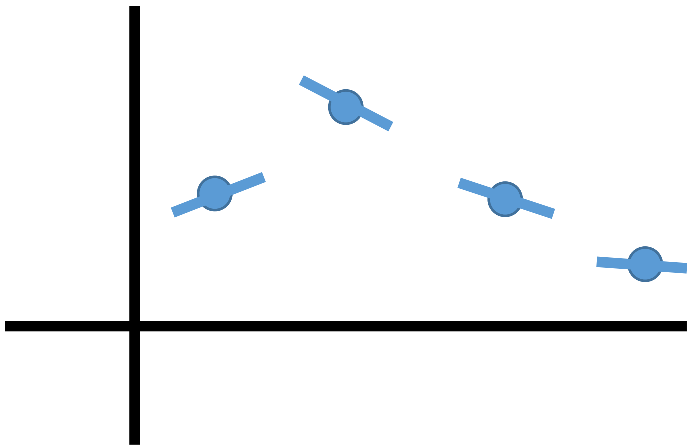
Interpolation – Efficiently Computing Splines

CS370 – Lecture 7

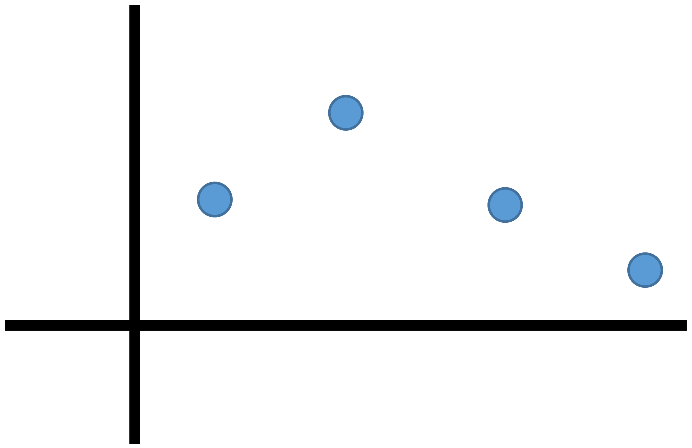
May 16, 2016

Two strategies for smooth curves

Hermite interpolation: Given points and their slopes, fit a curve. Gives matching 1st derivatives between intervals.



Cubic spline interpolation: Given points only, fit a curve with matching 1st and 2nd derivatives across intervals.



Computing Cubic Splines – Cost

Cubic splines gave us a linear system, $Ax = b$, of size $(4n - 4)^2$ for n points.

Basic algorithms (i.e., Gaussian elimination) for linear systems take $O(N^3)$ time for N unknowns.

For the special case of cubic splines, one can do **much** better – $\mathrm{O}(N)$. *i.e., linear* in the unknowns. We will start working towards this…

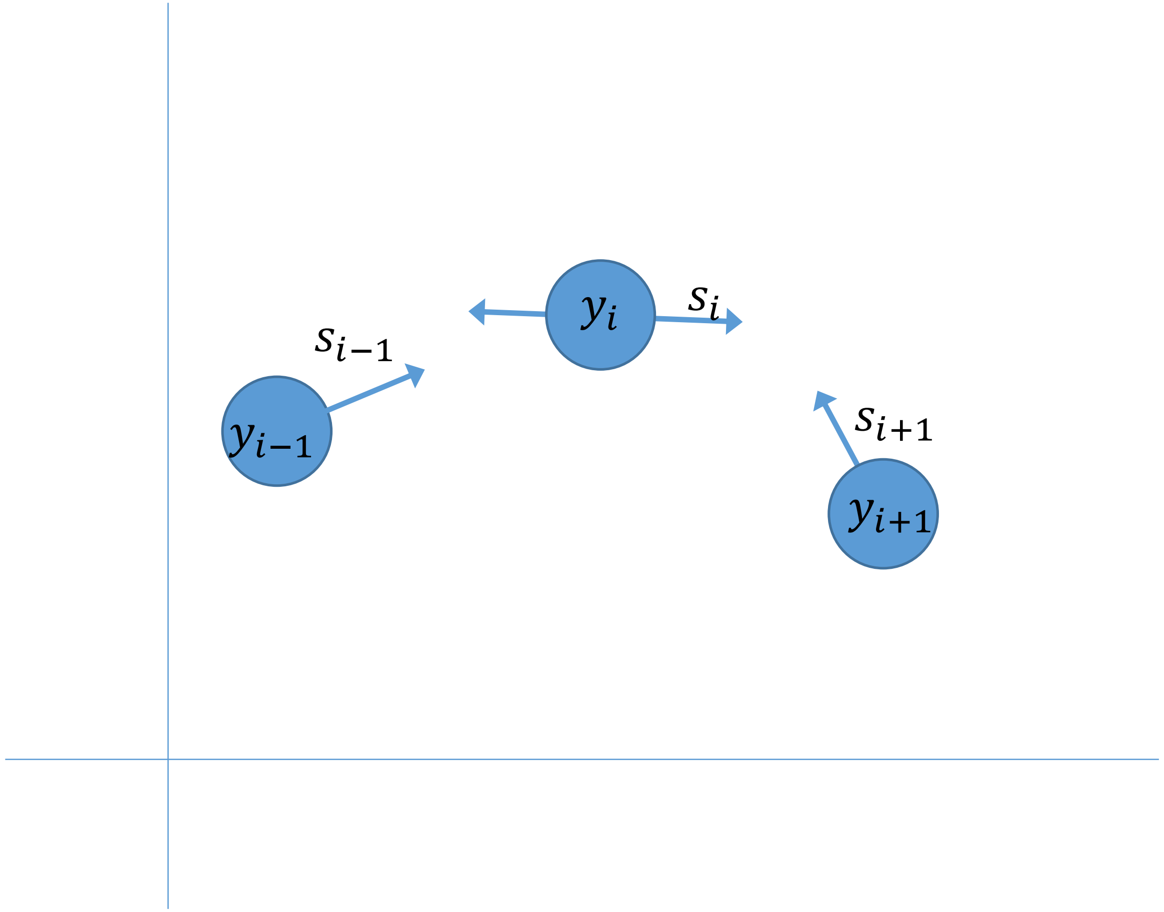
Cubic splines from Hermite interpolation

Strategy: Use Hermite interpolation as a tool to build a cubic spline!

1. Express our unknown polynomial using the Hermite interpolant equations.
2. Treat the s_i values (1^{st} derivatives at nodes) as ***unknowns***.
3. Solve for s_i that give continuous 2^{nd} derivatives – this forces our interpolant to satisfy the def'n of a cubic spline.
4. Given s_i, plug into the direct Hermite interpolation eq'ns to recover the polynomial coefficients: a_i, b_i, c_i, d_i.

Reminder: Hermite interpolation equations

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$



$$a_i = y_i$$

$$b_i = s_i$$

$$c_i = \frac{3y_i' - 2s_i - s_{i+1}}{\Delta x_i}$$

$$d_i = \frac{s_{i+1} + s_i - 2y_i'}{(\Delta x_i)^2}$$

$$\Delta x_i = x_{i+1} - x_i$$

$$y_i' = \frac{y_{i+1} - y_i}{\Delta x_i}$$

We will solve for a set of s_i that give continuous 2nd derivatives, so the ***result is a cubic spline***.

Recovering a Cubic Spline

For cubic splines, we had three sets of equations (ignoring ends).

1. Values match at all interval endpoints.
2. First derivatives match at interior points.
3. Second derivatives match at interior points.

Satisfied already by using the Hermite closed form! (items 1–2)

Must be enforced explicitly by solving. (item 3)

To achieve item (3) we need to find s_i that give $S_i''(x) = S_{i+1}''(x)$.

Let’s see how…

Computing Cubic Splines – Equation Summary

Solve the following equations for the unknown s_i variables:

Interior nodes ($i = 2 \ \dots \ n-1$):

$$\Delta x_i s_{i-1} + 2(\Delta x_{i-1} + \Delta x_i) s_i + \Delta x_{i-1} \mathrm{s}_{\mathrm{i+1}} = 3(\Delta x_i y'_{i-1} + \Delta x_{i-1} y'_i)$$

Clamped BC ($i = 1 \text{ or } n$):

$$s_1 = s_1^*, \qquad s_n = s_n^*$$

Free BC ($i = 1 \text{ or } n$):

$$s_1 + \frac{s_2}{2} = \frac{3}{2} y'_1, \qquad \frac{s_{n-1}}{2} + s_n = \frac{3}{2} y'_{n-1}$$

Computing Cubic Splines - Matrix Form

Interior nodes ($i \ = 2 \ \dots \ n-1$):

$$\Delta x_i s_{i-1} + 2(\Delta x_{i-1} + \Delta x_i) s_i + \Delta x_{i-1} \mathrm{s}_{\mathrm{i+1}} = 3(\Delta x_i y'_{i-1} + \Delta x_{i-1} y'_i)$$

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How does this look in matrix form $T\vec{s} = \vec{r}$?

What are the matrix T and RHS vector $\vec{r}$?

$$\begin{bmatrix} & & \\ & T & \\ & & \end{bmatrix} \begin{bmatrix} s_1 \\ \dots \\ s_i \\ \dots \\ s_n \end{bmatrix} = \begin{bmatrix} r_1 \\ \dots \\ r_i \\ \dots \\ r_n \end{bmatrix}$$

Matrix Form - Interior rows

$$\Delta x_i s_{i-1} + 2(\Delta x_{i-1} + \Delta x_i)s_i + \Delta x_{i-1} s_{i+1} = 3(\Delta x_i y'_{i-1} + \Delta x_{i-1} y'_i)$$

This gives active entries for row *i:*

$$T_{i,i-1} = \Delta x_i$$
$$T_{i,i} = 2(\Delta x_{i-1} + \Delta x_i)$$
$$T_{i,i+1} = \Delta x_{i-1}$$

with all other columns 0

$$T_{i,k} = 0\,;k \neq i-1, i, i+1.$$

The right hand side is

$$r_i = 3(\Delta x_i y'_{i-1} + \Delta x_{i-1} y'_i).$$

Matrix Form – Clamped BC

Left end:

$$s_1 = s_1^* \longrightarrow \begin{gathered} T_{1,1} = 1 \\ T_{1,k} = 0, k \neq 1 \\ \text{and} \\ r_1 = s_1^* \end{gathered}$$

Right end:

$$s_n = s_n^* \longrightarrow \begin{gathered} T_{n,n} = 1 \\ T_{n,k} = 0, k \neq n \\ \text{and} \\ r_n = s_n^* \end{gathered}$$

Matrix Form – Free/Natural BC

Left end:

$$s_1 + \frac{s_2}{2} = \frac{3}{2} y_1' \longrightarrow$$

$$T_{1,1} = 1$$
$$T_{1,2} = \frac{1}{2}$$
$$T_{1,k} = 0, k \neq 1,2$$
and
$$r_1 = \frac{3}{2} y_1'$$

Right end:

$$\frac{s_{n-1}}{2} + s_n = \frac{3}{2} y_{n-1}' \longrightarrow$$

$$T_{n,n} = 1,$$
$$T_{n,n-1} = \frac{1}{2}$$
$$T_{n,k} = 0, k \neq n, n-1$$
and
$$r_n = \frac{3}{2} y_{n-1}'$$

Example:

What is the linear system for s_i to fit a spline to the 4 points $(0,1), (2,1), (3,3), (4,-1)$ with clamped BC of $s_1 = 1$ and $s_4 = -1$?

Equation Summary:

Interior nodes ($i = 2 \ldots n-1$):

$$\Delta x_i s_{i-1} + 2(\Delta x_{i-1} + \Delta x_i)s_i + \Delta x_{i-1} s_{i+1} = 3(\Delta x_i y'_{i-1} + \Delta x_{i-1} y'_i)$$

Clamped BC ($i = 1$ or n):

$$s_1 = s_1^*, \qquad s_n = s_n^*$$

Free BC ($i = 1$ *or* n):

$$s_1 + \frac{s_2}{2} = \frac{3}{2} y'_1, \qquad \frac{s_{n-1}}{2} + s_n = \frac{3}{2} y'_{n-1}$$

$$\Delta x_i = x_{i+1} - x_i$$

$$y'_i = \frac{y_{i+1} - y_i}{\Delta x_i}$$

Recovering the spline

Finally, given the correct solution, $\vec{s}$, to the system, how do we recover the desired piecewise polynomial?

Plug the s_i back into the Hermite interpolation equations for each interval.

Summary: Hermite interpolation equations:

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

$$a_i = y_i$$

$$b_i = s_i$$

$$c_i = \frac{3y_i' - 2s_i - s_{i+1}}{\Delta x_i}$$

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