Lecture 7

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Buckingham's Pi Theorem

Informal explanation: We know equations need to be dimensionally consistent. Buckingham's Pi Theorem is designed to make predictions about solutions based on this. Roughly....Any relationshop between N physical quantities involving r effective dimensions can be completely described in terms of N-r dimensionless quantities.

Example 1: The sky-diver problem

Coinsider the problem of finding the terminal velocity of an object in free fall. If we know that this quantitiy, V, should depend only on:

- m (mass of object)
- g (gravitational acceleration)
- α (drag coefficient, assuming $F_{air} = \alpha v$)

There are N = 4 quantities (V, m, g, α). How many dimensions are there?

- $[V] = LT^{-1}$
- $[m] = M$
- $[g] = LT^{-2}$
- $[\alpha] = \frac{[force]}{[velocity]} = \frac{MLT^{-2}}{LT^{-1}} = MT^{-1}$

We have M, L, And T so r = 3

We should therfore be able to construct N-r = 1 dimensionless variable. We'll call it Π

Π is used because it must be constructed as:

$$m^{P_m} g^{P_g} \alpha^{P_\alpha} V^{P_V}$$

It's a product of powers of our quantities. What could Π be? By inspection:

$$\Pi = \frac{mg}{\alpha V}$$

We could also use $\frac{\alpha V}{mg}$ or $\frac{\alpha^2 V^2}{m^2 g^2}$ but those are just pwoers of the original.

By Buckingham's Π theorem, the solution can be expressed using just Π. It must therefore be of the form Π = c where c is a dimensionless constant

$$\frac{mg}{\alpha V} = C$$
$$V = C_1 \frac{mg}{\alpha}$$

Example 2: Radioactive Decay

The mass m, of a radioactive substance remaining after a time t, deends on the initial mass m_0 and a decay rate k (where $[k] = T^{-1}$). The dimensions:

- $[m] = M$

t = T

- $[m_0] = M$
- $[k]] = T^{-1}$

We can construct $N - r = 4 - 2 = 2$ independent dimensionless quantitites $\Pi_1 = \frac{m}{m_0}$, $\Pi_2 = kt$
The solution can be completely described with these 2:

$$\Pi_1 = f(\Pi_2)$$
$$\frac{m}{m_0} = f(kt)$$
$$m = m_0 f(kt)$$
$$m(t) = m_0 e^{-kt}$$

Example 3: Size of a crater (from the problem set) We are told that the important quantities are

- E = energy of explosion

- ρ = density of soil
- g = gravitational acceleration
- l = length scale of the crater (diameter or radius)

To produce a crater twice as large, how much more energy is needed?

- $[E] = ML^2T^{-2}$
- $[\rho] = ML^{-3}$
- $[g] = LT^{-2}$
- $[l] = L$

$$\Pi = \frac{E}{\rho g l^4}$$

The solution has the form $\Pi = C$ ie $E = C\rho g l^4$
Replacing l with 2l shows that we need 16 times more energy

Buckingham's Π Theorem more rigorously

As motivation for making this more rigorous, consider this:
Example:
Consider the RC circuit again. We have:
(Considering the charge q(t))

- $[q] = Q$
- $[t] = T$
- $[v] = ML^2T^{-2}Q^{-1}$
- $[R] = MC^2TQ^{-2}$
- $[C] = M^{-1}L^{-2}T^2Q^2$

Fig7.1

5 Quantities, 4 dimensions would seem to imply 5 - 4 = 1 dimensionless quantity. However we have two! ($\frac{t}{RC}$ and $\frac{q}{VC}$)
What's wrong? There are really only three dimensions here: Q, T, and ML^2. (Better use Q,T,V for voltage).
To formalize the logic, note that any dimensionless quantitiy in this problem must be constructed as:

$$\Pi = q^{P_q} t^{P_t} V^{P_V} R^{P_R} C^{P_C}$$

We must have $[\Pi] = 1$ so

$$1 = [q]^{P_q} [t]^{P_t} [V]^{P_V} [R]^{P_R} [C]^{P_C}$$
$$1 = Q^{P_q} T^{P_t} [ML^2T^{-2}Q^{-1}]^{P_V} [ML^2T^{-1}Q^{-2}]^{P_R} [M^{-1}L^{-2}T^2Q^2]^{P_C}$$
$$1 = M^{P_V+P_R-P_C} L^{2P_V+2P_R-2P_C} T^{P_t-2P_V-P_R+2P_C} Q^{P_q-P_v-2P_R+2P_C}$$

Therefore:

- $P_V + P_R - P_C = 0$
- $2P_V + 2P_R - 2P_C = 0$
- $P_t - 2P_V - P_R + 2P_C$
- $P_q - P_V - 2P_R + 2P_C$

ie:
$$\begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 1 & -2 & -1 & 2 \\ 1 & 0 & -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} P_q \\ P_t \\ P_V \\ P_R \\ P_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
We call this the dimensional matrix, D

Buckingham's Π theorem

A Relationship between N physical quantities whose dimensional matrix has rank r, can be completely described in terms of N - r independent dimensionless quantities

We can construct D by inspection:

Eg1:

Suppose:

- $[v] = M^{V_M} L^{V_L}$
- $[w] = M^{W_M} L^{W_L}$
- $[x] = M^{x_M} L^{x_L}$

Then: $D = \begin{bmatrix} V_M & W_M & x_M \\ V_L & W_L & x_L \end{bmatrix}$

We can also use D to construct the variables Π_i. Reduce D to echelon form:

$$D = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_q \\ P_t \\ P_V \\ P_R \\ P_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow P_q - P_R + P_C = 0$$

$$P_t + P_R = 0$$

$$P_V + P_R - P_C = 0$$

Eg2

Set $P_R = 0$, $P_C = 1$? Then $P_q = -1$, $P_t = 0$, $P_V = 1$
These give $\Pi = \frac{VC}{q}$
Set $P_R = 1$, $P_C = 0$? Then $P_q = 1$, $P_t = -1$, $P_V = -1$
These give $\Pi = \frac{Rq}{Vt}$
Set $P_R = 1$, $P_C = 1$? Then $P_q = 0$, $P_t = -1$, $P_C = 0$

$$\rightarrow \Pi = \frac{RC}{t}$$

Atomic Bomb Example

(Sir Geoffrey Taylor 1947)

Assumption that 5 quantities are needed:

- t: time since detonation $[t] = T$
- ρ density of air $[\rho] = ML^{-3}$
- E : Energy released $[E] = ML^2T^{-2}$
- P : atmospheric pressure $[P] = ML^{-1}T^{-2}$
- R : radius of shockwave $[R] = L$

$$D = \begin{array}{c} \\ (M) \\ (L) \\ (T) \end{array} \begin{array}{c} \begin{array}{ccccc} P_t & P_\rho & P_E & P_p & P_R \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & -3 & 2 & -1 & 1 \\ 1 & 0 & -2 & -2 & 0 \end{bmatrix} \end{array}$$ This reduces to:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-6}{5} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{3}{5} & \frac{-1}{5} \\ 0 & 0 & 1 & \frac{2}{5} & \frac{1}{5} \end{bmatrix} N - r = 5 - 3 = 2$$

$$P_t = \frac{6}{5}P_p - \frac{2}{5}P_R$$

$$P_\rho = \frac{-3}{5}P_p + \frac{1}{5}P_R$$

$$P_E = \frac{-2}{5}P_P - \frac{1}{5}P_R$$