Efficient cubic splines

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Start from Hermite formulas:

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

The 2nd derivative is:

$$\begin{aligned} S_i''(x) &= 2c_i + 6d_i(x - x_i) \\ &= 2\frac{(3y_i' - 2s_i - s_{i+1})}{\Delta X_i} + 6\frac{(s_i + s_{i+1} - 2y_i')}{\Delta X_i^2}(X - X_i) \end{aligned}$$

To force matching 2nd derivatives, set $S_i''(X_{i+1}) = S_{i+1}''(X_{i+1})$ for $i = 1$ to $n-2$:

$$S_i''(X_{i+1}) = \frac{2(3y_i' - 2s_i - s_{i+1})}{\Delta X_i} + \frac{6(s_i + s_{i+1} - 2y_i')}{\Delta X_i^{\cancel{2}}}\overbrace{\cancel{(X_{i+1} - X_i)}}^{\Delta X_i}$$

$$S_{i+1}''(X_{i+1}) = \frac{2(3y_{i+1}' - 2s_{i+1} - s_{i+2})}{\Delta X_{i+1}} + \cancel{\frac{6(s_{i+1} + s_{i+2} - 2y_{i+1}')}{(\Delta X_{i+1})^2}\underbrace{(X_{i+1} - X_{i+1})}_{0}}$$

Equating gives:

$$\frac{2s_i + 4s_{i+1} - 6y_i'}{\Delta X_i} = \frac{6y_{i+1}' - 4s_{i+1} - 2s_{i+2}}{\Delta X_{i+1}}$$

Simplifying gives:

$$\Delta X_{i+1}\, s_i + 2(\Delta X_i + \Delta X_{i+1})\, s_{i+1} + \Delta X_i\, s_{i+2} = 3(\Delta X_{i+1}\, y_i' + \Delta X_i\, y_{i+1}')$$

Shift index by 1 to give one equation per interior node for $i = 2$ to $n-1$.

$$\Delta X_i\, s_{i-1} + 2(\Delta X_{i-1} + \Delta X_i)\, s_i + \Delta X_{i-1}\, s_{i+1} = 3(\Delta X_i\, y_{i-1}' + \Delta X_{i-1}\, y_i')$$

Boundary Conditions

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Need 2 more equations, for $i=1$ and n.

1) Clamped (specified slopes): $\left.\begin{array}{l} S_1'(X_1) = s_1 = s_1^* \\ S_{n-1}'(X_n) = s_n = s_n^* \end{array}\right\}$ given

2) Natural/free (zero 2nd derivative):

$$S_1''(x_1) = 0 \quad \text{and} \quad S_{n-1}''(X_n) = 0.$$

$$S_1''(X_1) = 2c_1 + \cancel{6d_1(X_1 - X_1)}^{\to 0} = 0$$

$$\therefore \; C_1 = \frac{3y_1' - 2s_1 - s_2}{\Delta X_1} = 0$$

This simplifies to $s_1 + \frac{1}{2}s_2 = \frac{3}{2}y_1'$.

Likewise for $S_{n-1}''(X_n) = 0$ gives

$$\frac{s_{n-1}}{2} + s_n = \frac{3}{2}y_{n-1}'$$

for the other boundary. (Try as an exercise.)

Example Problem

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Data points: $(0,1), (2,1), (3,3), (4,-1)$ w/ $S_1 = 1$ and $S_4 = -1$.

$$\Delta X_1 = 2 - 0 = 2 \qquad y_1' = \frac{1-1}{2} = 0$$
$$\Delta X_2 = 3 - 2 = 1 \qquad y_2' = 3 - 1 = 2$$
$$\Delta X_3 = 1 \qquad y_3' = -1 - 3 = -4$$

Construct rows $i = 1$ to 4:

$i=1$: $S_1 = 1 \rightarrow T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \quad r_1 = 1$

$i=2$:

$$\Delta X_2 S_1 + 2(\Delta X_2 + \Delta X_1) S_2 + \Delta X_1 S_3 = 3(\Delta X_2 y_1' + \Delta X_1 y_2')$$
$$(1) S_1 + 2(3) S_2 + 2 S_3 = 3(1 \cdot 0 + 2 \cdot 2)$$

$$\therefore T_2 = \begin{bmatrix} 1 & 6 & 2 & 0 \end{bmatrix} \qquad r_2 = 12$$

etc...

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad r = \begin{bmatrix} 1 \\ 12 \\ -6 \\ -1 \end{bmatrix}$$

This is the final system.