LU factorization with pivoting

Find $PA=LU$ factorization of $\begin{bmatrix} 1 & 4 & 5 \\ -2 & 3 & 3 \\ 3 & 0 & 6 \end{bmatrix}$.

Set $P = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to start.

$$\begin{bmatrix} 1 & 4 & 5 \\ -2 & 3 & 3 \\ 3 & 0 & 6 \end{bmatrix} \xrightarrow[P=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}]{\text{Swap R1 \& R3}} \begin{bmatrix} 3 & 0 & 6 \\ -2 & 3 & 3 \\ 1 & 4 & 5 \end{bmatrix} \xrightarrow[R_3 := R_3 - (1/3)R_1]{R_2 := R_2 - (-\frac{2}{3})R_1} \begin{bmatrix} 3 & 0 & 6 \\ \boxed{-2/3} & 3 & 7 \\ \boxed{1/3} & 4 & 3 \end{bmatrix}$$

$$\xrightarrow[P=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}]{\text{Swap } R_2 \,\&\, R_3} \begin{bmatrix} 3 & 0 & 6 \\ \boxed{1/3} & 4 & 3 \\ \boxed{-2/3} & 3 & 7 \end{bmatrix} \xrightarrow{R_3 := R_3 - (\frac{3}{4})R_2} \begin{bmatrix} 3 & 0 & 6 \\ \boxed{1/3} & 4 & 3 \\ \boxed{-2/3} & \boxed{3/4} & \frac{19}{4} \end{bmatrix}$$

Note: factors were swapped too!

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/3 & 3/4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 0 & 6 \\ 0 & 4 & 3 \\ 0 & 0 & 19/4 \end{bmatrix}$$

Now solve $Ax=b$ for x with $b = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$ using $PA=LU$.

Process: Compute Pb,
Solve $Lz = Pb$,
Solve $Ux = z$.
Done.

$$Pb = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

Solve $Lz = Pb$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/3 & 3/4 & 1 \end{bmatrix} z = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \longrightarrow z = \begin{bmatrix} -3 \\ 5 \\ -19/4 \end{bmatrix}$$

Solve $Ux = z$

$$\begin{bmatrix} 3 & 0 & 6 \\ 0 & 4 & 3 \\ 0 & 0 & \frac{19}{4} \end{bmatrix} x = \begin{bmatrix} -3 \\ 5 \\ -19/4 \end{bmatrix} \longrightarrow x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

FLOP count for LU factorization

$$\sum_{k=1}^{n}\sum_{i=k+1}^{n}\sum_{j=k+1}^{n} 2 = 2\sum_{k=1}^{n}\sum_{i=k+1}^{n}\left(n-(k+1)+1\right)$$

$$= 2\sum_{k=1}^{n}\sum_{i=k+1}^{n}(n-k)$$

$$= 2\sum_{k=1}^{n}(n-k)^2$$

$$= 2\sum_{k=1}^{n}\left(n^2-2nk+k^2\right)$$

Use the ~~arithmetic~~ sum identities:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= 2n^3 - 4n\left(\frac{n(n+1)}{2}\right) + 2\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 2n^3 - 2n^3 - 2n^2 + \frac{4n^3+6n^2+2n}{6}$$

$$= \frac{2n^3}{3} - n^2 + \frac{n}{3} = \frac{2n^3}{3} + O(n^2)$$

For an *exact* flop count, we should also add the flops outside the innermost loop. However, in this case they are $O(n^2)$ and don't change the leading order term.