Find the DFT of data

$$f_n = \cos\left(\frac{2\pi n}{N}\right) \text{ for } n = 0 \dots N-1.$$

DFT is $$F_k = \frac{1}{N}\sum_{n=0}^{N-1} \cos\left(\frac{2\pi n}{N}\right) W^{-nk}.$$

By Euler's formula, $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, so

$$F_k = \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{2}\left(e^{\frac{i2\pi n}{N}} + e^{\frac{-i2\pi n}{N}}\right) W^{-nk}.$$ Since $W = e^{\frac{i2\pi}{N}}$, we have

$$= \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{2}\left(W^n + W^{-n}\right) W^{-nk}$$

$$= \frac{1}{2N}\sum_{n=0}^{N-1}\left(W^{n(1-k)} + W^{-n(1+k)}\right)$$

3 cases to consider: $k=1$, $k=N-1$, all other k.

For $k=1$:

$$F_1 = \frac{1}{2N}\sum_{n=0}^{N-1}\left(\cancelto{1}{W^0} + W^{-2n}\right) = \frac{1}{2} + \frac{1}{2N}\sum_{n=0}^{N-1}\left(W^{-2}\right)^n$$

Using $\displaystyle\sum_{j=0}^{N-1} x^j = \frac{x^N - 1}{x-1}$ gives $$F_1 = \frac{1}{2} + \frac{1}{2N}\left(\cancelto{0}{\frac{(W^{-2})^N - 1}{W^{-2} - 1}}\right) = \frac{1}{2}$$

since W^{-2} is an N^{th} root of unity.

For $k = N-1$:

$$F_{N-1} = \frac{1}{2N}\sum_{n=0}^{N-1}\left(W^{n(2-N)} + \cancelto{1}{W^{-nN}}\right) = \frac{1}{2} + \cancelto{0}{\frac{1}{2N}\sum_{n=0}^{N-1}\left(W^{2-N}\right)^n} = \frac{1}{2}$$

Else, both terms give 0.

$$\therefore\ F_k = \begin{cases} 1/2 & \text{for } k = 1, N-1 \\ 0 & \text{for all other } k \in [0, N-1]. \end{cases}$$

F_k is periodic and doubly infinite

Given F_k for $k \in [0, N-1]$, then for $k \in (-\infty, \infty)$, F_k is one of the existing coefficients

We can express an arbitrary k as $k = mN + p$ where $p \in [0, N-1]$, i.e. $k \equiv p \pmod{N}$

Then $W^{-k} = e^{\frac{-2\pi i}{N}(mN+p)}$

$$= e^{-2\pi i m} e^{\frac{-2\pi i p}{N}} = (1) \cdot W^{-p}$$

Hence $F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n W^{-nk} = \frac{1}{N} \sum_{n=0}^{N-1} f_n W^{-np} = F_p$.