Orthogonality identity derivation

The identity is:

$$\sum_{j=0}^{N-1} W^{jk} W^{-jl} = \sum_{j=0}^{N-1} W^{j(k-l)} = N\delta_{k,l}.$$

Derivation:

Two cases: ① $k=l$ and ② $k\neq l$. (Assume $k,l \in [0, N-1]$, for now.)

① For $k=l$, $\sum_{j=0}^{N-1} W^{j(0)} = \sum_{j=0}^{N-1} 1 = N$

② For $k\neq l$, we'll use the following fact

$$x^N - 1 = (x-1)\underbrace{(x^{N-1} + x^{N-2} + \dots + x + 1)}_{\sum_{j=0}^{N-1} x^j}$$

$$\therefore \sum_{j=0}^{N-1} x^j = \frac{x^N - 1}{x-1}.$$ (Just a geometric series.)

Applying this to our problem...

$$\sum_{j=0}^{N-1} W^{j(k-l)} = \sum_{j=0}^{N-1} \left(W^{k-l}\right)^j = \frac{\left(W^{k-l}\right)^N - 1}{W^{k-l} - 1}.$$

Since $k\neq l$, what is ~~then~~ the numerator? Zero!

$(W^{k-l})^N = (W^N)^{k-l} = 1$ since W is an N^{th} root of unity, $W^N = 1$.

$$\therefore \sum_{j=0}^{N-1} W^{j(k-l)} = \frac{1-1}{W^{k-l}-1} = 0$$ for $k\neq l$. That's it!

Determining F_k coefficients

Similar to earlier continuous case, use orthogonality to find the Fourier coefficients.

We have
$$f_n = \sum_{j=0}^{N-1} F_j W^{nj}$$

Multiply by W^{-nk} and sum over $[0, N-1]$.

$$\sum_{n=0}^{N-1} (f_n) W^{-nk} = \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} F_j W^{nj} W^{-nk}$$ Combine exponents...

$$= \sum_{j=0}^{N-1} F_j \sum_{n=0}^{N-1} W^{n(j-k)}$$ Apply orthogonality identity...

$$= \sum_{j=0}^{N-1} F_j N \delta_{j,k}$$ All but k^{th} iteration give 0, so...

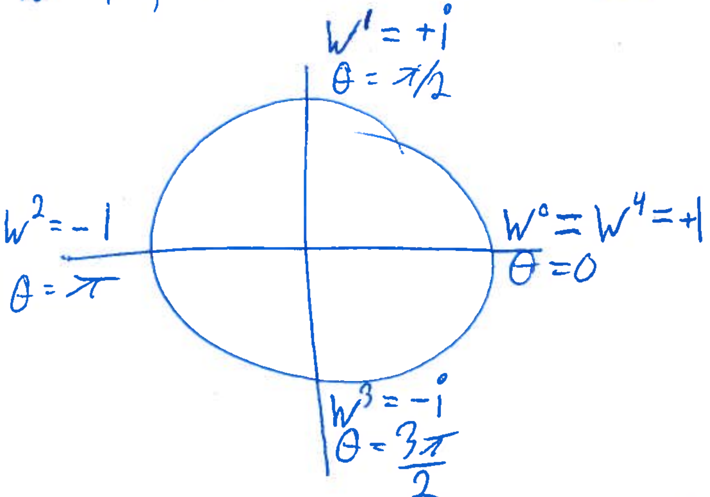
$$= F_k N$$

$$\therefore F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n W^{-nk}$$

Given data f_n, we can now find all F_k, for $k \in [0, N-1]$.

$F = [-2, 2+i, -2, 2-i]$. Find f_n for $n = 0$ to 3.

$N = 4$, so $W = e^{\frac{2\pi i}{N}} = e^{\frac{2\pi i}{4}} = e^{\frac{\pi i}{2}}$.



$$f_n = \sum_{k=0}^{N-1} F_k W^{nk}$$

$n = 0$: $f_0 = \sum_{k=0}^{N-1} F_k W^0 = \sum_{k=0}^{N-1} F_k = -2 + 2 + i - 2 + 2 - i$

$f_0 = 0$

$n = 1$: $f_1 = \sum_{k=0}^{N-1} F_k W^k = (-2)\cdot W^0 + (2+i)W^1 + (-2)W^2 + (2-i)W^3$

$= (-2)\cdot 1 + (2+i)\cdot i + (-2)(-1) + (2-i)(-i)$

$= -2 + 2i - 1 + 2 - 2i - 1$

$= -2$

$n = 2$: $f_2 = \sum_{k=0} F_k W^{2k} = (-2)\cdot W^0 + (2+i)W^2 + (-2)W^4 + (2-i)W^6$

$= (-2)\cdot 1 + (2+i)(-1) + (-2)\cdot 1 + (2-i)(-1)$

$= -2 - 2 - i - 2 - 2 + i$

$= -8$

$n = 3$: $f_3 = 2$. (Try it.)