Lecture

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Missed a class in here...

Solving Homogeneous Vector DEs Continued

Case3: Repeated Eigenvalues

Example:
$\overrightarrow{x}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \overrightarrow{x}$

$det(A - \lambda I) = det(\begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix})$
$= (4-\lambda)(2-\lambda) + 1$
$= \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$
So $\lambda = 3$

Setting $(A - \lambda I)\overrightarrow{V} = \overrightarrow{0}$ gives
$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $v_1 + v_2 = 0$ and we may use $\overrightarrow{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\rightarrow$ One solution is $\overrightarrow{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$

What's a second one? Try multipying by t?

IF $\overrightarrow{x_1} = \overrightarrow{v} e^{\lambda t}$ then we guess:
$\overrightarrow{x_2} = \overrightarrow{v} te^{\lambda t}$?
This gives $\overrightarrow{x_2}' = \overrightarrow{v} e^{\lambda t} + \lambda \overrightarrow{v} te^{\lambda t}$
ie. $\overrightarrow{v} e^{\lambda t} = \overrightarrow{0}$ which is not true!

What else could we try?

Consider: we have guessed: $\overrightarrow{x_2} = \overrightarrow{v} te^{\lambda t} = \begin{bmatrix} v_1 te^{\lambda t} \\ v_2 te^{\lambda t} \end{bmatrix}$
So:

$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} = \begin{bmatrix} c_1v_1e^{\lambda t} + c_2v_1te^{\lambda t} \\ c_1v_2e^{\lambda t} + c_2v_2te^{\lambda t} \end{bmatrix}$

We <u>want</u>
$\overrightarrow{x} = \begin{bmatrix} \alpha_1e^{\lambda t} + \alpha_2te^{\lambda t} \\ \alpha_3e^{\lambda t} + \alpha_4te^{\lambda t} \end{bmatrix}$
Where α_{1-4} are related in some way so that there are only two arbitrary constants

Our guess is too restrictive - it requires a particular relationship between fourc oncstants, one wayt to relax this restriction is this:
For $\overrightarrow{x_2}$, we guess $\overrightarrow{x_2} = \overrightarrow{v}te^{\lambda t} + \overrightarrow{u}e^{\lambda t}$ where $\overrightarrow{u}$ is a vector to be determined

(This way $\overrightarrow{x}(t) = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} = \begin{bmatrix} (c_1v_1 + c_2u_1)e^{\lambda t} + c_2v_1te^{\lambda t} \\ (c_1v_2 + c_2u_2)e^{\lambda t} + v_2v_2te^{\lambda t} \end{bmatrix}$
What is $\overrightarrow{u}$?

Plug $\overrightarrow{x_2}$ into the DE: $\overrightarrow{x}' = A\overrightarrow{x}$ becomes
$\overrightarrow{v}e^{\lambda t} + \lambda\overrightarrow{v}te^{\lambda t} = A\overrightarrow{v}te^{\lambda t} + A\overrightarrow{u}e^{\lambda t}$
$\rightarrow A\overrightarrow{u} - \lambda\overrightarrow{u} = \overrightarrow{v}$
$\rightarrow (A - \lambda I)\overrightarrow{(}u) = \overrightarrow{v}$

We can solve this for $\overrightarrow{u}$ In fact, $\overrightarrow{u}$ is called a generalized eigenvector. Note that $(A - \lambda I)^2\overrightarrow{u} = (A - \lambda I)\overrightarrow{v} = \overrightarrow{0}$
Conclusion: The guess $\overrightarrow{x_2} = \overrightarrow{v}te^{\lambda t} + \overrightarrow{u}e^{\lambda t}$ works for $\overrightarrow{u}$ is a generalized eigenvector (of order 2)

Back to the example....

We had $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$
$(A - \lambda I) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$
To find $\overrightarrow{u}$ solve $(A0\lambda I)\overrightarrow{u} = \overrightarrow{v}$
ie:
$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$u_1 + u_2 = 1$

Setting $u_2 = 0$ gives $u_1 = 1$ so $\overrightarrow{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ So... $\overrightarrow{x_1} = \overrightarrow{v} e^{\lambda t}$
$\overrightarrow{x_2} = \overrightarrow{v} t e^{\lambda t} + \overrightarrow{u} e^{\lambda t}$

$\rightarrow$ the general solution is:

$$\begin{aligned} \overrightarrow{x} &= c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2} \\ &= c_1 \overrightarrow{v} e^{\lambda t} + c_2 [\overrightarrow{v} t e^{\lambda t} + \overrightarrow{u} e^{\lambda t}] \\ &= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 [\begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} \\ &= \begin{bmatrix} (c_1 + c_2) e^{3t} + c_2 t e^{3t} \\ -c_1 e^{3t} - c_2 t e^{3t} \end{bmatrix} \end{aligned}$$