Deriving (Inverse) Discrete Fourier Transform (notation)

Our approximation is

$$f(t) = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k e^{\left(\frac{2\pi i k t}{T}\right)}.$$

For the n^{th} point, $t_n = \frac{nT}{N}$. Setting $t = t_n$, then

$$f_n = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k e^{\left(\frac{2\pi i k t_n}{T}\right)} = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k e^{\left[\frac{2\pi i k}{\cancel{T}}\left(\frac{n\cancel{T}}{N}\right)\right]} = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} c_k e^{\left(\frac{i 2\pi n k}{N}\right)}.$$

For convenience later, let's manipulate this to a sum over $0, \ldots, N-1$.

Split the sum in 2 parts:

$$f_n = \sum_{k=0}^{\frac{N}{2}} c_k e^{\left(\frac{i 2\pi n k}{N}\right)} + \sum_{k=-\frac{N}{2}+1}^{-1} c_k e^{\left(\frac{i 2\pi n k}{N}\right)}$$

Apply a change of variables in the 2nd term, defining $j = N + k$:

$$\sum_{k=-\frac{N}{2}+1}^{-1} c_k e^{\left(\frac{i 2\pi n k}{N}\right)} = \sum_{j=\frac{N}{2}+1}^{N-1} c_{j-N} e^{\left[\frac{i 2\pi n (j-N)}{N}\right]} = \sum_{j=\frac{N}{2}+1}^{N-1} c_{j-N} e^{\frac{i 2\pi n j}{N}} \cdot \underbrace{\cancel{e^{\frac{-i 2\pi n N}{N}}}}_{1}.$$

The previous step used

$$e^{\frac{-i 2\pi n \cancel{N}}{\cancel{N}}} = e^{-i 2\pi n} = \cos(2\pi n) - i \underbrace{\cancel{\sin(2\pi n)}}_{0} = 1$$

We will define our C_j coefficients to be periodic $C_{j\pm N} = C_j$. Then, we have

$$\sum_{j=\frac{N}{2}+1}^{N-1} C_{j-N} e^{\left(\frac{i2\pi nj}{N}\right)} = \sum_{j=\frac{N}{2}+1}^{N-1} C_j e^{\left(\frac{i2\pi nj}{N}\right)}$$

Finally, plugging into our expression for f_n yields

$$f_n = \sum_{k=0}^{\frac{N}{2}} C_k e^{\left(\frac{i2\pi nk}{N}\right)} + \sum_{j=\frac{N}{2}+1}^{N-1} C_j e^{\left(\frac{i2\pi nj}{N}\right)}$$

$$= \sum_{k=0}^{N-1} C_k e^{\left(\frac{i2\pi nk}{N}\right)}$$ by combining summations.

These C_k are the N discrete Fourier transform coefficients, for the N input data points f_n. We rename the C_k as F_k.

$$f_n = \sum_{k=0}^{N-1} F_k e^{\left(\frac{i2\pi nk}{N}\right)} = \sum_{k=0}^{N-1} F_k W^{nk}$$

where we also defined $W = e^{\left(\frac{i2\pi}{N}\right)}$ for brevity.