Amath 250 Lecture 2

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<u>Seperable Equations</u>

A Differential equation $\frac{dy}{dx} = f(x, y)$ said to be <u>seperable</u> if $f(x)$ can be factored as $f(x, y) = g(x)h(y)$

In these cases we have $\frac{dy}{dx} = g(x)h(y)$, and dividing by $h(y)$ gives:

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

Integrating both sides with respect to x gives

$$\int \frac{1}{h(y)}\frac{dy}{dx}dx = \int g(x)dx$$

That is:

$$\int \frac{1}{h(y)}dy = \int g(x)dx$$

If we can find antiderivatives and solve for y, we'll have our solution.
Aside (substitution?):

$$\int \frac{1}{h(y)}\frac{dy}{dx}$$

let u = y $du = \frac{dy}{dx}dx$

$$= \int \frac{1}{h(u)}du$$

$$= \int \frac{h(y)}{d}y$$

Example: solve $\frac{dy}{dx} = e^{x+y}$

This is $\frac{dy}{dx} = d^x e^y$
Trating dy and dx as differentials and seperating the variables we get $\frac{dy}{e^y} = e^x dx$

Summing both sides, we have $\int e^{-y}dy = \int e^{y}dx$

Integrate:

$$-e^{-y} = e^{x} + c$$

sovle for y:

$$e^{-y} = -e^{x} + c_1 (c_1 = -c)$$
$$-y = ln(c_1 - e^{x})$$
$$y = -ln(c_1 - e^{x})$$
$$= ln[\frac{1}{c_1 - e^{x}}]$$

Check our answer?:

$$y = ln[\frac{1}{c_1 - e^{x}}]$$
$$(c_1 - e^{x})[-(\frac{1}{c_1 - e^{x}})]^2(-e)^x = \frac{e^{x}}{c_1 - e^{x}}$$
$$e^{x+y} = e^{x}e^{y} = e^{x}[\frac{1}{c_1 - e^{x}}] = \frac{e^{x}}{c_1 - e^{x}} = \frac{dy}{dx}$$

What if we need the particular solution passing through (0,0)?
From $-e^{-y} = e^{x} + c$, setting x = y = 0 gives $c = -2$ so $c_1 = 2$, and $y = ln[\frac{1}{2-e^{x}}]$
You will always have multiple curves, we will want to find the curve that solves for the points given. See image 2.1, the red line is the correct one.

One problem to watch for

When seperating variables, we may losee certain "singular" solutions.
Eg. Consider: $\frac{dy}{dx} = -4xy^2$
We may write (if $y! = 0$)

$$\int \frac{dy}{y^2} = -\int 4xdx$$

$$\frac{-1}{y} = -2x^2 + c$$

$$y = \frac{1}{2x^2 - c}$$

This is the general solution, and yet y = 0 is also a solution

First-Order Linear Equations

A first-order Differential equation is said to be linear if it is of the form $a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$

To motivate our method of solution, consider the special case where $a_0x = a_1'(x)$.

$$a_1y' + a_1'y = f(x)$$

We recognize the LHS as $\frac{d}{dx}[a_1(x)y(x)]$

$$\frac{d}{dx}[a_1y] = f(x)$$

$$a_1y = \int f(x)dx$$

$$y(x) = \frac{1}{a_1(x)}\int f(x)dx$$

What if $a_0 \neq a_1'$? We can actually create this structure! We'll multiply through by another function I(x) (an integrating factor).

Step 1 Divide trhough by $a_1(x)$ to get:

$$\frac{dy}{dx} + k(x)y = g(x)$$

Where $k = \frac{a_0}{a_1}$ and $g = \frac{f}{a_1}$

This is referred to as the standard form of a linear first-order differential equation

Step 2 Multiply through by the (unknown) factor I(x)

$$I(x)\frac{dy}{dx} + I(x)k(x)y = I(x)g(x)$$

Now we want $I(x)k(x)$ to be equal to $\frac{dI}{dx}$
ie:

$$\frac{dI}{dx} = KI$$
$$\int \frac{dI}{I} = \int K(x)dx$$
$$ln|I| = \int K(x)dx$$
$$|I| = e^{\int K(x)dx}$$
$$I = e^{\int K(x)dx}$$

(aside) $e^{H(x)+c} = e^c e^{H(x)}$ so we can remove the absolutes?
We now have:

$$\frac{d}{dx}[Iy] = I(x)g(x)$$
$$y = \frac{1}{I(x)} \int I(x)g(x)dx$$

where $I = e^{\int k(x)dx}$

<u>Jan 9th:</u>

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

1. Put the equation in standard form: $\frac{dy}{dx} + k(x)y = g(x)$

2. Find the integrating factor
   $I(x) = e^{\int k(x)dx}$
   $= e^{K(x)+C}$
   $= e^c e^{K(x)}$

3. Factor this into the equation, and recall that the LHS must be $\frac{d}{dx}(I(x)y(x))$
   $I(x)\frac{dy}{dx} + I(x)k(x)y = I(x)g(x)$
   $\frac{d}{dx}(Iy) = I(x)g(x)$

examples:

1.

$\frac{dy}{dx} + xy = x$ (already in standard form)

An integrating factor is $I(x) = e^{\int k(x)dx} = e^{\int xdx} = e^{\frac{1}{2}x^2}$

Incorporating this, we have:

$$e^{\frac{1}{2}x^2}\frac{dy}{dx} + xe^{\frac{1}{2}x^2}y = xe^{\frac{1}{2}x^2}$$

$$\frac{d}{dx}(e^{\frac{1}{2}x^2}y) = xe^{\frac{1}{2}x^2}$$

Make sure we have the correct integrating factor, make sure that the above is the derivative of the line below. CHECK THIS ALWAYS.

$$e^{\frac{1}{2}x^2}y = \int xe^{\frac{1}{2}x^2}dx$$

$$= e^{\frac{1}{2}x^2} + c$$

$$=> y = 1 + Ce^{-\frac{1}{2}x^2}$$

2.

$$xlnx\frac{dy}{dx} + y - x^3lnx = 0$$

(assume $x > 1$)

In standard form:

$$\frac{dy}{dx} + \frac{y}{xlnx} = x^2$$

Int. Factor: $I(x) = e^{\int \frac{1}{xlnx}dx}$

$= |lnx| = lnx$(since x > 1) (see below how we found this)

$$\int \frac{1}{xlnx}dx$$

$$= \int \frac{1}{u}du$$

$$= ln|u| + c$$

$$= ln|lnx| + c$$

$$lnx\frac{dy}{dx} + \frac{y}{x} = x^2 lnx$$

$$\frac{d}{dx}((lnx)y) = x^2 lnx$$

Check?

$$ylnx = \int x^2 lnx dx$$

$$ylnx = uv - \int vdu$$

$$= \frac{1}{3}x^3 lnx - \int \frac{x^2}{3}dx$$

$$= \frac{1}{3}x^3 lnx = \frac{x^3}{9} + C$$

$$y = \frac{1}{3}x^3 - \frac{x^3}{9lnx} + \frac{c}{lnx}$$

where above, $u = lnx$ $du = \frac{1}{x}dx$ $dv = x^2 dx$ $v = \frac{1}{3}x^3$