Solving Inhomogeneous Linear Equations

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Feb 6th, 2017

Method 1: Method of Undetermined Coefficients

Eg1:

$$y'' + 2y' - 3y = e^x$$

Find y_h the character equation is $m^2 + 2m - 3 = 0$, m = 1, -3

$$y_h = C_1e^x + C_2e^{-3x}$$

For y_p we guess $y = Axe^x$
$y' = Ae^x + Axe^x$
$y'' = 2Ae^x + Axe^x$

Plug this into the DE: $(2Ae^x + Axe^x) + 2(Ae^x + Axe^x) - 3Axe^x = e^x$
$\rightarrow 4A = 1$, so $A = \frac{1}{4}$
and $y = C_1e^x + C_2e^{-3x} + \frac{1}{4xe^x}$

Method 2: Variation of Parameters

Given: $y'' + p(x)y'q(x)y = F(x)$
With homogeneous solution:
$y_h = C_1y_1(x) + C_2y_2(x)$
We assume that the solution can be written as:
$y = u_1(x)y_1(x) + u_2(x)y_2(x)$

Differentiate:
$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$

We have the freedom to impose a condition, since we have 2 unknown functions and only one condition (the DE).
As the second condition set:

$$u_1'y_1 + u_2'y_2 = 0$$
$$\rightarrow y' = u_1y_1' + u_2y_2'$$

Differentiate:

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

Plug y'', y', y into the DE:

$$(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(x)(u_1y_1' + u_2y_2') + g(x)(u_1y_1 + u_2y_2) = F(x)$$
$$\to u_1'y_1' + u_2'y_2' + u_1[y_1'' + p(x)y_1' + q(x)y_1] + u_2[y_2'' + p(x)y_2' + q(x)y_2] = F(x)$$
$$\to u_1'y_1' + u_2'y_2' = F(x)$$

We were able to get rid of the bracketed parts because we are looking at our inhomogeneous case which means the original equation is equal to 0

We can solve for u_1' and u_2' and then integrate!

Example1

$y'' + 2y' - 3y = e^x$

We know $y_h = C_1e^x + C_2e^{-3x}$
We let: $y = u_1(X)e^x + u_2(x)e^{-3x}$
To find u_1, u_2 solve:

$$u_1'y_1 + u_2'y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = F(x)$$

ie:

$$u_1'e^x + u_2'e^{-3x} = 0$$
$$u_1'e^x - 3u_2'e^{-3x} = e^x$$

Isolate one of the unknowns.
Subtract:

$$4u_2'e^{-3x} = -e^x$$
$$\to u_2' = \frac{-1}{4}e^{4x}$$

$$\rightarrow u_2 = \frac{-1}{16}e^{4x} + C_2$$

Now solve for u_1

$$\begin{aligned} u_1' e^x &= -u_2' e^{-3x} \\ &= \frac{1}{4}e^x \\ u_1' &= \frac{1}{4} \\ u_1 &= \frac{1}{4}x + C_1 \end{aligned}$$

Plug these into the above equations:

$$\begin{aligned} y &= u_1 e^x + u_2 e^{-3x} \\ &= (\frac{1}{4}x + C_1)e^x + (\frac{-1}{16}e^{4x} + C_2)e^{-3x} \\ &= C_1 e^x + C_2 e^{-3x} + \frac{1}{4}xe^x - \frac{1}{16}e^x \\ &= C_3 e^x + C_2 e^{-3x} + \frac{1}{4}xe^x \end{aligned}$$

Example2

Solve the IVP $y'' + 4y = \frac{1}{x}$ for $y(\pi) = 0$ and $y'(\pi) = 0$
Here $y_h = C_1 cos(2x) + C_2 sin(2x)$
We look for $y = u_1(x)cos(2x) + u_2 sin(2x)$
Solve: $u_1' cos(2x) + u_2' sin(2x) = 0$
$-2u_1' sin(2x) + 2u_2' cos(2x) = \frac{1}{x}$
Multiply 1 by $2cos(2x)$ and 2 by $sin(2x)$ then subtract:

$$\begin{aligned} u_1'(2cos^2(2x) + 2sin^2(2x) &= \frac{-sin(2x)}{x} \\ \rightarrow u_1' &= \frac{-sin(2x)}{2x} \\ \rightarrow u_1 &= -\int_\pi^x \frac{sin(2t)}{2t}dt + C_1 \end{aligned}$$

To find u_2? Equation 1 gives:

$$\begin{aligned} u_2' sin(2x) &= -u_1' cos(2x) \\ &= \frac{sin(2x)cos(2x)}{2x} \\ \rightarrow u_2' &= \frac{cos(2x)}{2x} \\ \rightarrow u_2 &= \int_{\pi}^{x} \frac{cos(2t)}{2t} dt \end{aligned}$$

The full solution is:

$$y = C_1 cos(2x) + C_2 sin(2x) + cos(2x) \int_{pi}^{x} \frac{sin(2t)}{2t} dt - sin(2x) \int_{\pi}^{x} \frac{cos(2t)}{2t} dt$$

To use the IC's, we need the derivitive:

$$y' = -2C_1 sin(2x) + 2C_2 cos(2x) + \frac{cos(2x)sin(2x)}{2x} - 2sin(2x) \int_{\pi}^{x} \frac{sin(2t)}{2t} dt - \frac{sin(2x)cos(2x)}{2x} - cos(2x)$$

$y(\pi) = C_1 = 0$ and $y'(\pi) = 2C_2 = 0$ which means $C_1 = C_2 = 0$

$$\rightarrow y = \frac{cos(2x)}{2} \int_{\pi}^{x} \frac{sin(2t)}{t} dt - \frac{sin(2x)}{2} \int_{\pi}^{x} \frac{cos(2t)}{t} dt$$