Aliasing

Our Fourier series of a continuous signal was

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{\frac{i2\pi tk}{T}} \text{ for a period } T.$$

If we sample at discrete points $t_n = n\Delta t = nT/N$ then

$$f_n = f(t_n) = \sum_{k=-\infty}^{\infty} C_k e^{\frac{i2\pi kn}{N}} = \sum_{k=-\infty}^{\infty} C_k W^{nk}.$$ This is exact

for arbitrarily high frequencies. $\therefore$ C_k are the exact Fourier coefficients for f, since we consider $k \in (-\infty, \infty)$.

But, IDFT uses the approximation

$$f_n = \sum_{k=-\frac{N}{2}+1}^{N/2} F_k W^{nk}, \text{ ie. } k \in \left[-\frac{N}{2}+1, \frac{N}{2}\right].$$

[Same as $[0, N-1]$ by periodicity of F_k.]

We have only N F_k coefficients.

How do C_k and F_k relate?

The DFT gives

$$F_\ell = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n W^{-n\ell}.$$

We plug in our exact f_n expansion in terms of C_k, to find a relation with F_k.

$$F_\ell = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} W^{-n\ell} \sum_{k=-\infty}^{\infty} C_k W^{nk} = \sum_{k=-\infty}^{\infty} \frac{C_k}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} W^{n(k-\ell)}$$

The $W^{n(k-\ell)}$ term hints at using orthogonality. But, we need to adapt it for $k \in (-\infty, \infty)$ instead of $[0, N-1]$.

Identity is:

$$\sum_{p=-\frac{N}{2}+1}^{\frac{N}{2}} W^{p(k-\ell)} = N(\delta_{k,\ell} + \delta_{k,\ell+N} + \delta_{k,\ell-N} + \delta_{k,\ell+2N} + \delta_{k,\ell-2N} + \dots)$$

Why? Since if $\ell = k + jN$ (i.e. $\ell \equiv k \pmod N$)

then $W^{p(\ell-k)} = W^{pjN} = 1$ since W is an nth root of unity.

Using this identity, we have

$$F_\ell = \sum_{k=-\infty}^{\infty} \frac{c_k}{\cancel{N}} \cancel{N} (\delta_{k,\ell} + \delta_{k,\ell+N} + \dots) = c_\ell + c_{\ell+N} + c_{\ell-N} + c_{\ell+2N} + c_{\ell-2N} + \dots$$

$\therefore$ The DFT coefficients F_k are sums of true Fourier series coeffs c_k including arbitrary high frequencies.

Actual high frequencies c_k for $k \notin [\frac{N}{2}-1, N/2]$ contribute to ("alias as") low frequencies F_k for $k \in [\frac{N}{2}-1, \frac{N}{2}]$.

Finding Correlation via FFT

Definitions:

Correlation vector $\phi_n = \frac{1}{N} \sum_{l=0}^{N-1} y_{l+n} z_l$ ①

FFT of input satisfies (by def)

$y_{l+n} = \sum_{m=0}^{N-1} Y_m W^{m(l+n)}$ ② $z_l = \sum_{r=0}^{N-1} Z_r W^{rl}$ ③

where $Z = FFT(z)$, $Y = FFT(y)$.

FFT of correlation $\Phi = FFT(\phi)$ satisfies

$$\Phi_k = \frac{1}{N} \sum_{n=0}^{N-1} \phi_n W^{-nk} \quad ④$$

We'll write Φ_k in terms of Y_k and Z_k.

Plug ① into ④:

$$\Phi_k = \frac{1}{N} \sum_{n=0}^{N-1} W^{-nk} \left(\frac{1}{N} \sum_{l=0}^{N-1} y_{l+n} z_l \right) = \frac{1}{N^2} \sum_n \sum_l y_{l+n} z_l W^{-nk}$$

Plug ② and ③ in for y and z.

$$\Phi_k = \frac{1}{N^2} \sum_n \sum_l \left[\left(\sum_m Y_m W^{m(l+n)} \right) \left(\sum_r Z_r W^{rl} \right) W^{-nk} \right]$$

Now, simplify!

$$\Phi_k = \frac{1}{N^2} \sum_n \sum_l \sum_r \sum_m Y_m Z_r W^{ml} W^{mn} W^{rl} W^{-nk}$$

$$= \frac{1}{N^2} \sum_n \sum_r \sum_m \left[Y_m Z_r W^{-n(k-m)} \sum_l W^{l(r+m)} \right]$$ Use orthogonality identity. ~~case~~ $r = N-m$ case.

$$= \frac{1}{N^2} \sum_n \sum_m \sum_r Y_m Z_r W^{-n(k-m)} N \delta_{r,N-m}$$ Use δ to eliminate sum.

$$= \frac{1}{N} \sum_n \sum_m Y_m Z_{N-m} W^{-n(k-m)}$$

$$= \frac{1}{N} \sum_m Y_m Z_{N-m} \sum_n W^{-n(k-m)}$$ Use orthogonality again.

$$= \frac{1}{N} \sum_m Y_m Z_{N-m} N \delta_{k,m}$$ Eliminate sum via δ.

$$\Phi_k = Y_k Z_{N-k}$$

For real data, Z is conjugate symmetric so

$Z_k = \overline{Z_{N-k}}$ or $\overline{Z_k} = Z_{N-k}$.

Hence $\Phi_k = Y_k \overline{Z_k}$.

$\therefore$ FFT of correlation can be found by multiplying pairs of entries in the FFTs of the data.