Amath 250 Lecture 3

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The Existence/Uniqueness theorem (for 1st order IVPs)

The IVP:

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0$$

Has a unique solutino provided that both f(x,y) and $f_y(x,y)$ are continuous within a neighbourhood of (x_0, y_0)

Corollary: If f and f_y are cts in a region of the xy-plane, then the various solution curves of $y' = f(x,y)$ will not intersect in that region.

Sketching Families of Solutions

If you have problems that are too difficult - you can look at the derivitive. Using the DE we already have info about the derivitive and we can use that info directly.

We can often tell a lot about he graphs of the solutions from the DE itself.

Example 1: Consider the equation:
$\frac{dy}{dx} = y^2 - 4$

If we set the $\frac{dy}{dx} = 0$ we get $y = +/-2$
This means that:

- the constant functions $y = +/-2$ are solutions (They are called "equilibrium solutions)
- none of the other solutions have any critical points
- $\frac{dy}{dx} > 0$ iff $|y| > 2$ and $\frac{dy}{dx} < 0$ iff $|y| < 2$
- in fact, $\frac{dy}{dx} \neq 0$ when $y \approx +/-2$, and $|\frac{dy}{dx}|$ increases with distance from $y = +/-2$

Inflection points?
If $\frac{dy}{dx} = y^2 - 4$ then $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} = 2y(y^2 - 4)$
There is a line of inlection points along the x-axis

Of course we <u>can</u> solve this DE.

$$\frac{dy}{dx} = y^2 - 4$$
$$\int \frac{dy}{y^2-4} = \int dx$$
$$\frac{1}{4}\int (\frac{1}{y-2} - \frac{1}{y+2})dy = x + c_1$$
$$ln|y-2| - ln|y+2| = 4x + 4c_1$$
$$ln|\frac{y-2}{y+2} = 4x + c_2$$
$$|\frac{y-2}{y+2}| = e^{4x+c_2}$$
$$= e^{c_2}e^{4x}$$
$$\frac{y-2}{y+2} = c_3e^{4x}$$
$$y - 2 = c_3(y+2)e^{4x}$$
$$y(1 - c_3e^{4x}) = 2 + 2c_3e^{4x}$$
$$y = 2(\frac{1+Ce^{4x}}{1-Ce^{4x}})$$

Partial fraction decomposition for above:

$$\frac{1}{y^2-4} = \frac{A}{y+2} + \frac{B}{y-2}$$
$$1 = A(y-2) + B(y+2)$$

Setting $y = 2$ gives $B = \frac{1}{4}$
Setting $y = -2$ gives $A = -\frac{1}{4}$

The solution will have verticle asymptotes when:

$$1 - Ce^{4x} = 0$$

ie $x = \frac{1}{4}ln(\frac{1}{c}) = -\frac{1}{4}lnC$
(see figure 3.1)

Guidelines for sketching

- solve the DE if possible
- Identify any "exceptional" solutions. These may be singular solutions or equilibrium solutions, or other. for linear equations, setting $C = 0$ will usually yield an exceptional solution
- Consider the behaviour of other solutions as $x \to \pm\infty$ or near vertical asymptotes
- For more detail, find out where $y' = 0$. This will give you a curve, called the horizontal isocline, on which every solution to the DE has a horizontal tangent
- (optional: do the same with y'')

Example:
Consider the DE $\frac{dx}{dy} = y - x^2$
In standard form: $y' - y = -x^2$
Int. Factor: $I(x) = e^{\int K(x)dx} = e^{-\int dx}$
Factor this in:

$$e^{-x}\frac{dy}{dx} - ye^{-x} = -x^2e6{-}x$$

$$\frac{d}{dx}ye^{-x} = -x^2e^{-x}$$

$$ye^{-x} = -\int x^2e^{-x}dx$$

$$= ... = e^{-x}(x^2 + 2x + 2) + C$$

Need to take double integral above, but we skip that step

$$y = x^2 2x + 2 + Ce^x$$

Exceptional solutions? $C = 0$ gives a parabola: $y = x^2 + 2x + 2 = (x+1)^2 + 1$

(See figure 3.2)

How do the other solutions relate to this?
Well, $Ce^x \to 0$ as $x \to -\infty$, so every solution approaches the parabola asymptotically as $x \to -\infty$.
As $x \to \infty$, $Ce^x \to \pm\infty$ (Depending on the sign of C), so the olutions diverge from the parabola

Critical points? (horizontal isocline?)

$$\frac{dy}{dx} = 0 \to y = x^2$$

(see figure 3.3) the red is the isocline and that is where points will change direction/where there will be critical points.

Another Sketching Example:
Consider $\frac{dy}{dx} = (xy)^{\frac{2}{3}}$
Solve? This is seperable:

$$\frac{dy}{y^{\frac{2}{3}}} = x^{\frac{2}{3}} dx$$

$$\to \int y^{\frac{-2}{3}} dy = \int x^{\frac{2}{3}} dx$$

$$\to 3y^{\frac{1}{3}} = \frac{3}{5} x^{\frac{5}{3}} + c_1$$

$$\to y = (\frac{1}{5} x^{\frac{5}{3}} + c)^3$$

Exceptional solutions? $C = 0$ gives $y = \frac{1}{125} x^5$
$y = 0$ is also a solution (it's singular)
See figure 3.4
Notice that the two lines overlap.

$f(x,y) = (xy)^{\frac{2}{3}}$ is the cts on $\Re^2$, but $f_y(x,y) = \frac{2}{3}x^{\frac{2}{3}}y^{\frac{1}{3}} = \frac{2x^{\frac{2}{3}}}{3y^{\frac{1}{3}}}$ is not cts when $y = 0$

Other solutions ($C \neq 0$) should start to resemble the $c = 0$ solution as $x \to \pm\infty$. Isocline?
$\frac{dy}{dx} = 0 \iff x = 0$ or $y = 0$