Fourier Transforms – Discrete Fourier Transform Derivation

CS370 Lecture 19 – February 15, 2017

Discrete Fourier Transform

To be useful, the operations we need are:

1. Convert time-domain data f_n to frequency-domain F_k.
2. Convert frequency-domain data F_k to time-domain f_n.

In many applications, we do (1), perform some processing on F_k, then perform (2) to get back new modified data.

So far we can do (2).

[Inverse] Discrete Fourier Transform

Last day, we derived the expression for what is actually the ***inverse*** discrete Fourier transform:

Given the discrete Fourier coefficients F_k, the data f_n are recovered as:

$$f_n = \sum_{k=0}^{N-1} F_k W^{nk} \quad \text{for} \quad W = e^{2\pi i/N}.$$

Today we'll (1) derive the (forward) discrete Fourier transform to find F_k and (2) time permitting, work through an example of inverse DFT.

Another orthogonality identity

To find the Fourier coefficients, we'll need another useful property of our Nth roots of unity:

$$\sum_{j=0}^{N-1} W^{jk}\, W^{-jl} = \sum_{j=0}^{N-1} W^{j(k-l)} = N\delta_{k,l}$$

assuming (for now) that $k, l \in [0, N-1]$.

The symbol $\delta_{k,l}$ indicates the Kronecker delta satisfying:

$$\delta_{k,l} = \begin{cases} 0; & k \neq l \\ 1; & k = l \end{cases}$$

Discrete Fourier Transform

This identity will allow us to work out the reverse direction (f_n to F_k).

The approach is similar to how we found the continuous Fourier series coefficients c_k for continuous functions.

We will work through this derivation, and then consider an example problem.

Derivation!

A Discrete Fourier Transform (DFT) pair

We can now convert any data set into its Fourier coefficients, and back.

Inverse DFT:

$$f_n = \sum_{k=0}^{N-1} F_k W^{nk}$$

DFT:

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n W^{-nk} \,.$$

The discrete Fourier transform is **invertible**.

IDFT Definition: $f_n = \sum_{k=0}^{N-1} F_k W^{nk}$ for $W = e^{2\pi i/N}$

Example Problem

Given 4 discrete Fourier coefficients F_k, find the corresponding 4 data points f_n.

Let $F = \begin{pmatrix} -2 \\ 2+i \\ -2 \\ 2-i \end{pmatrix}$. What is the vector f?

Equivalently:

If $F_0 = -2, F_1 = 2+i, F_2 = -2, F_3 = 2-i$, with $N = 4$, find f_0, f_1, f_2, and f_3.

Next time…

Next day, we'll:

- look at some additional properties of the DFT.
- try to develop more intuition for the meaning of Fourier coefficients of discrete data.