Determining coefficient a_0

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt)$$

Integrate over $[0, 2\pi]$:

$$\int_0^{2\pi} f(t)\,dt = a_0 \int_0^{2\pi} dt + \underbrace{\cancel{\sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos(kt)\,dt}^{\,0} + \cancel{\sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin(kt)\,dt}^{\,0}}_{\text{cancelled by orthogonality identities}}$$

Therefore

$$a_0 = \frac{\int_0^{2\pi} f(t)\,dt}{\int_0^{2\pi} dt} = \frac{1}{2\pi} \int_0^{2\pi} f(t)\,dt$$

i.e. a_0 is the average value of f over $[0, 2\pi]$.

Determining Coefficient a_ℓ (for integer $\ell > 0$)

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt)$$

Multiply by $\cos(\ell t)$ and integrate over $[0, 2\pi]$

$$\int_0^{2\pi} f(t)\cos(\ell t)\,dt = \underbrace{\int_0^{2\pi} a_0 \cos(\ell t)\,dt}_{0 \text{ by orthogonality}} + \sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos(kt)\cos(\ell t)\,dt$$

$$+ \underbrace{\cancel{\sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin(kt)\cos(\ell t)\,dt}}_{0 \text{ by orthogonality}}$$

Use one more orthogonality identity

$$\int_0^{2\pi} \cos(kt)\cos(k't)\,dt = \begin{cases} 0 & \text{if } k \neq k' \\ \pi & \text{if } k = k' \end{cases}$$

$\therefore$ Only the $k=\ell$ case of the summation survives. Rest are 0.

Hence $$\int_0^{2\pi} f(t)\cos(\ell t)\,dt = a_\ell \int_0^{2\pi} \cos^2(\ell t)\,dt = a_\ell \cdot \pi$$

$$\therefore a_\ell = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(\ell t)\,dt$$