# Solving Inhomogeneous Vector DE's

Graham Cooper

March 20th, 2017

## Solving Inhomogenous Vector DE's

Thre options (at least) for solving 2nd-order linear equations:

1. Method of Undetermined coefficients
2. Variation of Parameters
3. Laplace transform

All of these can be adopted for vector DE's

## 1 Method of Undetermined coefficients

This is the easiest method for very simple forcing terms $\overrightarrow{f}(t)$ (eg constants or forms like: $\begin{bmatrix} k_1e^{at} \\ k_2e^{at} \end{bmatrix}$)
It gets more difficult when $\overrightarrow{f}(t)$ is more complicated

### Example1

Solve:
$\overrightarrow{x} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \overrightarrow{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
Solution: Solve $\overrightarrow{x}' = A\overrightarrow{x}$ first $\overrightarrow{x_h} = C_1e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

To find a particular solution we guess that:
$\overrightarrow{x_p} = \begin{bmatrix} A \\ B \end{bmatrix}$

Plug this into the DE:
$\overrightarrow{0} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

ie:
$A + B = -2$ and $2B = -1$ so $B = \frac{-1}{2}$ and $A = \frac{3}{2}$

$$\rightarrow \overrightarrow{x}(t) = \overrightarrow{x_h} + \overrightarrow{x_p} = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{-3}{2} \\ \frac{-1}{2} \end{bmatrix}$$

## 2) Variation of Parameters

Consider the equation: $\overrightarrow{x}' = A\overrightarrow{x} + \overrightarrow{f}(t)$
If hte solution to the homogeneous equation $\overrightarrow{x}' = A\overrightarrow{x}$ is $\overrightarrow{x_h} = C_1\overrightarrow{x_1} + C_2\overrightarrow{x_2}$ then we assume that the solution can be expressed as: $\overrightarrow{x} = u_1(t)\overrightarrow{x_1}(t) + u_2(t)\overrightarrow{x_2}(t)$

Differentiating: $\overrightarrow{x} = u_1\overrightarrow{x_1} + u_2\overrightarrow{x_2}$
Gives: $\overrightarrow{x}' = u_1'\overrightarrow{x_1} + u_1\overrightarrow{x}' + u_2'\overrightarrow{x_2} + u_2\overrightarrow{x_2}'$ so:
$\overrightarrow{x}' = A\overrightarrow{x} + \overrightarrow{f}(t) \rightarrow$
$u_1'\overrightarrow{x_1} + u_1'\overrightarrow{x_1}' + u_2'\overrightarrow{x_2} + u_2\overrightarrow{x_2}' = Au_1\overrightarrow{x_1} + Au_2\overrightarrow{x_2} + A\overrightarrow{f}$

Now by assumption: $\overrightarrow{x_1}' = A\overrightarrow{x}$ and $\overrightarrow{x_2}' = A\overrightarrow{x_2}$ so $u_1'\overrightarrow{x_1} + u_2'\overrightarrow{x_2} = A\overrightarrow{f}$

In component form:

$u_1'x_{11} + u_2'x_{21} = Af_1$
$u_1'x_{12} + u_2'x_{22} = Af_2$

We will always be able to solve this for $u_1'$ and $u_2'$ because $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ are linearly independent functions, so the matrix $\begin{bmatrix} \overrightarrow{x_1} & \overrightarrow{x_2} \end{bmatrix}$ is invertible.

Integrating gives $u_1$ and $u_2$ and hence $\overrightarrow{x}$

**Example:**

Solve $\overrightarrow{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \overrightarrow{x} + \begin{bmatrix} t \\ e_t \end{bmatrix}$
Given that $\overrightarrow{x_h} = C_1 \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + C_2 \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}$

Note: to use the method of undetermined coefficients, we would guess:
$\overrightarrow{x_p} = \begin{bmatrix} A_1t + B_1 + C_1e^t \\ A_2t + B_2 + C_2e^t \end{bmatrix}$

Using var. of parameters we set $\overrightarrow{x} = u_1\overrightarrow{x_1} + u_2\overrightarrow{x_2}$ and sove $u_1\overrightarrow{x_1} + u_2\overrightarrow{x_2} = \overrightarrow{f}$

ie:
$u_1' \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + u_2 \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} = \begin{bmatrix} t \\ e_t \end{bmatrix}$
$u_1'e^{3t} + u_2'e^{-t} = t$ 1)
$2u_1'e^{3t} - 2u_2'e^{-t} = e^t$ 2)

$2 * 1) + 2)$ gives $4u_1'e^{3t} = 2t + e^t$
$2 * 1) - 2)$ gives $4u_2'e^{-t} = t - e^t$

$\rightarrow u_1' = \frac{1}{2}te^{-3t} - \frac{1}{4}e^{-2t}$
$\rightarrow u_1(t) = \int \frac{1}{2}t^{-3t}dt - \int \frac{1}{4}e^{-2t}dt$
let $u = t$ $dv = e^{-3t}dt$ and $du = dt$ $v = \frac{-1}{3}e^{-3t}$
$= \frac{1}{2}[\frac{-1}{3}te^{-3t} + \int \frac{1}{3}e^{-3t}dt] - \frac{1}{4}\int e^{-2t}dt$
$= \frac{-1}{6}te^{-3t} - \frac{1}{18}e^{-3t} + \frac{1}{8}e^{-2t} + C_1$

Meanwhile $u_2' = \frac{1}{2}te^{t} - \frac{1}{4}e^{2t}$
$\rightarrow u_2 = \ldots$
$= \frac{1}{2}te^t - \frac{1}{2}e^t - \frac{1}{8}e^{2t} + C_2$

Therefore $\overrightarrow{x} = u_1\overrightarrow{x_1} + u_2\overrightarrow{x_2}$
$= (\frac{-1}{6}te^{-3t} - \frac{1}{18}e^{-3t} + \frac{1}{8}e^{-2t} + C_1) \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + (\frac{1}{2}te^t - \frac{1}{2}e^t - \frac{1}{8}e^{2t} + C_2) \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$
$= \ldots$ SImplifications
$= c_1e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}t - \frac{5}{9} - \frac{1}{4}e^t \\ \frac{-4}{3}t + \frac{8}{9} \end{bmatrix}$

# The Fundamental Matrix

There is a more convenient way to express general solutions to vector DEs using this. Definition The fundamental matrix of a DE $\overrightarrow{x}' = A\overrightarrow{x}$ is the matrix $\Phi(t)$ such that $\overrightarrow{x}(t) = \Phi(t)\overrightarrow{x}(0)$ That is: $\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

We can always construct this matrix from the general solution; notice that the tsolution to the special case $\overrightarrow{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $\overrightarrow{x}(t) = \begin{bmatrix} \Phi_{11}(t) \\ \Phi_{12}(t) \end{bmatrix}$

The solution to teh case $\overrightarrow{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\overrightarrow{x}(t) = \begin{bmatrix} \Phi_{12}(t) \\ \Phi_{22}(t) \end{bmatrix}$

## Example

The DE $\overrightarrow{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \overrightarrow{x}$ has solution: