

Amath 250 Lecture 3

Graham Cooper

January 9th, 2017

The Existence/Uniqueness theorem (for 1st order IVPs)

The IVP:

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

Has a unique solution provided that both $f(x, y)$ and $f_y(x, y)$ are continuous within a neighbourhood of (x_0, y_0)

Corollary: If f and f_y are cts in a region of the xy -plane, then the various solution curves of $y' = f(x, y)$ will not intersect in that region.

Sketching Families of Solutions

If you have problems that are too difficult - you can look at the derivative. Using the DE we already have info about the derivative and we can use that info directly.

We can often tell a lot about the graphs of the solutions from the DE itself.

Example 1: Consider the equation:

$$\frac{dy}{dx} = y^2 - 4$$

If we set the $\frac{dy}{dx} = 0$ we get $y = +/ - 2$

This means that:

- the constant functions $y = +/ - 2$ are solutions (They are called "equilibrium solutions")
- none of the other solutions have any critical points
- $\frac{dy}{dx} > 0$ iff $|y| > 2$ and $\frac{dy}{dx} < 0$ iff $|y| < 2$
- in fact, $\frac{dy}{dx} \neq 0$ when $y \approx +/ - 2$, and $|\frac{dy}{dx}|$ increases with distance from $y = +/ - 2$

Inflection points?

If $\frac{dy}{dx} = y^2 - 4$ then $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} = 2y(y^2 - 4)$

There is a line of inflection points along the x-axis

Of course we can solve this DE.

$$\frac{dy}{dx} = y^2 - 4$$

$$\int \frac{dy}{y^2 - 4} = \int dx$$

$$\frac{1}{4} \int \left(\frac{1}{y-2} - \frac{1}{y+2} \right) dy = x + c_1$$

$$\ln|y-2| - \ln|y+2| = 4x + 4c_1$$

$$\ln \left| \frac{y-2}{y+2} \right| = 4x + c_2$$

$$\left| \frac{y-2}{y+2} \right| = e^{4x+c_2}$$

$$= e^{c_2} e^{4x}$$

$$\frac{y-2}{y+2} = c_3 e^{4x}$$

$$y-2 = c_3(y+2)e^{4x}$$

$$y(1 - c_3 e^{4x}) = 2 + 2c_3 e^{4x}$$

$$y = 2 \left(\frac{1 + C e^{4x}}{1 - C e^{4x}} \right)$$

Partial fraction decomposition for above:

$$\frac{1}{y^2 - 4} = \frac{A}{y+2} + \frac{B}{y-2}$$

$$1 = A(y-2) + B(y+2)$$

Setting $y = 2$ gives $B = \frac{1}{4}$

Setting $y = -2$ gives $A = -\frac{1}{4}$

The solution will have vertical asymptotes when:

$$1 - Ce^{4x} = 0$$

ie $x = \frac{1}{4}\ln(\frac{1}{C}) = -\frac{1}{4}\ln C$
(see figure 3.1)

Guidelines for sketching

- solve the DE if possible
- Identify any "exceptional" solutions. These may be singular solutions or equilibrium solutions, or other. for linear equations, setting $C = 0$ will usually yield an exceptional solution
- Consider the behaviour of other solutions as $x \rightarrow \pm\infty$ or near vertical asymptotes
- For more detail, find out where $y' = 0$. This will give you a curve, called the horizontal isocline, on which every solution to the DE has a horizontal tangent
- (optional: do the same with y'')

Example:

Consider the DE $\frac{dy}{dx} = y - x^2$

In standard form: $y' - y = -x^2$

Int. Factor: $I(x) = e^{\int K(x)dx} = e^{-\int dx}$

Factor this in:

$$e^{-x} \frac{dy}{dx} - ye^{-x} = -x^2 e^{-x}$$

$$\frac{d}{dx} ye^{-x} = -x^2 e^{-x}$$

$$ye^{-x} = - \int x^2 e^{-x} dx$$

$$= \dots = e^{-x}(x^2 + 2x + 2) + C$$

Need to take double integral above, but we skip that step

$$y = x^2 + 2x + 2 + Ce^x$$

Exceptional solutions? $C = 0$ gives a parabola: $y = x^2 + 2x + 2 = (x+1)^2 + 1$

(See figure 3.2)

How do the other solutions relate to this?

Well, $Ce^x \rightarrow 0$ as $x \rightarrow -\infty$, so every solution approaches the parabola asymptotically as $x \rightarrow -\infty$.

As $x \rightarrow \infty$, $Ce^x \rightarrow \pm\infty$ (Depending on the sign of C), so the solutions diverge from the parabola

Critical points? (horizontal isocline?)

$$\frac{dy}{dx} = 0 \rightarrow y = x^2$$

(see figure 3.3) the red is the isocline and that is where points will change direction/where there will be critical points.