FFT derivation

One DFT can be written as two (shorter) DFT's.

$$F_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n W^{-nk} = \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} f_n W^{-nk} + \frac{1}{N}\sum_{n=\frac{N}{2}}^{N-1} f_n W^{-nk}$$

Reindex 2nd sum using $m = n - \frac{N}{2}$:

$$F_k = \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} f_n W^{-nk} + \frac{1}{N}\sum_{m=\frac{N}{2}}^{\frac{N}{2}-1} f_{m+\frac{N}{2}} W^{-k(m+\frac{N}{2})}$$

(rename $m \to n$)

$$= \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} f_n W^{-nk} + \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} f_{n+\frac{N}{2}} W^{-nk} W^{-\frac{kN}{2}}$$

Combine the sums

$$= \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} \Big(f_n + f_{n+\frac{N}{2}} \underbrace{W^{-\frac{kN}{2}}}\Big) W^{-nk}$$

observe: $W^{-\frac{kN}{2}} = e^{\left(\frac{2\pi i}{N}\right)\left(-\frac{kN}{2}\right)} = e^{-k\pi i} = (-1)^k$

Hence two cases to treat:
k=even $(+1)$ and k=odd (-1).

Even: $$F_{2k} = \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} \left(f_n + f_{n+\frac{N}{2}}\right) W^{-2nk} \quad \text{for } k=0 \ldots \tfrac{N}{2}-1$$

Odd: $$F_{2k+1} = \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} \left(f_n - f_{n+\frac{N}{2}}\right) W^{-n(2k+1)}$$

$$= \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} \left[\left(f_n - f_{n+\frac{N}{2}}\right) W^{-n}\right] W^{-2nk} \quad \text{for } k=0 \ldots \tfrac{N}{2}-1$$

Next define two new vectors,

$$g_n = \frac{1}{2}\left(f_n + f_{n+\frac{N}{2}}\right) \quad , \quad h_n = \frac{1}{2}\left(f_n - f_{n+\frac{N}{2}}\right) W^{-n}$$

for $n = 0 \ldots \frac{N}{2} - 1$, and let $M = \frac{N}{2}$.

Then we observe

$$F_{2k} = \frac{2}{N} \sum_{n=0}^{\frac{N}{2}-1} g_n W_N^{-2nk} = \frac{1}{M} \sum_{n=0}^{M-1} g_n W_M^{-nk} = G_k$$

$$F_{2k} = \frac{2}{N} \sum_{n=0}^{\frac{N}{2}-1} h_n W^{-2nk} = \frac{1}{M} \sum_{n=0}^{M-1} h_n W_M^{-nk} = H_k$$

where we defined $W_N = e^{\frac{2\pi i}{N}}$ and $W_M = e^{\frac{2\pi i}{M}}$

so that $W_M = W_N^2$ (since $N = 2M$.)

Thuse we have composed the entries of F out of DFTs of 2 half-length vectors g and h.