TITLE

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End of last lecture (which I missed)

For $y'' + py' + qy = 0$ (p,q ϵR)
e^{mx} is a solution if $m = \frac{-p\pm\sqrt{p^2-4q}}{2}$
(solution to $m^2 + pm + q = 0$)
Case 1: $p^2 > 4q$ (2 real roots)

The two functions $e^{m_1 x}$ and $e^{m_2 x}$ using the value of m's from above ($\pm$)
This is the characteristics equation of the DE are both solutions. They are linearly independent, and so the general solution is:

Example: Solve the NP $y'' - y' - 2y = 0$, $y(0) = 0$, $y'(0) = 2$
Solution: The char. eqn is $m^2 - m - 2 = 0$ ie $(m-2)(m+1) = 0$, so $m = 2, -1$

The general solution is $y = C_1 e^{2x} + C_2 e^{-x}$ $(y' = 2C_1 e^{2x} - C_2 e^{-x})$
Enforcing hte initial conditions,
$y(0) = 0 \to 0 = C_1 + C_2$
$y'(0) = 2 \to 2 = 2C_1 - C_2$
$2 = 3C_1$ so $C_1 = \frac{2}{3}$, $C_2 = \frac{-2}{3}$
So $y = \frac{2}{3}(e^{2x} - 2^{-x})$

Case 2: $p^2 < 4q$ (Complex Roots)
Here we can again say that the soution is
$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$
WE can write this in terms of real-valued functions.

$$y = e^{\alpha x}[C_1 e^{i\beta x} + C_2 e^{-i\beta x}]$$
$$= e^{\alpha x}[C_1 * (cos(\beta x) + isin(\beta x) + C_2(cos(\beta x) - isin(\beta x)]$$
$$= e^{\alpha x}[(C_1 + C_2)cos(\beta x) + i(C_1 - C_2)sin(\beta x)]$$
$$= e^{\alpha x}(Acos(\beta x) + Bsin(\beta x))$$

Where $A = C_1 + C_2$ and $B = i(C_1 - C_2)$

Example: Find the general solution: $y'' + 2y' + 5y = 0$
Solution: The char. eqation is
$m^2 + 2m + 5 = 0$

$(m+1)^2+4=0$
$m=-2\pm 2i$
$\rightarrow y=e^{-x}(C_1cos(2x)+C_2sin(2x))$

Example: Solve the IVP $y''+4y'+5y=0$, $y(0)=2$, $y'(0)=1$
This time:
$m^2+4m+5=0$
$(m+2)^2+1=0$
$m=-2\pm i$
$\rightarrow y=e^{-2x}(C_1cos(x)+C_2sin(x))$

Enforce the ICs:

$$y'=-2e^{-2x}(C_1cos(x)+C_2sin(x))+e^{-2x}(-C_1sin(x)+C_2cos(x))$$

$y(0)=2\rightarrow C_1=2$
$y'(0)=1\rightarrow -2C_1+C_2=1\rightarrow C_2=5$

Thus, $y=e^{-2x}(2cos(x)+5sin(x))$

Case 3: $p^2=4q$ (Repeated roots)
If $p^2=4q$ we get only one exponential solution. There must be a second solution which si not an exponentional

Here's how it was found:
An equation with 2 identical roots should not differ much from an equation with 2 nearly identical roots.
Eg: Compare $y''+2y'+y=0$ to $y''+2.001y'+y=0$

So, suppose a DE has two roots, m and m + ϵ.
$y=C_1e^{mx}+C_2e^{m+\epsilon}x$

As $\epsilon\rightarrow 0$ It looks like we only get one family of solutions:
$y=(C_1+C_2)e6mx$ however, suppose: C_1 and C_2, depend on ϵ

Consider the oslution:
$y=\frac{1}{\epsilon}e^{m+\epsilon}x-\frac{1}{\epsilon}e^{mx}$
$=e^{mx}[\frac{e^{\epsilon x}-1}{\epsilon}]$

$lim_{\epsilon \to 0}(\frac{e^{\epsilon x}-1}{\epsilon}) = lim_{\epsilon \to 0}\frac{xe^{\epsilon x}}{1} = x$

Therefore xe^{mx} is a solution to the equation with $\epsilon = 0$. That is, the general solution for Case 3 is $y = C_1 e^{mx} + C_2 x e^{mx}$

Example: Solve $y'' + 6y' + 9y = 0$
Solution: $m^2 + 6m + 9 = 0 \iff (m+3)^2 = 0 \rightarrow m = -3$

The general solution is $y = C_1 e^{-3x} + C_2 x e^{-3x}$

Example: Solve $y'' + 4y' + 4y2e^{-2x} + 4x$
$y_h = C_1 e^{-2x} + C_2 x e^{-2x}$
$y_p = Ae^{-2x}$ will fail, so we usually multiply by x, but that will also fail. Try multiplying by x again!
$y_p = Ax^2 e^{-2x} + Bx + C$