CS370: Interpolation

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See figure 2.1

$$y = p(x)$$

We want to find a function p, such that the curve is 'nice' (where nice is piecewise polynomial or polynomial)

Given:
$(x_1, y_1), (x_2, y_2)..., (x_n, y_n)$ n points $x_1 < x_2 < ... < x_n$

Find a polynomial P(x) of degree $< n$
In general:

$$p(x) = c_1 + c_2 x + c_3 x^2 + ... + c_n x^{n-1}$$
$$p(x_1) = y_1$$
$$p(x_2) = y_2$$
$$...$$
$$p(x_n) = y_n$$

n unknowns, n equations (linear)

Example:
$(-1, 1), (1, 1), (2, 5), (4, 1)$
See figure 2.2

$$p(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$
$$p(-1) = c_1 - c_2 + c_3 - c_4 = 1$$
$$p(1) = c_1 + c_2 + c_3 + c_4 = 1$$
$$p(2) = c_1 + 2c_2 + 4c_3 + 8c_4 = 5$$
$$p(4) = c_1 + 4c_2 + 16c_3 + 64c_4 = 1$$

$$\left\{\begin{array}{cccccc} 1 & -1 & 1 & -1 & | & 1 \\ 1 & 1 & 1 & 1 & | & 1 \\ 1 & 2 & 4 & 8 & | & 5 \\ 1 & 4 & 16 & 64 & | & 1 \end{array}\right\} // \text{ Solve the matrix!!}$$

Now we are just writing out the solution...

$$\begin{aligned} p(x) &= c_1 + c_2x + c_3x^2 + c_4x^3 \\ &= 1 + b_2(x-1) + b_3(x-1)^2 + b_4(x-1)^3 \\ &= L_1(x) + L_2(x) + 5L_3(x) + L_4(x) \end{aligned}$$

$$\begin{aligned} L_1(x) &= \frac{(x-1)(x-2)(x-4)}{-30} \\ L_2(x) &= \frac{(x+1)(x-2)(x-4)}{6} \\ L_3(x) &= \frac{(x+1)(x-1)(x-4)}{-6} \\ L_4(x) &= \frac{(x+1)(x-1)(x-2)}{30} \end{aligned}$$

I think we are writing it out this way so that we can easily plug in the vlues and get the correct points??

Question:

1. Does an interpolating polynomial always exist?
2. If (1) is true then is the answer always unique?

$$\begin{aligned} p(x) &= c_1 + c_2x + ...c_nx^{n-1} \\ p(x_1) &= c_1 + c_2x_1 + ...c_nx_1^{n-1} \\ p(x_2) &= c_1 + c_2x_2 + ...c_nx_2^{n-1} \\ &... \\ p(x_n) &= c_1 + c_2x_n + ...c_nx_n^{n-1} \end{aligned}$$

$$\begin{Bmatrix} 1 & x_1 & x_1^2 & ... & x_1^{n-1} \\ 1 & x_2 & x_2^2 & ... & x_2^{n-1} \\ ... & ... & ... & ... & ... \\ 1 & x_n & x_n^2 & ... & x_n^{n-1} \end{Bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{Bmatrix}$$

The first matrix is the vandermonde (V) matrix.
V is invertable, $V \times \overrightarrow{c} = \overrightarrow{y}$
det $V \neq 0$ and det $V = \pi(x_i - x_j) \neq 0$ for $i < j$

Remember what a determinate is, remember what invertible is, but we will never be asked to do it.

$p(x)$

$$p(x) = q_1(x)(x - x_1) + y_1$$

$$p(x) = q_2(x)(x - x_2) + y_2$$

$$...$$

$$p(x) = q_n(x)(x - x_n) + y_n$$

Lagrange Polynomial

$$(x_1, y_1), (x_2, y_2)...(x_n, y_n)$$

$$p(x) = y_1 L_1(x) + y_2 L_2(x) + ... + y_n L_n(x)$$

$L_i(x_i) = 1, L_i(x_j) = 0$ for $i \neq j$ and $deg(L_i) = n - 1$
Lets construct L_1 using the above

$$L_1(x) = \frac{(x - x_2)(x - x_3)...(x - x_n)}{(x_1 - x_2)(x_1 - x_3)...(x_1 - x_n)}$$

$$L_i(x) = \frac{(x - x_1)...(x - x_{i-1})(x - x_{i+1})...(x - x_n)}{(x_i - x_1)...(x_i - x_{i-1})...(x_i - x_n)}$$

$L_i(x_i) = 1$ and $L_j(x_j) = 0$ where $j \neq i$

For A1 Q3 (January 13th) - figuring out the solution to the recurrence - and using the answer to help

$$??\boxed{I_n} \leftarrow I_{n-1} \leftarrow I_{n-2} \leftarrow ... \leftarrow I_0$$

$$\checkmark\boxed{\hat{I_n}} \leftarrow \hat{I_{n-1}} \leftarrow ... \leftarrow \hat{I_1} \leftarrow \hat{I_0}$$

$$e_n \leftarrow e_{n-1} \leftarrow ... \leftarrow e_1 \leftarrow e_0$$

$$e_n = (-\alpha)^n e_0$$

$$I_n? = formula(I_0) =$$

Using p?

$$??\boxed{p_n} \leftarrow p_{n-1}p_{n-2}, p_{n-2}p_{n-3}, ..., p_1, p_0$$

$p_n = as^n + bt^n$ and a, b depend on p_0, p_1

$$\checkmark\boxed{\hat{p_n}} \leftarrow \hat{p_{n-1}}\hat{p_{n-2}}..., \hat{p_1}\hat{p_0}$$

This line but with hats (I got lazy) $p_n = as^n + bt^n$ and a, b depend on p_0, p_1
solve for e_n

Recall from Jan 11th: (regoing over the start of this page)

Lagrange Form (again)

For $x_1, x_2, ...x_n$ distinct, construct $L_1(x), L_2(x)...L_n(x)$
Satisfying:

1. $L_i(x)$ has degree n-1
2. $L_i(x_i) = 1$
3. $L_i(x_j) = 0$ if $i \neq j$

How do we construct this:

$$L_1(x) = \frac{(x - x_2)(x - x_3)...(x - x_n)}{(x_1 - x_2)(x_1 - x_3)...(x_1 - x_n)}$$

We divide like this in order to get an equation that satisfies that if we plug in x_1 we will end up getting 1 as required, otherwise we will be getting a 0. This is actually pretty cool. Neat!

$$L_i(x) = \frac{(x - x_1)...(x - x_{i-i})(x - x_{i+1})...(x - x_n)}{(x_i - x_1)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)}$$

$$p(x) = y_1 L_1(x) + y_2 L_2(x) + ... + y_n L_n(x)$$

$$p(x_1) = y_1 1 + y_2 0 + ... + y_n 0 = y_1$$

$$...$$

$$p(x_n) = y_1 0 + y_2 0 + ... + y_n 1 = y_n$$

A question that he often has asked on midterms (and is almost 100% going to add it to ours):
Given: $x_1, x_2, x_3 x_4$ as $-1, 1, 2, 117, 412$
Form $p(x) = L_1(x) + L_2(x) + L_3(x) + L_4(x)$
Write $p(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$

Draw the graph!
Solve for the 4 numbers, and find what is y at each of the 4 points?
Then we find out that f(x) = 1 for each
Therefore the solution is p(x) = 1

Cubic Hermite Interpolation

Another type of interpolation

Given: (x_L, y_L) more on the left side and (x_R, y_R) on the right side, S_L slope of the left side, and S_R the slope of the right side

$p(x)$ has degree at most 3 since we have 4 uknowns
$p(x_L) = y_L$, $p(x_R) = y_R$, $p'(x_L) S_L$, $p'(x_R) = S_R$

$p(x) = c_1 + c_2(x - x_L) + c_3(x - x_L)^2 + c_4(x - x_L)^3$ $p'(x) = c_2 + 2c_3(x - x_L) + 3c_4(x - x_L)^2$ $p(x_L) = y_L \implies c_1$ $p'(x_L) = S_L \implies c_2$ $p(x_R) = y_R \implies c_1 + c_2 \Delta x + c_3 \Delta x^2 + c_4 \Delta x^3 = y_R$ $p'(x_R) = S_R \implies c_2 + 2c_3 \Delta x + 3c_4 \Delta x^2 = S_R$ where $\Delta x = x_R - x_L$

$$\left\{\begin{matrix} 1 & 0 & 0 & 0 & | & Y_L \\ 0 & 1 & 0 & 0 & | & S_L \\ 1 & \Delta x & \Delta x^2 & \Delta x^3 & | & Y_R \\ 0 & 1 & 2\Delta x & 3\Delta x^2 & | & S_L \end{matrix}\right\}$$

becomes

$$\left\{\begin{matrix} 1 & 0 & 0 & 0 & | & Y_L \\ 0 & 1 & 0 & 0 & | & S_L \\ 0 & 0 & 1 & 0 & | & \frac{3Y_R' - 2S_L - S_R}{\Delta x} \\ 0 & 0 & 0 & 1 & | & \frac{S_R + S_L - 2y_L'}{\Delta x^2} \end{matrix}\right\}$$

$$c_1 = y_L$$

$$c_2 = S_L$$

$$c_3 = \frac{3Y_R' - 2S_L - S_R}{\Delta x}$$

$$c_4 = \frac{S_R + S_L - 2y_L'}{\Delta x^2}$$

Sub into p(x)

$$p(x) = 3 - (x-1) + 3(x-1)^2 - (x-1)^3$$

From Jan 16th:
See image Interp1.1: He is showing that the polynomial (red line) could be bad, we want the green line instead.

Cubic Spline

Given: $(x_1, y_1), ..., (x_N, y_N)$ N points
$x_1 < x_2 < ... < x_{N-1} < x_N$

A cubic spline is a function S(x) defined on the interval $[x_1, x_N]$ which satisfies the following:
(see interp1.2 figure)

1. In each interval $[x_i, x_{i+1}]$ S(x) is a cubic polynomial. $S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$

2. S(x) interpolates the N points: $S(x_i) = y_i$

3. S'(x) is continuous

4. S"(x) is continuous

5. 2 other things??

Is this well defined?
How man unknowns? 4 per interval, N-1 intervals $\rightarrow 4N-4$ unknowns
How many conditions (equations)?
Condition(2) $\rightarrow$ 2 equations per interval $\rightarrow 2N-2$
$S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$
Condition(3) - 1 equation per interior point $\rightarrow N-2$
Condition(4) - 1 equation per interiour point $\rightarrow N-2$
In total we get $4N-6$ equations

Boundary Conditions

1. Natural cubic spline $S''(x_1) = 0$, $S''(x_N) = 0$

2. Clamped cubic spline $S'(x_1) = s_1$ and $S'(x_N) = s_N$ s_1, s_N are known

3. Periodic Cubic spline $S'(x_N) = S'(x_1)$ and $S''(x_N) = S''(x_1)$

4. Not-a-knot condition (Matlab default) $S'''(x)$ is continuous at x_2 and x_{N-1}

How do we compute a cubic spline?

Method 1:
Have $4N-4$ uknowns and $4N-4$ linear equations $\rightarrow$ sove via Gaussian elimination
This is a cost of: $O((4N-4)^3) = O(N^3)$

Method 2:
Think of the deriviatives $S_1, S_2, ...S_N$ as the unknowns. We will set up linear equations for these derivatives
Then:

1. This will give us $S_1(x), S_2(x), ..., S_{N-1}(x)$

2. We will solve linear system in O(N) operations

Given: (-2,1), (0,0), (1,3), (4,-1), (5,2)
Clamped
(Figure Interp1.3)

S(x) =

$$a_1 + b_1(x+2) + c_!(x+2)^2 + d_1(x+2)^3\_for\_-2 \leq x \leq 0$$
$$a_2 + b_2x + c_2x^2 + d_2x^3\_for\_0 \leq x \leq 1$$
$$a_3 + b_3(x-1) + c_3(x-1)^2 + d_3(x-1)^3\_for\_1 \leq x \leq 4$$
$$a_4 + b_4(x-4) + c_4(x-4)^2 + d_4(x-4)^3\_for\_4 \leq x \leq 5$$

16 uknowns and 16 equations
$a_1 = 1$, $a_2 = 0$, $a_3 = 3$, $a_4 = -1$
$a_1 + 2b_1 + 4c_1 + 8d_1 = 0$
$a_2 + b_2 + c_2 + d_2 = 3$
✓
✓
$b_1 + 4c_1 + 12d_1 = b_2$
$2c_1 + 12d_1 = 2c_2$
✓
✓
✓
✓
$b_1 = 1$
$b_4 = 2c_4 + 3d_4 = 0$

Go and do the assignment question that looks like this

Generic example:
Given: $(x_1, y_1), (x_2, y_2)..., (x_N, y_N)$
let $s_1, s_2, s_3...s_N$ denote the derivitive values of the spline S(x) at the points.
These are uknowns, but they exist.
Figure interp1.4
figure interp1.5 is a blown up of X_i

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

$a_i = y_i$
$b_i = s_i$
$c_i = \frac{3y_i' - 2s_i - s_{i+1}}{\Delta x_i}$
$d_i = \frac{s_i + s_{i+1} - 2y_i'}{\Delta x_i^2}$
$\Delta x_i = x_{i+1} - x_i$
$Y_i' = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$

Additional information:

$S''(x_1) = 0, S_1''(x_2) = S_2''(x_2), S_2''(x_3) = S_3''(x_3), ...S_{N-2}''(x_{N-1}) = S_{N-1}''(X_{N-1}), S_{N-1}''(x_N) = 0$

Equation 1:
$2c_1 = 0$ ie. $\frac{3y_1' - 2s_1 - s_2}{\Delta x_1} = 0$
$2s_1 + s_2 = 3y_1'$

We now want to set up a linear set of equations for the esses

$$\left\{\begin{matrix} * & * \\ ... & \\ ... & \\ ... & \\ ... & \\ * & * \end{matrix}\right\} \left\{\begin{matrix} s_1 \\ s_2 \\ s_3 \\ ... \\ s_{n-1} \\ s_N \end{matrix}\right\} = \left\{\begin{matrix} * \\ ... \\ ... \\ ... \\ ... \\ * \end{matrix}\right\}$$

$$c_{N-1} + 3d_{N-1}(x_N - x_{N-1}) = 0$$

$$3Y_{N-1}' - 2SN - 1 - S_N) + 3(S_{N-1} + S_N - 2Y_{N-1}') = 0$$

He deleted things on the bottom :(((

$$S_{N-1} + 2S_N = 3Y_{N-1}'$$

$$(x_i) = S_i''(x_i)$$

For $i = 2, 3, ...N - 1$

$$= 2c_i$$

$$= \frac{2(3Y_i' - 2s_i - s_{i+1})}{\Delta x_i}$$

$S''_{i-1}(x) = 2c_{i-1} + 6d_{i-1}(x - x_{i-1})$
$S_{i-1}(x) = a_{i-1} + b_{i-1}(x - x_i) + c_{i-1}(x0x_{i-1})^2 + d_{i-1}(x - x_{i-1})^3$

$c_{i-1} + 3d_{i-1}\Delta x_{i-1}$

$$\Delta x_i((3Y_{i-1}' - 2s_{i-1} - s_i) + 3(s_{i-1} + s_i - 2y_{i-1}')) = (3Y_i' - 2s_i - s_{i-1})\Delta x_{i-1}$$

$$\Delta x_i S_{i-1} + 2(\Delta x_i + \Delta x_{i-1})S_i$$

$$\Delta x_{i-1} S_{i+1} = 3y_{i-1}'\Delta x_i + 3y_i'\Delta x_{i-1}$$

The above is assignment question 5, but like wtf is going on...
Object is to find values for the S's in the matrix...

<u>From Jan 20th</u>

See slides.