_{kk=1}^{\bar{n}_v^e} a^r_{ii,kk,jj}(e) \partial_{h_{S(e,qq)}} r^c_{l(e,kk)}$$

$$- \sum_{kk=1}^{\bar{n}_v^e} \partial_{h_{S(e,qq)}} a^r_{ii,kk,jj}(e) \left[r^c_{l(e,kk)} + \frac{4}{3} r^c_{l(e,kk)}(t_{n-1}) - \frac{1}{3} r^c_{l(e,kk)}(t_{n-2}) \right]$$

$$- \sum_{kk=1}^{\bar{n}_v^e} a^z_{ii,kk,jj}(e) \partial_{h_{S(e,qq)}} z^c_{l(e,kk)}$$

$$\left. - \sum_{kk=1}^{\bar{n}_v^e} \partial_{h_{S(e,qq)}} a^z_{ii,kk,jj}(e) \left[z^c_{l(e,kk)} - \frac{4}{3} z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,kk)}(t_{n-2}) \right] \right\},$$

(26.201)

$$\partial_{h_{S(e,qq)}} \bar{\mathcal{M}}^{z,0d}_{e,ii} = -\frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_p^e} p_{l^p(e,jj)} \partial_{h_{S(e,qq)}} b^z_{jj,ii}(e). \tag{26.202}$$

from equation (26.125),

$$\partial_{h_{S_1(e_1,qq)}} \bar{\mathcal{M}}^{z,1}_{e_1,ii} = -\frac{4\Delta_t}{3} A \partial_{h_{S_1(e_1,qq)}} c_{ii,n_z,\partial_z \check{w}}(e) - \frac{2\Delta_t}{3} A \partial_{h_{S_1(e_1,qq)}} c_{ii,n_r,\partial_z \check{u}}(e)$$

$$-\frac{2\Delta_t}{3} A \partial_{h_{S_1(e_1,qq)}} c_{ii,n_r,\partial_r \check{w}}(e) + \frac{2\Delta_t}{3Ca} \sum_{jj\,=1}^{n_v^e} \sigma^1_{l_1^1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c^s_{jj,ii,t_z}(e)$$

$$- \frac{2\Delta_t}{3} \sum_{j\,=1}^{n_v} p^g_{l_1^1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_z}(e_1),$$

(26.203)

from equation (26.126),

$$\begin{aligned}
\partial_{h_{S_2(e_2,qq)}}\bar{\mathcal{M}}^{z,2}_{e_2,ii} &= \frac{2\Delta_t Be}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,t_r,t_z,\breve{u}}(e) \\
&+\frac{2\Delta_t Be}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,t_z,t_z,\breve{w}}(e) \\
&-\frac{4\Delta_t}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,n_r,n_r,n_z,\partial_r\breve{u}}(e)-\frac{4\Delta_t}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,n_r,n_z,n_z,\partial_z\breve{u}}(e) \\
&-\frac{4\Delta_t}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,n_r,n_z,n_z,\partial_r\breve{w}}(e)-\frac{4\Delta_t}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,n_z,n_z,n_z,\partial_z\breve{w}}(e) \\
&-\frac{4\Delta_t}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,t_r,t_z,n_r,\partial_r\breve{u}}(e)-\frac{2\Delta_t}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,t_r,t_z,n_z,\partial_z\breve{u}}(e) \\
&-\frac{2\Delta_t}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,t_z,t_z,n_r,\partial_z\breve{u}}(e)-\frac{2\Delta_t}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,t_z,t_z,n_r,\partial_r\breve{w}}(e) \\
&-\frac{2\Delta_t}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,t_r,t_z,n_z,\partial_r\breve{w}}(e)-\frac{4\Delta_t}{3}A\partial_{h_{S_2(e_2,qq)}}d_{ii,t_z,t_z,n_z,\partial_z\breve{w}}(e) \\
&-\frac{2\Delta_t Be}{3}\partial_A\sum_{jj=1}^{n_v}\tilde{u}^s_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_r,t_z}(e_2)-\frac{2\Delta_t Be}{3}\partial_A\sum_{jj=1}^{n_v}\tilde{w}^s_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_z,t_z}(e_2) \\
&+\frac{2\Delta_t Be}{3}\partial_A\sum_{jj=1}^{\bar{n}_v^{e_2}}\bar{u}_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_r,t_z}(e) \\
&+\frac{2\Delta_t Be}{3}\partial_A\sum_{jj=1}^{\bar{n}_v^{e_2}}\bar{w}_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_z,t_z}(e) \\
&-\frac{\Delta_t}{3Ca}\partial_A\sum_{jj=1}^{n_v}\sigma^2_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d^s_{ii,jj,t_z}(e_2)+\frac{2\Delta_t}{3}\partial_A\sum_{jj=1}^{\bar{n}_v^{e_2}}\lambda^2_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_z}(e),
\end{aligned} \tag{26.204}$$

and from equation (26.127),

$$\begin{aligned}
\partial_{h_{S_5(e_5,qq)}}\bar{\mathcal{M}}^{z,5}_{e_5,ii} = &-\frac{4\Delta_t}{3}A\partial_{h_{S_5(e_5,qq)}}g_{ii,n_z,\partial_z\breve{u}}-\frac{2\Delta_t}{3}A\partial_{h_{S_5(e_5,qq)}}g_{ii,n_r,\partial_z\breve{u}} \\
&-\frac{2\Delta_t}{3}A\partial_{h_{S_5(e_5,qq)}}g_{ii,n_r,\partial_r\breve{w}} \\
&\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^{e5}}\lambda^4_{l_5^5(e_5,jj)}\partial_{h_{S_5(e_5,qq)}}g_{ii,jj,n_z} \\
&+\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^{e}}\gamma^4_{l_5^5(e_5,jj)}\partial_{h_{S_5(e_5,qq)}}g_{ii,j,t_z}.
\end{aligned} \tag{26.205}$$

27. The continuity equation near an obtuse contact angle

We consider equation (24.3)

$$\partial_r \bar{u} + \partial_z \bar{w} = 0, \tag{27.1}$$

and we define

$$\bar{C}_i = \int_\Omega \psi_i \partial_r \bar{u} + \int_\Omega \psi_i \partial_z \bar{w}, \tag{27.2}$$

where i is an index that runs through the pressure node numbering. Substituting approximations (26.83) and (26.84) we have

$$\bar{\mathcal{C}}_i = \int_\Omega \psi_i \partial_r \left(\sum_{j=1}^{n_v} \bar{u}_j \phi_j \right) + \int_\Omega \psi_i \partial_z \left(\sum_{j=1}^{n_v} \bar{w}_j \phi_j \right), \tag{27.3}$$

where $\breve{C}_i$ results from the substitution of the approximation of $\bar{u}$ and $\bar{w}$ into $\bar{C}_i$.

We can re-write this as

$$\bar{\mathcal{C}}_i = \sum_{j=1}^{n_v} \hat{u}_j \int_\Omega \psi_i \partial_r \phi_j + \sum_{j=1}^{n_v} \bar{w}_j \int_\Omega \psi_i \partial_z \phi_j, \tag{27.4}$$

gathering the sums we have

$$\bar{\mathcal{C}}_i = \sum_{j=1}^{n_v} \left[\bar{u}_j \int_\Omega \psi_i \partial_r \phi_j + \bar{w}_j \int_\Omega \psi_i \partial_z \phi_j \right]. \tag{27.5}$$

We now express the integrals as a sum over the integrals on each element

$$\bar{\mathcal{C}}_i = \sum_{e=1}^{n_{\mathrm{el}}} \sum_{j=1}^{n_v} \left[\bar{u}_j \int_{\Omega_e} \psi_i \partial_r \phi_j + \bar{w}_j \int_{\Omega_e} \psi_i \partial_z \phi_j \right], \tag{27.6}$$

and moving to local numbering in variable j we have

$$\bar{\mathcal{C}}_i = \sum_{e=1}^{n_{\mathrm{el}}} \sum_{jj=1}^{n_v^e} \left[\bar{u}_{l(e,jj)} \int_{\Omega_e} \psi_i \partial_r \phi_{l(e,jj)} + \bar{w}_{l(e,jj)} \int_{\Omega_e} \psi_i \partial_z \phi_{l(e,jj)} \right]. \tag{27.7}$$

We now define

$$\bar{\mathcal{C}}_{e,ii} = \sum_{jj=1}^{n_v^e} \left[\bar{u}_{l(e,jj)} \underbrace{\int_{\Omega_e} \psi_{l^p(e,ii)} \partial_r \phi_{l(e,jj)}}_{b^r_{ii,jj}(e)} + \bar{w}_{l(e,jj)} \underbrace{\int_{\Omega_e} \psi_{l^p(e,ii)} \partial_z \phi_{l(e,jj)}}_{b^z_{ii,jj}(e)} \right], \tag{27.8}$$

and therefore

$$\bar{\mathcal{C}}_i = \sum_{\substack{e=1 \\ i=l^p(e,ii)}}^{n_{\mathrm{el}}} \bar{\mathcal{C}}_{e,ii}, \tag{27.9}$$

i.e.

$$\bar{\mathcal{C}}_i = \sum_{\substack{e=1 \\ i=l^p(e,ii)}}^{n_{\mathrm{el}}} \sum_{jj=1}^{n_v^e} \left[\bar{u}_{l(e,jj)} b^r_{ii,jj}(e) + \bar{w}_{l(e,jj)} b^z_{ii,jj}(e) \right]. \tag{27.10}$$

27.1. Jacobian terms

We now consider the derivatives of $\bar{\mathcal{C}}_i$ with respect to $\bar{u}_q$, $\bar{w}_q$ and h_q.

27.1.1. Derivatives of $\bar{\mathcal{C}}_i$ with respect to $\bar{u}_q$

$$\partial_{\bar{u}_q}\bar{\mathcal{C}}_i = \sum_{\substack{e=1\\ i=l^p(e,ii)}}^{n_{\text{el}}} \partial_{u_q}\left\{\sum_{jj=1}^{n_v^e}\left[\bar{u}_{l(e,jj)}b^r_{ii,jj}(e)+\bar{w}_{l(e,jj)}b^z_{ii,jj}(e)\right]\right\}, \tag{27.11}$$

moving the derivative into the sum and removing quantities that do not depend on u we have

$$\partial_{\bar{u}_q}\bar{\mathcal{C}}_i = \sum_{\substack{e=1\\ i=l^p(e,ii)}}^{n_{\text{el}}}\left\{\sum_{jj=1}^{n_v^e}\underbrace{\partial_{\bar{u}_q}\bar{u}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}b^r_{ii,jj}(e)\right\}, \tag{27.12}$$

i.e.

$$\boxed{\partial_{\bar{u}_q}\bar{\mathcal{C}}_i = \sum_{\substack{e=1\\ i=l^p(e,ii)\\ q=l(e,jj)}}^{n_{\text{el}}} b^r_{ii,jj}(e)}. \tag{27.13}$$

27.1.2. Derivatives of $\bar{\mathcal{C}}_i$ with respect to $\bar{w}_q$

$$\partial_{\bar{w}_q}\bar{\mathcal{C}}_i = \sum_{\substack{e=1\\ i=l^p(e,ii)}}^{n_{\text{el}}} \partial_{\bar{w}_q}\left\{\sum_{jj=1}^{n_v^e}\left[\bar{u}_{l(e,jj)}b^r_{ii,jj}(e)+\bar{w}_{l(e,jj)}b^z_{ii,jj}(e)\right]\right\}, \tag{27.14}$$

moving the derivative into the sum and removing quantities that do not depend on $\bar{u}$ we have

$$\partial_{\bar{w}_q}\bar{\mathcal{C}}_i = \sum_{\substack{e=1\\ i=l^p(e,ii)}}^{n_{\text{el}}}\left\{\sum_{jj=1}^{n_v^e}\underbrace{\partial_{\bar{w}_q}\bar{w}_{l(e,jj)}}_{\delta_{q,l(e,jj)}}b^z_{ii,jj}(e)\right\}, \tag{27.15}$$

i.e.

$$\boxed{\partial_{\bar{w}_q}\bar{\mathcal{C}}_i = \sum_{\substack{e=1\\ i=l^p(e,ii)\\ q=l(e,jj)}}^{n_{\text{el}}} b^z_{ii,jj}(e)}. \tag{27.16}$$

27.1.3. Derivatives of $\bar{\mathcal{C}}_i$ with respect to h_q

From equation (**??**) we have

$$\partial_{h_q}\bar{\mathcal{C}}_i = \sum_{\substack{e=1\\ i=l^p(e,ii)}}^{n_{\text{el}}} \partial_{h_q}\bar{\mathcal{C}}_{e,ii}. \tag{27.17}$$

Using local spine numbers we have

$$\partial_{h_q}\bar{\mathcal{C}}_i = \sum_{\substack{e=1\\ i=l^p(e,ii)}}^{n_{\text{el}}} \partial_{h_{S(e,qq)}}\bar{\mathcal{C}}_{e,ii}. \tag{27.18}$$

Therefore, using equation (27.8), we have

$$\partial_{h_{S(e,qq)}} \bar{\mathcal{C}}_{e,ii} = \sum_{jj=1}^{n_v^e} \left[\bar{u}_{l(e,jj)} \partial_{h_{S(e,qq)}} b^r_{ii,jj}(e) + \bar{w}_{l(e,jj)} \partial_{h_{S(e,qq)}} b^z_{ii,jj}(e) \right], \quad (27.19)$$

and hence

$$\boxed{\partial_{h_{S(e,qq)}} \bar{\mathcal{C}}_i = \sum_{\substack{e=1 \\ i=l^p(e,ii)}}^{n_{\mathrm{el}}} \sum_{\substack{jj=1 \\ q=S(e,qq)}}^{n_v^e} \left[\bar{u}_{l(e,jj)} \partial_{h_{S(e,qq)}} b^r_{ii,jj}(e) + \bar{w}_{l(e,jj)} \partial_{h_{S(e,qq)}} b^z_{ii,jj}(e) \right]}. \quad (27.20)$$

28. The slip condition on the liquid-solid interface (SC2) in the near-field

We recall equation (23.22)

$$\left[\boldsymbol{v}^{s_2} - \frac{1}{2}\left(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} + \boldsymbol{u}^s\right)\right] \cdot \left(\boldsymbol{I} - \boldsymbol{n}^2\boldsymbol{n}^2\right) = Es\,\nabla^s\sigma^2. \tag{28.1}$$

and we define the i-th SC2 residual as

$$\bar{\mathcal{S}}_i^2 = \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[\boldsymbol{v}^{s_2} - \frac{1}{2}\left(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} + \boldsymbol{u}^s\right)\right] \cdot \boldsymbol{t}^2 - Es \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \boldsymbol{t}^2 \cdot \nabla^s \sigma^2, \tag{28.2}$$

which, of course we wish to make identically null.

We thus have

$$\bar{\mathcal{S}}_i^2 = \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[\boldsymbol{v}^{s_2} \cdot \boldsymbol{t}^2 - \frac{1}{2}\bar{\boldsymbol{u}} \cdot \boldsymbol{t}^2 - \frac{1}{2}A\check{\boldsymbol{u}} \cdot \boldsymbol{t}^2 - \frac{1}{2}\boldsymbol{u}^s \cdot \boldsymbol{t}^2\right] - Es \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left(\partial_s \sigma^2\right) \boldsymbol{t}^2 \cdot \boldsymbol{t}^2, \tag{28.3}$$

i.e.

$$\begin{aligned}\bar{\mathcal{S}}_i^2 = &\int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[u^{s_2} t_r^2 + w^{s_2} t_z^2 - \frac{1}{2}\bar{u} t_r^2 - \frac{1}{2}\bar{w} t_z^2 - \frac{1}{2}A\check{u} t_r^2 - \frac{1}{2}A\check{w} t_z^2 - \frac{1}{2}u^s t_r^2 - \frac{1}{2}w^s t_z^2\right] \\ &- Es \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left(\partial_s \sigma^2\right),\end{aligned} \tag{28.4}$$

equivalently

$$\begin{aligned}\bar{\mathcal{S}}_i^2 = &\int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[u^{s_2} t_r^2\right] + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[w^{s_2} t_z^2\right] + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[-\frac{1}{2}\bar{u} t_r^2\right] + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[-\frac{1}{2}\bar{w} t_z^2\right] \\ &+ \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[-\frac{1}{2}A\check{u} t_r^2\right] + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[-\frac{1}{2}A\check{w} t_z^2\right] \\ &+ \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[-\frac{1}{2}u^s t_r^2\right] + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \left[-\frac{1}{2}w^s t_z^2\right] \\ &- Es \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \partial_s \sigma^2,\end{aligned} \tag{28.5}$$

i.e.

$$\begin{aligned}\bar{\mathcal{S}}_i^2 = &\int\limits_{\partial\Omega^{2,n}} \phi_i^2 u^{s_2} t_r^2 + \int\limits_{\partial\Omega^{2,n}} \phi_i^2 w^{s_2} t_z^2 - \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \bar{u} t_r^2 - \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \bar{w} t_z^2 \\ &- \frac{1}{2}A \int\limits_{\partial\Omega^{2,n}} \check{u} t_r^2 - \frac{1}{2}A \int\limits_{\partial\Omega^{2,n}} \check{w} t_z^2 - \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 u^s t_r^2 - \frac{1}{2} \int\limits_{\partial\Omega^{2,n}} \phi_i^2 w^s t_z^2 \\ &- Es \int\limits_{\partial\Omega^{2,n}} \phi_i^2 \partial_s \sigma^2,\end{aligned} \tag{28.6}$$

We consider the last integral on the right hand side above and we integrate by parts to obtain

$$\int_{\partial\Omega^2} \phi_i^2 \partial_s \sigma^2 = \phi_i^2 \sigma^2 \big|_{(r_c,z_c)}^{(r_o,z_o)} - \int_{\partial\Omega^2} \sigma^2 \partial_s \phi_i^2. \tag{28.7}$$

This yields

$$\begin{aligned}\bar{\mathcal{S}}_i^2 = {} & Es\,\phi_i^2(r_c,z_c)\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\sigma^2(r_o,z_o) + \int_{\partial\Omega^{2,n}} \phi_i^2 u^{s_2} t_r^2 \\ & + \int_{\partial\Omega^{2,n}} \phi_i^2 w^{s_2} t_z^2 - \frac{1}{2}\int_{\partial\Omega^{2,n}} \phi_i^2 \bar{u} t_r^2 - \frac{1}{2}\int_{\partial\Omega^{2,n}} \phi_i^2 \bar{w} t_z^2 \\ & - \frac{1}{2}A\int_{\partial\Omega^{2,n}} \check{u} t_r^2 - \frac{1}{2}A\int_{\partial\Omega^{2,n}} \check{w} t_z^2 - \frac{1}{2}\int_{\partial\Omega^{2,n}} \phi_i^2 u^s t_r^2 - \frac{1}{2}\int_{\partial\Omega^{2,n}} \phi_i^2 w^s t_z^2 \\ & + Es\int_{\partial\Omega^{2,n}} \sigma^2 \partial_s \phi_i^2,\end{aligned} \tag{28.8}$$

We recall the approximations

$$\bar{u} \approx \sum_{j=1}^{n_v} \bar{u}_j \phi_j, \tag{28.9}$$

$$\bar{w} \approx \sum_{j=1}^{n_v} \bar{w}_j \phi_j, \tag{28.10}$$

$$\sigma^2 \approx \sum_{j=1}^{n_v} \sigma_j^2 \phi_j^2, \tag{28.11}$$

$$u^s \approx \sum_{j=1}^{n_v} u_j^s \phi_j^2, \tag{28.12}$$

$$w^s \approx \sum_{j=1}^{n_v} w_j^s \phi_j^2; \tag{28.13}$$

$$u^{s_2} \approx \sum_{j=1}^{n_v} u_j^{s_2} \phi_j^2, \tag{28.14}$$

and

$$w^{s_2} \approx \sum_{j=1}^{n_v} w_j^{s_2} \phi_j^2. \tag{28.15}$$

We thus have

$$
\begin{aligned}
\bar{\mathcal{S}}_i^2 = {} & Es\,\phi_i^2(r_c,z_c)\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\sigma^2(r_o,z_o) - \frac{1}{2}A\int\limits_{\partial\Omega^{2,n}} \phi_i^2\check{u}t_r^2 - \frac{1}{2}A\int\limits_{\partial\Omega^{2,n}} \phi_i^2\check{w}t_z^2 \\
& + \int\limits_{\partial\Omega^{2,n}} \phi_i^2\left(\sum_{j=1}^{n_v} u_j^{s_2}\phi_j^2\right)t_r^2 + \int\limits_{\partial\Omega^{2,n}} \phi_i^2\left(\sum_{j=1}^{n_v} w_j^{s_2}\phi_j^2\right)t_z^2 \\
& - \frac{1}{2}\int\limits_{\partial\Omega^{2,n}} \phi_i^2\left(\sum_{j=1}^{n_v} \bar{u}_j\phi_j^2\right)t_r^2 - \frac{1}{2}\int\limits_{\partial\Omega^{2,n}} \phi_i^2\left(\sum_{j=1}^{n_v} \bar{w}_j\phi_j^2\right)t_z^2 \\
& - \frac{1}{2}\int\limits_{\partial\Omega^{2,n}} \phi_i^2\left(\sum_{j=1}^{n_v} u_j^{s}\phi_j^2\right)t_r^2 - \frac{1}{2}\int\limits_{\partial\Omega^{2,n}} \phi_i^2\left(\sum_{j=1}^{n_v} w_j^{s}\phi_j^2\right)t_z^2 \\
& + Es\int\limits_{\partial\Omega^{2,n}} \left(\sum_{j=1}^{n_v} \sigma_j^2\phi_j^2\right)\partial_s\phi_i^2.
\end{aligned}
\tag{28.16}
$$

Moving the integrals into the sums, we have

$$
\begin{aligned}
\mathcal{S}_i^2 = {} & Es\,\phi_i^2(r_c,z_c)\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\sigma^2(r_o,z_o) - \frac{A}{2}\int\limits_{\partial\Omega^{2,n}} \phi_i^2\check{u}t_r^2 - \frac{A}{2}\int\limits_{\partial\Omega^{2,n}} \phi_i^2\check{w}t_z^2 \\
& + \sum_{j=1}^{n_v} u_j^{s_2}\int\limits_{\partial\Omega^{2,n}} \phi_i^2\phi_j^2t_r^2 + \sum_{j=1}^{n_v} w_j^{s_2}\int\limits_{\partial\Omega^{2,n}} \phi_i^2\phi_j^2t_z^2 \\
& - \frac{1}{2}\sum_{j=1}^{n_v} \bar{u}_j\int\limits_{\partial\Omega^{2,n}} \phi_i^2\phi_j^2t_r^2 - \frac{1}{2}\sum_{j=1}^{n_v} \bar{w}_j\int\limits_{\partial\Omega^{2,n}} \phi_i^2\phi_j^2t_z^2 \\
& - \frac{1}{2}\sum_{j=1}^{n_v} u_j^{s}\int\limits_{\partial\Omega^{2,n}} \phi_i^2\phi_j^2t_r^2 - \frac{1}{2}\sum_{j=1}^{n_v} w_j^{s}\int\limits_{\partial\Omega^{2,n}} \phi_i^2\phi_j^2t_z^2 \\
& + Es\sum_{j=1}^{n_v} \sigma_j^2\int\limits_{\partial\Omega^{2,n}} \phi_j^2\partial_s\phi_i^2.
\end{aligned}
\tag{28.17}
$$

Decomposing the integrals into sums of integrals over each individual element and passing to local element node numbers we have

$$
\mathcal{S}_i^{2,r} = Es\,\phi_i^2(r_c,z_c)\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \mathcal{S}_{e_2,ii}^2, \tag{28.18}
$$

where

$$
\begin{aligned}
\mathcal{S}^2_{e_2,ii} = & -\frac{A}{2}\underbrace{\int\limits_{\partial\Omega^{2,n}} \phi^2_{l_2(e_2,ii)}\check{u}t^2_r}_{d_{ii,t_r,\check{u}}(e_2)} - \frac{A}{2}\underbrace{\int\limits_{\partial\Omega^{2,n}} \phi^2_{l_2(e_2,ii)}\check{w}t^2_z}_{d_{ii,t_z,\check{w}}(e_2)} \\
& + \sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,n}} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)}t^2_r}_{d_{ii,jj,t_r}(e_2)} \\
& + \sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,n}} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)}t^2_z}_{d_{ii,jj,t_z}(e_2)} \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \bar{u}_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,n}} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)}t^2_r}_{d_{ii,jj,t_r}(e_2)} \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \bar{w}_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,n}} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)}t^2_z}_{d_{ii,jj,t_z}(e_2)} \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} u^{s}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,n}} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)}t^2_r}_{d_{ii,jj,t_r}(e_2)} \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} w^{s}_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,n}} \phi^2_{l_2(e_2,ii)}\phi^2_{l_2(e_2,jj)}t^2_z}_{d_{ii,jj,t_z}(e_2)} \\
& + Es\sum_{jj=1}^{n_v^{2,e_2}} \sigma^2_{l_2^2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^{2,n}} \phi^2_{l_2(e_2,ii)}\partial_s\phi^2_{l_2(e_2,jj)}}_{d^s_{jj,ii}(e_2)}.
\end{aligned}
\tag{28.19}
$$

i.e.

$$
\mathcal{S}^{2,r}_i = Es\,\phi^2_i(r_c,z_c)\sigma^2(r_c,z_c) - Es\,\phi^2_i(r_o,z_o)\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n^2_{\mathrm{el}}} \mathcal{S}^2_{e_2,ii}, \tag{28.20}
$$

where

$$
\begin{aligned}
\mathcal{S}^2_{e_2,ii} = &-\frac{A}{2} d_{ii,t_r,\breve{u}}(e_2) - \frac{A}{2} d_{ii,t_z,\breve{w}}(e_2) \\
&+ \sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
&- \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} \bar{u}_{l_2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} \bar{w}_{l_2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
&- \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} u^{s}_{l_2^2(e_2,jj)} d_{ii,jj,t_r,t_r}(e_2) - \frac{1}{2} \sum_{jj=1}^{n_v^{2,e_2}} w^{s}_{l_2^2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
&+ Es \sum_{jj=1}^{n_v^{2,e_2}} \sigma^2_{l_2^2(e_2,jj)} d^s_{jj,ii}(e_2).
\end{aligned}
\tag{28.21}
$$

Summarising and re-arranging terms we have

$$
\mathcal{S}_i^{2,r} = Es\,\phi_i^2(r_c,z_c)\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1 \\ i=l_2(e_2,ii)}}^{n^2_{\text{el}}} \mathcal{S}^2_{e_2,ii}, \tag{28.22}
$$

where

$$
\begin{aligned}
\mathcal{S}^2_{e_2,ii} = &-\frac{A}{2}\left[d_{ii,t_r,\breve{u}}(e_2) + d_{ii,t_z,\breve{w}}(e_2)\right] \\
&+ \sum_{jj=1}^{n_v^{2,e_2}} \Bigg\{ d_{ii,jj,t_r}(e_2)\left[u^{s_2}_{l_2^2(e_2,jj)} - \frac{1}{2}\bar{u}_{l_2(e_2,jj)} - \frac{1}{2}u^{s}_{l_2^2(e_2,jj)}\right] \\
&\qquad + d_{ii,jj,t_z}(e_2)\left[w^{s_2}_{l_2^2(e_2,jj)} - \frac{1}{2}\bar{w}_{l_2(e_2,jj)} - \frac{1}{2}w^{s}_{l_2^2(e_2,jj)}\right] \\
&\qquad + Es\,\sigma^2_{l_2^2(e_2,jj)} d^s_{jj,ii}(e_2)\Bigg\}.
\end{aligned}
\tag{28.23}
$$

28.1. **Jacobian terms**

Here we find the derivative of $\mathcal{S}_i^2$ with respect to $\bar{u}_q$, $\bar{w}_q$, u^{s_2}, w^{s_2}, σ^2, θ_c, A and h_q.

28.1.1. **Derivatives of $\bar{\mathcal{S}}_i^2$ with respect to $\bar{u}_q$**

From equation (28.20) we have

$$\partial_{\bar{u}_q}\mathcal{S}_i^2 = Es\,\phi_i^2(r_c,z_c)\partial_{\bar{u}_q}\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\partial_{\bar{u}_q}\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{\bar{u}_q}\mathcal{S}_{e_2,ii}^2. \tag{28.24}$$

Now, from equation (28.21) we have

$$\begin{aligned}
\partial_{\bar{u}_q}\bar{\mathcal{S}}_{e_2,ii}^2 = &-\frac{A}{2}\partial_{\bar{u}_q}d_{ii,t_r,\breve{u}}(e_2) - \frac{A}{2}\partial_{\bar{u}_q}d_{ii,t_z,\breve{w}}(e_2)\\
&+\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{u}_q}u^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,t_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{u}_q}w^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,t_z}(e_2)\\
&-\frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{u}_q}\bar{u}_{l_2(e_2,jj)}d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{u}_q}\bar{w}_{l_2(e_2,jj)}d_{ii,jj,t_z}(e_2)\\
&-\frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{u}_q}u^{s}_{l_2^2(e_2,jj)}d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{u}_q}w^{s}_{l_2^2(e_2,jj)}d_{ii,jj,t_z}(e_2)\\
&+Es\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{u}_q}\sigma^2_{l_2^2(e_2,jj)}d^s_{jj,ii}(e_2),
\end{aligned}\tag{28.25}$$

i.e.

$$\partial_{\bar{u}_q}\mathcal{S}_{e_2,ii}^2 = -\frac{1}{2}d_{ii,jj,t_r}(e_2)|_{q=l_2(e_2,jj))}, \tag{28.26}$$

28.1.2. Derivatives with respect to $\bar{w}_q$

From equation (28.20) we have

$$\partial_{\bar{w}_q}\mathcal{S}_i^{2,r} = Es\,\phi_i^2(r_c,z_c)\partial_{\bar{w}_q}\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\partial_{\bar{w}_q}\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{\bar{w}_q}\mathcal{S}_{e_2,ii}^2. \tag{28.27}$$

Now, from equation (28.21) we have

$$\begin{aligned}
\partial_{\bar{w}_q}\mathcal{S}_{e_2,ii}^2 = &-\frac{A}{2}\partial_{\bar{w}_q}d_{ii,t_r,\check{u}}(e_2) - \frac{A}{2}\partial_{\bar{w}_q}d_{ii,t_z,\check{w}}(e_2)\\
&+\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{w}_q}u^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,t_r}(e_2)\\
&+\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{w}_q}w^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,t_z}(e_2)\\
&-\frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{w}_q}\bar{u}_{l_2(e_2,jj)}d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{w}_q}\bar{w}_{l_2(e_2,jj)}d_{ii,jj,t_z}(e_2)\\
&-\frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{w}_q}u^{s}_{l_2^2(e_2,jj)}d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{w}_q}w^{s}_{l_2^2(e_2,jj)}d_{ii,jj,t_z}(e_2)\\
&+Es\sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{w}_q}\sigma^2_{l_2^2(e_2,jj)}d^s_{jj,ii}(e_2),
\end{aligned} \tag{28.28}$$

i.e.

$$\partial_{\bar{w}_q}\bar{\mathcal{S}}_{e_2,ii}^2 = -\frac{1}{2}d_{ii,jj,t_z}(e_2)|_{q=l_2(e_2,jj)}. \tag{28.29}$$

28.1.3. **Derivatives with respect to $u_q^{s_2}$**

From equation (28.20) we have

$$\partial_{u_q^{s_2}} \mathcal{S}_i^{2,r} = Es\,\phi_i^2(r_c, z_c)\partial_{u_q^{s_2}} \sigma^2(r_c, z_c) - Es\,\phi_i^2(r_o, z_o)\partial_{u_q^{s_2}} \sigma^2(r_o, z_o) + \sum_{\substack{e_2=1 \\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{u_q^{s_2}} \mathcal{S}_{e_2,ii}^2. \tag{28.30}$$

Now, from equation (28.21) we have

$$\begin{aligned}
\partial_{u_q^{s_2}} \mathcal{S}_{e_2,ii}^2 = {} & -\frac{A}{2}\partial_{u_q^{s_2}} d_{ii,t_r,\check{u}}(e_2) - \frac{A}{2}\partial_{u_q^{s_2}} d_{ii,t_z,\check{w}}(e_2) \\
& + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} u_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,t_r}(e_2) \\
& + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} w_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,t_z}(e_2) \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \bar{u}_{l_2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \bar{w}_{l_2(e_2,jj)} d_{ii,jj,t_z}(e_2) \\
& - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} u_{l_2^2(e_2,jj)}^{s} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} w_{l_2^2(e_2,jj)}^{s} d_{ii,jj,t_z}(e_2) \\
& + Es\sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \sigma_{l_2^2(e_2,jj)}^2 d_{jj,ii}^s(e_2),
\end{aligned} \tag{28.31}$$

i.e.

$$\partial_{u_q^{s_2}} \mathcal{S}_{e_2,ii}^2 = d_{ii,jj,t_r}(e_2)|_{q=l_2^2(e_2,jj)}. \tag{28.32}$$

28.1.4. **Derivatives with respect to $w_q^{s_2}$**

From equation (28.20) we have

$$\partial_{w_q^{s_2}} \mathcal{S}_i^{2,r} = Es\,\phi_i^2(r_c,z_c)\partial_{w_q^{s_2}}\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\partial_{w_q^{s_2}}\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{w_q^{s_2}}\mathcal{S}_{e_2,ii}^2. \tag{28.33}$$

Now, from equation (28.21) we have

$$\begin{aligned}
\partial_{w_q^{s_2}} \mathcal{S}_{e_2,ii}^2 = &-\frac{A}{2}\partial_{w_q} d_{ii,t_r,\breve{u}}(e_2) - \frac{A}{2}\partial_{w_q} d_{ii,t_z,\breve{w}}(e_2)\\
&+ \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} u_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,t_r}(e_2)\\
&+ \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} w_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,t_z}(e_2)\\
&- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \bar{u}_{l_2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \bar{w}_{l_2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\
&- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} u_{l_2^2(e_2,jj)}^{s} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} w_{l_2^2(e_2,jj)}^{s} d_{ii,jj,t_z}(e_2)\\
&+ Es \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \sigma_{l_2^2(e_2,jj)}^2 d_{jj,ii}^s(e_2),
\end{aligned} \tag{28.34}$$

i.e.

$$\partial_{w_q^{s_2}} \mathcal{S}_{e_2,ii}^2 = d_{ii,jj,t_z}(e_2)|_{q=l_2^2(e_2,jj)}. \tag{28.35}$$

28.1.5. **Derivatives with respect to σ_q^2**

From equation (28.20) we have

$$\partial_{\sigma_q^2}\mathcal{S}_i^2 = Es\,\phi_i^2(r_c,z_c)\partial_{\sigma_q^2}\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\partial_{\sigma_q^2}\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{\sigma_q^2}\mathcal{S}_{e_2,ii}^2. \tag{28.36}$$

Now, from equation (28.21) we have

$$\begin{aligned}
\partial_{\sigma_q^2}\mathcal{S}_{e_2,ii}^2 = &-\frac{A}{2}\partial_{\sigma_q^2}d_{ii,t_r,\check{u}}(e_2) - \frac{A}{2}\partial_{\sigma_q^2}d_{ii,t_z,\check{w}}(e_2)\\
&+\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\sigma_q^2}u^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,t_r}(e_2)\\
&+\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\sigma_q^2}w^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,t_z}(e_2)\\
&-\frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\sigma_q^2}\bar{u}_{l_2(e_2,jj)}d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\sigma_q^2}\bar{w}_{l_2(e_2,jj)}d_{ii,jj,t_z}(e_2)\\
&-\frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\sigma_q^2}u^{s}_{l_2^2(e_2,jj)}d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\sigma_q^2}w^{s}_{l_2^2(e_2,jj)}d_{ii,jj,t_z}(e_2)\\
&+Es\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\sigma_q^2}\sigma^2_{l_2^2(e_2,jj)}d^s_{jj,ii}(e_2),
\end{aligned} \tag{28.37}$$

i.e.

$$\partial_{\sigma_q^2}\bar{\mathcal{S}}_{e_2,ii}^2 = Es\,d^s_{jj,ii}(e_2)\big|_{q=l_2^2(e_2,jj)}. \tag{28.38}$$

28.1.6. Derivatives of $\bar{\mathcal{S}}_i^2$ with respect to θ_c

From equation (28.20) we have

$$\partial_{\theta_c}\mathcal{S}_i^{2,r} = Es\,\phi_i^2(r_c,z_c)\partial_{\theta_c}\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\partial_{\theta_c}\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{\theta_c}\mathcal{S}_{e_2,ii}^2. \tag{28.39}$$

Now, from (28.21) we have

$$\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{S}}_{e_2,ii}^2 = &-\frac{A}{2}\partial_{\theta_c}d_{ii,t_r,\breve{u}}(e_2) - \frac{A}{2}\partial_{\theta_c}d_{ii,t_z,\breve{w}}(e_2)\\
&+\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\theta_c}u_{l_2^2(e_2,jj)}^{s_2}d_{ii,jj,t_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}}\partial_{\theta_c}w_{l_2^2(e_2,jj)}^{s_2}d_{ii,jj,t_z}(e_2)\\
&-\frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\theta_c}\bar{u}_{l_2(e_2,jj)}d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\theta_c}\bar{w}_{l_2(e_2,jj)}d_{ii,jj,t_z}(e_2)\\
&-\frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\theta_c}u_{l_2^2(e_2,jj)}^{s}d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\theta_c}w_{l_2^2(e_2,jj)}^{s}d_{ii,jj,t_z}(e_2)\\
&+Es\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\theta_c}\sigma_{l_2^2(e_2,jj)}^2 d_{jj,ii}^s(e_2),
\end{aligned} \tag{28.40}$$

i.e.

$$\partial_{\theta_c}\bar{\mathcal{S}}_{e_2,ii}^2 = -\frac{A}{2}\left[\partial_{\theta_c}d_{ii,t_r,\breve{u}}(e_2) + \partial_{\theta_c}d_{ii,t_z,\breve{w}}(e_2)\right], \tag{28.41}$$

28.1.7. Derivatives with respect to A

From equation (28.20) we have

$$\partial_A \mathcal{S}_i^2 = Es\,\phi_i^2(r_c, z_c)\partial_A \sigma^2(r_c, z_c) - Es\,\phi_i^2(r_o, z_o)\partial_A \sigma^2(r_o, z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_A \mathcal{S}_{e_2,ii}^2. \quad (28.42)$$

Now, from equation (28.21) we have

$$\begin{aligned}\partial_A \mathcal{S}_{e_2,ii}^2 = &-\partial_A \frac{A}{2} d_{ii,t_r,\check{u}}(e_2) - \partial_A \frac{A}{2} d_{ii,t_z,\check{w}}(e_2)\\ &+ \sum_{jj=1}^{n_v^{2,e_2}} \partial_A u_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,t_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_A w_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,t_z}(e_2)\\ &- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_A \bar{u}_{l_2(e_2,jj)} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_A \bar{w}_{l_2(e_2,jj)} d_{ii,jj,t_z}(e_2)\\ &- \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_A u_{l_2^2(e_2,jj)}^{s} d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \partial_A w_{l_2^2(e_2,jj)}^{s} d_{ii,jj,t_z}(e_2)\\ &+ Es \sum_{jj=1}^{n_v^{2,e_2}} \partial_A \sigma_{l_2^2(e_2,jj)}^2 d_{jj,ii}^s(e_2),\end{aligned} \quad (28.43)$$

i.e.

$$\partial_A \mathcal{S}_{e_2,ii}^2 = -\frac{1}{2} d_{ii,t_r,\check{u}}(e_2) - \frac{1}{2} d_{ii,t_z,\check{w}}(e_2), \quad (28.44)$$

i.e.

$$\partial_A \mathcal{S}_{e_2,ii}^2 = -\frac{1}{2}\left[d_{ii,t_r,\check{u}}(e_2) + d_{ii,t_z,\check{w}}(e_2)\right]. \quad (28.45)$$

28.1.8. Derivatives of S_i^2 with respect to h_q

From equation (28.20) we have

$$\partial_{h_q}\mathcal{S}_i^{2,r} = Es\,\phi_i^2(r_c,z_c)\partial_{h_q}\sigma^2(r_c,z_c) - Es\,\phi_i^2(r_o,z_o)\partial_{h_q}\sigma^2(r_o,z_o) + \sum_{\substack{e_2=1\\ i=l_2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\mathrm{el}}^2} \partial_{S_2(e_2,qq)}\mathcal{S}_{e_2,ii}^2. \tag{28.46}$$

Now, from equation (28.21) we have

$$\begin{aligned}\partial_{h_q}\mathcal{S}_{e_2,ii}^2 = & -\frac{A}{2}\partial_{h_q}d_{ii,t_r,\breve{u}}(e_2) - \frac{A}{2}\partial_{h_q}d_{ii,t_z,\breve{w}}(e_2)\\ & + \sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)}\partial_{h_q}d_{ii,jj,t_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)}\partial_{h_q}d_{ii,jj,t_z}(e_2)\\ & - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \bar{u}_{l_2(e_2,jj)}\partial_{h_q}d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} \bar{w}_{l_2(e_2,jj)}\partial_{h_q}d_{ii,jj,t_z}(e_2)\\ & - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} u^{s}_{l_2^2(e_2,jj)}\partial_{h_q}d_{ii,jj,t_r}(e_2) - \frac{1}{2}\sum_{jj=1}^{n_v^{2,e_2}} w^{s}_{l_2^2(e_2,jj)}\partial_{h_q}d_{ii,jj,t_z}(e_2)\\ & + Es\sum_{jj=1}^{n_v^{2,e_2}} \sigma^2_{l_2^2(e_2,jj)}\partial_{h_q}d^s_{jj,ii}(e_2),\end{aligned} \tag{28.47}$$

passing to local spine numbers and re-arranging terms we have

$$\begin{aligned}\partial_{h_{S_2(e_2,qq)}}\mathcal{S}_{e_2,ii}^2 = & -\frac{A}{2}\left[\partial_{h_{S_2(e_2,qq)}}d_{ii,t_r,\breve{u}}(e_2) + \partial_{h_{S_2(e_2,qq)}}d_{ii,t_z,\breve{w}}(e_2)\right]\\ & + \sum_{jj=1}^{n_v^{2,e_2}}\Bigg\{\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_r}(e_2)\left[u^{s_2}_{l_2^2(e_2,jj)} - \frac{1}{2}\bar{u}_{l_2(e_2,jj)} - \frac{1}{2}u^{s}_{l_2^2(e_2,jj)}\right]\\ & \qquad + \partial_{h_{S_2(e_2,qq)}}d_{ii,jj,t_z}(e_2)\left[w^{s_2}_{l_2^2(e_2,jj)} - \frac{1}{2}\bar{w}_{l_2(e_2,jj)} - \frac{1}{2}w^{s}_{l_2^2(e_2,jj)}\right]\\ & \qquad + Es\,\sigma^2_{l_2^2(e_2,jj)}\underbrace{\partial_{h_{S_2(e_2,qq)}}d^s_{jj,ii}(e_2)}_{=0}\Bigg\},\end{aligned} \tag{28.48}$$

29. Impermeability condition near obtuse contact angle (I2)

The impermeability equation in the near field has the exact same form as in the far field, so we need not repeat all the derivations from section 8 here.

30. The mass exchange condition on boundary 2 (MEC2) in the near-field

We recall MEC2 which in the near field is given by equation (23.24) which states

$$(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{v}^{s_2}) \cdot \boldsymbol{n}^2 = Fs\,(\rho^{s_2} - Ds\,)\,. \tag{30.1}$$

i.e.

$$(\bar{u} + A\check{u} - u^{s_2})n_r^2 + (\bar{w} + A\check{w} - w^{s_2})n_z^2 - Fs\,\rho^{s_2} + Fs\,Ds\, = 0, \tag{30.2}$$

and define the i-th MEC2 residual as

$$\begin{aligned}\bar{E}_i^2 = &\int\limits_{\partial\Omega^2} \phi_i^2 \bar{u} n_r^2 + \int\limits_{\partial\Omega^2} \phi_i^2 \bar{w} n_z^2 + A\int\limits_{\partial\Omega^2} \phi_i^2 \check{u} n_r^2 + A\int\limits_{\partial\Omega^2} \phi_i^2 \check{w} n_z^2 \\ &- \int\limits_{\partial\Omega^2} \phi_i^2 u^{s_2} n_r^2 - \int\limits_{\partial\Omega^2} \phi_i^2 w^{s_2} n_z^2 - Fs\int\limits_{\partial\Omega^2} \phi_i^2 \rho^{s_2} + Fs\,Ds\int\limits_{\partial\Omega^2} \phi_i^2,\end{aligned} \tag{30.3}$$

where i is an index that runs through the boundary 2 node numbering.

We substitute approximations

$$\bar{u} \approx \sum_{j=1}^{n_v} \bar{u}_j \phi_j, \tag{30.4}$$

$$\bar{w} \approx \sum_{j=1}^{n_v} \bar{w}_j \phi_j, \tag{30.5}$$

$$u^{s_2} \approx \sum_{j=1}^{n_v} u_j^{s_2} \phi_j^2, \tag{30.6}$$

$$w^{s_2} \approx \sum_{j=1}^{n_v} w_j^{s_2} \phi_j \tag{30.7}$$

and

$$\rho^{s_2} \approx \sum_{j=1}^{n_v} \rho_j^{s_2} \phi_j. \tag{30.8}$$

into the residual equation above and obtain

$$\begin{aligned}\bar{\mathcal{E}}_i^2 = &\int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} \bar{u}_j \phi_j\right) n_r^2 + \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} \bar{w}_j \phi_j\right) n_z^2 \\ &+ A\int\limits_{\partial\Omega^2} \phi_i^2 \check{u} n_r^2 + A\int\limits_{\partial\Omega^2} \phi_i^2 \check{w} n_z^2 - \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} u_j^{s_2} \phi_j^2\right) n_r^2 \\ &- \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} w_j^{s_2} \phi_j\right) n_z^2 - Fs\int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} \rho_j^{s_2} \phi_j\right) + Fs\,Ds\int\limits_{\partial\Omega^2} \phi_i^2.\end{aligned} \tag{30.9}$$

Moving the integrals into the sums and re-arranging terms we have

$$\bar{\mathcal{E}}_i^2 = Fs\,Ds \int\limits_{\partial\Omega^2} \phi_i^2 + A \int\limits_{\partial\Omega^2} \phi_i^2 \check{u} n_r^2 + A \int\limits_{\partial\Omega^2} \phi_i^2 \check{w} n_z^2 + \sum_{j=1}^{n_v} \bar{u}_j \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_r^2 + \sum_{j=1}^{n_v} \bar{w}_j \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_z^2$$
$$- \sum_{j=1}^{n_v} u_j^{s_2} \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_r^2 - \sum_{j=1}^{n_v} w_j^{s_2} \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2 n_z^2 - Fs \sum_{j=1}^{n_v} \rho_j^{s_2} \int\limits_{\partial\Omega^2} \phi_i^2 \phi_j^2. \tag{30.10}$$

Decomposing the integrals into sums of integrals over each line-element and using local node-numbers we have

$$\bar{\mathcal{E}}_i^2 = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \bar{\mathcal{E}}_{e_2,ii}^2, \tag{30.11}$$

where

$$\bar{\mathcal{E}}_{e_2,ii}^2 = Fs\,Ds \underbrace{\int\limits_{\partial\Omega^2} \phi_{l_2(e_2,ii)}^2}_{d_{ii}(e_2)} + A \underbrace{\int\limits_{\partial\Omega^2} \phi_{l_2(e_2,ii)}^2 \check{u} n_r^2}_{d_{ii,n_r,\check{u}}(e_2)} + A \underbrace{\int\limits_{\partial\Omega^2} \phi_{l_2(e_2,ii)}^2 \check{w} n_z^2}_{d_{ii,n_z,\check{w}}(e_2)}$$
$$+ \sum_{jj=1}^{n_v^{2,e_2}} \bar{u}_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^2} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2 n_r^2}_{d_{ii,jj,n_r}(e_2)} + \sum_{jj=1}^{n_v^{2,e_2}} \bar{w}_{l_2(e_2,jj)} \underbrace{\int\limits_{\partial\Omega^2} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2 n_z^2}_{d_{ii,jj,n_z}(e_2)}$$
$$- \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2^2(e_2,jj)}^{s_2} \underbrace{\int\limits_{\partial\Omega^2} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2 n_r^2}_{d_{ii,jj,n_r}(e_2)} - \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2^2(e_2,jj)}^{s_2} \underbrace{\int\limits_{\partial\Omega^2} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2 n_z^2}_{d_{ii,jj,n_z}(e_2)}$$
$$- Fs \sum_{jj=1}^{n_v^{2,e_2}} \rho_{l_2^2(e_2,jj)}^{s_2} \underbrace{\int\limits_{\partial\Omega^2} \phi_{l_2(e_2,ii)}^2 \phi_{l_2(e_2,jj)}^2}_{d_{ii,jj}(e_2)}, \tag{30.12}$$

i.e.

$$\bar{\mathcal{E}}_{e_2,ii}^2 = Fs\,Ds\, d_{ii}(e_2) + A d_{ii,n_r,\check{u}}(e_2) + A d_{ii,n_z,\check{w}}(e_2)$$
$$+ \sum_{jj=1}^{n_v^{2,e_2}} \bar{u}_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \bar{w}_{l_2(e_2,jj)} d_{ii,jj,n_z}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} u_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,n_r}(e_2)$$
$$- \sum_{jj=1}^{n_v^{2,e_2}} w_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,n_z}(e_2) - Fs \sum_{jj=1}^{n_v^{2,e_2}} \rho_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj}(e_2). \tag{30.13}$$

Summarising and re-arranging terms we have

$$\bar{\mathcal{E}}_i^2 = \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \bar{\mathcal{E}}_{e_2,ii}^2, \tag{30.14}$$

where

$$\begin{aligned}\bar{\mathcal{E}}_{e_2,ii}^2 = {} & Fs\,Ds\,d_{ii}(e_2) + A\left[d_{ii,n_r,\breve{u}}(e_2) + d_{ii,n_z,\breve{w}}(e_2)\right]\\ & + \sum_{jj=1}^{n_v^{2,e_2}} \left\{ d_{ii,jj,n_r}(e_2)\left[\bar{u}_{l_2(e_2,jj)} - u^{s_2}_{l_2^2(e_2,jj)}\right]\right.\\ & \qquad + d_{ii,jj,n_z}(e_2)\left[\bar{w}_{l_2(e_2,jj)} - w^{s_2}_{l_2^2(e_2,jj)}\right]\\ & \qquad \left. - Fs\,\rho^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj}(e_2)\right\}.\end{aligned} \tag{30.15}$$

30.1. Jacobian terms

We now calculate the derivatives of $\bar{\mathcal{E}}_i^2$ with respect to $\bar{u}_q$, $\bar{w}_q$, $u_q^{s_2}$, $w_q^{s_2}$, $\rho_q^{s_2}$, θ_c, A and h_q.

30.1.1. Derivatives of $\bar{\mathcal{E}}_i^2$ with respect to $\bar{u}_q$

Using equation (30.11) we have

$$\partial_{\bar{u}_q}\bar{\mathcal{E}}_i^2 = \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{\bar{u}_q}\bar{\mathcal{E}}_{e_2,ii}^2, \tag{30.16}$$

and from equation (30.13) we have

$$\begin{aligned}
\partial_{\bar{u}_q}\bar{\mathcal{E}}_{e_2,ii}^2 = {} & Fs\, Ds\, \partial_{\bar{u}_q} d_{ii}(e_2) + A\partial_{\bar{u}_q} d_{ii,n_r,\breve{u}}(e_2) + A\partial_{\bar{u}_q} d_{ii,n_z,\breve{w}}(e_2) \\
& + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{u}_q}\bar{u}_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{u}_q}\bar{w}_{l_2(e_2,jj)} d_{ii,jj,n_z}(e_2) \\
& - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{u}_q} u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{u}_q} w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e_2) \\
& - Fs \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\bar{u}_q}\rho^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj}(e_2),
\end{aligned} \tag{30.17}$$

i.e.

$$\partial_{\bar{u}_q}\mathcal{E}_{e_2,ii}^2 = d_{ii,jj,n_r}(e_2)|_{q=l_2(e_2,jj)}. \tag{30.18}$$

30.1.2. Derivatives of $\bar{\mathcal{E}}_i^2$ with respect to $\bar{w}_q$

Using equation (30.11) we have

$$\partial_{\bar{w}_q}\bar{\mathcal{E}}_i^2 = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{\bar{w}_q}\bar{\mathcal{E}}_{e_2,ii}^2, \tag{30.19}$$

and from equation (30.13) we have

$$\begin{aligned}\partial_{\bar{w}_q}\bar{\mathcal{E}}_{e_2,ii}^2 = {} & Fs\,Ds\,\partial_{\bar{w}_q}d_{ii}(e_2) + A\partial_{\bar{w}_q}d_{ii,n_r,\breve{u}}(e_2) + A\partial_{\bar{w}_q}d_{ii,n_z,\breve{w}}(e_2)\\ & + \sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{w}_q}\bar{u}_{l_2(e_2,jj)}d_{ii,jj,n_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{w}_q}\bar{w}_{l_2(e_2,jj)}d_{ii,jj,n_z}(e_2)\\ & - \sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{w}_q}u^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,n_r}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{w}_q}w^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj,n_z}(e_2)\\ & - Fs\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\bar{w}_q}\rho^{s_2}_{l_2^2(e_2,jj)}d_{ii,jj}(e_2),\end{aligned} \tag{30.20}$$

i.e.

$$\partial_{\bar{w}_q}\bar{\mathcal{E}}_{e_2,ii}^2 = d_{ii,jj,n_z}(e_2)|_{q=l_2(e_2,jj)}. \tag{30.21}$$

30.1.3. Derivatives of $\bar{\mathcal{E}}_i^2$ with respect to $u_q^{s_2}$

Using equation (30.11) we have

$$\partial_{u_q^{s_2}} \bar{\mathcal{E}}_i^2 = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{u_q^{s_2}} \bar{\mathcal{E}}_{e_2,ii}^2, \tag{30.22}$$

and from equation (30.13) we have

$$\begin{aligned}
\partial_{u_q^{s_2}} \bar{\mathcal{E}}_{e_2,ii}^2 = {} & Fs\, Ds\, \partial_{u_q^{s_2}} d_{ii}(e_2) + A\partial_{u_q^{s_2}} d_{ii,n_r,\breve{u}}(e_2) + A\partial_{\bar{w}_q} d_{ii,n_z,\breve{w}}(e_2) \\
& + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} u_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} w_{l_2(e_2,jj)} d_{ii,jj,n_z}(e_2) \\
& - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \bar{u}_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,n_r}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \bar{w}_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,n_z}(e_2) \\
& - Fs \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj}(e_2),
\end{aligned} \tag{30.23}$$

i.e.

$$\partial_{u_q^{s_2}} \bar{\mathcal{E}}_{e_2,ii}^2 = -d_{ii,jj,n_r}(e_2)|_{q=l_2(e_2,jj)}. \tag{30.24}$$

30.1.4. **Derivatives of $bar\mathcal{E}_i^2$ with respect to $w_q^{s_2}$**

Using equation (30.11) we have

$$\partial_{w_q^{s_2}} \bar{\mathcal{E}}_i^2 = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{w_q^{s_2}} \bar{\mathcal{E}}_{e_2,ii}^2, \tag{30.25}$$

and from equation (30.13) we have

$$\begin{aligned} \partial_{w_q^{s_2}} \bar{\mathcal{E}}_{e_2,ii}^2 = {} & Fs\, Ds\, \partial_{w_q^{s_2}} d_{ii}(e_2) + A\partial_{w_q^{s_2}} d_{ii,n_r,\breve{u}}(e_2) + A\partial_{w_q^{s_2}} d_{ii,n_z,\breve{w}}(e_2) \\ & + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \bar{u}_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \bar{w}_{l_2(e_2,jj)} d_{ii,jj,n_z}(e_2) \\ & - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} u_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,n_r}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} w_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,n_z}(e_2) \\ & - Fs \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj}(e_2), \end{aligned} \tag{30.26}$$

i.e.

$$\partial_{w_q^{s_2}} \bar{\mathcal{E}}_{e_2,ii}^2 = -d_{ii,jj,n_z}(e_2)|_{q=l_2(e_2,jj)}. \tag{30.27}$$

30.1.5. **Derivatives of $\mathcal{E}_i^2$ with respect to $\rho_q^{s_2}$**

Using equation (30.11) we have

$$\partial_{\rho_q^{s_2}} \bar{\mathcal{E}}_i^2 = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \partial_{\rho_q^{s_2}} \bar{\mathcal{E}}_{e_2,ii}^2, \tag{30.28}$$

and from equation (30.13) we have

$$\begin{aligned}
\partial_{\rho_q^{s_2}} \bar{\mathcal{E}}_{e_2,ii}^2 = {} & Fs\, Ds\, \partial_{\rho_q^{s_2}} d_{ii}(e_2) + A\partial_{w_q^{s_2}} d_{ii,n_r,\check{u}}(e_2) + A\partial_{w_q^{s_2}} d_{ii,n_z,\check{w}}(e_2) \\
& + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \bar{u}_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \bar{w}_{l_2(e_2,jj)} d_{ii,jj,n_z}(e_2) \\
& - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} u_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,n_r}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} w_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj,n_z}(e_2) \\
& - Fs \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj}(e_2),
\end{aligned} \tag{30.29}$$

i.e.

$$\partial_{\rho_q^{s_2}} \bar{\mathcal{E}}_{e_2,ii}^2 = -Fs\, d_{ii,jj}(e_2)|_{q=l_2(e_2,jj)}. \tag{30.30}$$

30.1.6. Derivatives of $\mathcal{E}_i^2$ with respect to θ_c

Using equation (30.11) we have

$$\partial_{\theta_c}\bar{\mathcal{E}}_i^2 = \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{\theta_c}\bar{\mathcal{E}}_{e_2,ii}^2, \tag{30.31}$$

and from equation (30.13) we have

$$\begin{aligned}\partial_{\theta_c}\bar{\mathcal{E}}_{e_2,ii}^2 = {} & Fs\,Ds\,\partial_{\theta_c}d_{ii}(e_2) + A\partial_{\theta_c}d_{ii,n_r,\breve{u}}(e_2) + A\partial_{\theta_c}d_{ii,n_z,\breve{w}}(e_2)\\ & + \sum_{jj=1}^{n_v^{2,e_2}} \bar{u}_{l_2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,n_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_A\bar{w}_{l_2(e_2,jj)}d_{ii,jj,n_z}(e_2)\\ & - \sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,n_r}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)}\partial_{\theta_c}d_{ii,jj,n_z}(e_2)\\ & - Fs\sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)}\partial_{\theta_c}d_{ii,jj}(e_2),\end{aligned} \tag{30.32}$$

i.e.

$$\partial_{\theta_c}\bar{\mathcal{E}}_{e_2,ii}^2 = A\partial_{\theta_c}d_{ii,n_r,\breve{u}}(e_2) + A\partial_{\theta_c}d_{ii,n_z,\breve{w}}(e_2). \tag{30.33}$$

30.1.7. Derivatives of $\mathcal{E}_i^2$ with respect to A

Using equation (30.11) we have

$$\partial_A \bar{\mathcal{E}}_i^2 = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_A \bar{\mathcal{E}}_{e_2,ii}^2, \tag{30.34}$$

and from equation (30.13) we have

$$\begin{aligned}\partial_A \bar{\mathcal{E}}_{e_2,ii}^2 = {} & Fs\,Ds\,\partial_A d_{ii}(e_2) + \partial_A A d_{ii,n_r,\check{u}}(e_2) + \partial_A A d_{ii,n_z,\check{w}}(e_2) \\ & + \sum_{jj=1}^{n_v^{2,e_2}} \partial_A \bar{u}_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) + \sum_{jj=1}^{n_v^{2,e_2}} \partial_A \bar{w}_{l_2(e_2,jj)} d_{ii,jj,n_z}(e_2) \\ & - \sum_{jj=1}^{n_v^{2,e_2}} \partial_A u^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_r}(e_2) - \sum_{jj=1}^{n_v^{2,e_2}} \partial_A w^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e_2) \\ & - Fs \sum_{jj=1}^{n_v^{2,e_2}} \partial_A \rho^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj}(e_2),\end{aligned} \tag{30.35}$$

i.e.

$$\partial_A \bar{\mathcal{E}}_{e_2,ii}^2 = d_{ii,n_r,\check{u}}(e_2) + d_{ii,n_z,\check{w}}(e_2). \tag{30.36}$$

30.1.8. **Derivatives of $\bar{\mathcal{E}}_i^2$ with respect to h_q**

Using equation (30.11) we have

$$\partial_{h_q}\bar{\mathcal{E}}_i^2 = \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\mathrm{el}}^2} \partial_{h_{S_2(e_2,qq)}}\bar{\mathcal{E}}_{e_2,ii}^2, \tag{30.37}$$

and from equation (30.13) we have

$$\begin{aligned}\partial_{h_{S_2(e_2,qq)}}\bar{\mathcal{E}}_{e_2,ii}^2 = {} & Fs\,Ds\,\partial_{h_{S_2(e_2,qq)}}d_{ii}(e_2)+\partial_{h_{S_2(e_2,qq)}}Ad_{ii,n_r,\check{u}}(e_2)+\partial_{h_{S_2(e_2,qq)}}Ad_{ii,n_z,\check{w}}(e_2)\\ & + \sum_{jj=1}^{n_v^{2,e_2}} \bar{u}_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2)\\ & + \sum_{jj=1}^{n_v^{2,e_2}} \bar{w}_{l_2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_z}(e_2)\\ & - \sum_{jj=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2)\\ & - \sum_{jj=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_z}(e_2)\\ & - Fs\sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj}(e_2),\end{aligned} \tag{30.38}$$

i.e.

$$\begin{aligned}\partial_{h_{S_2(e_2,qq)}}\bar{\mathcal{E}}_{e_2,ii}^2 = {} & Fs\,Ds\,\partial_{h_{S_2(e_2,qq)}}d_{ii}(e_2)\\ & + A\left[\partial_{h_{S_2(e_2,qq)}}d_{ii,n_r,\check{u}}(e_2)+\partial_{h_{S_2(e_2,qq)}}d_{ii,n_z,\check{w}}(e_2)\right]\\ & + \sum_{jj=1}^{n_v^{2,e_2}}\left[\partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_r}(e_2)\left\{\bar{u}_{l_2(e_2,jj)}-u^{s_2}_{l_2^2(e_2,jj)}\right\}\right.\\ & \qquad + \partial_{h_{S_2(e_2,qq)}}d_{ii,jj,n_z}(e_2)\left\{\bar{w}_{l_2(e_2,jj)}-w^{s_2}_{l_2^2(e_2,jj)}\right\}\\ & \qquad \left. - Fs\,\rho^{s_2}_{l_2^2(e_2,jj)}\partial_{h_{S_2(e_2,qq)}}d_{ii,jj}(e_2)\right].\end{aligned} \tag{30.39}$$

31. The density transport equation on boundary 2 (DTC2) in the near field

We recall equation (23.25) given by

$$Ts\,\{\partial_t\rho^{s_2}+\rho^{s_2}\nabla^s\cdot\boldsymbol{c}+\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]\}=Ds\,-\rho^{s_2}. \tag{31.1}$$

i.e.

$$Ts\,\partial_t\rho^{s_2}+Ts\,\rho^{s_2}t_r^2\partial_s\partial_t r^c+Ts\,\rho^{s_2}t_z^2\partial_s\partial_t z^c+Ts\,\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=Ds\,-\rho^{s_2}, \tag{31.2}$$

where ∂_s is the derivative with respect to the arc-length (which increases in the direction in which the tangent points).

We thus define the i-th DTC2 residual as

$$\begin{aligned}D_i^2=Ts\int\limits_{\partial\Omega^2}\phi_i^2\partial_t\rho^{s_2}+Ts\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s\partial_t r^c+Ts\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s\partial_t z^c\\+Ts\int\limits_{\partial\Omega^2}\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]-Ds\int\limits_{\partial\Omega^2}\phi_i^2+\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}.\end{aligned} \tag{31.3}$$

We consider now the term

$$Ts\int\limits_{\partial\Omega^2}\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})], \tag{31.4}$$

and we recall the vector calculus identity

$$\nabla^s\cdot(\phi\boldsymbol{A})=\boldsymbol{A}\cdot\nabla^s\phi+\phi\nabla^s\cdot\boldsymbol{A} \tag{31.5}$$

Using this identity with $\phi=\phi_i^2$ and $\boldsymbol{A}=\rho^{s_2}\left(\boldsymbol{v}^{s-2}-\boldsymbol{c}\right)$, we have

$$\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\right]=\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2+\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]\,, \tag{31.6}$$

i.e.

$$\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2+\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\right], \tag{31.7}$$

equivalently

$$\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\right]-\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2, \tag{31.8}$$

i.e.

$$\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\right]-\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2. \tag{31.9}$$

We now separate the normal and tangential components of $\boldsymbol{v}^{s_2}$ and $\boldsymbol{c}$, obtaining

$$\begin{aligned}\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\left(\boldsymbol{v}_\parallel^{s_2}-\boldsymbol{c}_\parallel\right)\right]+\nabla^s\cdot\left[\phi_i^2\rho^{s_2}\,\underbrace{(v_\perp^{s_2}-c_\perp)}_{=0}\,\boldsymbol{n}^2\right]\\-\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2,\end{aligned} \tag{31.10}$$

where the underbraced factor is equal to zero by the impermeability condition.

Taking this into the integral above, we have

$$\begin{aligned}Ts\int\limits_{\partial\Omega^2}\phi_i^2\nabla^s\cdot[\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})]=-Ts\int\limits_{C^2}\boldsymbol{m}^2\cdot\left[\phi_i^2\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\right]\\-Ts\int\limits_{\partial\Omega^2}\rho^{s_2}\,(\boldsymbol{v}^{s_2}-\boldsymbol{c})\cdot\nabla^s\phi_i^2,\end{aligned} \tag{31.11}$$

where we have applied the surface divergence theorem to the first term on the right-hand side above.

Here we notice that in the 2D case which we are considering, the boundary of $\partial\Omega^2$, given by C^2 is simply the end points of boundary 2, where the appropriate conditions are to be applied.

$$\begin{aligned} Ts \int_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = -Ts \int_{\partial\Omega^2} & \rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right) \cdot \nabla^s \phi_i^2 \\ & - Ts\, \phi_i^2(c)\rho_c^{s_2}\left(\boldsymbol{v}_c^{s_2} \cdot \boldsymbol{m}_c^2 - \boldsymbol{c}_c \cdot \boldsymbol{m}_c^2\right) \\ & \underbrace{-Ts\, \phi_i^2(o)\rho_o^{s_2}\boldsymbol{v}_o^{s_2} \cdot \boldsymbol{m}_o^2 + Ts\, \phi_i^2(o)\rho_o^{s_2}\boldsymbol{c}_o \cdot \boldsymbol{m}_o^2}_{=0}, \end{aligned} \tag{31.12}$$

where the o sub-index stands for the origin, where there velocity of the coordinates and the surface are both zero. This yields

$$\begin{aligned} Ts \int_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = -Ts \int_{\partial\Omega^2} & \rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right) \cdot \nabla^s \phi_i^2 \\ & - Ts\, \phi_i^2(c)\rho_c^{s_2}\boldsymbol{v}_c^{s_2} \cdot \boldsymbol{m}_c^2 + Ts\, \phi_i^2(c)\rho_c^{s_2}\boldsymbol{c}_c \cdot \boldsymbol{m}_c^2, \end{aligned} \tag{31.13}$$

We notice here that we have not decomposed this equation into two parts (near-field and far-field), as it does not involve the bulk velocity variables, which are the only ones that require a separate treatment.

Re-writing the expression above we have

$$\begin{aligned} Ts \int_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = -Ts \int_{\partial\Omega^2} & \rho^{s_2}\left(\partial_s \phi_i^2\right)\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right) \cdot \boldsymbol{t}^2 \\ & - Ts\, \delta_{i,c}\rho_c^{s_2} u_c^{s_2} m_r^2(c) - Ts\, \delta_{i,c}\rho_c^{s_2} w_c^{s_2} m_z^2(c) \\ & + Ts\, \delta_{i,c}\rho_c^{s_2} m_r^2(c)\partial_t r_c^c + Ts\, \delta_{i,c}\rho_c^{s_2} m_z^2(c)\partial_t z_c^c, \end{aligned} \tag{31.14}$$

i.e.

$$\begin{aligned} Ts \int_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = -Ts \int_{\partial\Omega^2} & \rho^{s_2}\left(\partial_s \phi_i^2\right)\boldsymbol{v}^{s_2} \cdot \boldsymbol{t}^2 + Ts \int_{\partial\Omega^2} \rho^{s_2}\left(\partial_s \phi_i^2\right)\boldsymbol{c} \cdot \boldsymbol{t}^2 \\ & - Ts\, \delta_{i,c}\rho_c^{s_2} u_c^{s_2} m_r^2(c) - Ts\, \delta_{i,c}\rho_c^{s_2} w_c^{s_2} m_z^2(c) \\ & + Ts\, \delta_{i,c}\rho_c^{s_2} m_r^2(c)\partial_t r_c^c + Ts\, \delta_{i,c}\rho_c^{s_2} m_z^2(c)\partial_t z_c^c, \end{aligned} \tag{31.15}$$

which is

$$\begin{aligned} Ts \int_{\partial\Omega^2} \phi_i^2 \nabla^s \cdot \left[\rho^{s_2}\left(\boldsymbol{v}^{s_2} - \boldsymbol{c}\right)\right] = & -Ts \int_{\partial\Omega^2} \rho^{s_2} u^{s_2} t_r^2 \partial_s \phi_i^2 - Ts \int_{\partial\Omega^2} \rho^{s_2} w^{s_2} t_z^2 \partial_s \phi_i^2 \\ & + Ts \int_{\partial\Omega^2} \rho^{s_2} t_r^2 \left(\partial_s \phi_i^2\right) \partial_t r^c + Ts \int_{\partial\Omega^2} \rho^{s_2} t_z^2 \left(\partial_s \phi_i^2\right) \partial_t z^c \\ & - Ts\, \delta_{i,c}\rho_c^{s_2} u_c^{s_2} m_r^2(c) - Ts\, \delta_{i,c}\rho_c^{s_2} w_c^{s_2} m_z^2(c) \\ & + Ts\, \delta_{i,c}\rho_c^{s_2} m_r^2(c)\partial_t r_c^c + Ts\, \delta_{i,c}\rho_c^{s_2} m_z^2(c)\partial_t z_c^c, \end{aligned} \tag{31.16}$$

Taking this result into the residual equation 31.3 we have

$$
\begin{aligned}
D_i^2 = &-Ts\,\delta_{i,c}\rho_c^{s_2}u_c^{s_2}m_r^2(c) - Ts\,\delta_{i,c}\rho_c^{s_2}w_c^{s_2}m_z^2(c) \\
&+ Ts\,\delta_{i,c}\rho_c^{s_2}m_r^2(c)\partial_t r_c^c + Ts\,\delta_{i,c}\rho_c^{s_2}m_z^2(c)\partial_t z_c^c \\
&- Ts\int_{\partial\Omega^2}\rho^{s_2}u^{s_2}t_r^2\partial_s\phi_i^2 - Ts\int_{\partial\Omega^2}\rho^{s_2}w^{s_2}t_z^2\partial_s\phi_i^2 \\
&+ Ts\int_{\partial\Omega^2}\rho^{s_2}t_r^2\left(\partial_s\phi_i^2\right)\partial_t r^c + Ts\int_{\partial\Omega^2}\rho^{s_2}t_z^2\left(\partial_s\phi_i^2\right)\partial_t z^c \\
&+ Ts\int_{\partial\Omega^2}\phi_i^2\partial_t\rho^{s_2} + Ts\int_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s\partial_t r^c \\
&+ Ts\int_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s\partial_t z^c - Ds\int_{\partial\Omega^2}\phi_i^2 + \int_{\partial\Omega^2}\phi_i^2\rho^{s_2}.
\end{aligned}
\tag{31.17}
$$

We now recall the approximation

$$
\partial_t r^c \approx \frac{3r^c - 4r^c(t_{n-1}) + r^c(t_{n-2})}{2\Delta_t}
\tag{31.18}
$$

and

$$
\partial_t z^c \approx \frac{3z^c - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t};
\tag{31.19}
$$

and we introduce

$$
\partial_t \rho^{s_2} \approx \frac{3\rho^{s_2} - 4\rho^{s_2}(t_{n-1}) + \rho^{s_2}(t_{n-2})}{2\Delta_t}.
\tag{31.20}
$$

Substituting these approximations in the residual equation we have

$$
\begin{aligned}
\mathfrak{D}_i^2 = &-Ts\,\delta_{i,c}\rho_c^{s_2}u_c^{s_2}m_r^2(c) - Ts\,\delta_{i,c}\rho_c^{s_2}w_c^{s_2}m_z^2(c) \\
&+ Ts\,\delta_{i,c}\rho_c^{s_2}m_r^2(c)\frac{3r_c^c - 4r_c^c(t_{n-1}) + r_c^c(t_{n-2})}{2\Delta_t} \\
&+ Ts\,\delta_{i,c}\rho_c^{s_2}m_z^2(c)\frac{3z^c - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t} \\
&- Ts\int\limits_{\partial\Omega^2}\rho^{s_2}u^{s_2}t_r^2\partial_s\phi_i^2 - Ts\int\limits_{\partial\Omega^2}\rho^{s_2}w^{s_2}t_z^2\partial_s\phi_i^2 \\
&+ Ts\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2\frac{3r^c - 4r^c(t_{n-1}) + r^c(t_{n-2})}{2\Delta_t}\partial_s\phi_i^2 \\
&+ Ts\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2\frac{3z^c - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t}\partial_s\phi_i^2 \\
&+ Ts\int\limits_{\partial\Omega^2}\phi_i^2\frac{3\rho^{s_2} - 4\rho^{s_2}(t_{n-1}) + \rho^{s_2}(t_{n-2})}{2\Delta_t} \\
&+ Ts\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\frac{3\partial_s r^c - 4\partial_s r^c(t_{n-1}) + \partial_s r^c(t_{n-2})}{2\Delta_t} \\
&+ Ts\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\frac{3\partial_s z^c - 4\partial_s z^c(t_{n-1}) + \partial_s z^c(t_{n-2})}{2\Delta_t} - Ds\int\limits_{\partial\Omega^2}\phi_i^2 + \int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}.
\end{aligned}
\tag{31.21}
$$

Multiplying the residual equation by $2\Delta_t/3$ we have

$$
\begin{aligned}
\mathscr{D}_i^2 = &-\frac{2\Delta_t Ts}{3}\delta_{i,c}\rho_c^{s_2}u_c^{s_2}m_r^2(c) - \frac{2\Delta_t Ts}{3}\delta_{i,c}\rho_c^{s_2}w_c^{s_2}m_z^2(c)\\
&+ Ts\,\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c - \frac{4Ts}{3}\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c(t_{n-1}) + \frac{Ts}{3}\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c(t_{n-2})\\
&+ Ts\,\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c - \frac{4Ts}{3}\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c(t_{n-1}) + \frac{Ts}{3}\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c(t_{n-2})\\
&- \frac{2\Delta_t Ds}{3}\int\limits_{\partial\Omega^2}\phi_i^2 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}\\
&+ Ts\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2} - \frac{4Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}(t_{n-1}) + \frac{Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}(t_{n-2})\\
&- \frac{2\Delta_t Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}u^{s_2}t_r^2\partial_s\phi_i^2 - \frac{2\Delta_t Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}w^{s_2}t_z^2\partial_s\phi_i^2\\
&+ Ts\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2r^c\partial_s\phi_i^2 - \frac{4Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2r^c(t_{n-1})\partial_s\phi_i^2 + \frac{Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2r^c(t_{n-2})\partial_s\phi_i^2\\
&+ Ts\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2z^c\partial_s\phi_i^2 - \frac{4Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2z^c(t_{n-1})\partial_s\phi_i^2 + \frac{Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2z^c(t_{n-2})\partial_s\phi_i^2\\
&+ Ts\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c - \frac{4Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c(t_{n-1}) + \frac{Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c(t_{n-2})\\
&+ Ts\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c - \frac{4Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c(t_{n-1}) + \frac{Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c(t_{n-2}).
\end{aligned}
\tag{31.22}
$$

We now introduce the decomposition

$$
\begin{aligned}
\mathscr{D}_i^2 = &-\frac{2\Delta_t Ts}{3}\delta_{i,c}\rho_c^{s_2}u_c^{s_2}m_r^2(c) - \frac{2\Delta_t Ts}{3}\delta_{i,c}\rho_c^{s_2}w_c^{s_2}m_z^2(c)\\
&+ Ts\,\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c - \frac{4Ts}{3}\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c(t_{n-1}) + \frac{Ts}{3}\delta_{i,c}\rho_c^{s_2}m_r^2(c)r_c^c(t_{n-2})\\
&+ Ts\,\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c - \frac{4Ts}{3}\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c(t_{n-1}) + \frac{Ts}{3}\delta_{i,c}\rho_c^{s_2}m_z^2(c)z^c(t_{n-2})\\
&+ \mathscr{D}_i^{2,a} + \mathscr{D}_i^{2,b} + \mathscr{D}_i^{2,c},
\end{aligned}
\tag{31.23}
$$

where

$$
\mathscr{D}_i^{2,a} = -\frac{2\Delta_t Ds}{3}\int\limits_{\partial\Omega^2}\phi_i^2, \tag{31.24}
$$

$$
\mathscr{D}_i^{2,b} = \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2} + Ts\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2} - \frac{4Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}(t_{n-1}) + \frac{Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}(t_{n-2}), \tag{31.25}
$$

and

$$
\begin{aligned}
\mathscr{D}_i^{2,c} = &-\frac{2\Delta_t Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}u^{s_2}t_r^2\partial_s\phi_i^2 - \frac{2\Delta_t Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}w^{s_2}t_z^2\partial_s\phi_i^2 \\
&+ Ts\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2 r^c\partial_s\phi_i^2 - \frac{4Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2 r^c(t_{n-1})\partial_s\phi_i^2 + \frac{Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_r^2 r^c(t_{n-2})\partial_s\phi_i^2 \\
&+ Ts\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2 z^c\partial_s\phi_i^2 - \frac{4Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2 z^c(t_{n-1})\partial_s\phi_i^2 + \frac{Ts}{3}\int\limits_{\partial\Omega^2}\rho^{s_2}t_z^2 z^c(t_{n-2})\partial_s\phi_i^2 \\
&+ Ts\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c - \frac{4Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c(t_{n-1}) + \frac{Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_r^2\partial_s r^c(t_{n-2}) \\
&+ Ts\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c - \frac{4Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c(t_{n-1}) + \frac{Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\rho^{s_2}t_z^2\partial_s z^c(t_{n-2}).
\end{aligned}
\tag{31.26}
$$

Furthermore, we recall

$$\rho^{s_2} \approx \sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2, \tag{31.27}$$

$$\rho^{s_2}(t_{n-1}) \approx \sum_{j=1}^{n_v}\rho_j^{s_2}(t_{n-1})\phi_j^2, \tag{31.28}$$

$$\rho^{s_2}(t_{n-2}) \approx \sum_{j=1}^{n_v}\rho_j^{s_2}(t_{n-2})\phi_j^2, \tag{31.29}$$

$$r^c \approx \sum_{j=1}^{n_v}r_j^c\phi_j, \tag{31.30}$$

$$r^c(t_{n-1}) \approx \sum_{j=1}^{n_v}r_j^c(t_{n-1})\phi_j, \tag{31.31}$$

$$r^c(t_{n-2}) \approx \sum_{j=1}^{n_v}r_j^c(t_{n-2})\phi_j, \tag{31.32}$$

$$z^c \approx \sum_{j=1}^{n_v}z_j^c\phi_j \tag{31.33}$$

$$z^c(t_{n-1}) \approx \sum_{j=1}^{n_v}z_j^c(t_{n-1})\phi_j \tag{31.34}$$

$$z^c(t_{n-2}) \approx \sum_{j=1}^{n_v}z_j^c(t_{n-2})\phi_j \tag{31.35}$$

$$u^{s_2} \approx \sum_{j=1}^{n_v} u_j^{s_2} \phi_j^2 \tag{31.36}$$

and

$$w^{s_2} \approx \sum_{j=1}^{n_v} w_j^{s_2} \phi_j. \tag{31.37}$$

Substituting the approximations above into the residual we have where

$$\mathcal{D}_i^{2,a} = -\frac{2\Delta_t Ds}{3} \int\limits_{\partial\Omega^2} \phi_i^2, \tag{31.38}$$

$$\begin{aligned}
\mathcal{D}_i^{2,b} = \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} \rho_j^{s_2} \phi_j^2 \right) \\
+ Ts \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} \rho_j^{s_2} \phi_j^2 \right) \\
- \frac{4Ts}{3} \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} \rho_j^{s_2} (t_{n-1}) \phi_j^2 \right) \\
+ \frac{Ts}{3} \int\limits_{\partial\Omega^2} \phi_i^2 \left(\sum_{j=1}^{n_v} \rho_j^{s_2} (t_{n-2}) \phi_j^2 \right),
\end{aligned} \tag{31.39}$$

and

$$
\begin{aligned}
\mathcal{D}_i^{2,c} = & -\frac{2\Delta_t Ts}{3}\int\limits_{\partial\Omega^2}\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)\left(\sum_{k=1}^{n_v}u_k^{s_2}\phi_k^2\right)t_r^2\partial_s\phi_i^2 \\
& -\frac{2\Delta_t Ts}{3}\int\limits_{\partial\Omega^2}\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)\left(\sum_{k=1}^{n_v}w_k^{s_2}\phi_k^2\right)t_z^2\partial_s\phi_i^2 \\
& + Ts\int\limits_{\partial\Omega^2}\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_r^2\left(\sum_{k=1}^{n_v}r_k^c\phi_k\right)\partial_s\phi_i^2 \\
& -\frac{4Ts}{3}\int\limits_{\partial\Omega^2}\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_r^2\left(\sum_{k=1}^{n_v}r_k^c(t_{n-1})\phi_k\right)\partial_s\phi_i^2 \\
& +\frac{Ts}{3}\int\limits_{\partial\Omega^2}\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_r^2\left(\sum_{k=1}^{n_v}r_k^c(t_{n-2})\phi_k\right)\partial_s\phi_i^2 \\
& + Ts\int\limits_{\partial\Omega^2}\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_z^2\left(\sum_{k=1}^{n_v}z_k^c\phi_k\right)\partial_s\phi_i^2 \\
& -\frac{4Ts}{3}\int\limits_{\partial\Omega^2}\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_z^2\left(\sum_{k=1}^{n_v}z_k^c(t_{n-1})\phi_k\right)\partial_s\phi_i^2 \\
& +\frac{Ts}{3}\int\limits_{\partial\Omega^2}\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_z^2\left(\sum_{k=1}^{n_v}z_k^c(t_{n-2})\phi_k\right)\partial_s\phi_i^2 \\
& + Ts\int\limits_{\partial\Omega^2}\phi_i^2\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_r^2\left(\sum_{k=1}^{n_v}r_k^c\partial_s\phi_k\right) \\
& -\frac{4Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_r^2\left(\sum_{k=1}^{n_v}r_k^c(t_{n-1})\partial_s\phi_k\right) \\
& +\frac{Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_r^2\left(\sum_{k=1}^{n_v}r_k^c(t_{n-2})\partial_s\phi_k\right) \\
& + Ts\int\limits_{\partial\Omega^2}\phi_i^2\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_z^2\left(\sum_{k=1}^{n_v}z_k^c\partial_s\phi_k\right) \\
& -\frac{4Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_z^2\left(\sum_{k=1}^{n_v}z_k^c(t_{n-1})\partial_s\phi_k\right) \\
& +\frac{Ts}{3}\int\limits_{\partial\Omega^2}\phi_i^2\left(\sum_{j=1}^{n_v}\rho_j^{s_2}\phi_j^2\right)t_z^2\left(\sum_{k=1}^{n_v}z_k^c(t_{n-2})\partial_s\phi_k\right).
\end{aligned}
\tag{31.40}
$$

$$\mathcal{D}_i^{2,a} = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \mathcal{D}_{e_2,ii}^{2,a}, \tag{31.41}$$

$$\mathcal{D}_i^{2,b} = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \mathcal{D}_{e_2,ii}^{2,b} \tag{31.42}$$

and

$$\mathcal{D}_i^{2,c} = \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2} \mathcal{D}_{e_2,ii}^{2,c}; \tag{31.43}$$

where

$$\mathcal{D}_{e_2,ii}^{2,a} = -\frac{2\Delta_t Ds}{3} \underbrace{\int_{\partial\Omega^2} \phi_i^2}_{d_{ii}(e_2)}, \tag{31.44}$$

$$\begin{aligned}
\mathcal{D}_{e_2,ii}^{2,b} = \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho_{l_2^2(e_2,jj)}^{s_2} \underbrace{\int_{\partial\Omega^2} \phi_i^2 \phi_j^2}_{d_{ii,jj}(e_2)} \\
+Ts \sum_{jj=1}^{n_v^{2,e_2}} \rho_{l_2^2(e_2,jj)}^{s_2} \underbrace{\int_{\partial\Omega^2} \phi_i^2 \phi_j^2}_{d_{ii,jj}(e_2)} \\
-\frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho_{l_2^2(e_2,jj)}^{s_2}(t_{n-1}) \underbrace{\int_{\partial\Omega^2} \phi_i^2 \phi_j^2}_{d_{ii,jj}(e_2)} \\
+\frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho_{l_2^2(e_2,jj)}^{s_2}(t_{n-2}) \underbrace{\int_{\partial\Omega^2} \phi_i^2 \phi_j^2}_{d_{ii,jj}(e_2)},
\end{aligned} \tag{31.45}$$

and

$$
\begin{aligned}
\mathcal{D}^{2,c}_{e_2,ii} = & -\frac{2\Delta_t Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}u^{s_2}_{l_2^2(e_2,kk)}\underbrace{\int_{\partial\Omega^2} t_r^2\phi_j^2\phi_k^2\partial_s\phi_i^2}_{d^s_{jj,kk,ii,t_r}(e_2)} \\
& -\frac{2\Delta_t Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}w^{s_2}_{l_2^2(e_2,kk)}\underbrace{\int_{\partial\Omega^2} t_z^2\phi_j^2\phi_k^2\partial_s\phi_i^2}_{d^s_{jj,kk,ii,t_z}(e_2)} \\
& +Ts\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}\underbrace{\int_{\partial\Omega^2} t_r^2\phi_j^2\phi_k\partial_s\phi_i^2}_{d^s_{jj,kk,ii,t_r}(e_2)} - \frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}(t_{n-1})\underbrace{\int_{\partial\Omega^2} t_r^2\phi_j^2\phi_k \ldots}_{d^s_{jj,kk,ii,t_r}\ldots} \\
& +\frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}(t_{n-2})\underbrace{\int_{\partial\Omega^2} t_r^2\phi_j^2\phi_k\partial_s\phi_i^2}_{d^s_{jj,kk,ii,t_r}(e_2)} \\
& +Ts\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}\underbrace{\int_{\partial\Omega^2} t_z^2\phi_j^2\phi_k\partial_s\phi_i^2}_{d^s_{jj,kk,ii,t_z}(e_2)} - \frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}(t_{n-1})\underbrace{\int_{\partial\Omega^2} t_z^2\phi_j^2\phi_k \ldots}_{d^s_{jj,kk,ii,t_z}\ldots} \\
& +\frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}(t_{n-2})\underbrace{\int_{\partial\Omega^2} t_z^2\phi_j^2\phi_k\partial_s\phi_i^2}_{d^s_{jj,kk,ii,t_z}(e_2)} \\
& +Ts\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}\underbrace{\int_{\partial\Omega^2} \phi_i^2\phi_j^2\partial_s\phi_k^2 t_r^2}_{d^s_{ii,jj,kk,t_r}(e_2)} - \frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}(t_{n-1})\underbrace{\int_{\partial\Omega^2} \phi_i^2\phi_j^2\partial_s \ldots}_{d^s_{ii,jj,kk,t_r}\ldots} \\
& +\frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}(t_{n-2})\underbrace{\int_{\partial\Omega^2} \phi_i^2\phi_j^2\partial_s\phi_k^2 t_r^2}_{d^s_{ii,jj,kk,t_r}(e_2)} \\
& +Ts\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}\underbrace{\int_{\partial\Omega^2} \phi_i^2\phi_j^2\partial_s\phi_k^2 t_z^2}_{d^s_{ii,jj,kk,t_z}(e_2)} - \frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}(t_{n-1})\underbrace{\int_{\partial\Omega^2} \phi_i^2\phi_j^2 \ldots}_{d^s_{ii,jj,kk}\ldots} \\
& +\frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}(t_{n-2})\underbrace{\int_{\partial\Omega^2} \phi_i^2\phi_j^2\partial_s\phi_k^2 t_z^2}_{d^s_{ii,jj,kk,t_z}(e_2)}\,.
\end{aligned}
\tag{31.46}
$$

Re-writing we have

$$\mathcal{D}^{2,a}_{e_2,ii} = -\frac{2\Delta_t Ds}{3} d_{ii}(e_2), \tag{31.47}$$

$$\begin{aligned}\mathcal{D}^{2,b}_{e_2,ii} = \frac{2\Delta_t}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj}(e_2) \\ + Ts \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)} d_{ii,jj}(e_2) \\ - \frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)}(t_{n-1}) d_{ii,jj}(e_2) \\ + \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)}(t_{n-2}) d_{ii,jj}(e_2),\end{aligned} \tag{31.48}$$

and

$$\begin{aligned}
\mathcal{D}^{2,c}_{e_2,ii} ={}& -\frac{2\Delta_t Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}u^{s_2}_{l_2^2(e_2,kk)}d^{s}_{jj,kk,ii,t_r}(e_2)\\
&-\frac{2\Delta_t Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}w^{s_2}_{l_2^2(e_2,kk)}d^{s}_{jj,kk,ii,t_z}(e_2)\\
&+Ts\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}d^{s}_{jj,kk,ii,t_r}(e_2)\\
&-\frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}(t_{n-1})d^{s}_{jj,kk,ii,t_r}(e_2)\\
&+\frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}(t_{n-2})d^{s}_{jj,kk,ii,t_r}(e_2)\\
&+Ts\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}d^{s}_{jj,kk,ii,t_z}(e_2)\\
&-\frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}(t_{n-1})d^{s}_{jj,kk,ii,t_z}(e_2)\\
&+\frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}(t_{n-2})d^{s}_{jj,kk,ii,t_z}(e_2)\\
&+Ts\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}d^{s}_{ii,jj,kk,t_r}(e_2)\\
&-\frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}(t_{n-1})d^{s}_{ii,jj,kk,t_r}(e_2)\\
&+\frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}r^{c}_{l_2(e_2,kk)}(t_{n-2})d^{s}_{ii,jj,kk,t_r}(e_2)\\
&+Ts\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}d^{s}_{ii,jj,kk,t_z}(e_2)\\
&-\frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}(t_{n-1})d^{s}_{ii,jj,kk,t_z}(e_2)\\
&+\frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\rho^{s_2}_{l_2^2(e_2,jj)}\sum_{kk=1}^{n_v^{2,e_2}}z^{c}_{l_2(e_2,kk)}(t_{n-2})d^{s}_{ii,jj,kk,t_z}(e_2).
\end{aligned}\tag{31.49}$$

Summarising and re-arranging we have

$$
\begin{aligned}
\mathcal{D}_i^2 = &-\frac{2\Delta_t Ts}{3}\delta_{i,c}\rho_c^{s_2}\left[u_c^{s_2}m_r^2(c)+w_c^{s_2}m_z^2(c)\right] \\
&+Ts\,\delta_{i,c}\rho_c^{s_2}m_r^2(c)\left[r_c^c-\frac{4}{3}r_c^c(t_{n-1})+\frac{1}{3}r_c^c(t_{n-2})\right] \\
&+Ts\,\delta_{i,c}\rho_c^{s_2}m_z^2(c)\left[z_c^c-\frac{4}{3}z_c^c(t_{n-1})+\frac{1}{3}z_c^c(t_{n-2})\right] \\
&+\sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2}\mathcal{D}_{e_2,ii}^{2,a}+\sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2}\mathcal{D}_{e_2,ii}^{2,b}+\sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\mathrm{el}}^2}\mathcal{D}_{e_2,ii}^{2,c},
\end{aligned}
\tag{31.50}
$$

with

$$
\mathcal{D}_{e_2,ii}^{2,a} = -\frac{2\Delta_t Ds}{3}d_{ii}(e_2), \tag{31.51}
$$

$$
\begin{aligned}
\mathcal{D}_{e_2,ii}^{2,b} = \sum_{j=1}^{n_v} d_{ii,jj}(e_2)\Bigg\{&\frac{2\Delta_t}{3}\rho^{s_2}_{l_2^2(e_2,jj)} \\
&+Ts\left[\rho^{s_2}_{l_2^2(e_2,jj)}-\frac{4}{3}\rho^{s_2}_{l_2^2(e_2,jj)}(t_{n-1})+\frac{1}{3}\rho^{s_2}_{l_2^2(e_2,jj)}(t_{n-2})\right]\Bigg\},
\end{aligned}
\tag{31.52}
$$

and

$$
\begin{aligned}
\mathcal{D}_{e_2,ii}^{2,c} = \sum_{jj\,=1}^{n_v^{2,e_2}} Ts\,\rho^{s_2}_{l_2^2(e_2,jj)}\Bigg\{&-\frac{2\Delta_t}{3}\underbrace{\sum_{kk=1}^{n_v^{2,e_2}}\left[u^{s_2}_{l_2^2(e_2,kk)}d^s_{jj,kk,ii,t_r}(e_2)+w^{s_2}_{l_2^2(e_2,kk)}d^s_{jj,kk,ii,t_z}(e_2)\right]}_{A^{s_2}_{ii,jj}(e_2)} \\
&+\underbrace{\sum_{kk=1}^{n_v^{2,e_2}}d^s_{jj,kk,ii,t_r}(e_2)\left[r^c_{l_2(e_2,kk)}-\frac{4}{3}r^c_{l_2(e_2,kk)}(t_{n-1})+\frac{1}{3}r^c_{l_2(e_2,kk)}(t_{n-2})\right]}_{B^{s_2}_{ii,jj}(e_2)} \\
&+\underbrace{\sum_{kk=1}^{n_v^{2,e_2}}d^s_{jj,kk,ii,t_z}(e_2)\left[z^c_{l_2(e_2,kk)}-\frac{4}{3}z^c_{l_2(e_2,kk)}(t_{n-1})+\frac{1}{3}z^c_{l_2(e_2,kk)}(t_{n-2})\right]}_{C^{s_2}_{ii,jj}(e_2)} \\
&+\underbrace{\sum_{kk=1}^{n_v^{2,e_2}}d^s_{ii,jj,kk,t_r}(e_2)\left[r^c_{l_2(e_2,kk)}-\frac{4}{3}r^c_{l_2(e_2,kk)}(t_{n-1})+\frac{1}{3}r^c_{l_2(e_2,kk)}(t_{n-2})\right]}_{D^{s_2}_{ii,jj}(e_2)} \\
&+\underbrace{\sum_{kk=1}^{n_v^{2,e_2}}d^s_{ii,jj,kk,t_z}(e_2)\left[z^c_{l_2(e_2,kk)}-\frac{4}{3}z^c_{l_2(e_2,kk)}(t_{n-1})+\frac{1}{3}z^c_{l_2(e_2,kk)}(t_{n-2})\right]}_{E^{s_2}_{ii,jj}(e_2)}\Bigg\}.
\end{aligned}
\tag{31.53}
$$

31.1. Jacobian terms

Here we find the derivatives of $\mathcal{D}_i^2$ with respect to $\rho_q^{s_2}$, $u_q^{s_2}$, $w_q^{s_2}$ and h_q.

31.1.1. Derivatives of $\mathcal{D}_i^2$ with respect to $\rho_q^{s_2}$

Using equations (31.50) we have

$$\begin{aligned}
\partial_{\rho_q^{s_2}} \mathcal{D}_i^2 = &-\frac{2\Delta_t Ts}{3}\delta_{i,c} u_c^{s_2} m_r^2(c)\partial_{\rho_q^{s_2}}\rho_c^{s_2} - \frac{2\Delta_t Ts}{3}\delta_{i,c} w_c^{s_2} m_z^2(c)\partial_{\rho_q^{s_2}}\rho_c^{s_2}\\
&+ Ts\,\delta_{i,c} m_r^2(c) r_c^c \partial_{\rho_q^{s_2}}\rho_c^{s_2} - \frac{4Ts}{3}\delta_{i,c} m_r^2(c) r_c^c(t_{n-1})\partial_{\rho_q^{s_2}}\rho_c^{s_2}\\
&+ \frac{Ts}{3}\delta_{i,c} m_r^2(c) r_c^c(t_{n-2})\partial_{\rho_q^{s_2}}\rho_c^{s_2}\\
&+ Ts\,\delta_{i,c} m_z^2(c) z^c \partial_{\rho_q^{s_2}}\rho_c^{s_2} - \frac{4Ts}{3}\delta_{i,c} m_z^2(c) z^c(t_{n-1})\partial_{\rho_q^{s_2}}\rho_c^{s_2}\\
&+ \frac{Ts}{3}\delta_{i,c} m_z^2(c) z^c(t_{n-2})\partial_{\rho_q^{s_2}}\rho_c^{s_2}\\
&+ \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{\rho_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,a} + \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{\rho_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,b} + \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{\rho_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,c},
\end{aligned} \tag{31.54}$$

i.e.

$$\begin{aligned}
\partial_{\rho_q^{s_2}} \mathcal{D}_i^2 = &-\frac{2\Delta_t Ts}{3}\delta_{i,c}\delta_{c,q} u_c^{s_2} m_r^2(c) - \frac{2\Delta_t Ts}{3}\delta_{i,c}\delta_{c,q} w_c^{s_2} m_z^2(c)\\
&+ Ts\,\delta_{i,c}\delta_{c,q} m_r^2(c) r_c^c - \frac{4Ts}{3}\delta_{i,c}\delta_{c,q} m_r^2(c) r_c^c(t_{n-1}) + \frac{Ts}{3}\delta_{i,c}\delta_{c,q} m_r^2(c) r_c^c(t_{n-2})\\
&+ Ts\,\delta_{i,c}\delta_{c,q} m_z^2(c) z^c - \frac{4Ts}{3}\delta_{i,c}\delta_{c,q} m_z^2(c) z^c(t_{n-1}) + \frac{Ts}{3}\delta_{i,c}\delta_{c,q} m_z^2(c) z^c(t_{n-2})\\
&\sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{\rho_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,a} + \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{\rho_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,b} + \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{\rho_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,c}.
\end{aligned} \tag{31.55}$$

From equation (31.47) we have

$$\partial_{\rho_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,a} = -\frac{2\Delta_t Ds}{3}\partial_{\rho_q^{s_2}} d_{ii}(e_2), \tag{31.56}$$

i.e.

$$\partial_{\rho_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,a} = 0. \tag{31.57}$$

From equation (31.48) we have

$$\begin{aligned}
\partial_{\rho_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,b} = &\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\partial_{\rho_q^{s_2}}\rho_j^{s_2} d_{ii,jj}(e_2) + Ts\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\rho_q^{s_2}}\rho_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj}(e_2)\\
&-\frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\rho_q^{s_2}}\rho_{l_2^2(e_2,jj)}^{s_2}(t_{n-1}) d_{ii,jj}(e_2)\\
&+\frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{\rho_q^{s_2}}\rho_{l_2^2(e_2,jj)}^{s_2}(t_{n-2}) d_{ii,jj}(e_2),
\end{aligned} \tag{31.58}$$

i.e.

$$\partial_{\rho_q^{s_2}} \mathcal{D}^{2,b}_{e_2,ii} = \frac{2\Delta_t}{3} d_{ii,jj}(e_2)|_{q=l_2^2(e_2,jj)} + Ts\, d_{ii,jj}(e_2)|_{q=l_2^2(e_2,jj)}. \tag{31.59}$$

From equation (31.49) we have

$$\begin{aligned}
\partial_{\rho_q^{s_2}} \mathcal{D}^{2,c}_{e_2,ii} =& -\frac{2\Delta_t Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,kk)} d^s_{jj,kk,ii,t_r}(e_2) \\
& - \frac{2\Delta_t Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,kk)} d^s_{jj,kk,ii,t_z}(e_2) \\
& + Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)} d^s_{jj,kk,ii,t_r}(e_2) \\
& - \frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{jj,kk,ii,t_r}(e_2) \\
& + \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{jj,kk,ii,t_r}(e_2) \\
& + Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)} d^s_{jj,kk,ii,t_z}(e_2) \\
& - \frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{jj,kk,ii,t_z}(e_2) \\
& + \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{jj,kk,ii,t_z}(e_2) \\
& + Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)} d^s_{ii,jj,kk,t_r}(e_2) \\
& - \frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{ii,jj,kk,t_r}(e_2) \\
& + \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{ii,jj,kk,t_r}(e_2) \\
& + Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)} d^s_{ii,jj,kk,t_z}(e_2) \\
& - \frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{ii,jj,kk,t_z}(e_2) \\
& + \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{\rho_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{ii,jj,kk,t_z}(e_2).
\end{aligned} \tag{31.60}$$

i.e.

$$
\begin{aligned}
\partial_{\rho_q^{s_2}} \mathcal{D}^{2,c}_{e_2,ii} = &- \frac{2\Delta_t Ts}{3} \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,kk)} d^s_{jj,kk,ii,t_r}(e_2) \\
&- \frac{2\Delta_t Ts}{3} \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,kk)} d^s_{jj,kk,ii,t_z}(e_2) \\
&+ Ts \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)} d^s_{jj,kk,ii,t_r}(e_2) \\
&- \frac{4Ts}{3} \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{jj,kk,ii,t_r}(e_2) \\
&+ \frac{Ts}{3} \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{jj,kk,ii,t_r}(e_2) \\
&+ Ts \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)} d^s_{jj,kk,ii,t_z}(e_2) \\
&- \frac{4Ts}{3} \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{jj,kk,ii,t_z}(e_2) \\
&+ \frac{Ts}{3} \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{jj,kk,ii,t_z}(e_2) \\
&+ Ts \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)} d^s_{ii,jj,kk,t_r}(e_2) - \frac{4Ts}{3} \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{ii,jj,kk,t_r}(e_2) \\
&+ \frac{Ts}{3} \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{ii,jj,kk,t_r}(e_2) \\
&+ Ts \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)} d^s_{ii,jj,kk,t_z}(e_2) - \frac{4Ts}{3} \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{ii,jj,kk,t_z}(e_2) \\
&+ \frac{Ts}{3} \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{ii,jj,kk,t_z}(e_2).
\end{aligned}
\tag{31.61}
$$

Equivalently,

$$\partial_{\rho_q^{s_2}} \mathcal{D}^{2,c}_{e_2,ii} = \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} Ts \left\{ -\frac{2\Delta_t}{3} \left[u^{s_2}_{l_2^2(e_2,kk)} d^s_{jj,kk,ii,t_r}(e_2) + w^{s_2}_{l_2^2(e_2,kk)} d^s_{jj,kk,ii,t_z}(e_2) \right] \right.$$
$$+ d^s_{jj,kk,ii,t_r}(e_2) \left[r^c_{l_2(e_2,kk)} - \frac{4}{3} r^c_{l_2(e_2,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l_2(e_2,kk)}(t_{n-2}) \right]$$
$$+ d^s_{jj,kk,ii,t_z}(e_2) \left[z^c_{l_2(e_2,kk)} - \frac{4}{3} z^c_{l_2(e_2,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l_2(e_2,kk)}(t_{n-2}) \right]$$
$$+ d^s_{ii,jj,kk,t_r}(e_2) \left[r^c_{l_2(e_2,kk)} - \frac{4}{3} r^c_{l_2(e_2,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l_2(e_2,kk)}(t_{n-2}) \right]$$
$$\left. + d^s_{ii,jj,kk,t_z}(e_2) \left[z^c_{l_2(e_2,kk)} - \frac{4}{3} z^c_{l_2(e_2,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l_2(e_2,kk)}(t_{n-2}) \right] \right\}, \tag{31.62}$$

i.e.

$$\partial_{\rho_q^{s_2}} \mathcal{D}^{2,c}_{e_2,ii} = \sum_{\substack{kk=1 \\ q=l_2^2(e_2,jj)}}^{n_v^{2,e_2}} Ts \left\{ -\frac{2\Delta_t}{3} \left[u^{s_2}_{l_2^2(e_2,kk)} d^s_{jj,kk,ii,t_r}(e_2) + w^{s_2}_{l_2^2(e_2,kk)} d^s_{jj,kk,ii,t_z}(e_2) \right] \right.$$
$$+ \left[d^s_{jj,kk,ii,t_r}(e_2) + d_{ii,jj,kk,t_r}(e_2) \right] \left[r^c_{l_2(e_2,kk)} - \frac{4}{3} r^c_{l_2(e_2,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l_2(e_2,kk)}(t_{n-2}) \right]$$
$$\left. + \left[d^s_{jj,kk,ii,t_z}(e_2) + d_{ii,jj,kk,t_z}(e_2) \right] \left[z^c_{l_2(e_2,kk)} - \frac{4}{3} z^c_{l_2(e_2,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l_2(e_2,kk)}(t_{n-2}) \right] \right\}. \tag{31.63}$$

31.1.2. **Derivatives of $\mathcal{D}_i^2$ with respect to $u_q^{s_2}$**

Using equations (31.50) we have

$$\begin{aligned}
\partial_{u_q^{s_2}} \mathcal{D}_i^2 = &-\frac{2\Delta_t Ts}{3}\delta_{i,c}\rho_c^{s_2} m_r^2(c)\partial_{u_q^{s_2}} u_c^{s_2} - \frac{2\Delta_t Ts}{3}\delta_{i,c}\partial_{u_q^{s_2}}\rho_c^{s_2} w_c^{s_2} m_z^2(c) \\
&+ Ts\,\delta_{i,c}\partial_{u_q^{s_2}}\rho_c^{s_2} m_r^2(c) r_c^c - \frac{4Ts}{3}\delta_{i,c}\partial_{u_q^{s_2}}\rho_c^{s_2} m_r^2(c) r_c^c(t_{n-1}) \\
&+ \frac{Ts}{3}\delta_{i,c}\partial_{u_q^{s_2}}\rho_c^{s_2} m_r^2(c) r_c^c(t_{n-2}) \\
&+ Ts\,\delta_{i,c}\partial_{u_q^{s_2}}\rho_c^{s_2} m_z^2(c) z^c - \frac{4Ts}{3}\delta_{i,c}\partial_{u_q^{s_2}}\rho_c^{s_2} m_z^2(c) z^c(t_{n-1}) \\
&+ \frac{Ts}{3}\delta_{i,c}\partial_{u_q^{s_2}}\rho_c^{s_2} m_z^2(c) z^c(t_{n-2}) \\
&+ \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{u_q^{s_2}} \mathcal{D}_{e_2,ii}^{2,a} + \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{u_q^{s_2}} \mathcal{D}_{e_2,ii}^{2,b} + \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{u_q^{s_2}} \mathcal{D}_{e_2,ii}^{2,c},
\end{aligned} \tag{31.64}$$

i.e.

$$\begin{aligned}
\partial_{u_q^{s_2}} \mathcal{D}_i^2 = &-\frac{2\Delta_t Ts}{3}\delta_{i,c}\delta_{c,q}\rho_c^{s_2} m_r^2(c) \\
&+ \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{u_q^{s_2}} \mathcal{D}_{e_2,ii}^{2,a} + \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{u_q^{s_2}} \mathcal{D}_{e_2,ii}^{2,b} + \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{u_q^{s_2}} \mathcal{D}_{e_2,ii}^{2,c},
\end{aligned} \tag{31.65}$$

From equation (31.47) we have

$$\partial_{u_q^{s_2}} \mathcal{D}_{e_2,ii}^{2,a} = -\frac{2\Delta_t Ds}{3}\partial_{u_q^{s_2}} d_{ii}(e_2), \tag{31.66}$$

i.e.

$$\partial_{u_q^{s_2}} \mathcal{D}_{e_2,ii}^{2,a} = 0. \tag{31.67}$$

From equation (31.48) we have

$$\begin{aligned}
\partial_{u_q^{s_2}} \mathcal{D}_{e_2,ii}^{2,b} = &\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\partial_{u_q^{s_2}}\rho_j^{s_2} d_{ii,jj}(e_2) \\
&+ Ts\sum_{jj=1}^{n_v^{2,e_2}}\partial_{u_q^{s_2}}\rho_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj}(e_2) - \frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{u_q^{s_2}}\rho_{l_2^2(e_2,jj)}^{s_2}(t_{n-1}) d_{ii,jj}(e_2) \\
&+ \frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{u_q^{s_2}}\rho_{l_2^2(e_2,jj)}^{s_2}(t_{n-2}) d_{ii,jj}(e_2),
\end{aligned} \tag{31.68}$$

i.e.

$$\partial_{u_q^{s_2}} \mathcal{D}_{e_2,ii}^{2,b} = 0. \tag{31.69}$$

From equation (31.49) we have

$$
\begin{aligned}
\partial_{u_q^{s_2}} \mathcal{D}^{2,c}_{e_2,ii} = &- \frac{2\Delta_t Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} d^s_{jj,kk,ii,t_r}(e_2) \partial_{u_q^{s_2}} u^{s_2}_{l_2^2(e_2,kk)} \\
&- \frac{2\Delta_t Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,kk)} d^s_{jj,kk,ii,t_z}(e_2) \\
&+ Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)} d^s_{jj,kk,ii,t_r}(e_2) \\
&- \frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{jj,kk,ii,t_r}(e_2) \\
&+ \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{jj,kk,ii,t_r}(e_2) \\
&+ Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)} d^s_{jj,kk,ii,t_z}(e_2) \\
&- \frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{jj,kk,ii,t_z}(e_2) \\
&+ \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{jj,kk,ii,t_z}(e_2) \\
&+ Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)} d^s_{ii,jj,kk,t_r}(e_2) \\
&- \frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{ii,jj,kk,t_r}(e_2) \\
&+ \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{ii,jj,kk,t_r}(e_2) \\
&+ Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)} d^s_{ii,jj,kk,t_z}(e_2) \\
&- \frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{ii,jj,kk,t_z}(e_2) \\
&+ \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{u_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{ii,jj,kk,t_z}(e_2).
\end{aligned}
\tag{31.70}
$$

i.e.

$$\partial_{u_q^{s_2}} \mathcal{D}^{2,c}_{e_2,ii} = \sum_{\substack{jj=1 \\ q=l_2^2(e_2.kk)}}^{n_v^{2,e_2}} -\frac{2\Delta_t Ts}{3} \rho^{s_2}_{l_2^2(e_2,jj)} d^s_{jj,kk,ii,t_r}(e_2). \tag{31.71}$$

31.1.3. **Derivatives of $\mathcal{D}_i^2$ with respect to $w_q^{s_2}$**

Using equations (31.50) we have

$$\begin{aligned}
\partial_{w_q^{s_2}} \mathcal{D}_i^2 = & -\frac{2\Delta_t Ts}{3}\delta_{i,c}\partial_{w_q^{s_2}}\rho_c^{s_2} m_r^2(c) u_c^{s_2} - \frac{2\Delta_t Ts}{3}\delta_{i,c}\rho_c^{s_2} m_z^2(c)\partial_{w_q^{s_2}} w_c^{s_2} \\
& + Ts\,\delta_{i,c}\partial_{w_q^{s_2}}\rho_c^{s_2} m_r^2(c) r_c^c - \frac{4Ts}{3}\delta_{i,c}\partial_{w_q^{s_2}}\rho_c^{s_2} m_r^2(c) r_c^c(t_{n-1}) \\
& + \frac{Ts}{3}\delta_{i,c}\partial_{w_q^{s_2}}\rho_c^{s_2} m_r^2(c) r_c^c(t_{n-2}) + Ts\,\delta_{i,c}\partial_{w_q^{s_2}}\rho_c^{s_2} m_z^2(c) z^c \\
& - \frac{4Ts}{3}\delta_{i,c}\partial_{w_q^{s_2}}\rho_c^{s_2} m_z^2(c) z^c(t_{n-1}) + \frac{Ts}{3}\delta_{i,c}\partial_{w_q^{s_2}}\rho_c^{s_2} m_z^2(c) z^c(t_{n-2}) \\
& + \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{w_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,a} + \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{w_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,b} + \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{w_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,c},
\end{aligned} \tag{31.72}$$

i.e.

$$\begin{aligned}
\partial_{w_q^{s_2}} \mathcal{D}_i^2 = & -\frac{2\Delta_t Ts}{3}\delta_{i,c}\delta_{c,q}\rho_c^{s_2} m_z^2(c) \\
& + \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{w_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,a} + \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{w_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,b} + \sum_{\substack{e_2=1 \\ i=l_2^2(e_2,ii)}}^{n_{\text{el}}^2} \partial_{w_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,c},
\end{aligned} \tag{31.73}$$

From equation (31.47) we have

$$\partial_{w_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,a} = -\frac{2\Delta_t Ds}{3}\partial_{w_q^{s_2}} d_{ii}(e_2), \tag{31.74}$$

i.e.

$$\partial_{w_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,a} = 0. \tag{31.75}$$

From equation (31.48) we have

$$\begin{aligned}
\partial_{w_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,b} = & \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\partial_{w_q^{s_2}}\rho_j^{s_2} d_{ii,jj}(e_2) \\
& + Ts\sum_{jj=1}^{n_v^{2,e_2}}\partial_{w_q^{s_2}}\rho_{l_2^2(e_2,jj)}^{s_2} d_{ii,jj}(e_2) - \frac{4Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{w_q^{s_2}}\rho_{l_2^2(e_2,jj)}^{s_2}(t_{n-1}) d_{ii,jj}(e_2) \\
& + \frac{Ts}{3}\sum_{jj=1}^{n_v^{2,e_2}}\partial_{w_q^{s_2}}\rho_{l_2^2(e_2,jj)}^{s_2}(t_{n-2}) d_{ii,jj}(e_2),
\end{aligned} \tag{31.76}$$

i.e.

$$\partial_{w_q^{s_2}}\mathcal{D}_{e_2,ii}^{2,b} = 0. \tag{31.77}$$

From equation (31.49) we have

$$
\begin{aligned}
\partial_{w_q^{s_2}} \mathcal{D}^{2,c}_{e_2,ii} = & -\frac{2\Delta_t Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} d^s_{jj,kk,ii,t_r}(e_2) \partial_{w_q^{s_2}} u^{s_2}_{l_2^2(e_2,kk)} \\
& -\frac{2\Delta_t Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} d^s_{jj,kk,ii,t_z}(e_2) \partial_{w_q^{s_2}} w^{s_2}_{l_2^2(e_2,kk)} \\
& + Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)} d^s_{jj,kk,ii,t_r}(e_2) \\
& -\frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{jj,kk,ii,t_r}(e_2) \\
& + \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{jj,kk,ii,t_r}(e_2) \\
& + Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)} d^s_{jj,kk,ii,t_z}(e_2) \\
& -\frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{jj,kk,ii,t_z}(e_2) \\
& + \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{jj,kk,ii,t_z}(e_2) \\
& + Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)} d^s_{ii,jj,kk,t_r}(e_2) \\
& -\frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{ii,jj,kk,t_r}(e_2) \\
& + \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{ii,jj,kk,t_r}(e_2) \\
& + Ts \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)} d^s_{ii,jj,kk,t_z}(e_2) \\
& -\frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-1}) d^s_{ii,jj,kk,t_z}(e_2) \\
& + \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \partial_{w_q^{s_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-2}) d^s_{ii,jj,kk,t_z}(e_2).
\end{aligned}
\tag{31.78}
$$

i.e.

$$\partial_{w_q^{s_2}} \mathcal{D}^{2,c}_{e_2,ii} = \sum_{\substack{jj=1 \\ q=l_2^2(e_2,kk)}}^{n_v^{2,e_2}} -\frac{2\Delta_t Ts}{3} \rho^{s_2}_{l_2^2(e_2,jj)} d^s_{jj,kk,ii,t_z}(e_2). \tag{31.79}$$

31.1.4. **Derivatives of $\mathcal{D}_i^2$ with respect to h_q**

Using equations (31.50) we have

$$
\begin{aligned}
\partial_{h_q}\mathcal{D}_i^2 = &-\frac{2\Delta_t Ts}{3}\delta_{i,c}\rho_c^{s_2}u_c^{s_2}\partial_{h_q}m_r^2(c) - \frac{2\Delta_t Ts}{3}\delta_{i,c}\rho_c^{s_2}w_c^{s_2}\partial_{h_q}m_z^2(c)\\
&+ Ts\,\delta_{i,c}\rho_c^{s_2}m_r^2(c)\partial_{h_q}r_c^c - \frac{4Ts}{3}\delta_{i,c}\rho_c^{s_2}r_c^c(t_{n-1})\partial_{h_q}m_r^2(c)\\
&+ \frac{Ts}{3}\delta_{i,c}\rho_c^{s_2}r_c^c(t_{n-2})\partial_{h_q}m_r^2(c)\\
&+ Ts\,\delta_{i,c}\rho_c^{s_2}m_z^2(c)\partial_{h_q}z_c^c - \frac{4Ts}{3}\delta_{i,c}\rho_c^{s_2}z^c(t_{n-1})\partial_{h_q}m_z^2(c)\\
&+ \frac{Ts}{3}\delta_{i,c}\rho_c^{s_2}z^c(t_{n-2})\partial_{h_q}m_z^2(c)\\
&+ \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\mathrm{el}}^2} \partial_{h_{S_2(e_2,qq)}}\mathcal{D}_{e_2,ii}^{2,a} + \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\mathrm{el}}^2} \partial_{h_{S_2(e_2,qq)}}\mathcal{D}_{e_2,ii}^{2,b}\\
&+ \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\mathrm{el}}^2} \partial_{h_{S_2(e_2,qq)}}\mathcal{D}_{e_2,ii}^{2,c},
\end{aligned}
\tag{31.80}
$$

i.e.

$$
\begin{aligned}
\partial_{h_q}\mathcal{D}_i^2 = &\,Ts\,\delta_{i,c}\rho_c^{s_2}m_r^2(c)\partial_{h_q}r_c^c + Ts\,\delta_{i,c}\rho_c^{s_2}m_z^2(c)\partial_{h_q}z_c^c\\
&+ \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\mathrm{el}}^2} \partial_{h_{S_2(e_2,qq)}}\mathcal{D}_{e_2,ii}^{2,a} + \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\mathrm{el}}^2} \partial_{h_{S_2(e_2,qq)}}\mathcal{D}_{e_2,ii}^{2,b}\\
&+ \sum_{\substack{e_2=1\\ i=l_2^2(e_2,ii)\\ q=S_2(e_2,qq)}}^{n_{\mathrm{el}}^2} \partial_{h_{S_2(e_2,qq)}}\mathcal{D}_{e_2,ii}^{2,c}.
\end{aligned}
\tag{31.81}
$$

Observation: Notice that for wetting on a flat surface we have

$$
\begin{aligned}
\partial_{h_q}\boldsymbol{m}^2 &= \partial_{h_q}\left(\boldsymbol{m}^2\cos(\theta_c) + \boldsymbol{n}^2(c)\sin(\theta_c)\right)\\
&= 0;
\end{aligned}
\tag{31.82}
$$

however, for the case of liquid spreading on a smooth but non-planar surface, we have that $\boldsymbol{m}^2$ and $\boldsymbol{n}^2(c)$ are functions of the length of the first spine (the wetted length).

Now, from equation (31.47) we have

$$
\partial_{h_{S_2(e_2,qq)}}\mathcal{D}_{e_2,ii}^{2,a} = -\frac{2\Delta_t Ds}{3}\partial_{h_{S_2(e_2,qq)}}d_{ii}(e_2). \tag{31.83}
$$

From equation (31.48) we have

$$\partial_{h_{S_2(e_2,qq)}} \mathcal{D}^{2,b}_{e_2,ii} = \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \rho^{s_2}_j \partial_{h_{S_2(e_2,qq)}} d_{ii,jj}(e_2) \tag{31.84}$$

$$+ Ts \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)} \partial_{h_{S_2(e_2,qq)}} d_{ii,jj}(e_2)$$

$$- \frac{4Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)}(t_{n-1}) \partial_{h_{S_2(e_2,qq)}} d_{ii,jj}(e_2)$$

$$+ \frac{Ts}{3} \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)}(t_{n-2}) \partial_{h_{S_2(e_2,qq)}} d_{ii,jj}(e_2),$$

i.e.

$$\partial_{h_{S_2(e_2,qq)}} \mathcal{D}^{2,b}_{e_2,ii} = \sum_{jj=1}^{n_v^{2,e_2}} \partial_{h_{S_2(e_2,qq)}} d_{ii,jj}(e_2) \left\{ \frac{2\Delta_t}{3} \rho^{s_2}_{l_2^2(e_2,jj)} \right. \tag{31.85}$$

$$\left. + Ts \left[\rho^{s_2}_{l_2^2(e_2,jj)} - \frac{4}{3} \rho^{s_2}_{l_2^2(e_2,jj)}(t_{n-1}) + \frac{1}{3} \rho^{s_2}_{l_2^2(e_2,jj)}(t_{n-2}) \right] \right\}.$$

From equation (31.49) we have

$$
\begin{aligned}
\partial_{h_{S_2(e_2,qq)}} \mathcal{D}^{2,c}_{e_2,ii} = \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)} Ts \Bigg\{ & -\frac{2\Delta_t}{3} \sum_{kk=1}^{n_v^{2,e_2}} u^{s_2}_{l_2^2(e_2,kk)} \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_r}(e_2) \\
& - \frac{2\Delta_t}{3} \sum_{kk=1}^{n_v^{2,e_2}} w^{s_2}_{l_2^2(e_2,kk)} \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_z}(e_2) \\
& + \sum_{kk=1}^{n_v^{2,e_2}} d^s_{jj,kk,ii,t_r}(e_2) \partial_{h_{S_2(e_2,qq)}} r^c_{l_2(e_2,kk)} + \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)} \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_r}(e_2) \\
& - \frac{4}{3} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-1}) \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_r}(e_2) \\
& + \frac{1}{3} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-2}) \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_r}(e_2) \\
& + \sum_{kk=1}^{n_v^{2,e_2}} d^s_{jj,kk,ii,t_z}(e_2) \partial_{h_{S_2(e_2,qq)}} z^c_{l_2(e_2,kk)} + \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)} \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_z}(e_2) \\
& - \frac{4}{3} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-1}) \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_z}(e_2) \\
& + \frac{1}{3} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-2}) \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_z}(e_2) \\
& + \sum_{kk=1}^{n_v^{2,e_2}} d^s_{ii,jj,kk,t_r}(e_2) \partial_{h_{S_2(e_2,qq)}} r^c_{l_2(e_2,kk)} + \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)} \partial_{h_{S_2(e_2,qq)}} d^s_{ii,jj,kk,t_r}(e_2) \\
& - \frac{4}{3} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-1}) \partial_{h_{S_2(e_2,qq)}} d^s_{ii,jj,kk,t_r}(e_2) \\
& + \frac{1}{3} \sum_{kk=1}^{n_v^{2,e_2}} r^c_{l_2(e_2,kk)}(t_{n-2}) \partial_{h_{S_2(e_2,qq)}} d^s_{ii,jj,kk,t_r}(e_2) \\
& + \sum_{kk=1}^{n_v^{2,e_2}} d^s_{ii,jj,kk,t_z}(e_2) \partial_{h_{S_2(e_2,qq)}} z^c_{l_2(e_2,kk)} + \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)} \partial_{h_{S_2(e_2,qq)}} d^s_{ii,jj,kk,t_z}(e_2) \\
& - \frac{4}{3} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-1}) \partial_{h_{S_2(e_2,qq)}} d^s_{ii,jj,kk,t_z}(e_2) \\
& + \frac{1}{3} \sum_{kk=1}^{n_v^{2,e_2}} z^c_{l_2(e_2,kk)}(t_{n-2}) \partial_{h_{S_2(e_2,qq)}} d^s_{ii,jj,kk,t_z}(e_2) \Bigg\}.
\end{aligned}
\tag{31.86}
$$

i.e.

$$
\begin{aligned}
\partial_{h_{S_2(e_2,qq)}} \mathcal{D}^{2,c}_{e_2,ii} = \sum_{jj=1}^{n_v^{2,e_2}} \rho^{s_2}_{l_2^2(e_2,jj)} Ts \sum_{kk=1}^{n_v^{2,e_2}} \Bigg\{ -\frac{2\Delta_t}{3} \Big[& u^{s_2}_{l_2^2(e_2,kk)} \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_r}(e_2) \\
& + w^{s_2}_{l_2^2(e_2,kk)} \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_z}(e_2) \Big] \\
& + d^s_{jj,kk,ii,t_r}(e_2) \partial_{h_{S_2(e_2,qq)}} r^c_{l_2(e_2,kk)} \\
+ \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_r}(e_2) & \left[r^c_{l_2(e_2,kk)} - \frac{4}{3} r^c_{l_2(e_2,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l_2(e_2,kk)}(t_{n-2}) \right] \\
& + d^s_{jj,kk,ii,t_z}(e_2) \partial_{h_{S_2(e_2,qq)}} z^c_{l_2(e_2,kk)} \\
+ \partial_{h_{S_2(e_2,qq)}} d^s_{jj,kk,ii,t_z}(e_2) & \left[z^c_{l_2(e_2,kk)} - \frac{4}{3} z^c_{l_2(e_2,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l_2(e_2,kk)}(t_{n-2}) \right] \\
& + d^s_{ii,jj,kk,t_r}(e_2) \partial_{h_{S_2(e_2,qq)}} r^c_{l_2(e_2,kk)} \\
+ \partial_{h_{S_2(e_2,qq)}} d^s_{ii,jj,kk,t_r}(e_2) & \left[r^c_{l_2(e_2,kk)} - \frac{4}{3} r^c_{l_2(e_2,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l_2(e_2,kk)}(t_{n-2}) \right] \\
& + d^s_{ii,jj,kk,t_z}(e_2) \partial_{h_{S_2(e_2,qq)}} z^c_{l_2(e_2,kk)} \\
+ \partial_{h_{S_2(e_2,qq)}} d^s_{ii,jj,kk,t_z}(e_2) & \left[z^c_{l_2(e_2,kk)} - \frac{4}{3} z^c_{l_2(e_2,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l_2(e_2,kk)}(t_{n-2}) \right] \Bigg\}.
\end{aligned}
\tag{31.87}
$$

32. The $\sigma - \rho$ state equation on boundary 2 (TDC2) in the near field

Residuals for this equation are identical to those in the far-field so we will not repeat here the derivations of section 11.

33. The slip condition on boundary 1 (SC1) in the near-field

We recall equation (23.16) which states

$$\left(\boldsymbol{v}^{s_1} - \bar{\boldsymbol{u}} - A\check{\boldsymbol{u}}\right) \cdot \left(\boldsymbol{I} - \boldsymbol{n}^1\boldsymbol{n}^1\right) = \frac{1 + 4Eg\,Bg}{4Bg} \nabla^s \sigma^1. \tag{33.1}$$

We define the i-th SC1 residual as

$$\bar{S}_i^1 = \int\limits_{\partial\Omega^1} \phi_i^1 \left(\boldsymbol{v}^{s_1} - \bar{\boldsymbol{u}} - A\check{\boldsymbol{u}}\right) \cdot \boldsymbol{t}^1 - \frac{1 + 4Eg\,Bg}{4Bg} \int\limits_{\partial\Omega^1} \phi_i^1 \boldsymbol{t}^1 \cdot \nabla^s \sigma^1, \tag{33.2}$$

i.e.

$$\bar{S}_i^1 = \int\limits_{\partial\Omega^1} \phi_i^1 \boldsymbol{v}^{s_1} \cdot \boldsymbol{t}^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \bar{\boldsymbol{u}} \cdot \boldsymbol{t}^1 - \int\limits_{\partial\Omega^1} \phi_i^1 A\check{\boldsymbol{u}} \cdot \boldsymbol{t}^1 - \frac{1 + 4Eg\,Bg}{4Bg} \int\limits_{\partial\Omega^1} \phi_i^1 \left(\partial_s \sigma^1\right) \boldsymbol{t}^1 \cdot \boldsymbol{t}^1, \tag{33.3}$$

equivalently

$$\begin{aligned} \bar{S}_i^1 = {} & \int\limits_{\partial\Omega^1} \phi_i^1 u^{s_1} t_r^1 + \int\limits_{\partial\Omega^1} \phi_i^1 w^{s_1} t_z^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \bar{u} t_r^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \bar{w} t_z^1 \\ & - A \int\limits_{\partial\Omega^1} \phi_i^1 \check{u} t_r^1 - A \int\limits_{\partial\Omega^1} \phi_i^1 \check{w} t_z^1 - \frac{1 + 4Eg\,Bg}{4Bg} \int\limits_{\partial\Omega^1} \phi_i^1 \partial_s \sigma^1. \end{aligned} \tag{33.4}$$

We consider the last integral on the right hand side above and we integrate by parts to obtain

$$- \int\limits_{\partial\Omega^1} \phi_i^1 \partial_s \sigma^1 = -\phi_i^1 \sigma^1 \big|_{(r_c, z_c)}^{(r_J, z_J)} + \int\limits_{\partial\Omega^1} \sigma^1 \partial_s \phi_i^1. \tag{33.5}$$

This yields

$$\begin{aligned} \bar{S}_i^1 = {} & \int\limits_{\partial\Omega^1} \phi_i^1 u^{s_1} t_r^1 + \int\limits_{\partial\Omega^1} \phi_i^1 w^{s_1} t_z^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \bar{u} t_r^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \bar{w} t_z^1 \\ & - A \int\limits_{\partial\Omega^1} \phi_i^1 \check{u} t_r^1 - A \int\limits_{\partial\Omega^1} \phi_i^1 \check{w} t_z^1 + \frac{1 + 4Eg\,Bg}{4Bg} \int\limits_{\partial\Omega^1} \sigma^1 \partial_s \phi_i^1 \\ & + \frac{1 + 4Eg\,Bg}{4Bg} \phi_i^1(r_c, z_c) \sigma^1(r_c, z_c) - \frac{1 + 4Eg\,Bg}{4Bg} \phi_i^1(r_J, z_J) \sigma^1(r_J, z_J). \end{aligned} \tag{33.6}$$

We now recall the approximations

$$\bar{u} \approx \sum_{j=1}^{n_v} \bar{u}_j \phi_j, \tag{33.7}$$

$$\bar{w} \approx \sum_{j=1}^{n_v} \bar{w}_j \phi_j \tag{33.8}$$

and

$$\sigma^1 \approx \sum_{j=1}^{n_v} \sigma_j^1 \phi_j \tag{33.9}$$

and we introduce

$$u^{s_1} \approx \sum_{j=1}^{n_v} u_j^{s_1} \phi_j \tag{33.10}$$

and

$$w^{s_1} \approx \sum_{j=1}^{n_v} w_j^{s_1} \phi_j. \tag{33.11}$$

Substituting these approximations into the residual equation we have

$$\begin{aligned}
\bar{\mathcal{S}}_i^1 = & \frac{1 + 4Eg\,Bg}{4Bg} \phi_i^1(r_c, z_c)\sigma^1(r_c, z_c) \\
& - \frac{1 + 4Eg\,Bg}{4Bg} \phi_i^1(r_a, z_a)\sigma^1(r_a, z_a) - A \int\limits_{\partial\Omega^1} \phi_i^1 \check{u} t_r^1 - A \int\limits_{\partial\Omega^1} \phi_i^1 \check{w} t_z^1 \\
& \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} u_j^{s_1} \phi_j \right) t_r^1 + \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} w_j^{s_1} \phi_j \right) t_z^1 \\
& - \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} \bar{u}_j \phi_j \right) t_r^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} \bar{w}_j \phi_j \right) t_z^1 \\
& + \frac{1 + 4Eg\,Bg}{4Bg} \int\limits_{\partial\Omega^1} \left(\sum_{j=1}^{n_v} \sigma_j^1 \phi_j \right) \partial_s \phi_i^1.
\end{aligned} \tag{33.12}$$

Moving the integrals into the sum we have

$$\begin{aligned}
\bar{\mathcal{S}}_i^1 = & \frac{1 + 4Eg\,Bg}{4Bg} \phi_i^1(r_c, z_c)\sigma^1(r_c, z_c) \\
& - \frac{1 + 4Eg\,Bg}{4Bg} \phi_i^1(r_a, z_a)\sigma^1(r_a, z_a) - A \int\limits_{\partial\Omega^1} \phi_i^1 \check{u} t_r^1 - A \int\limits_{\partial\Omega^1} \phi_i^1 \check{w} t_z^1 \\
& + \sum_{j=1}^{n_v} u_j^{s_1} \int\limits_{\partial\Omega^1} \phi_i^1 \phi_j^1 t_r^1 + \sum_{j=1}^{n_v} w_j^{s_1} \int\limits_{\partial\Omega^1} \phi_i^1 \phi_j^1 t_z^1 \\
& - \sum_{j=1}^{n_v} \bar{u}_j \int\limits_{\partial\Omega^1} \phi_i^1 \phi_j^1 t_r^1 - \sum_{j=1}^{n_v} \bar{w}_j \int\limits_{\partial\Omega^1} \phi_i^1 \phi_j^1 t_z^1 \\
& + \frac{1 + 4Eg\,Bg}{4Bg} \sum_{j=1}^{n_v} \sigma_j^1 \int\limits_{\partial\Omega^1} \phi_j^1 \partial_s \phi_i^1.
\end{aligned} \tag{33.13}$$

Decomposing the integral into sums over line-elements and passing to local node numbers we have

$$\bar{\mathcal{S}}_i^1 = \frac{1 + 4Eg\,Bg}{4Bg} \phi_i^1(r_c, z_c)\sigma^1(r_c, z_c) - \frac{1 + 4Eg\,Bg}{4Bg} \phi_i^1(r_a, z_a)\sigma^1(r_a, z_a) + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1} \mathcal{S}_{e_1,ii}^1, \tag{33.14}$$

where

$$
\begin{aligned}
\bar{\mathcal{S}}^1_{e_1,ii} = &-A \underbrace{\int\limits_{\partial\Omega^1} \phi^1_i \check{u} t^1_r}_{c_{ii,t_r,\check{u}}} - A \underbrace{\int\limits_{\partial\Omega^1} \phi^1_i \check{w} t^1_z}_{c_{ii,t_z,\check{w}}} \\
&+ \sum_{jj=1}^{n_v^{1,e_1}} u^{s_1}_{l^1_1(e_1,jj)} \underbrace{\int\limits_{\partial\Omega^1} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} t^1_r}_{c_{ii,jj,t_r}(e_1)} + \sum_{jj=1}^{n_v^{1,e_1}} w^{s_1}_{l^1_1(e_1,jj)} \underbrace{\int\limits_{\partial\Omega^1} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} t^1_z}_{c_{ii,jj,t_z}(e_1)} \\
&- \sum_{jj=1}^{n_v^{1,e_1}} \bar{u}_{l_1(e_1,jj)} \underbrace{\int\limits_{\partial\Omega^1} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} t^1_r}_{c_{ii,jj,t_r}(e_1)} - \sum_{jj=1}^{n_v^{1,e_1}} \bar{w}_{l_1(e_1,jj)} \underbrace{\int\limits_{\partial\Omega^1} \phi^1_{l_1(e_1,ii)} \phi^1_{l_1(e_1,jj)} t^1_z}_{c_{ii,jj,t_z}(e_1)} \\
&+ \frac{1+4Eg\,Bg}{4Bg} \sum_{jj=1}^{n_v^{1,e_1}} \sigma^1_{l^1_1(e_1,jj)} \underbrace{\int\limits_{\partial\Omega^1} \phi^1_{l_1(e_1,jj)} \partial_s \phi^1_{l_1(e_1,ii)}}_{c^s_{jj,ii}(e_1)},
\end{aligned}
\tag{33.15}
$$

i.e.

$$
\begin{aligned}
\mathcal{S}^1_{e_1,ii} = &-A c_{ii,t_r,\check{u}} - A c_{ii,t_z,\check{w}} \\
&+ \sum_{jj=1}^{n_v^{1,e_1}} u^{s_1}_{l^1_1(e_1,jj)} c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} w^{s_1}_{l^1_1(e_1,jj)} c_{ii,jj,t_z}(e_1) \\
&- \sum_{jj=1}^{n_v^{1,e_1}} \bar{u}_{l_1(e_1,jj)} c_{ii,jj,t_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \bar{w}_{l_1(e_1,jj)} c_{ii,jj,t_z}(e_1) \\
&- \frac{1+4Eg\,Bg}{4Bg} \sum_{jj=1}^{n_v^{1,e_1}} \sigma^1_{l^1_1(e_1,jj)} c^s_{jj,ii}(e_1).
\end{aligned}
\tag{33.16}
$$

Summarising and re-arraging terms we have

$$
\bar{\mathcal{S}}^1_i = \sum_{\substack{e_1=1 \\ i=l^1_1(e_1,ii)}}^{n^1_{\mathrm{el}}} \mathcal{S}^1_{e_1,ii}, \tag{33.17}
$$

where

$$
\begin{aligned}
\mathcal{S}^1_{e_1,ii} = &-A \left[c_{ii,t_r,\check{u}} + c_{ii,t_z,\check{w}}\right] \\
&+ \sum_{jj=1}^{n_v^{1,e_1}} \left\{ c_{ii,jj,t_r}(e_1) \left[u^{s_1}_{l^1_1(e_1,jj)} - \bar{u}_{l_1(e_1,jj)}\right] \right. \\
&\qquad + c_{ii,jj,t_z}(e_1) \left[w^{s_1}_{l^1_1(e_1,jj)} - \bar{w}_{l_1(e_1,jj)}\right] \\
&\qquad \left. + \frac{1+4Eg\,Bg}{4Bg} \sigma^1_{l^1_1(e_1,jj)} c^s_{jj,ii}(e_1) \right\}.
\end{aligned}
\tag{33.18}
$$

33.1. Jacobian terms

Here we find the derivatives of $\bar{\mathcal{S}}_i^1$ with respect to $\bar{u}_q$, $\bar{w}_q$, $u_q^{s_1}$, $w_q^{s_1}$, σ_q^1, θ_c, A and h_q.

33.1.1. Derivatives of $\bar{\mathcal{S}}_i^1$ with respect to $\bar{u}_q$

Using equation (33.14) we have

$$\begin{aligned}\partial_{\bar{u}_q}\bar{\mathcal{S}}_i^1 = {} & \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{u_q}\sigma^1(r_c,z_c) \\ & - \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{u_q}\sigma^1(r_a,z_a) + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1} \partial_{\bar{u}_q}\bar{\mathcal{S}}_{e_1,ii}^1.\end{aligned} \tag{33.19}$$

Form equation (33.16) we have

$$\begin{aligned}\partial_{\bar{u}_q}\bar{\mathcal{S}}_{e_1,ii}^1 = {} & -\partial_{\bar{u}_q}Ac_{ii,t_r,\check{u}} - \partial_{\bar{u}_q}Ac_{ii,t_z,\check{w}} \\ & + \sum_{jj=1}^{n_v^{1,e_1}}\partial_{\bar{u}_q}u^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}}\partial_{\bar{u}_q}w^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,t_z}(e_1) \\ & - \sum_{jj=1}^{n_v^{1,e_1}}c_{ii,jj,t_r}(e_1)\partial_{\bar{u}_q}\bar{u}_{l_1(e_1,jj)} - \sum_{jj=1}^{n_v^{1,e_1}}\partial_{\bar{u}_q}\bar{w}_{l_1(e_1,jj)}c_{ii,jj,t_z}(e_1) \\ & + \frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{\bar{u}_q}\sigma^1_{l_1^1(e_1,jj)}c^s_{jj,ii}(e_1),\end{aligned} \tag{33.20}$$

i.e.

$$\partial_{\bar{u}_q}\bar{\mathcal{S}}_{e_1,ii}^1 = -c_{ii,jj,t_r}(e_1)|_{q=l_1(e_1,jj)}. \tag{33.21}$$

33.1.2. Derivatives of $\bar{\mathcal{S}}_i^1$ with respect to $\bar{w}_q$

Using equation (33.14) we have

$$\begin{aligned}\partial_{\bar{w}_q}\bar{\mathcal{S}}_i^1 = {} & \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{w_q}\sigma^1(r_c,z_c) \\ & - \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{w_q}\sigma^1(r_a,z_a) + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_{\bar{w}_q}\bar{\mathcal{S}}_{e_1,ii}^1.\end{aligned} \tag{33.22}$$

Form equation (33.16) we have

$$\begin{aligned}\partial_{\bar{w}_q}\bar{\mathcal{S}}_{e_1,ii}^1 = {} & -\partial_{\bar{w}_q}Ac_{ii,t_r,\check{u}} - \partial_{\bar{w}_q}Ac_{ii,t_z,\check{w}} \\ & + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{w}_q}u^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{w}_q}w^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,t_z}(e_1) \\ & - \sum_{jj=1}^{n_v^{1,e_1}} c_{ii,jj,t_r}(e_1)\partial_{\bar{w}_q}\bar{u}_{l_1(e_1,jj)} - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{w}_q}\bar{w}_{l_1(e_1,jj)}c_{ii,jj,t_z}(e_1) \\ & + \frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{w}_q}\sigma^1_{l_1^1(e_1,jj)}c^s_{jj,ii}(e_1),\end{aligned} \tag{33.23}$$

i.e.

$$\partial_{\bar{w}_q}\bar{\mathcal{S}}_{e_1,ii}^1 = -c_{ii,jj,t_z}(e_1)|_{q=l_1(e_1,jj)}. \tag{33.24}$$

33.1.3. Derivatives of $\bar{\mathcal{S}}_i^1$ with respect to $u_q^{s_1}$

Using equation (33.14) we have

$$\begin{aligned}\partial_{u_q^{s_1}}\bar{\mathcal{S}}_i^1 = {} & \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{u_q^{s_1}}\sigma^1(r_c,z_c) \\ & - \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{u_q^{s_1}}\sigma^1(r_a,z_a) + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_{u_q^{s_1}}\bar{\mathcal{S}}_{e_1,ii}^1.\end{aligned} \tag{33.25}$$

Form equation (33.16) we have

$$\begin{aligned}\partial_{u_q^{s_1}}\bar{\mathcal{S}}_{e_1,ii}^1 = {} & -\partial_{u_q^{s_1}}Ac_{ii,t_r,\breve{u}} - \partial_{u_q^{s_1}}Ac_{ii,t_z,\breve{w}} \\ & + \sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q^{s_1}}u_{l_1^1(e_1,jj)}^{s_1}c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q^{s_1}}w_{l_1^1(e_1,jj)}^{s_1}c_{ii,jj,t_z}(e_1) \\ & - \sum_{jj=1}^{n_v^{1,e_1}}c_{ii,jj,t_r}(e_1)\partial_{u_q^{s_1}}u_{l_1(e_1,jj)} - \sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q^{s_1}}w_{l_1(e_1,jj)}c_{ii,jj,t_z}(e_1) \\ & + \frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q^{s_1}}\sigma_{l_1^1(e_1,jj)}^1 c_{jj,ii}^s(e_1),\end{aligned} \tag{33.26}$$

i.e.

$$\partial_{u_q^{s_1}}\bar{\mathcal{S}}_{e_1,ii}^1 = c_{ii,jj,t_r}(e_1)|_{q=l_1(e_1,jj)}. \tag{33.27}$$

33.1.4. Derivatives of S_i^1 with respect to $w_q^{s_1}$

Using equation (33.14) we have

$$
\begin{aligned}
\partial_{w_q^{s_1}} \bar{\mathcal{S}}_i^1 = {} & \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{w_q^{s_1}}\sigma^1(r_c,z_c) \\
& - \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{w_q^{s_1}}\sigma^1(r_a,z_a) + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_{w_q^{s_1}} \mathcal{S}_{e_1,ii}^1 .
\end{aligned}
\tag{33.28}
$$

Form equation (33.16) we have

$$
\begin{aligned}
\partial_{w_q^{s_1}} \bar{\mathcal{S}}_{e_1,ii}^1 = {} & -\partial_{w_q^{s_1}} Ac_{ii,t_r,\check{u}} - \partial_{w_q^{s_1}} Ac_{ii,t_z,\check{w}} \\
& + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} u_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} w_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,t_z}(e_1) \\
& - \sum_{jj=1}^{n_v^{1,e_1}} c_{ii,jj,t_r}(e_1) \partial_{w_q^{s_1}} \bar{u}_{l_1(e_1,jj)} - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \bar{w}_{l_1(e_1,jj)} c_{ii,jj,t_z}(e_1) \\
& + \frac{1+4Eg\,Bg}{4Bg} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \sigma_{l_1^1(e_1,jj)}^1 c_{jj,ii}^s(e_1),
\end{aligned}
\tag{33.29}
$$

i.e.

$$
\partial_{w_q^{s_1}} \bar{\mathcal{S}}_{e_1,ii}^1 = c_{ii,jj,t_z}(e_1)|_{q=l_1(e_1,jj)} . \tag{33.30}
$$

33.1.5. Derivatives of S_i^1 with respect to σ_q^1

Using equation (12.14) we have

$$\begin{aligned}\partial_{\sigma_q^1}\bar{\mathcal{S}}_i^1 &= \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{\sigma_q^1}\sigma^1(r_c,z_c)\\ &\quad-\frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{\sigma_q^1}\sigma^1(r_a,z_a)+\sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\sigma_q^1}\bar{\mathcal{S}}_{e_1,ii}^1.\end{aligned}\tag{33.31}$$

Form equation (12.16) we have

$$\begin{aligned}\partial_{\sigma_q^1}\bar{\mathcal{S}}_{e_1,ii}^1 &= -\partial_{\sigma_q^1}Ac_{ii,t_r,\breve{u}}-\partial_{\sigma_q^1}Ac_{ii,t_z,\breve{w}}\\ &\quad+\sum_{jj=1}^{n_v^{1,e_1}}u_{l_1^1(e_1,jj)}^{s_1}\partial_{\sigma_q^1}c_{ii,jj,t_r}(e_1)+\sum_{jj=1}^{n_v^{1,e_1}}w_{l_1^1(e_1,jj)}^{s_1}\partial_{\sigma_q^1}c_{ii,jj,t_z}(e_1)\\ &\quad-\sum_{jj=1}^{n_v^{1,e_1}}\bar{u}_{l_1(e_1,jj)}\partial_{\sigma_q^1}c_{ii,jj,t_r}(e_1)-\sum_{jj=1}^{n_v^{1,e_1}}\bar{w}_{l_1(e_1,jj)}\partial_{\sigma_q^1}c_{ii,jj,t_z}(e_1)\\ &\quad+\frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}}c_{jj,ii}^s(e_1)\partial_{\sigma_q^1}\sigma_{l_1^1(e_1,jj)}^1,\end{aligned}\tag{33.32}$$

i.e.

$$\partial_{\sigma_q^1}\bar{\mathcal{S}}_{e_1,ii}^1=\frac{1+4Eg\,Bg}{4Bg}c_{jj,ii}^s(e_1)|_{q=l_1^1(e_1,jj)},\tag{33.33}$$

33.1.6. **Derivatives of S_i^1 with respect to θ_c**

Using equation (33.14) we have

$$\partial_{\theta_c}\bar{\mathcal{S}}_i^1 = \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{\theta_c}\sigma^1(r_c,z_c) - \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{\theta_c}\sigma^1(r_a,z_a) + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_{\theta_c}\bar{\mathcal{S}}_{e_1,ii}^1. \tag{33.34}$$

Form equation (33.16) we have

$$\begin{aligned}\partial_{\theta_c}\bar{\mathcal{S}}_{e_1,ii}^1 = &-A\partial_{\theta_c}c_{ii,t_r,\breve{u}} - A\partial_{\theta_c}c_{ii,t_z,\breve{w}}\\ &+\sum_{jj=1}^{n_v^{1,e_1}} u_{l_1^1(e_1,jj)}^{s_1}\partial_{\theta_c}c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1^1(e_1,jj)}^{s_1}\partial_{\theta_c}c_{ii,jj,t_z}(e_1)\\ &-\sum_{jj=1}^{n_v^{1,e_1}} \bar{u}_{l_1(e_1,jj)}\partial_{\theta_c}c_{ii,jj,t_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \bar{w}_{l_1(e_1,jj)}\partial_{\theta_c}c_{ii,jj,t_z}(e_1)\\ &+\frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}}\sigma_{l_1^1(e_1,jj)}^1\partial_{\theta_c}c_{jj,ii}^s(e_1),\end{aligned} \tag{33.35}$$

i.e.

$$\partial_{\theta_c}\bar{\mathcal{S}}_{e_1,ii}^1 = -A\left[\partial_{\theta_c}c_{ii,t_r,\breve{u}} + \partial_{\theta_c}c_{ii,t_z,\breve{w}}\right]. \tag{33.36}$$

33.1.7. Derivatives of S_i^1 with respect to A

Using equation (33.14) we have

$$\begin{aligned}\partial_A \bar{\mathcal{S}}_i^1 = {}& \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_A\sigma^1(r_c,z_c) \\ & - \frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_A\sigma^1(r_a,z_a) + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_A\bar{\mathcal{S}}_{e_1,ii}^1.\end{aligned} \tag{33.37}$$

Form equation (33.16) we have

$$\begin{aligned}\partial_A\bar{\mathcal{S}}_{e_1,ii}^1 = {}& -\partial_A A c_{ii,t_r,\breve{u}} - \partial_A A c_{ii,t_z,\breve{w}} \\ & + \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1^1(e_1,jj)}^{s_1}\partial_A c_{ii,jj,t_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1^1(e_1,jj)}^{s_1}\partial_A c_{ii,jj,t_z}(e_1) \\ & - \sum_{jj=1}^{n_v^{1,e_1}} \bar{u}_{l_1(e_1,jj)}\partial_A c_{ii,jj,t_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \bar{w}_{l_1(e_1,jj)}\partial_A c_{ii,jj,t_z}(e_1) \\ & + \frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}} c_{jj,ii}^s(e_1)\partial_A\sigma_{l_1^1(e_1,jj)}^1,\end{aligned} \tag{33.38}$$

i.e.

$$\partial_A\bar{\mathcal{S}}_{e_1,ii}^1 = -c_{ii,t_r,\breve{u}} - c_{ii,t_z,\breve{w}}. \tag{33.39}$$

33.1.8. Derivatives of S_i^1 with respect to h_q

Using equation (12.14) and local spine numbers we have

$$\begin{aligned}\partial_{h_q}\bar{\mathcal{S}}_i^1 = &\frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_c,z_c)\partial_{h_q}\sigma^1(r_c,z_c)\\ &-\frac{1+4Eg\,Bg}{4Bg}\phi_i^1(r_a,z_a)\partial_{h_q}\sigma^1(r_a,z_a)+\sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\mathrm{el}}^1}\partial_{h_{S_1(e_1,qq)}}\bar{\mathcal{S}}_{e_1,ii}^1.\end{aligned}\tag{33.40}$$

Form equation (33.16) we have

$$\begin{aligned}\partial_{h_{S_1(e_1,qq)}}\bar{\mathcal{S}}_{e_1,ii}^1 = &-\partial_{h_{S_1(e_1,qq)}}Ac_{ii,t_r,\check{u}}-\partial_{h_{S_1(e_1,qq)}}Ac_{ii,t_z,\check{w}}\\ &+\sum_{jj=1}^{n_v^{1,e_1}}u^{s_1}_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_r}(e_1)+\sum_{jj=1}^{n_v^{1,e_1}}w^{s_1}_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_z}(e_1)\\ &-\sum_{jj=1}^{n_v^{1,e_1}}\bar{u}_{l_1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_r}(e_1)-\sum_{jj=1}^{n_v^{1,e_1}}\bar{w}_{l_1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_z}(e_1)\\ &+\frac{1+4Eg\,Bg}{4Bg}\sum_{jj=1}^{n_v^{1,e_1}}\sigma^1_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c^s_{jj,ii}(e_1),\end{aligned}\tag{33.41}$$

i.e.

$$\begin{aligned}\partial_{h_{S_1(e_1,qq)}}\bar{\mathcal{S}}_{e_1,ii}^1 = &-A\left[\partial_{h_{S_1(e_1,qq)}}c_{ii,t_r,\check{u}}+\partial_{h_{S_1(e_1,qq)}}c_{ii,t_z,\check{w}}\right]\\ &+\sum_{jj=1}^{n_v^{1,e_1}}\Bigg[\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_r}(e_1)\left\{u^{s_1}_{l_1^1(e_1,jj)}-\bar{u}_{l_1(e_1,jj)}\right\}\\ &+\partial_{h_{S_1(e_1,qq)}}c_{ii,jj,t_z}(e_1)\left\{w^{s_1}_{l_1^1(e_1,jj)}-\bar{w}_{l_1(e_1,jj)}\right\}\\ &+\frac{1+4Eg\,Bg}{4Bg}\sigma^1_{l_1^1(e_1,jj)}\underbrace{\partial_{h_{S_1(e_1,qq)}}c^s_{jj,ii}(e_1)}_{=0}\Bigg].\end{aligned}\tag{33.42}$$

34. Kinematic boundary condition (KBC) in the near field

This is identical to the far-field case discussed in section (13), so we will not repeat the derivations here.

35. Mass transfer condition on boundary 1 (MEC1) in the near-field

We recall MEC1 which in the near field is given by equation (23.18) which states

$$(\bar{\boldsymbol{u}} + A\check{\boldsymbol{u}} - \boldsymbol{v}^{s_1}) \cdot \boldsymbol{n}^1 = Fg\,(\rho^{s_1} - Dg\,)\,. \tag{35.1}$$

i.e.

$$(\bar{u} + A\check{u} - u^{s_1})n_r^1 + (\bar{w} + A\check{w} - w^{s_1})n_z^1 - Fg\,\rho^{s_1} + Fg\,Dg\, = 0, \tag{35.2}$$

and define the i-th MEC1 residual as

$$\begin{aligned}\bar{E}_i^1 = &\int\limits_{\partial\Omega^1} \phi_i^1 \bar{u} n_r^1 + \int\limits_{\partial\Omega^1} \phi_i^1 \bar{w} n_z^1 + A \int\limits_{\partial\Omega^1} \phi_i^1 \check{u} n_r^1 + A \int\limits_{\partial\Omega^1} \phi_i^1 \check{w} n_z^1 \\ &- \int\limits_{\partial\Omega^1} \phi_i^1 u^{s_1} n_r^1 - \int\limits_{\partial\Omega^1} \phi_i^1 w^{s_1} n_z^1 - Fg \int\limits_{\partial\Omega^1} \phi_i^1 \rho^{s_1} + Fg\,Dg \int\limits_{\partial\Omega^1} \phi_i^1,\end{aligned} \tag{35.3}$$

where i is an index that runs through the boundary 1 node numbering.

We substitute approximations

$$\bar{u} \approx \sum_{j=1}^{n_v} \bar{u}_j \phi_j, \tag{35.4}$$

$$\bar{w} \approx \sum_{j=1}^{n_v} \bar{w}_j \phi_j, \tag{35.5}$$

$$u^{s_1} \approx \sum_{j=1}^{n_v} u_j^{s_1} \phi_j^1, \tag{35.6}$$

$$w^{s_1} \approx \sum_{j=1}^{n_v} w_j^{s_1} \phi_j^1 \tag{35.7}$$

and

$$\rho^{s_1} \approx \sum_{j=1}^{n_v} \rho_j^{s_1} \phi_j^1. \tag{35.8}$$

into the residual equation above and obtain

$$\begin{aligned}\bar{\mathcal{E}}_i^1 = &\int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} \bar{u}_j \phi_j^1\right) n_r^1 + \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} \bar{w}_j \phi_j^1\right) n_z^1 \\ &+ A \int\limits_{\partial\Omega^1} \phi_i^1 \check{u} n_r^1 + A \int\limits_{\partial\Omega^1} \phi_i^1 \check{w} n_z^1 - \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} u_j^{s_1} \phi_j^1\right) n_r^1 \\ &- \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} w_j^{s_1} \phi_j^1\right) n_z^2 - Fg \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} \rho_j^{s_1} \phi_j^1\right) + Fg\,Dg \int\limits_{\partial\Omega^1} \phi_i^1.\end{aligned} \tag{35.9}$$

Moving the integrals into the sums and re-arranging terms we have

$$\bar{\mathcal{E}}_i^1 = Fg\,Dg \int\limits_{\partial\Omega^1} \phi_i^1 + A \int\limits_{\partial\Omega^1} \phi_i^1 \check{u} n_r^1 + A \int\limits_{\partial\Omega^1} \phi_i^1 \check{w} n_z^1 + \sum_{j=1}^{n_v} \bar{u}_j \int\limits_{\partial\Omega^1} \phi_i^1 \phi_j^1 n_r^1 + \sum_{j=1}^{n_v} \bar{w}_j \int\limits_{\partial\Omega^1} \phi_i^1 \phi_j^1 n_z^1$$
$$- \sum_{j=1}^{n_v} u_j^{s_1} \int\limits_{\partial\Omega^1} \phi_i^1 \phi_j^1 n_r^1 - \sum_{j=1}^{n_v} w_j^{s_1} \int\limits_{\partial\Omega^1} \phi_i^1 \phi_j^1 n_z^1 - Fg \sum_{j=1}^{n_v} \rho_j^{s_1} \int\limits_{\partial\Omega^1} \phi_i^1 \phi_j^1. \tag{35.10}$$

Decomposing the integrals into sums of integrals over each line-element and using local node-numbers we have

$$\bar{\mathcal{E}}_i^1 = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \bar{\mathcal{E}}_{e_1,ii}^1, \tag{35.11}$$

where

$$\bar{\mathcal{E}}_{e_1,ii}^1 = Fg\,Dg \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1}_{c_{ii}(e_1)} + A \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \check{u} n_r^1}_{c_{ii,n_r,\check{u}}(e_1)} + A \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \check{w} n_z^1}_{c_{ii,n_z,\check{w}}(e_1)}$$
$$+ \sum_{jj=1}^{n_v^{1,e_1}} \bar{u}_{l_1(e_1,jj)} \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \phi_{l_1(e_1,jj)}^1 n_r^1}_{c_{ii,jj,n_r}(e_1)} + \sum_{jj=1}^{n_v^{1,e_1}} \bar{w}_{l_1(e_1,jj)} \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \phi_{l_1(e_1,jj)}^1 n_z^1}_{c_{ii,jj,n_z}(e_1)}$$
$$- \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1^1(e_1,jj)}^{s_1} \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \phi_{l_1(e_1,jj)}^1 n_r^1}_{c_{ii,jj,n_r}(e_1)} - \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1^1(e_1,jj)}^{s_1} \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \phi_{l_1(e_1,jj)}^1 n_z^1}_{c_{ii,jj,n_z}(e_1)}$$
$$-Fg \sum_{jj=1}^{n_v^{1,e_1}} \rho_{l_1^1(e_1,jj)}^{s_1} \underbrace{\int\limits_{\partial\Omega^1} \phi_{l_1(e_1,ii)}^1 \phi_{l_1(e_1,jj)}^1}_{c_{ii,jj}(e_1)}, \tag{35.12}$$

i.e.

$$\bar{\mathcal{E}}_{e_1,ii}^1 = Fg\,Dg\,c_{ii}(e_1) + Ac_{ii,n_r,\check{u}}(e_1) + Ac_{ii,n_z,\check{w}}(e_1)$$
$$+ \sum_{jj=1}^{n_v^{1,e_1}} \bar{u}_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \bar{w}_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} u_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,n_r}(e_1)$$
$$- \sum_{jj=1}^{n_v^{1,e_1}} w_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,n_z}(e_1) - Fg \sum_{jj=1}^{n_v^{1,e_1}} \rho_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj}(e_1). \tag{35.13}$$

Summarising and re-arranging terms we have

$$\bar{\mathcal{E}}^1_i = \sum_{\substack{e_1=1\\ i=l^1_1(e_1,ii)}}^{n^1_{\mathrm{el}}} \bar{\mathcal{E}}^1_{e_1,ii}, \tag{35.14}$$

where

$$\begin{aligned}\bar{\mathcal{E}}^1_{e_1,ii} = Fg\, Dg\, c_{ii}(e_1) + A\left[c_{ii,n_r,\breve{u}}(e_1) + c_{ii,n_z,\breve{w}}(e_1)\right]\\ + \sum_{jj=1}^{n_v^{1,e_1}} \Big\{ c_{ii,jj,n_r}(e_1)\left[\bar{u}_{l_1(e_1,jj)} - u^{s_1}_{l^1_1(e_1,jj)}\right]\\ + c_{ii,jj,n_z}(e_1)\left[\bar{w}_{l_1(e_1,jj)} - w^{s_1}_{l^1_1(e_1,jj)}\right]\\ - Fg\,\rho^{s_1}_{l^1_1(e_1,jj)} c_{ii,jj}(e_1)\Big\}.\end{aligned} \tag{35.15}$$

35.1. Jacobian terms

We now calculate the derivatives of $\bar{\mathcal{E}}_i^1$ with respect to $\bar{u}_q$, $\bar{w}_q$, $u_q^{s_1}$, $w_q^{s_1}$, $\rho_q^{s_1}$, θ_c, A and h_q.

35.1.1. Derivatives of $\bar{\mathcal{E}}_i^1$ with respect to $\bar{u}_q$

Using equation (35.11) we have

$$\partial_{\bar{u}_q}\bar{\mathcal{E}}_i^1 = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1} \partial_{\bar{u}_q}\bar{\mathcal{E}}_{e_1,ii}^1, \tag{35.16}$$

and from equation (35.13) we have

$$\begin{aligned}\partial_{\bar{u}_q}\bar{\mathcal{E}}_{e_1,ii}^1 = {} & Fg\,Dg\,\partial_{\bar{u}_q}c_{ii}(e_1) + A\partial_{\bar{u}_q}c_{ii,n_r,\breve{u}}(e_1) + A\partial_{\bar{u}_q}c_{ii,n_z,\breve{w}}(e_1) \\ & + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{u}_q}\bar{u}_{l_1(e_1,jj)}c_{ii,jj,n_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{u}_q}\bar{w}_{l_1(e_1,jj)}c_{ii,jj,n_z}(e_1) \\ & - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{u}_q}u^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,n_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{u}_q}w^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj,n_z}(e_1) \\ & - Fg\sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{u}_q}\rho^{s_1}_{l_1^1(e_1,jj)}c_{ii,jj}(e_1),\end{aligned} \tag{35.17}$$

i.e.

$$\partial_{\bar{u}_q}\mathcal{E}_{e_1,ii}^1 = c_{ii,jj,n_r}(e_1)|_{q=l_1(e_1,jj)}. \tag{35.18}$$

35.1.2. **Derivatives of $\bar{\mathcal{E}}^1_i$ with respect to $\bar{w}_q$**

Using equation (35.11) we have

$$\partial_{\bar{w}_q}\bar{\mathcal{E}}^1_i = \sum_{\substack{e_1=1\\ i=l^1_1(e_1,ii)}}^{n^1_{\mathrm{el}}} \partial_{\bar{w}_q}\bar{\mathcal{E}}^1_{e_1,ii}, \tag{35.19}$$

and from equation (35.13) we have

$$\begin{aligned}\partial_{\bar{w}_q}\bar{\mathcal{E}}^1_{e_1,ii} = {}& Fg\,Dg\,\partial_{\bar{w}_q}c_{ii}(e_1) + A\partial_{\bar{w}_q}c_{ii,n_r,\breve{u}}(e_1) + A\partial_{\bar{w}_q}c_{ii,n_z,\breve{w}}(e_1)\\ & + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{w}_q}\bar{u}_{l_1(e_1,jj)}c_{ii,jj,n_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{w}_q}\bar{w}_{l_1(e_1,jj)}c_{ii,jj,n_z}(e_1)\\ & - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{w}_q}u^{s_1}_{l^1_1(e_1,jj)}c_{ii,jj,n_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{w}_q}w^{s_1}_{l^1_1(e_1,jj)}c_{ii,jj,n_z}(e_1)\\ & - Fg\sum_{jj=1}^{n_v^{1,e_1}} \partial_{\bar{w}_q}\rho^{s_1}_{l^1_1(e_1,jj)}c_{ii,jj}(e_1),\end{aligned} \tag{35.20}$$

i.e.

$$\partial_{\bar{w}_q}\bar{\mathcal{E}}^1_{e_1,ii} = c_{ii,jj,n_z}(e_1)|_{q=l_1(e_1,jj)}. \tag{35.21}$$

35.1.3. Derivatives of $\bar{\mathcal{E}}_i^1$ with respect to $u_q^{s_1}$

Using equation (35.11) we have

$$\partial_{u_q^{s_1}} \bar{\mathcal{E}}_i^1 = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_{u_q^{s_1}} \bar{\mathcal{E}}_{e_1,ii}^1, \tag{35.22}$$

and from equation (35.13) we have

$$\begin{aligned}
\partial_{u_q^{s_1}} \bar{\mathcal{E}}_{e_1,ii}^1 = {} & Fg\, Dg\, \partial_{u_q^{s_1}} c_{ii}(e_1) + A\partial_{u_q^{s_1}} c_{ii,n_r,\check{u}}(e_1) + A\partial_{\bar{w}_q} c_{ii,n_z,\check{w}}(e_1) \\
& + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} u_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} w_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1) \\
& - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \bar{u}_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,n_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \bar{w}_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,n_z}(e_1) \\
& - Fg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj}(e_1),
\end{aligned} \tag{35.23}$$

i.e.

$$\partial_{u_q^{s_1}} \bar{\mathcal{E}}_{e_1,ii}^1 = -c_{ii,jj,n_r}(e_1)|_{q=l_1(e_1,jj)}. \tag{35.24}$$

35.1.4. Derivatives of $bar\mathcal{E}_i^1$ with respect to $w_q^{s_1}$

Using equation (35.11) we have

$$\partial_{w_q^{s_1}} \bar{\mathcal{E}}_i^1 = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\rm el}^1} \partial_{w_q^{s_1}} \bar{\mathcal{E}}_{e_1,ii}^1, \tag{35.25}$$

and from equation (35.13) we have

$$\begin{aligned}\partial_{w_q^{s_1}} \bar{\mathcal{E}}_{e_1,ii}^1 = {} & Fg\, Dg\, \partial_{w_q^{s_1}} c_{ii}(e_1) + A\partial_{w_q^{s_1}} c_{ii,n_r,\breve{u}}(e_1) + A\partial_{w_q^{s_1}} c_{ii,n_z,\breve{w}}(e_1) \\ & + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \bar{u}_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \bar{w}_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1) \\ & - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} u^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj,n_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} w^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj,n_z}(e_1) \\ & - Fg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj}(e_1),\end{aligned} \tag{35.26}$$

i.e.

$$\partial_{w_q^{s_1}} \bar{\mathcal{E}}_{e_1,ii}^1 = -c_{ii,jj,n_z}(e_1)|_{q=l_1(e_1,jj)}. \tag{35.27}$$

35.1.5. **Derivatives of $\mathcal{E}_i^1$ with respect to $\rho_q^{s_1}$**

Using equation (35.11) we have

$$\partial_{\rho_q^{s_1}} \bar{\mathcal{E}}_i^1 = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_{\rho_q^{s_1}} \bar{\mathcal{E}}_{e_1,ii}^1, \tag{35.28}$$

and from equation (35.13) we have

$$\begin{aligned}
\partial_{\rho_q^{s_1}} \bar{\mathcal{E}}_{e_1,ii}^1 = {} & Fg\, Dg\, \partial_{\rho_q^{s_1}} c_{ii}(e_1) + A\partial_{w_q^{s_1}} c_{ii,n_r,\breve{u}}(e_1) + A\partial_{w_q^{s_1}} c_{ii,n_z,\breve{w}}(e_1) \\
& + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \bar{u}_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \bar{w}_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1) \\
& - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} u_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,n_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} w_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,n_z}(e_1) \\
& - Fg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj}(e_1),
\end{aligned} \tag{35.29}$$

i.e.

$$\partial_{\rho_q^{s_1}} \bar{\mathcal{E}}_{e_1,ii}^1 = c_{ii,jj}(e_1)|_{q=l_1(e_1,jj)}. \tag{35.30}$$

35.1.6. **Derivatives of $\mathcal{E}_i^1$ with respect to θ_c**

Using equation (35.11) we have

$$\partial_{\theta_c}\bar{\mathcal{E}}_i^1 = \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \partial_{\theta_c}\bar{\mathcal{E}}_{e_1,ii}^1, \tag{35.31}$$

and from equation (35.13) we have

$$\begin{aligned}
\partial_{\theta_c}\bar{\mathcal{E}}_{e_1,ii}^1 = Fg\,Dg\,\partial_{\theta_c}c_{ii}(e_1) + A\partial_{\theta_c}c_{ii,n_r,\check{u}}(e_1) + A\partial_{\theta_c}c_{ii,n_z,\check{w}}(e_1)\\
+ \sum_{jj=1}^{n_v^{1,e_1}} \bar{u}_{l_1(e_1,jj)}\partial_{\theta_c}c_{ii,jj,n_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \bar{w}_{l_1(e_1,jj)}\partial_{\theta_c}c_{ii,jj,n_z}(e_1)\\
- \sum_{jj=1}^{n_v^{1,e_1}} u^{s_1}_{l_1^1(e_1,jj)}\partial_{\theta_c}c_{ii,jj,n_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} w^{s_1}_{l_1^1(e_1,jj)}\partial_{\theta_c}c_{ii,jj,n_z}(e_1)\\
- Fg\sum_{jj=1}^{n_v^{1,e_1}} \rho^{s_1}_{l_1^1(e_1,jj)}\partial_{\theta_c}c_{ii,jj}(e_1),
\end{aligned} \tag{35.32}$$

i.e.

$$\partial_{\theta_c}\bar{\mathcal{E}}_{e_1,ii}^1 = A\left[\partial_{\theta_c}c_{ii,n_r,\check{u}}(e_1) + \partial_{\theta_c}c_{ii,n_z,\check{w}}(e_1)\right]. \tag{35.33}$$

35.1.7. Derivatives of $\mathcal{E}_i^1$ with respect to A

Using equation (35.11) we have

$$\partial_A \bar{\mathcal{E}}_i^1 = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1} \partial_A \bar{\mathcal{E}}_{e_1,ii}^1, \tag{35.34}$$

and from equation (35.13) we have

$$\begin{aligned} \partial_A \bar{\mathcal{E}}_{e_1,ii}^1 = {} & Fg\, Dg\, \partial_A c_{ii}(e_1) + \partial_A A c_{ii,n_r,\breve{u}}(e_1) + \partial_A A c_{ii,n_z,\breve{w}}(e_1) \\ & + \sum_{jj=1}^{n_v^{1,e_1}} \partial_A \bar{u}_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1) + \sum_{jj=1}^{n_v^{1,e_1}} \partial_A \bar{w}_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1) \\ & - \sum_{jj=1}^{n_v^{1,e_1}} \partial_A u_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,n_r}(e_1) - \sum_{jj=1}^{n_v^{1,e_1}} \partial_A w_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj,n_z}(e_1) \\ & - Fg \sum_{jj=1}^{n_v^{1,e_1}} \partial_A \rho_{l_1^1(e_1,jj)}^{s_1} c_{ii,jj}(e_1), \end{aligned} \tag{35.35}$$

i.e.

$$\partial_A \bar{\mathcal{E}}_{e_1,ii}^1 = c_{ii,n_r,\breve{u}}(e_1) + c_{ii,n_z,\breve{w}}(e_1). \tag{35.36}$$

35.1.8. Derivatives of $\bar{\mathcal{E}}^1_i$ with respect to h_q

Using equation (35.11) we have

$$\partial_{h_q} \bar{\mathcal{E}}^1_i = \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n^1_{\rm el}} \partial_{h_{S_1(e_1,qq)}} \bar{\mathcal{E}}^1_{e_1,ii}, \tag{35.37}$$

and from equation (35.13) we have

$$\begin{aligned}\partial_{h_{S_1(e_1,qq)}} \bar{\mathcal{E}}^1_{e_1,ii} = Fg\, Dg\, \partial_{h_{S_1(e_1,qq)}} c_{ii}(e_1) + \partial_{h_{S_1(e_1,qq)}} A c_{ii,n_r,\check{u}}(e_1) + \partial_{h_{S_1(e_1,qq)}} A c_{ii,n_z,\check{w}}(e_1)\\ + \sum_{jj=1}^{n_v^{1,e_1}} \bar{u}_{l_1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_r}(e_1)\\ + \sum_{jj=1}^{n_v^{1,e_1}} \bar{w}_{l_1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_z}(e_1)\\ - \sum_{jj=1}^{n_v^{1,e_1}} u^{s_1}_{l_1^1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_r}(e_1)\\ - \sum_{jj=1}^{n_v^{1,e_1}} w^{s_1}_{l_1^1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_z}(e_1)\\ - Fg \sum_{jj=1}^{n_v^{1,e_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c_{ii,jj}(e_1),\end{aligned} \tag{35.38}$$

i.e.

$$\begin{aligned}\partial_{h_{S_1(e_1,qq)}} \bar{\mathcal{E}}^1_{e_1,ii} = Fg\, Dg\, \partial_{h_{S_1(e_1,qq)}} c_{ii}(e_1) + A\left[\partial_{h_{S_1(e_1,qq)}} c_{ii,n_r,\check{u}}(e_1) + \partial_{h_{S_1(e_1,qq)}} c_{ii,n_z,\check{w}}(e_1)\right]\\ + \sum_{jj=1}^{n_v^{1,e_1}} \Big[\partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_r}(e_1) \left\{\bar{u}_{l_1(e_1,jj)} - u^{s_1}_{l_1^1(e_1,jj)}\right\}\\ + \partial_{h_{S_1(e_1,qq)}} c_{ii,jj,n_z}(e_1) \left\{\bar{w}_{l_1(e_1,jj)} - w^{s_1}_{l_1^1(e_1,jj)}\right\}\\ - Fg\, \rho^{s_1}_{l_1^1(e_1,jj)} \partial_{h_{S_1(e_1,qq)}} c_{ii,jj}(e_1)\Big].\end{aligned} \tag{35.39}$$

36. The density transport equation on boundary 1 (DTC1) in the near field

We recall equation (23.19) given by

$$Tg\,\{\partial_t\rho^{s_1}+\rho^{s_1}\nabla^s\cdot\boldsymbol{c}+\nabla^s\cdot[\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})]\}=Dg\,-\rho^{s_1}. \tag{36.1}$$

i.e.

$$Tg\,\partial_t\rho^{s_1}+Tg\,\rho^{s_1}t_r^1\partial_s\partial_t r^c+Tg\,\rho^{s_1}t_z^2\partial_s\partial_t z^c+Tg\,\nabla^s\cdot[\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})]=Dg\,-\rho^{s_1}. \tag{36.2}$$

We thus define the i-th DTC1 residual as

$$\begin{aligned}D_i^1=Tg\int\limits_{\partial\Omega^1}\phi_i^1\partial_t\rho^{s_1}+Tg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_r^1\partial_t\partial_s r^c+Tg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_z^1\partial_t\partial_s z^c\\+Tg\int\limits_{\partial\Omega^1}\phi_i^1\nabla^s\cdot[\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})]-Dg\int\limits_{\partial\Omega^1}\phi_i^1+\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}.\end{aligned} \tag{36.3}$$

We consider now the term

$$Tg\int\limits_{\partial\Omega^1}\phi_i^1\nabla^s\cdot[\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})], \tag{36.4}$$

and we recall the vector calculus identity

$$\nabla^s\cdot(\phi\boldsymbol{A})=\boldsymbol{A}\cdot\nabla^s\phi+\phi\nabla^s\cdot\boldsymbol{A}, \tag{36.5}$$

i.e.

$$\phi\nabla^s\cdot\boldsymbol{A}=\nabla^s\cdot(\phi\boldsymbol{A})-\boldsymbol{A}\cdot\nabla^s\phi. \tag{36.6}$$

Using this identity with $\phi=\phi_i^1$ and $\boldsymbol{A}=\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})$, we have

$$\phi_i^1\nabla^s\cdot[\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})]=\nabla^s\cdot\left[\phi_i^1\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})\right]-\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})\cdot\nabla^s\phi_i^1. \tag{36.7}$$

Separating the tangential and normal components of $\boldsymbol{v}^{s_1}$ and $\boldsymbol{c}$ in the first term of the RHS, we have

$$\begin{aligned}\phi_i^1\nabla^s\cdot[\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})]=\nabla^s\cdot\left[\phi_i^1\rho^{s_1}\left(\boldsymbol{v}_{\parallel}^{s_1}-\boldsymbol{c}_{\parallel}\right)\right]+\nabla^s\cdot\left[\phi_i^1\rho^{s_1}\,\underbrace{(v_{\perp}^{s_1}-c_{\perp})}_{=0}\,\boldsymbol{n}^1\right]\\-\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})\cdot\nabla^s\phi_i^1,\end{aligned} \tag{36.8}$$

where the underbraced factor is zero by the KBC.

Therefore

$$\begin{aligned}&Tg\int\limits_{\partial\Omega^1}\phi_i^1\nabla^s\cdot[\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})]\\&=-Tg\int\limits_{\partial\Omega^1}\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})\cdot\nabla^s\phi_i^1-Tg\int\limits_{C^1}\boldsymbol{m}^1\cdot\left[\phi_i^1\rho^{s_1}\,(\boldsymbol{v}^{s_1}-\boldsymbol{c})\right],\end{aligned} \tag{36.9}$$

where we have applied the surface divergence theorem to the second term on the right-hand side above. Here we notice that in the 2D case which we are considering, the boundary of $\partial\Omega^1$, given by C^1 is simply the end points of boundary 1. Where the appropriate conditions are to be applied.

This yields

$$
\begin{aligned}
Tg \int\limits_{\partial\Omega^1} \phi_i^1 \nabla^s \cdot \left[\rho^{s_1}\left(\boldsymbol{v}^{s_1}-\boldsymbol{c}\right)\right] = &-Tg \int\limits_{\partial\Omega^1} \rho^{s_1}\left(\boldsymbol{v}^{s_1}-\boldsymbol{c}\right)\cdot\nabla^s\phi_i^1 \\
&- Tg\,\phi_i^1(c)\rho_c^{s_1}\boldsymbol{v}_c^{s_1}\cdot\boldsymbol{m}_c^1 + Tg\,\phi_i^1(c)\rho_c^{s_1}\boldsymbol{c}_c\cdot\boldsymbol{m}_c^1 \\
&\underbrace{-Tg\,\phi_i^1(a)\rho_a^{s_1}\boldsymbol{v}_a^{s_1}\cdot\boldsymbol{m}_a^1 + Tg\,\phi_i^1(a)\rho_a^{s_1}\boldsymbol{c}_a\cdot\boldsymbol{m}_a^1}_{=0},
\end{aligned}
\tag{36.10}
$$

where the a sub-index stands for the apex, where the tangential velocity of the coordinates and the surface are both zero. We notice here that we have not decomposed this equation into two parts (near-field and far-field), as it does not involve the bulk velocity variables, which are the only ones that require a separate treatment.

Re-writing the expression above we have

$$
\begin{aligned}
Tg \int\limits_{\partial\Omega^1} \phi_i^1 \nabla^s \cdot \left[\rho^{s_1}\left(\boldsymbol{v}^{s_1}-\boldsymbol{c}\right)\right] = &-Tg \int\limits_{\partial\Omega^1} \rho^{s_1}\left(\partial_s\phi_i^1\right)\left(\boldsymbol{v}^{s_1}-\boldsymbol{c}\right)\cdot\boldsymbol{t}^1 \\
&- Tg\,\delta_{i,c}\rho_c^{s_1}u_c^{s_1}m_r^1(c) - Tg\,\delta_{i,c}\rho_c^{s_1}w_c^{s_1}m_z^1(c) \\
&+ Tg\,\delta_{i,c}\rho_c^{s_1}m_r^1(c)\partial_t r_c^c + Tg\,\delta_{i,c}\rho_c^{s_1}m_z^1(c)\partial_t z_c^c,
\end{aligned}
\tag{36.11}
$$

i.e.

$$
\begin{aligned}
Tg \int\limits_{\partial\Omega^2} \phi_i^1 \nabla^s \cdot \left[\rho^{s_1}\left(\boldsymbol{v}^{s_1}-\boldsymbol{c}\right)\right] = &-Tg \int\limits_{\partial\Omega^1} \rho^{s_1}\left(\partial_s\phi_i^2\right)\boldsymbol{v}^{s_1}\cdot\boldsymbol{t}^1 + Tg \int\limits_{\partial\Omega^1} \rho^{s_1}\left(\partial_s\phi_i^1\right)\boldsymbol{c}\cdot\boldsymbol{t}^1 \\
&- Tg\,\delta_{i,c}\rho_c^{s_1}u_c^{s_1}m_r^1(c) - Tg\,\delta_{i,c}\rho_c^{s_1}w_c^{s_1}m_z^1(c) \\
&+ Tg\,\delta_{i,c}\rho_c^{s_1}m_r^1(c)\partial_t r_c^c + Tg\,\delta_{i,c}\rho_c^{s_1}m_z^1(c)\partial_t z_c^c,
\end{aligned}
\tag{36.12}
$$

which is

$$
\begin{aligned}
Tg \int\limits_{\partial\Omega^1} \phi_i^1 \nabla^s \cdot \left[\rho^{s_1}\left(\boldsymbol{v}^{s_1}-\boldsymbol{c}\right)\right] = &-Tg \int\limits_{\partial\Omega^1} \rho^{s_1}u^{s_1}t_r^1\partial_s\phi_i^1 - Tg \int\limits_{\partial\Omega^1} \rho^{s_1}w^{s_1}t_z^1\partial_s\phi_i^1 \\
&+ Tg \int\limits_{\partial\Omega^1} \rho^{s_1}t_r^1\left(\partial_s\phi_i^1\right)\partial_t r^c + Tg \int\limits_{\partial\Omega^1} \rho^{s_1}t_z^1\left(\partial_s\phi_i^1\right)\partial_t z^c \\
&- Tg\,\delta_{i,c}\rho_c^{s_1}u_c^{s_1}m_r^1(c) - Tg\,\delta_{i,c}\rho_c^{s_1}w_c^{s_1}m_z^1(c) \\
&+ Tg\,\delta_{i,c}\rho_c^{s_1}m_r^1(c)\partial_t r_c^c + Tg\,\delta_{i,c}\rho_c^{s_1}m_z^1(c)\partial_t z_c^c,
\end{aligned}
\tag{36.13}
$$

Taking this result into the residual equation we have

$$
\begin{aligned}
D_i^1 = & -Tg\,\delta_{i,c}\rho_c^{s_1}u_c^{s_1}m_r^1(c) - Tg\,\delta_{i,c}\rho_c^{s_1}w_c^{s_1}m_z^1(c) \\
& + Tg\,\delta_{i,c}\rho_c^{s_1}m_r^1(c)\partial_t r_c^c + Tg\,\delta_{i,c}\rho_c^{s_1}m_z^1(c)\partial_t z_c^c \\
& - Tg\int\limits_{\partial\Omega^1}\rho^{s_1}u^{s_1}t_r^1\partial_s\phi_i^1 - Tg\int\limits_{\partial\Omega^1}\rho^{s_1}w^{s_1}t_z^1\partial_s\phi_i^1 \\
& + Tg\int\limits_{\partial\Omega^1}\rho^{s_1}t_r^1\left(\partial_s\phi_i^1\right)\partial_t r^c + Tg\int\limits_{\partial\Omega^1}\rho^{s_1}t_z^1\left(\partial_s\phi_i^1\right)\partial_t z^c \\
& + Tg\int\limits_{\partial\Omega^1}\phi_i^1\partial_t\rho^{s_1} + Tg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_r^1\partial_t\partial_s r^c \\
& + Tg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_z^1\partial_t\partial_s z^c - Dg\int\limits_{\partial\Omega^1}\phi_i^1 + \int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}.
\end{aligned}
\tag{36.14}
$$

We now recall the approximations

$$
\partial_t r^c \approx \frac{3r^c - 4r^c(t_{n-1}) + r^c(t_{n-2})}{2\Delta_t} \tag{36.15}
$$

and

$$
\partial_t z^c \approx \frac{3z^c - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t}; \tag{36.16}
$$

and we introduce

$$
\partial_t \rho^{s_1} \approx \frac{3\rho^{s_1} - 4\rho^{s_1}(t_{n-1}) + \rho^{s_1}(t_{n-2})}{2\Delta_t}, \tag{36.17}
$$

$$
\partial_t\partial_s r^c \approx \frac{3\partial_s r^c - 4\partial_s r^c(t_{n-1}) + \partial_s r^c(t_{n-2})}{2\Delta_t} \tag{36.18}
$$

and

$$
\partial_t\partial_s z^c \approx \frac{3\partial_s z^c - 4\partial_s z^c(t_{n-1}) + \partial_s z^c(t_{n-2})}{2\Delta_t}; \tag{36.19}
$$

Substituting these approximations in the residual equation we have

$$
\begin{aligned}
\mathfrak{D}_i^1 = & -Tg\,\delta_{i,c}\rho_c^{s_1} u_c^{s_1} m_r^1(c) - Tg\,\delta_{i,c}\rho_c^{s_1} w_c^{s_1} m_z^1(c) \\
& + Tg\,\delta_{i,c}\rho_c^{s_1} m_r^1(c)\frac{3r_c^c - 4r_c^c(t_{n-1}) + r_c^c(t_{n-2})}{2\Delta_t} \\
& + Tg\,\delta_{i,c}\rho_c^{s_1} m_z^1(c)\frac{3z^c - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t} \\
& - Tg \int\limits_{\partial\Omega^1} \rho^{s_1} u^{s_1} t_r^1 \partial_s \phi_i^1 - Tg \int\limits_{\partial\Omega^1} \rho^{s_1} w^{s_1} t_z^1 \partial_s \phi_i^1 \\
& + Tg \int\limits_{\partial\Omega^1} \rho^{s_1} t_r^1 \frac{3r^c - 4r^c(t_{n-1}) + r^c(t_{n-2})}{2\Delta_t} \partial_s \phi_i^1 \\
& + Tg \int\limits_{\partial\Omega^1} \rho^{s_1} t_z^1 \frac{3z^c - 4z^c(t_{n-1}) + z^c(t_{n-2})}{2\Delta_t} \partial_s \phi_i^1 \\
& + Tg \int\limits_{\partial\Omega^1} \phi_i^1 \frac{3\rho^{s_1} - 4\rho^{s_1}(t_{n-1}) + \rho^{s_1}(t_{n-2})}{2\Delta_t} \\
& + Tg \int\limits_{\partial\Omega^1} \phi_i^1 \rho^{s_1} t_r^1 \frac{3\partial_s r^c - 4\partial_s r^c(t_{n-1}) + \partial_s r^c(t_{n-2})}{2\Delta_t} \\
& + Tg \int\limits_{\partial\Omega^1} \phi_i^1 \rho^{s_1} t_z^1 \frac{3\partial_s z^c - 4\partial_s z^c(t_{n-1}) + \partial_s z^c(t_{n-2})}{2\Delta_t} - Dg \int\limits_{\partial\Omega^1} \phi_i^1 + \int\limits_{\partial\Omega^1} \phi_i^1 \rho^{s_1}.
\end{aligned}
\tag{36.20}
$$

Multiplying the residual equation by $2\Delta_t/3$ we have

$$
\begin{aligned}
\mathscr{D}_i^1 = &-\frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}u_c^{s_1}m_r^1(c) - \frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}w_c^{s_1}m_z^1(c) \\
&+ Tg\,\delta_{i,c}\rho_c^{s_1}m_r^1(c)r_c^c - \frac{4Tg}{3}\delta_{i,c}\rho_c^{s_1}m_r^1(c)r_c^c(t_{n-1}) + \frac{Tg}{3}\delta_{i,c}\rho_c^{s_1}m_r^1(c)r_c^c(t_{n-2}) \\
&+ Tg\,\delta_{i,c}\rho_c^{s_1}m_z^1(c)z^c - \frac{4Tg}{3}\delta_{i,c}\rho_c^{s_1}m_z^1(c)z^c(t_{n-1}) + \frac{Tg}{3}\delta_{i,c}\rho_c^{s_1}m_z^1(c)z^c(t_{n-2}) \\
&- \frac{2\Delta_t Dg}{3}\int\limits_{\partial\Omega^1}\phi_i^1 + \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1} \\
&+ Tg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1} - \frac{4Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}(t_{n-1}) + \frac{Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}(t_{n-2}) \\
&- \frac{2\Delta_t Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}u^{s_1}t_r^1\partial_s\phi_i^1 - \frac{2\Delta_t Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}w^{s_1}t_z^1\partial_s\phi_i^1 \\
&+ Tg\int\limits_{\partial\Omega^1}\rho^{s_1}t_r^1r^c\partial_s\phi_i^1 - \frac{4Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}t_r^1r^c(t_{n-1})\partial_s\phi_i^1 + \frac{Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}t_r^1r^c(t_{n-2})\partial_s\phi_i^1 \\
&+ Tg\int\limits_{\partial\Omega^1}\rho^{s_1}t_z^1z^c\partial_s\phi_i^1 - \frac{4Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}t_z^1z^c(t_{n-1})\partial_s\phi_i^1 + \frac{Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}t_z^1z^c(t_{n-2})\partial_s\phi_i^1 \\
&+ Tg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_r^1\partial_s r^c - \frac{4Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_r^1\partial_s r^c(t_{n-1}) + \frac{Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_r^1\partial_s r^c(t_{n-2}) \\
&+ Tg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_z^1\partial_s z^c - \frac{4Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_z^1\partial_s z^c(t_{n-1}) + \frac{Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_z^1\partial_s z^c(t_{n-2}).
\end{aligned}
\tag{36.21}
$$

We now introduce the decomposition

$$
\begin{aligned}
\mathscr{D}_i^1 = &-\frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}u_c^{s_1}m_r^1(c) - \frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}w_c^{s_1}m_z^1(c) \\
&+ Tg\,\delta_{i,c}\rho_c^{s_1}m_r^1(c)r_c^c - \frac{4Tg}{3}\delta_{i,c}\rho_c^{s_1}m_r^1(c)r_c^c(t_{n-1}) + \frac{Tg}{3}\delta_{i,c}\rho_c^{s_1}m_r^1(c)r_c^c(t_{n-2}) \\
&+ Tg\,\delta_{i,c}\rho_c^{s_1}m_z^1(c)z^c - \frac{4Tg}{3}\delta_{i,c}\rho_c^{s_1}m_z^1(c)z^c(t_{n-1}) + \frac{Tg}{3}\delta_{i,c}\rho_c^{s_1}m_z^1(c)z^c(t_{n-2}) \\
&+ \mathscr{D}_i^{1,a} + \mathscr{D}_i^{1,b} + \mathscr{D}_i^{1,c},
\end{aligned}
\tag{36.22}
$$

where

$$
\mathscr{D}_i^{1,a} = -\frac{2\Delta_t Dg}{3}\int\limits_{\partial\Omega^g}\phi_i^1, \tag{36.23}
$$

$$
\mathscr{D}_i^{1,b} = \frac{2\Delta_t}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1} + Tg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1} - \frac{4Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}(t_{n-1}) + \frac{Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}(t_{n-2}),
\tag{36.24}
$$

and

$$
\begin{aligned}
\mathscr{D}_i^{1,c} = &-\frac{2\Delta_t Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}u^{s_1}t_r^1\partial_s\phi_i^1 - \frac{2\Delta_t Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}w^{s_1}t_z^1\partial_s\phi_i^1 \\
&+ Tg\int\limits_{\partial\Omega^1}\rho^{s_1}t_r^1 r^c\partial_s\phi_i^1 - \frac{4Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}t_r^1 r^c(t_{n-1})\partial_s\phi_i^1 + \frac{Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}t_r^1 r^c(t_{n-2})\partial_s\phi_i^1 \\
&+ Tg\int\limits_{\partial\Omega^1}\rho^{s_1}t_z^1 z^c\partial_s\phi_i^1 - \frac{4Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}t_z^1 z^c(t_{n-1})\partial_s\phi_i^1 + \frac{Tg}{3}\int\limits_{\partial\Omega^1}\rho^{s_1}t_z^1 z^c(t_{n-2})\partial_s\phi_i^1 \\
&+ Tg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_r^1\partial_s r^c - \frac{4Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_r^1\partial_s r^c(t_{n-1}) + \frac{Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_r^1\partial_s r^c(t_{n-2}) \\
&+ Tg\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_z^1\partial_s z^c - \frac{4Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_z^1\partial_s z^c(t_{n-1}) + \frac{Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\rho^{s_1}t_z^1\partial_s z^c(t_{n-2}).
\end{aligned}
\tag{36.25}
$$

Furthermore, we recall

$$\rho^{s_1} \approx \sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1, \tag{36.26}$$

$$\rho^{s_1}(t_{n-1}) \approx \sum_{j=1}^{n_v}\rho_j^{s_1}(t_{n-1})\phi_j^1, \tag{36.27}$$

$$\rho^{s_1}(t_{n-2}) \approx \sum_{j=1}^{n_v}\rho_j^{s_1}(t_{n-2})\phi_j^1, \tag{36.28}$$

$$r^c \approx \sum_{j=1}^{n_v} r_j^c\phi_j, \tag{36.29}$$

$$r^c(t_{n-1}) \approx \sum_{j=1}^{n_v} r_j^c(t_{n-1})\phi_j, \tag{36.30}$$

$$r^c(t_{n-2}) \approx \sum_{j=1}^{n_v} r_j^c(t_{n-2})\phi_j, \tag{36.31}$$

$$z^c \approx \sum_{j=1}^{n_v} z_j^c\phi_j \tag{36.32}$$

$$z^c(t_{n-1}) \approx \sum_{j=1}^{n_v} z_j^c(t_{n-1})\phi_j \tag{36.33}$$

$$z^c(t_{n-2}) \approx \sum_{j=1}^{n_v} z_j^c(t_{n-2})\phi_j \tag{36.34}$$

$$u^{s_1} \approx \sum_{j=1}^{n_v} u_j^{s_1} \phi_j^1 \tag{36.35}$$

and

$$w^{s_1} \approx \sum_{j=1}^{n_v} w_j^{s_1} \phi_j. \tag{36.36}$$

Substituting the approximations above into the residual we have where

$$\mathcal{D}_i^{1,a} = -\frac{2\Delta_t Dg}{3} \int\limits_{\partial\Omega^1} \phi_i^1, \tag{36.37}$$

$$\begin{aligned} \mathcal{D}_i^{1,b} = \frac{2\Delta_t}{3} \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} \rho_j^{s_1} \phi_j^1 \right) \\ + Tg \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} \rho_j^{s_1} \phi_j^1 \right) \\ - \frac{4Tg}{3} \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} \rho_j^{s_1}(t_{n-1}) \phi_j^1 \right) \\ + \frac{Tg}{3} \int\limits_{\partial\Omega^1} \phi_i^1 \left(\sum_{j=1}^{n_v} \rho_j^{s_1}(t_{n-2}) \phi_j^1 \right), \end{aligned} \tag{36.38}$$

and

$$
\begin{aligned}
\mathcal{D}_i^{1,c} =& -\frac{2\Delta_t Tg}{3}\int\limits_{\partial\Omega^1}\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)\left(\sum_{k=1}^{n_v}u_k^{s_1}\phi_k^1\right)t_r^1\partial_s\phi_i^1\\
&-\frac{2\Delta_t Tg}{3}\int\limits_{\partial\Omega^1}\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)\left(\sum_{k=1}^{n_v}w_k^{s_1}\phi_k^1\right)t_z^1\partial_s\phi_i^1\\
&+Tg\int\limits_{\partial\Omega^1}\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_r^1\left(\sum_{k=1}^{n_v}r_k^c\phi_k\right)\partial_s\phi_i^1\\
&-\frac{4Tg}{3}\int\limits_{\partial\Omega^1}\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_r^1\left(\sum_{k=1}^{n_v}r_k^c(t_{n-1})\phi_k\right)\partial_s\phi_i^1\\
&+\frac{Tg}{3}\int\limits_{\partial\Omega^1}\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_r^1\left(\sum_{k=1}^{n_v}r_k^c(t_{n-2})\phi_k^1\right)\partial_s\phi_i^1\\
&+Tg\int\limits_{\partial\Omega^1}\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_z^1\left(\sum_{k=1}^{n_v}z_k^c\phi_k^1\right)\partial_s\phi_i^1\\
&-\frac{4Tg}{3}\int\limits_{\partial\Omega^1}\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_z^1\left(\sum_{k=1}^{n_v}z_k^c(t_{n-1})\phi_k\right)\partial_s\phi_i^1\\
&+\frac{Tg}{3}\int\limits_{\partial\Omega^1}\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_z^1\left(\sum_{k=1}^{n_v}z_k^c(t_{n-2})\phi_k\right)\partial_s\phi_i^1\\
&+Tg\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_r^1\left(\sum_{k=1}^{n_v}r_k^c\partial_s\phi_k\right)\\
&-\frac{4Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_r^1\left(\sum_{k=1}^{n_v}r_k^c(t_{n-1})\partial_s\phi_k\right)\\
&+\frac{Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_r^1\left(\sum_{k=1}^{n_v}r_k^c(t_{n-2})\partial_s\phi_k\right)\\
&+Tg\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_z^1\left(\sum_{k=1}^{n_v}z_k^c\partial_s\phi_k\right)\\
&-\frac{4Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_z^1\left(\sum_{k=1}^{n_v}z_k^c(t_{n-1})\partial_s\phi_k\right)\\
&+\frac{Tg}{3}\int\limits_{\partial\Omega^1}\phi_i^1\left(\sum_{j=1}^{n_v}\rho_j^{s_1}\phi_j^1\right)t_z^1\left(\sum_{k=1}^{n_v}z_k^c(t_{n-2})\partial_s\phi_k\right).
\end{aligned}
\tag{36.39}
$$

Moving the integrals into the sums, decomposing the integrals in sums of integrals of each line-element and passing to local node numbers we have

$$\mathcal{D}_i^{1,a} = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \mathcal{D}_{e_1,ii}^{1,a}, \tag{36.40}$$

$$\mathcal{D}_i^{1,b} = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \mathcal{D}_{e_1,ii}^{1,b} \tag{36.41}$$

and

$$\mathcal{D}_i^{1,c} = \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1} \mathcal{D}_{e_1,ii}^{1,c}; \tag{36.42}$$

where

$$\mathcal{D}_{e_1,ii}^{1,a} = -\frac{2\Delta_t Dg}{3} \underbrace{\int_{\partial\Omega_{e_1}^1} \phi_i^1}_{d_{ii}(e_1)}, \tag{36.43}$$

$$\begin{aligned} \mathcal{D}_{e_1,ii}^{1,b} = \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \rho_{l_1^1(e_1,jj)}^{s_1} \underbrace{\int_{\partial\Omega_{e_1}^1} \phi_i^1 \phi_j^1}_{c_{ii,jj}(e_1)} \\ + Tg \sum_{jj=1}^{n_v^{1,e_1}} \rho_{l_1^1(e_1,jj)}^{s_1} \underbrace{\int_{\partial\Omega_{e_1}^1} \phi_i^1 \phi_j^1}_{c_{ii,jj}(e_1)} \\ - \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \rho_{l_1^1(e_1,jj)}^{s_1}(t_{n-1}) \underbrace{\int_{\partial\Omega_{e_1}^1} \phi_i^1 \phi_j^1}_{c_{ii,jj}(e_1)} \\ + \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \rho_{l_1^1(e_1,jj)}^{s_1}(t_{n-2}) \underbrace{\int_{\partial\Omega_{e_1}^1} \phi_i^1 \phi_j^1}_{c_{ii,jj}(e_1)}, \end{aligned} \tag{36.44}$$

and

$$
\begin{aligned}
\mathcal{D}^{1,c}_{e_1,ii} =& -\frac{2\Delta_t Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}u^{s_1}_{l_1^1(e_1,kk)}\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_r\phi^1_j\phi^1_k\partial_s\phi^1_i}_{c^s_{jj,kk,ii,t_r}(e_1)} \qquad (36.45)\\
&-\frac{2\Delta_t Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}w^{s_1}_{l_1^1(e_1,kk)}\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_z\phi^1_j\phi^1_k\partial_s\phi^1_i}_{c^s_{jj,kk,ii,t_z}(e_1)}\\
&+Tg\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_r\phi^1_j\phi^1_k\partial_s\phi^1_i}_{c^s jj,kk,ii,t_r(e_1)}-\frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-1})\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_r\phi^1_j\phi^1_k\partial_s\phi^1_i}_{c^s_{jj,kk,ii,t_r}(e_1)}\\
&+\frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-2})\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_r\phi^1_j\phi^1_k\partial_s\phi^1_i}_{c^s_{jj,kk,ii,t_r}(e_1)}\\
&+Tg\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_z\phi^1_j\phi^1_k\partial_s\phi^1_i}_{c^s_{jj,kk,ii,t_z}(e_1)}-\frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-1})\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_z\phi^1_j\phi^1_k\partial_s\phi^1_i}_{c^s_{jj,kk,ii,t_z}(e_1)}\\
&+\frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-2})\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_z\phi^1_j\phi^1_k\partial_s\phi^1_i}_{c^s_{jj,kk,ii,t_z}(e_1)}\\
&+Tg\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}\underbrace{\int_{\partial\Omega^1_{e_1}} \phi^1_i\phi^1_j\partial_s\phi^1_k t^1_r}_{c^s_{ii,jj,kk,t_r}(e_1)}-\frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-1})\underbrace{\int_{\partial\Omega^1_{e_1}} \phi^1_i\phi^1_j\partial_s\phi^1_k t^1_r}_{c^s_{ii,jj,kk,t_r}(e_1)}\\
&+\frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-2})\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_r\phi^1_i\phi^1_j\partial_s\phi^1_k}_{c^s_{ii,jj,kk,t_r}(e_1)}\\
&+Tg\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_z\phi^1_i\phi^1_j\partial_s\phi^1_k}_{c^s_{ii,jj,kk,t_z}(e_1)}-\frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-1})\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_z\phi^1_i\phi^1_j\partial_s\phi^1_k}_{c^s_{ii,jj,kk,t_z}(e_1)}\\
&+\frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-2})\underbrace{\int_{\partial\Omega^1_{e_1}} t^1_z\phi^1_i\phi^1_j\partial_s\phi^1_k}_{c^s_{ii,jj,kk,t_z}(e_1)}\,.
\end{aligned}
$$

Re-writing we have

$$\mathcal{D}^{1,a}_{e_1,ii} = -\frac{2\Delta_t Dg}{3} c_{ii}(e_1), \tag{36.46}$$

$$\begin{aligned}\mathcal{D}^{1,b}_{e_1,ii} = \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \rho^{s_1}_{l^1_1(e_1,jj)} c_{ii,jj}(e_1) \\ + Tg \sum_{jj=1}^{n_v^{1,e_1}} \rho^{s_1}_{l^1_1(e_1,jj)} c_{ii,jj}(e_1) \\ - \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \rho^{s_1}_{l^1_1(e_1,jj)}(t_{n-1}) c_{ii,jj}(e_1) \\ + \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \rho^{s_1}_{l^1_1(e_1,jj)}(t_{n-2}) c_{ii,jj}(e_1),\end{aligned} \tag{36.47}$$

and

$$
\begin{aligned}
\mathcal{D}^{1,c}_{e_1,ii} =& -\frac{2\Delta_t Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}u^{s_1}_{l_1^1(e_1,kk)}c^s_{jj,kk,ii,t_r}(e_1)\\
&-\frac{2\Delta_t Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}w^{s_1}_{l_1^1(e_1,kk)}c^s_{jj,kk,ii,t_z}(e_1)\\
&+Tg\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}d^s_{jj,kk,ii,t_r}(e_1)\\
&-\frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-1})d^1_{jj,kk,ii,t_r}(e_1)\\
&+\frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-2})c^s_{jj,kk,ii,t_r}(e_1)\\
&+Tg\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}c^s_{jj,kk,ii,t_z}(e_1)\\
&-\frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-1})c^s_{jj,kk,ii,t_z}(e_1)\\
&+\frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-2})c^s_{jj,kk,ii,t_z}(e_1)\\
&+Tg\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}c^s_{ii,jj,kk,t_r}(e_1)\\
&-\frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-1})c^s_{ii,jj,kk,t_r}(e_1)\\
&+\frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-2})c^s_{ii,jj,kk,t_r}(e_1)\\
&+Tg\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}c^s_{ii,jj,kk,t_z}(e_1)\\
&-\frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-1})c^s_{ii,jj,kk,t_z}(e_1)\\
&+\frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-2})c^s_{ii,jj,kk,t_z}(e_1).
\end{aligned}
\tag{36.48}
$$

Summarising and re-arranging we have

$$\begin{aligned}\mathcal{D}_i^1 = &-\frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}\left[u_c^{s_1}m_r^1(c)+w_c^{s_1}m_z^1(c)\right]\\ &+Tg\,\delta_{i,c}\rho_c^{s_1}m_r^1(c)\left[r_c^c-\frac{4}{3}r_c^c(t_{n-1})+\frac{1}{3}r_c^c(t_{n-2})\right]\\ &+Tg\,\delta_{i,c}\rho_c^{s_1}m_z^1(c)\left[z^c-\frac{4}{3}z^c(t_{n-1})+\frac{1}{3}z^c(t_{n-2})\right]\\ &+\sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1}\mathcal{D}_{e_1,ii}^{1,a}+\sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1}\mathcal{D}_{e_1,ii}^{1,b}+\sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1}\mathcal{D}_{e_1,ii}^{1,c},\end{aligned}\tag{36.49}$$

with

$$\mathcal{D}_{e_1,ii}^{1,a}=-\frac{2\Delta_t Dg}{3}c_{ii}(e_1),\tag{36.50}$$

$$\begin{aligned}\mathcal{D}_{e_1,ii}^{1,b}=\sum_{j=1}^{n_v}c_{ii,jj}(e_1)\Bigg\{&\frac{2\Delta_t}{3}\rho_{l_1^1(e_1,jj)}^{s_1}\\ &+Tg\left[\rho_{l_1^1(e_1,jj)}^{s_1}-\frac{4}{3}\rho_{l_1^1(e_1,jj)}^{s_1}(t_{n-1})+\frac{1}{3}\rho_{l_1^1(e_1,jj)}^{s_1}(t_{n-2})\right]\Bigg\},\end{aligned}\tag{36.51}$$

and

$$\begin{aligned}\mathcal{D}_{e_1,ii}^{1,c}=\sum_{jj=1}^{n_v^{1,e_1}}Tg\,\rho_{l_1^1(e_1,jj)}^{s_1}\Bigg\{&-\frac{2\Delta_t}{3}\underbrace{\sum_{kk=1}^{n_v^{1,e_1}}\left[u_{l_1^1(e_1,kk)}^{s_1}c_{jj,kk,ii,t_r}^{s}(e_1)+w_{l_1^1(e_1,kk)}^{s_1}c_{jj,kk,ii,t_z}^{s}(e_1)\right]}_{A_{ii,jj}^{s_1}(e_2)}\\ &+\underbrace{\sum_{kk=1}^{n_v^{1,e_1}}c_{jj,kk,ii,t_r}^{s}(e_1)\left[r_{l_1(e_1,kk)}^c-\frac{4}{3}r_{l_1(e_1,kk)}^c(t_{n-1})+\frac{1}{3}r_{l_1(e_1,kk)}^c(t_{n-2})\right]}_{B_{ii,jj}^{s_1}(e_2)}\\ &+\underbrace{\sum_{kk=1}^{n_v^{1,e_1}}c_{jj,kk,ii,t_z}^{s}(e_1)\left[z_{l_1(e_1,kk)}^c-\frac{4}{3}z_{l_1(e_1,kk)}^c(t_{n-1})+\frac{1}{3}z_{l_1(e_1,kk)}^c(t_{n-2})\right]}_{C_{ii,jj}^{s_1}(e_2)}\\ &+\underbrace{\sum_{kk=1}^{n_v^{1,e_1}}c_{ii,jj,kk,t_r}^{s}(e_1)\left[r_{l_1(e_1,kk)}^c-\frac{4}{3}r_{l_1(e_1,kk)}^c(t_{n-1})+\frac{1}{3}r_{l_1(e_1,kk)}^c(t_{n-2})\right]}_{D_{ii,jj}^{s_1}(e_2)}\\ &+\underbrace{\sum_{kk=1}^{n_v^{1,e_1}}c_{ii,jj,kk,t_z}^{s}(e_1)\left[z_{l_1(e_1,kk)}^c-\frac{4}{3}z_{l_1(e_1,kk)}^c(t_{n-1})+\frac{1}{3}z_{l_1(e_1,kk)}^c(t_{n-2})\right]}_{E_{ii,jj}^{s_1}(e_2)}\Bigg\}.\end{aligned}\tag{36.52}$$

36.1. Jacobian terms

Here we find the derivatives of $\mathcal{D}_i^1$ with respect to $\rho_q^{s_1}$, $u_q^{s_1}$, $w_q^{s_1}$, θ_c and h_q.

36.1.1. Derivatives of $\mathcal{D}_i^1$ with respect to $\rho_q^{s_1}$

Using equations (36.49) we have

$$\begin{aligned}\partial_{\rho_q^{s_1}}\mathcal{D}_i^1 = &-\frac{2\Delta_t Tg}{3}\delta_{i,c}\partial_{\rho_q^{s_1}}\rho_c^{s_1}u_c^{s_1}m_r^1(c) - \frac{2\Delta_t Tg}{3}\delta_{i,c}\partial_{\rho_q^{s_1}}\rho_c^{s_1}w_c^{s_1}m_z^1(c)\\ &+ Tg\,\delta_{i,c}\partial_{\rho_q^{s_1}}\rho_c^{s_1}m_r^1(c)r_c^c - \frac{4Tg}{3}\delta_{i,c}\partial_{\rho_q^{s_1}}\rho_c^{s_1}m_r^1(c)r_c^c(t_{n-1})\\ &+ \frac{Tg}{3}\delta_{i,c}\partial_{\rho_q^{s_1}}\rho_c^{s_1}m_r^1(c)r_c^c(t_{n-2})\\ &+ Tg\,\delta_{i,c}\partial_{\rho_q^{s_1}}\rho_c^{s_1}m_z^1(c)z^c - \frac{4Tg}{3}\delta_{i,c}\partial_{\rho_q^{s_1}}\rho_c^{s_1}m_z^1(c)z^c(t_{n-1})\\ &+ \frac{Tg}{3}\delta_{i,c}\partial_{\rho_q^{s_1}}\rho_c^{s_1}m_z^1(c)z^c(t_{n-2})\\ &+ \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\rho_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\rho_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\rho_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,c},\end{aligned} \tag{36.53}$$

i.e.

$$\begin{aligned}\partial_{\rho_q^{s_1}}\mathcal{D}_i^1 = &\frac{2\Delta_t Tg}{3}\delta_{i,c}\delta_{q,c}u_c^{s_1}m_r^1(c) + \frac{2\Delta_t Tg}{3}\delta_{i,c}\delta_{q,c}w_c^{s_1}m_z^1(c)\\ &- Tg\,\delta_{i,c}\delta_{q,c}m_r^1(c)\left[r_c^c - \frac{4}{3}r_c^c(t_{n-1}) + \frac{1}{3}r_c^c(t_{n-2})\right]\\ &- Tg\,\delta_{i,c}\delta_{q,c}m_z^1(c)\left[z^c - \frac{4}{3}z^c(t_{n-1}) + \frac{1}{3}z^c(t_{n-2})\right]\\ &+ \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\rho_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\rho_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\rho_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,c},\end{aligned} \tag{36.54}$$

equivalently

$$\begin{aligned}\partial_{\rho_q^{s_1}}\mathcal{D}_i^1 = Tg\,\delta_{i,c}\delta_{q,c}\Bigg\{&\frac{2\Delta_t}{3}\left[u_c^{s_1}m_r^1(c) + w_c^{s_1}m_z^1(c)\right]\\ &- m_r^1(c)\left[r_c^c - \frac{4}{3}r_c^c(t_{n-1}) + \frac{1}{3}r_c^c(t_{n-2})\right]\\ &- m_z^1(c)\left[z^c - \frac{4}{3}z^c(t_{n-1}) + \frac{1}{3}z^c(t_{n-2})\right]\Bigg\}\\ &+ \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\rho_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\rho_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\rho_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,c},\end{aligned} \tag{36.55}$$

From equation (36.46) we have

$$\partial_{\rho_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a} = -\frac{2\Delta_t Dg}{3}\partial_{\rho_q^{s_1}}c_{ii}(e_1), \tag{36.56}$$

i.e.

$$\partial_{\rho_q^{s_1}} \mathcal{D}^{1,a}_{e_1,ii} = 0. \tag{36.57}$$

From equation (36.47) we have

$$\partial_{\rho_q^{s_1}} \mathcal{D}^{1,b}_{e_1,ii} = \frac{2\Delta_t}{3} \sum_{j=1}^{n_v} \partial_{\rho_q^{s_1}} \rho_j^{s_1} c_{ii,jj}(e_1) + Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} c_{ii,jj}(e_1) \tag{36.58}$$

$$- \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)}(t_{n-1}) c_{ii,jj}(e_1)$$

$$+ \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)}(t_{n-2}) c_{ii,jj}(e_1),$$

i.e.

$$\partial_{\rho_q^{s_1}} \mathcal{D}^{1,b}_{e_1,ii} = \frac{2\Delta_t}{3} c_{ii,jj}(e_1)|_{q=l_1^1(e_1,jj)} + Tg\, c_{ii,jj}(e_1)|_{q=l_1^1(e_1,jj)}, \tag{36.59}$$

equivalently

$$\partial_{\rho_q^{s_1}} \mathcal{D}^{1,b}_{e_1,ii} = c_{ii,jj}(e_1)|_{q=l_1^1(e_1,jj)} \left[\frac{2\Delta_t}{3} + Tg \right]. \tag{36.60}$$

From equation (36.48) we have

$$
\begin{aligned}
\partial_{\rho_q^{s_1}} \mathcal{D}^{1,c}_{e_1,ii} =& -\frac{2\Delta_t Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} u^{s_1}_{l_1^1(e_1,kk)} c^s_{jj,kk,ii,t_r}(e_1) \\
& -\frac{2\Delta_t Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} w^{s_1}_{l_1^1(e_1,kk)} c^s_{jj,kk,ii,t_z}(e_1) \\
& + Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)} c^s_{jj,kk,ii,t_r}(e_1) \\
& -\frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{jj,kk,ii,t_r}(e_1) \\
& +\frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{jj,kk,ii,t_r}(e_1) \\
& + Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)} c^s_{jj,kk,ii,t_z}(e_1) \\
& -\frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{jj,kk,ii,t_z}(e_1) \qquad (36.61)\\
& +\frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{jj,kk,ii,t_z}(e_1) \\
& + Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)} c^s_{ii,jj,kk,t_r}(e_1) \\
& -\frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{ii,jj,kk,t_r}(e_1) \\
& +\frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{ii,jj,kk,t_r}(e_1) \\
& + Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)} c^s_{ii,jj,kk,t_z}(e_1) \\
& -\frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{ii,jj,kk,t_z}(e_1) \\
& +\frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{\rho_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{ii,jj,kk,t_z}(e_1).
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\partial_{\rho_q^{s_1}} \mathcal{D}^{1,c}_{e_1,ii} ={}& -\frac{2\Delta_t Tg}{3} \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} u^{s_1}_{l_1^1(e_1,kk)} c^s_{jj,kk,ii,t_r}(e_1) \\
&- \frac{2\Delta_t Tg}{3} \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} w^{s_1}_{l_1^1(e_1,kk)} c^s_{jj,kk,ii,t_z}(e_1) \\
&+ Tg \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)} c^s_{jj,kk,ii,t_r}(e_1) \\
&- \frac{4Tg}{3} \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{jj,kk,ii,t_r}(e_1) \\
&+ \frac{Tg}{3} \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{jj,kk,ii,t_r}(e_1) \\
&+ Tg \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)} c^s_{jj,kk,ii,t_z}(e_1) \\
&- \frac{4Tg}{3} \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{jj,kk,ii,t_z}(e_1) \\
&+ \frac{Tg}{3} \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{jj,kk,ii,t_z}(e_1) \\
&+ Tg \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)} c^s_{ii,jj,kk,t_r}(e_1) - \frac{4Tg}{3} \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{ii,jj,kk,t_r}(e_1) \\
&+ \frac{Tg}{3} \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{ii,jj,kk,t_r}(e_1) \\
&+ Tg \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)} c^s_{ii,jj,kk,t_z}(e_1) - \frac{4Tg}{3} \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{ii,jj,kk,t_z}(e_1) \\
&+ \frac{Tg}{3} \sum_{\substack{kk=1\\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{ii,jj,kk,t_z}(e_1).
\end{aligned}
\tag{36.62}
$$

Equivalently,

$$
\begin{aligned}
\partial_{\rho_q^{s_1}} \mathcal{D}^{1,c}_{e_1,ii} \;=\; \sum_{\substack{kk=1 \\ q=l_1^1(e_1,jj)}}^{n_v^{1,e_1}} & Tg \left\{ -\frac{2\Delta_t}{3} \left[u^{s_1}_{l_1^1(e_1,kk)} c^s_{jj,kk,ii,t_r}(e_1) + w^{s_1}_{l_1^1(e_1,kk)} c^s_{jj,kk,ii,t_z}(e_1) \right] \right. \\
& + c^s_{jj,kk,ii,t_r}(e_1) \left[r^c_{l_1(e_1,kk)} - \frac{4}{3} r^c_{l_1(e_1,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l_1(e_1,kk)}(t_{n-2}) \right] \\
& + c^s_{jj,kk,ii,t_z}(e_1) \left[z^c_{l_1(e_1,kk)} - \frac{4}{3} z^c_{l_1(e_1,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l_1(e_1,kk)}(t_{n-2}) \right] \\
& + c^s_{ii,jj,kk,t_r}(e_1) \left[r^c_{l_1(e_1,kk)} - \frac{4}{3} r^c_{l_1(e_1,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l_1(e_1,kk)}(t_{n-2}) \right] \\
& \left. + c^s_{ii,jj,kk,t_z}(e_1) \left[z^c_{l_1(e_1,kk)} - \frac{4}{3} z^c_{l_1(e_1,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l_1(e_1,kk)}(t_{n-2}) \right] \right\}.
\end{aligned}
\tag{36.63}
$$

36.1.2. Derivatives of $\mathcal{D}_i^1$ with respect to $u_q^{s_1}$

Using equations (36.49) we have

$$\begin{aligned}\partial_{u_q^{s_1}}\mathcal{D}_i^1 = &-\frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}m_r^1(c)\partial_{u_q^{s_1}}u_c^{s_1} - \frac{2\Delta_t Tg}{3}\delta_{i,c}\partial_{u_q^{s_1}}\rho_c^{s_1}w_c^{s_1}m_z^1(c)\\ &+Tg\,\delta_{i,c}\partial_{u_q^{s_1}}\rho_c^{s_1}m_r^1(c)r_c^c - \frac{4Tg}{3}\delta_{i,c}\partial_{u_q^{s_1}}\rho_c^{s_1}m_r^1(c)r_c^c(t_{n-1})\\ &+\frac{Tg}{3}\delta_{i,c}\partial_{u_q^{s_1}}\rho_c^{s_1}m_r^1(c)r_c^c(t_{n-2})\\ &+Tg\,\delta_{i,c}\partial_{u_q^{s_1}}\rho_c^{s_1}m_z^1(c)z^c - \frac{4Tg}{3}\delta_{i,c}\partial_{u_q^{s_1}}\rho_c^{s_1}m_z^1(c)z^c(t_{n-1})\\ &+\frac{Tg}{3}\delta_{i,c}\partial_{u_q^{s_1}}\rho_c^{s_1}m_z^1(c)z^c(t_{n-2})\\ &+\sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1}\partial_{u_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1}\partial_{u_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1}\partial_{u_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,c},\end{aligned}\tag{36.64}$$

i.e.

$$\begin{aligned}\partial_{u_q^{s_1}}\mathcal{D}_i^1 = &-\frac{2\Delta_t Tg}{3}\delta_{i,c}\delta_{q,c}\rho_c^{s_1}m_r^1(c) + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1}\partial_{u_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a}\\ &+\sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1}\partial_{u_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\text{el}}^1}\partial_{u_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,c},\end{aligned}\tag{36.65}$$

From equation (31.47) we have

$$\partial_{u_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a} = -Dg\,\partial_{u_q^{s_1}}d_{ii}(e_1),\tag{36.66}$$

i.e.

$$\partial_{u_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a} = 0.\tag{36.67}$$

From equation (36.47) we have

$$\begin{aligned}\partial_{u_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} = &\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\partial_{u_q^{s_1}}\rho_j^{s_1}d_{ii,jj}(e_1) + Tg\sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q^{s_1}}\rho_{l_1^1(e_1,jj)}^{s_1}d_{ii,jj}(e_1)\\ &-\frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q^{s_1}}\rho_{l_1^1(e_1,jj)}^{s_1}(t_{n-1})d_{ii,jj}(e_1)\\ &+\frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{u_q^{s_1}}\rho_{l_1^1(e_1,jj)}^{s_1}(t_{n-2})d_{ii,jj}(e_1),\end{aligned}\tag{36.68}$$

i.e.

$$\partial_{u_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} = 0.\tag{36.69}$$

From equation (36.48) we have

$$
\begin{aligned}
\partial_{u_q^{s_1}} \mathcal{D}^{1,c}_{e_1,ii} = &- \frac{2\Delta_t Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} c^s_{jj,kk,ii,t_r}(e_1) \partial_{u_q^{s_1}} u^{s_1}_{l_1^1(e_1,kk)} \\
&- \frac{2\Delta_t Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} w^{s_1}_{l_1^1(e_1,kk)} c^s_{jj,kk,ii,t_z}(e_1) \\
&+ Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)} c^s_{jj,kk,ii,t_r}(e_1) \\
&- \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{jj,kk,ii,t_r}(e_1) \\
&+ \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{jj,kk,ii,t_r}(e_1) \\
&+ Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)} c^s_{jj,kk,ii,t_z}(e_1) \\
&- \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{jj,kk,ii,t_z}(e_1) \qquad (36.70) \\
&+ \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{jj,kk,ii,t_z}(e_1) \\
&+ Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)} c^s_{ii,jj,kk,t_r}(e_1) \\
&- \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{ii,jj,kk,t_r}(e_1) \\
&+ \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{ii,jj,kk,t_r}(e_1) \\
&+ Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)} c^s_{ii,jj,kk,t_z}(e_1) \\
&- \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{ii,jj,kk,t_z}(e_1) \\
&+ \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{u_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{ii,jj,kk,t_z}(e_1).
\end{aligned}
$$

i.e.

$$\partial_{u_q^{s_1}} \mathcal{D}^{1,c}_{e_1,ii} = \sum_{\substack{jj=1 \\ q=l_1^1(e_1.kk)}}^{n_v^{1,e_1}} -\frac{2\Delta_t Tg}{3} \rho^{s_1}_{l_1^1(e_1,jj)} c^s_{jj,kk,ii,t_r}(e_1). \tag{36.71}$$

36.1.3. **Derivatives of $\mathcal{D}_i^1$ with respect to $w_q^{s_1}$**

Using equations (36.49) we have

$$\begin{aligned}
\partial_{w_q^{s_1}}\mathcal{D}_i^1 = &-\frac{2\Delta_t Tg}{3}\delta_{i,c}\partial_{w_q^{s_1}}\rho_c^{s_1}m_r^1(c)u_c^{s_1} - \frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}m_z^1(c)\partial_{w_q^{s_1}}w_c^{s_1}\\
&+ Tg\,\delta_{i,c}\partial_{w_q^{s_1}}\rho_c^{s_1}m_r^1(c)r_c^c - \frac{4Tg}{3}\delta_{i,c}\partial_{w_q^{s_1}}\rho_c^{s_1}m_r^1(c)r_c^c(t_{n-1})\\
&+ \frac{Tg}{3}\delta_{i,c}\partial_{w_q^{s_1}}\rho_c^{s_1}m_r^1(c)r_c^c(t_{n-2})\\
&+ Tg\,\delta_{i,c}\partial_{w_q^{s_1}}\rho_c^{s_1}m_z^1(c)z^c - \frac{4Tg}{3}\delta_{i,c}\partial_{w_q^{s_1}}\rho_c^{s_1}m_z^1(c)z^c(t_{n-1})\\
&+ \frac{Tg}{3}\delta_{i,c}\partial_{w_q^{s_1}}\rho_c^{s_1}m_z^1(c)z^c(t_{n-2})\\
&+ \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\rm el}^1}\partial_{w_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\rm el}^1}\partial_{w_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\rm el}^1}\partial_{w_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,c},
\end{aligned} \tag{36.72}$$

i.e.

$$\begin{aligned}
\partial_{w_q^{s_1}}\mathcal{D}_i^1 = &-\frac{2\Delta_t Tg}{3}\delta_{i,c}\delta_{q,c}\rho_c^{s_1}m_z^1(c) + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\rm el}^1}\partial_{w_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a}\\
&+ \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\rm el}^1}\partial_{w_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)}}^{n_{\rm el}^1}\partial_{w_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,c},
\end{aligned} \tag{36.73}$$

From equation (36.46) we have

$$\partial_{w_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a} = -Dg\,\partial_{w_q^{s_1}}d_{ii}(e_1), \tag{36.74}$$

i.e.

$$\partial_{w_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,a} = 0. \tag{36.75}$$

From equation (36.47) we have

$$\begin{aligned}
\partial_{w_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} = &\frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\partial_{w_q^{s_1}}\rho_j^{s_1}d_{ii,jj}(e_1) + Tg\sum_{jj=1}^{n_v^{1,e_1}}\partial_{w_q^{s_1}}\rho_{l_1^1(e_1,jj)}^{s_1}d_{ii,jj}(e_1)\\
&- \frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{w_q^{s_1}}\rho_{l_1^1(e_1,jj)}^{s_1}(t_{n-1})d_{ii,jj}(e_1)\\
&+ \frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\partial_{w_q^{s_1}}\rho_{l_1^1(e_1,jj)}^{s_1}(t_{n-2})d_{ii,jj}(e_1),
\end{aligned} \tag{36.76}$$

i.e.

$$\partial_{w_q^{s_1}}\mathcal{D}_{e_1,ii}^{1,b} = 0. \tag{36.77}$$

From equation (36.48) we have

$$
\begin{aligned}
\partial_{w_q^{s_1}} \mathcal{D}^{1,c}_{e_1,ii} =& -\frac{2\Delta_t Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} c^s_{jj,kk,ii,t_r}(e_1) \partial_{w_q^{s_1}} u^{s_1}_{l_1^1(e_1,kk)} \\
& - \frac{2\Delta_t Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} c^s_{jj,kk,ii,t_z}(e_1) \partial_{w_q^{s_1}} w^{s_1}_{l_1^1(e_1,kk)} \\
& + Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)} c^s_{jj,kk,ii,t_r}(e_1) \\
& - \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{jj,kk,ii,t_r}(e_1) \\
& + \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{jj,kk,ii,t_r}(e_1) \\
& + Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)} c^s_{jj,kk,ii,t_z}(e_1) \\
& - \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{jj,kk,ii,t_z}(e_1) \\
& + \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{jj,kk,ii,t_z}(e_1) \\
& + Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)} c^s_{ii,jj,kk,t_r}(e_1) \\
& - \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{ii,jj,kk,t_r}(e_1) \\
& + \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{ii,jj,kk,t_r}(e_1) \\
& + Tg \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)} c^s_{ii,jj,kk,t_z}(e_1) \\
& - \frac{4Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-1}) c^s_{ii,jj,kk,t_z}(e_1) \\
& + \frac{Tg}{3} \sum_{jj=1}^{n_v^{1,e_1}} \partial_{w_q^{s_1}} \rho^{s_1}_{l_1^1(e_1,jj)} \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-2}) c^s_{ii,jj,kk,t_z}(e_1).
\end{aligned}
\tag{36.78}
$$

i.e.

$$\partial_{w_q^{s_1}} \mathcal{D}^{1,c}_{e_1,ii} = \sum_{\substack{jj=1\\ q=l_1^1(e_1,kk)}}^{n_v^{1,e_1}} -\frac{2\Delta_t Tg}{3} \rho^{s_1}_{l_1^1(e_1,jj)} c^s_{jj,kk,ii,t_z}(e_1). \tag{36.79}$$

36.1.4. Derivatives of $\mathcal{D}_i^1$ with respect to θ_c

Using equations (36.49) we have

$$\begin{aligned}
\partial_{\theta_c}\mathcal{D}_i^1 = & -\frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}u_c^{s_1}\partial_{\theta_c}m_r^1(c) - \frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}w_c^{s_1}\partial_{\theta_c}m_z^1(c) \\
& + Tg\,\delta_{i,c}\rho_c^{s_1}r_c^c\partial_{\theta_c}m_r^1(c) - \frac{4Tg}{3}\delta_{i,c}\rho_c^{s_1}r_c^c(t_{n-1})\partial_{\theta_c}m_r^1(c) \\
& + \frac{Tg}{3}\delta_{i,c}\rho_c^{s_1}r_c^c(t_{n-2})\partial_{\theta_c}m_r^1(c) \\
& + Tg\,\delta_{i,c}\rho_c^{s_1}z_c^c\partial_{\theta_c}m_z^1(c) - \frac{4Tg}{3}\delta_{i,c}\rho_c^{s_1}z^c(t_{n-1})\partial_{\theta_c}m_z^1(c) \\
& + \frac{Tg}{3}\delta_{i,c}\rho_c^{s_1}z^c(t_{n-2})\partial_{\theta_c}m_z^1(c) \\
& + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,a} + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,c}.
\end{aligned} \tag{36.80}$$

i.e.

$$\begin{aligned}
\partial_{\theta_c}\mathcal{D}_i^1 = & \frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}\left[u_c^{s_1}\partial_{\theta_c}m_r^1(c) + w_c^{s_1}\partial_{\theta_c}m_z^1(c)\right] \\
& - Tg\,\delta_{i,c}\rho_c^{s_1}\partial_{\theta_c}m_r^1(c)\left[r_c^c - \frac{4}{3}r_c^c(t_{n-1}) + \frac{1}{3}r_c^c(t_{n-2})\right] \\
& - Tg\,\delta_{i,c}\rho_c^{s_1}\partial_{\theta_c}m_z^1(c)\left[z_c^c - \frac{4}{3}z^c(t_{n-1}) + \frac{1}{3}z^c(t_{n-2})\right] \\
& + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,a} + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,c},
\end{aligned} \tag{36.81}$$

equivalently

$$\begin{aligned}
\partial_{\theta_c}\mathcal{D}_i^1 = & Tg\,\delta_{i,c}\rho_c^{s_1}\left\{\frac{2\Delta_t}{3}\left[u_c^{s_1}\partial_{\theta_c}m_r^1(c) + w_c^{s_1}\partial_{\theta_c}m_z^1(c)\right]\right. \\
& - \partial_{\theta_c}m_r^1(c)\left[r_c^c - \frac{4}{3}r_c^c(t_{n-1}) + \frac{1}{3}r_c^c(t_{n-2})\right] \\
& \left. - \partial_{\theta_c}m_z^1(c)\left[z_c^c - \frac{4}{3}z^c(t_{n-1}) + \frac{1}{3}z^c(t_{n-2})\right]\right\} \\
& + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,a} + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1 \\ i=l_1^1(e_1,ii)}}^{n_{\mathrm{el}}^1}\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,c}.
\end{aligned} \tag{36.82}$$

From equation (36.46) we have

$$\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,a} = -Dg\,\partial_{\theta_c}c_{ii}(e_1), \tag{36.83}$$

i.e.

$$\partial_{\theta_c}\mathcal{D}_{e_1,ii}^{1,a} = 0. \tag{36.84}$$

From equation (36.47) we have

$$\begin{aligned}\partial_{\theta_c}\mathcal{D}^{1,b}_{e_1,ii} = {} & \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\rho^{s_1}_j\partial_{\theta_c}c_{ii,jj}(e_1) \\ & + Tg\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\partial_{\theta_c}c_{ii,jj}(e_1) - \frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}(t_{n-1})\partial_{\theta_c}c_{ii,jj}(e_1) \\ & + \frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}(t_{n-2})\partial_{\theta_c}c_{ii,jj}(e_1),\end{aligned} \tag{36.85}$$

i.e.

$$\partial_{\theta_c}\mathcal{D}^{1,b}_{e_1,ii} = 0. \tag{36.86}$$

From equation (36.48) we have

$$\begin{aligned}\partial_{\theta_c}\mathcal{D}^{1,c}_{e_1,ii} = \sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}Tg\Bigg\{ & -\frac{2\Delta_t}{3}\sum_{kk=1}^{n_v^{1,e_1}}u^{s_1}_{l_1^1(e_1,kk)}\partial_{\theta_c}c^s_{jj,kk,ii,t_r}(e_1) \\ & - \frac{2\Delta_t}{3}\sum_{kk=1}^{n_v^{1,e_1}}w^{s_1}_{l_1^1(e_1,kk)}\partial_{\theta_c}c^s_{jj,kk,ii,t_z}(e_1) \\ + \sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}\partial_{\theta_c}c^s_{jj,kk,ii,t_r}(e_1) & - \frac{4}{3}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-1})\partial_{\theta_c}c^s_{jj,kk,ii,t_r}(e_1) \\ & + \frac{1}{3}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-2})\partial_{\theta_c}c^s_{jj,kk,ii,t_r}(e_1) \\ + \sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}\partial_{\theta_c}c^s_{jj,kk,ii,t_z}(e_1) & - \frac{4}{3}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-1})\partial_{\theta_c}c^s_{jj,kk,ii,t_z}(e_1) \\ & + \frac{1}{3}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-2})\partial_{\theta_c}c^s_{jj,kk,ii,t_z}(e_1) \\ + \sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}\partial_{\theta_c}c_{ii,jj,kk,t_r}(e_1) & - \frac{4}{3}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-1})\partial_{\theta_c}c_{ii,jj,kk,t_r}(e_1) \\ & + \frac{1}{3}\sum_{kk=1}^{n_v^{1,e_1}}r^c_{l_1(e_1,kk)}(t_{n-2})\partial_{\theta_c}c_{ii,jj,kk,t_r}(e_1) \\ + \sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}\partial_{\theta_c}c_{ii,jj,kk,t_z}(e_1) & - \frac{4}{3}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-1})\partial_{\theta_c}c_{ii,jj,kk,t_z}(e_1) \\ & + \frac{1}{3}\sum_{kk=1}^{n_v^{1,e_1}}z^c_{l_1(e_1,kk)}(t_{n-2})\partial_{\theta_c}c_{ii,jj,kk,t_z}(e_1)\Bigg\}.\end{aligned} \tag{36.87}$$

i.e.

$$\partial_{\theta_c}\mathcal{D}^{1,c}_{e_1,ii} = 0. \tag{36.88}$$

36.1.5. Derivatives of $\mathcal{D}_i^1$ with respect to h_q

Using equations (36.49) we have

$$\begin{aligned}
\partial_{h_q}\mathcal{D}_i^1 = &-\frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}u_c^{s_1}\partial_{h_q}m_r^1(c) - \frac{2\Delta_t Tg}{3}\delta_{i,c}\rho_c^{s_1}w_c^{s_1}\partial_{h_q}m_z^1(c)\\
&+ Tg\,\delta_{i,c}\rho_c^{s_1}m_r^1(c)\partial_{h_q}r_c^c - \frac{4Tg}{3}\delta_{i,c}\rho_c^{s_1}r_c^c(t_{n-1})\partial_{h_q}m_r^1(c)\\
&+ \frac{Tg}{3}\delta_{i,c}\rho_c^{s_1}r_c^c(t_{n-2})\partial_{h_q}m_r^1(c)\\
&+ Tg\,\delta_{i,c}\rho_c^{s_1}m_z^1(c)\partial_{h_q}z_c^c - \frac{4Tg}{3}\delta_{i,c}\rho_c^{s_1}z^c(t_{n-1})\partial_{h_q}m_z^1(c)\\
&+ \frac{Tg}{3}\delta_{i,c}\rho_c^{s_1}z^c(t_{n-2})\partial_{h_q}m_z^1(c)\\
&+ \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\mathrm{el}}^1}\partial_{h_{S_1(e_1,qq)}}\mathcal{D}_{e_1,ii}^{1,a} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\mathrm{el}}^1}\partial_{h_{S_1(e_1,qq)}}\mathcal{D}_{e_1,ii}^{1,b}\\
&+ \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\mathrm{el}}^1}\partial_{h_{S_1(e_1,qq)}}\mathcal{D}_{e_1,ii}^{1,c}.
\end{aligned} \tag{36.89}$$

i.e.

$$\begin{aligned}
\partial_{h_q}\mathcal{D}_i^1 = &\,Tg\,\delta_{i,c}\rho_c^{s_1}m_r^1(c)\partial_{h_q}r_c^c + Tg\,\delta_{i,c}\rho_c^{s_1}m_z^1(c)\partial_{h_q}z_c^c + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\mathrm{el}}^1}\partial_{h_{S_1(e_1,qq)}}\mathcal{D}_{e_1,ii}^{1,a}\\
&+ \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\mathrm{el}}^1}\partial_{h_{S_1(e_1,qq)}}\mathcal{D}_{e_1,ii}^{1,b} + \sum_{\substack{e_1=1\\ i=l_1^1(e_1,ii)\\ q=S_1(e_1,qq)}}^{n_{\mathrm{el}}^1}\partial_{h_{S_1(e_1,qq)}}\mathcal{D}_{e_1,ii}^{1,c}.
\end{aligned} \tag{36.90}$$

Observation: The derivatives of $\boldsymbol{m}^1$ with respect to the spine lengths here are zero, however in the general case of a smooth but non-planar surface, this would depend on the length of the first spine.

From equation (36.46) we have

$$\partial_{h_{S_1(e_1,qq)}}\mathcal{D}_{e_1,ii}^{1,a} = -\frac{2\Delta_t Dg}{3}\partial_{h_{S_1(e_1,qq)}}c_{ii}(e_1). \tag{36.91}$$

From equation (36.47) we have

$$\partial_{h_{S_1(e_1,qq)}}\mathcal{D}^{1,b}_{e_1,ii} = \frac{2\Delta_t}{3}\sum_{j=1}^{n_v}\rho_j^{s_1}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj}(e_1) \tag{36.92}$$

$$+ Tg\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj}(e_1)$$

$$-\frac{4Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}(t_{n-1})\partial_{h_{S_1(e_1,qq)}}c_{ii,jj}(e_1)$$

$$+\frac{Tg}{3}\sum_{jj=1}^{n_v^{1,e_1}}\rho^{s_1}_{l_1^1(e_1,jj)}(t_{n-2})\partial_{h_{S_1(e_1,qq)}}c_{ii,jj}(e_1),$$

i.e.

$$\partial_{h_{S_1(e_1,qq)}}\mathcal{D}^{1,b}_{e_1,ii} = \sum_{jj=1}^{n_v^{1,e_1}}\partial_{h_{S_1(e_1,qq)}}c_{ii,jj}(e_1)\left\{\frac{2\Delta_t}{3}\rho^{s_1}_{l_1^1(e_1,jj)}\right. \tag{36.93}$$

$$\left. + Tg\left[\rho^{s_1}_{l_1^1(e_1,jj)} - \frac{4}{3}\rho^{s_1}_{l_1^1(e_1,jj)}(t_{n-1}) + \frac{1}{3}\rho^{s_1}_{l_1^1(e_1,jj)}(t_{n-2})\right]\right\}.$$

From equation (36.48) we have

$$
\begin{aligned}
\partial_{h_{S_1(e_1,qq)}}\mathcal{D}^{1,c}_{e_1,ii} = \sum_{jj=1}^{n_v^{1,e_1}} Tg\,\rho^{s_1}_{l_1^1(e_1,jj)} \Bigg\{ & -\frac{2\Delta_t}{3}\sum_{kk=1}^{n_v^{1,e_1}} u^{s_1}_{l_1^1(e_1,kk)}\partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_r}(e_1) \\
& - \frac{2\Delta_t}{3}\sum_{kk=1}^{n_v^{1,e_1}} c^s_{jj,kk,ii,t_z}(e_1)\partial_{h_{S_1(e_1,qq)}} w^{s_1}_{l_1^1(e_1,kk)} \\
& + \sum_{kk=1}^{n_v^{1,e_1}} c^s_{jj,kk,ii,t_r}(e_1)\partial_{h_{S_1(e_1,qq)}} r^c_{l_1(e_1,kk)} + \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}\partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_r}(e_1) \\
& - \frac{4}{3}\sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-1})\partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_r}(e_1) \\
& + \frac{1}{3}\sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-2})\partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_r}(e_1) \\
& + \sum_{kk=1}^{n_v^{1,e_1}} c^s_{jj,kk,ii,t_z}(e_1)\partial_{h_{S_1(e_1,qq)}} z^c_{l_1(e_1,kk)} + \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}\partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_z}(e_1) \\
& - \frac{4}{3}\sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-1})\partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_z}(e_1) \\
& + \frac{1}{3}\sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-2})\partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_z}(e_1) \\
& + \sum_{kk=1}^{n_v^{1,e_1}} c^s_{ii,jj,kk,t_r}(e_1)\partial_{h_{S_1(e_1,qq)}} r^c_{l_1(e_1,kk)} + \sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}\partial_{h_{S_1(e_1,qq)}} c^s_{ii,jj,kk,t_r}(e_1) \\
& - \frac{4}{3}\sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-1})\partial_{h_{S_1(e_1,qq)}} c^s_{ii,jj,kk,t_r}(e_1) \\
& + \frac{1}{3}\sum_{kk=1}^{n_v^{1,e_1}} r^c_{l_1(e_1,kk)}(t_{n-2})\partial_{h_{S_1(e_1,qq)}} c^s_{ii,jj,kk,t_r}(e_1) \\
& + \sum_{kk=1}^{n_v^{1,e_1}} c^s_{ii,jj,kk,t_z}(e_1)\partial_{h_{S_1(e_1,qq)}} z^c_{l_1(e_1,kk)} + \sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}\partial_{h_{S_1(e_1,qq)}} c^s_{ii,jj,kk,t_z}(e_1) \\
& - \frac{4}{3}\sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-1})\partial_{h_{S_1(e_1,qq)}} c^s_{ii,jj,kk,t_z}(e_1) \\
& + \frac{1}{3}\sum_{kk=1}^{n_v^{1,e_1}} z^c_{l_1(e_1,kk)}(t_{n-2})\partial_{h_{S_1(e_1,qq)}} c^s_{ii,jj,kk,t_z}(e_1) \Bigg\}.
\end{aligned}
\tag{36.94}
$$

i.e.

$$
\begin{aligned}
\partial_{h_{S_1(e_1,qq)}} \mathcal{D}^{1,c}_{e_1,ii} = \sum_{jj=1}^{n_v^{1,e_1}} Tg\, \rho^{s_1}_{l_1^1(e_1,jj)} \Bigg\{ & -\frac{2\Delta_t}{3} \sum_{kk=1}^{n_v^{1,e_1}} \Big[u^{s_1}_{l_1^1(e_1,kk)} \partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_r}(e_1) \\
& + w^{s_1}_{l_1^1(e_1,kk)} \partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_z}(e_1) \Big] \\
& + \sum_{kk=1}^{n_v^{1,e_1}} \Big(c^s_{jj,kk,ii,t_r}(e_1) \partial_{h_{S_1(e_1,qq)}} r^c_{l_1(e_1,kk)} \\
& + \partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_r}(e_1) \Big[r^c_{l_1(e_1,kk)} - \frac{4}{3} r^c_{l_1(e_1,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l_1(e_1,kk)}(t_{n-2}) \Big] \Big) \\
& + \sum_{kk=1}^{n_v^{1,e_1}} \Big(c^s_{jj,kk,ii,t_z}(e_1) \partial_{h_{S_1(e_1,qq)}} z^c_{l_1(e_1,kk)} \\
& + \partial_{h_{S_1(e_1,qq)}} c^s_{jj,kk,ii,t_z}(e_1) \Big[z^c_{l_1(e_1,kk)} - \frac{4}{3} z^c_{l_1(e_1,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l_1(e_1,kk)}(t_{n-2}) \Big] \Big) \\
& + \sum_{kk=1}^{n_v^{1,e_1}} \Big(c^s_{ii,jj,kk,t_r}(e_1) \partial_{h_{S_1(e_1,qq)}} r^c_{l_1(e_1,kk)} \\
& + \partial_{h_{S_1(e_1,qq)}} c^s_{ii,jj,kk,t_r}(e_1) \Big[r^c_{l_1(e_1,kk)} - \frac{4}{3} r^c_{l_1(e_1,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l_1(e_1,kk)}(t_{n-2}) \Big] \Big) \\
& + \sum_{kk=1}^{n_v^{1,e_1}} \Big(c^s_{ii,jj,kk,t_z}(e_1) \partial_{h_{S_1(e_1,qq)}} z^c_{l_1(e_1,kk)} \\
& + \partial_{h_{S_1(e_1,qq)}} c^s_{ii,jj,kk,t_z}(e_1) \Big[z^c_{l_1(e_1,kk)} - \frac{4}{3} z^c_{l_1(e_1,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l_1(e_1,kk)}(t_{n-2}) \Big] \Big) \Bigg\}.
\end{aligned}
\tag{36.95}
$$

37. The $\sigma - \rho$ state equation on boundary 1 (TDC1) in the near field

These residuals are identical to those in the far-field, so the full derivation from section 16 will not be repeated here.

38. Young's equation (CAC)

We recall equation (23.26) which states the contact angle condition CAC, given by

$$\sigma_c^1 \cos\theta_c + \sigma_c^2 = So\,. \tag{38.1}$$

The residual equation associated to it is

$$\mathcal{Y} = \sigma_c^1 \cos\theta_c + \sigma_c^2 - So\,, \tag{38.2}$$

where the sub-index c indicates that the function is evaluated at the contact line.

38.1. Jacobian terms

Here we find the derivative of $\mathcal{Y}$ with respect to σ^1, σ^2 and θ_c.

38.1.1. Derivatives of $\mathcal{Y}$ with respect to σ_q^1

From equation (38.2) we have

$$\partial_{\sigma_q^1}\mathcal{Y} = \partial_{\sigma_q^1}\sigma_c^1 \cos\theta_c + \partial_{\sigma_q^1}\sigma_c^2 - \partial_{\sigma_q^1} So\,, \tag{38.3}$$

i.e.

$$\partial_{\sigma_q^1}\mathcal{Y} = \delta_{q,c}\cos\theta_c. \tag{38.4}$$

38.1.2. Derivatives of $\mathcal{Y}$ with respect to σ_q^2

From equation (38.2) we have

$$\partial_{\sigma_q^2}\mathcal{Y} = \partial_{\sigma_q^2}\sigma_c^1 \cos\theta_c + \partial_{\sigma_q^2}\sigma_c^2 - \partial_{\sigma_q^2} So\,, \tag{38.5}$$

i.e.

$$\partial_{\sigma_q^2}\mathcal{Y} = \delta_{q,c}. \tag{38.6}$$

38.1.3. Derivatives of $\mathcal{Y}$ with respect to θ_c

From equation (38.2) we have

$$\partial_{\theta_c}\mathcal{Y} = \partial_{\theta_c}\sigma_c^1 \cos\theta_c + \partial_{\theta_c}\sigma_c^2 - \partial_{\theta_c} So\,, \tag{38.7}$$

i.e.

$$\partial_{\sigma_q^2}\mathcal{Y} = -\delta_{q,c}\sigma_c^1 \sin\theta_c. \tag{38.8}$$

39. Mass balance at contact line (MBC)

From equation (2.58), we have

$$\rho^{s_1}\left(\boldsymbol{v}^{s_1}-\boldsymbol{c}\right)\cdot\boldsymbol{m}^1+\rho^{s_2}\left(\boldsymbol{v}^{s_2}-\boldsymbol{c}\right)\cdot\boldsymbol{m}^2=0., \tag{39.1}$$

i.e.

$$\rho^{s_1}\boldsymbol{v}^{s_1}\cdot\boldsymbol{m}^1-\rho^{s_1}\boldsymbol{c}\cdot\boldsymbol{m}^1+\rho^{s_2}\boldsymbol{v}^{s_2}\cdot\boldsymbol{m}^2-\rho^{s_2}\boldsymbol{c}\cdot\boldsymbol{m}^2=0, \tag{39.2}$$

or, equivalently,

$$\begin{aligned}&\rho^{s_1}\left(u^{s_1}m_r^1+w^{s_1}m_z^1\right)-\rho^{s_1}\left(m_r^1\partial_t r^c+m_z^1\partial_t z^c\right)\\&\quad+\rho^{s_2}\left(u^{s_2}m_r^2+w^{s_2}m_z^2\right)-\rho^{s_2}\left(m_r^2\partial_t r^c+m_z^2\partial_t z^c\right)=0.\end{aligned} \tag{39.3}$$

Re-writing we have

$$\begin{aligned}B=&\,\rho^{s_1}u^{s_1}m_r^1+\rho^{s_1}w^{s_1}m_z^1-\rho^{s_1}m_r^1\partial_t r^c-\rho^{s_1}m_z^1\partial_t z^c\\&+\rho^{s_2}u^{s_2}m_r^2+\rho^{s_2}w^{s_2}m_z^2-\rho^{s_2}m_r^2\partial_t r^c-\rho^{s_2}m_z^2\partial_t z^c.\end{aligned} \tag{39.4}$$

We now recall the BDF2 approximation of $\partial_t r^c$ and $\partial_t z^c$, and we substitute them above obtaining

$$\begin{aligned}\mathfrak{B}=&\,\rho^{s_1}u^{s_1}m_r^1+\rho^{s_1}w^{s_1}m_z^1\\&-\rho^{s_1}m_r^1\frac{3r^c-4r^c(t_n-1)+r^c(t_n-2)}{2\Delta_t}-\rho^{s_1}m_z^1\frac{3z^c-4z^c(t_n-1)+z^c(t_n-2)}{2\Delta_t}\\&+\rho^{s_2}u^{s_2}m_r^2+\rho^{s_2}w^{s_2}m_z^2\\&-\rho^{s_2}m_r^2\frac{3r^c-4r^c(t_n-1)+r^c(t_n-2)}{2\Delta_t}-\rho^{s_2}m_z^2\frac{3z^c-4z^c(t_n-1)+z^c(t_n-2)}{2\Delta_t}.\end{aligned} \tag{39.5}$$

Multiplying this equation by $2\Delta_t/3$, we have

$$\begin{aligned}\mathcal{B}=&\,\frac{2\Delta_t}{3}\rho^{s_1}u^{s_1}m_r^1+\frac{2\Delta_t}{3}\rho^{s_1}w^{s_1}m_z^1\\&-\rho^{s_1}m_r^1r^c+\frac{4}{3}\rho^{s_1}m_r^1r^c(t_n-1)-\frac{1}{3}\rho^{s_1}m_r^1r^c(t_n-2)\\&-\rho^{s_1}m_z^1z^c+\frac{4}{3}\rho^{s_1}m_z^1z^c(t_n-1)-\frac{1}{3}\rho^{s_1}m_z^1z^c(t_n-2)\\&+\frac{2\Delta_t}{3}\rho^{s_2}u^{s_2}m_r^2+\frac{2\Delta_t}{3}\rho^{s_2}w^{s_2}m_z^2\\&-\rho^{s_2}m_r^2r^c+\frac{4}{3}\rho^{s_2}m_r^2r^c(t_n-1)-\frac{1}{3}\rho^{s_2}m_r^2r^c(t_n-2)\\&-\rho^{s_2}m_z^2z^c+\frac{4}{3}\rho^{s_2}m_z^2z^c(t_n-1)-\frac{1}{3}\rho^{s_2}m_z^2z^c(t_n-2).\end{aligned} \tag{39.6}$$

Grouping terms we have

$$
\begin{aligned}
\mathcal{B} = \frac{2\Delta_t}{3}\rho^{s_1}\left[u^{s_1}m_r^1 + w^{s_1}m_z^1\right] \\
&- \rho^{s_1}m_r^1\left[r^c - \frac{4}{3}r^c(t_n-1) + \frac{1}{3}r^c(t_n-2)\right] \\
&- \rho^{s_1}m_z^1\left[z^c - \frac{4}{3}z^c(t_n-1) + \frac{1}{3}z^c(t_n-2)\right] \\
&+ \frac{2\Delta_t}{3}\rho^{s_2}\left[u^{s_2}m_r^2 + w^{s_2}m_z^2\right] \\
&- \rho^{s_2}m_r^2\left[r^c - \frac{4}{3}r^c(t_n-1) + \frac{1}{3}r^c(t_n-2)\right] \\
&- \rho^{s_2}m_z^2\left[z^c - \frac{4}{3}z^c(t_n-1) + \frac{1}{3}z^c(t_n-2)\right].
\end{aligned}
\tag{39.7}
$$

39.1. **Jacobian terms**

Here we find the derivatives of the mass balance equation with respect to u^{s_2}, w^{s_2}, ρ^{s_2}, u^{s_1}, w^{s_1}, ρ^{s_1}, h_q and θ_c.

39.1.1. **Derivatives with respect to $u_q^{s_2}$**

From equation (39.6) we have

$$\begin{aligned}\partial_{u_q^{s_2}}\mathcal{B} = {}& \frac{2\Delta_t}{3}\partial_{u_q^{s_2}}\rho^{s_1}u^{s_1}m_r^1 + \frac{2\Delta_t}{3}\partial_{u_q^{s_2}}\rho^{s_1}w^{s_1}m_z^1 \\ & - \partial_{u_q^{s_2}}\rho^{s_1}m_r^1 r^c + \frac{4}{3}\partial_{u_q^{s_2}}\rho^{s_1}m_r^1 r^c(t_n-1) - \frac{1}{3}\partial_{u_q^{s_2}}\rho^{s_1}m_r^1 r^c(t_n-2) \\ & - \partial_{u_q^{s_2}}\rho^{s_1}m_z^1 z^c + \frac{4}{3}\partial_{u_q^{s_2}}\rho^{s_1}m_z^1 z^c(t_n-1) - \frac{1}{3}\partial_{u_q^{s_2}}\rho^{s_1}m_z^1 z^c(t_n-2) \\ & + \frac{2\Delta_t}{3}\rho^{s_2}m_r^2\partial_{u_q^{s_2}}u^{s_2} + \frac{2\Delta_t}{3}\partial_{u_q^{s_2}}\rho^{s_2}w^{s_2}m_z^2 \\ & - \partial_{u_q^{s_2}}\rho^{s_2}m_r^2 r^c + \frac{4}{3}\partial_{u_q^{s_2}}\rho^{s_2}m_r^2 r^c(t_n-1) - \frac{1}{3}\partial_{u_q^{s_2}}\rho^{s_2}m_r^2 r^c(t_n-2) \\ & - \partial_{u_q^{s_2}}\rho^{s_2}m_z^2 z^c + \frac{4}{3}\partial_{u_q^{s_2}}\rho^{s_2}m_z^2 z^c(t_n-1) - \frac{1}{3}\partial_{u_q^{s_2}}\rho^{s_2}m_z^2 z^c(t_n-2)\end{aligned} \tag{39.8}$$

i.e.

$$\partial_{u_q^{s_2}}\mathcal{B} = \frac{2\Delta_t}{3}m_r^2\rho^{s_2}\delta_{q,c_2}. \tag{39.9}$$

39.1.2. **Derivatives with respect to** $w_q^{s_2}$

From equation (39.6) we have

$$\begin{aligned}\partial_{w_q^{s_2}}\mathcal{B} = {} & \frac{2\Delta_t}{3}\partial_{w_q^{s_2}}\rho^{s_1}u^{s_1}m_r^1 + \frac{2\Delta_t}{3}\partial_{w_q^{s_2}}\rho^{s_1}w^{s_1}m_z^1 \\ & - \partial_{w_q^{s_2}}\rho^{s_1}m_r^1r^c + \frac{4}{3}\partial_{w_q^{s_2}}\rho^{s_1}m_r^1r^c(t_n-1) - \frac{1}{3}\partial_{w_q^{s_2}}\rho^{s_1}m_r^1r^c(t_n-2) \\ & - \partial_{w_q^{s_2}}\rho^{s_1}m_z^1z^c + \frac{4}{3}\partial_{w_q^{s_2}}\rho^{s_1}m_z^1z^c(t_n-1) - \frac{1}{3}\partial_{w_q^{s_2}}\rho^{s_1}m_z^1z^c(t_n-2) \\ & + \frac{2\Delta_t}{3}\partial_{w_q^{s_2}}\rho^{s_2}u^{s_2}m_r^2 + \frac{2\Delta_t}{3}\rho^{s_2}m_z^2\partial_{w_q^{s_2}}w^{s_2} \\ & - \partial_{w_q^{s_2}}\rho^{s_2}m_r^2r^c + \frac{4}{3}\partial_{w_q^{s_2}}\rho^{s_2}m_r^2r^c(t_n-1) - \frac{1}{3}\partial_{w_q^{s_2}}\rho^{s_2}m_r^2r^c(t_n-2) \\ & - \partial_{w_q^{s_2}}\rho^{s_2}m_z^2z^c + \frac{4}{3}\partial_{w_q^{s_2}}\rho^{s_2}m_z^2z^c(t_n-1) - \frac{1}{3}\partial_{w_q^{s_2}}\rho^{s_2}m_z^2z^c(t_n-2),\end{aligned} \tag{39.10}$$

i.e.

$$\partial_{w_q^{s_2}}\mathcal{B} = \frac{2\Delta_t}{3}m_z^2\rho^{s_2}\delta_{q,c_2}. \tag{39.11}$$

39.1.3. Derivatives with respect to $\rho_q^{s_2}$

From equation (39.6) we have

$$\begin{aligned}\partial_{\rho_q^{s_2}}\mathcal{B} = {} & \frac{2\Delta_t}{3}\partial_{\rho_q^{s_2}}\rho^{s_1}u^{s_1}m_r^1 + \frac{2\Delta_t}{3}\partial_{\rho_q^{s_2}}\rho^{s_1}w^{s_1}m_z^1 \\ & - \partial_{\rho_q^{s_2}}\rho^{s_1}m_r^1r^c + \frac{4}{3}\partial_{\rho_q^{s_2}}\rho^{s_1}m_r^1r^c(t_n-1) - \frac{1}{3}\partial_{\rho_q^{s_2}}\rho^{s_1}m_r^1r^c(t_n-2) \\ & - \partial_{\rho_q^{s_2}}\rho^{s_1}m_z^1z^c + \frac{4}{3}\partial_{\rho_q^{s_2}}\rho^{s_1}m_z^1z^c(t_n-1) - \frac{1}{3}\partial_{\rho_q^{s_2}}\rho^{s_1}m_z^1z^c(t_n-2) \\ & + \frac{2\Delta_t}{3}u^{s_2}m_r^2\partial_{\rho_q^{s_2}}\rho^{s_2} + \frac{2\Delta_t}{3}m_z^2w^{s_2}\partial_{\rho_q^{s_2}}\rho^{s_2} \\ & - m_r^2r^c\partial_{\rho_q^{s_2}}\rho^{s_2} + \frac{4}{3}m_r^2r^c(t_n-1)\partial_{\rho_q^{s_2}}\rho^{s_2} - \frac{1}{3}m_r^2r^c(t_n-2)\partial_{\rho_q^{s_2}}\rho^{s_2} \\ & - m_z^2z^c\partial_{\rho_q^{s_2}}\rho^{s_2} + \frac{4}{3}m_z^2z^c(t_n-1)\partial_{\rho_q^{s_2}}\rho^{s_2} - \frac{1}{3}m_z^2z^c(t_n-2)\partial_{\rho_q^{s_2}}\rho^{s_2},\end{aligned} \tag{39.12}$$

i.e.

$$\begin{aligned}\partial_{\rho_q^{s_2}}\mathcal{B} = \delta_{q,c_2}\Bigg\{ & \frac{2\Delta_t}{3}u^{s_2}m_r^2 + \frac{2\Delta_t}{3}m_z^2w^{s_2} \\ & - m_r^2r^c + \frac{4}{3}m_r^2r^c(t_n-1) - \frac{1}{3}m_r^2r^c(t_n-2) \\ & - m_z^2z^c + \frac{4}{3}m_z^2z^c(t_n-1) - \frac{1}{3}m_z^2z^c(t_n-2)\Bigg\},\end{aligned} \tag{39.13}$$

or, equivalently,

$$\begin{aligned}\partial_{\rho_q^{s_2}}\mathcal{B} = \delta_{q,c_2}\Bigg\{ & \frac{2\Delta_t}{3}\left[u^{s_2}m_r^2 + w^{s_2}m_z^2\right] \\ & - m_r^2\left[r^c - \frac{4}{3}r^c(t_n-1) + \frac{1}{3}r^c(t_n-2)\right] \\ & - m_z^2\left[z^c - \frac{4}{3}z^c(t_n-1) + \frac{1}{3}z^c(t_n-2)\right]\Bigg\}.\end{aligned} \tag{39.14}$$

39.1.4. **Derivatives with respect to** $u_q^{s_1}$

From equation (39.4) we have

$$
\begin{aligned}
\partial_{u_q^{s_1}}\mathcal{B} = {} & \frac{2\Delta_t}{3}\rho^{s_1} m_r^1 \partial_{u_q^{s_1}} u^{s_1} + \frac{2\Delta_t}{3}\partial_{u_q^{s_1}}\rho^{s_1} w^{s_1} m_z^1 \\
& - \partial_{u_q^{s_1}}\rho^{s_1} m_r^1 r^c + \frac{4}{3}\partial_{u_q^{s_1}}\rho^{s_1} m_r^1 r^c(t_n-1) - \frac{1}{3}\partial_{u_q^{s_1}}\rho^{s_1} m_r^1 r^c(t_n-2) \\
& - \partial_{u_q^{s_1}}\rho^{s_1} m_z^1 z^c + \frac{4}{3}\partial_{u_q^{s_1}}\rho^{s_1} m_z^1 z^c(t_n-1) - \frac{1}{3}\partial_{u_q^{s_1}}\rho^{s_1} m_z^1 z^c(t_n-2) \\
& + \frac{2\Delta_t}{3}\partial_{u_q^{s_1}}\rho^{s_2} u^{s_2} m_r^2 + \frac{2\Delta_t}{3}\partial_{u_q^{s_1}}\rho^{s_2} m_z^2 w^{s_2} \\
& - \partial_{u_q^{s_1}}\rho^{s_2} m_r^2 r^c + \frac{4}{3}\partial_{u_q^{s_1}}\rho^{s_2} m_r^2 r^c(t_n-1) - \frac{1}{3}\partial_{u_q^{s_1}}\rho^{s_2} m_r^2 r^c(t_n-2) \\
& - \partial_{u_q^{s_1}}\rho^{s_2} m_z^2 z^c + \frac{4}{3}\partial_{u_q^{s_1}}\rho^{s_2} m_z^2 z^c(t_n-1) - \frac{1}{3}\partial_{u_q^{s_1}}\rho^{s_2} m_z^2 z^c(t_n-2),
\end{aligned}
\tag{39.15}
$$

i.e.

$$
\partial_{u_q^{s_1}}\mathcal{B} = \frac{2\Delta_t}{3} m_r^1 \rho^{s_1}\delta_{q,c_1}. \tag{39.16}
$$

39.1.5. Derivatives with respect to $w_q^{s_1}$

From equation (39.6) we have

$$\begin{aligned}
\partial_{w_q^{s_1}}\mathcal{B} = {} & \frac{2\Delta_t}{3}\partial_{w_q^{s_1}}\rho^{s_1}m_r^1u^{s_1} + \frac{2\Delta_t}{3}\rho^{s_1}m_z^1\partial_{w_q^{s_1}}w^{s_1} \\
& - \partial_{w_q^{s_1}}\rho^{s_1}m_r^1r^c + \frac{4}{3}\partial_{w_q^{s_1}}\rho^{s_1}m_r^1r^c(t_n-1) - \frac{1}{3}\partial_{w_q^{s_1}}\rho^{s_1}m_r^1r^c(t_n-2) \\
& - \partial_{w_q^{s_1}}\rho^{s_1}m_z^1z^c + \frac{4}{3}\partial_{w_q^{s_1}}\rho^{s_1}m_z^1z^c(t_n-1) - \frac{1}{3}\partial_{w_q^{s_1}}\rho^{s_1}m_z^1z^c(t_n-2) \\
& + \frac{2\Delta_t}{3}\partial_{w_q^{s_1}}\rho^{s_2}u^{s_2}m_r^2 + \frac{2\Delta_t}{3}\partial_{w_q^{s_1}}\rho^{s_2}m_z^2w^{s_2} \\
& - \partial_{w_q^{s_1}}\rho^{s_2}m_r^2r^c + \frac{4}{3}\partial_{w_q^{s_1}}\rho^{s_2}m_r^2r^c(t_n-1) - \frac{1}{3}\partial_{w_q^{s_1}}\rho^{s_2}m_r^2r^c(t_n-2) \\
& - \partial_{w_q^{s_1}}\rho^{s_2}m_z^2z^c + \frac{4}{3}\partial_{w_q^{s_1}}\rho^{s_2}m_z^2z^c(t_n-1) - \frac{1}{3}\partial_{w_q^{s_1}}\rho^{s_2}m_z^2z^c(t_n-2),
\end{aligned} \tag{39.17}$$

i.e.

$$\partial_{w_q^{s_1}}\mathcal{B} = \frac{2\Delta_t}{3}m_z^1\rho^{s_1}\delta_{q,c_1}. \tag{39.18}$$

39.1.6. **Derivatives with respect to $\rho_q^{s_1}$**

From equation (39.6) we have

$$\begin{aligned}
\partial_{\rho_q^{s_1}}\mathcal{B} = {} & \frac{2\Delta_t}{3} m_r^1 u^{s_1}\partial_{\rho_q^{s_1}}\rho^{s_1} + \frac{2\Delta_t}{3} m_z^1 w^{s_1}\partial_{\rho_q^{s_1}}\rho^{s_1} \\
& - m_r^1 r^c \partial_{\rho_q^{s_1}}\rho^{s_1} + \frac{4}{3} m_r^1 r^c(t_n-1)\partial_{\rho_q^{s_1}}\rho^{s_1} - \frac{1}{3} m_r^1 r^c(t_n-2)\partial_{\rho_q^{s_1}}\rho^{s_1} \\
& - m_z^1 z^c \partial_{\rho_q^{s_1}}\rho^{s_1} + \frac{4}{3} m_z^1 z^c(t_n-1)\partial_{\rho_q^{s_1}}\rho^{s_1} - \frac{1}{3} m_z^1 z^c(t_n-2)\partial_{\rho_q^{s_1}}\rho^{s_1} \\
& + \frac{2\Delta_t}{3}\partial_{\rho_q^{s_1}}\rho^{s_2} u^{s_2} m_r^2 + \frac{2\Delta_t}{3}\partial_{\rho_q^{s_1}}\rho^{s_2} m_z^2 w^{s_2} \\
& - \partial_{\rho_q^{s_1}}\rho^{s_2} m_r^2 r^c + \frac{4}{3}\partial_{\rho_q^{s_1}}\rho^{s_2} m_r^2 r^c(t_n-1) - \frac{1}{3}\partial_{\rho_q^{s_1}}\rho^{s_2} m_r^2 r^c(t_n-2) \\
& - \partial_{\rho_q^{s_1}}\rho^{s_2} m_z^2 z^c + \frac{4}{3}\partial_{\rho_q^{s_1}}\rho^{s_2} m_z^2 z^c(t_n-1) - \frac{1}{3}\partial_{\rho_q^{s_1}}\rho^{s_2} m_z^2 z^c(t_n-2),
\end{aligned} \tag{39.19}$$

i.e.

$$\begin{aligned}
\partial_{\rho_q^{s_1}}\mathcal{B} = \delta_{q,c_1}\Bigg\{ & \frac{2\Delta_t}{3} m_r^1 u^{s_1} + \frac{2\Delta_t}{3} m_z^1 w^{s_1} \\
& - m_r^1 r^c + \frac{4}{3} m_r^1 r^c(t_n-1) - \frac{1}{3} m_r^1 r^c(t_n-2) \\
& - m_z^1 z^c + \frac{4}{3} m_z^1 z^c(t_n-1) - \frac{1}{3} m_z^1 z^c(t_n-2)\Bigg\},
\end{aligned} \tag{39.20}$$

or, equivalently,

$$\begin{aligned}
\partial_{\rho_q^{s_1}}\mathcal{B} = \delta_{q,c_1}\Bigg\{ & \frac{2\Delta_t}{3}\left[m_r^1 u^{s_1} + m_z^1 w^{s_1}\right] \\
& - m_r^1\left[r^c + \frac{4}{3} r^c(t_n-1) - \frac{1}{3} r^c(t_n-2)\right] \\
& - m_z^1\left[z^c + \frac{4}{3} z^c(t_n-1) - \frac{1}{3} z^c(t_n-2)\right]\Bigg\}.
\end{aligned} \tag{39.21}$$

39.1.7. **Derivatives with respect to** h_q

From equation (39.6) we have

$$\begin{aligned}
\partial_{h_q}\mathcal{B} = {} & \frac{2\Delta_t}{3}\partial_{h_q} m_r^1 u^{s_1}\rho^{s_1} + \frac{2\Delta_t}{3}\partial_{h_q} m_z^1 w^{s_1}\rho^{s_1} \\
& - \rho^{s_1} m_r^1 \partial_{h_q} r^c + \frac{4}{3}\partial_{h_q} m_r^1 r^c(t_n-1)\rho^{s_1} - \frac{1}{3}\partial_{h_q} m_r^1 r^c(t_n-2)\rho^{s_1} \\
& - \rho^{s_1} m_z^1 \partial_{h_q} z^c + \frac{4}{3}\partial_{h_q} m_z^1 z^c(t_n-1)\rho^{s_1} - \frac{1}{3}\partial_{h_q} m_z^1 z^c(t_n-2)\rho^{s_1} \\
& + \frac{2\Delta_t}{3}\partial_{h_q}\rho^{s_2} u^{s_2} m_r^2 + \frac{2\Delta_t}{3}\partial_{h_q}\rho^{s_2} m_z^2 w^{s_2} \\
& - \rho^{s_2} m_r^2 \partial_{h_q} r^c + \frac{4}{3}\partial_{h_q}\rho^{s_2} m_r^2 r^c(t_n-1) - \frac{1}{3}\partial_{h_q}\rho^{s_2} m_r^2 r^c(t_n-2) \\
& - \rho^{s_2} m_z^2 \partial_{h_q} z^c + \frac{4}{3}\partial_{h_q}\rho^{s_2} m_z^2 z^c(t_n-1) - \frac{1}{3}\partial_{h_q}\rho^{s_2} m_z^2 z^c(t_n-2),
\end{aligned} \tag{39.22}$$

i.e.

$$\partial_{h_q}\mathcal{B} = -\delta_{q,c}\left\{\rho^{s_1}\left[m_r^1\partial_{h_q} r^c + m_z^1\partial_{h_q} z^c\right] + \rho^{s_2}\left[m_r^2\partial_{h_q} r^c + m_z^2\partial_{h_q} z^c\right]\right\}. \tag{39.23}$$

39.1.8. **Derivative with respect to θ_c**

From equation (39.6) we have

$$
\begin{aligned}
\partial_{\theta_c}\mathcal{B} = {} & \frac{2\Delta_t}{3}u^{s_1}\rho^{s_1}\partial_{\theta_c}m_r^1 + \frac{2\Delta_t}{3}w^{s_1}\rho^{s_1}\partial_{\theta_c}m_z^1 \\
& - \rho^{s_1}r^c\partial_{\theta_c}m_r^1 + \frac{4}{3}r^c(t_n-1)\rho^{s_1}\partial_{\theta_c}m_r^1 - \frac{1}{3}r^c(t_n-2)\rho^{s_1}\partial_{\theta_c}m_r^1 \\
& - \rho^{s_1}z^c\partial_{\theta_c}m_z^1 + \frac{4}{3}\rho^{s_1}z^c(t_n-1)\partial_{\theta_c}m_z^1 - \frac{1}{3}\rho^{s_1}z^c(t_n-2)\partial_{\theta_c}m_z^1 \\
& + \frac{2\Delta_t}{3}\rho^{s_2}u^{s_2}\partial_{\theta_c}m_r^2 + \frac{2\Delta_t}{3}\rho^{s_2}w^{s_2}\partial_{\theta_c}m_z^2 \\
& - \partial_{\theta_c}\rho^{s_2}m_r^2r^c + \frac{4}{3}\partial_{\theta_c}\rho^{s_2}m_r^2r^c(t_n-1) - \frac{1}{3}\partial_{\theta_c}\rho^{s_2}m_r^2r^c(t_n-2) \\
& - \partial_{\theta_c}\rho^{s_2}m_z^2z^c + \frac{4}{3}\partial_{\theta_c}\rho^{s_2}m_z^2z^c(t_n-1) - \frac{1}{3}\partial_{\theta_c}\rho^{s_2}m_z^2z^c(t_n-2),
\end{aligned}
\tag{39.24}
$$

i.e.

$$
\begin{aligned}
\partial_{\theta_c}\mathcal{B} = {} & \frac{2\Delta_t}{3}\rho^{s_1}\left[u^{s_1}\partial_{\theta_c}m_r^1 + w^{s_1}\partial_{\theta_c}m_z^1\right] \\
& - \rho^{s_1}\partial_{\theta_c}m_r^1\left[r^c + \frac{4}{3}r^c(t_n-1) - \frac{1}{3}r^c(t_n-2)\right] \\
& - \rho^{s_1}\partial_{\theta_c}m_z^1\left[z^c + \frac{4}{3}z^c(t_n-1) - \frac{1}{3}z^c(t_n-2)\right],
\end{aligned}
\tag{39.25}
$$

which is

$$
\begin{aligned}
\partial_{\theta_c}\mathcal{B} = {} & \frac{2\Delta_t}{3}\rho^{s_1}\left[u^{s_1}\partial_{\theta_c}(-\cos(\theta_c)) + w^{s_1}\partial_{\theta_c}(\sin(\theta_c))\right] \\
& - \rho^{s_1}\partial_{\theta_c}(-\cos(\theta_c))\left[r^c + \frac{4}{3}r^c(t_n-1) - \frac{1}{3}r^c(t_n-2)\right] \\
& - \rho^{s_1}\partial_{\theta_c}(\sin(\theta_c))\left[z^c + \frac{4}{3}z^c(t_n-1) - \frac{1}{3}z^c(t_n-2)\right],
\end{aligned}
\tag{39.26}
$$

or, equivalently,

$$
\begin{aligned}
\partial_{\theta_c}\mathcal{B} = {} & \frac{2\Delta_t}{3}\rho^{s_1}\left[u^{s_1}(\sin(\theta_c)) + w^{s_1}(\cos(\theta_c))\right] \\
& - \rho^{s_1}(\sin(\theta_c))\left[r^c + \frac{4}{3}r^c(t_n-1) - \frac{1}{3}r^c(t_n-2)\right] \\
& - \rho^{s_1}(\cos(\theta_c))\left[z^c + \frac{4}{3}z^c(t_n-1) - \frac{1}{3}z^c(t_n-2)\right].
\end{aligned}
\tag{39.27}
$$

40. Pressure limit condition

$$\lim_{\zeta \to 0} \partial_\theta p = 0 \tag{40.1}$$

41. Compatibility of r-velocity

At the separatrix of the domains we have

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + A\check{u}. \tag{41.1}$$

We thus impose

$$u_{l_s(ii)} - \bar{u}_{l_s(ii)} - A\check{u}_{l_s(ii)} = 0, \tag{41.2}$$

at each velocity node on the separatrix.

41.1. *Jacobian terms*

41.2. *Derivatives of C_i^u with respect to h_q*

$$\partial_{h_q} C_i^u = \partial_{h_q} u_{b^5(ii)} - \partial_{h_q} \bar{u}_{b^4(ii)} - A\partial_{h_q} \check{u}_{b^4(ii)} \tag{41.3}$$

i.e.

$$\partial_{h_q} C_i^u = -A\left[\partial_r \check{u}_{b^4(ii)} \partial_{h_q} r + \partial_z \check{u}_{b^4(ii)} \partial_{h_q} z\right] \tag{41.4}$$

42. Compatibility of z-velocity

$$\boldsymbol{w} = \bar{\boldsymbol{w}} + A\check{w}, \tag{42.1}$$

43. Equation summary for obtuse contact angle flow

43.1. r-momentum residuals

We recall equations (25.117) and (25.129) form which we have

$$\bar{\mathcal{M}}_i^r = \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\text{el}}} \left[\bar{\mathcal{M}}_{e,ii}^{r,0a} + \bar{\mathcal{M}}_{e,ii}^{r,0b} + \bar{\mathcal{M}}_{e,ii}^{r,0c}\right]}_{\bar{\mathcal{M}}_i^{r,0}} \tag{43.1}$$

$$+ \underbrace{\sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\text{el}}^1} \bar{\mathcal{M}}_{e_1,ii}^{r,1} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_r^1(r_c,z_c)}{Ca} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_d,z_d)\phi_i(r_d,z_d)m_r^1(r_d,z_d)}{Ca}}_{\bar{\mathcal{M}}_i^{r,1}}$$

$$+ \underbrace{\sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\text{el}}^2} \bar{\mathcal{M}}_{e,ii}^{r,2}}_{\bar{\mathcal{M}}_i^{r,2}} + \underbrace{\sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\text{el}}^4} \bar{\mathcal{M}}_{e_4,ii}^{r,4}}_{\bar{\mathcal{M}}_i^{r,4}},$$

where; form equation (25.138), we have

$$\bar{\mathcal{M}}_{e,ii}^{r,0a} = \frac{2\Delta_t}{3}\left\{-St\, a_{ii,g_r}(e) + Re\, (A)^2\left[a_{ii,\breve{u},\partial_r\breve{u}}(e) + a_{ii,\breve{w},\partial_z\breve{u}}(e)\right]\right\} + Re\, A a_{ii,\breve{u}}(e), \tag{43.2}$$

from equation (25.131) we have

$$\begin{aligned}
\bar{\mathcal{M}}_{e,ii}^{r,0b} &= \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}\left[a_{ii,jj}^{z,r}(e) + Re\, A a_{ii,jj,\partial_z\breve{u}}(e)\right]\\
&+ \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)}\left\{2a_{ii,jj}^{r,r}(e) + a_{ii,jj}^{z,z}(e) + Re\, A\left[a_{ii,jj,\breve{u}}^r(e) + a_{ii,jj,\breve{w}}^z(e) + a_{ii,jj,\partial_r\breve{u}}(e)\right]\right\}\\
&+ Re\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}(e)\bar{u}_{l(e,jj)} + Re\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}(e)\left[-\frac{4}{3}u_{l(e,jj)}(t_{n-1}) + \frac{1}{3}u_{l(e,jj)}(t_{n-2})\right]\\
&- Re\, A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\breve{u}}(e)\left[r_{l(e,jj)}^c - \frac{4}{3}r_{l(e,jj)}^c(t_{n-1}) + \frac{1}{3}r_{l(e,jj)}^c(t_{n-2})\right]\\
&- Re\, A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\breve{u}}(e)\left[z_{l(e,jj)}^c - \frac{4}{3}z_{l(e,jj)}^c(t_{n-1}) - \frac{1}{3}z_{l(e,jj)}^c(t_{n-2})\right]\\
&- \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_p^e} p_{l^p(e,jj)}b_{jj,ii}^r(e),
\end{aligned} \tag{43.3}$$

from equation (25.132) we have

$$\bar{\mathcal{M}}^{r,0c}_{e,ii} = \sum_{jj=1}^{\bar{n}_v^e} Re\,\bar{u}_{l(e,jj)} \left\{ \frac{2\Delta_t}{3} \underbrace{\sum_{kk=1}^{\bar{n}_v^e} \left[\bar{u}_{l(e,kk)} a^r_{ii,kk,jj}(e) + \bar{w}_{l(e,kk)} a^z_{ii,kk,jj}(e) \right]}_{\bar{A}_{ii,jj}(e)} \right.$$
$$- \underbrace{\sum_{kk=1}^{\bar{n}_v^e} a^r_{ii,kk,jj}(e) \left[r^c_{l(e,kk)} - \frac{4}{3} r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,kk)}(t_{n-2}) \right]}_{B_{ij}} \tag{43.4}$$
$$\left. - \underbrace{\sum_{kk=1}^{\bar{n}_v^e} a^z_{ii,kk,jj}(e) \left[z^c_{l(e,kk)} - \frac{4}{3} z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,kk)}(t_{n-2}) \right]}_{C_{ij}} \right\},$$

from equation (25.142) we have

$$\bar{\mathcal{M}}^{r,1}_{e_1,ii} = -\frac{2\Delta_t}{3} \left\{ A \left[2c_{ii,n_r,\partial_r\breve{u}}(e_1) + c_{ii,n_z,\partial_z\breve{u}}(e_1) + c_{ii,n_z,\partial_r\breve{w}}(e_1) \right] \right.$$
$$\left. - \frac{1}{Ca} \sum_{jj=1}^{\bar{n}_v^{e_1}} \sigma^1_{l^1_1(e_1,jj)} c^s_{jj,ii,t_r}(e_1) \right\}, \tag{43.5}$$

from equation (25.143) we have

$$\bar{\mathcal{M}}^{r,2}_{e_2,ii} = \frac{2\Delta_t}{3} \left\{ Be \left[-d_{ii,t_r,t_r,u^s}(e_2) - d_{ii,t_r,t_z,w^s}(e_2) + A \left\{ d_{ii,t_r,t_r,\breve{u}}(e_2) + d_{ii,t_r,t_z,\breve{w}}(e_2) \right\} \right] \right.$$
$$- A \left[2d_{ii,n_r,n_r,n_r,\partial_r\breve{u}}(e_2) + 2d_{ii,n_r,n_r,n_z,\partial_z\breve{u}}(e_2) + 2d_{ii,n_r,n_r,n_z,\partial_r\breve{w}}(e_2) \right.$$
$$+ 2d_{ii,n_r,n_z,n_z,\partial_z\breve{w}}(e_2) + 2d_{ii,t_r,t_r,n_r,\partial_r\breve{u}}(e_2) + d_{ii,t_r,t_r,n_z,\partial_z\breve{u}}(e_2)$$
$$+ d_{ii,t_r,t_z,n_r,\partial_z\breve{u}}(e_2) + d_{ii,t_r,t_r,n_r,\partial_r\breve{w}}(e_2) + d_{ii,t_r,t_r,n_z,\partial_r\breve{w}}(e_2)$$
$$\left. \left. + 2d_{ii,t_r,t_z,n_z,\partial_z\breve{w}}(e_2) \right] \right\}$$
$$+ \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \left\{ Be \left[\bar{u}_{l_2(e_2,jj)} d_{ii,jj,t_r,t_r}(e_2) + \bar{w}_{l_2(e_2,jj)} d_{ii,jj,t_r,t_z}(e_2) \right] \right.$$
$$\left. + \lambda^2_{l^2_2(e_2,jj)} d_{ii,jj,n_r}(e_2) \right\}, \tag{43.6}$$

and from equation (**??**) we have

$$\bar{\mathcal{M}}^{r,4}_{e,ii} = -\frac{2\Delta_t}{3} \left\{ A \left[2g_{ii,n_r,\partial_r\breve{u}}(e_4) + g_{ii,n_z,\partial_r\breve{w}}(e_4) + g_{ii,n_z,\partial_z\breve{u}}(e_4) \right] \right.$$
$$\left. - \sum_{jj=1}^{\bar{n}_v^e} \lambda^4_{l_4(e_4,jj)} g_{ii,jj,n_r}(e_4) - \sum_{jj=1}^{n_v} \gamma^4_{l_4(e_4,jj)} g_{ii,jj,t_r}(e_4) \right\}; \tag{43.7}$$

43.2. z-momentum residuals

We recall equation (26.105) which states

$$\bar{\mathcal{M}}_i^z = \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}_{e,ii}^{z,0a}}_{\bar{\mathcal{M}}_i^{z,0a}} + \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}_{e,ii}^{z,0b}}_{\bar{\mathcal{M}}_i^{z,0b}} + \underbrace{\sum_{\substack{e=1\\ i=l(e,ii)}}^{\bar{n}_{\mathrm{el}}} \bar{\mathcal{M}}_{e,ii}^{z,0c}}_{\bar{\mathcal{M}}_i^{z,0c}} \tag{43.8}$$

$$+ \underbrace{\frac{2\Delta_t}{3}\frac{\sigma^1(r_c,z_c)\phi_i(r_c,z_c)m_z^1(r_c,z_c)}{Ca} + \frac{2\Delta_t}{3}\frac{\sigma^1(r_d,z_d)\phi_i(r_d,z_d)m_z^1(r_d,z_d)}{Ca} + \sum_{\substack{e_1=1\\ i=l_1(e,ii)}}^{\bar{n}_{\mathrm{el}}^1} \bar{\mathcal{M}}_{e_1,ii}^{z,1}}_{\bar{\mathcal{M}}_i^{z,1}}$$

$$+ \underbrace{\sum_{\substack{e_2=1\\ i=l_2(e,ii)}}^{\bar{n}_{\mathrm{el}}^2} \bar{\mathcal{M}}_{e,ii}^{z,2}}_{\bar{\mathcal{M}}_i^{z,2}} + \underbrace{\sum_{\substack{e_4=1\\ i=l_4(e,ii)}}^{\bar{n}_{\mathrm{el}}^4} \bar{\mathcal{M}}_{e_4,ii}^{z,4}}_{\bar{\mathcal{M}}_i^{z,4}},$$

where, from equation (26.121) we have

$$\bar{\mathcal{M}}_{e,ii}^{z,0a} = \frac{2\Delta_t}{3}\left\{-St\, a_{ii,g_z}(e) + Re\, (A)^2\left[a_{ii,\check{u},\partial_r\check{w}}(e) + a_{ii,\check{w},\partial_z\check{w}}(e)\right]\right\} + Re\, A a_{ii,\check{w}}(e), \tag{43.9}$$

from equation (26.122) we have

$$\begin{aligned}
\bar{\mathcal{M}}_{e,ii}^{z,0b} &= \frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} \bar{u}_{l(e,jj)}\left[a_{ii,jj}^{r,z}(e) + Re\, A a_{ii,jj,\partial_r\check{w}}(e)\right] \\
&+\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)}\left\{a_{ii,jj}^{r,r}(e) + 2a_{ii,jj}^{z,z}(e) + Re\, A\left[a_{ii,jj,\check{u}}^{r}(e) + a_{ii,jj,\check{w}}^{z}(e) + a_{ii,jj,\partial_z\check{w}}(e)\right]\right\} \\
&+Re\sum_{jj=1}^{\bar{n}_v^e} \bar{w}_{l(e,jj)} a_{ii,jj}(e) + \frac{Re}{3}\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj}(e)\left[-4w_{l(e,jj)}(t_{n-1}) + w_{l(e,jj)}(t_{n-2})\right] \\
&-Re\, A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_r\check{w}}(e)\left[r_{l(e,jj)}^c - \frac{4}{3}r_{l(e,jj)}^c(t_{n-1}) + \frac{1}{3}r_{l(e,jj)}^c(t_{n-2})\right] \\
&-Re\, A\sum_{jj=1}^{\bar{n}_v^e} a_{ii,jj,\partial_z\check{w}}(e)\left[z_{l(e,jj)}^c - \frac{4}{3}z_{l(e,jj)}^c(t_{n-1}) + \frac{1}{3}z_{l(e,jj)}^c(t_{n-2})\right] \\
&-\frac{2\Delta_t}{3}\sum_{jj=1}^{\bar{n}_p^e} p_{l^p(e,jj)} b_{jj,ii}^z(e),
\end{aligned} \tag{43.10}$$

from equation(26.123) we have

$$\bar{\mathcal{M}}_i^{z,0c} = \sum_{jj=1}^{\bar{n}_v^e} Re\,\bar{w}_{l(e,jj)} \left\{ \frac{2\Delta_t}{3} \underbrace{\sum_{kk=1}^{\bar{n}_v^e} \left[\bar{u}_{l(e,kk)} a^r_{ii,kk,jj}(e) + \bar{w}_{l(e,kk)} a^z_{ii,kk,jj}(e)\right]}_{\bar{A}_{i,j}} \right.$$
$$- \underbrace{\sum_{kk=1}^{\bar{n}_v^e} a^r_{ii,kk,jj}(e) \left[r^c_{l(e,kk)} - \frac{4}{3} r^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} r^c_{l(e,kk)}(t_{n-2})\right]}_{B_{i,j}} \tag{43.11}$$
$$\left. - \underbrace{\sum_{kk=1}^{\bar{n}_v^e} a^z_{ii,kk,jj}(e) \left[z^c_{l(e,kk)} - \frac{4}{3} z^c_{l(e,kk)}(t_{n-1}) + \frac{1}{3} z^c_{l(e,kk)}(t_{n-2})\right]}_{C_{ij}} \right\},$$

from equation (26.125) we have

$$\begin{aligned} \bar{\mathcal{M}}_{e_1,ii}^{z,1} &= -\frac{2\Delta_t}{3} A \left[2c_{ii,n_z,\partial_z \breve{w}}(e) + c_{ii,n_r,\partial_r \breve{w}}(e) + c_{ii,n_r,\partial_z \breve{u}}(e)\right] \\ &\quad + \frac{2\Delta_t}{3Ca} \sum_{jj=1}^{n_v^e} \sigma^1_{l_1^1(e_1,jj)} c^s_{jj,ii,t_z}(e), \end{aligned} \tag{43.12}$$

from (26.126) we have

$$\begin{aligned} \bar{\mathcal{M}}_{e_2,ii}^{z,2} = \frac{2\Delta_t}{3} \{ & Be\left[-d_{ii,t_r,t_z,u^s}(e) - d_{ii,t_z,t_z,w^s}(e) + A\left(d_{ii,t_r,t_z,\breve{u}}(e) + d_{ii,t_z,t_z,\breve{w}}(e)\right)\right] \\ & - A\left[2d_{ii,n_r,n_r,n_z,\partial_r \breve{u}}(e) + 2d_{ii,n_r,n_z,n_z,\partial_z \breve{u}}(e) + 2d_{ii,n_r,n_z,n_z,\partial_r \breve{w}}(e)\right. \\ & + 2d_{ii,n_z,n_z,n_z,\partial_z \breve{w}}(e) + 2d_{ii,t_r,t_z,n_r,\partial_r \breve{u}}(e) + d_{ii,t_r,t_z,n_z,\partial_z \breve{u}}(e) \\ & + d_{ii,t_z,t_z,n_r,\partial_z \breve{u}}(e) + d_{ii,t_z,t_z,n_r,\partial_r \breve{w}}(e) + d_{ii,t_r,t_z,n_z,\partial_r \breve{w}}(e) \\ & \left. + 2d_{ii,t_z,t_z,n_z,\partial_z \breve{w}}(e)\right]\} \\ + \frac{2\Delta_t}{3} \sum_{j=1}^{\bar{n}_v^{e2}} \Big\{ & Be\left[\bar{u}_{l_2(e_2,jj)} d_{ii,jj,t_r,t_z}(e) + \bar{w}_{l_2(e_2,jj)} d_{ii,jj,t_z,t_z}(e)\right] \\ & + \lambda^2_{l_2^2(e_2,jj)} d_{ii,jj,n_z}(e) \Big\}, \end{aligned} \tag{43.13}$$

and from (**??**)

$$\begin{aligned} \bar{\mathcal{M}}_{e_4,ii}^{z,4} &= -\frac{4\Delta_t}{3} A g_{ii,n_z,\partial_z \breve{u}} - \frac{2\Delta_t}{3} A g_{ii,n_r,\partial_z \breve{u}} - \frac{2\Delta_t}{3} A g_{ii,n_r,\partial_r \breve{w}} \\ &\quad \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e4}} \lambda^4_{l_4^4(e_4,jj)} g_{ii,jj,n_z} \\ &\quad + \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^e} \gamma^4_{l_4^4(e_4,jj)} g_{ii,j,t_z}. \end{aligned} \tag{43.14}$$

43.3. Continuity residuals

From equation (27.10) we have

$$\bar{\mathcal{C}}_i = \sum_{\substack{e=1\\ i=l^p(e,ii)}}^{n_{\mathrm{el}}} \sum_{jj=1}^{n_v^e} \left[\bar{u}_{l(e,jj)} b^r_{ii,jj}(e) + \bar{w}_{l(e,jj)} b^z_{ii,jj}(e)\right]. \tag{43.15}$$

43.4. KBC residuals

We recall equations (**??**) and (**??**), which imply that

$$\begin{aligned}\bar{\mathcal{K}}_i = \sum_{\substack{e_1=1\\ i=l_1(e_1,ii)}}^{n^1_{\mathrm{el}}} &\left\{\frac{2\Delta_t}{3} A\left[c_{ii,n_r,\breve{u}}(e_1) + c_{ii,n_z,\breve{w}}(e_1)\right]\right.\\ &+ \frac{2\Delta_t}{3} \sum_{jj=1}^{\bar{n}_v^{e_1}} \left[\bar{u}_{l_1(e_1,jj)} c_{ii,jj,n_r}(e_1) + \bar{w}_{l_1(e_1,jj)} c_{ii,jj,n_z}(e_1)\right]\\ &- \sum_{j=1}^{n_v^{e_1}} c_{ii,jj,n_r}(e_1)\left[r^c_{l_1(e_1,jj)} - \frac{4}{3} r^c_{l_1(e_1,jj)}(t_{n-1}) + \frac{1}{3} r^c_{l_1(e_1,jj)}(t_{n-2})\right]\\ &\left.- \sum_{jj=1}^{n_v^{e_1}} c_{ii,jj,n_z}(e_1)\left[z^c_{l_1(e_1,jj)} - \frac{4}{3} z^c_{l_1(e_1,jj)}(t_{n-1}) + \frac{1}{3} z^c_{l_1(e_1,jj)}(t_{n-2})\right]\right\}.\end{aligned} \tag{43.16}$$

43.5. Impermeability residuals

We recall equation (**??**), which states

$$\begin{aligned}\breve{I}_i = \sum_{\substack{e_2=1\\ i=l_2^2(e,ii)}}^{n^2_{\mathrm{el}}} &\left\{A\left[d_{\breve{u},n_r,ii}(e_2) + d_{\breve{w},n_z,ii}(e_2)\right]\right.\\ &\left.+ \sum_{jj=1}^{n_v^{e_2}} \left[\bar{u}_{l_2(e_2,jj)} d_{ii,jj,n_r}(e_2) + w_{l_2(e,jj)} d_{ii,jj,n_z}(e_2)\right]\right\}.\end{aligned} \tag{43.17}$$

44. System Jacobian for split-domain formulation

$$
\left[\begin{array}{ccccccccccccc|cc|ccccccccccccc|c|c}
\partial_{\bar{\boldsymbol{u}}}\bar{\mathcal{M}}^r & \partial_{\bar{\boldsymbol{w}}}\bar{\mathcal{M}}^r & \partial_{\boldsymbol{p}_\mathrm{n}}\bar{\mathcal{M}}^r & \partial_{\boldsymbol{\lambda}^2_\mathrm{n}}\bar{\mathcal{M}}^r & 0 & 0 & 0 & \partial_{\boldsymbol{\sigma}^2_\mathrm{n}}\bar{\mathcal{M}}^r & \partial_{\boldsymbol{h}_\mathrm{n}}\bar{\mathcal{M}}^r & 0 & 0 & 0 & \partial_{\boldsymbol{\sigma}^1_\mathrm{n}}\bar{\mathcal{M}}^r & \partial_{\boldsymbol{\lambda}^4}\bar{\mathcal{M}}^r & \partial_{\boldsymbol{\gamma}^4}\bar{\mathcal{M}}^r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{h_\mathrm{f}}\bar{\mathcal{M}}^r & 0 & 0 & 0 & 0 & \partial_{\theta_c}\bar{\mathcal{M}}^r & \partial_A\bar{\mathcal{M}}^r \\
\partial_{\bar{\boldsymbol{u}}}\bar{\mathcal{M}}^z & \partial_{\bar{\boldsymbol{w}}}\bar{\mathcal{M}}^z & \partial_{\boldsymbol{p}_\mathrm{n}}\bar{\mathcal{M}}^z & \partial_{\boldsymbol{\lambda}^2_\mathrm{n}}\bar{\mathcal{M}}^z & 0 & 0 & 0 & \partial_{\boldsymbol{\sigma}^2_\mathrm{n}}\bar{\mathcal{M}}^r & \partial_{\boldsymbol{h}_\mathrm{n}}\bar{\mathcal{M}}^z & 0 & 0 & 0 & \partial_{\boldsymbol{\sigma}^1_\mathrm{n}}\bar{\mathcal{M}}^r & \partial_{\boldsymbol{\lambda}^4}\bar{\mathcal{M}}^z & \partial_{\boldsymbol{\gamma}^4}\bar{\mathcal{M}}^z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{h_\mathrm{f}}\bar{\mathcal{M}}^z & 0 & 0 & 0 & 0 & \partial_{\theta_c}\bar{\mathcal{M}}^r & \partial_A\bar{\mathcal{M}}^z \\
\partial_{\bar{\boldsymbol{u}}}\mathcal{C}_\mathrm{n} & \partial_{\bar{\boldsymbol{w}}}\mathcal{C}_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{C}_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}_\mathrm{f}}\mathcal{C}_\mathrm{n} & \partial_{\boldsymbol{w}_\mathrm{f}}\mathcal{C}_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{f}}\mathcal{C}_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_{\bar{\boldsymbol{u}}}\mathcal{S}^2_\mathrm{n} & \partial_{\bar{\boldsymbol{w}}}\mathcal{S}^2_\mathrm{n} & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_\mathrm{n}}\mathcal{S}^2_\mathrm{n} & \partial_{\boldsymbol{w}^{s_2}_\mathrm{n}}\mathcal{S}^2_\mathrm{n} & 0 & \partial_{\boldsymbol{\sigma}^2_\mathrm{n}}\mathcal{S}^2_\mathrm{n} & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{S}^2_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}_f}\mathcal{S}^2_\mathrm{n} & \partial_{\boldsymbol{w}_f}\mathcal{S}^2_\mathrm{n} & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_f}\mathcal{S}^2_\mathrm{n} & \partial_{\boldsymbol{w}^{s_2}_f}\mathcal{S}^2_\mathrm{n} & 0 & \partial_{\boldsymbol{\sigma}^2_f}\mathcal{S}^2_\mathrm{n} & \partial_{\boldsymbol{h}_f}\mathcal{S}^2_\mathrm{n} & 0 & 0 & 0 & 0 & \partial_{\theta_c}\mathcal{S}^2_\mathrm{n} & \partial_A\mathcal{S}^2_\mathrm{n} \\
0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_\mathrm{n}}\mathcal{I}_\mathrm{n} & \partial_{\boldsymbol{w}^{s_2}_\mathrm{n}}\mathcal{I}_\mathrm{n} & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{I}_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_\mathrm{f}}\mathcal{I}_\mathrm{n} & \partial_{\boldsymbol{w}^{s_2}_\mathrm{f}}\mathcal{I}_\mathrm{n} & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{f}}\mathcal{I}_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_{\bar{\boldsymbol{u}}}\mathcal{E}^2_\mathrm{n} & \partial_{\bar{\boldsymbol{w}}}\mathcal{E}^2_\mathrm{n} & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_\mathrm{n}}\mathcal{E}^2_\mathrm{n} & \partial_{\boldsymbol{w}^{s_2}_\mathrm{n}}\mathcal{E}^2_\mathrm{n} & \partial_{\boldsymbol{\rho}^{s_2}_\mathrm{n}}\mathcal{E}^2_\mathrm{n} & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{E}^2_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}_f}\bar{E}^2_\mathrm{n} & \partial_{\boldsymbol{w}_f}\mathcal{E}^2_\mathrm{n} & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_f}\mathcal{E}^2_\mathrm{n} & \partial_{\boldsymbol{w}^{s_2}_f}\mathcal{E}^2_\mathrm{n} & \partial_{\boldsymbol{\rho}^{s_2}_f}\mathcal{E}^2_\mathrm{n} & 0 & \partial_{\boldsymbol{h}_f}\mathcal{E}^2_\mathrm{n} & 0 & 0 & 0 & 0 & \partial_{\theta_c}\mathcal{E}^2_\mathrm{n} & \partial_A\mathcal{E}^2_\mathrm{n} \\
0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_\mathrm{n}}\mathcal{D}^2_\mathrm{n} & \partial_{\boldsymbol{w}^{s_2}_\mathrm{n}}\mathcal{D}^2_\mathrm{n} & \partial_{\boldsymbol{\rho}^{s_2}_\mathrm{n}}\mathcal{D}^2_\mathrm{n} & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{D}^2_\mathrm{n} & \partial_{\boldsymbol{u}^{s_1}_\mathrm{n}}\mathcal{D}^2_\mathrm{n} & \partial_{\boldsymbol{w}^{s_1}_\mathrm{n}}\mathcal{D}^2_\mathrm{n} & \partial_{\boldsymbol{\rho}^{s_1}_\mathrm{n}}\mathcal{D}^2_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_f}\mathcal{D}^2_\mathrm{n} & \partial_{\boldsymbol{w}^{s_2}_f}\mathcal{D}^2_\mathrm{n} & \partial_{\boldsymbol{\rho}^{s_2}_f}\mathcal{D}^2_\mathrm{n} & 0 & \partial_{\boldsymbol{h}_\mathrm{f}}\mathcal{D}^2_\mathrm{n} & 0 & 0 & 0 & 0 & \partial_{\theta_c}\mathcal{D}^2_\mathrm{n} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{\rho}^{s_2}_\mathrm{n}}\mathcal{T}^2_\mathrm{n} & \partial_{\boldsymbol{\sigma}^2_\mathrm{n}}\mathcal{T}^2_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial_{\bar{\boldsymbol{u}}}\mathcal{S}^1_\mathrm{n} & \partial_{\bar{\boldsymbol{w}}}\mathcal{S}^1_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{S}^1_\mathrm{n} & \partial_{\boldsymbol{u}^{s_1}_\mathrm{n}}\mathcal{S}^1_\mathrm{n} & \partial_{\boldsymbol{w}^{s_1}_\mathrm{n}}\mathcal{S}^1_\mathrm{n} & 0 & \partial_{\boldsymbol{\sigma}^1_\mathrm{n}}\mathcal{S}^1_\mathrm{n} & 0 & 0 & \partial_{\boldsymbol{u}_f}\mathcal{S}^1_\mathrm{n} & \partial_{\boldsymbol{w}_f}\mathcal{S}^1_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_f}\mathcal{S}^1_\mathrm{n} & \partial_{\boldsymbol{u}^{s_1}_f}\mathcal{S}^1_\mathrm{n} & \partial_{\boldsymbol{w}^{s_1}_f}\mathcal{S}^1_\mathrm{n} & 0 & \partial_{\boldsymbol{\sigma}^1_f}\mathcal{S}^1_\mathrm{n} & \partial_{\theta_c}\mathcal{S}^1_\mathrm{n} & \partial_A\mathcal{S}^1_\mathrm{n} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{K}_\mathrm{n} & \partial_{\boldsymbol{u}^{s_1}_\mathrm{n}}\mathcal{K}_\mathrm{n} & \partial_{\boldsymbol{w}^{s_1}_\mathrm{n}}\mathcal{K}_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{f}}\mathcal{K}_\mathrm{n} & \partial_{\boldsymbol{u}^{s_1}_\mathrm{f}}\mathcal{K}_\mathrm{n} & \partial_{\boldsymbol{w}^{s_1}_\mathrm{f}}\mathcal{K}_\mathrm{n} & 0 & 0 & 0 & 0 \\
\partial_{\bar{\boldsymbol{u}}}\mathcal{E}^1_\mathrm{n} & \partial_{\bar{\boldsymbol{w}}}\mathcal{E}^1_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{E}^1_\mathrm{n} & \partial_{\boldsymbol{u}^{s_1}_\mathrm{n}}\mathcal{E}^1_\mathrm{n} & \partial_{\boldsymbol{w}^{s_1}_\mathrm{n}}\mathcal{E}^1_\mathrm{n} & \partial_{\boldsymbol{\rho}^{s_1}_\mathrm{n}}\mathcal{E}^1_\mathrm{n} & 0 & 0 & 0 & \partial_{\boldsymbol{u}_f}\mathcal{E}^1_\mathrm{n} & \partial_{\boldsymbol{w}_f}\mathcal{E}^1_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_f}\mathcal{E}^1_\mathrm{n} & \partial_{\boldsymbol{u}^{s_1}_f}\mathcal{E}^1_\mathrm{n} & \partial_{\boldsymbol{w}^{s_1}_f}\mathcal{E}^1_\mathrm{n} & \partial_{\boldsymbol{\rho}^{s_1}_f}\mathcal{E}^1_\mathrm{n} & 0 & \partial_{\theta_c}\mathcal{E}^1_\mathrm{n} & \partial_A\mathcal{E}^1_\mathrm{n} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{D}^1_\mathrm{n} & \partial_{\boldsymbol{u}^{s_1}_\mathrm{n}}\mathcal{D}^1_\mathrm{n} & \partial_{\boldsymbol{w}^{s_1}_\mathrm{n}}\mathcal{D}^1_\mathrm{n} & \partial_{\boldsymbol{\rho}^{s_1}_\mathrm{n}}\mathcal{D}^1_\mathrm{n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{f}}\mathcal{D}^1_\mathrm{n} & \partial_{\boldsymbol{u}^{s_1}_f}\mathcal{D}^1_\mathrm{n} & \partial_{\boldsymbol{w}^{s_1}_f}\mathcal{D}^1_\mathrm{n} & \partial_{\boldsymbol{\rho}^{s_1}_f}\mathcal{D}^1_\mathrm{n} & 0 & \partial_{\theta_c}\mathcal{D}^1_\mathrm{n} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{\rho}^{s_1}_\mathrm{n}}\bar{\mathcal{T}}^1 & \partial_{\boldsymbol{\sigma}^1_\mathrm{n}}\bar{\mathcal{T}}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\partial_{\bar{\boldsymbol{u}}}\mathcal{C}^u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{C}^u & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}_\mathrm{f}}\mathcal{C}^u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\theta_c}\mathcal{C}^u & \partial_A\mathcal{C}^u \\
0 & \partial_{\bar{\boldsymbol{w}}}\mathcal{C}^w & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{C}^w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{w}_\mathrm{f}}\mathcal{C}^w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\theta_c}\mathcal{C}^w & \partial_A\mathcal{C}^w \\
\hline
0 & 0 & \partial_{\boldsymbol{p}_\mathrm{n}}\mathcal{M}^r & \partial_{\boldsymbol{\lambda}^2_\mathrm{n}}\mathcal{M}^r & 0 & 0 & 0 & \partial_{\sigma^2_\mathrm{n}}\mathcal{M}^r & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{M}^r & 0 & 0 & 0 & \partial_{\sigma^1_\mathrm{n}}\mathcal{M}^r & \partial_{\boldsymbol{\lambda}^4}\mathcal{M}^r & \partial_{\boldsymbol{\gamma}^4}\mathcal{M}^r & \partial_{\boldsymbol{u}_\mathrm{f}}\mathcal{M}^r & \partial_{\boldsymbol{w}_\mathrm{f}}\mathcal{M}^r & \partial_{\boldsymbol{p}_\mathrm{f}}\mathcal{M}^r & \partial_{\boldsymbol{\lambda}^2_\mathrm{f}}\mathcal{M}^r & 0 & 0 & 0 & \partial_{\sigma^2_\mathrm{f}}\mathcal{M}^r & \partial_{\boldsymbol{h}_\mathrm{f}}\mathcal{M}^r & 0 & 0 & 0 & \partial_{\sigma^1_\mathrm{f}}\mathcal{M}^r & 0 & 0 \\
0 & 0 & \partial_{\boldsymbol{p}_\mathrm{n}}\mathcal{M}^z & \partial_{\boldsymbol{\lambda}^2_\mathrm{n}}\mathcal{M}^z & 0 & 0 & 0 & \partial_{\sigma^2_\mathrm{n}}\mathcal{M}^z & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{M}^z & 0 & 0 & 0 & \partial_{\sigma^1_\mathrm{n}}\mathcal{M}^z & \partial_{\boldsymbol{\lambda}^4}\mathcal{M}^z & \partial_{\boldsymbol{\gamma}^4}\mathcal{M}^z & \partial_{\boldsymbol{u}_\mathrm{f}}\mathcal{M}^z & \partial_{\boldsymbol{w}_\mathrm{f}}\mathcal{M}^z & \partial_{\boldsymbol{p}_\mathrm{f}}\mathcal{M}^z & \partial_{\boldsymbol{\lambda}^2_\mathrm{f}}\mathcal{M}^z & 0 & 0 & 0 & \partial_{\sigma^2_\mathrm{f}}\mathcal{M}^z & \partial_{\boldsymbol{h}_\mathrm{f}}\mathcal{M}^z & 0 & 0 & 0 & \partial_{\sigma^1_\mathrm{f}}\mathcal{M}^z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{C}_f & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}_\mathrm{f}}\mathcal{C}_f & \partial_{\boldsymbol{w}_\mathrm{f}}\mathcal{C}_f & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{f}}\mathcal{C}_f & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_\mathrm{n}}\mathcal{S}^2_f & \partial_{\boldsymbol{w}^{s_2}_\mathrm{n}}\mathcal{S}^2_f & 0 & \partial_{\sigma^2_\mathrm{n}}\mathcal{S}^2_f & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{S}^2_f & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}_f}\mathcal{S}^2_f & \partial_{\boldsymbol{w}_f}\mathcal{S}^2_f & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_f}\mathcal{S}^2_f & \partial_{\boldsymbol{w}^{s_2}_f}\mathcal{S}^2_f & 0 & \partial_{\boldsymbol{\sigma}^2_f}\mathcal{S}^2_f & \partial_{\boldsymbol{h}_f}\mathcal{S}^2_f & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_\mathrm{n}}\mathcal{I}_f & \partial_{\boldsymbol{w}^{s_2}_\mathrm{n}}\mathcal{I}_f & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{I}_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_f}\mathcal{I}_f & \partial_{\boldsymbol{w}^{s_2}_f}\mathcal{I}_f & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{f}}\mathcal{I}_f & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_\mathrm{n}}\mathcal{E}^2_f & \partial_{\boldsymbol{w}^{s_2}_\mathrm{n}}\mathcal{E}^2_f & \partial_{\boldsymbol{\rho}^{s_2}_\mathrm{n}}\mathcal{E}^2_f & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{E}^2_f & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}_f}\mathcal{E}^2_f & \partial_{\boldsymbol{w}_f}\mathcal{E}^2_f & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_f}\mathcal{E}^2_f & \partial_{\boldsymbol{w}^{s_2}_f}\mathcal{E}^2_f & \partial_{\boldsymbol{\rho}^{s_2}_f}\mathcal{E}^2_f & 0 & \partial_{\boldsymbol{h}_f}\mathcal{E}^2_f & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_\mathrm{n}}\mathcal{D}^2_f & \partial_{\boldsymbol{w}^{s_2}_\mathrm{n}}\mathcal{D}^2_f & \partial_{\boldsymbol{\rho}^{s_2}_\mathrm{n}}\mathcal{D}^2_f & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{D}^2_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{u}^{s_2}_f}\mathcal{D}^2_f & \partial_{\boldsymbol{w}^{s_2}_f}\mathcal{D}^2_f & \partial_{\boldsymbol{\rho}^{s_2}_f}\mathcal{D}^2_f & 0 & \partial_{\boldsymbol{h}_f}\mathcal{D}^2_f & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{\rho}^{s_2}_f}\mathcal{T}^2_f & \partial_{\boldsymbol{\sigma}^2_f}\mathcal{T}^2_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}}\mathcal{S}^1_f & \partial_{\boldsymbol{u}^{s_1}_\mathrm{n}}\mathcal{S}^1_f & \partial_{\boldsymbol{w}^{s_1}_\mathrm{n}}\mathcal{S}^1_f & 0 & \partial_{\boldsymbol{\sigma}^1_\mathrm{n}}\mathcal{S}^1_f & 0 & 0 & \partial_{\boldsymbol{u}_f}\mathcal{S}^1_f & \partial_{\boldsymbol{w}_f}\mathcal{S}^1_f & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_f}\mathcal{S}^1_f & \partial_{\boldsymbol{u}^{s_1}_f}\mathcal{S}^1_f & \partial_{\boldsymbol{w}^{s_1}_f}\mathcal{S}^1_f & 0 & \partial_{\boldsymbol{\sigma}^1_f}\mathcal{S}^1_f & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{K}_f & \partial_{\boldsymbol{u}^{s_2}_\mathrm{n}}\mathcal{K}_f & \partial_{\boldsymbol{w}^{s_2}_\mathrm{n}}\mathcal{K}_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{f}}\mathcal{K}_f & \partial_{\boldsymbol{u}^{s_1}_f}\mathcal{K}_f & \partial_{\boldsymbol{w}^{s_1}_f}\mathcal{K}_f & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{E}^1_f & \partial_{\boldsymbol{u}^{s_1}_\mathrm{n}}\mathcal{E}^1_f & \partial_{\boldsymbol{w}^{s_1}_\mathrm{n}}\mathcal{E}^1_f & \partial_{\boldsymbol{\rho}^{s_1}_\mathrm{n}}\mathcal{E}^1_f & 0 & 0 & 0 & \partial_{\boldsymbol{u}_f}\mathcal{E}^1_f & \partial_{\boldsymbol{w}_f}\mathcal{E}^1_f & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_f}\mathcal{E}^1_f & \partial_{\boldsymbol{u}^{s_1}_f}\mathcal{E}^1_f & \partial_{\boldsymbol{w}^{s_1}_f}\mathcal{E}^1_f & \partial_{\boldsymbol{\rho}^{s_1}_f}\mathcal{E}^1_f & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_\mathrm{n}}\mathcal{D}^1_f & \partial_{\boldsymbol{u}^{s_1}_\mathrm{n}}\mathcal{D}^1_f & \partial_{\boldsymbol{w}^{s_1}_\mathrm{n}}\mathcal{D}^1_f & \partial_{\boldsymbol{\rho}^{s_1}_\mathrm{n}}\mathcal{D}^1_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{h}_f}\mathcal{D}^1_f & \partial_{\boldsymbol{u}^{s_1}_f}\mathcal{D}^1_f & \partial_{\boldsymbol{w}^{s_1}_f}\mathcal{D}^1_f & \partial_{\boldsymbol{\rho}^{s_1}_f}\mathcal{D}^1_f & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\boldsymbol{\rho}^{s_1}_f}\mathcal{T}^1_f & \partial_{\boldsymbol{\sigma}^1_f}\mathcal{T}^1_f & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\sigma^2_\mathrm{n}}\mathcal{Y} & 0 & 0 & 0 & 0 & \partial_{\sigma^1_\mathrm{n}}\mathcal{Y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{\theta_c}\mathcal{Y} & 0 \\
\hline
0 & 0 & \partial_{\boldsymbol{p}_\mathrm{n}}\mathcal{C}^p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\tag{44.1}
$$

The matrix above is organised to guarantee that all diagonal blocks are square. This is not necessary but it helps when thinking of the balance between the number of equations and unknowns. Moreover, we have organised the block matrix in super-blocks or meta-blocks, i.e. blocks of blocks, in order to better illustrate how it is to be assembled.

The first meta-block column corresponds to all the unknowns in the near field, the second to the variables that are shared by the near and far fields through the matching conditions at the separatrix spine. The third column corresponds to variables defined on the far field only and the fourth column contains a single variable, namely A, which is necessary to specify the amplitude of the eigen-solution.

Rows are organised on the basis of an entirely analogue principle though, naturally, dealing with equations rather than variables. That is to say, the first meta-block row contains Jacobians of equations defined on the near field, the second contains the Jacobians of those equations that define the matching conditions at the separatrix spine, the third row corresponds to equations define only in the far field and the last row contains that Jacobian of a single equation (i.e. the equation gradient) which is there to guarantee that the radial nature of the pressure distribution is preserved in the limit near the contact line (see Sprittles & Shikhmurzaev (2011*b*)).

It is important to highlight that, regarding the assembly of the system of equations and its Jacobian, the treatment of the momentum residuals and the velocity variables is rather different from the rest of the equations and unknowns of the problem. Indeed, there are no velocity variables that play a role in the near and far field momentum equations simultaneously. The only link between them is established by the compatibility conditions, which are considered in a separate meta-block. That is to say, we have two velocity variables defined on each node of the separatrix (one for each sub-domain), one being the $\bar{\cdot}$ variable on the near field and the other being the full velocity.

On the other hand, all remaining variables defined on the separatrix (i.e. p, λ^4, γ^4, λ^2, h) are shared by the corresponding residual equations on both sub-domains. Moreover, residual equations associated to nodes on the separatrix will have contributions from both sub-domains. There is room for deciding how to arrange these residuals split in the two domains. Here, we chose to place those residual equations in the first rows, together with the near-field equations.

This means, for instance, that the continuity residual associated to a pressure node on the separatrix spine will be placed in the first block of rows of the residual vector, and that it will be given by the sum of the contributions of the residuals on both sides. Consequently, the first row of meta-block in the Jacobian must include the derivatives of these residuals with respect to the far-field variables, as well as with respect to the near field variables.

The need for derivatives of these split residuals with respect to the far field variables is the origin of the blocks in the first row and third column meta-blocks. The observation above is critical to properly consider the derivatives of the continuity residuals associated to the pressure nodes on the separatrix with respect to the far field velocities and spine lengths. The same is true for one residual in the KBC and the impermeability equations.

We also chose that the variables which are shared amongst both sub-domains will be placed amongst those in the near field. Here, it is important to notice that residuals that are not associated to nodes on the separatrix, but their nodes share and element with those in the separatrix will have a contribution from the separatrix-node variables. Put differently, if a node is in the far field and close enough to the separatrix (though not on it), its residual will be in the third block of equations but its Jacobian will still include columns that correspond to those variables associated to nodes on the separatrix, which are placed amongst the first block of variables, together with the near field ones. This is the origin of the blocks in the third row and first column meta-block.

Finally, it is important to highlight that the residual equations that have contributions from both sub-domains, and were thus moved into the first group of equations, will also (quite naturally) depend on variables defined on the separatrix. This implies that some matrices in the first row and first column meta-block will have to have the contribution of these derivatives added to them. For the present system, this is only the case with the variable given by the length of the separatrix spine and the first spine (i.e. the length of the wetted area of the solid). This is because the three vector equations that have contributions from both sides (continuity, impermeability and KBC) only depend on the velocities (which are not shared variables) and the spine lengths, of which only the first spine and the separatrix spine are variables in the first column of meta-blocks.

Put simply, the first and last column of matrices $\partial_{h_n}\bar{\mathcal{C}}$, $\partial_{h_n}\bar{\mathcal{I}}$ and $\partial_{h_n}\bar{\mathcal{K}}$ will have to be incremented by the derivatives of the far field contribution to the rows that correspond to the separatrix.

This first and last column perturbation is also all that distinguishes the blocks $\partial_{h_n}\mathcal{M}^r$, $\partial_{h_n}\mathcal{M}^z$, $\partial_{h_n}\mathcal{C}_f$, $\partial_{h_n}\mathcal{I}_f$ and $\partial_{h_n}\mathcal{K}_f$ (in the last block column of the third row, first column meta-block) from zero matrices.

This might take some time and effort to be digested; however, it must be well understood as it is an essential part of mounting the Jacobian of the system.

45. Integrals over triangular elements for obtuse contact angles

Similarly as done in section 20, we now consider the terms that are added in the obtuse angle formulation.

From (25.130) we have

$$a_{g_r,ii}(e) = \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \hat{\boldsymbol{g}}_r, \tag{45.1}$$

which we can re-write as

$$a_{g_r,ii}(e) = \int_{E} \phi_{ii} \hat{\boldsymbol{g}}_r \det J_e, \tag{45.2}$$

with

$$\phi_{ii}(\cdot) = \phi_{l(e,ii)}(S_e(\cdot)), \tag{45.3}$$

where, as mentioned before, S_e maps points in the master element onto points in the element being considered. We highlight that ϕ_{ii} can be named in this way (with no reference to the original element, i.e. element number e) because once the integral is mapped to the master element, all information about the original element is stored in J_e (i.e. the Jacobian of S_e). That is to say, ϕ_{ii} no longer depends on the specific element.

Now, using Gaussian quadrature we have

$$\boxed{a_{g_r,ii}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\hat{\boldsymbol{g}}_r(pp) \det J_e(pp)}, \tag{45.4}$$

where we are using the notation $f(p)$ as a short version for $f(\xi_p, \eta_p)$, with (ξ_p, η_p) being the p-th Gaussian quadrature point, and $W(p)$ is the weight associated to the p-th Gaussian quadrature point (out of n_G points); and, as usual, double letter indexes are used to indicate local numbering.

Also from (25.130) we have

$$a_{ii,\check{u}}(e) = \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{u} \tag{45.5}$$

which we can re-write as

$$a_{ii,\check{u}}(e) = \int_{E} \check{u}(r_e(\xi,\eta), z_e(\xi,\eta))\phi_{ii} \det J_e, \tag{45.6}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,\check{u}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\check{u}(pp) \det J_e(pp)}, \tag{45.7}$$

where

$$\check{u}(pp) = \check{u}(r_e(\xi_{pp}, \eta_{pp}), z_e(\xi_{pp}, \eta_{pp})). \tag{45.8}$$

Also from (25.130) we have

$$a_{ii,\check{u}_{n-1}}(e) = \int_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{u}_{n-1} \tag{45.9}$$

which we can re-write as

$$a_{ii,\check{u}_{n-1}}(e) = \int\limits_E \check{u}_{n-1}(r_e(\xi,\eta), z_e(\xi,\eta))\phi_{ii} \det J_e, \tag{45.10}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,\check{u}_{n-1}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\check{u}_{n-1}(pp) \det J_e(pp)}. \tag{45.11}$$

From (25.130) we have

$$a_{ii,\check{u}_{n-2}}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\check{u}_{n-2} \tag{45.12}$$

which we can re-write as

$$a_{ii,\check{u}_{n-2}}(e) = \int\limits_E \check{u}_{n-2}(r_e(\xi,\eta), z_e(\xi,\eta))\phi_{ii} \det J_e, \tag{45.13}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,\check{u}_{n-2}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\check{u}_{n-2}(pp) \det J_e(pp)}, \tag{45.14}$$

where

$$\check{u}_{n-2}(pp) = \check{u}_{n-2}(r_e(\xi_{pp},\eta_{pp}), z_e(\xi_{pp},\eta_{pp})). \tag{45.15}$$

From (25.130) we have

$$a_{\check{u}\partial_r\check{u},ii}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\check{u}\partial_r\check{u}, \tag{45.16}$$

which we can re-write as

$$a_{\check{u}\partial_r\check{u},ii}(e) = \int\limits_E \phi_{ii}\check{u}\partial_r\check{u} \det J_e, \tag{45.17}$$

and using Gaussian quadrature we have

$$\boxed{a_{\check{u}\partial_r\check{u},ii}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\check{u}(pp)\partial_r\check{u}(pp) \det J_e(pp)}, \tag{45.18}$$

where

$$\check{u}(pp) = \check{u}(r_e(\xi_{pp},\eta_{pp}), z_e(\xi_{pp},\eta_{pp})), \tag{45.19}$$

and the same holds for the derivatives of $\check{u}$, and the obvious analogues for $\check{w}$ are used below.

From (25.130) we also have

$$a_{\check{w}\partial_z\check{u},ii}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\check{w}\partial_z\check{u}, \tag{45.20}$$

which we can re-write as

$$a_{\check{w}\partial_z\check{u},ii}(e) = \int\limits_E \phi_{ii}\check{w}\partial_z\check{u}\det J_e, \tag{45.21}$$

and using Gaussian quadrature we have

$$\boxed{a_{\check{w}\partial_z\check{u},ii}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\check{w}(pp)\partial_z\check{u}(pp)\det J_e(pp)}. \tag{45.22}$$

From (25.131) we also have $a_{ii,jj}(e)$, $a^{r,r}_{ii,jj}(e)$, $a^{z,z}_{ii,jj}(e)$, $a^{z,r}_{ii,jj}(e)$ and $b^{r}_{jj,ii}(e)$ which are identical to the ones used in the far field. Moreover, we have

$$a^r_{ii,jj,\check{u}}(e) = \int\limits_{\bar{\Omega}_e} \check{u}\phi_{l(e,ii)}\partial_r\phi_{l(e,jj)}, \tag{45.23}$$

which we can re-write as

$$a^r_{ii,jj,\check{u}}(e) = \int\limits_E \check{u}\phi_{ii}\left(\sum_{mm=1}^{6} T_{jj,mm}z_{e,mm}\right), \tag{45.24}$$

where we have cancelled the $\det J_e$ in the denominator of the expression for the derivative of $\phi_{l(e,jj)}$ with the one that corresponds to the Jacobian of the change of coordinates.

Using Gaussian quadrature we have

$$\boxed{a^r_{ii,jj,\check{u}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\check{u}(pp)\phi_{ii}(pp)\left(\sum_{mm=1}^{6} T_{jj,mm}(pp)z_{e,mm}\right)}. \tag{45.25}$$

From (25.131) we also have

$$a^z_{ii,jj,\check{w}}(e) = \int\limits_{\bar{\Omega}_e} \check{w}\phi_{l(e,ii)}\partial_z\phi_{l(e,jj)}, \tag{45.26}$$

which we can re-write as

$$a^z_{ii,jj,\check{w}}(e) = -\int\limits_E \check{w}\phi_{ii}\left(\sum_{mm=1}^{6} T_{jj,mm}r_{e,mm}\right). \tag{45.27}$$

Using Gaussian quadrature we have

$$\boxed{a^z_{ii,jj,\check{w}}(e) \approx -\sum_{pp=1}^{n_G} W(pp)\check{w}(pp)\phi_{ii}(pp)\left(\sum_{mm=1}^{6} T_{jj,mm}(pp)r_{e,mm}\right)}. \tag{45.28}$$

From (25.131) we also have

$$a_{ii,jj,\partial_r\check{u}}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}\partial_r\check{u}, \tag{45.29}$$

which we can re-write as

$$a_{ii,jj,\partial_r\check{u}}(e) = \int\limits_E \phi_{ii}\phi_{jj}\partial_r\check{u}\det J_e, \tag{45.30}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,jj,\partial_r\breve{u}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\phi_{jj}(pp)\partial_r\breve{u}(pp)\det J_e(pp)}. \tag{45.31}$$

From (25.131) we also have

$$a_{ii,jj,\partial_z\breve{u}}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\phi_{l(e,jj)}\partial_z\breve{u}, \tag{45.32}$$

which we can re-write as

$$a_{ii,jj,\partial_z\breve{u}}(e) = \int\limits_{E} \phi_{ii}\phi_{jj}\partial_z\breve{u}\det J_e, \tag{45.33}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,jj,\partial_z\breve{u}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\phi_{jj}(pp)\partial_z\breve{u}(pp)\det J_e(pp)}. \tag{45.34}$$

From (26.121) we have

$$a_{g_z,ii}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\hat{\boldsymbol{g}}_z, \tag{45.35}$$

which we can re-write as

$$a_{g_z,ii}(e) = \int\limits_{E} \phi_{ii}\hat{\boldsymbol{g}}_z\det J_e. \tag{45.36}$$

Using Gaussian quadrature we have

$$\boxed{a_{g_z,ii}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\hat{\boldsymbol{g}}_z(pp)\det J_e(pp)}. \tag{45.37}$$

Also from (26.121) we have

$$a_{ii,\breve{w}}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\breve{w}, \tag{45.38}$$

which we can re-write as

$$a_{ii,\breve{w}}(e) = \int\limits_{E} \phi_{ii}\breve{w}\det J_e, \tag{45.39}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,\breve{w}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\breve{w}(pp)\det J_e(pp)}. \tag{45.40}$$

From (26.121) we have

$$a_{ii,\breve{w}_{n-1}}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)}\breve{w}_{n-1}, \tag{45.41}$$

which we can re-write as

$$a_{ii,\check{w}_{n-1}}(e) = \int\limits_{E} \phi_{ii} \check{w}_{n-1} \det J_e, \tag{45.42}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,\check{w}_{n-1}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\check{w}_{n-1}(pp) \det J_e(pp)}. \tag{45.43}$$

From (26.121) we have

$$a_{ii,\check{w}_{n-2}}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{w}_{n-2}, \tag{45.44}$$

which we can re-write as

$$a_{ii,\check{w}_{n-2}}(e) = \int\limits_{E} \phi_{ii} \check{w}_{n-2} \det J_e, \tag{45.45}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,\check{w}_{n-2}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\check{w}_{n-2}(pp) \det J_e(pp)}. \tag{45.46}$$

Also from (26.121) we have

$$a_{ii,\check{u}\partial_r\check{w}}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{u}\partial_r\check{w}, \tag{45.47}$$

which we can re-write as

$$a_{ii,\check{u}\partial_r\check{w}}(e) = \int\limits_{E} \phi_{ii} \check{u}\partial_r\check{w} \det J_e, \tag{45.48}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,\check{u}\partial_r\check{w}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\check{u}(pp)\partial_r\check{w}(pp) \det J_e(pp)}. \tag{45.49}$$

From (26.121) we also have

$$a_{ii,\check{w}\partial_z\check{w}}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \check{w}\partial_z\check{w}, \tag{45.50}$$

which we can re-write as

$$a_{ii,\check{w}\partial_z\check{w}}(e) = \int\limits_{E} \phi_{ii} \check{w}\partial_z\check{w} \det J_e, \tag{45.51}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,\check{w}\partial_z\check{w}}(e) \approx \sum_{pp=1}^{n_G} W(p)\phi_{ii}(pp)\check{w}(pp)\partial_z\check{w}(pp) \det J_e(pp)}. \tag{45.52}$$

From (26.122) we also have

$$a_{ii,jj,\partial_r \check{w}}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,jj)} \partial_r \check{w}, \tag{45.53}$$

which we can re-write as

$$a_{ii,jj,\partial_r \check{w}}(e) = \int\limits_{E} \phi_{ii} \phi_{jj} \partial_r \check{w} \det J_e, \tag{45.54}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,jj,\partial_r \check{w}}(e) \approx \sum_{pp=1}^{n_G} W(pp) \phi_{ii}(pp) \phi_{jj}(pp) \partial_r \check{w}(pp) \det J_e(pp)}. \tag{45.55}$$

From (26.122) we also have

$$a_{ii,jj,\partial_z \check{w}}(e) = \int\limits_{\bar{\Omega}_e} \phi_{l(e,ii)} \phi_{l(e,jj)} \partial_z \check{w}, \tag{45.56}$$

which we can re-write as

$$a_{ii,jj,\partial_z \check{w}}(e) = \int\limits_{E} \phi_{ii} \phi_{jj} \partial_z \check{w} \det J_e, \tag{45.57}$$

and using Gaussian quadrature we have

$$\boxed{a_{ii,jj,\partial_z \check{w}}(e) \approx \sum_{pp=1}^{n_G} W(pp) \phi_{ii}(pp) \phi_{jj}(pp) \partial_z \check{w}(pp) \det J_e(pp)}. \tag{45.58}$$

45.1. **Derivatives of integrals over triangular elements**

The expressions given in section 45 contain all terms that depend of the coordinates of each element and that were added by the presence of the eigen-solution in the obtuse angle formulation. That is to say, the residuals are given by the product of these expressions by variables that (for the purpose of the resulting non-linear system of equations) do not depend of the variables (h) that determine the shape of our domain. Therefore, in order to calculate the derivatives of the residuals with respect to the lengths of the spines, we need to calculate the derivatives of these expressions.

Furthermore, in the integral expressions in section 45, all functions are independent of the location of the nodes, except for $r_{e,mm}$, $z_{e,mm}$, $\check{u}$, $\check{w}$, $\partial_r\check{u}$, $\partial_r\check{w}$, $\partial_z\check{u}$, $\partial_z\check{w}$ and $\det J_e$. We recall now the derivative of $\det J_e$

$$\partial_{h_q} \det J_e = \sum_{ii=1}^{6}\sum_{jj=1}^{6} \partial_{h_q}\left(r_{e,ii}T_{ii,jj}z_{e,jj}\right), \tag{45.59}$$

which yields

$$\boxed{\partial_{h_q} \det J_e = \sum_{ii=1}^{6}\sum_{jj=1}^{6} \left[\left(\partial_{h_q} r^e_{ii}\right)T_{ii,jj}z_{e,jj} + r_{e,ii}T_{ii,jj}\left(\partial_{h_q} z_{e,jj}\right)\right]}, \tag{45.60}$$

which reduces the problem to finding the derivatives of $r_{e,ii}$ and $z_{e,jj}$ with respect to each h_q. The way to calculate the latter two derivatives was presented in section 22.

Moreover, the derivatives of $\check{u}$, $\check{w}$, $\partial_r\check{u}$, $\partial_r\check{w}$, $\partial_z\check{u}$ and $\partial_z\check{w}$ with respect to h_q can be calculated using equations (B 1)-(B 32). We take derivatives of each of those expressions with respect to r and z, and using the chain rule, we can reduce the problem of finding these derivatives to that of finding $\partial_{h_q} r$ and $\partial_{h_q} z$.

45.1.1. **Derivatives of a terms**

From (20.17), we have

$$\partial_{h_q} a_{g_r,ii}(e) = \partial_{h_q} \int_E \phi_{ii}\hat{\boldsymbol{g}}_r \det J_e. \tag{45.61}$$

i.e.

$$\partial_{h_q} a_{g_r,ii}(e) = \int_E \phi_{ii}\hat{\boldsymbol{g}}_r \partial_{h_q} \det J_e. \tag{45.62}$$

and using Gaussian quadrature we have

$$\boxed{\partial_{h_q} a_{g_r,ii}(e) \approx \sum_{p=1}^{n_G} W(p)\phi_{ii}(p)\hat{\boldsymbol{g}}_r(p)\partial_{h_q} \det J_e(p)}. \tag{45.63}$$

From equation (45.6) we have

$$\partial_{h_q} a_{ii,\check{u}}(e) = \partial_{h_q} \int_E \check{u}(r_e(\xi,\eta), z_e(\xi,\eta))\phi_{ii} \det J_e, \tag{45.64}$$

i.e.

$$\partial_{h_q} a_{ii,\breve{u}}(e) = \int_E \phi_{ii} \partial_r \breve{u}(r_e(\xi,\eta), z_e(\xi,\eta)) \left(\partial_{h_q} r_e\right) \det J_e + \int_E \phi_{ii} \partial_z \breve{u}(r_e(\xi,\eta), z_e(\xi,\eta)) \left(\partial_{h_q} z_e\right) \det J_e + \int_E \phi_{ii} \breve{u}(r_e(\xi,\eta), z_e(\xi,\eta)) \left(\partial_{h_q} \det J_e\right). \quad (45.65)$$

Now, using Gaussian quadrature we have

$$\boxed{\begin{aligned}\partial_{h_q} a_{ii,\breve{u}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp) \Bigg\{ &\partial_r \breve{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp) \partial_{h_q} r_{e,mm}\right] \det J_e(pp) \\ &+ \partial_z \breve{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp) \partial_{h_q} z_{e,mm}\right] \det J_e(pp) \\ &+ \breve{u}(pp) \partial_{h_q} \det J_e(pp) \Bigg\}. \quad (45.66)\end{aligned}}$$

From equation (45.10) we have

$$\partial_{h_q} a_{ii,\breve{u}_{n-1}}(e) = \partial_{h_q} \int_E \breve{u}_{n-1}(r_e(\xi,\eta), z_e(\xi,\eta)) \phi_{ii} \det J_e, \quad (45.67)$$

i.e.

$$\partial_{h_q} a_{ii,\breve{u}_{n-1}}(e) = \int_E \phi_{ii} \breve{u}_{n-1}(r_e(\xi,\eta), z_e(\xi,\eta)) \partial_{h_q} \det J_e. \quad (45.68)$$

Now, using Gaussian quadrature we have

$$\boxed{\partial_{h_q} a_{ii,\breve{u}_{n-1}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp) \breve{u}_{n-1}(pp) \partial_{h_q} \det J_e(pp). \quad (45.69)}$$

From equation (45.13) we have

$$\partial_{h_q} a_{ii,\breve{u}_{n-2}}(e) = \partial_{h_q} \int_E \breve{u}_{n-2}(r_e(\xi,\eta), z_e(\xi,\eta)) \phi_{ii} \det J_e, \quad (45.70)$$

i.e.

$$\partial_{h_q} a_{ii,\breve{u}_{n-2}}(e) = \int_E \phi_{ii} \breve{u}_{n-2}(r_e(\xi,\eta), z_e(\xi,\eta)) \partial_{h_q} \det J_e. \quad (45.71)$$

Now, using Gaussian quadrature we have

$$\boxed{\partial_{h_q} a_{ii,\breve{u}_{n-2}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp) \breve{u}_{n-2}(pp) \partial_{h_q} \det J_e(pp). \quad (45.72)}$$

From equation (45.17) we have

$$\partial_{h_q} a_{\breve{u}\partial_r\breve{u},ii}(e) = \partial_{h_q} \int\limits_E \phi_{ii}\breve{u}\partial_r\breve{u} \det J_e, \tag{45.73}$$

which yields

$$\partial_{h_q} a_{\breve{u}\partial_r\breve{u},ii}(e) = \int\limits_E \phi_{ii} \left(\partial_{h_q}\breve{u}\right) \partial_r\breve{u} \det J_e + \int\limits_E \phi_{ii}\breve{u} \left(\partial_{h_q}\partial_r\breve{u}\right) \det J_e + \int\limits_E \phi_{ii}\breve{u}\partial_r\breve{u}\partial_{h_q} \det J_e, \tag{45.74}$$

i.e.

$$\begin{aligned}
\partial_{h_q} a_{\breve{u}\partial_r\breve{u},ii}(e) = &\int\limits_E \phi_{ii} \left(\partial_r\breve{u}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} r_{e,mm}\right)\right] \left(\partial_r\breve{u}\right) \det J_e \\
&+ \int\limits_E \phi_{ii} \left(\partial_z\breve{u}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} z_{e,mm}\right)\right] \left(\partial_r\breve{u}\right) \det J_e \\
&+ \int\limits_E \phi_{ii}\breve{u} \left(\partial_{rr}\breve{u}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} r_{e,mm}\right)\right] \det J_e \\
&+ \int\limits_E \phi_{ii}\breve{u} \left(\partial_{rz}\breve{u}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} z_{e,mm}\right)\right] \det J_e \\
&+ \int\limits_E \phi_{ii}\breve{u} \left(\partial_r\breve{u}\right) \partial_{h_q} \det J_e.
\end{aligned} \tag{45.75}$$

Now, using Gaussian quadrature we have

$$\boxed{\begin{aligned}
&\partial_{h_q} a_{\breve{u}\partial_r\breve{u},ii}(e) \\
&\quad\approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp) \Bigg\{ \partial_r\breve{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} r_{e,mm}\right] \partial_r\breve{u}(pp) \det J_e(pp) \\
&\qquad + \partial_z\breve{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} z_{e,mm}\right] \partial_r\breve{u}(pp) \det J_e(pp) \\
&\qquad + \breve{u}(pp)\partial_{rr}\breve{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} r_{e,mm}\right] \det J_e(pp) \\
&\qquad + \breve{u}(pp)\partial_{rz}\breve{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} z_{e,mm}\right] \det J_e(pp) \\
&\qquad + \breve{u}(pp)\partial_r\breve{u}(pp)\partial_{h_q} \det J_e(pp) \Bigg\}.
\end{aligned}} \tag{45.76}$$

From equation (45.21) we have

$$\partial_{h_q} a_{\breve{w}\partial_z\breve{u},ii}(e) = \partial_{h_q} \int\limits_E \phi_{ii}\breve{w}\partial_z\breve{u} \det J_e, \tag{45.77}$$

which yields

$$\partial_{h_q} a_{\check{w}\partial_z\check{u},ii}(e) = \int\limits_E \phi_{ii}\left(\partial_{h_q}\check{w}\right)\partial_z\check{u}\det J_e + \int\limits_E \phi_{ii}\check{w}\left(\partial_{h_q}\partial_z\check{u}\right)\det J_e + \int\limits_E \phi_{ii}\check{w}\partial_z\check{u}\partial_{h_q}\det J_e, \tag{45.78}$$

i.e.

$$\begin{aligned}\partial_{h_q} a_{\check{w}\partial_z\check{u},ii}(e) = &\int\limits_E \phi_{ii}\left(\partial_r\check{w}\right)\left[\sum_{mm=1}^{6}\phi_{mm}\left(\partial_{h_q} r_{e,mm}\right)\right]\left(\partial_z\check{u}\right)\det J_e \\ &+ \int\limits_E \phi_{ii}\left(\partial_z\check{w}\right)\left[\sum_{mm=1}^{6}\phi_{mm}\left(\partial_{h_q} z_{e,mm}\right)\right]\left(\partial_z\check{u}\right)\det J_e \\ &+ \int\limits_E \phi_{ii}\check{w}\left(\partial_{rz}\check{u}\right)\left[\sum_{mm=1}^{6}\phi_{mm}\left(\partial_{h_q} r_{e,mm}\right)\right]\det J_e \\ &+ \int\limits_E \phi_{ii}\check{w}\left(\partial_{zz}\check{u}\right)\left[\sum_{mm=1}^{6}\phi_{mm}\left(\partial_{h_q} z_{e,mm}\right)\right]\det J_e \\ &+ \int\limits_E \phi_{ii}\check{w}\left(\partial_z\check{u}\right)\partial_{h_q}\det J_e.\end{aligned} \tag{45.79}$$

Now, using Gaussian quadrature we have

$$\begin{aligned}&\partial_{h_q} a_{\check{w}\partial_z\check{u},ii}(e) \\ &\approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\Bigg\{\partial_r\check{w}(pp)\left[\sum_{mm=1}^{6}\phi_{mm}(pp)\partial_{h_q} r_{e,mm}\right]\partial_z\check{u}(pp)\det J_e(pp) \\ &\qquad + \partial_z\check{w}(pp)\left[\sum_{mm=1}^{6}\phi_{mm}(pp)\partial_{h_q} z_{e,mm}\right]\partial_z\check{u}(pp)\det J_e(pp) \\ &\qquad + \check{w}(pp)\partial_{rz}\check{u}(pp)\left[\sum_{mm=1}^{6}\phi_{mm}(pp)\partial_{h_q} r_{e,mm}\right]\det J_e(pp) \\ &\qquad + \check{w}(pp)\partial_{zz}\check{u}(pp)\left[\sum_{mm=1}^{6}\phi_{mm}(pp)\partial_{h_q} z_{e,mm}\right]\det J_e(pp) \\ &\qquad + \check{w}(pp)\partial_z\check{u}(pp)\partial_{h_q}\det J_e(pp)\Bigg\}.\end{aligned} \tag{45.80}$$

From equation (45.24) we have

$$\partial_{h_q} a^r_{\check{u},ii,jj}(e) = \partial_{h_q}\int\limits_E \check{u}\phi_{ii}\left(\sum_{mm=1}^{6} T_{jj,mm} z_{e,mm}\right), \tag{45.81}$$

which yields

$$\partial_{h_q} a^r_{\check{u},ii,jj}(e) = \int_E \phi_{ii} \left(\partial_{h_q} \check{u}\right) \left(\sum_{mm=1}^{6} T_{jj,mm} z_{e,mm} \right) + \int_E \phi_{ii} \check{u} \left(\sum_{mm=1}^{6} T_{jj,mm} \partial_{h_q} z_{e,mm} \right), \tag{45.82}$$

i.e.

$$\begin{aligned} \partial_{h_q} a^r_{\check{u},ii,jj}(e) = &\int_E \phi_{ii} \left(\partial_r \check{u}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \partial_{h_q} r_{e,mm} \right] \left(\sum_{mm=1}^{6} T_{jj,mm} z_{e,mm} \right) \\ &+ \int_E \phi_{ii} \left(\partial_z \check{u}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \partial_{h_q} z_{e,mm} \right] \left(\sum_{mm=1}^{6} T_{jj,mm} z_{e,mm} \right) \\ &+ \int_E \phi_{ii} \check{u} \left(\sum_{mm=1}^{6} T_{jj,mm} \partial_{h_q} z_{e,mm} \right). \end{aligned} \tag{45.83}$$

Now, using Gaussian quadrature we have

$$\boxed{\begin{aligned} &\partial_{h_q} a^r_{\check{u},ii,jj}(e) \\ &= \sum_{pp=1}^{n_G} W(pp) \phi_{ii} \Bigg\{ \partial_r \check{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp) \partial_{h_q} r_{e,mm} \right] \left(\sum_{mm=1}^{6} T_{jj,mm}(pp) z_{e,mm} \right) \\ &\qquad + \partial_z \check{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp) \partial_{h_q} z_{e,mm} \right] \left(\sum_{mm=1}^{6} T_{jj,mm}(pp) z_{e,mm} \right) \\ &\qquad + \check{u}(pp) \left(\sum_{mm=1}^{6} T_{jj,mm}(pp) \partial_{h_q} z_{e,mm} \right) \Bigg\}, \end{aligned} \quad (45.84)}$$

From equation (45.27) we have

$$\partial_{h_q} a^z_{\check{w},ii,jj}(e) = -\partial_{h_q} \int_E \phi_{ii} \check{w} \left(\sum_{mm=1}^{6} T_{jj,mm} r_{e,mm} \right), \tag{45.85}$$

which yields

$$\begin{aligned} &\partial_{h_q} a^z_{\check{w},ii,jj}(e) \\ &= -\int_E \phi_{ii} \left(\partial_{h_q} \check{w}\right) \left(\sum_{mm=1}^{6} T_{jj,mm} r_{e,mm} \right) - \int_E \phi_{ii} \check{w} \left(\sum_{mm=1}^{6} T_{jj,mm} \partial_{h_q} r_{e,mm} \right), \end{aligned} \tag{45.86}$$

i.e.

$$\begin{aligned}\partial_{h_q} a^z_{\breve{w},ii,jj}(e) = &-\int_E \phi_{ii}\left(\partial_r \breve{w}\right)\left[\sum_{mm=1}^{6} \phi_{mm}\partial_{h_q} r_{e,mm}\right]\left(\sum_{mm=1}^{6} T_{jj,mm} r_{e,mm}\right)\\ &-\int_E \phi_{ii}\left(\partial_z \breve{w}\right)\left[\sum_{mm=1}^{6} \phi_{mm}\partial_{h_q} z_{e,mm}\right]\left(\sum_{mm=1}^{6} T_{jj,mm} r_{e,mm}\right)\\ &-\int_E \phi_{ii}\breve{w}\left(\sum_{mm=1}^{6} T_{jj,mm}\partial_{h_q} r_{e,mm}\right).\end{aligned} \tag{45.87}$$

Now, using Gaussian quadrature we have

$$\begin{aligned}&\partial_{h_q} a^z_{\breve{w},ii,jj}(e)\\ &\quad= -\sum_{pp=1}^{n_G} \phi_{ii}(pp)\left\{\partial_r \breve{w}(pp)\left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} r_{e,mm}\right]\left(\sum_{mm=1}^{6} T_{jj,mm}(pp) r_{e,mm}\right)\right.\\ &\qquad\qquad + \partial_z \breve{w}(pp)\left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} z_{e,mm}\right]\left(\sum_{mm=1}^{6} T_{jj,mm}(pp) r_{e,mm}\right)\\ &\qquad\qquad \left. + \breve{w}(pp)\left(\sum_{mm=1}^{6} T_{jj,mm}(pp)\partial_{h_q} r_{e,mm}\right)\right\},\end{aligned} \tag{45.88}$$

From equation (45.30) we have

$$\partial_{h_q} a_{ii,jj,\partial_r \breve{u}}(e) = \partial_{h_q} \int_E \phi_{ii}\phi_{jj}\partial_r \breve{u} \det J_e, \tag{45.89}$$

which yields

$$\partial_{h_q} a_{ii,jj,\partial_r \breve{u}}(e) = \int_E \phi_{ii}\phi_{jj}\left(\partial_{h_q}\partial_r \breve{u}\right)\det J_e + \int_E \phi_{ii}\phi_{jj}\partial_r \breve{u}\partial_{h_q} \det J_e, \tag{45.90}$$

i.e.

$$\begin{aligned}\partial_{h_q} a_{ii,jj,\partial_r \breve{u}}(e) = &\int_E \phi_{ii}\phi_{jj}\left(\partial_{rr}\breve{u}\right)\left[\sum_{mm=1}^{6} \phi_{mm}\left(\partial_{h_q} r_{e,mm}\right)\right]\det J_e\\ &+\int_E \phi_{ii}\phi_{jj}\left(\partial_{rz}\breve{u}\right)\left[\sum_{mm=1}^{6} \phi_{mm}\left(\partial_{h_q} z_{e,mm}\right)\right]\det J_e\\ &+\int_E \phi_{ii}\phi_{jj}\left(\partial_r \breve{u}\right)\partial_{h_q} \det J_e.\end{aligned} \tag{45.91}$$

Now, using Gaussian quadrature we have

$$
\begin{aligned}
&\partial_{h_q} a_{ii,jj,\partial_r \breve{u}}(e) \\
&\quad \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\phi_{jj}(pp) \left\{ \partial_{rr}\breve{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} r_{e,mm} \right] \det J_e(pp) \right. \\
&\qquad \left. + \partial_{rz}\breve{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} z_{e,mm} \right] \det J_e(pp) + \partial_r \breve{u}(pp)\partial_{h_q} \det J_e(pp) \right\}.
\end{aligned} \tag{45.92}
$$

From equation (45.33) we have

$$\partial_{h_q} a_{\partial_z \breve{u},ii,jj}(e) = \partial_{h_q} \int_E \phi_{ii}\phi_{jj}\partial_z \breve{u} \det J_e, \tag{45.93}$$

which yields

$$\partial_{h_q} a_{\partial_z \breve{u},ii,jj}(e) = \int_E \phi_{ii}\phi_{jj} \left(\partial_{h_q}\partial_z \breve{u}\right) \det J_e + \int_E \phi_{ii}\phi_{jj}\partial_z \breve{u}\partial_{h_q} \det J_e, \tag{45.94}$$

i.e.

$$
\begin{aligned}
\partial_{h_q} a_{\partial_z \breve{u},ii,jj}(e) = &\int_E \phi_{ii}\phi_{jj} \left(\partial_{rz}\breve{u}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} r_{e,mm}\right) \right] \det J_e \\
&+ \int_E \phi_{ii}\phi_{jj} \left(\partial_{zz}\breve{u}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} z_{e,mm}\right) \right] \det J_e \\
&+ \int_E \phi_{ii}\phi_{jj} \left(\partial_z \breve{u}\right) \partial_{h_q} \det J_e.
\end{aligned} \tag{45.95}
$$

Now, using Gaussian quadrature we have

$$
\begin{aligned}
&\partial_{h_q} a_{\partial_z \breve{u},ii,jj}(e) \\
&\quad \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\phi_{jj}(pp) \left\{ \partial_{rz}\breve{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} r_{e,mm} \right] \det J_e(pp) \right. \\
&\qquad \left. + \partial_{zz}\breve{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} z_{e,mm} \right] \det J_e(pp) + \partial_z \breve{u}(pp)\partial_{h_q} \det J_e(pp) \right\}.
\end{aligned} \tag{45.96}
$$

From (20.42), we have

$$\partial_{h_q} a_{g_z,ii}(e) = \partial_{h_q} \int_E \phi_{ii}\hat{\boldsymbol{g}}_z \det J_e. \tag{45.97}$$

i.e.

$$\partial_{h_q} a_{g_z,ii}(e) = \int_E \phi_{ii}\hat{\boldsymbol{g}}_z \partial_{h_q} \det J_e. \tag{45.98}$$

and using Gaussian quadrature we have

$$\boxed{\partial_{h_q} a_{g_z,ii}(e) \approx \sum_{p=1}^{n_G} W(p)\phi_{ii}(p)\hat{\boldsymbol{g}}_z(p)\partial_{h_q} \det J_e(p)}. \tag{45.99}$$

From equation (45.39) we have

$$\partial_{h_q} a_{ii,\breve{w}}(e) = \partial_{h_q} \int_E \breve{w}(r_e(\xi,\eta), z_e(\xi,\eta))\phi_{ii} \det J_e, \tag{45.100}$$

i.e.

$$\begin{aligned}\partial_{h_q} a_{ii,\breve{w}}(e) = &\int_E \phi_{ii}\partial_r \breve{w}(r_e(\xi,\eta), z_e(\xi,\eta)) \left(\partial_{h_q} r_e\right) \det J_e \\ &+ \int_E \phi_{ii}\partial_z \breve{w}(r_e(\xi,\eta), z_e(\xi,\eta)) \left(\partial_{h_q} z_e\right) \det J_e \\ &+ \int_E \phi_{ii}\breve{w}(r_e(\xi,\eta), z_e(\xi,\eta)) \left(\partial_{h_q} \det J_e\right).\end{aligned} \tag{45.101}$$

Now, using Gaussian quadrature we have

$$\boxed{\begin{aligned}\partial_{h_q} a_{ii,\breve{w}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp) \Bigg\{ &\partial_r \breve{w}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} r_{e,mm}\right] \det J_e(pp) \\ &+ \partial_z \breve{w}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} z_{e,mm}\right] \det J_e(pp) \\ &+ \breve{w}(pp)\partial_{h_q} \det J_e(pp) \Bigg\}.\end{aligned}} \tag{45.102}$$

From equation (45.42) we have

$$\partial_{h_q} a_{ii,\breve{w}_{n-1}}(e) = \partial_{h_q} \int_E \breve{w}_{n-1}(r_e(\xi,\eta), z_e(\xi,\eta))\phi_{ii} \det J_e, \tag{45.103}$$

i.e.

$$\partial_{h_q} a_{ii,\breve{w}_{n-1}}(e) = \int_E \phi_{ii}\breve{w}_{n-1}(r_e(\xi,\eta), z_e(\xi,\eta))\partial_{h_q} \det J_e. \tag{45.104}$$

Now, using Gaussian quadrature we have

$$\boxed{\partial_{h_q} a_{ii,\breve{w}_{n-1}}(e) \approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\breve{w}_{n-1}(pp)\partial_{h_q} \det J_e(pp).} \tag{45.105}$$

From equation (45.45) we have

$$\partial_{h_q} a_{ii,\breve{w}_{n-2}}(e) = \partial_{h_q} \int_E \breve{w}_{n-2}(r_e(\xi,\eta), z_e(\xi,\eta))\phi_{ii} \det J_e, \tag{45.106}$$

i.e.

$$\partial_{h_q} a_{ii,\breve{w}_{n-2}}(e) = \int\limits_E \phi_{ii} \breve{w}_{n-2}(r_e(\xi,\eta), z_e(\xi,\eta)) \partial_{h_q} \det J_e. \tag{45.107}$$

Now, using Gaussian quadrature we have

$$\boxed{\partial_{h_q} a_{ii,\breve{w}_{n-2}}(e) \approx \sum_{pp=1}^{n_G} W(pp) \phi_{ii}(pp) \breve{w}_{n-2}(pp) \partial_{h_q} \det J_e(pp).} \tag{45.108}$$

From equation (45.48) we have

$$\partial_{h_q} a_{\breve{u}\partial_r\breve{w},ii}(e) = \partial_{h_q} \int\limits_E \phi_{ii} \breve{u} \partial_r \breve{w} \det J_e, \tag{45.109}$$

which yields

$$\begin{aligned} \partial_{h_q} a_{\breve{u}\partial_r\breve{w},ii}(e) = &\int\limits_E \phi_{ii} \left(\partial_{h_q} \breve{u}\right) \left(\partial_r \breve{w}\right) \det J_e + \int\limits_E \phi_{ii} \breve{u} \left(\partial_{h_q} \partial_r \breve{w}\right) \det J_e \\ &+ \int\limits_E \phi_{ii} \breve{u} \left(\partial_r \breve{w}\right) \partial_{h_q} \det J_e, \end{aligned} \tag{45.110}$$

i.e.

$$\begin{aligned} \partial_{h_q} a_{\breve{u}\partial_r\breve{w},ii}(e) = &\int\limits_E \phi_{ii} \left(\partial_r \breve{u}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} r_{e,mm}\right)\right] \left(\partial_r \breve{w}\right) \det J_e \\ &+ \int\limits_E \phi_{ii} \left(\partial_z \breve{u}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} z_{e,mm}\right)\right] \left(\partial_r \breve{w}\right) \det J_e \\ &+ \int\limits_E \phi_{ii} \breve{u} \left(\partial_{rr} \breve{w}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} r_{e,mm}\right)\right] \det J_e \\ &+ \int\limits_E \phi_{ii} \breve{u} \left(\partial_{rz} \breve{w}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} z_{e,mm}\right)\right] \det J_e \\ &+ \int\limits_E \phi_{ii} \breve{u} \left(\partial_r \breve{w}\right) \partial_{h_q} \det J_e. \end{aligned} \tag{45.111}$$

Now, using Gaussian quadrature we have

$$\begin{aligned}
&\partial_{h_q} a_{\check{u}\partial_r\check{w},ii}(e) \\
&\quad\approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp) \Bigg\{ \partial_r \check{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} r_{e,mm} \right] \partial_r \check{w}(pp) \det J_e(pp) \\
&\qquad + \partial_z \check{u}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} z_{e,mm} \right] \partial_r \check{w}(pp) \det J_e(pp) \\
&\qquad + \check{u}(pp)\partial_{rr}\check{w}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} r_{e,mm} \right] \det J_e(pp) \\
&\qquad + \check{u}(pp)\partial_{rz}\check{w}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} z_{e,mm} \right] \det J_e(pp) \\
&\qquad + \check{u}(pp)\partial_r \check{w}(pp)\partial_{h_q} \det J_e(pp) \Bigg\}.
\end{aligned} \tag{45.112}$$

From equation (45.51) we have

$$\partial_{h_q} a_{\check{w}\partial_z\check{w},ii}(e) = \partial_{h_q} \int_E \phi_{ii}\check{w}\partial_z\check{w} \det J_e, \tag{45.113}$$

which yields

$$\begin{aligned}
\partial_{h_q} a_{\check{w}\partial_z\check{w},ii}(e) &= \int_E \phi_{ii} \left(\partial_{h_q}\check{w}\right) \partial_z\check{w} \det J_e + \int_E \phi_{ii}\check{w} \left(\partial_{h_q}\partial_z\check{w}\right) \det J_e \\
&\quad + \int_E \phi_{ii}\check{w}\partial_z\check{w}\partial_{h_q} \det J_e,
\end{aligned} \tag{45.114}$$

i.e.

$$\begin{aligned}
\partial_{h_q} a_{\check{w}\partial_z\check{w},ii}(e) &= \int_E \phi_{ii} \left(\partial_r\check{w}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} r_{e,mm}\right) \right] \left(\partial_z\check{w}\right) \det J_e \\
&\quad + \int_E \phi_{ii} \left(\partial_z\check{w}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} z_{e,mm}\right) \right] \left(\partial_z\check{w}\right) \det J_e \\
&\quad + \int_E \phi_{ii}\check{w} \left(\partial_{rz}\check{w}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} r_{e,mm}\right) \right] \det J_e \\
&\quad + \int_E \phi_{ii}\check{w} \left(\partial_{zz}\check{w}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} z_{e,mm}\right) \right] \det J_e \\
&\quad + \int_E \phi_{ii}\check{w} \left(\partial_z\check{w}\right) \partial_{h_q} \det J_e.
\end{aligned} \tag{45.115}$$

Now, using Gaussian quadrature we have

$$
\begin{aligned}
&\partial_{h_q} a_{\breve{w}\partial_z\breve{w},ii}(e) \\
&\quad\approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\left\{\partial_r\breve{w}(pp)\left[\sum_{mm=1}^{6}\phi_{mm}(pp)\partial_{h_q}r_{e,mm}\right]\partial_z\breve{w}(pp)\det J_e(pp)\right.\\
&\qquad\qquad + \partial_z\breve{w}(pp)\left[\sum_{mm=1}^{6}\phi_{mm}(pp)\partial_{h_q}z_{e,mm}\right]\partial_z\breve{w}(pp)\det J_e(pp)\\
&\qquad\qquad + \breve{w}(pp)\partial_{rz}\breve{w}(pp)\left[\sum_{mm=1}^{6}\phi_{mm}(pp)\partial_{h_q}r_{e,mm}\right]\det J_e(pp)\\
&\qquad\qquad + \breve{w}(pp)\partial_{zz}\breve{w}(pp)\left[\sum_{mm=1}^{6}\phi_{mm}(pp)\partial_{h_q}z_{e,mm}\right]\det J_e(pp)\\
&\qquad\qquad \left. + \breve{w}(pp)\partial_z\breve{w}(pp)\partial_{h_q}\det J_e(pp)\right\}.
\end{aligned}
\tag{45.116}
$$

From equation (45.54) we have

$$
\partial_{h_q} a_{\partial_r\breve{w},ii,jj}(e) = \partial_{h_q}\int_E \phi_{ii}\phi_{jj}\left(\partial_r\breve{w}\right)\det J_e, \tag{45.117}
$$

which yields

$$
\partial_{h_q} a_{\partial_r\breve{w},ii,jj}(e) = \int_E \phi_{ii}\phi_{jj}\left(\partial_{h_q}\partial_r\breve{w}\right)\det J_e + \int_E \phi_{ii}\phi_{jj}\left(\partial_r\breve{w}\right)\partial_{h_q}\det J_e, \tag{45.118}
$$

i.e.

$$
\begin{aligned}
\partial_{h_q} a_{\partial_r\breve{w},ii,jj}(e) &= \int_E \phi_{ii}\phi_{jj}\left(\partial_{rr}\breve{w}\right)\left[\sum_{mm=1}^{6}\phi_{mm}\left(\partial_{h_q}r_{e,mm}\right)\right]\det J_e\\
&\quad + \int_E \phi_{ii}\phi_{jj}\left(\partial_{rz}\breve{w}\right)\left[\sum_{mm=1}^{6}\phi_{mm}\left(\partial_{h_q}z_{e,mm}\right)\right]\det J_e\\
&\quad + \int_E \phi_{ii}\phi_{jj}\left(\partial_r\breve{w}\right)\partial_{h_q}\det J_e.
\end{aligned}
\tag{45.119}
$$

Now, using Gaussian quadrature we have

$$
\begin{aligned}
&\partial_{h_q} a_{\partial_r\breve{w},ii,jj}(e) \\
&\quad\approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\phi_{jj}(pp)\left\{\partial_{rr}\breve{w}(pp)\left[\sum_{mm=1}^{6}\phi_{mm}(pp)\partial_{h_q}r_{e,mm}\right]\det J_e(pp)\right.\\
&\qquad \left. + \partial_{rz}\breve{w}(pp)\left[\sum_{mm=1}^{6}\phi_{mm}(pp)\partial_{h_q}z_{e,mm}\right]\det J_e(pp) + \partial_r\breve{w}(pp)\partial_{h_q}\det J_e(pp)\right\}.
\end{aligned}
\tag{45.120}
$$

From equation (45.57) we have

$$\partial_{h_q} a_{\partial_z \breve{w},ii,jj}(e) = \partial_{h_q} \int_E \phi_{ii}\phi_{jj} \left(\partial_z \breve{w}\right) \det J_e, \tag{45.121}$$

which yields

$$\partial_{h_q} a_{\partial_z \breve{w},ii,jj}(e) = \int_E \phi_{ii}\phi_{jj} \left(\partial_{h_q}\partial_z \breve{w}\right) \det J_e + \int_E \phi_{ii}\phi_{jj} \left(\partial_z \breve{w}\right) \partial_{h_q} \det J_e, \tag{45.122}$$

i.e.

$$\begin{aligned} \partial_{h_q} a_{\partial_z \breve{w},ii,jj}(e) = &\int_E \phi_{ii}\phi_{jj} \left(\partial_{rz} \breve{w}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} r_{e,mm}\right)\right] \det J_e \\ &+ \int_E \phi_{ii}\phi_{jj} \left(\partial_{zz} \breve{w}\right) \left[\sum_{mm=1}^{6} \phi_{mm} \left(\partial_{h_q} z_{e,mm}\right)\right] \det J_e \\ &+ \int_E \phi_{ii}\phi_{jj} \left(\partial_z \breve{w}\right) \partial_{h_q} \det J_e. \end{aligned} \tag{45.123}$$

Now, using Gaussian quadrature we have

$$\boxed{\begin{aligned} &\partial_{h_q} a_{\partial_z \breve{w},ii,jj}(e) \\ &\approx \sum_{pp=1}^{n_G} W(pp)\phi_{ii}(pp)\phi_{jj}(pp) \left\{ \partial_{rz}\breve{w}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} r_{e,mm}\right] \det J_e(pp) \right. \\ &\qquad \left. + \partial_{zz}\breve{w}(pp) \left[\sum_{mm=1}^{6} \phi_{mm}(pp)\partial_{h_q} z_{e,mm}\right] \det J_e(pp) + \partial_z \breve{w}(pp)\partial_{h_q} \det J_e(pp) \right\}. \end{aligned}} \tag{45.124}$$

46. Integrals over line elements for obtuse contact angles

We proceed as we did in section 21, now including the new terms.

46.1. The free-surface line elements

From (25.142) we have

$$c_{ii,n_r,\partial_r\check{u}}(e_1) = \int\limits_{\partial\Omega_{e_1}} n_r^1 \phi^1_{l_1(e_1,ii)} \partial_r \check{u}. \tag{46.1}$$

We recall that, in order to simplify our calculations when we consider the integral above and others on the same boundary, we will ensure that our line elements that lay on the free-surface always correspond to the side of the element containing the nodes of local number 2, 6 and 3 (see figure 10). Hence, line-elements on boundary 1 are easily parameterised by the variable ξ. See section 21 for more details.

We recall that the tangent to the line element is given by

$$\boldsymbol{t}^1 = \frac{(\partial_\xi r^1_{e_1}, \partial_\xi z^1_{e_1})}{\sqrt{\left(\partial_\xi r^1_{e_1}\right)^2 + \left(\partial_\xi z^1_{e_1}\right)^2}}, \tag{46.2}$$

where the tangent points in the direction of increasing ξ and $r^1_{e_1}$ is the r coordinate along on element e_1 on boundary 1. $r^1_{e_1}$ and its analogue for z are defined by the map $S^1_{e_1}$ which takes the interval $[-1,1]$ to the line element in boundary one, i.e.

$$(r^1_{e_1}, z^1_{e_1}) = S^1_{e_1}(\xi). \tag{46.3}$$

We also recall that

$$\boldsymbol{n}^1 = \frac{\alpha(-\partial_\xi z^1_{e_1}, \partial_\xi r^1_{e_1})}{\sqrt{\left(\partial_\xi r^1_{e_1}\right)^2 + \left(\partial_\xi z^1_{e_1}\right)^2}}, \tag{46.4}$$

where $\alpha = 1$ is the rotation is counter-clockwise and $\alpha = -1$ if the rotation is clockwise. On boundary 1 we decided to have the local line-element numbering so as to have $\alpha = 1$.

Moreover, we have

$$\boxed{\partial_\xi r^1_{e_1} = \sum_{jj=1}^{3} r^1_{e_1,jj} \partial_\xi \phi^1_{jj}}, \tag{46.5}$$

and

$$\boxed{\partial_\xi z^1_{e_1} = \sum_{jj=1}^{3} z^1_{e_1,jj} \partial_\xi \phi^1_{jj}}, \tag{46.6}$$

where we have once again used the fact that once we have mapped to the master element (in this case the interval $[-1,1]$) the interpolating functions ϕ no longer depend on the coordinate of the specific element to introduce the notation

$$\phi^1_{ii}(\cdot) = \phi^1_{l_1(e_1,jj)}(S^1_{e_1}(.)). \tag{46.7}$$

Furthermore, the derivatives with respect to the arc-length s, can be calculated using

$$\partial_s f = \partial_\xi f \partial_s \xi, \tag{46.8}$$

and we introduce

$$\boxed{J^1_{e_1} := \partial_\xi s = \sqrt{\left(\partial_\xi r^1_{e_1}\right)^2 + \left(\partial_\xi z^1_{e_1}\right)^2}}, \tag{46.9}$$

which is the determinant of the Jacobian of $S^1_{e_1}$.

We also highlight the the integral we are considering is a line integral and therefore when parameterising by ξ to actually perform the calculation we need to multiply the integrand by the derivative of the arc-length, yielding

$$c_{ii,n_r,\partial_r\check{u}}(e_1) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \frac{\partial_\xi z^1_{e_1}(\xi)}{J^1_{e_1}(\xi)} \phi^1_{ii}(\xi)\partial_r\check{u}\partial_\xi s. \tag{46.10}$$

We cancel $\partial_\xi s$ with $J^1_{e_1}$, which yields

$$c_{ii,n_r,\partial_r\check{u}}(e_1) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_r\check{u}. \tag{46.11}$$

Hence, using Gaussian quadrature we have

$$\boxed{c_{ii,n_r,\partial_r\check{u}}(e_1) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p)\partial_\xi z^1_{e_1}(p)\phi^1_{ii}(p)\partial_r\check{u}(p)}, \tag{46.12}$$

where we are using the notation $f(p)$ as a short version of $f(\xi_p)$, with ξ_p is the p-th Gaussian quadrature point and $W_l(p)$ is the p-th Gaussian quadrature weights (out of n_{lG} total points). Moreover,

$$\partial_r\check{u}(p) = \partial_r\check{u}(r(p), z(p)). \tag{46.13}$$

From (25.142) we also have

$$c_{ii,n_z,\partial_z\check{u}}(e_1) = \int\limits_{\partial\Omega_{e_1}} n^1_z \phi^1_{l_1(e_1,ii)}\partial_z\check{u}, \tag{46.14}$$

which in terms of the master line element is

$$c_{ii,n_z,\partial_z\check{u}}(e_1) = \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_z\check{u}, \tag{46.15}$$

where we have cancelled the Jacobian of the change of variables with the denominator in the expression for the normal.

Hence, using Gaussian quadrature we have

$$\boxed{c_{ii,n_z,\partial_z\check{u}}(e_1) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p)\partial_\xi r^1_{e_1}(p)\phi^1_{ii}(p)\partial_z\check{u}(p)}, \tag{46.16}$$

where we are using the notations described above.

From (25.142) we also have

$$c_{ii,n_z,\partial_r\check{w}}(e_1) = \int\limits_{\partial\Omega_{e_1}} n^1_z \phi^1_{l_1(e_1,ii)}\partial_r\check{w}, \tag{46.17}$$

which in terms of the master line element is

$$c_{ii,n_z,\partial_r\check{w}}(e_1) = \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_r\check{w}, \tag{46.18}$$

where we have cancelled the Jacobian of the change of variables with the denominator in the expression for the normal.

Hence, using Gaussian quadrature we have

$$\boxed{c_{ii,n_z,\partial_z\breve{u}}(e_1) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p)\partial_\xi r^1_{e_1}(p)\phi^1_{ii}(p)\partial_r\breve{w}(p)}, \tag{46.19}$$

where we are using the notations described above.

From (26.125) we have

$$c_{ii,n_z,\partial_z\breve{w}}(e_1) = \int_{\partial\Omega_{e_1}} n^1_z\phi^1_{l_1(e_1,ii)}\partial_z\breve{w}, \tag{46.20}$$

which in terms of the master line element is

$$c_{ii,n_z,\partial_z\breve{w}}(e_1) = \alpha \int_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_z\breve{w}, \tag{46.21}$$

where we have cancelled the Jacobian of the change of variables with the denominator in the expression for the normal.

Hence, using Gaussian quadrature we have

$$\boxed{c_{ii,n_z,\partial_z\breve{w}}(e_1) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p)\partial_\xi r^1_{e_1}(p)\phi^1_{ii}(p)\partial_z\breve{w}(p)}. \tag{46.22}$$

From (26.125) we have

$$c_{ii,n_r,\partial_z\breve{u}}(e_1) = \int_{\partial\Omega_{e_1}} n^1_r\phi^1_{l_1(e_1,ii)}\partial_z\breve{u}, \tag{46.23}$$

which in terms of the master line element is

$$c_{ii,n_r,\partial_z\breve{u}}(e_1) = -\alpha \int_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_z\breve{u}, \tag{46.24}$$

where we have cancelled the Jacobian of the change of variables with the denominator in the expression for the normal.

Hence, using Gaussian quadrature we have

$$\boxed{c_{ii,n_z,\partial_z\breve{u}}(e_1) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p)\partial_\xi z^1_{e_1}(p)\phi^1_{ii}(p)\partial_z\breve{u}(p)}. \tag{46.25}$$

From (26.125) we have

$$c_{ii,n_r,\partial_r\breve{w}}(e_1) = \int_{\partial\Omega_{e_1}} n^1_r\phi^1_{l_1(e_1,ii)}\partial_r\breve{w}, \tag{46.26}$$

which in terms of the master line element is

$$c_{ii,n_r,\partial_r\check{w}}(e_1) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_r\check{w}, \tag{46.27}$$

where we have cancelled the Jacobian of the change of variables with the denominator in the expression for the normal.

Hence, using Gaussian quadrature we have

$$\boxed{c_{ii,n_r,\partial_r\check{w}}(e_1) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p)\partial_\xi z^1_{e_1}(p)\phi^1_{ii}(p)\partial_r\check{w}(p)}. \tag{46.28}$$

From equation (**??**) we have

$$c_{ii,n_r,\check{u}}(e_1) = \int\limits_{\partial\Omega_{e_1}} n^1_r\phi^1_{l_1(e_1,ii)}\check{u}, \tag{46.29}$$

which in terms of the master line element is

$$c_{ii,n_r,\check{u}}(e_1) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi)\phi^1_{ii}(\xi)\check{u}, \tag{46.30}$$

where we have cancelled the Jacobian of the change of variables with the denominator in the expression for the normal.

Hence, using Gaussian quadrature we have

$$\boxed{c_{ii,n_r,\check{u}}(e_1) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p)\partial_\xi z^1_{e_1}(p)\phi^1_{ii}(p)\check{u}(p)}. \tag{46.31}$$

From equation (**??**) we have

$$c_{ii,n_z,\check{w}}(e_1) = \int\limits_{\partial\Omega_{e_1}} n^1_z\phi^1_{l_1(e_1,ii)}\check{w}, \tag{46.32}$$

which in terms of the master line element is

$$c_{ii,n_z,\check{w}}(e_1) = \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\check{w}, \tag{46.33}$$

where we have cancelled the Jacobian of the change of variables with the denominator in the expression for the normal.

Hence, using Gaussian quadrature we have

$$\boxed{c_{ii,n_z,\check{w}}(e_1) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p)\partial_\xi r^1_{e_1}(p)\phi^1_{ii}(p)\check{w}(p)}. \tag{46.34}$$

46.2. The liquid-solid boundary line elements

Here we follow what was done in section 21.2, adding the expressions for the extra terms.

We recall that we can also arrange all local node numbering to guarantee that every side of a triangular element that falls on boundary 2 is the side containing local nodes 1, 5 and 2 (see figure 10). This allows us to have a natural parametrisation of these line elements using variable η. See section 21.2 for further details.

Now, from (25.143) we have

$$d_{ii,t_r,t_r,\check{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_r^2 t_r^2 \phi^2_{l_2(e_2,ii)} \check{u}, \tag{46.35}$$

where we have

$$\boldsymbol{t}^2 = \frac{(\partial_\eta r^2_{e_2}, \partial_\eta z^2_{e_2})}{\sqrt{\left(\partial_\eta r^2_{e_2}\right)^2 + \left(\partial_\eta z^2_{e_2}\right)^2}}, \tag{46.36}$$

which yields a tangent vector that points in the direction of increasing η.

Naturally, we also have

$$\boxed{\partial_\eta r^2_{e_2} = \sum_{jj=1}^{3} r^2_{e_2,jj} \partial_\eta \phi^1_{jj}}, \tag{46.37}$$

$$\boxed{\partial_\eta z^2_{e_2} = \sum_{jj=1}^{3} z^2_{e_2,jj} \partial_\eta \phi^1_{jj}} \tag{46.38}$$

and

$$\boxed{J^2_{e_2} := \partial_\eta s = \sqrt{\left(\partial_\eta r\right)^2 + \left(\partial_\eta z\right)^2}}. \tag{46.39}$$

Hence, we can re-write (46.35) as

$$d_{ii,t_r,t_r,\check{u}}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2}{J^2_{e_2}} \phi^2_{ii} \check{u}, \tag{46.40}$$

where we have cancelled one of the square roots from the denominator of the tangent vector components with the $\partial_\eta s$.

Now, using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_r,\check{u}}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))^2}{J^2_{e_2}(p)} \phi^2_{ii}(p) \check{u}(p)}. \tag{46.41}$$

From (25.143) we also have

$$d_{ii,t_r,t_z,\check{w}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_r^2 t_z^2 \phi^2_{l_2(e_2,ii)} \check{w}, \tag{46.42}$$

and moving to the master line element we have

$$d_{ii,t_r,t_z,\check{w}}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{J^2_{e_2}} \phi^2_{ii} \check{w}. \tag{46.43}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_z,\check{w}}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))(\partial_\eta z^2_{e_2}(p))}{J^2_{e_2}(p)} \phi^2_{ii}(p)\check{w}(p)}. \tag{46.44}$$

From (25.143) we also have

$$d_{ii,t_r,t_r,u^s}(e_2) = \int\limits_{\partial\Omega_{e_2}} t^2_r t^2_r \phi^2_{l_2(e_2,ii)} u^s, \tag{46.45}$$

and moving to the master line element we have

$$d_{ii,t_r,t_r,u^s}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta r^2_{e_2})}{J^2_{e_2}} \phi^2_{ii} u^s. \tag{46.46}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_r,u^s}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))(\partial_\eta r^2_{e_2}(p))}{J^2_{e_2}(p)} \phi^2_{ii}(p)u^s(p)}. \tag{46.47}$$

From (25.143) we also have

$$d_{ii,t_r,t_z,w^s}(e_2) = \int\limits_{\partial\Omega_{e_2}} t^2_r t^2_z \phi^2_{l_2(e_2,ii)} w^s, \tag{46.48}$$

and moving to the master line element we have

$$d_{ii,t_r,t_z,w^s}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{J^2_{e_2}} \phi^2_{ii} w^s. \tag{46.49}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_z,w^s}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))(\partial_\eta z^2_{e_2}(p))}{J^2_{e_2}(p)} \phi^2_{ii}(p)w^s(p)}. \tag{46.50}$$

From (25.143) we also have

$$d_{ii,n_r,n_r,n_r,\partial_r\check{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} n^2_r n^2_r n^2_r \phi^2_{l_2(e_2,ii)} \partial_r \check{u}, \tag{46.51}$$

and moving to the master line element we have

$$d_{ii,n_r,n_r,n_r,\partial_r\check{u}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^3}{(J^2_{e_2})^2} \phi^2_{ii} \partial_r \check{u}. \tag{46.52}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,n_r,n_r,n_r,\partial_r\check{u}}(e_2) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p))^3}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p)\check{w}(p)}. \tag{46.53}$$

From (25.143) we also have

$$d_{ii,n_r,n_r,n_z,\partial_z\check{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} n_r^2 n_r^2 n_z^2 \phi^2_{l_2(e_2,ii)} \partial_z \check{u}, \tag{46.54}$$

and moving to the master line element we have

$$d_{ii,n_r,n_r,n_z,\partial_z\check{u}}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2 (\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z \check{u}. \tag{46.55}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,n_r,n_r,n_z,\partial_z\check{u}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p))^2 (\partial_\eta r^2_{e_2}(p))}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p) \partial_z \check{u}(p)}. \tag{46.56}$$

From (25.143) we also have

$$d_{ii,n_r,n_r,n_z,\partial_r\check{w}}(e_2) = \int\limits_{\partial\Omega_{e_2}} n_r^2 n_r^2 n_z^2 \phi^2_{l_2(e_2,ii)} \partial_r \check{w}, \tag{46.57}$$

and moving to the master line element we have

$$d_{ii,n_r,n_r,n_z,\partial_r\check{w}}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2 (\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii} \partial_r \check{w}. \tag{46.58}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,n_r,n_r,n_z,\partial_r\check{w}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p))^2 (\partial_\eta r^2_{e_2}(p))}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p) \partial_r \check{w}(p)}. \tag{46.59}$$

From (25.143) we also have

$$d_{ii,n_r,n_z,n_z,\partial_z\check{w}}(e_2) = \int\limits_{\partial\Omega_{e_2}} n_r^2 n_z^2 n_z^2 \phi^2_{l_2(e_2,ii)} \partial_z \check{w}, \tag{46.60}$$

and moving to the master line element we have

$$d_{ii,n_r,n_z,n_z,\partial_z\check{w}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2}) (\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z \check{w}. \tag{46.61}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,n_r,n_z,n_z,\partial_z\check{w}}(e_2) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p)) (\partial_\eta r^2_{e_2}(p))^2}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p) \partial_z \check{w}(p)}. \tag{46.62}$$

From (25.143) we also have

$$d_{ii,t_r,t_r,n_r,\partial_r\check{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_r^2 t_r^2 n_r^2 \phi^2_{l_2(e_2,ii)} \partial_r \check{u}, \tag{46.63}$$

and moving to the master line element we have

$$d_{ii,t_r,t_r,n_r,\partial_r\check{u}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii}\partial_r\check{u}. \tag{46.64}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_r,n_r,\partial_r\check{u}}(e_2) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))^2(\partial_\eta z^2_{e_2}(p))}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p)\partial_r\check{u}(p)}. \tag{46.65}$$

From (25.143) we also have

$$d_{ii,t_r,t_r,n_z,\partial_z\check{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t^2_r t^2_r n^2_z \phi^2_{l_2(e_2,ii)}\partial_z\check{u}, \tag{46.66}$$

and moving to the master line element we have

$$d_{ii,t_r,t_r,n_z,\partial_z\check{u}}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2} \phi^2_{ii}\partial_z\check{u}. \tag{46.67}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_r,n_z,\partial_z\check{u}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))^3}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p)\partial_z\check{u}(p)}. \tag{46.68}$$

From (25.143) we also have

$$d_{ii,t_r,t_z,n_r,\partial_z\check{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t^2_r t^2_z n^2_r \phi^2_{l_2(e_2,ii)}\partial_z\check{u}, \tag{46.69}$$

and moving to the master line element we have

$$d_{ii,t_r,t_z,n_r,\partial_z\check{u}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})^2}{(J^2_{e_2})^2} \phi^2_{ii}\partial_z\check{u}. \tag{46.70}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_z,n_r,\partial_z\check{u}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))(\partial_\eta z^2_{e_2}(p))^2}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p)\partial_z\check{u}(p)}. \tag{46.71}$$

From (25.143) we also have

$$d_{ii,t_r,t_r,n_r,\partial_r\check{w}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t^2_r t^2_r n^2_r \phi^2_{l_2(e_2,ii)}\partial_r\check{w}, \tag{46.72}$$

and moving to the master line element we have

$$d_{ii,t_r,t_r,n_r,\partial_r\check{w}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii}\partial_r\check{w}. \tag{46.73}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_r,n_r,\partial_r\check{w}}(e_2) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))^2(\partial_\eta z^2_{e_2}(p))}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p)\partial_r\check{w}(p)}. \tag{46.74}$$

From (25.143) we also have

$$d_{ii,t_r,t_r,n_z,\partial_r\check{w}}(e_2) = \int_{\partial\Omega_{e_2}} t_r^2 t_r^2 n_z^2 \phi^2_{l_2(e_2,ii)} \partial_r\check{w}, \tag{46.75}$$

and moving to the master line element we have

$$d_{ii,t_r,t_r,n_z,\partial_r\check{w}}(e_2) = \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2} \phi^2_{ii} \partial_r\check{w}. \tag{46.76}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_r,n_z,\partial_r\check{w}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))^3}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p)\partial_r\check{w}(p)}. \tag{46.77}$$

From (25.143) we also have

$$d_{ii,t_r,t_z,n_z,\partial_z\check{w}}(e_2) = \int_{\partial\Omega_{e_2}} t_r^2 t_z^2 n_z^2 \phi^2_{l_2(e_2,ii)} \partial_z\check{w}, \tag{46.78}$$

and moving to the master line element we have

$$d_{ii,t_r,t_z,n_z,\partial_z\check{w}}(e_2) = \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z\check{w}. \tag{46.79}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_z,n_z,\partial_z\check{w}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))^2(\partial_\eta z^2_{e_2}(p))}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p)\partial_z\check{w}(p)}. \tag{46.80}$$

From (26.126) we have

$$d_{ii,t_r,t_z,\check{u}}(e_2) = \int_{\partial\Omega_{e_2}} t_r^2 t_z^2 \phi^2_{l_2(e_2,ii)} \check{u}, \tag{46.81}$$

which we can write in terms of the master line element as

$$d_{ii,t_r,t_z,\check{u}}(e_2) = \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{J^2_{e_2}} \phi^2_{ii} \check{u}, \tag{46.82}$$

where we have cancelled one of the square roots from the denominator of the tangent vector components with the $\partial_\eta s$.

Now, using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_z,\check{u}}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))(\partial_\eta z^2_{e_2}(p))}{J^2_{e_2}(p)} \phi^2_{ii}(p)\check{u}(p)}. \tag{46.83}$$

From (26.126) we also have

$$d_{ii,t_z,t_z,\breve{w}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_z^2 t_z^2 \phi^2_{l_2(e_2,ii)} \breve{w}, \tag{46.84}$$

and moving to the master line element we have

$$d_{ii,t_z,t_z,\breve{w}}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2}{J^2_{e_2}} \phi^2_{ii} \breve{w}. \tag{46.85}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_z,t_z,\breve{w}}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p))^2}{J^2_{e_2}(p)} \phi^2_{ii}(p) \breve{w}(p)}. \tag{46.86}$$

From (26.126) we also have

$$d_{ii,t_r,t_z,u^s}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_r^2 t_z^2 \phi^2_{l_2(e_2,ii)} u^s, \tag{46.87}$$

and moving to the master line element we have

$$d_{ii,t_r,t_z,u^s}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{J^2_{e_2}} \phi^2_{ii} u^s. \tag{46.88}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_z,u^s}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))(\partial_\eta z^2_{e_2}(p))}{J^2_{e_2}(p)} \phi^2_{ii}(p) u^s}. \tag{46.89}$$

From (26.126) we also have

$$d_{ii,t_z,t_z,w^s}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_z^2 t_z^2 \phi^2_{l_2(e_2,ii)} w^s, \tag{46.90}$$

and moving to the master line element we have

$$d_{ii,t_z,t_z,w^s}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2}{J^2_{e_2}} \phi^2_{ii} w^s. \tag{46.91}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_z,t_z,w^s}(e_2) \approx \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p))^2}{J^2_{e_2}(p)} \phi^2_{ii}(p) w^s}. \tag{46.92}$$

From (26.126) we also have

$$d_{ii,n_r,n_r,n_z,\partial_r\breve{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} n_r^2 n_r^2 n_z^2 \phi^2_{l_2(e_2,ii)} \partial_r \breve{u}, \tag{46.93}$$

and moving to the master line element we have

$$d_{ii,n_r,n_r,n_z,\partial_r\breve{u}}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r\breve{u}. \tag{46.94}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,n_r,n_r,n_z,\partial_r\breve{u}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p))^2(\partial_\eta r^2_{e_2}(p))}{(J^2_{e_2}(p))^2}\phi^2_{ii}(p)\breve{w}(p)}. \tag{46.95}$$

From (26.126) we also have

$$d_{ii,n_r,n_z,n_z,\partial_z\breve{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} n^2_r n^2_z n^2_z \phi^2_{l_2(e_2,ii)}\partial_z\breve{u}, \tag{46.96}$$

and moving to the master line element we have

$$d_{ii,n_r,n_z,n_z,\partial_z\breve{u}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z\breve{u}. \tag{46.97}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,n_r,n_z,n_z,\partial_z\breve{u}}(e_2) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p))(\partial_\eta r^2_{e_2}(p))^2}{(J^2_{e_2}(p))^2}\phi^2_{ii}(p)\partial_z\breve{u}(p)}. \tag{46.98}$$

From (26.126) we also have

$$d_{ii,n_r,n_z,n_z,\partial_r\breve{w}}(e_2) = \int\limits_{\partial\Omega_{e_2}} n^2_r n^2_z n^2_z \phi^2_{l_2(e_2,ii)}\partial_r\breve{w}, \tag{46.99}$$

and moving to the master line element we have

$$d_{ii,n_r,n_z,n_z,\partial_r\breve{w}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r\breve{w}. \tag{46.100}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,n_r,n_z,n_z,\partial_r\breve{w}}(e_2) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p))(\partial_\eta r^2_{e_2}(p))^2}{(J^2_{e_2}(p))^2}\phi^2_{ii}(p)\partial_r\breve{w}(p)}. \tag{46.101}$$

From (26.126) we also have

$$d_{ii,n_z,n_z,n_z,\partial_z\breve{w}}(e_2) = \int\limits_{\partial\Omega_{e_2}} n^2_z n^2_z n^2_z \phi^2_{l_2(e_2,ii)}\partial_z\breve{w}, \tag{46.102}$$

and moving to the master line element we have

$$d_{ii,n_z,n_z,n_z,\partial_z\breve{w}}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z\breve{w}. \tag{46.103}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,n_z,n_z,n_z,\partial_z\check{w}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))^3}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p) \partial_z \check{w}(p)}. \tag{46.104}$$

From (26.126) we also have

$$d_{ii,t_r,t_z,n_r,\partial_r\check{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_r^2 t_z^2 n_r^2 \phi^2_{l_2(e_2,ii)} \partial_r \check{u}, \tag{46.105}$$

and moving to the master line element we have

$$d_{ii,t_r,t_z,n_r,\partial_r\check{u}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})^2}{(J^2_{e_2})^2} \phi^2_{ii} \partial_r \check{u}. \tag{46.106}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_z,n_r,\partial_r\check{u}}(e_2) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))(\partial_\eta z^2_{e_2}(p))^2}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p) \partial_r \check{u}(p)}. \tag{46.107}$$

From (26.126) we also have

$$d_{ii,t_r,t_z,n_z,\partial_z\check{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_r^2 t_z^2 n_z^2 \phi^2_{l_2(e_2,ii)} \partial_z \check{u}, \tag{46.108}$$

and moving to the master line element we have

$$d_{ii,t_r,t_z,n_z,\partial_z\check{u}}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z \check{u}. \tag{46.109}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_z,n_z,\partial_z\check{u}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))^2(\partial_\eta z^2_{e_2}(p))}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p) \partial_z \check{u}(p)}. \tag{46.110}$$

From (26.126) we also have

$$d_{ii,t_z,t_z,n_r,\partial_z\check{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_z^2 t_z^2 n_r^2 \phi^2_{l_2(e_2,ii)} \partial_z \check{u}, \tag{46.111}$$

and moving to the master line element we have

$$d_{ii,t_z,t_z,n_r,\partial_z\check{u}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^3}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z \check{u}. \tag{46.112}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_z,t_z,n_r,\partial_z\check{u}}(e_2) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p))^3}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p) \partial_z \check{u}(p)}. \tag{46.113}$$

From (26.126) we also have

$$d_{ii,t_z,t_z,n_r,\partial_r\breve{w}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_z^2 t_z^2 n_r^2 \phi^2_{l_2(e_2,ii)} \partial_r \breve{w}, \tag{46.114}$$

and moving to the master line element we have

$$d_{ii,t_z,t_z,n_r,\partial_r\breve{w}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^3}{(J^2_{e_2})^2} \phi^2_{ii} \partial_r \breve{w}. \tag{46.115}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_z,t_z,n_r,\partial_r\breve{w}}(e_2) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta z^2_{e_2}(p))^3}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p) \partial_r \breve{w}(p)}. \tag{46.116}$$

From (26.126) we also have

$$d_{t_r,t_z,n_z,\partial_r\breve{w},ii}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_r^2 t_z^2 n_z^2 \phi^2_{l_2(e_2,ii)} \partial_r \breve{w}, \tag{46.117}$$

and moving to the master line element we have

$$d_{ii,t_r,t_z,n_z,\partial_r\breve{w}}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2 (\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii} \partial_r \breve{w}. \tag{46.118}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_r,t_z,n_z,\partial_r\breve{w}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))^2 (\partial_\eta z^2_{e_2}(p))}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p) \partial_r \breve{w}(p)}. \tag{46.119}$$

From (26.126) we also have

$$d_{ii,t_z,t_z,n_z,\partial_z\breve{w}}(e_2) = \int\limits_{\partial\Omega_{e_2}} t_z^2 t_z^2 n_z^2 \phi^2_{l_2(e_2,ii)} \partial_z \breve{w}, \tag{46.120}$$

and moving to the master line element we have

$$d_{ii,t_z,t_z,n_z,\partial_z\breve{w}}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})^2}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z \breve{w}. \tag{46.121}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,t_z,t_z,n_z,\partial_z\breve{w}}(e_2) \approx \alpha \sum_{p=1}^{n_{lG}} W_l(p) \frac{(\partial_\eta r^2_{e_2}(p))(\partial_\eta z^2_{e_2}(p))^2}{(J^2_{e_2}(p))^2} \phi^2_{ii}(p) \partial_z \breve{w}(p)}. \tag{46.122}$$

From (**??**) we also have

$$d_{ii,n_r,\breve{u}}(e_2) = \int\limits_{\partial\Omega_{e_2}} n_r^2 \breve{u} \phi^2_{l_2(e_2,ii)}, \tag{46.123}$$

and moving to the master line element we have

$$d_{ii,n_r,\check{u}}(e_2) = -\alpha \int_{\eta=-1}^{\eta=1} (\partial_\eta z^2_{e_2})\check{u}\phi^2_{ii}. \tag{46.124}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,n_r,\check{u}}(e_2) \approx -\alpha \sum_{p=1}^{n_{lG}} W_l(p)(\partial_\eta z^2_{e_2}(p))\check{u}(p)\phi^2_{ii}(p)}. \tag{46.125}$$

From (**??**) we also have

$$d_{ii,n_z,\check{w}}(e_2) = \int_{\partial\Omega_{e_2}} n_z^2 \check{w}\phi^2_{l_2(e_2,ii)}, \tag{46.126}$$

and moving to the master line element we have

$$d_{ii,n_z,\check{w}}(e_2) = \alpha \int_{\eta=-1}^{\eta=1} (\partial_\eta r^2_{e_2})\check{w}\phi^2_{ii}. \tag{46.127}$$

Using Gaussian quadrature we have

$$\boxed{d_{ii,n_z,\check{w}}(e_2) \approx \alpha \sum_{p=1}^{n_{l_G}} W_l(p)(\partial_\eta r^2_{e_2}(p))\check{w}(p)\phi^2_{ii}(p)}. \tag{46.128}$$

46.3. The separatrix boundary line elements

As was discussed in section 21.3, a consequence of our prior choice of local numbering is that the line elements along boundary 4 must correspond to the side of the master element that contains nodes 3, 4 and 1 (see figure 10). We then choose to parameterise these line elements using variable ξ. See section 21.3 for more details.

From equation (**??**) we have

$$g_{ii,n_r,\partial_r\check{u}}(e_4) = \int_{\partial\Omega_{e_3}} \phi_{l_4(e_4,ii)} n_r^4 \partial_r \check{u}, \tag{46.129}$$

which in terms of the master line element is given by

$$g_{ii,n_r,\partial_r\check{u}}(e_4) = -\alpha \int_{\xi=-1}^{\xi=1} \phi_{ii}^4 \partial_\xi z_{e_4}^4 \partial_r \check{u}, \tag{46.130}$$

where we have cancelled the denominator of the expression for the tangent with the Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{g_{ii,n_r,\partial_r\check{u}}(e_4) \approx -\alpha \sum_{p=1}^{n_{lG}} \phi_{ii}^4(p) \partial_\xi z_{e_4}^4(p) \partial_r \check{u}(p)}. \tag{46.131}$$

From equation (**??**) we have

$$g_{ii,n_z,\partial_r\check{w}}(e_4) = \int_{\partial\Omega_{e_3}} \phi_{l_4(e_4,ii)} n_z^4 \partial_r \check{w}, \tag{46.132}$$

which in terms of the master line element is given by

$$g_{ii,n_z,\partial_r\check{w}}(e_4) = \alpha \int_{\xi=-1}^{\xi=1} \partial_\xi r_{e_4}^4 \phi_{ii}^4 \partial_r \check{w}, \tag{46.133}$$

where we have cancelled the denominator of the expression for the tangent with the Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{g_{ii,n_z,\partial_r\check{w}}(e_4) \approx \alpha \sum_{p=1}^{n_{lG}} \phi_{ii}^4(p) \partial_\xi r_{e_4}^4(p) \partial_r \check{w}(p)}. \tag{46.134}$$

From equation (**??**) we have

$$g_{ii,n_z,\partial_z\check{u}}(e_4) = \int_{\partial\Omega_{e_3}} \phi_{l_4(e_4,ii)} n_z^4 \partial_z \check{u}, \tag{46.135}$$

which in terms of the master line element is given by

$$g_{ii,n_z,\partial_z\check{u}}(e_4) = \alpha \int_{\xi=-1}^{\xi=1} \phi_{ii}^4 \partial_\xi r_{e_4}^4 \partial_z \check{u}, \tag{46.136}$$

where we have cancelled the denominator of the expression for the tangent with the

Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{g_{ii,n_z,\partial_z\check{u}}(e_4) \approx \alpha \sum_{p=1}^{n_{lG}} \phi^4_{ii}(p)\partial_\xi r^4_{e_4}(p)\partial_z\check{u}(p)}. \tag{46.137}$$

From equation (**??**) we have

$$g_{ii,jj,n_r}(e_4) = \int\limits_{\partial\Omega_{e_3}} \phi_{l_4(e_4,ii)}\phi_{l_4(e_4,jj)}n^4_r, \tag{46.138}$$

which in terms of the master line element is given by

$$g_{ii,jj,n_r}(e_4) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \phi^4_{ii}\phi^4_{jj}\partial_\xi z^4_{e_4}, \tag{46.139}$$

where we have cancelled the denominator of the expression for the tangent with the Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{g_{ii,jj,n_r}(e_4) \approx -\alpha \sum_{p=1}^{n_{lG}} \phi^4_{ii}(p)\phi^4_{jj}(p)\partial_\xi z^4_{e_4}(p)}. \tag{46.140}$$

From equation (**??**) we have

$$g_{ii,jj,t_r}(e_4) = \int\limits_{\partial\Omega_{e_3}} \phi_{l_4(e_4,ii)}\phi_{l_4(e_4,jj)}t^4_r, \tag{46.141}$$

which in terms of the master line element is given by

$$g_{ii,jj,t_r}(e_4) = \int\limits_{\xi=-1}^{\xi=1} \phi^4_{ii}\phi^4_{jj}\partial_\xi r^4_{e_4}, \tag{46.142}$$

where we have cancelled the denominator of the expression for the tangent with the Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{g_{ii,jj,n_r}(e_4) \approx \sum_{p=1}^{n_{lG}} \partial_\xi t^4_{e_4}(p)\psi^4_{jj}(p)\phi^4_{ii}(p)}. \tag{46.143}$$

From equation (**??**) we have

$$g_{ii,n_z,\partial_z\check{u}}(e_4) = \int\limits_{\partial\Omega_{e_3}} n^4_z\phi_{l_4(e_4,ii)}\partial_z\check{u}, \tag{46.144}$$

which in terms of the master line element is given by

$$g_{ii,n_z,\partial_z\check{u}}(e_4) = \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^4_{e_4}\phi^4_{ii}\partial_z\check{u}, \tag{46.145}$$

where we have cancelled the denominator of the expression for the tangent with the

Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{g_{ii,n_z,\partial_z\check{u}}(e_4) \approx \alpha \sum_{p=1}^{n_{lG}} \partial_\xi r_{e_4}^4(p)\phi_{ii}^4(p)\partial_z\check{u}(p)}. \tag{46.146}$$

From equation (??) we have

$$g_{ii,n_r,\partial_z\check{u}}(e_4) = \int\limits_{\partial\Omega_{e_3}} n_r^4\phi_{l_4(e_4,ii)}\partial_z\check{u}, \tag{46.147}$$

which in terms of the master line element is given by

$$g_{ii,n_r,\partial_z\check{u}}(e_4) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z_{e_4}^4\phi_{ii}^4\partial_z\check{u}, \tag{46.148}$$

where we have cancelled the denominator of the expression for the tangent with the Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{g_{ii,n_r,\partial_z\check{u}}(e_4) \approx -\alpha \sum_{p=1}^{n_{lG}} \phi_{ii}^4(p)\partial_\xi z_{e_4}^4(p)\partial_z\check{u}(p)}. \tag{46.149}$$

From equation (??) we have

$$g_{ii,n_r,\partial_r\check{w}}(e_4) = \int\limits_{\partial\Omega_{e_3}} n_r^4\phi_{l_4(e_4,ii)}\partial_r\check{w}, \tag{46.150}$$

which in terms of the master line element is given by

$$g_{ii,n_r,\partial_r\check{w}}(e_4) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z_{e_4}^4\phi_{ii}^4\partial_r\check{w}, \tag{46.151}$$

where we have cancelled the denominator of the expression for the tangent with the Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{g_{ii,n_r,\partial_r\check{w}}(e_4) \approx -\alpha \sum_{p=1}^{n_{lG}} \partial_\xi z_{e_4}^4(p)\phi_{ii}^4(p)\partial_r\check{w}(p)}. \tag{46.152}$$

From equation (??) we have

$$g_{ii,jj,n_z}(e_4) = \int\limits_{\partial\Omega_{e_3}} \phi_{l_4(e_4,ii)}\phi_{l_4(e_4,jj)}n_z^4, \tag{46.153}$$

which in terms of the master line element is given by

$$g_{ii,jj,n_z}(e_4) = \alpha \int\limits_{\xi=-1}^{\xi=1} \phi_{ii}^4\phi_{jj}^4\partial_\xi r_{e_4}^4, \tag{46.154}$$

where we have cancelled the denominator of the expression for the tangent with the

Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{g_{ii,jj,n_z}(e_4) \approx \alpha \sum_{p=1}^{n_{lG}} \phi^4_{ii}(p)\phi^4_{jj}(p)\partial_\xi r^4_{e_4}(p)}. \tag{46.155}$$

From equation (**??**) we have

$$g_{ii,jj,t_z}(e_4) = \int\limits_{\partial\Omega_{e_3}} \phi_{l_4(e_4,ii)}\phi_{l_4(e_4,jj)}t^4_z, \tag{46.156}$$

which in terms of the master line element is given by

$$g_{ii,jj,t_z}(e_4) = \int\limits_{\xi=-1}^{\xi=1} \phi^4_{ii}\phi^4_{jj}\partial_\xi z^4_{e_4}, \tag{46.157}$$

where we have cancelled the denominator of the expression for the tangent with the Jacobian of the change of coordinates. Using Gaussian quadrature we have

$$\boxed{g_{ii,jj,t_z}(e_4) \approx \sum_{p=1}^{n_{lG}} \phi^4_{ii}(p)\phi^4_{jj}(p)\partial_\xi z^4_{e_4}(p)}. \tag{46.158}$$

46.4. Derivatives of line element integrals near the contact line

46.5. Derivatives of line-element integrals

We recall that

$$J_{e_i}^i(p) = \sqrt{\left(\partial_{\xi_i} r_{e_i}^i(p)\right)^2 + \left(\partial_{\xi_i} z_{e_i}^i(p)\right)^2}, \tag{46.159}$$

where $\xi_i = \xi$ for $i = 1, 4$ and $\xi_i = \eta$ for $i = 2$; and we notice that

$$\partial_{h_q} J_{e_i}^i(p) = \frac{1}{2}\frac{1}{J_{e_i}^i(p)}\left[2\partial_{\xi_i} r_{e_i}^i(p)\partial_{h_q}\left(\partial_{\xi_i} r_{e_i}^i(p)\right) + 2\partial_{\xi_i} z_{e_i}^i(p)\partial_{h_q}\left(\partial_{\xi_i} z_{e_i}^i(p)\right)\right], \tag{46.160}$$

i.e.

$$\boxed{\partial_{h_q} J_{e_i}^i(p) = \frac{\left[\partial_{\xi_i} r_{e_i}^i(p)\partial_{h_q}\left(\partial_{\xi_i} r_{e_i}^i(p)\right) + \partial_{\xi_i} z_{e_i}^i(p)\partial_{h_q}\left(\partial_{\xi_i} z_{e_i}^i(p)\right)\right]}{J_{e_i}^i(p)}}, \tag{46.161}$$

which reduces the problem of finding the derivative of the Jacobian to finding

$$\partial_{h_q}\left(\partial_{\xi_i} r_{e_i}^i\right) = \partial_{h_q}\partial_{\xi_i}\left(\sum_{mm=1}^{3} r_{e_i,mm}^i \phi_{mm}^i\right), \tag{46.162}$$

i.e.

$$\partial_{h_q}\left(\partial_{\xi_i} r_{e_i}^i\right) = \partial_{h_q}\left(\sum_{mm=1}^{3} r_{e_i,mm}^i \partial_{\xi_i}\phi_{mm}^i\right), \tag{46.163}$$

which yields

$$\boxed{\partial_{h_q}\left(\partial_{\xi_i} r_{e_i}^i\right) = \sum_{mm=1}^{3}\left(\partial_{\xi_i}\phi_{mm}^i\right)\left(\partial_{h_q} r_{e_i,mm}^i\right)}; \tag{46.164}$$

and, similarly,

$$\boxed{\partial_{h_q}\left(\partial_{\xi_i} z_{e_i}^i\right) = \sum_{mm=1}^{3}\left(\partial_{\xi_i}\phi_{mm}^i\right)\left(\partial_{h_q} z_{e_i,mm}^i\right)}. \tag{46.165}$$

46.5.1. Derivatives of c terms

From equation (46.11) we have

$$\partial_{h_q} c_{ii,n_r,\partial_r\check{u}}(e_1) = -\alpha\partial_{h_q}\int_{\xi=-1}^{\xi=1}\partial_\xi z_{e_1}^1(\xi)\phi_{ii}^1(\xi)\partial_r\check{u}. \tag{46.166}$$

or equivalently

$$\partial_{h_q} c_{ii,n_r,\partial_r\check{u}}(e_1) = -\alpha\int_{\xi=-1}^{\xi=1}\left[\partial_{h_q}\partial_\xi z_{e_1}^1(\xi)\right]\phi_{ii}^1(\xi)\partial_r\check{u} - \alpha\int_{\xi=-1}^{\xi=1}\partial_\xi z_{e_1}^1(\xi)\phi_{ii}^1(\xi)\partial_{h_q}\partial_r\check{u}. \tag{46.167}$$

i.e.

$$\partial_{h_q} c_{ii,n_r,\partial_r \check{u}}(e_1) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \left[\partial_{h_q} \partial_\xi z^1_{e_1}(\xi)\right] \phi^1_{ii}(\xi) \partial_r \check{u} - \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \left[\partial_{rr} \check{u}\right] \left[\sum_{mm=1}^{3} \phi^1_{mm}(\xi) \partial_{h_q} r_{e,mm}\right] - \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \left[\partial_{rz} \check{u}\right] \left[\sum_{mm=1}^{3} \phi^1_{mm}(\xi) \partial_{h_q} z_{e,mm}\right]. \quad (46.168)$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} c_{n_r,\partial_r \check{u},ii}(e_1) \approx -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp) \phi^1_{ii}(pp) \Bigg\{ \left[\partial_{h_q} \partial_\xi z^1_{e_1}(pp)\right] \partial_r \check{u}(pp) + \partial_\xi z^1_{e_1}(pp) \left[\partial_{rr} \check{u}(pp)\right] \left[\sum_{mm=1}^{3} \phi^1_{mm}(pp) \partial_{h_q} r_{e,mm}\right] + \partial_\xi z^1_{e_1}(pp) \left[\partial_{rz} \check{u}(pp)\right] \left[\sum_{mm=1}^{3} \phi^1_{mm}(pp) \partial_{h_q} z_{e,mm}\right] \Bigg\}. \quad (46.169)$$

From equation (46.15) we have

$$\partial_{h_q} c_{ii,n_z,\partial_z \check{u}}(e_1) = \alpha \partial_{h_q} \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_z \check{u}, \quad (46.170)$$

or equivalently

$$\partial_{h_q} c_{ii,n_z,\partial_z \check{u}}(e_1) = \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_{h_q} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_z \check{u} + \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{h_q} \partial_z \check{u}. \quad (46.171)$$

i.e.

$$\partial_{h_q} c_{n_z,\partial_z \check{u},ii}(e_1) = \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_{h_q} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_z \check{u} + \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{rz} \check{u} \left[\sum_{mm=1}^{3} \phi^1_{mm}(\xi) \partial_{h_q} r_{e,mm}\right] + \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{zz} \check{u} \left[\sum_{mm=1}^{3} \phi^1_{mm}(\xi) \partial_{h_q} z_{e,mm}\right]. \quad (46.172)$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} c_{n_z,\partial_z\breve{u},ii}(e_1) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^1_{ii}(pp) \left\{ \partial_{h_q}\partial_\xi r^1_{e_1}(pp)\partial_z\breve{u}(pp) \right. + \partial_\xi r^1_{e_1}(\xi)\partial_{rz}\breve{u}\left[\sum_{mm=1}^{3}\phi^1_{mm}(\xi)\partial_{h_q} r_{e,mm}\right] \left. + \partial_\xi r^1_{e_1}(pp)\partial_{zz}\breve{u}(pp)\left[\sum_{mm=1}^{3}\phi^1_{mm}(pp)\partial_{h_q} z_{e,mm}\right]\right\}. \tag{46.173}$$

From equation (46.18) we have

$$\partial_{h_q} c_{ii,n_z,\partial_r\breve{w}}(e_1) = \alpha\partial_{h_q}\int_{\xi=-1}^{\xi=1}\partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_r\breve{w}, \tag{46.174}$$

or equivalently

$$\partial_{h_q} c_{ii,n_z,\partial_r\breve{w}}(e_1) = \alpha\int_{\xi=-1}^{\xi=1}\partial_{h_q}\partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_r\breve{w} + \alpha\int_{\xi=-1}^{\xi=1}\partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_{h_q}\partial_r\breve{w}, \tag{46.175}$$

i.e.

$$\partial_{h_q} c_{ii,n_z,\partial_r\breve{w}}(e_1) = \alpha\int_{\xi=-1}^{\xi=1}\partial_{h_q}\partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_r\breve{w} + \alpha\int_{\xi=-1}^{\xi=1}\partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_{rr}\breve{w}\left[\sum_{mm=1}^{3}\phi^1_{mm}(\xi)\partial_{h_q} r_{e,mm}\right] + \alpha\int_{\xi=-1}^{\xi=1}\partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3}\phi^1_{mm}(\xi)\partial_{h_q} z_{e,mm}\right], \tag{46.176}$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} c_{ii,n_z,\partial_r\breve{w}}(e_1) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^1_{ii}(pp) \left\{ \partial_{h_q}\partial_\xi r^1_{e_1}(pp)\partial_r\breve{w}(pp) \right. + \partial_\xi r^1_{e_1}(pp)\partial_{rr}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi^1_{mm}(pp)\partial_{h_q} r_{e,mm}\right] \left. + \partial_\xi r^1_{e_1}(pp)\partial_{rz}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi^1_{mm}(pp)\partial_{h_q} z_{e,mm}\right]\right\}. \tag{46.177}$$

From equation (46.21) we have

$$\partial_{h_q} c_{ii,n_z,\partial_z\breve{w}}(e_1) = \alpha\partial_{h_q}\int_{\xi=-1}^{\xi=1}\partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_z\breve{w}, \tag{46.178}$$

or equivalently

$$\partial_{h_q} c_{ii,n_z,\partial_z \breve{w}}(e_1) = \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_{h_q} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_z \breve{w} + \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{h_q} \partial_z \breve{w}, \quad (46.179)$$

i.e.

$$\begin{aligned}
\partial_{h_q} c_{ii,n_z,\partial_z \breve{w}}(e_1) = \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_{h_q} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_z \breve{w} \\
+ \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{rz} \breve{w} \left[\sum_{mm=1}^{3} \phi^1_{mm}(\xi) \partial_{h_q} r_{e,mm} \right] \\
+ \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{zz} \breve{w} \left[\sum_{mm=1}^{3} \phi^1_{mm}(\xi) \partial_{h_q} z_{e,mm} \right],
\end{aligned} \quad (46.180)$$

Using Gaussian quadrature, this yields

$$\begin{aligned}
\partial_{h_q} c_{ii,n_z,\partial_r \breve{w}}(e_1) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp) \phi^1_{ii}(pp) \Bigg\{ \partial_{h_q} \partial_\xi r^1_{e_1}(pp) \partial_r \breve{w}(pp) \\
+ \partial_\xi r^1_{e_1}(pp) \partial_{rr} \breve{w}(pp) \left[\sum_{mm=1}^{3} \phi^1_{mm}(pp) \partial_{h_q} r_{e,mm} \right] \\
+ \partial_\xi r^1_{e_1}(pp) \partial_{rz} \breve{w}(pp) \left[\sum_{mm=1}^{3} \phi^1_{mm}(pp) \partial_{h_q} z_{e,mm} \right] \Bigg\}.
\end{aligned} \quad (46.181)$$

From equation (46.24) we have

$$\partial_{h_q} c_{ii,n_r,\partial_z \breve{u}}(e_1) = -\alpha \partial_{h_q} \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_z \breve{u}, \quad (46.182)$$

or equivalently

$$\partial_{h_q} c_{ii,n_r,\partial_z \breve{u}}(e_1) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \partial_{h_q} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_z \breve{u} - \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{h_q} \partial_z \breve{u}, \quad (46.183)$$

i.e.

$$\partial_{h_q} c_{ii,n_r,\partial_z \check{u}}(e_1) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \partial_{h_q} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_z \check{u} - \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{rz} \check{u} \left[\sum_{mm=1}^{3} \phi^1_{mm}(\xi) \partial_{h_q} r_{e,mm} \right] - \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{zz} \check{u} \left[\sum_{mm=1}^{3} \phi^1_{mm}(\xi) \partial_{h_q} z_{e,mm} \right], \tag{46.184}$$

Using Gaussian quadrature, this yields

$$\boxed{\partial_{h_q} c_{ii,n_r,\partial_z \check{u}}(e_1) \approx -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp) \phi^1_{ii}(pp) \left\{ \partial_{h_q} \partial_\xi z^1_{e_1}(pp) \partial_z \check{u}(pp) + \partial_\xi z^1_{e_1}(pp) \partial_{rz} \check{u}(pp) \left[\sum_{mm=1}^{3} \phi^1_{mm}(pp) \partial_{h_q} r_{e,mm} \right] + \partial_\xi z^1_{e_1}(pp) \partial_{zz} \check{u}(pp) \left[\sum_{mm=1}^{3} \phi^1_{mm}(pp) \partial_{h_q} z_{e,mm} \right] \right\}.} \tag{46.185}$$

From equation (46.27) we have

$$\partial_{h_q} c_{ii,n_r,\partial_r \check{w}}(e_1) = -\alpha \partial_{h_q} \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_r \check{w}, \tag{46.186}$$

or equivalently

$$\partial_{h_q} c_{ii,n_r,\partial_r \check{w}}(e_1) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \partial_{h_q} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_r \check{w} - \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{h_q} \partial_r \check{w}, \tag{46.187}$$

i.e.

$$\partial_{h_q} c_{ii,n_r,\partial_r \check{w}}(e_1) = -\alpha \int\limits_{\xi=-1}^{\xi=1} \partial_{h_q} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_r \check{w} - \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{rr} \check{w} \left[\sum_{mm=1}^{3} \phi^1_{mm}(\xi) \partial_{h_q} r_{e,mm} \right] - \alpha \int\limits_{\xi=-1}^{\xi=1} \partial_\xi z^1_{e_1}(\xi) \phi^1_{ii}(\xi) \partial_{rz} \check{w} \left[\sum_{mm=1}^{3} \phi^1_{mm}(\xi) \partial_{h_q} z_{e,mm} \right], \tag{46.188}$$

Using Gaussian quadrature, this yields

$$\begin{aligned}\partial_{h_q} c_{ii,n_r,\partial_r \breve{w}}(e_1) \approx -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^1(pp) \Bigg\{ & \partial_{h_q}\partial_\xi z_{e_1}^1(\xi)\partial_r \breve{w} \\ & + \partial_\xi z_{e_1}^1(\xi)\partial_{rr}\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^1(\xi)\partial_{h_q} r_{e,mm}\right] \\ & + \partial_\xi z_{e_1}^1(\xi)\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^1(\xi)\partial_{h_q} z_{e,mm}\right]\Bigg\}. \end{aligned} \tag{46.189}$$

From equation (46.30) we have

$$\partial_{h_q} c_{ii,n_r,\breve{u}}(e_1) = -\alpha\partial_{h_q}\int_{\xi=-1}^{\xi=1}\partial_\xi z_{e_1}^1(\xi)\phi_{ii}^1(\xi)\breve{u}, \tag{46.190}$$

or equivalently

$$\partial_{h_q} c_{ii,n_r,\partial_r \breve{w}}(e_1) = -\alpha\int_{\xi=-1}^{\xi=1}\partial_{h_q}\partial_\xi z_{e_1}^1(\xi)\phi_{ii}^1(\xi)\partial_r\breve{w} - \alpha\int_{\xi=-1}^{\xi=1}\partial_\xi z_{e_1}^1(\xi)\phi_{ii}^1(\xi)\partial_{h_q}\partial_r\breve{w}, \tag{46.191}$$

i.e.

$$\begin{aligned}\partial_{h_q} c_{ii,n_r,\partial_r \breve{w}}(e_1) = -\alpha & \int_{\xi=-1}^{\xi=1}\partial_{h_q}\partial_\xi z_{e_1}^1(\xi)\phi_{ii}^1(\xi)\partial_r\breve{w} \\ & - \alpha\int_{\xi=-1}^{\xi=1}\partial_\xi z_{e_1}^1(\xi)\phi_{ii}^1(\xi)\partial_{rr}\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^1(\xi)\partial_{h_q} r_{e,mm}\right] \\ & - \alpha\int_{\xi=-1}^{\xi=1}\partial_\xi z_{e_1}^1(\xi)\phi_{ii}^1(\xi)\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^1(\xi)\partial_{h_q} z_{e,mm}\right]. \end{aligned} \tag{46.192}$$

Using Gaussian quadrature, this yields

$$\begin{aligned}\partial_{h_q} c_{ii,n_r,\partial_r \breve{w}}(e_1) \approx -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^1(pp) \Bigg\{ & \partial_{h_q}\partial_\xi z_{e_1}^1(pp)\partial_r \breve{w}(pp) \\ & + \partial_\xi z_{e_1}^1(pp)\partial_{rr}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^1(pp)\partial_{h_q} r_{e,mm}\right] \\ & + \partial_\xi z_{e_1}^1(\xi)\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^1(pp)\partial_{h_q} z_{e,mm}\right]\Bigg\}. \end{aligned} \tag{46.193}$$

From equation (46.33) we have

$$\partial_{h_q} c_{ii,n_z,\breve{w}}(e_1) = \alpha\partial_{h_q}\int_{\xi=-1}^{\xi=1}\partial_\xi r_{e_1}^1(\xi)\phi_{ii}^1(\xi)\breve{w}, \tag{46.194}$$

or equivalently

$$\partial_{h_q} c_{ii,n_z,\breve{w}}(e_1) = \alpha \int_{\xi=-1}^{\xi=1} \partial_{h_q}\partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\breve{w} + \alpha \int_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_{h_q}\breve{w}, \quad (46.195)$$

i.e.

$$\begin{aligned}\partial_{h_q} c_{ii,n_z,\breve{w}}(e_1) = \alpha &\int_{\xi=-1}^{\xi=1} \partial_{h_q}\partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\breve{w} \\ &+ \alpha \int_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_r\breve{w}\left[\sum_{mm=1}^{3}\phi^1_{mm}(\xi)\partial_{h_q} r_{e,mm}\right] \quad (46.196)\\ &+ \alpha \int_{\xi=-1}^{\xi=1} \partial_\xi r^1_{e_1}(\xi)\phi^1_{ii}(\xi)\partial_z\breve{w}\left[\sum_{mm=1}^{3}\phi^1_{mm}(\xi)\partial_{h_q} z_{e,mm}\right].\end{aligned}$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}\partial_{h_q} c_{ii,n_z,\breve{w}}(e_1) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^1_{ii}(pp)\Bigg\{&\partial_{h_q}\partial_\xi r^1_{e_1}(\xi)\breve{w} \\ &+ \partial_\xi r^1_{e_1}(\xi)\partial_r\breve{w}\left[\sum_{mm=1}^{3}\phi^1_{mm}(\xi)\partial_{h_q} r_{e,mm}\right] \quad (46.197)\\ &+ \partial_\xi r^1_{e_1}(\xi)\partial_z\breve{w}\left[\sum_{mm=1}^{3}\phi^1_{mm}(\xi)\partial_{h_q} z_{e,mm}\right]\Bigg\}.\end{aligned}}$$

46.5.2. Derivatives of d terms

From equation (46.40) we have

$$\partial_{h_q} d_{ii,t_r,t_r,\breve{u}}(e_2) = \partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2}{J^2_{e_2}} \phi^2_{ii} \breve{u}, \tag{46.198}$$

or equivalently

$$\begin{aligned} \partial_{h_q} d_{ii,t_r,t_r,\breve{u}}(e_2) = & \int\limits_{\eta=-1}^{\eta=1} \phi^2_{ii} \frac{2\partial_\eta r^2_{e_2}(\eta)\partial_{h_q}\partial_\eta r^2_{e_2}(\eta)}{J^2_{e_2}} \breve{u} \\ & + \int\limits_{\eta=-1}^{\eta=1} \phi^2_{ii} \frac{(\partial_\eta r^2_{e_2})^2}{J^2_{e_2}} \partial_r \breve{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta) \partial_{h_q} r^2_{e_2,mm} \right] \\ & + \int\limits_{\eta=-1}^{\eta=1} \phi^2_{ii} \frac{(\partial_\eta r^2_{e_2})^2}{J^2_{e_2}} \partial_z \breve{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta) \partial_{h_q} z^2_{e_2,mm} \right] \\ & - \int\limits_{\eta=-1}^{\eta=1} \phi^2_{ii} \frac{(\partial_\eta r^2_{e_2})^2}{\left(J^2_{e_2}\right)^2} \breve{u} \partial_{h_q} J^2_{e_2}. \end{aligned} \tag{46.199}$$

Using Gaussian quadrature, this yields

$$\begin{aligned} \partial_{h_q} d_{ii,t_r,t_r,\breve{u}}(e_2) \approx \sum_{pp=1}^{n_{lG}} W_{l_G}(pp) \phi^2_{ii}(pp) \Bigg\{ & \frac{2\partial_\eta r^2_{e_2}(\eta)\partial_{h_q}\partial_\eta r^2_{e_2}(\eta)}{J^2_{e_2}} \breve{u} \\ & + \frac{(\partial_\eta r^2_{e_2})^2}{J^2_{e_2}} \partial_r \breve{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta) \partial_{h_q} r^2_{e_2,mm} \right] \\ + \frac{(\partial_\eta r^2_{e_2})^2}{J^2_{e_2}} \partial_z \breve{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta) \partial_{h_q} z^2_{e_2,mm} \right] & - \frac{(\partial_\eta r^2_{e_2})^2}{\left(J^2_{e_2}\right)^2} \breve{u} \partial_{h_q} J^2_{e_2} \Bigg\}. \end{aligned} \tag{46.200}$$

From equation (46.43) we have

$$\partial_{h_q} d_{ii,t_r,t_z,\breve{w}}(e_2) = \partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{J^2_{e_2}} \phi^2_{ii} \breve{w}, \tag{46.201}$$

or equivalently

$$\begin{aligned}\partial_{h_q} d_{ii,t_r,t_z,\breve{w}}(e_2) = &\int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_{h_q}\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\breve{w} + \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\breve{w}\\ &+ \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\partial_r\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right]\\ &+ \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\partial_z\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right]\\ &- \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_{h_q}\breve{w}\partial_{h_q} J_{e_2}^2.\end{aligned} \tag{46.202}$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}\partial_{h_q} d_{ii,t_r,t_z,\breve{w}}(e_2) \approx \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\Bigg\{&\frac{(\partial_{h_q}\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\breve{w} + \frac{(\partial_\eta r_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\breve{w}\\ &+ \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\partial_r\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right]\\ &+ \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\partial_z\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right]\\ &- \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\breve{w}\partial_{h_q} J_{e_2}^2\Bigg\}.\end{aligned} \quad (46.203)}$$

From equation (46.46) we have

$$\partial_{h_q} d_{ii,t_r,t_r,u^s}(e_2) = \partial_{h_q}\int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)^2}{J_{e_2}^2}\phi_{ii}^2 u^s, \tag{46.204}$$

or equivalently

$$\begin{aligned}\partial_{h_q} d_{ii,t_r,t_r,u^s}(e_2) = &\int\limits_{\eta=-1}^{\eta=1} \phi_{ii}^2\frac{2\partial_\eta r_{e_2}^2(\eta)\partial_{h_q}\partial_\eta r_{e_2}^2(\eta)}{J_{e_2}^2}u^s\\ &+ \int\limits_{\eta=-1}^{\eta=1} \phi_{ii}^2\frac{(\partial_\eta r_{e_2}^2)^2}{J_{e_2}^2}\partial_{h_q} u^s - \int\limits_{\eta=-1}^{\eta=1} \phi_{ii}^2\frac{(\partial_\eta r_{e_2}^2)^2}{\left(J_{e_2}^2\right)^2}u^s\partial_{h_q} J_{e_2}^2.\end{aligned} \tag{46.205}$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} d_{ii,t_r,t_r,\check{u}}(e_2) \approx \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\left\{\frac{2\partial_\eta r_{e_2}^2(pp)\partial_{h_q}\partial_{pp}r_{e_2}^2(\eta)}{J_{e_2}^2}u^s(pp)\right.$$
$$\left. + \frac{(\partial_\eta r_{e_2}^2)^2}{J_{e_2}^2}\partial_{h_q}u^s(pp) - \frac{(\partial_\eta r_{e_2}^2)^2}{\left(J_{e_2}^2\right)^2}u^s(pp)\partial_{h_q}J_{e_2}^2\right\}. \tag{46.206}$$

From equation (46.49) we have

$$\partial_{h_q} d_{ii,t_r,t_z,w^s}(e_2) = \partial_{h_q}\int_{\eta=-1}^{\eta=1}\frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2 w^s, \tag{46.207}$$

or equivalently

$$\partial_{h_q} d_{ii,t_r,t_z,w^s}(e_2) = \int_{\eta=-1}^{\eta=1}\frac{(\partial_{h_q}\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2 w^s + \int_{\eta=-1}^{\eta=1}\frac{(\partial_\eta r_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2 w^s$$
$$+ \int_{\eta=-1}^{\eta=1}\frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\partial_{h_q} w^s - \int_{\eta=-1}^{\eta=1}\frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2 w^s\partial_{h_q}J_{e_2}^2. \tag{46.208}$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} d_{ii,t_r,t_z,w^s}(e_2) \approx \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\left\{\frac{(\partial_{h_q}\partial_\eta r_{e_2}^2)(pp)(\partial_\eta z_{e_2}^2)(pp)}{J_{e_2}^2(pp)}w^s(pp)\right.$$
$$+ \frac{(\partial_\eta r_{e_2}^2)(pp)(\partial_{h_q}\partial_\eta z_{e_2}^2)(pp)}{J_{e_2}^2(pp)}w^s(pp)$$
$$+ \frac{(\partial_\eta r_{e_2}^2)(pp)(\partial_\eta z_{e_2}^2)(pp)}{J_{e_2}^2(pp)}\partial_{h_q}w^s(pp)$$
$$\left. - \frac{(\partial_\eta r_{e_2}^2)(pp)(\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2(pp))^2}w^s(pp)\partial_{h_q}J_{e_2}^2(pp)\right\}. \tag{46.209}$$

Observation: In the 2 integrals above the nature of the dependence of u^s and w^s on h_q is to be determined.

From equation (46.52) we have

$$\partial_{h_q} d_{ii,n_r,n_r,n_r,\partial_r\check{u}}(e_2) = -\alpha\partial_{h_q}\int_{\eta=-1}^{\eta=1}\frac{(\partial_\eta z_{e_2}^2)^3}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_r\check{u}, \tag{46.210}$$

or equivalently

$$\partial_{h_q} d_{ii,n_r,n_r,n_r,\partial_r \check{u}}(e_2) = -\alpha \int_{\eta=-1}^{\eta=1} \frac{3(\partial_\eta z^2_{e_2})^2(\partial_{h_q}\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r \check{u}$$

$$- \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rr}\check{u}\left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \tag{46.211}$$

$$- \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz}\check{u}\left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right]$$

$$+ 2\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^3}{(J^2_{e_2})^3}\phi^2_{ii}\partial_r \check{u}\partial_{h_q} J^2_{e_2}.$$

Using Gaussian quadrature, this yields

$$\begin{aligned}\partial_{h_q} d_{ii,n_r,n_r,n_r,\partial_r \check{u}}(e_2) \approx -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp)\Bigg\{ & 3\frac{(\partial_\eta z^2_{e_2})^2(\partial_{h_q}\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\partial_r\check{u} \\ & + \frac{(\partial_\eta z^2_{e_2})^3}{(J^2_{e_2})^2}\partial_{rr}\check{u}\left[\sum_{mm=1}^{3}\partial_{h_q} r^2_{e_2,mm}\right] \\ + \frac{(\partial_\eta z^2_{e_2})^3}{(J^2_{e_2})^2}\partial_{rz}\check{u}\left[\sum_{mm=1}^{3}\partial_{h_q} z^2_{e_2,mm}\right] & -2\frac{(\partial_\eta z^2_{e_2})^3}{(J^2_{e_2})^3}\partial_r\check{u}\partial_{h_q} J^2_{e_2}\Bigg\}.\end{aligned} \tag{46.212}$$

From equation (46.55) we have

$$\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_z \check{u}}(e_2) = \alpha\partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z \check{u}. \tag{46.213}$$

or equivalently

$$
\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_z \check{u}}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} & \frac{2(\partial_\eta z_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2} \phi_{ii}^2 \partial_z \check{u} \\
& + \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_{h_q}\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2} \phi_{ii}^2 \partial_z \check{u} \\
& + \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2} \phi_{ii}^2 \partial_{rz} \check{u} \left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta) \partial_{h_q} r_{e_2,mm}^2 \right] \\
& + \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2} \phi_{ii}^2 \partial_{zz} \check{u} \left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta) \partial_{h_q} z_{e_2,mm}^2 \right] \\
& - 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^3} \phi_{ii}^2 \partial_z \check{u} \partial_{h_q} J_{e_2}^2 .
\end{aligned}
\tag{46.214}
$$

Using Gaussian quadrature, this yields

$$
\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_z \check{u}}(e_2) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp) \phi_{ii}^2(pp) \Bigg\{ & 2\frac{(\partial_\eta z_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2} \partial_z \check{u} \\
& + \frac{(\partial_\eta z_{e_2}^2)^2(\partial_{h_q}\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2} \partial_z \check{u} \\
& + \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2} \partial_{rz} \check{u} \left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta) \partial_{h_q} r_{e_2,mm}^2 \right] \\
& + \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2} \phi_{ii}^2 \partial_{zz} \check{u} \left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta) \partial_{h_q} z_{e_2,mm}^2 \right] \\
& - 2\frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^3} \phi_{ii}^2 \partial_z \check{u} \partial_{h_q} J_{e_2}^2 \Bigg\} .
\end{aligned}
\tag{46.215}
$$

From equation (46.58) we have

$$
\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_r \check{w}}(e_2) = \alpha \partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2} \phi_{ii}^2 \partial_r \check{w}. \tag{46.216}
$$

or equivalently

$$
\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_r \breve{w}}(e_2) = \alpha & \int_{\eta=-1}^{\eta=1} 2\frac{(\partial_\eta z_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2 \partial_r \breve{w} \\
& + \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_{h_q}\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2 \partial_r \breve{w} \\
& + \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2 \partial_{rr} \breve{w} \left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right] \\
& + \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2 \partial_{rz} \breve{w} \left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right] \\
& - 2\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^3}\phi_{ii}^2 \partial_r \breve{w} \partial_{h_q} J_{e_2}^2 .
\end{aligned}
\tag{46.217}
$$

Using Gaussian quadrature, this yields

$$
\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_r \breve{w}}(e_2) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp) \Bigg\{ & 2\frac{(\partial_\eta z_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2 \partial_r \breve{w} \\
& + \frac{(\partial_\eta z_{e_2}^2)^2(\partial_{h_q}\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2 \partial_r \breve{w} \\
& + \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2 \partial_{rr} \breve{w} \left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right] \\
& + \frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2 \partial_{rz} \breve{w} \left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right] \\
& - 2\frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^3}\phi_{ii}^2 \partial_r \breve{w} \partial_{h_q} J_{e_2}^2 \Bigg\}.
\end{aligned}
\tag{46.218}
$$

From equation (46.61) we have

$$
\partial_{h_q} d_{ii,n_r,n_z,n_z,\partial_z \breve{w}}(e_2) = -\alpha \partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)(\partial_\eta r_{e_2}^2)^2}{(J_{e_2}^2)^2}\phi_{ii}^2 \partial_z \breve{w}, \tag{46.219}
$$

or equivalently

$$
\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_z \breve{w}}(e_2) = \alpha \int_{\eta=-1}^{\eta=1} & 2\frac{(\partial_\eta z^2_{e_2})(\partial_{h_q}\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r \breve{w} \\
& + \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r \breve{w} \\
& + \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz} \breve{w} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\
& + \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{zz} \breve{w} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \\
& - 2\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^3}\phi^2_{ii}\partial_r \breve{w}\partial_{h_q} J^2_{e_2}.
\end{aligned}
\tag{46.220}
$$

Using Gaussian quadrature, this yields

$$
\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_z \breve{w}}(e_2) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp) \Bigg\{ & 2\frac{(\partial_\eta z^2_{e_2})(\partial_{h_q}\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_z \breve{w} \\
& + \frac{(\partial_\eta z^2_{e_2})^2(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_z \breve{w} \\
& + \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_{rz} \breve{w} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\
& + \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_{zz} \breve{w} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \\
& - 2\frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^3}\partial_z \breve{w}\partial_{h_q} J^2_{e_2} \Bigg\}.
\end{aligned}
\tag{46.221}
$$

From equation (46.64) we have

$$
\partial_{h_q} d_{ii,t_r,t_r,n_r,\partial_r \breve{u}}(e_2) = -\alpha\partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r \breve{u}, \tag{46.222}
$$

or equivalently

$$\partial_{h_q} d_{ii,t_r,t_r,n_r,\partial_r\check{u}}(e_2) = -\alpha \int_{\eta=-1}^{\eta=1} 2\frac{(\partial_\eta r^2_{e_2})(\partial_{h_q}\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r\check{u}$$

$$- \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_{h_q}\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r\check{u}$$

$$- \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rr}\check{u}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right]$$

$$- \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz}\check{u}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right]$$

$$+ 2\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^3}\phi^2_{ii}\partial_r\check{u}\partial_{h_q} J^2_{e_2}. \tag{46.223}$$

Using Gaussian quadrature, this yields

$$\begin{aligned}\partial_{h_q} d_{ii,t_r,t_r,n_r,\partial_r\check{u}}(e_2) \approx -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp)\Bigg\{ & 2\frac{(\partial_\eta r^2_{e_2})(\partial_{h_q}\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\partial_r\check{u} \\ & + \frac{(\partial_\eta r^2_{e_2})^2(\partial_{h_q}\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\partial_r\check{u} \\ & + \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\partial_{rr}\check{u}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\ & + \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\partial_{rz}\check{u}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \\ & - 2\frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^3}\partial_r\check{u}\partial_{h_q} J^2_{e_2}\Bigg\}.\end{aligned} \tag{46.224}$$

From equation (46.67) we have

$$\partial_{h_q} d_{ii,t_r,t_r,n_z,\partial_z\check{u}}(e_2) = \alpha\partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z\check{u}, \tag{46.225}$$

or equivalently

$$\partial_{h_q} d_{ii,t_r,t_r,n_z,\partial_z \check{u}}(e_2) = \alpha \int_{\eta=-1}^{\eta=1} 3\frac{(\partial_\eta r^2_{e_2})^2(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z \check{u}$$
$$+ \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz}\check{u}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \tag{46.226}$$
$$+ \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{zz}\check{u}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right]$$
$$- 2\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^3}\phi^2_{ii}\partial_z \check{u}\partial_{h_q} J^2_{e_2}.$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}\partial_{h_q} d_{ii,t_r,t_r,n_z,\partial_z \check{u}}(e_2) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp)\Bigg\{&3\frac{(\partial_\eta r^2_{e_2})^2(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_z \check{u}\\ &+\frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\partial_{rz}\check{u}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right]\\ &+\frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\partial_{zz}\check{u}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right]\\ &-2\frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^3}\partial_z \check{u}\partial_{h_q} J^2_{e_2}\Bigg\}.\end{aligned}} \tag{46.227}$$

From equation (46.73) we have

$$\partial_{h_q} d_{ii,t_r,t_r,n_r,\partial_r \check{w}}(e_2) = -\alpha\partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r \check{w}, \tag{46.228}$$

or equivalently

$$\begin{aligned}\partial_{h_q} d_{ii,t_r,t_r,n_r,\partial_r\check{w}}(e_2) = &-\alpha \int\limits_{\eta=-1}^{\eta=1} 2\frac{(\partial_\eta r^2_{e_2})(\partial_{h_q}\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r\check{w}\\
&- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_{h_q}\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r\check{w}\\
&- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rr}\check{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right]\\
&- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz}\check{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right]\\
&+ 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^3}\phi^2_{ii}\partial_r\check{w}\partial_{h_q} J^2_{e_2}.\end{aligned} \tag{46.229}$$

Using Gaussian quadrature, this yields

$$\begin{aligned}&\partial_{h_q} d_{ii,t_r,t_r,n_r,\partial_r\check{w}}(e_2)\\
&\approx -\alpha\sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp)\left\{2\frac{(\partial_\eta r^2_{e_2})(pp)(\partial_{h_q}\partial_\eta r^2_{e_2})(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)}\partial_r\check{w}(pp)\right.\\
&\qquad + \frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_{h_q}\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)}\partial_r\check{u}(pp)\\
&\qquad + \frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)}\partial_{rr}\check{w}(pp)\left[\sum_{mm=1}^{3}\phi^2_{mm}(pp)\partial_{h_q} r^2_{e_2,mm}\right]\\
&\qquad + \frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)}\partial_{rz}\check{w}(pp)\left[\sum_{mm=1}^{3}\phi^2_{mm}(pp)\partial_{h_q} z^2_{e_2,mm}\right]\\
&\qquad \left.- 2\frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^3(pp)}\partial_r\check{w}(pp)\partial_{h_q} J^2_{e_2}(pp)\right\}.\end{aligned} \tag{46.230}$$

From equation (46.76) we have

$$\partial_{h_q} d_{ii,t_r,t_r,n_z,\partial_r\check{w}}(e_2) = \alpha\partial_{h_q}\int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r\check{w}, \tag{46.231}$$

or equivalently

$$
\begin{aligned}
\partial_{h_q} d_{ii,t_r,t_r,n_z,\partial_r \breve{w}}(e_2) = \alpha \int_{\eta=-1}^{\eta=1} 3\frac{(\partial_\eta r^2_{e_2})^2 (\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r \breve{w} \\
+ \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rr}\breve{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\
+ \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \\
- 2\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^3}\phi^2_{ii}\partial_r \breve{w}\partial_{h_q} J^2_{e_2}.
\end{aligned}
\tag{46.232}
$$

Using Gaussian quadrature, this yields

$$
\begin{aligned}
\partial_{h_q} d_{ii,t_r,t_r,n_z,\partial_r \breve{w}}(e_2) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp)\Bigg\{ 3\frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_{h_q}\partial_\eta r^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)}\partial_r\breve{w}(pp) \\
+ \frac{(\partial_\eta r^2_{e_2})^3(pp)}{(J^2_{e_2})^2(pp)}\partial_{rr}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi^2_{mm}(pp)\partial_{h_q} r^2_{e_2,mm}\right] \\
+ \frac{(\partial_\eta r^2_{e_2})^3(pp)}{(J^2_{e_2})^2(pp)}\partial_{rz}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi^2_{mm}(pp)\partial_{h_q} z^2_{e_2,mm}\right] \\
- 2\frac{(\partial_\eta r^2_{e_2})^3(pp)}{(J^2_{e_2})^3(pp)}\partial_r\breve{w}(pp)\partial_{h_q} J^2_{e_2}(pp)\Bigg\}.
\end{aligned}
\tag{46.233}
$$

From equation (46.79) we have

$$
\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_z \breve{w}}(e_2) = \alpha\partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z\breve{w}, \tag{46.234}
$$

or equivalently

$$\begin{aligned}
\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_z\breve{w}}(e_2) = \alpha \int_{\eta=-1}^{\eta=1} & 2\frac{(\partial_\eta r^2_{e_2})(\partial_{h_q}\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z\breve{w} \\
&+ \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_{h_q}\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z\breve{w} \\
&+ \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\
&+ \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{zz}\breve{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \\
&- 2\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^3}\phi^2_{ii}\partial_z\breve{w}\partial_{h_q} J^2_{e_2}.
\end{aligned} \tag{46.235}$$

Using Gaussian quadrature, this yields

$$\begin{aligned}
&\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_z\breve{w}}(e_2) \\
&\approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp)\left\{2\frac{(\partial_\eta r^2_{e_2})(pp)(\partial_{h_q}\partial_\eta r^2_{e_2})(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)}\partial_z\breve{w}(pp)\right. \\
&\qquad + \frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_{h_q}\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)}\partial_z\breve{w}(pp) \\
&\qquad + \frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)}\partial_{rz}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi^2_{mm}(pp)\partial_{h_q} r^2_{e_2,mm}\right] \\
&\qquad + \frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)}\partial_{zz}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi^2_{mm}(pp)\partial_{h_q} z^2_{e_2,mm}\right] \\
&\qquad \left. - 2\frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^3(pp)}\partial_z\breve{w}(pp)\partial_{h_q} J^2_{e_2}(pp)\right\}.
\end{aligned} \tag{46.236}$$

From equation (46.82) we have

$$. \tag{46.237}$$

$$\partial_{h_q} d_{ii,t_r,t_z,\breve{u}}(e_2) = \partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{J^2_{e_2}}\phi^2_{ii}\breve{u}, \tag{46.238}$$

or equivalently

$$\partial_{h_q} d_{ii,t_r,t_z,\breve{u}}(e_2) = \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_{h_q}\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\breve{u} + \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\breve{u}$$

$$+ \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\partial_r\breve{u}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right] \tag{46.239}$$

$$+ \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\partial_z\breve{u}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right]$$

$$- \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\breve{u}\partial_{h_q} J_{e_2}^2.$$

Using Gaussian quadrature, this yields

$$\begin{aligned}
\partial_{h_q} d_{ii,t_r,t_z,\breve{u}}(e_2) = &\\
\approx \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp) \Bigg\{ & \frac{(\partial_{h_q}\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\breve{u} + \frac{(\partial_\eta r_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\breve{u} \\
& + \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\partial_r\breve{u}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right] \\
& + \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\partial_z\breve{u}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right] \\
& - \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\breve{u}\partial_{h_q} J_{e_2}^2\Bigg\}.
\end{aligned} \tag{46.240}$$

From equation (46.85) we have

$$\partial_{h_q} d_{ii,t_z,t_z,\breve{w}}(e_2) = \partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2}{J_{e_2}^2}\phi_{ii}^2\breve{w}, \tag{46.241}$$

or equivalently

$$\begin{aligned}\partial_{h_q} d_{ii,t_z,t_z,\breve{w}}(e_2) = &\int\limits_{\eta=-1}^{\eta=1} 2\frac{(\partial_\eta z_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\breve{w} \\ &+ \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2}{J_{e_2}^2}\phi_{ii}^2\partial_r\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right] \\ &+ \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2}{J_{e_2}^2}\phi_{ii}^2\partial_z\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right] \\ &- \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2}{(J_{e_2}^2)^2}\phi_{ii}^2\breve{w}\partial_{h_q} J_{e_2}^2.\end{aligned} \tag{46.242}$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}\partial_{h_q} d_{ii,t_z,t_z,\breve{w}}(e_2) \approx \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\Bigg\{&2\frac{(\partial_\eta z_{e_2}^2)(pp)(\partial_{h_q}\partial_\eta z_{e_2}^2)(pp)}{J_{e_2}^2(pp)}\breve{w}(pp) \\ &+ \frac{(\partial_\eta z_{e_2}^2(pp))^2}{J_{e_2}^2(pp)}\partial_r\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^2(pp)\partial_{h_q} r_{e_2,mm}^2\right] \\ &+ \frac{(\partial_\eta z_{e_2}^2(pp))^2}{J_{e_2}^2(pp)}\partial_z\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^2(pp)\partial_{h_q} z_{e_2,mm}^2\right] \\ &- \frac{(\partial_\eta z_{e_2}^2(pp))^2}{(J_{e_2}^2(pp))^2}\breve{w}(pp)\partial_{h_q} J_{e_2}^2(pp)\Bigg\}.\end{aligned}} \tag{46.243}$$

From equation (46.88) we have

$$. \tag{46.244}$$

$$\partial_{h_q} d_{ii,t_r,t_z,u^s}(e_2) = \partial_{h_q}\int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2 u^s, \tag{46.245}$$

or equivalently

$$\begin{aligned}\partial_{h_q} d_{ii,t_r,t_z,u^s}(e_2) = &\int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_{h_q}\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2 u^s + \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2 u^s \\ &+ \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2\partial_{h_q} u^s - \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2 u^s\partial_{h_q} J_{e_2}^2.\end{aligned} \tag{46.246}$$

Using Gaussian quadrature, this yields

$$
\begin{aligned}
\partial_{h_q} d_{ii,t_r,t_z,u^s}(e_2) = \\
&\approx \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\left\{\frac{(\partial_{h_q}\partial_\eta r_{e_2}^2)(pp)(\partial_\eta z_{e_2}^2)(pp)}{J_{e_2}^2}u^s(pp)\right.\\
&\quad + \frac{(\partial_\eta r_{e_2}^2)(pp)(\partial_{h_q}\partial_\eta z_{e_2}^2)(pp)}{J_{e_2}^2(pp)}u^s(pp)\\
&\quad + \frac{(\partial_\eta r_{e_2}^2)(pp)(\partial_\eta z_{e_2}^2)(pp)}{J_{e_2}^2(pp)}\partial_{h_q}u^s(pp)\\
&\quad \left. - \frac{(\partial_\eta r_{e_2}^2)(pp)(\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2(pp))^2}u^s(pp)\partial_{h_q}J_{e_2}^2(pp)\right\}.
\end{aligned}
\tag{46.247}
$$

From equation (46.91) we have

$$
\partial_{h_q} d_{ii,t_z,t_z,w^s}(e_2) = \partial_{h_q}\int_{\eta=-1}^{\eta=1}\frac{(\partial_\eta z_{e_2}^2)^2}{J_{e_2}^2}\phi_{ii}^2 w^s,
\tag{46.248}
$$

or equivalently

$$
\begin{aligned}
\partial_{h_q} d_{ii,t_z,t_z,w^s}(e_2) = \int_{\eta=-1}^{\eta=1} 2\frac{(\partial_\eta z_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)}{J_{e_2}^2}\phi_{ii}^2 w^s \\
+ \int_{\eta=-1}^{\eta=1}\frac{(\partial_\eta z_{e_2}^2)^2}{J_{e_2}^2}\phi_{ii}^2\partial_{h_q}w^s - \int_{\eta=-1}^{\eta=1}\frac{(\partial_\eta z_{e_2}^2)^2}{(J_{e_2}^2)^2}\phi_{ii}^2 w^s\partial_{h_q}J_{e_2}^2.
\end{aligned}
\tag{46.249}
$$

Using Gaussian quadrature, this yields

$$
\begin{aligned}
\partial_{h_q} d_{ii,t_z,t_z,w^s}(e_2) \approx \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\left\{2\frac{(\partial_\eta z_{e_2}^2)(pp)(\partial_{h_q}\partial_\eta z_{e_2}^2)(pp)}{J_{e_2}^2(pp)}w^s(pp)\right.\\
\left. + \frac{(\partial_\eta z_{e_2}^2(pp))^2}{J_{e_2}^2(pp)}\partial_{h_q}w^s(pp) - \frac{(\partial_\eta z_{e_2}^2(pp))^2}{(J_{e_2}^2(pp))^2}w^s(pp)\partial_{h_q}J_{e_2}^2(pp)\right\}.
\end{aligned}
\tag{46.250}
$$

From equation (46.94) we have

$$
\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_r\breve{u}}(e_2) = \alpha\partial_{h_q}\int_{\eta=-1}^{\eta=1}\frac{(\partial_\eta z_{e_2}^2)^2(\partial_\eta r_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_r\breve{u},
\tag{46.251}
$$

or equivalently

$$\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_r \breve{u}}(e_2) = \alpha & \int\limits_{\eta=-1}^{\eta=1} 2\frac{(\partial_\eta z^2_{e_2})(\partial_{h_q}\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r \breve{u} \\
& + \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r \breve{u} \\
& + \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rr} \breve{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\
& + \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz} \breve{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \\
& - 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^3}\phi^2_{ii}\partial_r \breve{u}\partial_{h_q} J^2_{e_2}.
\end{aligned} \tag{46.252}$$

Using Gaussian quadrature, this yields

$$\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_r,n_z,\partial_r \breve{u}}(e_2) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp) \Bigg\{ & 2\frac{(\partial_\eta z^2_{e_2})(\partial_{h_q}\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_r \breve{u} \\
& + \frac{(\partial_\eta z^2_{e_2})^2(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_r \breve{u} \\
& + \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_{rr} \breve{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\
& + \frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_{rz} \breve{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \\
& - 2\frac{(\partial_\eta z^2_{e_2})^2(\partial_\eta r^2_{e_2})}{(J^2_{e_2})^3}\partial_r \breve{u}\partial_{h_q} J^2_{e_2} \Bigg\}.
\end{aligned} \tag{46.253}$$

From equation (46.97) we have

$$\partial_{h_q} d_{ii,n_r,n_z,n_z,\partial_z \breve{u}}(e_2) = -\alpha\partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z \breve{u}. \tag{46.254}$$

or equivalently

$$
\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_z,n_z,\partial_z \check{u}}(e_2) = &-\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_{h_q}\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z \check{u} \\
&- 2\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z \check{u} \\
&- \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2} \phi^2_{ii} \partial_{rz} \check{u} \\
&- \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2} \phi^2_{ii} \partial_{zz} \check{u} \\
&+ 2\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^3} \phi^2_{ii} \partial_z \check{u} \partial_{h_q} J^2_{e_2}.
\end{aligned}
\tag{46.255}
$$

Using Gaussian quadrature, this yields

$$
\boxed{
\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_z,n_z,\partial_z \check{u}}(e_2) = -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp) \phi^2_{ii}(pp) \Bigg\{ &\frac{(\partial_{h_q}\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2} \partial_z \check{u} \\
&+ 2\frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2} \partial_z \check{u} \\
&+ \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2} \partial_{rz} \check{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta) \partial_{h_q} r^2_{e_2,mm} \right] \\
&+ \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2} \partial_{zz} \check{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta) \partial_{h_q} z^2_{e_2,mm} \right] \\
&- 2\frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^3} \partial_z \check{u} \partial_{h_q} J^2_{e_2} \Bigg\}.
\end{aligned}
\quad (46.256)
}
$$

From equation (46.100) we have

$$
\partial_{h_q} d_{ii,n_r,n_z,n_z,\partial_r \check{w}}(e_2) = -\alpha \partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2} \phi^2_{ii} \partial_r \check{w}. \tag{46.257}
$$

or equivalently

$$\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_z,n_z,\partial_r \check{w}}(e_2) = &-\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_{h_q}\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r \check{w} \\
&- 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r \check{w} \\
&- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rr} \check{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\
&- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz} \check{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \\
&+ 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^3}\phi^2_{ii}\partial_r \check{w}\partial_{h_q} J^2_{e_2}.
\end{aligned} \tag{46.258}$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}
\partial_{h_q} d_{ii,n_r,n_z,n_z,\partial_r \check{w}}(e_2) \approx -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp) \Bigg\{ &\frac{(\partial_{h_q}\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2}\partial_r \check{w} \\
&+ 2\frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_r \check{w} \\
&+ \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2}\partial_{rr} \check{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\
&+ \frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^2}\partial_{rz} \check{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \\
&- 2\frac{(\partial_\eta z^2_{e_2})(\partial_\eta r^2_{e_2})^2}{(J^2_{e_2})^3}\partial_r \check{w}\partial_{h_q} J^2_{e_2}\Bigg\}.
\end{aligned}} \tag{46.259}$$

From equation (46.103) we have

$$\partial_{h_q} d_{ii,n_z,n_z,n_z,\partial_z \check{w}}(e_2) = \alpha\partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z \check{w}, \tag{46.260}$$

or equivalently

$$\partial_{h_q} d_{ii,n_z,n_z,n_z,\partial_z\check{w}}(e_2) = 3\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z\check{w}$$

$$+\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz}\check{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \tag{46.261}$$

$$+\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{zz}\check{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right]$$

$$-2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^3}\phi^2_{ii}\partial_z\check{w}\partial_{h_q}J^2_{e_2}.$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}\partial_{h_q} d_{ii,n_z,n_z,n_z,\partial_z\check{w}}(e_2) \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp)\Bigg\{&3\frac{(\partial_\eta r^2_{e_2})^2(\partial_{h_q}\partial_\eta r^2_{e_2})}{(J^2_{e_2})^2}\partial_z\check{w}\\ &+\frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\partial_{rz}\check{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right]\\ &+\frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^2}\partial_{zz}\check{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right]\\ &-2\frac{(\partial_\eta r^2_{e_2})^3}{(J^2_{e_2})^3}\partial_z\check{w}\partial_{h_q}J^2_{e_2}\Bigg\}.\end{aligned} \quad (46.262)}$$

From equation (46.106) we have

$$\partial_{h_q} d_{ii,t_r,t_z,n_r,\partial_r\check{u}}(e_2) = -\alpha\partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})^2}{(J^2_{e_2})^2}\phi^2_{ii}\partial_r\check{u}, \tag{46.263}$$

or equivalently

$$\begin{aligned}\partial_{h_q} d_{ii,t_r,t_z,n_r,\partial_r\check{u}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} &\frac{(\partial_{h_q}\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)^2}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_r\check{u} \\ &- 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)(\partial_{h_q}\partial_\eta z_{e_2}^2)^2}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_r\check{u} \\ &- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)^2}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_{rr}\check{u}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right] \\ &- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)^2}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_{rz}\check{u}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right] \\ &+ 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)^2}{(J_{e_2}^2)^3}\phi_{ii}^2\partial_r\check{u}\partial_{h_q} J_{e_2}^2.\end{aligned} \tag{46.264}$$

Using Gaussian quadrature, this yields

$$\begin{aligned}&\partial_{h_q} d_{ii,t_r,t_z,n_r,\partial_r\check{u}}(e_2) \\ &\approx -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\left\{\frac{(\partial_{h_q}\partial_\eta r_{e_2}^2(pp))(\partial_\eta z_{e_2}^2)^2(pp)}{(J_{e_2}^2)^2(pp)}\partial_r\check{u}(pp)\right. \\ &\qquad + 2\frac{(\partial_\eta r_{e_2}^2)(pp)(\partial_\eta z_{e_2}^2)(pp)(\partial_{h_q}\partial_\eta z_{e_2}^2(pp))}{(J_{e_2}^2)^2(pp)}\partial_r\check{u}(pp) \\ &\quad + \frac{(\partial_\eta r_{e_2}^2(pp))(\partial_\eta z_{e_2}^2(pp))^2}{(J_{e_2}^2(pp))^2}\partial_{rr}\check{u}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^2(pp)\partial_{h_q} r_{e_2,mm}^2\right] \\ &\quad + \frac{(\partial_\eta r_{e_2}^2(pp))(\partial_\eta z_{e_2}^2(pp))^2}{(J_{e_2}^2(pp))^2}\partial_{rz}\check{u}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^2(pp)\partial_{h_q} z_{e_2,mm}^2\right] \\ &\qquad \left. - 2\frac{(\partial_\eta r_{e_2}^2(pp))(\partial_\eta z_{e_2}^2(pp))^2}{(J_{e_2}^2(pp))^3}\partial_r\check{u}(pp)\partial_{h_q} J_{e_2}^2(pp)\right\}.\end{aligned} \tag{46.265}$$

From equation (46.109) we have

$$\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_z\check{u}}(e_2) = \alpha\partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)^2(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_z\check{u}, \tag{46.266}$$

or equivalently

$$
\begin{aligned}
\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_z \check{u}}(e_2) = 2\alpha & \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_{h_q}\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z \check{u} \\
& + \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_{h_q}\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z \check{u} \\
& + \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii} \partial_{rz} \check{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta) \partial_{h_q} r^2_{e_2,mm} \right] \\
& + \alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2} \phi^2_{ii} \partial_{zz} \check{u} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta) \partial_{h_q} z^2_{e_2,mm} \right] \\
& - 2\alpha \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})^2(\partial_\eta z^2_{e_2})}{(J^2_{e_2})^3} \phi^2_{ii} \partial_z \check{u} \partial_{h_q} J^2_{e_2}.
\end{aligned}
\tag{46.267}
$$

Using Gaussian quadrature, this yields

$$
\begin{aligned}
& \partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_z \check{u}}(e_2) \\
& \quad \approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp) \phi^2_{ii}(pp) \left\{ 2 \frac{(\partial_\eta r^2_{e_2})(pp)(\partial_{h_q}\partial_\eta r^2_{e_2})(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)} \partial_z \check{u}(pp) \right. \\
& \qquad + \frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_{h_q}\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)} \partial_z \check{u}(pp) \\
& \qquad + \frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)} \partial_{rz} \check{u}(pp) \left[\sum_{mm=1}^{3} \phi^2_{mm}(pp) \partial_{h_q} r^2_{e_2,mm} \right] \\
& \qquad + \frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)} \partial_{zz} \check{u}(pp) \left[\sum_{mm=1}^{3} \phi^2_{mm}(pp) \partial_{h_q} z^2_{e_2,mm} \right] \\
& \qquad \left. - 2 \frac{(\partial_\eta r^2_{e_2})^2(pp)(\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^3(pp)} \partial_z \check{u}(pp) \partial_{h_q} J^2_{e_2}(pp) \right\}.
\end{aligned}
\tag{46.268}
$$

From equation (46.112) we have

$$
\partial_{h_q} d_{ii,t_z,t_z,n_r,\partial_z \check{u}}(e_2) = -\alpha \partial_{h_q} \int_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z^2_{e_2})^3}{(J^2_{e_2})^2} \phi^2_{ii} \partial_z \check{u}, \tag{46.269}
$$

or equivalently

$$\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_z\check{u}}(e_2) = -3\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_{h_q}\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_z\check{u}$$

$$- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^3}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_{rz}\check{u}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right] \tag{46.270}$$

$$- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^3}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_{zz}\check{u}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right]$$

$$+ 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^3}{(J_{e_2}^2)^3}\phi_{ii}^2\partial_z\check{u}\partial_{h_q} J_{e_2}^2.$$

Using Gaussian quadrature, this yields

$$\begin{aligned}
&\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_z\check{u}}(e_2) \\
&\approx -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\left\{3\frac{(\partial_\eta z_{e_2}^2)^2(pp)(\partial_{h_q}\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2)^2(pp)}\partial_z\check{u}(pp)\right. \\
&\quad + \frac{(\partial_\eta r_{e_2}^2)^2(pp)(\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2)^2(pp)}\partial_{rz}\check{u}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^2(pp)\partial_{h_q} r_{e_2,mm}^2\right] \\
&\quad + \frac{(\partial_\eta z_{e_2}^2)^3(pp)}{(J_{e_2}^2)^2(pp)}\partial_{zz}\check{u}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^2(pp)\partial_{h_q} z_{e_2,mm}^2\right] \\
&\quad \left. - 2\frac{(\partial_\eta z_{e_2}^2)^3(pp)}{(J_{e_2}^2)^3(pp)}\partial_z\check{u}(pp)\partial_{h_q} J_{e_2}^2(pp)\right\}.
\end{aligned} \tag{46.271}$$

From equation (46.115) we have

$$\partial_{h_q} d_{ii,t_z,t_z,n_r,\partial_r\check{w}}(e_2) = -\alpha\partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^3}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_r\check{w}, \tag{46.272}$$

or equivalently

$$\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_r\breve{w}}(e_2) = -3\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^2(\partial_{h_q}\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_r\breve{w}$$

$$- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^3}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_{rr}\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right] \tag{46.273}$$

$$- \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^3}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right]$$

$$+ 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta z_{e_2}^2)^3}{(J_{e_2}^2)^3}\phi_{ii}^2\partial_r\breve{w}\partial_{h_q} J_{e_2}^2.$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}
&\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_r\breve{w}}(e_2)\\
&\approx -\alpha\sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\left\{3\frac{(\partial_\eta z_{e_2}^2)^2(pp)(\partial_{h_q}\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2)^2(pp)}\partial_r\breve{w}(pp)\right.\\
&\quad+\frac{(\partial_\eta r_{e_2}^2)^2(pp)(\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2)^2(pp)}\partial_{rr}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^2(pp)\partial_{h_q} r_{e_2,mm}^2\right] \qquad (46.274)\\
&\qquad+\frac{(\partial_\eta z_{e_2}^2)^3(pp)}{(J_{e_2}^2)^2(pp)}\partial_{rz}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^2(pp)\partial_{h_q} z_{e_2,mm}^2\right]\\
&\qquad\qquad\left.-2\frac{(\partial_\eta z_{e_2}^2)^3(pp)}{(J_{e_2}^2)^3(pp)}\partial_r\breve{w}(pp)\partial_{h_q} J_{e_2}^2(pp)\right\}.
\end{aligned}}$$

From equation (46.118) we have

$$\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_r\breve{w}}(e_2) = \alpha\partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)^2(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_r\breve{w}, \tag{46.275}$$

or equivalently

$$\begin{aligned}\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_r\check{w}}(e_2) = 2\alpha \int\limits_{\eta=-1}^{\eta=1} &\frac{(\partial_\eta r_{e_2}^2)(\partial_{h_q}\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_r\check{w}\\
&+ \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)^2(\partial_{h_q}\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_r\check{w}\\
&+ \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)^2(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_{rr}\check{w}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right]\\
&+ \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)^2(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_{rz}\check{w}\left[\sum_{mm=1}^{3}\phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right]\\
&- 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)^2(\partial_\eta z_{e_2}^2)}{(J_{e_2}^2)^3}\phi_{ii}^2\partial_r\check{w}\partial_{h_q} J_{e_2}^2.
\end{aligned} \tag{46.276}$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}&\partial_{h_q} d_{ii,t_r,t_z,n_z,\partial_r\check{w}}(e_2)\\
&\approx \alpha\sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\left\{2\frac{(\partial_\eta r_{e_2}^2)(pp)(\partial_{h_q}\partial_\eta r_{e_2}^2)(pp)(\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2)^2(pp)}\partial_r\check{w}(pp)\right.\\
&\qquad + \frac{(\partial_\eta r_{e_2}^2)^2(pp)(\partial_{h_q}\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2)^2(pp)}\partial_r\check{w}(pp)\\
&\qquad + \frac{(\partial_\eta r_{e_2}^2)^2(pp)(\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2)^2(pp)}\partial_{rr}\check{w}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^2(pp)\partial_{h_q} r_{e_2,mm}^2\right]\\
&\qquad + \frac{(\partial_\eta r_{e_2}^2)^2(pp)(\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2)^2(pp)}\partial_{rz}\check{w}(pp)\left[\sum_{mm=1}^{3}\phi_{mm}^2(pp)\partial_{h_q} z_{e_2,mm}^2\right]\\
&\qquad \left. - 2\frac{(\partial_\eta r_{e_2}^2)^2(pp)(\partial_\eta z_{e_2}^2)(pp)}{(J_{e_2}^2)^3(pp)}\partial_r\check{w}(pp)\partial_{h_q} J_{e_2}^2(pp)\right\}.
\end{aligned}} \tag{46.277}$$

From equation (46.121) we have

$$\partial_{h_q} d_{ii,t_z,t_z,n_z,\partial_z\check{w}}(e_2) = \alpha\partial_{h_q}\int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r_{e_2}^2)(\partial_\eta z_{e_2}^2)^2}{(J_{e_2}^2)^2}\phi_{ii}^2\partial_z\check{w}, \tag{46.278}$$

or equivalently

$$
\begin{aligned}
\partial_{h_q} d_{ii,t_z,t_z,n_z,\partial_z\breve{w}}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} & \frac{(\partial_{h_q}\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})^2}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z\breve{w} \\
& + 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})(\partial_{h_q}\partial_\eta z^2_{e_2})}{(J^2_{e_2})^2}\phi^2_{ii}\partial_z\breve{w} \\
& + \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})^2}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\
& + \alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})^2}{(J^2_{e_2})^2}\phi^2_{ii}\partial_{zz}\breve{w}\left[\sum_{mm=1}^{3}\phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \\
& - 2\alpha \int\limits_{\eta=-1}^{\eta=1} \frac{(\partial_\eta r^2_{e_2})(\partial_\eta z^2_{e_2})^2}{(J^2_{e_2})^3}\phi^2_{ii}\partial_z\breve{w}\partial_{h_q} J^2_{e_2}.
\end{aligned}
\tag{46.279}
$$

Using Gaussian quadrature, this yields

$$
\boxed{
\begin{aligned}
&\partial_{h_q} d_{ii,t_z,t_z,n_z,\partial_z\breve{w}}(e_2) \\
&\approx \alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi^2_{ii}(pp)\left\{\frac{(\partial_{h_q}\partial_\eta r^2_{e_2})(pp)(\partial_\eta z^2_{e_2})^2(pp)}{(J^2_{e_2})^2(pp)}\partial_z\breve{w}(pp)\right. \\
&\qquad + 2\frac{(\partial_\eta r^2_{e_2})(pp)(\partial_\eta z^2_{e_2})(pp)(\partial_{h_q}\partial_\eta z^2_{e_2})(pp)}{(J^2_{e_2})^2(pp)}\partial_z\breve{w}(pp) \\
&\quad + \frac{(\partial_\eta r^2_{e_2})(pp)(\partial_\eta z^2_{e_2})^2(pp)}{(J^2_{e_2})^2(pp)}\partial_{rz}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi^2_{mm}(pp)\partial_{h_q} r^2_{e_2,mm}\right] \\
&\quad + \frac{(\partial_\eta r^2_{e_2})(pp)(\partial_\eta z^2_{e_2})^2(pp)}{(J^2_{e_2})^2(pp)}\partial_{zz}\breve{w}(pp)\left[\sum_{mm=1}^{3}\phi^2_{mm}(pp)\partial_{h_q} z^2_{e_2,mm}\right] \\
&\qquad \left. - 2\frac{(\partial_\eta r^2_{e_2})(pp)(\partial_\eta z^2_{e_2})^2(pp)}{(J^2_{e_2})^3(pp)}\partial_z\breve{w}(pp)\partial_{h_q} J^2_{e_2}(pp)\right\}.
\end{aligned}
}
\tag{46.280}
$$

From equation (46.124) we have

$$
\partial_{h_q} d_{ii,n_r,\breve{u}}(e_2) = -\alpha\partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} (\partial_\eta z^2_{e_2})\breve{u}\phi^2_{ii}, \tag{46.281}
$$

or equivalently

$$\begin{aligned}\partial_{h_q} d_{ii,n_r,\check{u}}(e_2) = -\alpha \int\limits_{\eta=-1}^{\eta=1} (\partial_{h_q}\partial_\eta z_{e_2}^2)\check{u}\phi_{ii}^2 \\ - \alpha \int\limits_{\eta=-1}^{\eta=1} (\partial_\eta z_{e_2}^2)\partial_r \check{u}\left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right]\phi_{ii}^2 \\ - \alpha \int\limits_{\eta=-1}^{\eta=1} (\partial_\eta z_{e_2}^2)\partial_z \check{u}\left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right]\phi_{ii}^2.\end{aligned} \tag{46.282}$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}\partial_{h_q} d_{ii,n_r,\check{u}}(e_2) \approx -\alpha \sum_{pp=1}^{n_{lG}} W_{l_G}(pp)\phi_{ii}^2(pp)\Bigg\{(\partial_{h_q}\partial_\eta z_{e_2}^2)\check{u} \\ + (\partial_\eta z_{e_2}^2)\partial_r \check{u}\left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right] \\ + (\partial_\eta z_{e_2}^2)\partial_z \check{u}\left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right]\Bigg\}.\end{aligned} \quad (46.283)}$$

From equation (46.127) we have

$$\partial_{h_q} d_{ii,n_z,\check{w}}(e_2) = \alpha \partial_{h_q} \int\limits_{\eta=-1}^{\eta=1} (\partial_\eta r_{e_2}^2)\check{w}\phi_{ii}^2, \tag{46.284}$$

or equivalently

$$\begin{aligned}\partial_{h_q} d_{n_z,\check{w},ii}(e_2) = \alpha \int\limits_{\eta=-1}^{\eta=1} (\partial_{h_q}\partial_\eta r_{e_2}^2)\check{w}\phi_{ii}^2 \\ + \alpha \int\limits_{\eta=-1}^{\eta=1} (\partial_\eta r_{e_2}^2)\partial_r \check{w}\left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta)\partial_{h_q} r_{e_2,mm}^2\right]\phi_{ii}^2 \\ + \alpha \int\limits_{\eta=-1}^{\eta=1} (\partial_\eta r_{e_2}^2)\partial_z \check{w}\left[\sum_{mm=1}^{3} \phi_{mm}^2(\eta)\partial_{h_q} z_{e_2,mm}^2\right]\phi_{ii}^2.\end{aligned} \tag{46.285}$$

Using Gaussian quadrature, this yields

$$
\begin{aligned}
\partial_{h_q} d_{n_z,\breve{w},ii}(e_2) \approx \alpha \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi^2_{ii}(pp) \Bigg\{ & (\partial_{h_q}\partial_\eta r^2_{e_2})\breve{w} \\
& + (\partial_\eta r^2_{e_2})\partial_r \breve{w} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} r^2_{e_2,mm}\right] \\
& + (\partial_\eta r^2_{e_2})\partial_z \breve{w} \left[\sum_{mm=1}^{3} \phi^2_{mm}(\eta)\partial_{h_q} z^2_{e_2,mm}\right] \Bigg\}.
\end{aligned}
\tag{46.286}
$$

46.5.3. Derivatives of g terms

From equation (46.130) we have

$$\partial_{h_q} g_{n_r,\partial_r\check{u},ii}(e_4) = -\alpha\partial_{h_q}\int\limits_{\xi=-1}^{\xi=1}\partial_\xi z^4_{e_4}(\xi)\phi^4_{ii}\partial_r\check{u}, \tag{46.287}$$

or equivalently

$$\begin{aligned}\partial_{h_q} g_{ii,n_r,\partial_r\check{u}}(e_4) = &-\alpha\int\limits_{\xi=-1}^{\xi=1}\partial_{h_q}\partial_\xi z^4_{e_4}(\xi)\phi^4_{ii}\partial_r\check{u}(\xi)\\ &-\alpha\int\limits_{\xi=-1}^{\xi=1}\partial_\xi z^4_{e_4}(\xi)\phi^4_{ii}(\partial_{rr}\check{u})\left[\sum_{mm=1}^{3}\phi^2_{mm}(\xi)\partial_{h_q} r^2_{e_2,mm}\right]\\ &-\alpha\int\limits_{\xi=-1}^{\xi=1}\partial_\xi z^4_{e_4}(\xi)\phi^4_{ii}(\partial_{rz}\check{u})\left[\sum_{mm=1}^{3}\phi^2_{mm}(\xi)\partial_{h_q} z^2_{e_2,mm}\right].\end{aligned} \tag{46.288}$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}\partial_{h_q} g_{ii,n_r,\partial_r\check{u}}(e_4) \approx -\alpha\sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi^4_{ii}(pp)&\Big\{\partial_{h_q}\partial_\xi z^4_{e_4}(pp)\partial_r\check{u}(pp)\\ &+\partial_\xi z^4_{e_4}(pp)\partial_{rr}\check{u}(pp)\left[\sum_{mm=1}^{3}\phi^4_{mm}(pp)\partial_{h_q} r^4_{e_4,mm}\right]\\ &+\partial_\xi z^4_{e_4}(pp)\partial_{rz}\check{u}(pp)\left[\sum_{mm=1}^{3}\phi^4_{mm}(pp)\partial_{h_q} z^4_{e_4,mm}\right]\Big\}.\end{aligned}} \tag{46.289}$$

From equation (46.133) we have

$$\partial_{h_q} g_{ii,n_z,\partial_r\check{w}}(e_4) = \alpha\partial_{h_q}\int\limits_{\xi=-1}^{\xi=1}\partial_\xi r^4_{e_4}\phi^4_{ii}\partial_r\check{w}, \tag{46.290}$$

or equivalently

$$\begin{aligned}\partial_{h_q} g_{ii,n_z,\partial_r\check{w}}(e_4) = \alpha&\int\limits_{\xi=-1}^{\xi=1}\partial_{h_q}\partial_\xi r^4_{e_4}(\xi)\phi^4_{ii}(\xi)\partial_r\check{w}\\ &+\alpha\int\limits_{\xi=-1}^{\xi=1}\partial_\xi r^4_{e_4}(\xi)\phi^4_{ii}(\xi)\partial_{rr}\check{w}\left[\sum_{mm=1}^{3}\phi^4_{mm}(\xi)\partial_{h_q} r^4_{e_4,mm}\right]\\ &+\alpha\int\limits_{\xi=-1}^{\xi=1}\partial_\xi r^4_{e_4}(\xi)\phi^4_{ii}(\xi)\partial_{rz}\check{w}\left[\sum_{mm=1}^{3}\phi^4_{mm}(\xi)\partial_{h_q} z^4_{e_4,mm}\right].\end{aligned} \tag{46.291}$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} g_{ii,n_z,\partial_r \breve{w}}(e_4) \approx \alpha \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi_{ii}^4(pp) \left\{ \partial_{h_q}\partial_\xi r_{e_4}^4(pp)\partial_r \breve{w} + \partial_\xi r_{e_4}^4(pp)\partial_{rr}\breve{w}\left[\sum_{mm=1}^{3} \phi_{mm}^4(pp)\partial_{h_q} r_{e_4,mm}^4\right] + \partial_\xi r_{e_4}^4(pp)\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3} \phi_{mm}^4(pp)\partial_{h_q} z_{e_4,mm}^4\right]\right\}. \tag{46.292}$$

From equation (46.136) we have

$$\partial_{h_q} g_{ii,n_z,\partial_z \breve{u}}(e_4) = \alpha \partial_{h_q} \int_{\xi=-1}^{\xi=1} \partial_\xi r_{e_4}^4 \phi_{ii}^4 \partial_z \breve{u}, \tag{46.293}$$

or equivalently

$$\partial_{h_q} g_{ii,n_z,\partial_z \breve{u}}(e_4) = \alpha \int_{\xi=-1}^{\xi=1} \partial_{h_q}\partial_\xi r_{e_4}^4(\xi)\phi_{ii}^4(\xi)\partial_z \breve{u} + \alpha \int_{\xi=-1}^{\xi=1} \partial_\xi r_{e_4}^4(\xi)\phi_{ii}^4(\xi)\partial_{rz}\breve{u}\left[\sum_{mm=1}^{3} \phi_{mm}^4(\xi)\partial_{h_q} r_{e_4,mm}^4\right] + \alpha \int_{\xi=-1}^{\xi=1} \partial_\xi r_{e_4}^4(\xi)\phi_{ii}^4(\xi)\partial_{zz}\breve{u}\left[\sum_{mm=1}^{3} \phi_{mm}^4(\xi)\partial_{h_q} r_{e_4,mm}^4\right]. \tag{46.294}$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} g_{ii,n_z,\partial_r \breve{w}}(e_4) \approx \alpha \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi_{ii}^4(pp) \left\{ \partial_{h_q}\partial_\xi r_{e_4}^4(pp)\partial_r \breve{w} + \partial_\xi r_{e_4}^4(pp)\partial_{rr}\breve{w}\left[\sum_{mm=1}^{3} \phi_{mm}^4(pp)\partial_{h_q} r_{e_4,mm}^4\right] + \partial_\xi r_{e_4}^4(pp)\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3} \phi_{mm}^4(pp)\partial_{h_q} z_{e_4,mm}^4\right]\right\}. \tag{46.295}$$

From equation (46.139) we have

$$\partial_{h_q} g_{ii,jj,n_r}(e_4) = -\alpha \partial_{h_q} \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\phi_{jj}^4(\xi)\partial_\xi z_{e_4}^4(\xi), \tag{46.296}$$

or equivalently

$$\partial_{h_q} g_{ii,jj,n_r}(e_4) = -\alpha \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\phi_{jj}^4(\xi)\partial_{h_q}\partial_\xi z_{e_4}^4(\xi). \tag{46.297}$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} g_{ii,jj,n_r}(e_4) \approx -\alpha \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi_{ii}^4(pp)\phi_{jj}^4(pp)\partial_{h_q}\partial_\xi z_{e_4}^4(pp). \quad (46.298)$$

From equation (46.142) we have

$$\partial_{h_q} g_{ii,jj,t_r}(e_4) = \partial_{h_q} \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\phi_{jj}^4(\xi)\partial_\xi r_{e_4}^4(\xi), \quad (46.299)$$

or equivalently

$$\partial_{h_q} g_{ii,jj,t_r}(e_4) = \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\phi_{jj}^4(\xi)\partial_{h_q}\partial_\xi r_{e_4}^4(\xi). \quad (46.300)$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} g_{ii,jj,t_r}(e_4) \approx \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi_{ii}^4(pp)\phi_{jj}^4(pp)\partial_{h_q}\partial_\xi r_{e_4}^4(pp). \quad (46.301)$$

From equation (46.136) we have

$$\partial_{h_q} g_{ii,n_z,\partial_z \breve{u}}(e_4) = \alpha\partial_{h_q} \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\partial_\xi r_{e_4}^4(\xi)\partial_z \breve{u}, \quad (46.302)$$

or equivalently

$$\begin{aligned} \partial_{h_q} g_{ii,n_z,\partial_z \breve{u}}(e_4) = \alpha &\int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\partial_{h_q}\partial_\xi r_{e_4}^4(\xi)\partial_z \breve{u} \\ &+ \alpha \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\partial_\xi r_{e_4}^4(\xi)\partial_{rz} \breve{u}\left[\sum_{mm=1}^{3} \phi_{mm}^4(\xi)\partial_{h_q} r_{e_4,mm}^4\right] \\ &+ \alpha \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\partial_\xi r_{e_4}^4(\xi)\partial_{zz} \breve{u}\left[\sum_{mm=1}^{3} \phi_{mm}^4(\xi)\partial_{h_q} r_{e_4,mm}^4\right]. \end{aligned} \quad (46.303)$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} g_{ii,n_z,\partial_z \breve{u}}(e_4) \approx \alpha \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi_{ii}^4(pp) \left\{ \partial_{h_q}\partial_\xi r_{e_4}^4(pp)\partial_z \breve{u}(pp) \right.$$
$$+ \partial_\xi r_{e_4}^4(pp)\partial_{rz}\breve{u} \left[\sum_{mm=1}^{3} \phi_{mm}^4(pp)\partial_{h_q} r_{e_4,mm}^4 \right] \quad (46.304)$$
$$\left. + \partial_\xi r_{e_4}^4(pp)\partial_{zz}\breve{u} \left[\sum_{mm=1}^{3} \phi_{mm}^4(pp)\partial_{h_q} z_{e_4,mm}^4 \right] \right\}.$$

From equation (46.136) we have

$$\partial_{h_q} g_{ii,n_z,\partial_z \breve{u}}(e_4) = \alpha \partial_{h_q} \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\partial_\xi r_{e_4}^4(\xi)\partial_z \breve{u}, \quad (46.305)$$

or equivalently

$$\partial_{h_q} g_{ii,n_z,\partial_z \breve{u}}(e_4) = \alpha \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\partial_{h_q}\partial_\xi r_{e_4}^4(\xi)\partial_z \breve{u}$$
$$+ \alpha \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\partial_\xi r_{e_4}^4(\xi)\partial_{rz}\breve{u} \left[\sum_{mm=1}^{3} \phi_{mm}^4(\xi)\partial_{h_q} r_{e_4,mm}^4 \right] \quad (46.306)$$
$$+ \alpha \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\partial_\xi r_{e_4}^4(\xi)\partial_{zz}\breve{u} \left[\sum_{mm=1}^{3} \phi_{mm}^4(\xi)\partial_{h_q} r_{e_4,mm}^4 \right].$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} g_{ii,n_z,\partial_z \breve{u}}(e_4) \approx \alpha \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi_{ii}^4(pp) \left\{ \partial_{h_q}\partial_\xi r_{e_4}^4(pp)\partial_z \breve{u}(pp) \right.$$
$$+ \partial_\xi r_{e_4}^4(pp)\partial_{rz}\breve{u} \left[\sum_{mm=1}^{3} \phi_{mm}^4(pp)\partial_{h_q} r_{e_4,mm}^4 \right] \quad (46.307)$$
$$\left. + \partial_\xi r_{e_4}^4(pp)\partial_{zz}\breve{u} \left[\sum_{mm=1}^{3} \phi_{mm}^4(pp)\partial_{h_q} z_{e_4,mm}^4 \right] \right\}.$$

From equation (46.148) we have

$$\partial_{h_q} g_{ii,n_r,\partial_z \breve{u}}(e_4) = -\alpha \partial_{h_q} \int_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\partial_\xi z_{e_4}^4(\xi)\partial_z \breve{u}, \quad (46.308)$$

or equivalently

$$\begin{aligned}\partial_{h_q} g_{ii,n_r,\partial_z \breve{u}}(e_4) = -\alpha \int\limits_{\xi=-1}^{\xi=1} & \phi_{ii}^4(\xi)\partial_{h_q}\partial_\xi z_{e_4}^4(\xi)\partial_z \breve{u} \\ -\alpha \int\limits_{\xi=-1}^{\xi=1} & \phi_{ii}^4(\xi)\partial_\xi z_{e_4}^4(\xi)\partial_{rz}\breve{u}\left[\sum_{mm=1}^{3}\phi_{mm}^4(\xi)\partial_{h_q} r_{e_4,mm}^4\right] \\ -\alpha \int\limits_{\xi=-1}^{\xi=1} & \phi_{ii}^4(\xi)\partial_\xi z_{e_4}^4(\xi)\partial_{zz}\breve{u}\left[\sum_{mm=1}^{3}\phi_{mm}^4(\xi)\partial_{h_q} r_{e_4,mm}^4\right].\end{aligned} \tag{46.309}$$

Using Gaussian quadrature, this yields

$$\boxed{\begin{aligned}\partial_{h_q} g_{ii,n_r,\partial_z \breve{u}}(e_4) \approx -\alpha \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi_{ii}^4(pp) \Bigg\{ & \partial_{h_q}\partial_\xi z_{e_4}^4(pp)\partial_z \breve{u}(pp) \\ & + \partial_\xi z_{e_4}^4(pp)\partial_{rz}\breve{u}\left[\sum_{mm=1}^{3}\phi_{mm}^4(pp)\partial_{h_q} r_{e_4,mm}^4\right] \\ & + \partial_\xi z_{e_4}^4(pp)\partial_{zz}\breve{u}\left[\sum_{mm=1}^{3}\phi_{mm}^4(pp)\partial_{h_q} z_{e_4,mm}^4\right]\Bigg\}.\end{aligned}} \tag{46.310}$$

From equation (46.151) we have

$$\partial_{h_q} g_{ii,n_r,\partial_r \breve{w}}(e_4) = -\alpha\partial_{h_q} \int\limits_{\xi=-1}^{\xi=1} \phi_{ii}^4(\xi)\partial_\xi z_{e_4}^4(\xi)\partial_r \breve{w}, \tag{46.311}$$

or equivalently

$$\begin{aligned}\partial_{h_q} g_{ii,n_r,\partial_r \breve{w}}(e_4) = -\alpha \int\limits_{\xi=-1}^{\xi=1} & \phi_{ii}^4(\xi)\partial_{h_q}\partial_\xi z_{e_4}^4(\xi)\partial_r \breve{w} \\ -\alpha \int\limits_{\xi=-1}^{\xi=1} & \phi_{ii}^4(\xi)\partial_\xi z_{e_4}^4(\xi)\partial_{rr}\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^4(\xi)\partial_{h_q} r_{e_4,mm}^4\right] \\ -\alpha \int\limits_{\xi=-1}^{\xi=1} & \phi_{ii}^4(\xi)\partial_\xi z_{e_4}^4(\xi)\partial_{rz}\breve{w}\left[\sum_{mm=1}^{3}\phi_{mm}^4(\xi)\partial_{h_q} r_{e_4,mm}^4\right].\end{aligned} \tag{46.312}$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} g_{ii,n_r,\partial_r \breve{w}}(e_4) \approx -\alpha \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi^4_{ii}(pp) \Bigg\{ \partial_{h_q}\partial_\xi z^4_{e_4}(pp)\partial_r \breve{w}(pp) + \partial_\xi z^4_{e_4}(pp)\partial_{rr}\breve{w} \left[\sum_{mm=1}^{3} \phi^4_{mm}(pp)\partial_{h_q} r^4_{e_4,mm} \right] + \partial_\xi z^4_{e_4}(pp)\partial_{rz}\breve{w} \left[\sum_{mm=1}^{3} \phi^4_{mm}(pp)\partial_{h_q} z^4_{e_4,mm} \right] \Bigg\}. \quad (46.313)$$

From equation (46.154) we have

$$\partial_{h_q} g_{ii,jj,n_z}(e_4) = \alpha \partial_{h_q} \int_{\xi=-1}^{\xi=1} \phi^4_{ii}(\xi)\phi^4_{jj}(\xi)\partial_\xi r^4_{e_4}(\xi), \quad (46.314)$$

or equivalently

$$\partial_{h_q} g_{ii,jj,n_z}(e_4) = \alpha \int_{\xi=-1}^{\xi=1} \phi^4_{ii}(\xi)\phi^4_{jj}(\xi)\partial_{h_q}\partial_\xi r^4_{e_4}(\xi). \quad (46.315)$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} g_{ii,jj,n_z}(e_4) \approx \alpha \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi^4_{ii}(pp)\phi^4_{jj}(pp)\partial_{h_q}\partial_\xi r^4_{e_4}(pp). \quad (46.316)$$

From equation (46.157) we have

$$\partial_{h_q} g_{ii,jj,t_z}(e_4) = \partial_{h_q} \int_{\xi=-1}^{\xi=1} \phi^4_{ii}(\xi)\phi^4_{jj}(\xi)\partial_\xi z^4_{e_4}(\xi), \quad (46.317)$$

or equivalently

$$\partial_{h_q} g_{ii,jj,t_z}(e_4) = \int_{\xi=-1}^{\xi=1} \phi^4_{ii}(\xi)\phi^4_{jj}(\xi)\partial_{h_q}\partial_\xi z^4_{e_4}(\xi). \quad (46.318)$$

Using Gaussian quadrature, this yields

$$\partial_{h_q} g_{ii,jj,t_z}(e_4) \approx \sum_{pp=1}^{n_{l_G}} W_{l_G}(pp)\phi^4_{ii}(pp)\phi^4_{jj}(pp)\partial_{h_q}\partial_\xi z^4_{e_4}(pp). \quad (46.319)$$

47. Singular element at contact line

It was shown in Sprittles & Shikhmurzaev (2011*a*) that, for the boundary conditions here considered, the pressure has a logarithmic singularity at the contact line. This represents a problem for our present formulation when approximating the pressure in the vicinity of the contact line if, as described above, we approximate the solution using piece-wise smooth polynomials. A solution to a very similar problem was given in Sprittles & Shikhmurzaev (2011*b*), where boundary conditions were similar. In principle, the same solution should work in this case provided we continue to deal with an acute contact angle.

The solution given in Sprittles & Shikhmurzaev (2011*b*) consists in defining function

$$\psi_c^* = \psi_c \ln(\sqrt{r^2 + z^2}), \tag{47.1}$$

and using this function as the interpolating function for pressure associated to node c (i.e. the contact line), instead of using ψ_c (as the prior sections would suggest).

Introducing this change, requires that we treat the element at the contact line (i.e. element 1 in our numbering) in a different way from the other triangular elements when considering the sum per elements of the residual and Jacobian contribution. More specifically, we need to provide a different expression for $b^r_{ii,jj}(e)$, $b^z_{ii,jj}(e)$, $\partial_{h_q} b^r_{ii,jj}(e)$ and $\partial_{h_q} b^z_{ii,jj}(e)$, for $e = 1$.

From equation (20.44)

$$b^r_{jj,ii}(e) = \int\limits_{\Omega_e} \psi_{l^p(e,jj)} \partial_r \phi_{l(e,ii)}, \tag{47.2}$$

and for $e = 1$ and $jj = 2$, we have $l^p(e, jj) = c$. That is to say, we are dealing with ψ_{jj} been the pressure-interpolating function associated to the contact line. Therefore we re-define

$$b^r_{jj=2,ii}(e = 1) = \int\limits_{\Omega_e} \psi^*_{l^p(e,jj)} \partial_r \phi_{l(e,ii)}, \tag{47.3}$$

and using local node numbering and equation (20.11) we have

$$b^r_{jj=2,ii}(e = 1) = \int\limits_{E} \psi^*_{jj} \left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm} \right), \tag{47.4}$$

where

$$\psi^*_{jj} = \psi_{jj} \ln \left(\sqrt{ \left(\sum_{kk=1}^{6} r_{e,kk} \phi_{kk} \right)^2 + \left(\sum_{kk=1}^{6} z_{e,kk} \phi_{kk} \right)^2 } \right). \tag{47.5}$$

As advised in Sprittles & Shikhmurzaev (2011*b*), we use Gaussian quadrature with at least 16 points to approximate this integration, yielding

$$\boxed{b^r_{jj=2,ii}(e = 1) \approx \sum_{pp=1}^{n_G} \left[\psi^*_{jj}(pp) \left(\sum_{mm=1}^{6} T_{ii,mm}(pp) z_{e,mm} \right) \right]}, \tag{47.6}$$

where we once again have used the abbreviated notation $f(pp) = f(\xi_{pp}, \eta_{pp})$; and we also use

$$\boxed{\psi^*_{jj}(pp) = \psi_{jj}(pp)\ln\left(\sqrt{\left(\sum_{kk=1}^{6} r_{e,kk}\phi_{kk}(pp)\right)^2 + \left(\sum_{kk=1}^{6} z_{e,kk}\phi_{kk}(pp)\right)^2}\right)}. \quad (47.7)$$

Similarly, using equation (20.48), we re-define

$$b^z_{jj=2,ii}(e=1) = -\int\limits_E \psi^*_{jj}\left(\sum_{mm=1}^{6} T_{ii,mm} r_{e,mm}\right), \quad (47.8)$$

which, using Gaussian quadrature, yields

$$\boxed{b^z_{jj=2,ii}(e=1) \approx -\sum_{pp=1}^{n_G}\left[\psi^*_{jj}(pp)\left(\sum_{mm=1}^{6} T_{ii,mm}(pp) r_{e,mm}\right)\right]}. \quad (47.9)$$

We consider now $\partial_{h_q} b^r_{jj=2,ii}(e=1)$ and $\partial_{h_q} b^z_{jj=2,ii}(e=1)$, which involve the newly introduce function ψ^*. We highlight that, in contrast with function ψ_2, function ψ^*_2 actually depends on $r_{e=1,kk}$ and $z_{e=1,kk}$ (with $kk = 1,\ldots,6$); and these functions in turn depend on h_q (for those q-indexed spines that affect the shape of the first element). Therefore, our re-definition of $b^r_{jj=2,ii}(e=1)$ and $b^z_{jj=2,ii}(e=1)$ implies that their derivatives with respect to h_q will have a different expression, which is given in what follows.

From equation (47.4) we have

$$\partial_{h_q} b^r_{jj\ =2,ii}(e=1) = \partial_{h_q}\int\limits_E \psi^*_{jj}\left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm}\right), \quad (47.10)$$

i.e.

$$\begin{aligned}\partial_{h_q} b^r_{jj\ =2,ii}(e=1) = &\int\limits_E \left[\left(\partial_{h_q}\psi^*_{jj}\right)\left(\sum_{mm=1}^{6} T_{ii,mm} z_{e,mm}\right)\right] \\ &+ \int\limits_E \psi^*_{jj}\left(\sum_{mm=1}^{6} T_{ii,mm}\partial_{h_q} z_{e,mm}\right),\end{aligned} \quad (47.11)$$

where

$$\partial_{h_q}\psi^*_{jj} = \psi_{jj}\frac{\left(\sum\limits_{kk=1}^{6} r_{e,kk}\phi_{kk}\right)\left(\sum\limits_{kk=1}^{6}\phi_{kk}\partial_{h_q} r_{e,kk}\right) + \left(\sum\limits_{kk=1}^{6} z_{e,kk}\phi_{kk}\right)\left(\sum\limits_{kk=1}^{6}\phi_{kk}\partial_{h_q} z_{e,kk}\right)}{\left(\sum\limits_{kk=1}^{6} r_{e,kk}\phi_{kk}\right)^2 + \left(\sum\limits_{kk=1}^{6} z_{e,kk}\phi_{kk}\right)^2}. \quad (47.12)$$

Using Gaussian quadrature we have

$$\partial_{h_q} b^r_{jj\ =2,ii}(e=1) \approx \sum_{pp=1}^{n_G} \left\{ \left(\partial_{h_q}\psi^*_{jj}(pp)\right) \left(\sum_{mm=1}^{6} T_{ii,mm}(pp) z_{e,mm} \right) \right\} + \sum_{pp=1}^{n_G} \left\{ \psi^*_{jj}(pp) \left(\sum_{mm=1}^{6} T_{ii,mm}(pp) \partial_{h_q} z_{e,mm} \right) \right\}, \tag{47.13}$$

where

$$\partial_{h_q}\psi^*_{jj}(pp) = \psi_{jj}(pp) \left[\frac{\left(\sum_{kk=1}^{6} r_{e,kk}\phi_{kk}(pp)\right)\left(\sum_{kk=1}^{6} \phi_{kk}(pp)\partial_{h_q} r_{e,kk}\right)}{\left(\sum_{kk=1}^{6} r_{e,kk}\phi_{kk}(pp)\right)^2 + \left(\sum_{kk=1}^{6} z_{e,kk}\phi_{kk}(pp)\right)^2} + \frac{\left(\sum_{kk=1}^{6} z_{e,kk}\phi_{kk}(pp)\right)\left(\sum_{kk=1}^{6} \phi_{kk}(pp)\partial_{h_q} z_{e,kk}\right)}{\left(\sum_{kk=1}^{6} r_{e,kk}\phi_{kk}(pp)\right)^2 + \left(\sum_{kk=1}^{6} z_{e,kk}\phi_{kk}(pp)\right)^2} \right]. \tag{47.14}$$

Similarly, from equation (47.8) we have

$$\partial_{h_q} b^z_{jj\ =2,ii}(e=1) = -\partial_{h_q} \int_E \psi^*_{jj} \left(\sum_{mm=1}^{6} T_{ii,mm} r_{e,mm} \right), \tag{47.15}$$

i.e.

$$\partial_{h_q} b^z_{jj\ =2,ii}(e=1) = -\int_E \left[\left(\partial_{h_q}\psi^*_{jj}\right) \left(\sum_{mm=1}^{6} T_{ii,mm} r_{e,mm} \right) \right] - \int_E \psi^*_{jj} \left(\sum_{mm=1}^{6} T_{ii,mm} \partial_{h_q} r_{e,mm} \right). \tag{47.16}$$

Using Gaussian quadrature we have

$$\partial_{h_q} b^z_{jj\ =2,ii}(e=1) \approx -\sum_{pp=1}^{n_G} \left\{ \left(\partial_{h_q}\psi^*_{jj}(pp)\right) \left(\sum_{mm=1}^{6} T_{ii,mm}(pp) r_{e,mm} \right) \right\} - \sum_{pp=1}^{n_G} \left\{ \psi^*_{jj}(pp) \left(\sum_{mm=1}^{6} T_{ii,mm}(pp) \partial_{h_q} r_{e,mm} \right) \right\}. \tag{47.17}$$

Appendix A. Exact solution to the Stokes equation in a wedge with no tangential stress on the boundaries

We follow Sprittles & Shikhmurzaev (2011 *a*) and consider the flow of an incompressible Newtonian fluid with uniform density in a wedge-shaped region. We use polar coordinates (ζ, θ), where $\zeta = \sqrt{r^2+z^2}$ and $\theta = \arctan(z/r)$, with the origin on the contact line and the fluid occupying the region given by $0 \le \theta \le \theta_c$. The radial and azimuthal velocity components are respectively given by v_ζ and v_θ. The law of conservation of mass is given by

$$\partial_\zeta(\zeta v_\zeta) + \partial_\theta v_\theta = 0, \tag{A 1}$$

radial conservation of momentum is given by

$$\partial_\zeta p = \Delta v_\zeta - \frac{v_\zeta}{\zeta} - \frac{2}{\zeta^2}\partial_\theta v_\theta \tag{A 2}$$

and azimuthal conservation of momentum is given by

$$\frac{1}{\zeta}\partial_\theta p = \Delta v_\theta - \frac{v_\theta}{\zeta^2} + \frac{2}{\zeta^2}\partial_\theta v_\zeta, \tag{A 3}$$

where

$$\Delta = \partial_{\zeta\zeta} + \frac{1}{\zeta}\partial_\zeta + \frac{2}{\zeta^2}\partial_{\theta\theta}. \tag{A 4}$$

At $\theta = 0$, the flow satisfies the impermeability condition

$$v_\theta(\zeta, \theta = 0) = 0, \tag{A 5}$$

and the condition of no tangential stress

$$\partial_\theta v_\zeta = 0. \tag{A 6}$$

At $\theta = \theta_c$ we impose the kinematic boundary condition

$$v_\theta = 0, \tag{A 7}$$

and

$$\partial_\theta v_\zeta = 0. \tag{A 8}$$

We introduce the stream function $\psi(\zeta, \theta)$, with

$$v_\zeta = \frac{1}{\zeta}\partial_\theta \psi \tag{A 9}$$

and

$$v_\theta = -\partial_\zeta \psi. \tag{A 10}$$

Naturally, the substitution of (A 9) and (A 10) into (A 1) shows it is identically satisfied. Now, substituting them into (A 2) we have

$$\partial_\zeta p = \partial_{\zeta\zeta}\left(\frac{1}{\zeta}\partial_\theta\psi\right) + \frac{1}{\zeta}\partial_\zeta\left(\frac{1}{\zeta}\partial_\theta\psi\right) + \frac{1}{\zeta^2}\partial_{\theta\theta}\left(\frac{1}{\zeta}\partial_\theta\psi\right) - \frac{1}{\zeta^2}\left(\frac{1}{\zeta}\partial_\theta\psi\right) - \frac{2}{\zeta^2}\partial_\theta\left(-\partial_\zeta\psi\right) \tag{A 11}$$

which implies

$$\partial_\zeta p = \partial_\zeta\left(-\frac{1}{\zeta^2}\partial_\theta\psi + \frac{1}{\zeta}\partial_{\zeta\theta}\psi\right) + \frac{1}{\zeta}\left(-\frac{1}{\zeta^2}\partial_\theta\psi + \frac{1}{\zeta}\partial_{\zeta\theta}\psi\right) + \frac{1}{\zeta^3}\partial_{\theta\theta\theta}\psi - \frac{1}{\zeta^3}\partial_\theta\psi + \frac{2}{\zeta^2}\partial_{\zeta\theta}\psi \tag{A 12}$$

i.e.

$$\begin{aligned}\partial_\zeta p = &\frac{2}{\zeta^3}\partial_\theta\psi - \frac{1}{\zeta^2}\partial_{\zeta\theta}\psi - \frac{1}{\zeta^2}\partial_{\zeta\theta}\psi + \frac{1}{\zeta}\partial_{\zeta\zeta\theta}\psi - \frac{1}{\zeta^3}\partial_\theta\psi \\ &+ \frac{1}{\zeta^2}\partial_{\zeta\theta}\psi + \frac{1}{\zeta^3}\partial_{\theta\theta\theta}\psi - \frac{1}{\zeta^3}\partial_\theta\psi + \frac{2}{\zeta^2}\partial_{\zeta\theta}\psi,\end{aligned} \tag{A 13}$$

grouping similar terms and re-arranging we have

$$\partial_\zeta p = \frac{1}{\zeta^3}\partial_{\theta\theta\theta}\psi + \frac{1}{\zeta}\partial_{\zeta\zeta\theta}\psi + \frac{1}{\zeta^2}\partial_{\zeta\theta}\psi. \tag{A 14}$$

Taking one derivative with respect to θ of the equation above and dividing both sides by ζ, we have

$$\frac{1}{\zeta}\partial_{\zeta\theta}p = \frac{1}{\zeta^4}\partial_{\theta\theta\theta\theta}\psi + \frac{1}{\zeta^2}\partial_{\zeta\zeta\theta\theta}\psi + \frac{1}{\zeta^3}\partial_{\zeta\theta\theta}\psi. \tag{A 15}$$

Now, substituting equations (A 9) and (A 10) into (A 3), we have

$$\frac{1}{\zeta}\partial_\theta p = \partial_{\zeta\zeta}\left(-\partial_\zeta\psi\right) + \frac{1}{\zeta}\partial_\zeta\left(-\partial_\zeta\psi\right) + \frac{1}{\zeta^2}\partial_{\theta\theta}\left(-\partial_\zeta\psi\right) - \frac{1}{\zeta^2}\left(-\partial_\zeta\psi\right) + \frac{2}{\zeta^2}\partial_\theta\left(\frac{1}{\zeta}\partial_\theta\psi\right), \tag{A 16}$$

i.e.

$$\frac{1}{\zeta}\partial_\theta p = -\partial_{\zeta\zeta\zeta}\psi - \frac{1}{\zeta}\partial_{\zeta\zeta}\psi - \frac{1}{\zeta^2}\partial_{\zeta\theta\theta}\psi + \frac{1}{\zeta^2}\partial_\zeta\psi + \frac{2}{\zeta^3}\partial_{\theta\theta}\psi, \tag{A 17}$$

multiplying the equation above by ζ, we have

$$\partial_\theta p = -\zeta\partial_{\zeta\zeta\zeta}\psi - \partial_{\zeta\zeta}\psi - \frac{1}{\zeta}\partial_{\zeta\theta\theta}\psi + \frac{1}{\zeta}\partial_\zeta\psi + \frac{2}{\zeta^2}\partial_{\theta\theta}\psi, \tag{A 18}$$

and the taking derivatives with respect to ζ

$$\begin{aligned}\partial_{\zeta\theta}p = &-\partial_{\zeta\zeta\zeta}\psi - \zeta\partial_{\zeta\zeta\zeta\zeta}\psi - \partial_{\zeta\zeta\zeta}\psi + \frac{1}{\zeta^2}\partial_{\zeta\theta\theta}\psi - \frac{1}{\zeta}\partial_{\zeta\zeta\theta\theta}\psi \\ &- \frac{1}{\zeta^2}\partial_\zeta\psi + \frac{1}{\zeta}\partial_{\zeta\zeta}\psi - \frac{4}{\zeta^3}\partial_{\theta\theta}\psi + \frac{2}{\zeta^2}\partial_{\zeta\theta\theta}\psi,\end{aligned} \tag{A 19}$$

and dividing both sides by ζ we have

$$\begin{aligned}\frac{1}{\zeta}\partial_{\zeta\theta}p = &-\frac{1}{\zeta}\partial_{\zeta\zeta\zeta}\psi - \partial_{\zeta\zeta\zeta\zeta}\psi - \frac{1}{\zeta}\partial_{\zeta\zeta\zeta}\psi + \frac{1}{\zeta^3}\partial_{\zeta\theta\theta}\psi - \frac{1}{\zeta^2}\partial_{\zeta\zeta\theta\theta}\psi \\ &- \frac{1}{\zeta^3}\partial_\zeta\psi + \frac{1}{\zeta^2}\partial_{\zeta\zeta}\psi - \frac{4}{\zeta^4}\partial_{\theta\theta}\psi + \frac{2}{\zeta^3}\partial_{\zeta\theta\theta}\psi.\end{aligned} \tag{A 20}$$

Re-arraging terms we have

$$\frac{1}{\zeta}\partial_{\zeta\theta}p = -\partial_{\zeta\zeta\zeta\zeta}\psi - \frac{1}{\zeta^2}\partial_{\zeta\zeta\theta\theta}\psi - \frac{2}{\zeta}\partial_{\zeta\zeta\zeta}\psi + \frac{3}{\zeta^3}\partial_{\zeta\theta\theta}\psi + \frac{1}{\zeta^2}\partial_{\zeta\zeta}\psi - \frac{4}{\zeta^4}\partial_{\theta\theta}\psi - \frac{1}{\zeta^3}\partial_\zeta\psi. \tag{A 21}$$

We subtract equation (A 20) from (A 15) and obtain

$$\begin{aligned}&\frac{1}{\zeta^4}\partial_{\theta\theta\theta\theta}\psi + \frac{1}{\zeta^2}\partial_{\zeta\zeta\theta\theta}\psi + \frac{1}{\zeta^3}\partial_{\zeta\theta\theta}\psi + \partial_{\zeta\zeta\zeta\zeta}\psi + \frac{1}{\zeta^2}\partial_{\zeta\zeta\theta\theta}\psi \\ &+ \frac{2}{\zeta}\partial_{\zeta\zeta\zeta}\psi - \frac{3}{\zeta^3}\partial_{\zeta\theta\theta}\psi - \frac{1}{\zeta^2}\partial_{\zeta\zeta}\psi + \frac{4}{\zeta^4}\partial_{\theta\theta}\psi + \frac{1}{\zeta^3}\partial_\zeta\psi = 0.\end{aligned} \tag{A 22}$$

Re-arranging terms we have

$$\begin{aligned}\partial_{\zeta\zeta\zeta\zeta}\psi+\frac{2}{\zeta^2}\partial_{\zeta\zeta\theta\theta}\psi+\frac{1}{\zeta^4}\partial_{\theta\theta\theta\theta}\psi\\+\frac{2}{\zeta}\partial_{\zeta\zeta\zeta}\psi-\frac{2}{\zeta^3}\partial_{\zeta\theta\theta}\psi-\frac{1}{\zeta^2}\partial_{\zeta\zeta}\psi+\frac{4}{\zeta^4}\partial_{\theta\theta}\psi+\frac{1}{\zeta^3}\partial_{\zeta}\psi=0.\end{aligned} \tag{A 23}$$

Adding and subtracting convenient terms we have

$$\begin{aligned}\partial_{\zeta\zeta\zeta\zeta}\psi+\frac{1}{\zeta^2}\partial_{\zeta\zeta\theta\theta}\psi+\frac{2}{\zeta}\partial_{\zeta\zeta\zeta}\psi-\frac{3}{\zeta^3}\partial_{\zeta\theta\theta}\psi-\frac{1}{\zeta^2}\partial_{\zeta\zeta}\psi+\frac{4}{\zeta^4}\partial_{\theta\theta}\psi\\+\frac{1}{\zeta^3}\partial_{\zeta}\psi+\underbrace{\frac{1}{\zeta^2}\partial_{\zeta\zeta\theta\theta}\psi+\frac{1}{\zeta^3}\partial_{\zeta\theta\theta}\psi+\frac{1}{\zeta^4}\partial_{\theta\theta\theta\theta}\psi}_{\frac{1}{\zeta^2}\partial_{\theta\theta}\Delta\psi}=0.\end{aligned} \tag{A 24}$$

Re-writing we have

$$\begin{aligned}\partial_{\zeta\zeta\zeta\zeta}\psi+\frac{1}{\zeta^2}\partial_{\zeta\zeta\theta\theta}\psi+\frac{1}{\zeta}\partial_{\zeta\zeta\zeta}\psi-\frac{4}{\zeta^3}\partial_{\zeta\theta\theta}\psi-\frac{2}{\zeta^2}\partial_{\zeta\zeta}\psi+\frac{6}{\zeta^4}\partial_{\theta\theta}\psi+\frac{2}{\zeta^3}\partial_{\zeta}\psi\\+\underbrace{\frac{1}{\zeta}\partial_{\zeta\zeta\zeta}\psi}_{\frac{1}{\zeta}\partial_{\zeta}(\partial_{\zeta\zeta}\psi)}\underbrace{-\frac{1}{\zeta^3}\partial_{\zeta}\psi+\frac{1}{\zeta^2}\partial_{\zeta\zeta}\psi}_{\frac{1}{\zeta}\partial_{\zeta}\left(\frac{1}{\zeta}\partial_{\zeta}\psi\right)}\underbrace{-\frac{2}{\zeta^4}\partial_{\theta\theta}\psi+\frac{1}{\zeta^3}\partial_{\zeta\theta\theta}\psi}_{\frac{1}{\zeta}\partial_{\zeta}\left(\frac{1}{\zeta^2}\partial_{\theta\theta}\psi\right)}+\frac{1}{\zeta^2}\partial_{\theta\theta}\Delta\psi=0.\end{aligned} \tag{A 25}$$

The three under-braced terms are equal to $\frac{1}{\zeta}\partial_{\zeta}\Delta\psi$, therefore we get

$$\begin{aligned}\underbrace{\partial_{\zeta\zeta\zeta\zeta}\psi}_{\partial_{\zeta\zeta}(\partial_{\zeta\zeta}\psi)}+\underbrace{\frac{2}{\zeta^3}\partial_{\zeta}\psi-\frac{1}{\zeta^2}\partial_{\zeta\zeta}\psi-\frac{1}{\zeta^2}\partial_{\zeta\zeta}\psi+\frac{1}{\zeta}\partial_{\zeta\zeta\zeta}\psi}_{\partial_{\zeta}\left(-\frac{1}{\zeta^2}\partial_{\zeta}\psi+\frac{1}{\zeta}\partial_{\zeta\zeta}\psi\right)}\\+\underbrace{\frac{6}{\zeta^4}\partial_{\theta\theta}\psi-\frac{2}{\zeta^3}\partial_{\zeta\theta\theta}\psi-\frac{2}{\zeta^3}\partial_{\zeta\theta\theta}\psi+\frac{1}{\zeta^2}\partial_{\zeta\zeta\theta\theta}\psi}_{\partial_{\zeta}\left(-\frac{2}{\zeta^3}\partial_{\theta\theta}\psi+\frac{1}{\zeta^2}\partial_{\zeta\theta\theta}\psi\right)}+\frac{1}{\zeta}\partial_{\zeta}\Delta\psi+\frac{1}{\zeta^2}\partial_{\theta\theta}\Delta\psi=0.\end{aligned} \tag{A 26}$$

That is

$$\begin{aligned}\partial_{\zeta\zeta}(\partial_{\zeta\zeta}\psi)+\underbrace{\partial_{\zeta}\left(-\frac{1}{\zeta^2}\partial_{\zeta}\psi+\frac{1}{\zeta}\partial_{\zeta\zeta}\psi\right)}_{\partial_{\zeta\zeta}\left(\frac{1}{\zeta}\partial_{\zeta}\psi\right)}\\+\underbrace{\partial_{\zeta}\left(-\frac{2}{\zeta^3}\partial_{\theta\theta}\psi+\frac{1}{\zeta^2}\partial_{\zeta\theta\theta}\psi\right)}_{\partial_{\zeta\zeta}\left(\frac{1}{\zeta^2}\partial_{\theta\theta}\psi\right)}+\frac{1}{\zeta}\partial_{\zeta}\Delta\psi+\frac{1}{\zeta^2}\partial_{\theta\theta}\Delta\psi=0,\end{aligned} \tag{A 27}$$

which reveals that equation (A 23) is just the bi-harmonic equation

$$\Delta^2\psi=0, \tag{A 28}$$

and which, from equations (A 5), (A 6), (A 7) and (A 8), must be subject to the boundary conditions

$$\psi(\zeta,\theta=0)=0, \tag{A 29}$$

$$\partial_{\theta\theta}\psi(\zeta,\theta=0)=0, \tag{A 30}$$

$$\psi(\zeta,\theta=\theta_c)=0, \tag{A 31}$$

and

$$\partial_{\theta\theta}\psi(\zeta,\theta=\theta_c)=0. \tag{A 32}$$

We consider solution candidates to the bi-harmonic equation which are of the form

$$\psi(\zeta,\theta)=\zeta^{\lambda}F(\theta). \tag{A 33}$$

Substituting this into equation (A 23) we have

$$\begin{aligned}\frac{1}{\zeta^4}\partial_{\theta\theta\theta\theta}(\zeta^\lambda F)+\frac{2}{\zeta^2}\partial_{\zeta\zeta\theta\theta}(\zeta^\lambda F)+\partial_{\zeta\zeta\zeta\zeta}(\zeta^\lambda F)-\frac{2}{\zeta^3}\partial_{\zeta\theta\theta}(\zeta^\lambda F)\\ +\frac{2}{\zeta}\partial_{\zeta\zeta\zeta}(\zeta^\lambda F)+\frac{4}{\zeta^4}\partial_{\theta\theta}(\zeta^\lambda F)-\frac{1}{\zeta^2}\partial\zeta\zeta(\zeta^\lambda F)+\frac{1}{\zeta^3}\partial_{\zeta}(\zeta^\lambda F)=0,\end{aligned} \tag{A 34}$$

i.e.

$$\begin{aligned}\zeta^{\lambda-4}F''''+\frac{2}{\zeta^2}F''\lambda(\lambda-1)\zeta^{\lambda-2}+F\lambda(\lambda-1)(\lambda-2)(\lambda-3)\zeta^{\lambda-4}-\frac{2}{\zeta^3}F''\lambda\zeta^{\lambda-1}\\ +\frac{2}{\zeta}F\lambda(\lambda-1)(\lambda-2)\zeta^{\lambda-3}+\frac{4}{\zeta^4}F''\zeta^{\lambda}-\frac{1}{\zeta^2}F\lambda(\lambda-1)\zeta^{\lambda-2}+\frac{1}{\zeta^3}F\lambda\zeta^{\lambda-1}=0,\end{aligned} \tag{A 35}$$

which yields

$$\begin{aligned}\zeta^{\lambda-4}F''''+2\lambda(\lambda-1)\zeta^{\lambda-4}F''+\lambda(\lambda-1)(\lambda-2)(\lambda-3)\zeta^{\lambda-4}F-2\lambda\zeta^{\lambda-4}F''\\ +2\lambda(\lambda-1)(\lambda-2)\zeta^{\lambda-4}F+4\zeta^{\lambda-4}F''-\lambda(\lambda-1)\zeta^{\lambda-4}F+\lambda\zeta^{\lambda-4}F=0.\end{aligned} \tag{A 36}$$

Re-arranging we have

$$\begin{aligned}\zeta^{\lambda-4}\left\{F''''+\left[2\lambda(\lambda-1)-2\lambda+4\right]F''\right.\\ \left.+\lambda\left[(\lambda-1)(\lambda-2)(\lambda-3)+2(\lambda-1)(\lambda-2)-(\lambda-1)+1\right]F\right\}=0,\end{aligned} \tag{A 37}$$

hence

$$F''''+2\left[\lambda(\lambda-1)-\lambda+2\right]F''+\lambda\left[(\lambda-1)(\lambda-2)\left\{(\lambda-3)+2\right\}-\lambda+2\right]F=0, \tag{A 38}$$

i.e.

$$F''''+2\left[\lambda^2-2\lambda+2\right]F''+\lambda\left[(\lambda-1)^2(\lambda-2)-(\lambda-2)\right]F=0, \tag{A 39}$$

which can be re-written as

$$F''''+2\left[\lambda^2-2\lambda+2\right]F''+\lambda(\lambda-2)\underbrace{\left[(\lambda-1)^2-1\right]}_{\lambda^2-2\lambda}F=0, \tag{A 40}$$

i.e.

$$F''''+2\left[\lambda^2-2\lambda+2\right]F''+\lambda^2(\lambda-2)^2F=0. \tag{A 41}$$

Equivalently

$$D^4F+\left[2\lambda^2-4\lambda+4\right]D^2F+\lambda^2(\lambda-2)^2F=0, \tag{A 42}$$

re-arranging e have

$$D^4F+\lambda^2D^2F+(\lambda-2)^2\,D^2F+\lambda^2(\lambda-2)^2F=0, \tag{A 43}$$

i.e.

$$D^2\left(D^2+\lambda^2\right)F+(\lambda-2)^2\left(D^2+\lambda^2\right)F=0. \tag{A 44}$$

Consequently

$$\left(D^2+\lambda^2\right)\left(D^2+(\lambda-2)^2\right)F=0. \tag{A 45}$$

When $\lambda \neq 2$ we have

$$F=A\sin(\lambda\theta)+B\cos(\lambda\theta)+C\sin\left((\lambda-2)\theta\right)+D\cos\left((\lambda-2)\theta\right), \tag{A 46}$$

and for $\lambda=2$ we have

$$F=A\sin(\lambda\theta)+B\cos(\lambda\theta)+C\theta+D. \tag{A 47}$$

Verifying boundary condition (A 29) we have

$$\psi(\zeta,\theta=0)=\zeta^\lambda\left(B+D\right)=0, \tag{A 48}$$

which implies

$$D=-B \tag{A 49}$$

For $\lambda \neq 2$, condition (A 30) yields

$$\partial_{\theta\theta}\psi(\zeta,\theta=0)=-\zeta^\lambda\lambda^2B-\zeta^\lambda\left(\lambda-2\right)^2B=0, \tag{A 50}$$

i.e.

$$-\lambda^2\cancel{(B+D)}-\left(-4\lambda+4\right)B=0, \tag{A 51}$$

which, unless $\lambda=1$ implies $B=D=0$. In the case $\lambda=2$, we simply have

$$\partial_{\theta\theta}\psi(\zeta,\theta=0)=-\zeta^\lambda\lambda^2B=0, \tag{A 52}$$

i.e.

$$B=0. \tag{A 53}$$

In so far, for $\lambda \neq 2$ and $\lambda \neq 1$ we have

$$F=A\sin(\lambda\theta)+C\sin\left((\lambda-2)\theta\right), \tag{A 54}$$

for $\lambda=2$

$$F=A\sin(\lambda\theta)+C\theta, \tag{A 55}$$

and for $\lambda=1$

$$F=A\sin(\lambda\theta)+B\cos(\lambda\theta)+C\sin\left((\lambda-2)\theta\right)-B\cos\left((\lambda-2)\theta\right). \tag{A 56}$$

We now verify condition (A 31) which for $\lambda \neq 2$ and $\lambda \neq 1$ yields

$$\psi(\zeta,\theta_c)=\zeta^\lambda\left(A\sin(\lambda\theta_c)+C\sin\left((\lambda-2)\theta_c\right)\right)=0, \tag{A 57}$$

i.e.

$$A\sin(\lambda\theta_c)+C\sin\left((\lambda-2)\theta_c\right)=0, \tag{A 58}$$

and from (A 32) we have

$$\partial_{\theta\theta}\psi(\zeta,\theta_c)=\zeta^\lambda\left(-A\lambda^2\sin(\lambda\theta_c)-C(\lambda-2)^2\sin\left((\lambda-2)\theta\right)\right)=0, \tag{A 59}$$

i.e.

$$-\lambda^2\underbrace{\left(A\sin(\lambda\theta_c)+C\sin\left((\lambda-2)\theta_c\right)\right)}_{=0}+(-4\lambda+4)C\sin\left((\lambda-2)\theta_c\right)=0, \tag{A 60}$$

so, unless $\lambda=2+k\pi/\theta_c$ for an integer k, we have

$$C=0. \tag{A 61}$$

Assuming $\lambda \neq 2 + k\pi/\theta_c$, we have from (A 58)

$$A \sin(\lambda\theta_c) = 0, \tag{A 62}$$

which implies that either $A = 0$ (i.e. the trivial solution $\check{\psi} = 0$) or

$$\lambda = k\pi/\theta_c, \tag{A 63}$$

with integer k. We'll take $k = 1$. This yields

$$\psi = A\zeta^{\frac{\pi}{\theta_c}} \sin(\pi\frac{\theta}{\theta_c}). \tag{A 64}$$

We highlight that the solution is only determined up to a scaling factor A. For convenience we define

$$\check{\psi} = \zeta^{\frac{\pi}{\theta_c}} \sin(\pi\frac{\theta}{\theta_c}). \tag{A 65}$$

The cases $\lambda = 2$, $\lambda = 1$ and $\lambda = 2 + k\pi/\theta_c$ are to be dealt with separately, but are not of direct interest in our present application.

For simplicity and added generality, the derivation above was done for a wedge contained between $\theta = 0$ and $\theta = \theta_c$; however, the model we are using actually describes the fluid as being between $\theta = \pi - \theta_c + \theta_s$ and $\theta = \pi + \theta_s$, where θ_s is the angle the outward pointing tangent to the liquid-solid interface at the contact line makes with the r axis. Consequently, the actual solution that we are interested in is

$$\psi = \zeta^\lambda \sin\left(\lambda(\theta + \theta_c + \theta_s - \pi)\right), \tag{A 66}$$

i.e.

$$\psi = -\zeta^\lambda \sin\left(\lambda(\pi - \theta_c - \theta_s - \theta)\right), \tag{A 67}$$

We highlight that the horizontal and vertical components of velocity for this stream function are given by

$$\begin{aligned}
\check{u} &= \partial_z\psi \\
&= \partial_\zeta\psi\partial_z\zeta + \partial_\theta\psi\partial_z\theta \\
&= -\left[\lambda\zeta^{\lambda-1}\sin\left(\lambda(\pi-\theta_c-\theta_s-\theta)\right)\right]\partial_z\left(\sqrt{r^2+z^2}\right) \\
&\quad - \left[-\lambda\zeta^\lambda\cos\left(\lambda(\pi-\theta_c-\theta_s-\theta)\right)\right]\partial_z\left(\arctan\left(\frac{z}{r}\right)\right) \\
&= -\left[\lambda\zeta^{\lambda-1}\sin\left(\lambda(\pi-\theta_c-\theta_s-\theta)\right)\right]\left(\frac{z}{\sqrt{r^2+z^2}}\right) \\
&\quad - \left[-\lambda\zeta^\lambda\cos\left(\lambda(\pi-\theta_c-\theta_s-\theta)\right)\right]\left(\frac{r}{r^2+z^2}\right) \\
&= -\lambda z(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
&\quad + \lambda r(r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right),
\end{aligned} \tag{A 68}$$

and

$$
\begin{aligned}
\check{w} &= -\partial_r \psi \\
&= -\partial_\zeta \psi \partial_r \zeta - \partial_\theta \psi \partial_r \theta \\
&= \left[\lambda \zeta^{\lambda-1} \sin\left(\lambda(\pi - \theta_c - \theta_s - \theta)\right)\right] \partial_r \left(\sqrt{r^2 + z^2}\right) \\
&\quad + \left[-\lambda \zeta^{\lambda} \cos\left(\lambda(\pi - \theta_c - \theta_s - \theta)\right)\right] \partial_r \left(\arctan\left(\frac{z}{r}\right)\right) \\
&= \left[\lambda \zeta^{\lambda-1} \sin\left(\lambda(\pi - \theta_c - \theta_s - \theta)\right)\right] \left(\frac{r}{\sqrt{r^2 + z^2}}\right) \\
&\quad + \left[-\lambda \zeta^{\lambda} \cos\left(\lambda(\pi - \theta_c - \theta_s - \theta)\right)\right] \left(-\frac{z}{r^2 + z^2}\right) \\
&= \lambda r (r^2 + z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \left\{\pi - \theta_c - \theta_s - \arctan\left(\frac{z}{r}\right)\right\}\right) \\
&\quad + \lambda z (r^2 + z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \left\{\pi - \theta_c - \theta_s - \arctan\left(\frac{z}{r}\right)\right\}\right);
\end{aligned}
\tag{A 69}
$$

where $\theta_s \leq \arctan(z/r) \leq \pi + \theta_s$.

It is also important to mention that substituting the solution into equations (A 2) and (A 3), we see that the resulting flow has constant pressure everywhere on the domain and it is only determined up to translation by a constant, which we conveniently select to be $\check{p} = 0$.

Appendix B. Eigen-solution velocities and their derivatives

To calculate the derivatives of the velocities of the eigen-solution required above, we recall (see equations (A 68) and (A 69) in Appendix A) that

$$\begin{aligned}\breve{u} = &-\lambda z(r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\ &+\lambda r(r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right),\end{aligned} \tag{B 1}$$

and

$$\begin{aligned}\breve{w} = &\lambda r(r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\ &+\lambda z(r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right),\end{aligned} \tag{B 2}$$

where $\theta_s \le \arctan(z/r) \le \pi+\theta_s$ and

$$\lambda = \pi/\theta_c. \tag{B 3}$$

From (B 1), we have

$$\begin{aligned}\partial_r\breve{u} = &-\lambda z\left[\frac{\lambda-2}{2}\left(r^2+z^2\right)^{\frac{\lambda-4}{2}} 2r\right] \sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\ &-\lambda z(r^2+z^2)^{\frac{\lambda-2}{2}}\left[\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\lambda\frac{1}{1+\frac{z^2}{r^2}}\frac{z}{r^2}\right] \\ &+\lambda\left[(r^2+z^2)^{\frac{\lambda-2}{2}}+r\frac{\lambda-2}{2}(r^2+z^2)^{\frac{\lambda-4}{2}}2r\right]\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\ &+\lambda r(r^2+z^2)^{\frac{\lambda-2}{2}}\left[-\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\lambda\frac{1}{1+\frac{z^2}{r^2}}\frac{z}{r^2}\right],\end{aligned} \tag{B 4}$$

i.e.

$$\boxed{\begin{aligned}\partial_r\breve{u} = &-\lambda\left(\lambda-2\right)rz\left(r^2+z^2\right)^{\frac{\lambda-4}{2}} \sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\ &-\lambda^2 z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\ &+\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\ &+\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\ &-\lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right).\end{aligned}} \tag{B 5}$$

Similarly, we have

$$\begin{aligned}\partial_z \breve{u} = &-\lambda\left[(r^2+z^2)^{\frac{\lambda-2}{2}} + z\frac{\lambda-2}{2}\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}2z\right]\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &-\lambda z(r^2+z^2)^{\frac{\lambda-2}{2}}\left[\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\lambda\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right)\right]\\ &+\lambda r\left[\frac{\lambda-2}{2}(r^2+z^2)^{\frac{\lambda-4}{2}}2z\right]\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &+\lambda r(r^2+z^2)^{\frac{\lambda-2}{2}}\left[-\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\lambda\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right)\right],\end{aligned} \tag{B 6}$$

i.e.

$$\boxed{\begin{aligned}\partial_z \breve{u} = &-\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &-\lambda(\lambda-2)z^2\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &+\lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &+\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &+\lambda^2 r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right).\end{aligned} \quad \text{(B 7)}}$$

From (B 2) we have

$$\begin{aligned}\partial_r \breve{w} = &\lambda\left[(r^2+z^2)^{\frac{\lambda-2}{2}} + r\frac{\lambda-2}{2}(r^2+z^2)^{\frac{\lambda-4}{2}}2r\right]\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &+\lambda r(r^2+z^2)^{\frac{\lambda-2}{2}}\left[\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\lambda\frac{1}{1+\frac{z^2}{r^2}}\frac{z}{r^2}\right]\\ &+\lambda z\left[\frac{\lambda-2}{2}(r^2+z^2)^{\frac{\lambda-4}{2}}2r\right]\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &+\lambda z(r^2+z^2)^{\frac{\lambda-2}{2}}\left[-\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\lambda\frac{1}{1+\frac{z^2}{r^2}}\frac{z}{r^2}\right],\end{aligned} \tag{B 8}$$

i.e.

$$\boxed{\begin{aligned}\partial_r \breve{w} = &\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &+\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &+\lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &+\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\\ &-\lambda^2 z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right),\end{aligned} \quad \text{(B 9)}}$$

Similarly, we have

$$
\begin{aligned}
\partial_z \breve{w} = \partial_z &\left[\lambda r (r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\right] \\
&+ \partial_z \left[\lambda z (r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\right],
\end{aligned}
\tag{B 10}
$$

i.e.

$$
\begin{aligned}
\partial_z \breve{w} = \lambda r \partial_z &\left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right] \sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
&+ \lambda r (r^2+z^2)^{\frac{\lambda-2}{2}} \partial_z \left[\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\right] \\
&+ \lambda \partial_z [z] (r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
&+ \lambda z \partial_z \left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right] \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
&+ \lambda z (r^2+z^2)^{\frac{\lambda-2}{2}} \partial_z \left[\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\right],
\end{aligned}
\tag{B 11}
$$

or, equivalently,

$$
\begin{aligned}
\partial_z \breve{w} = \lambda r &\frac{\lambda-2}{\cancel{2}} (r^2+z^2)^{\frac{\lambda-4}{2}} \cancel{2} z \sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
&+ \lambda r (r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \lambda \frac{1}{1+\frac{r^2}{z^2}} \left(-\frac{1}{r}\right) \\
&+ \lambda \partial_z [z] (r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
&+ \lambda z \partial_z \left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right] \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
&+ \lambda z (r^2+z^2)^{\frac{\lambda-2}{2}} \partial_z \left[\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right)\right],
\end{aligned}
\tag{B 12}
$$

which is

$$
\begin{aligned}
\partial_z \breve{w} = \lambda r z &(\lambda-2) (r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
&+ \lambda r (r^2+z^2)^{\frac{\lambda-2}{2}} \left[\cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \lambda \frac{1}{1+\frac{z^2}{r^2}} \left(-\frac{1}{r}\right)\right] \\
&+ \lambda \left[(r^2+z^2)^{\frac{\lambda-2}{2}} + z \frac{\lambda-2}{2} (r^2+z^2)^{\frac{\lambda-4}{2}} 2z\right] \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
&+ \lambda z (r^2+z^2)^{\frac{\lambda-2}{2}} \left[-\sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \lambda \frac{1}{1+\frac{z^2}{r^2}} \left(-\frac{1}{r}\right)\right],
\end{aligned}
\tag{B 13}
$$

i.e.

$$
\begin{aligned}
\partial_z \breve{w} = {} & \lambda(\lambda-2) r z (r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
& - \lambda^2 r^2 (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
& + \lambda (r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
& + \lambda(\lambda-2) z^2 (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right) \\
& + \lambda^2 r z (r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda\left\{\pi-\theta_c-\theta_s-\arctan\left(\frac{z}{r}\right)\right\}\right).
\end{aligned}
\tag{B 14}
$$

The second order derivatives of the velocity components are as follows. From equation (B 5) we have

$$
\begin{aligned}
\partial_{rr} \breve{u} = \partial_r \Bigg\{ & \lambda(\lambda-2) r z \left(r^2+z^2\right)^{\frac{\lambda-4}{2}} \sin\left(\lambda\left[\pi+\theta_s-\arctan\left(\frac{z}{r}\right)\right]\right) \\
& + \lambda^2 z^2 (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda\left[\pi+\theta_s-\arctan\left(\frac{z}{r}\right)\right]\right) \\
& - \lambda\left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right] \cos\left(\lambda\left[\pi+\theta_s-\arctan\left(\frac{z}{r}\right)\right]\right) \\
& - \lambda(\lambda-2) r^2 (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda\left[\pi+\theta_s-\arctan\left(\frac{z}{r}\right)\right]\right) \\
& + \lambda^2 r z (r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda\left[\pi+\theta_s-\arctan\left(\frac{z}{r}\right)\right]\right) \Bigg\}.
\end{aligned}
\tag{B 15}
$$

$$
\begin{aligned}
\partial_{rr}\breve{u} = {} & \lambda\left(\lambda-2\right)z\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda\left(\lambda-2\right)rz\frac{\lambda-4}{\cancel{2}}\left(r^2+z^2\right)^{\frac{\lambda-6}{2}}\cancel{2}r\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda\left(\lambda-2\right)rz\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}(-z)(-1)\frac{1}{r^2} \\
& + \lambda^2z^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}r\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}(-z)(-1)\frac{1}{r^2} \\
& - \lambda\frac{\lambda-2}{\cancel{2}}\left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right]\cancel{2}r\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda\left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right]\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}(-z)(-1)\frac{1}{r^2} \\
& - \lambda(\lambda-2)2r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)r^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}r\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}(-z)(-1)\frac{1}{r^2} \\
& + \lambda^2z(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2rz\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}r\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}(-z)(-1)\frac{1}{r^2},
\end{aligned}
\tag{B 16}
$$

i.e.

$$
\begin{aligned}
\partial_{rr}\check{u} = {} & \lambda\left(\lambda-2\right)z\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& + \lambda\left(\lambda-2\right)\left(\lambda-4\right)r^2 z\left(r^2+z^2\right)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& + \lambda^2\left(\lambda-2\right)rz^2\left(r^2+z^2\right)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& + \lambda^2(\lambda-4)rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& - \lambda^3 z^3(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& - \lambda(\lambda-2)r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& + \lambda^2 z(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& - 2\lambda(\lambda-2)r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& - \lambda(\lambda-2)(\lambda-4)r^3(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& + \lambda^2(\lambda-2)r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& + \lambda^2 z(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& + \lambda^2(\lambda-4)r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& + \lambda^3 rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right),
\end{aligned}
\tag{B 17}
$$

and

$$
\begin{aligned}
\partial_{rz}\check{u} = \partial_z \Bigg\{ & \lambda\left(\lambda-2\right)rz\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& + \lambda^2 z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) - \lambda\left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right]\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& - \lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& + \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\Bigg\}.
\end{aligned}
\tag{B 18}
$$

$$
\begin{aligned}
\partial_{rz}\breve{u} = {} & \lambda\left(\lambda-2\right)r\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda\left(\lambda-2\right)rz\frac{\lambda-4}{\cancel{2}}\left(r^2+z^2\right)^{\frac{\lambda-6}{2}}\cancel{2}z\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda\left(\lambda-2\right)rz\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& + \lambda^2 2z(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 z^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& - \lambda\frac{\lambda-2}{\cancel{2}}\left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right]\cancel{2}z\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda\left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right]\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& - \lambda(\lambda-2)r^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& + \lambda^2 r(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 rz\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right).
\end{aligned}
\tag{B 19}
$$

i.e.

$$
\begin{aligned}
\partial_{rz}\check{u} &= \lambda(\lambda-2)\,r\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda(\lambda-2)(\lambda-4)rz^2\left(r^2+z^2\right)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2(\lambda-2)\,r^2z\left(r^2+z^2\right)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+2\lambda^2z(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2(\lambda-4)z^3(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^3rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)z(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2r(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)(\lambda-4)r^2z(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2(\lambda-2)r^3(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2r(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2(\lambda-4)rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^3r^2z(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right).
\end{aligned}
\tag{B 20}
$$

Similarly, from (B 7), we have

$$
\begin{aligned}
\partial_{zz}\check{u} = \partial_z\Big\{&\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda(\lambda-2)z^2\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\Big\},
\end{aligned}
\tag{B 21}
$$

i.e.

$$
\begin{aligned}
\partial_{zz}\breve{u} = {} & \lambda\frac{\lambda-2}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-4}{2}}\cancel{2}z\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& + \lambda(\lambda-2)2z\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(\lambda-2)z^2\frac{\lambda-4}{\cancel{2}}\left(r^2+z^2\right)^{\frac{\lambda-6}{2}}\cancel{2}z\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(\lambda-2)z^2\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& - \lambda^2 r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 rz\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& - \lambda(\lambda-2)r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)rz\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& - \lambda^2 r^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right),
\end{aligned}
\tag{B 22}
$$

which yields

$$
\begin{aligned}
\partial_{zz}\check{u} = {} & \lambda(\lambda-2)z(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda^2 r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& +2\lambda(\lambda-2)z\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& +\lambda(\lambda-2)(\lambda-4)z^3\left(r^2+z^2\right)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda^2(\lambda-2)rz^2\left(r^2+z^2\right)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda^2 r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda^2(\lambda-4)rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda^3 r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda(\lambda-2)r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda(\lambda-2)(\lambda-4)rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda^2(\lambda-2)r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda^2(\lambda-4)r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& +\lambda^3 r^3(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right),
\end{aligned}
\tag{B 23}
$$

From equation (B 9), we have

$$
\begin{aligned}
\partial_{rr}\check{w} = \partial_r\Big\{ & -\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& -\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& +\lambda^2 z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\Big\},
\end{aligned}
\tag{B 24}
$$

i.e.

$$
\begin{aligned}
\partial_{rr}\breve{w} = &-\lambda\frac{\lambda-2}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-4}{2}}\cancel{2}r\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\,(-z)\,(-1)\frac{1}{r^2}\\
&-\lambda(\lambda-2)2r(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)r^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}r\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\,(-z)\,(-1)\frac{1}{r^2}\\
&-\lambda^2z(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2rz\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}r\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\,(-z)\,(-1)\frac{1}{r^2}\\
&-\lambda(\lambda-2)z(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)rz\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}r\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\,(-z)\,(-1)\frac{1}{r^2}\\
&+\lambda^2z^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}r\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\,(-z)\,(-1)\frac{1}{r^2},
\end{aligned}
\tag{B 25}
$$

which yields

$$
\begin{aligned}
\partial_{rr}\breve{w} = &-\lambda(\lambda-2)r(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2 z(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-2\lambda(\lambda-2)r(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)(\lambda-4)r^3(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2(\lambda-2)r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2 z(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2(\lambda-4)r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^3 r z^2(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)z(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)(\lambda-4)r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2(\lambda-2)r z^2(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2(\lambda-4)r z^2(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^3 z^3(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right),
\end{aligned}
\tag{B 26}
$$

and

$$
\begin{aligned}
\partial_{rz}\breve{w} = \partial_z\Big\{&-\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2 z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\Big\},
\end{aligned}
\tag{B 27}
$$

i.e.

$$
\begin{aligned}
\partial_{rz}\breve{w} = &-\lambda\frac{\lambda-2}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-4}{2}}\cancel{2}z\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right)\\
&-\lambda(\lambda-2)r^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right)\\
&-\lambda^2r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2rz\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right)\\
&-\lambda(\lambda-2)r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)rz\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right)\\
&+\lambda^2 2z(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2z^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right),
\end{aligned}
\tag{B 28}
$$

which yields

$$
\begin{aligned}
\partial_{rz}\breve{w} = & -\lambda(\lambda-2)z(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)(\lambda-4)r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2(\lambda-2)r^3(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2(\lambda-4)rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^3 r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)r(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)(\lambda-4)rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2(\lambda-2)r^2 z(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + 2\lambda^2 z(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2(\lambda-4)z^3(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^3 rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right),
\end{aligned} \tag{B 29}
$$

Finally, from equation (B 14), we have

$$
\begin{aligned}
\partial_{zz}\breve{w} = \partial_z\Bigg\{ & -\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\Bigg\},
\end{aligned} \tag{B 30}
$$

i.e.

$$
\begin{aligned}
\partial_{zz}\breve{w} = & -\lambda(\lambda-2)r(r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)rz\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& + \lambda^2 r^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& - \lambda\frac{\lambda-2}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-4}{2}}\cancel{2}z \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\lambda \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& - \lambda(\lambda-2)2z(r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)z^2\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(\lambda-2)z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right) \\
& - \lambda^2 r(r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 rz\frac{\lambda-4}{\cancel{2}}(r^2+z^2)^{\frac{\lambda-6}{2}}\cancel{2}z \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\lambda \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\frac{1}{1+\frac{z^2}{r^2}}\left(-\frac{1}{r}\right),
\end{aligned}
\tag{B 31}
$$

which yields

$$
\begin{aligned}
\partial_{zz}\breve{w} = &-\lambda(\lambda-2)r(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)(\lambda-4)rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2(\lambda-2)r^2z(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2(\lambda-4)r^2z(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^3r^3(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)z(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2r(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-2\lambda(\lambda-2)z(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)(\lambda-4)z^3(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2(\lambda-2)rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2r(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2(\lambda-4)rz^2(r^2+z^2)^{\frac{\lambda-6}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^3r^2z(r^2+z^2)^{\frac{\lambda-6}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right).
\end{aligned}
\tag{B 32}
$$

Furthermore, we will also need to calculate derivatives with respect to λ of the velocities and velocity gradients of the eigen-solution. We recall equation B 1, which states that

$$
\breve{u} = \lambda z(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) - \lambda r(r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right), \tag{B 33}
$$

from where we have

$$
\begin{aligned}
&\partial_\lambda\breve{u}\\
&= \partial_\lambda\left[\lambda z(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right] - \partial_\lambda\left[\lambda r(r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right]
\end{aligned}
\tag{B 34}
$$

i.e.

$$
\begin{aligned}
\partial_\lambda\breve{u} = &\, z(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda z\partial_\lambda\left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right]\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\partial_\lambda\lambda z(r^2+z^2)^{\frac{\lambda-2}{2}}\partial_\lambda\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-r(r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda r\partial_\lambda\left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right]\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda r(r^2+z^2)^{\frac{\lambda-2}{2}}\partial_\lambda\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right),
\end{aligned}
\tag{B 35}
$$

or, equivalently,

$$\begin{aligned}\partial_\lambda \breve{u} &= z(r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &+ \lambda z(r^2+z^2)^{\frac{\lambda-2}{2}} \ln\left(r^2+z^2\right)\frac{1}{2} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &+ \lambda z(r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \arctan\left(-\frac{z}{r}\right) \\ &- r(r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda r(r^2+z^2)^{\frac{\lambda-2}{2}} \ln\left(r^2+z^2\right)\frac{1}{2} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &+ \lambda r(r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \arctan\left(-\frac{z}{r}\right).\end{aligned} \tag{B 36}$$

Similarly, we recall equation B 2, which states that

$$\breve{w} = -\lambda r(r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) - \lambda z(r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right), \tag{B 37}$$

from where we have

$$\begin{aligned}\partial_\lambda \breve{w} &= -\partial_\lambda \left[\lambda r(r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\ &- \partial_\lambda \left[\lambda z(r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right],\end{aligned} \tag{B 38}$$

i.e.

$$\begin{aligned}\partial_\lambda \breve{w} &= -\partial_\lambda \left[\lambda r\right] (r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda r \partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right] \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda r(r^2+z^2)^{\frac{\lambda-2}{2}} \partial_\lambda \left[\sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\ &- \partial_\lambda \left[\lambda z\right] (r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda z \partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right] \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda z(r^2+z^2)^{\frac{\lambda-2}{2}} \partial_\lambda \left[\cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right],\end{aligned} \tag{B 39}$$

or, equivalently,

$$\begin{aligned}\partial_\lambda \breve{w} &= -r(r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda r(r^2+z^2)^{\frac{\lambda-2}{2}} \ln\left(r^2+z^2\right)\frac{1}{2} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda r(r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \arctan\left(-\frac{z}{r}\right) \\ &- z(r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda z(r^2+z^2)^{\frac{\lambda-2}{2}} \frac{\ln\left(r^2+z^2\right)}{2} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &+ \lambda z(r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \arctan\left(-\frac{z}{r}\right).\end{aligned} \tag{B 40}$$

We also recall equation B 5

$$
\begin{aligned}
\partial_r \breve{u} = {} & \lambda\left(\lambda-2\right) rz\left(r^2+z^2\right)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 z^2 (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) - \lambda (r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2) r^2 (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 rz (r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right),
\end{aligned}
\tag{B 41}
$$

from where we have

$$
\begin{aligned}
\partial_{r\lambda} \breve{u} = {} & \partial_\lambda \left[\lambda\left(\lambda-2\right) rz\left(r^2+z^2\right)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\
& + \partial_\lambda \left[\lambda^2 z^2 (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\
& - \partial_\lambda \left[\lambda (r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\
& - \partial_\lambda \left[\lambda(\lambda-2) r^2 (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\
& + \partial_\lambda \left[\lambda^2 rz (r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right],
\end{aligned}
\tag{B 42}
$$

i.e.

$$
\begin{aligned}
\partial_{r\lambda} \breve{u} = {} & \partial_\lambda \left[\lambda\left(\lambda-2\right) rz\right] \left(r^2+z^2\right)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda\left(\lambda-2\right) rz \partial_\lambda \left[\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\right] \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda\left(\lambda-2\right) rz\left(r^2+z^2\right)^{\frac{\lambda-4}{2}} \partial_\lambda \left[\sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\
& + \partial_\lambda \left[\lambda^2 z^2\right] (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 z^2 \partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right] \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 z^2 (r^2+z^2)^{\frac{\lambda-4}{2}} \partial_\lambda \left[\cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\
& - \partial_\lambda \left[\lambda\right] (r^2+z^2)^{\frac{\lambda-2}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda \partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right] \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda (r^2+z^2)^{\frac{\lambda-2}{2}} \partial_\lambda \left[\cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\
& - \partial_\lambda \left[\lambda(\lambda-2) r^2\right] (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2) r^2 \partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right] \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2) r^2 (r^2+z^2)^{\frac{\lambda-4}{2}} \partial_\lambda \left[\cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\
& + \partial_\lambda \left[\lambda^2 rz\right] (r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 rz \partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right] \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 rz (r^2+z^2)^{\frac{\lambda-4}{2}} \partial_\lambda \left[\sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right].
\end{aligned}
\tag{B 43}
$$

or, equivalently,

$$
\begin{aligned}
\partial_{r\lambda}\breve{u} = {} & 2\,(\lambda-1)\,rz\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& +\lambda\,(\lambda-2)\,rz\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& +\lambda\,(\lambda-2)\,rz\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right)\\
& +2\lambda z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& +\lambda^2z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& -\lambda^2z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right)\\
& -\left(r^2+z^2\right)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& -\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& +\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right)\\
& -2(\lambda-1)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& -\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& +\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right)\\
& +2\lambda rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& +\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& +\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right),
\end{aligned}
\tag{B 44}
$$

We also recall equation B 7

$$
\begin{aligned}
\partial_z\breve{u} = {} & \lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)+\lambda(\lambda-2)z^2\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& -\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& -\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
& -\lambda^2r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right),
\end{aligned}
\tag{B 45}
$$

from where we have

$$
\begin{aligned}
\partial_{z\lambda}\check{u} = {} & \partial_\lambda\left[\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right] \\
& + \partial_\lambda\left[\lambda(\lambda-2)z^2\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right] \\
& - \partial_\lambda\left[\lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right] \\
& - \partial_\lambda\left[\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right] \\
& - \partial_\lambda\left[\lambda^2 r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right],
\end{aligned}
\tag{B 46}
$$

i.e.

$$
\begin{aligned}
\partial_{z\lambda}\check{u} = {} & (r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right) \\
& + 2(\lambda-1)z^2\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(\lambda-2)z^2\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(\lambda-2)z^2\left(r^2+z^2\right)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right) \\
& - 2\lambda rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right) \\
& - 2(\lambda-1)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right) \\
& - 2\lambda r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right),
\end{aligned}
\tag{B 47}
$$

We also recall equation B 9

$$\partial_r \breve{w} = -\lambda(r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) - \lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)$$
$$- \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)$$
$$- \lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)$$
$$+ \lambda^2 z^2(r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right), \tag{B 48}$$

from where we have

$$\begin{aligned} \partial_{r\lambda} \breve{w} = &-\partial_\lambda \left[\lambda(r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\ &- \partial_\lambda \left[\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\ &- \partial_\lambda \left[\lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\ &- \partial_\lambda \left[\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\ &+ \partial_\lambda \left[\lambda^2 z^2(r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right], \end{aligned} \tag{B 49}$$

i.e.

$$\begin{aligned} \partial_{r\lambda} \breve{w} = &-\partial_\lambda [\lambda] (r^2+z^2)^{\frac{\lambda-2}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda\partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right] \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda(r^2+z^2)^{\frac{\lambda-2}{2}} \partial_\lambda \left[\sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\ &- \partial_\lambda \left[\lambda(\lambda-2)r^2\right] (r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda(\lambda-2)r^2\partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right] \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}} \partial_\lambda \left[\sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\ &- \partial_\lambda \left[\lambda^2 rz\right] (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda^2 rz\partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right] \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}} \partial_\lambda \left[\cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\ &- \partial_\lambda \left[\lambda(\lambda-2)rz\right] (r^2+z^2)^{\frac{\lambda-4}{2}} \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda(\lambda-2)rz\partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right] \cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &- \lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}} \partial_\lambda \left[\cos\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right] \\ &+ \partial_\lambda \left[\lambda^2 z^2\right] (r^2+z^2)^{\frac{\lambda-4}{2}} \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &+ \lambda^2 z^2\partial_\lambda \left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right] \sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right) \\ &+ \lambda^2 z^2(r^2+z^2)^{\frac{\lambda-4}{2}} \partial_\lambda \left[\sin\left(\lambda \arctan\left(-\frac{z}{r}\right)\right)\right], \end{aligned} \tag{B 50}$$

or, equivalently,

$$
\begin{aligned}
\partial_{r\lambda}\breve{w} = &-(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right)\\
&-2(\lambda-1)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right)\\
&-2\lambda rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right)\\
&-2(\lambda-1)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right)\\
&+2\lambda z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right),
\end{aligned}
\tag{B 51}
$$

We also recall equation B 14

$$
\begin{aligned}
\partial_z\breve{w} = &-\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right),
\end{aligned}
\tag{B 52}
$$

from where we have

$$
\begin{aligned}
\partial_{z\lambda}\breve{w} = &-\partial_\lambda\left[\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right]\\
&+\partial_\lambda\left[\lambda^2r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right]\\
&-\partial_\lambda\left[\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right]\\
&-\partial_\lambda\left[\lambda(\lambda-2)z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right]\\
&-\partial_\lambda\left[\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right],
\end{aligned}
\tag{B 53}
$$

i.e.

$$
\begin{aligned}
\partial_{z\lambda}\breve{w} = &-\partial_\lambda\left[\lambda(\lambda-2)rz\right](r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)rz\partial_\lambda\left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right]\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\partial_\lambda\left[\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right]\\
&+\partial_\lambda\left[\lambda^2r^2\right](r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2r^2\partial_\lambda\left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right]\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&+\lambda^2r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\partial_\lambda\left[\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right]\\
&-\partial_\lambda\left[\lambda\right](r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda\partial_\lambda\left[(r^2+z^2)^{\frac{\lambda-2}{2}}\right]\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\partial_\lambda\left[\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right]\\
&-\partial_\lambda\left[\lambda(\lambda-2)z^2\right](r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)z^2\partial_\lambda\left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right]\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda(\lambda-2)z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\partial_\lambda\left[\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right]\\
&-\partial_\lambda\left[\lambda^2rz\right](r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2rz\partial_\lambda\left[(r^2+z^2)^{\frac{\lambda-4}{2}}\right]\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\\
&-\lambda^2rz(r^2+z^2)^{\frac{\lambda-4}{2}}\partial_\lambda\left[\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\right],
\end{aligned}
\tag{B 54}
$$

or, equivalently,

$$
\begin{aligned}
\partial_{z\lambda}\breve{w} = {}& -2(\lambda-1)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right) \\
& + 2\lambda r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda^2 r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 r^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right) \\
& - (r^2+z^2)^{\frac{\lambda-2}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(r^2+z^2)^{\frac{\lambda-2}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right) \\
& - 2(\lambda-1)z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda(\lambda-2)z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& + \lambda(\lambda-2)z^2(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right) \\
& - 2\lambda rz(r^2+z^2)^{\frac{\lambda-4}{2}}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\frac{\ln\left(r^2+z^2\right)}{2}\sin\left(\lambda\arctan\left(-\frac{z}{r}\right)\right) \\
& - \lambda^2 rz(r^2+z^2)^{\frac{\lambda-4}{2}}\cos\left(\lambda\arctan\left(-\frac{z}{r}\right)\right)\arctan\left(-\frac{z}{r}\right).
\end{aligned} \tag{B 55}
$$

Here we highlight that the origin is assumed to be at the contact line, and therefore translation (and possibly a rotation) of the original (r, z) coordinates is likely to be needed before replacing the independent variables in the expressions above. Moreover, when calculating derivatives with respect to the r and z variables as given everywhere else in the text, extra terms are needed, which arise from the translation (and possibly rotation of the coordinates assumed above).

Appendix C. Asymptotic solution in a wedge with Navier slip on one side and no tangential stress on the other

We follow Sprittles & Shikhmurzaev (2011 *a*) and consider the flow of an incompressible Newtonian fluid with uniform density in a wedge-shaped region. We use polar coordinates (r, θ), with the origin on the contact line and the fluid occupying the region given by $0 \le \theta \le \theta_c$. The radial and azimuthal velocity components are respectively given by v_ζ and v. The law of conservation of mass is given by

$$\partial_r(r v_\zeta) + \partial_\theta v = 0, \tag{C 1}$$

radial conservation of momentum is given by

$$\partial_r p = \Delta v_\zeta - \frac{v_\zeta}{r} - \frac{2}{r^2}\partial_\theta v \tag{C 2}$$

and azimuthal conservation of momentum is given by

$$\frac{1}{r}\partial_\theta p = \Delta v - \frac{v}{r^2} + \frac{2}{r^2}\partial_\theta v_\zeta, \tag{C 3}$$

where

$$\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{2}{r^2}\partial_{\theta\theta}. \tag{C 4}$$

At $\theta = 0$, the flow satisfies the impermeability condition

$$v(r, \theta = 0) = 0, \tag{C 5}$$

the Navier slip condition

$$\partial_\theta v_\zeta = rBe\,(v_\zeta - 1). \tag{C 6}$$

At $\theta = \theta_c$ we impose the kinematic boundary condition

$$v = 0, \tag{C 7}$$

and

$$\partial_\theta v_\zeta = 0. \tag{C 8}$$

bi-harmonic equation

$$\Delta^2 \psi = 0, \tag{C 9}$$

and which, from equations (A 5), (A 6), (A 7) and (A 8), must be subject to the boundary conditions

$$\psi(r, \theta = 0) = 0, \tag{C 10}$$

$$\partial_{\theta\theta}\psi(r, \theta = 0) = rBe\,\left(\partial_\theta \psi(r, \theta = 0) - r\right), \tag{C 11}$$

$$\psi(r, \theta = \theta_c) = 0, \tag{C 12}$$

and

$$\partial_{\theta\theta}\psi(r, \theta = \theta_c) = 0. \tag{C 13}$$

We consider solution candidates to the bi-harmonic equation which are of the form

$$\psi(r, \theta) = r^\lambda F(\theta). \tag{C 14}$$

Substituting this into equation (A 23) we have

$$\begin{aligned}&\frac{1}{r^4}\partial_{\theta\theta\theta\theta}(r^\lambda F) + \frac{2}{r^2}\partial_{rr\theta\theta}(r^\lambda F) + \partial_{rrrr}(r^\lambda F) - \frac{2}{r^3}\partial_{r\theta\theta}(r^\lambda F)\\ &\quad + \frac{2}{r}\partial_{rrr}(r^\lambda F) + \frac{4}{r^4}\partial_{\theta\theta}(r^\lambda F) - \frac{1}{r^2}\partial rr(r^\lambda F) + \frac{1}{r^3}\partial_r(r^\lambda F) = 0,\end{aligned} \tag{C 15}$$

i.e.

$$
\begin{aligned}
&r^{\lambda-4}F'''' + \frac{2}{r^2}F''\lambda(\lambda-1)r^{\lambda-2} + F\lambda(\lambda-1)(\lambda-2)(\lambda-3)r^{\lambda-4} - \frac{2}{r^3}F''\lambda r^{\lambda-1} \\
&\quad + \frac{2}{r}F\lambda(\lambda-1)(\lambda-2)r^{\lambda-3} + \frac{4}{r^4}F''r^{\lambda} - \frac{1}{r^2}F\lambda(\lambda-1)r^{\lambda-2} + \frac{1}{r^3}F\lambda r^{\lambda-1} = 0,
\end{aligned} \tag{C 16}
$$

which yields

$$
\begin{aligned}
&r^{\lambda-4}F'''' + 2\lambda(\lambda-1)r^{\lambda-4}F'' + \lambda(\lambda-1)(\lambda-2)(\lambda-3)r^{\lambda-4}F - 2\lambda r^{\lambda-4}F'' \\
&\quad + 2\lambda(\lambda-1)(\lambda-2)r^{\lambda-4}F + 4r^{\lambda-4}F'' - \lambda(\lambda-1)r^{\lambda-4}F + \lambda r^{\lambda-4}F = 0.
\end{aligned} \tag{C 17}
$$

Re-arranging we have

$$
\begin{aligned}
&r^{\lambda-4}\left\{F'''' + \left[2\lambda(\lambda-1) - 2\lambda + 4\right]F''\right. \\
&\quad \left. + \lambda\left[(\lambda-1)(\lambda-2)(\lambda-3) + 2(\lambda-1)(\lambda-2) - (\lambda-1) + 1\right]F\right\} = 0,
\end{aligned} \tag{C 18}
$$

hence

$$
F'''' + 2\left[\lambda(\lambda-1) - \lambda + 2\right]F'' + \lambda\left[(\lambda-1)(\lambda-2)\left\{(\lambda-3)+2\right\} - \lambda + 2\right]F = 0, \tag{C 19}
$$

i.e.

$$
F'''' + 2\left[\lambda^2 - 2\lambda + 2\right]F'' + \lambda\left[(\lambda-1)^2(\lambda-2) - (\lambda-2)\right]F = 0, \tag{C 20}
$$

which can be re-written as

$$
F'''' + 2\left[\lambda^2 - 2\lambda + 2\right]F'' + \lambda(\lambda-2)\underbrace{\left[(\lambda-1)^2 - 1\right]}_{\lambda^2-2\lambda}F = 0, \tag{C 21}
$$

i.e.

$$
F'''' + 2\left[\lambda^2 - 2\lambda + 2\right]F'' + \lambda^2(\lambda-2)^2F = 0. \tag{C 22}
$$

REFERENCES

Shikhmurzaev, Yu D 1993 The moving contact line on a smooth solid surface. *International Journal of Multiphase Flow* **19** (4), 589–610.

Shikhmurzaev, Yulii D 1997 Moving contact lines in liquid/liquid/solid systems. *Journal of Fluid Mechanics* **334**, 211–249.

Shikhmurzaev, Yulii D 2007 *Capillary flows with forming interfaces*. CRC Press.

Shikhmurzaev, Yulii D 2020 Moving contact lines and dynamic contact angles: a 'litmus test'for mathematical models, accomplishments and new challenges. *The European Physical Journal Special Topics* **229** (10), 1945–1977.

Sprittles, JE & Shikhmurzaev, YD 2012*a* The dynamics of liquid drops and their interaction with solids of varying wettabilities. *Physics of Fluids* **24** (8), 082001.

Sprittles, James E & Shikhmurzaev, YD 2013 Finite element simulation of dynamic wetting flows as an interface formation process. *Journal of Computational Physics* **233**, 34–65.

Sprittles, James E & Shikhmurzaev, Yulii D 2011*a* Viscous flow in domains with corners: Numerical artifacts, their origin and removal. *Computer methods in applied mechanics and engineering* **200** (9-12), 1087–1099.

Sprittles, James E & Shikhmurzaev, Yulii D 2011*b* Viscous flows in corner regions: Singularities and hidden eigensolutions. *International Journal for Numerical Methods in Fluids* **65** (4), 372–382.

Sprittles, James E & Shikhmurzaev, Yulii D 2012*b* Finite element framework for describing dynamic wetting phenomena. *International Journal for Numerical Methods in Fluids* **68** (10), 1257–1298.

Zhang, Linbo, Cui, Tao & Liu, Hui 2009 A set of symmetric quadrature rules on triangles and tetrahedra. *Journal of Computational Mathematics* pp. 89–96.