Fourier Transforms – DFT Properties and Visualization

CS370 Lecture 21 – Feb 27, 2017

Midterm reminder

Tomorrow (Tuesday) evening at 7pm.

Location: Based on the *first letter of your surname* (i.e. last name), you go to the following room for the midterm:

A-M -> DC 1351
N-T -> MC 2017
U-Z -> MC 2054

Please don't miss it! That would make me very sad.

Recall: Discrete Fourier Transform

We have developed the Discrete Fourier Transform pair:

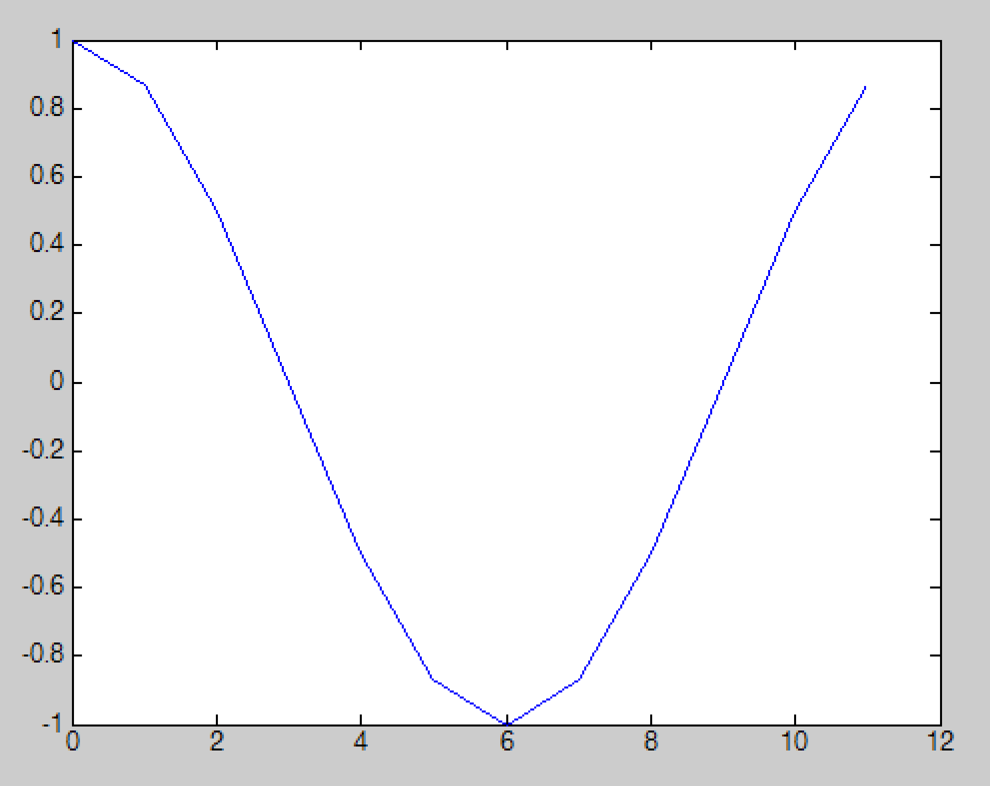
$$f_n = \sum_{k=0}^{N-1} F_k W^{nk} \qquad \text{and} \qquad F_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n W^{-nk}$$

where $W = e^{\frac{2\pi i}{N}}$, recalling Euler's formula $e^{i\theta} = \cos\theta + i\,\sin\theta$.

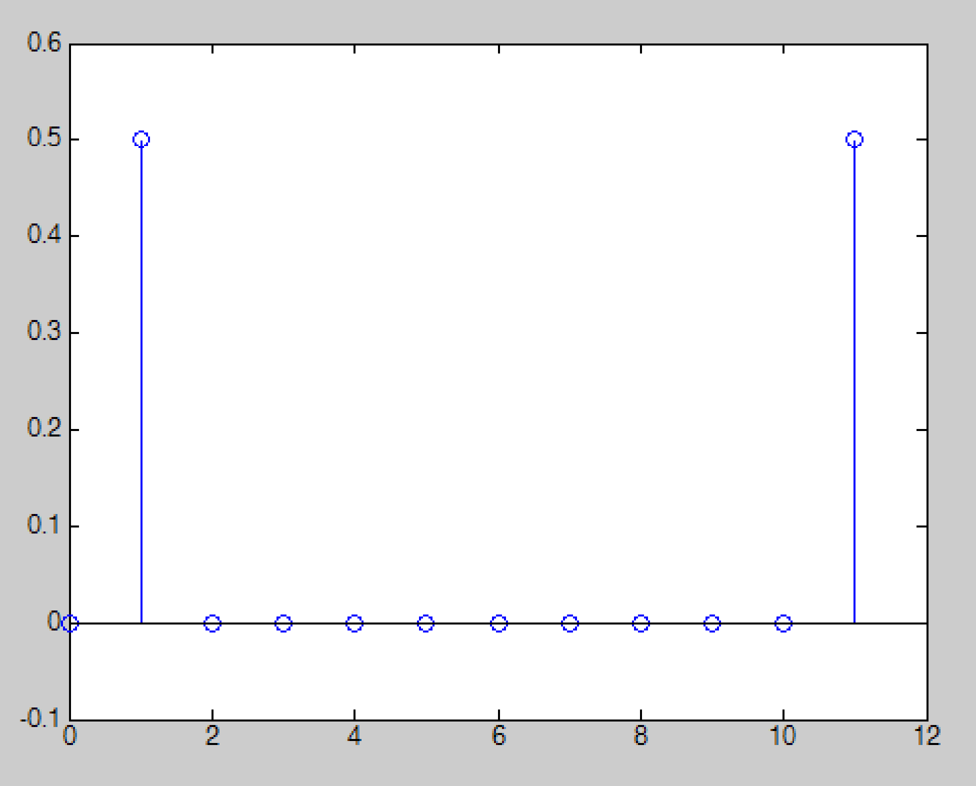
This allows a perfect transformation between N "time-domain" data points f_n and N "frequency-domain" Fourier coefficients F_k.

The Fourier view emphasizes the frequencies present in the data.

Example 5.1: DFT of N=12 points for $\cos\left(\frac{2\pi n}{N}\right)$



Original Data f_n



Resulting Fourier coefficients F_k

Two Properties of the DFT

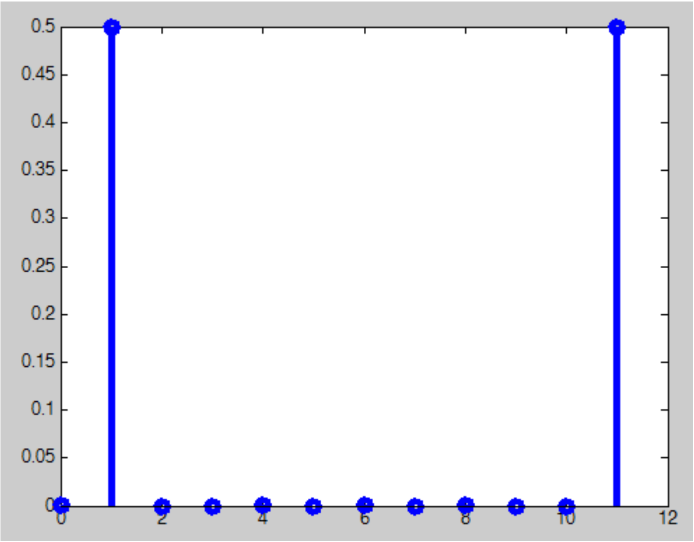
As consequences of the properties of Nth roots of unity…

1. The sequence $\{F_k\}$ is doubly infinite and periodic.

i.e., If we allow $k < 0$ or $k > N - 1$, the F_k coefficients repeat.

2. *Conjugate symmetry*: If data f_n is *real,* $F_k = \overline{F_{N-k}}$.

Hence the $|F_k|$ are symmetric about $k = \frac{N}{2}$, as we saw in the cos example's power spectra.



F_k plot for $\cos\left(\frac{2\pi n}{N}\right)$ exhibits conjugate symmetry.

Discrete Fourier Transform

Typically, we want to learn/achieve something by processing the data.

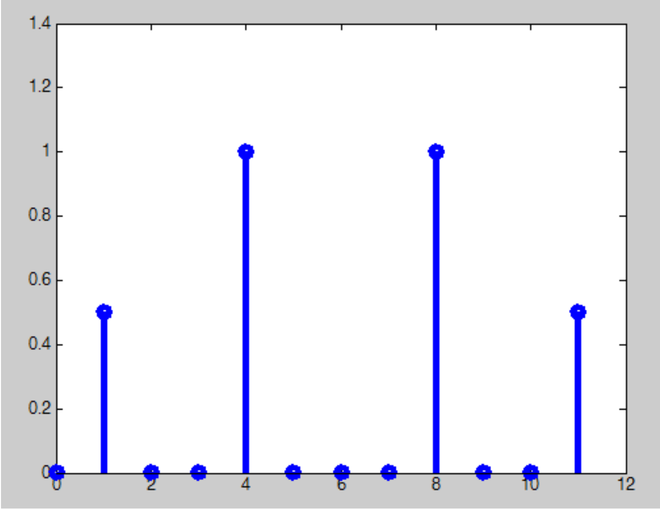
(Image data, audio samples, prices, intensities, etc.)

In theory, the time-domain data tells us everything!

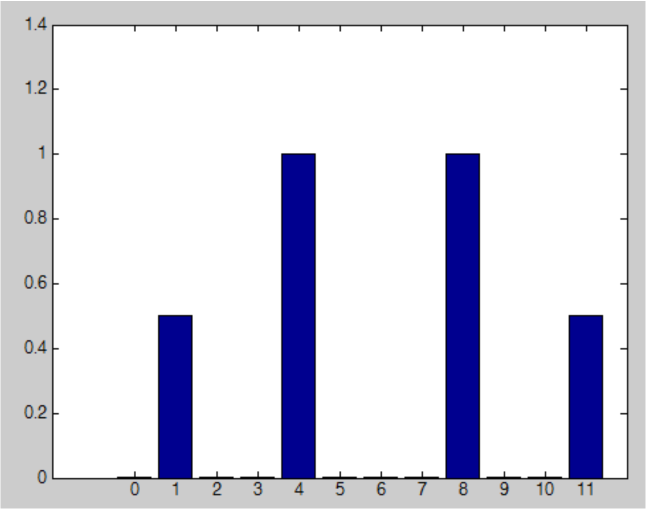
In practice, Fourier coefficients provide easier access to useful insights/information for certain problems.

Power/Fourier Spectrum

The "power spectrum" visualizes the Fourier coefficients by plotting their moduli/magnitudes $|F_k|$, e.g., using Matlab.



stem(abs(F))



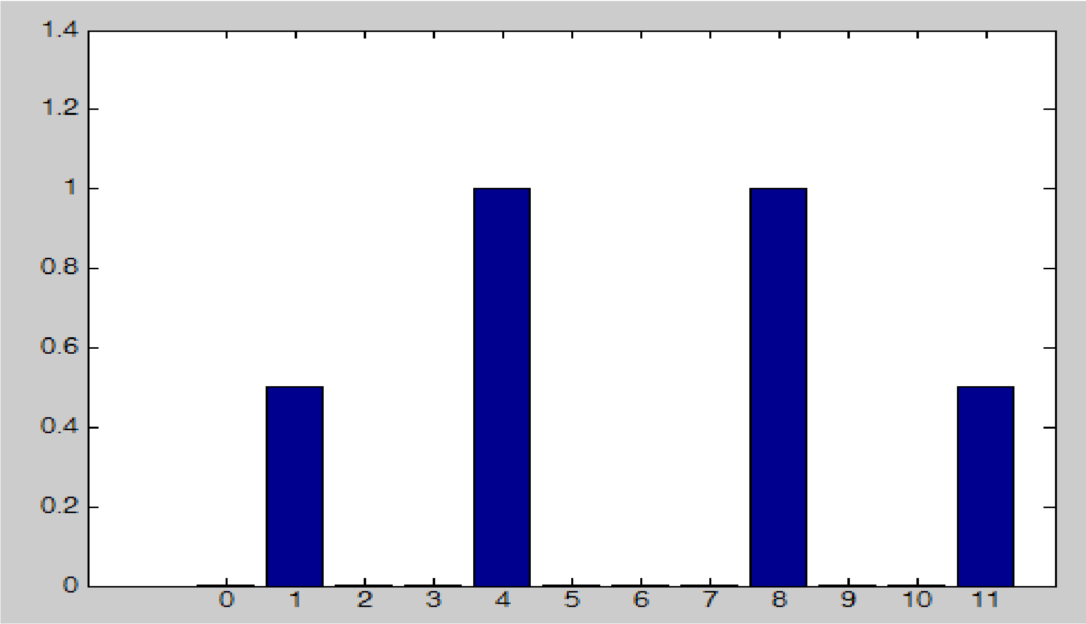
bar(abs(F))

This expresses the magnitude of the frequencies, but ignores phase.

(Note: other sources often call $|F_k|^2$ the power spectrum.)

Frequency Information

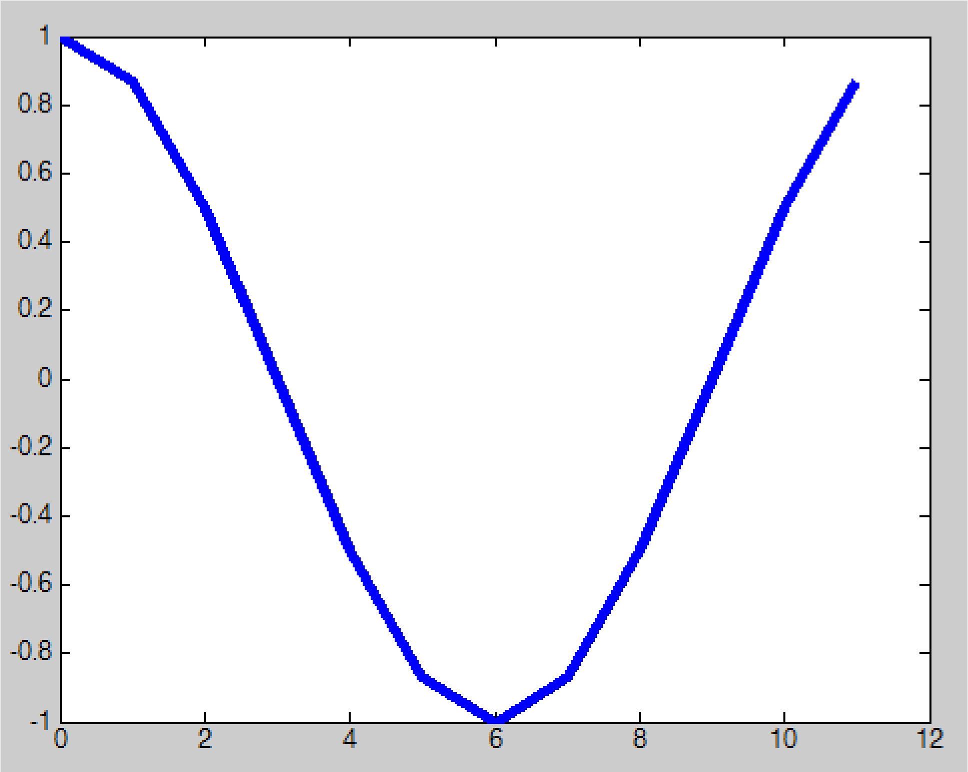
Frequencies of corresponding waves **increase** for index k = 0 to N/2 and then **decrease** again from N/2+1 to N-1.



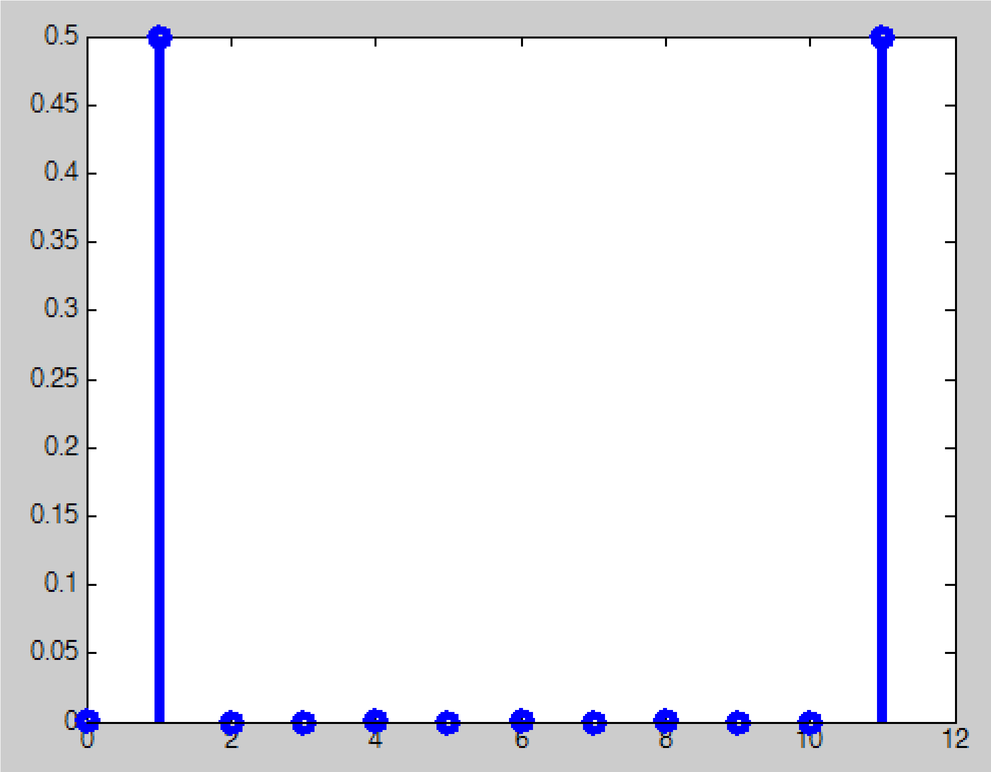
Low High Low

Example Cosine (Notes, Example 5.1)

Last time, we saw a (rather artificial) example, $\cos\left(\frac{2\pi n}{N}\right)$.



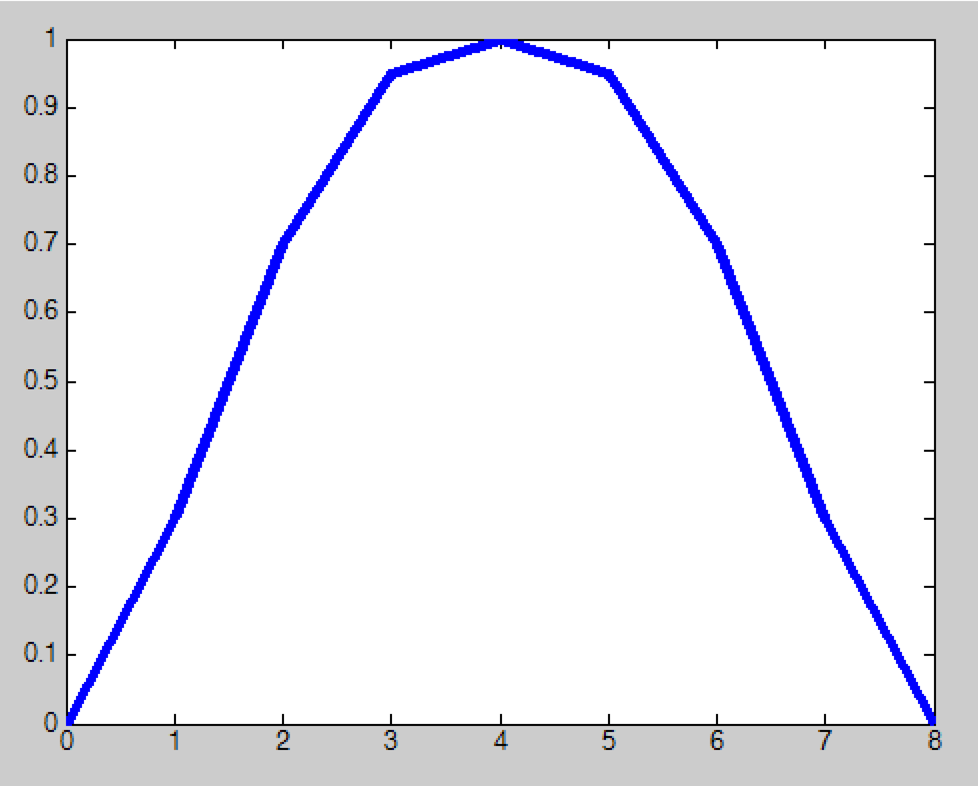
f_n



F_k

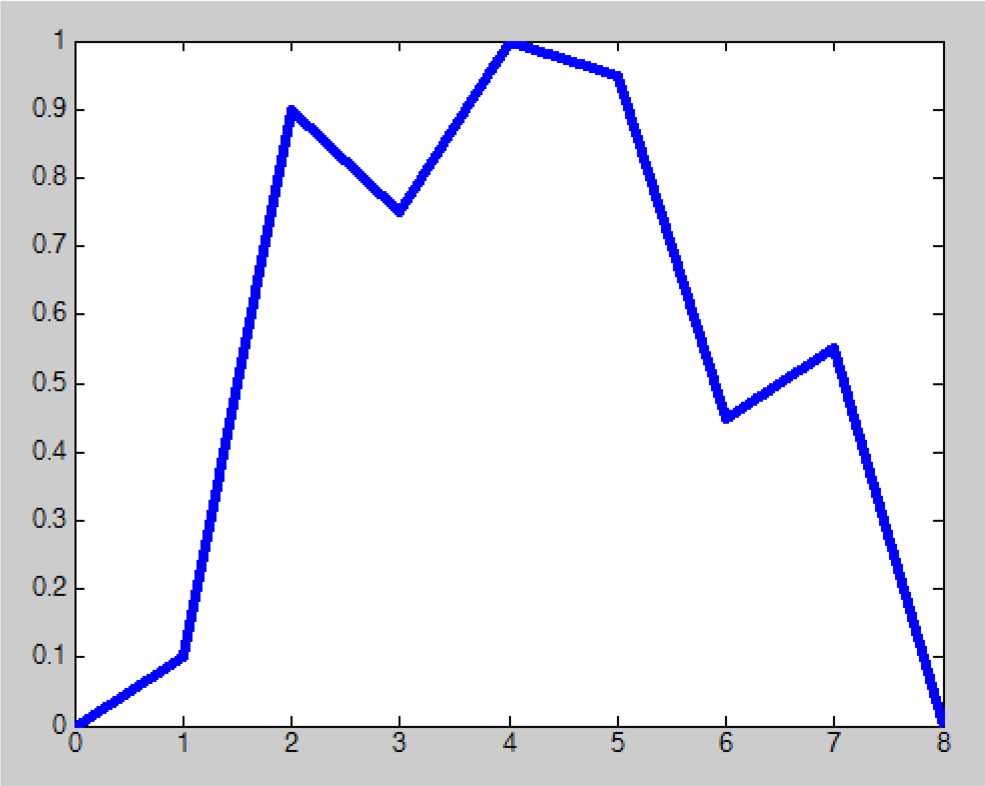
Discrete Data Examples

What if our data is less artificial and more irregular? (N=9 here.)



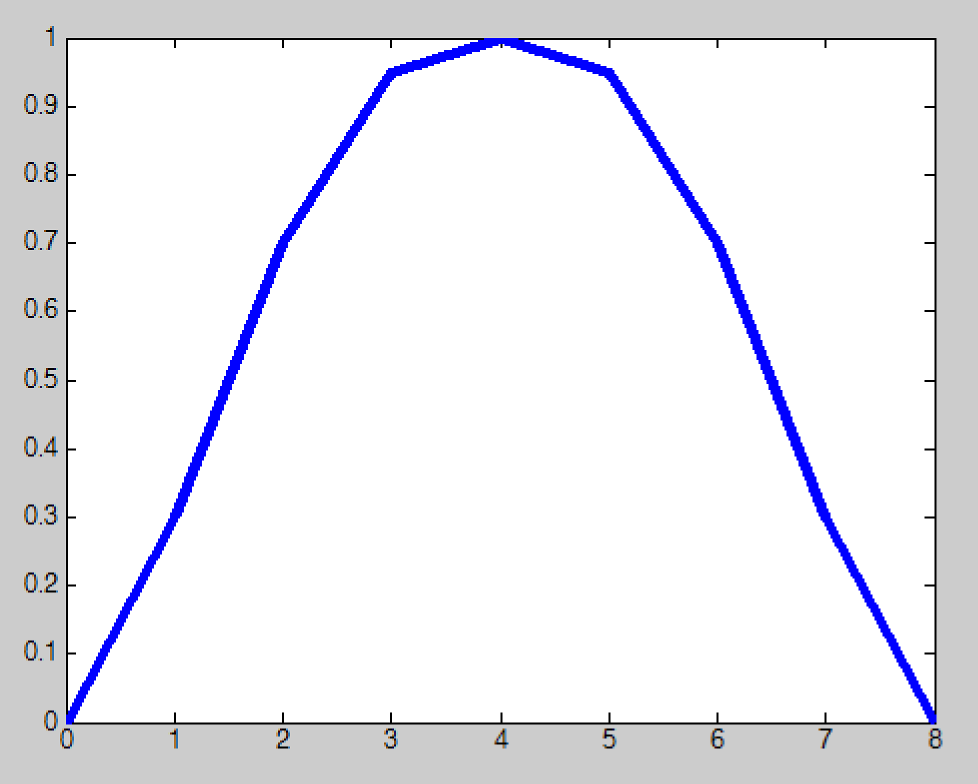
A smooth hump - data.

or

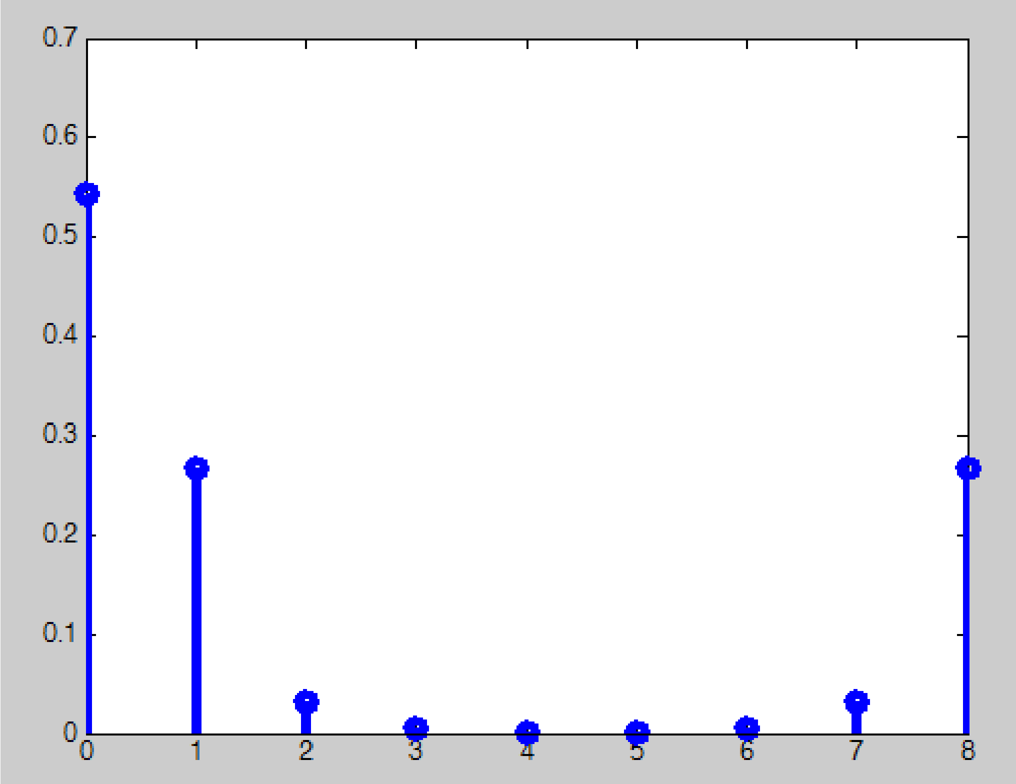


A rough hump - data.

Discrete Data – A Smooth Hump

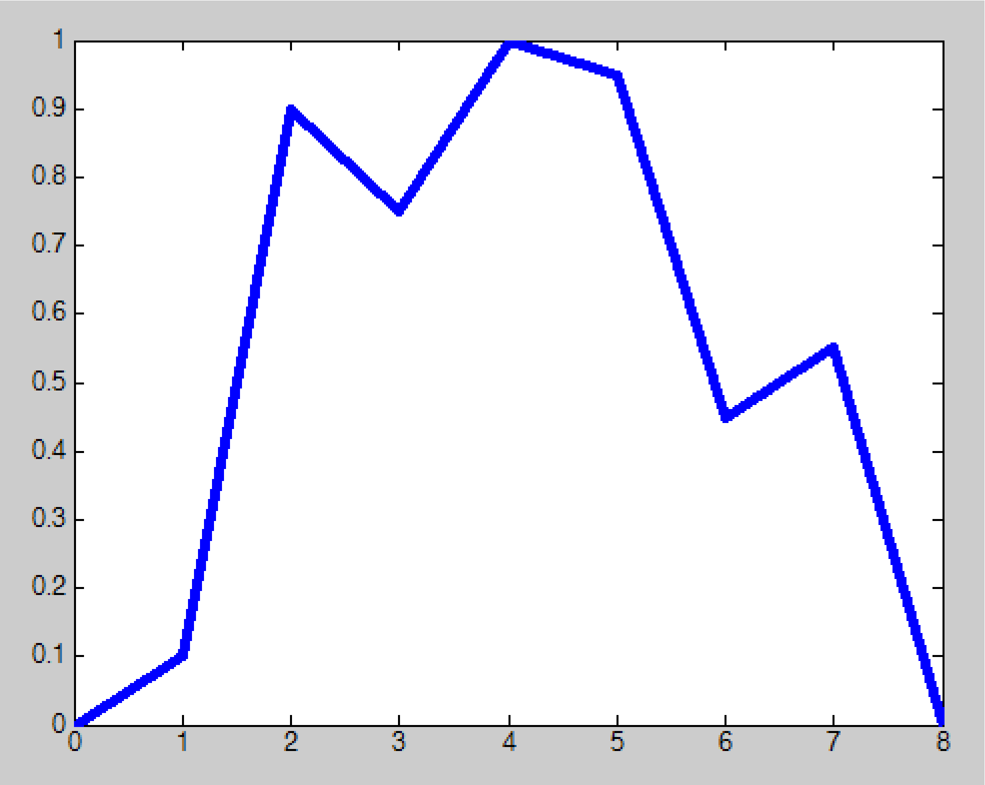


Data

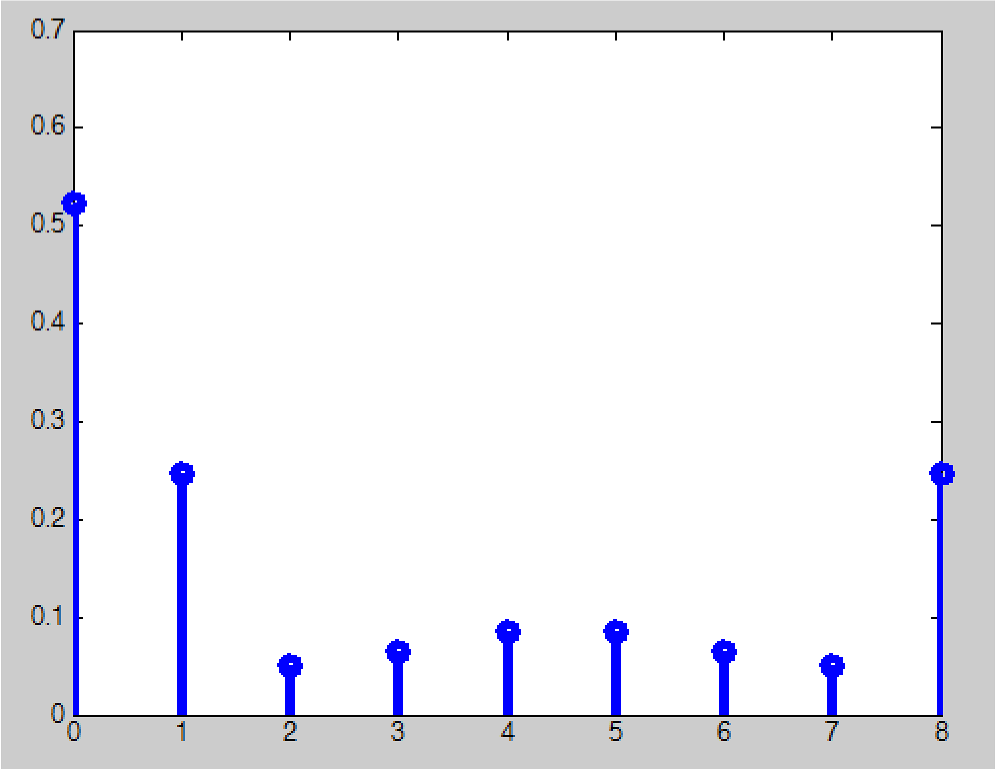


Fourier coefficients.

Discrete Data – A Rough Hump



Data



Fourier coefficients.

Discrete Data - Observations

- Coefficient F_0 is always the average of data values, $F_0 = \frac{1}{N}\sum_{n=0}^{N-1} f_n$. (Sometimes called the direct current or DC.)

- The smooth hump had one dominant frequency/wave; therefore one coefficient pair, F_1 and F_8, had large magnitude.

- The rough hump had more irregularity, so more active (higher) frequencies. Main hump still dominates, so low freqs F_1 and F_8 remain largest.

- The Fourier (F_k) plots are symmetric, since the data was real.

Matlab demo

Let’s explore the Fourier plots for various data.

Don’t forget:

- Matlab uses 1-based indexing, so $F_0 \to F(1)$
- Matlab puts the normalizing factor $\frac{1}{N}$ in the IDFT instead of DFT.