Lecture 10

Graham Cooper

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Oscillatory Systems

(Applications of 2nd-order linear DE's) If <u>MechanicalOscillators</u>

- simple pendulum (for small oscillations at least)
- a bouy or boat (floating object in water)
- cantilevers (something fixed at one end ie a building oscillating in the wind)

Spring(mechanical) Oscillator

As a "thought experiment", consider a mass attached to a spring, sliding on a frictionless surface. We will assume that the spring obey's Hooke's Law:
Force from spring = $F_{spring} = -kx$
s
We will also consider the effect of a damping force
$f_{damping} = -c\frac{dx}{dt}$ for $c > 0$
If we consider no other forces, then Newton's 2nd law of motion gives:
$F = ma \rightarrow -kx - c\frac{dx}{dt} = m\frac{d^2x}{dt}$

That is, $x''(t) + \frac{c}{m}x'(t) + \frac{k}{m}x = 0$

We can also consider the effect of an "external" force F(t)

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F(t)$$
$$\rightarrow x'' + \frac{c}{m}x' + \frac{k}{m}x = \frac{F(t)}{m}$$

Traditionally this is rewritten in terms of:

$\omega_0 = \sqrt{frackm}$ The natural frequency
and:
$\zeta = \frac{c}{2\omega_0 m} = \frac{c}{2\sqrt{km}}$ (the damping parameter)
Note: ζ is dimensionless!

$f(t) = \frac{F(t)}{m}$
With these, we have

$$x''(t) + 2\omega_0\zeta x'(t) + \omega_0^2 x(t) = f(t)$$

The Electrical Oscillator

Consider a circuit consisting of a source voltage, a resistor, a capacitor and an inductor in series
(see fig10.1) A variation of Faraday's law states that the voltage drop across the inductor is proportional to the derivative of the current.
$V_L(t) = L\frac{di}{dt}$ (L is called the inductance)
Combining this with our other laws we have:
$V(t) = V_R(t) + V_C(t) + V_L(t)$
$= Ri + \frac{1}{c}q + L\frac{di}{dt}$

So for the current we have:
$V'(t) = R\frac{di}{dt} + \frac{i}{c} + L\frac{d^2i}{dt^2}$
While for the charge we have:
$V(t) = R\frac{dq}{dt} + \frac{1}{c}q + L\frac{d^2q}{dt}$

Now if we let $\omega_0 = \frac{1}{\sqrt{LC}}$, $\zeta = \frac{R}{2\omega_0 L}$, $f(t) = \frac{V(t)}{L}$
We get:

$$q''(t) + 2\omega_0\zeta q'(t) + \omega_0^2 q(t) = f(t)$$

This is the same DE as fo the mechanical oscillator! We can draw an analogy

Compare:

$$mx'' + cx' + kx = F(t)$$

$$Lq'' + Rq' + \frac{1}{c}q = V(t)$$

We can see that displacement corresponds to charge (quantity of interest)
We can see that applied force corresponds to applied voltage (sources of energy)
we can see that spring corresponds to capacitor (storage of energy)
We can see that damper corresponds to resistor (appose the motion within

the system/lose energy))
we can see that mass corresponds to inductance (momentum)

Free Oscillations

With no external force/voltage we have:

$$y'' + 2\omega_0\zeta y' + \omega_0^2 y = 0$$

The characteristic equation is:

$$m^2 + 2\omega_0\zeta m + \omega^2 = 0$$

The roots are:

$$m = \frac{-2\omega_0\zeta \pm \sqrt{4\omega_0^2\zeta^2 - 4\omega_0^2}}{2}$$

$$m = -\omega_0\zeta \pm \omega_0\sqrt{\zeta^2 - 1}$$

Now we can see the reason for using ζ. The critical point between real and complex roots is $\zeta = 1$ ie. $C = 2\sqrt{km}$ or $R = L^{\frac{3}{2}}\sqrt{c}$

Case 1a $\zeta = 0$ (No damping at all)

Here we have:

$$y'' + \omega^2 y = 0$$

so $y = C_1 cos(\omega_0 t) + C_2 sin(\omega_0 t)$
now we see the reason for calling ω_0 the natural frequency:
If $y(0) = 0$ and $y'(0) = 0$ then $0 = c_1$ and $C_2\omega_0 = 0$ so $C_1 = C_2 = 0$ and $y = 0$

If $y(0) = y_0$ and $y'(0) = 0$ (initial displacement), we get:
$y_0 = C_1$ and $0 = C_2$, so $y = y_0 cos(\omega_0 t)$
We call this behaviour simple harmonic motion (SHM)
If $y(0) = 0$ and $y'(0) = v_0$ (initial velocity), we get:
$0 = C_1$ and $V_0 = C_2\omega_0$ so $C_2 = \frac{V_0}{\omega_0}$, and so $y = \frac{V_0}{\omega_0} sin(\omega_0 t)$

If $y_0 = y_0$ and $y'(0) = v_0$ then $y_0 = C_1$ and $v_0 = C_2\omega_0$

$$y = y_0 cos(\omega_0 t) + \frac{V_0}{\omega_0} sin(\omega_0 t)$$

This is still SHM; it can be expressed asa single cosine or sine function

How? Work backwards:

$$cos(A + B) = cos(A)cos(B) - sin(A)sin(B)$$
$$\to Rcos(\omega t - \phi) = R[cos(\omega_0 t)cos(\phi) + sin(\omega_0 t)sin(\phi)]$$
$$= (Rcos(\phi))cos(\omega_0 t) + (Rsin(\phi)sin(\omega_0 t)$$

Now ewe set: $Rcos(\phi) = C_1$ and $Rsin(\phi) = C_2$
Then: $R = \sqrt{C_1^2 + c_2^2}$ and $tan(\phi) = \frac{C_2}{C_1}$

So our general solution $y = C_1 cos(\omega_0 t) + C_2 sin(\omega_0 t)$ can also be expressed as $y = Rcos(\omega_0 t - \phi)$

Expressed as $y = Rcos(\omega_0 t - \phi)$
With this, $y = \sqrt{y_0^2 + \frac{v_0^2}{\omega_0^2}} cos(\omega_0 t - \phi)$ where $tan(\phi) = \frac{v_0}{y_0 \omega_0}$

Case 1b $zeta \epsilon (0, 1)$

In this case we have $m = -\omega_0 \zeta \pm \omega \sqrt{1 - \zeta^2} i$
(We are using: $\pm\sqrt{-a} = \pm ai$) so $y = e^{-\omega_0 \zeta t}[C_1 cos(\omega_0 \sqrt{1 - \zeta^2} t) + C_2 sin(\omega \sqrt{-\zeta^2} t)]$

Or we can write $y = Re^{-\omega_0 \zeta t} cos(\omega_0 \sqrt{1 - \zeta^2} t - \phi)$

The values of R and ϕ will depend on $y(0)$ and $y'(0)$ bein in general we get overdamped behaviour.

Note: The frequency is $\omega_0 \sqrt{1 - \zeta^2} i$ as ζ increases the oscillations slow down!

Case 2 $\zeta \epsilon (1, \infty)$

Here we get $m = -\omega_0 \zeta \pm \omega_0 \sqrt{\zeta^2 - 1}$ and $y = C_1 e^{-\omega_0 (\zeta + \sqrt{\zeta^2 - 1}) t} + C_2 e^{-\omega_0 (\zeta - \sqrt{\zeta^2 - 1}) t}$

We call this overdamped motion

Case 4 $\zeta = 1$

This is critical damping. The graphs look very similar to the overdamped case