Splines

Since we do not have control of what happens in between points, use piecewise polynomials

Local Control: being able to control individual sections of a curve.

Slow in Slow out: starts slowly, gets faster, stops slowly

oooo o o o o o ooo.

$$Q_i = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$

→ Points are not fixed.

Catmull-Rom ↗

B-splines have local control

- developed by using polynomials basis with desired properties
- No explicit formula for the basis functions

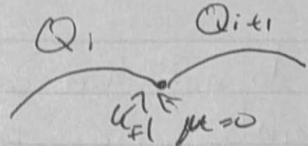
$$Q_i(u) = \sum_{k=0}^{3} P_{i+k} B_k(u) \qquad K = \# \text{ of blending functions}$$

Q_i = single segment of (cubic) spline defined over the interval $0 \le u \le 1$

$B_k(u)$ ... basis functions $(k = 0, 1, 2, 3)$

$P_i, P_{i+1}, P_{i+2}, P_{i+3}$ – control vertices, not fixed.

For C_0 continuity



$$Q_i(1) = Q_{i+1}(0)$$

$$\sum_{k=0}^{3} P_{i+k} B_k(1) = \sum_{k=0}^{3} P_{i+k+1} B_k(0)$$

for $i=0$

$$P_0 B_0(1) + P_1 B_1(1) + P_2 B_2(1) + P_3 B_3(1)$$
$$= P_1 B_0(0) + P_2 B_1(0) + P_3 B_2(0) + P_4 B_3(0)$$

$P_0 B_0(1)$ does not match points on RHS $\therefore = 0$
$P_4 B_3(0)$ does not match points on LHS $\therefore = 0$

Since the control vertices can take arbitrary values, this equation is satisfied by balancing its coefficients.

		C_1	C_2
	$B_0(1) = 0$	$B_0'(1) = 0$	$B_0''(1) = 0$
C_0	$B_1(1) = B_0(0)$	$B_1'(1) = B_0'(0)$	$B_1''(1) = B_0''(0)$
	$B_2(1) = B_1(0)$	$B_2'(1) = B_1'(0)$	$B_2''(1) = B_1''(0)$
	$B_3(1) = B_2(0)$	$B_3'(1) = B_2'(0)$	$B_3''(1) = B_2''(0)$
	$\cancel{B_4(1)} = 0 = B_3(0)$	$0 = B_3(0)$	$0 = B_3''(0)$

15 equations & 16 unknowns

$$B_0(\mu) = a_0\mu^3 + b_0\mu^2 + c_0\mu + d_0$$
$$B_1(\mu) = a_1\mu^3 + b_1\mu^2 + c_1\mu + d_1$$
$$B_2(\mu) = a_2\mu^3 + b_2\mu^2 + c_2\mu + d_2$$
$$B_3(\mu) = a_3\mu^3 + b_3\mu^2 + c_3\mu + d_3$$

16^{th} equation → bas functions must sum to 1
Convex hull property
$$B_0(\mu) + B_1(\mu) + B_2(\mu) + B_3(\mu) = 1$$

Splines

We solve the system to get the basis functions

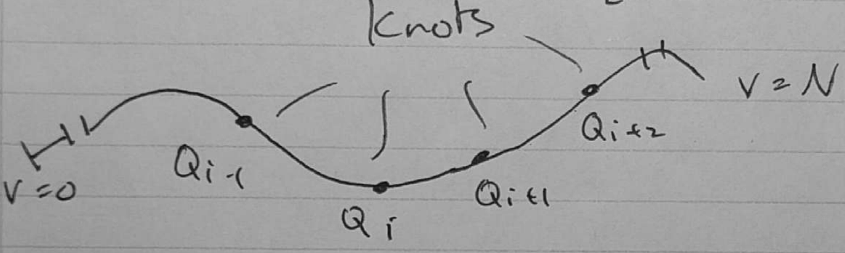
$B_0(u) = \frac{1}{6}(1-u)^3 \quad B_1(u) = \frac{1}{6}(3u^3 - 6u^2 + 4)$

$B_2(u) = \frac{1}{6}(-3u^3 + 3u^2 + 3u + 1)$

$B_3(u) = \frac{1}{6}u^3$

which comes from

$$Q_i(u) = [u^3 \; u^2 \; u \; 1] \, \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_i \\ P_{i+1} \\ P_{i+2} \\ P_{i+3} \end{bmatrix}$$



if knots are equally spaced (arc length) → uniform B-spline

otherwise we have non-uniform b-spline.

Two different normals due to two different vectors