Introduction to Dimensional Analysis

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In applications we'll want to keep track of the units of measurement. For cimplicity we'll speak instead of the "dimensions" being measured.
New notation: The dimensions of mass are mass - square brackets define the dimensions of something
$[mass] = M$, $[length] = L$, $[time] = T$ etc. We will add more later

We start with two axioms:

1. (D1) Physical quantities may only be added ,subtracted, or equated if they have the same dimensions.
2. (D2) Quantities of different dimensions can only be combined by multiplication and division, in which case we have $[AB] = [A][B]$ and $[\frac{A}{B}] = \frac{[A]}{[B]}$

We can define dimensions for any quatntites which we believe shoudl obey these rules
In Physichs applications, we have 5 dimensions:
M, L, T and $[temperature] = U$ and $[charge] = Q$
(We can also define our own dimensions $[money]$, or $[applies]$ and $[oranges]$ etc.)

We can use D2 to calculate dimensions of secondary quantities:
eg) $[speed] = \frac{[length]}{[time]} = LT^{-1}$
from: $v = \frac{ds}{dt} = \lim_{\Delta t} \frac{\Delta s}{\Delta t}$

$[acceleration] = [\frac{dv}{dt}] = \frac{[speed]}{[time]} = LT^{-2}$
$[force] = [mass][acceleration] = MLT^{-2}$
$[work] = [force][distance] = ML^2T^{-2}$
$[voltage] = \frac{[work]}{[charge]} = ML^2T^{-2}Q^{-1}$

Comment: Angles are.... <u>dimensionless</u>! (in radians, $\Theta = \frac{s}{r}$) with radius r and length of the side is s. so $[\Theta] = \frac{[s]}{[r]} = \frac{L}{L} = 1$

- There are some theoretical questions about what ahs dimensions and what doesn't (eg angles! - we'll treat angles as dimensionless, since $\Theta = \frac{s}{r}$)

- A consequence of D1 and D2 is that the input and ouput of any transcendental function must be dimensionless

Why? Suppose f(x) has a Maclaurin series. Then $f(x) = a_0 + a_1 x + a_2 x^2 + ...$

Eg: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...$

Application 1: Consistency checks.

<u>Example:</u> In the sky-diver problem, we started with $m\frac{dv}{dt} = -\alpha v - mg$

We know $[m] = M$, $[v] = LT^{-1}$, $[t] = T$, $[g] = LT^{-2}$
We can determine $[\alpha]$: the force due to air resistance is $[-\alpha v] = [force]$, so
$[\alpha] = \frac{[force]}{[v]} = \frac{MLT^{-2}}{LT^{-1}} = MT^{-1}$
Also $[mg] = MLT^{-2}$
$[m\frac{dv}{dt}] = \frac{MLT^{-1}}{T} = MLT^{-2}$

$\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{LT^{-1}}{T} = LT^{-2}$

Now we found the solution to be:
$v = \frac{mg}{\alpha}(e^{\frac{-\alpha t}{m}} - 1)$
Check? $[v] = LT^{-1}$
$[\frac{-\alpha}{m}t] = \frac{[\alpha][t]}{[m]} = \frac{(MT^{-1})T}{M} = 1$
$[\frac{mg}{\alpha}] = \frac{MLT^{-2}}{MT^{-1}} = LT^{-1}$

Application 2: Nondimentionalization of DEs

We may be able to introduce dimensionless variables, which may allow us to write our DE's in simpler forms.
Eg 1: For a pendulum, we might say that the period is a "characteristic time (for that pendulum), t_c. We could then define a dimensionless time variable, τ as $\tau = \frac{t}{t_c}$ (after 10 oscillations we'll have $\tau = 10$)

<u>**Procedure for Nondimensionalization**</u>

1. List the physical constants in the problem, and identify their dimensions.

2. Make a seperate list for the variables

3. Find combinations of the constants which have the same dimensions as the varaibles (do this for each variable). These will be the characteristic scales and we will then define $\tau = \frac{t}{t_c}$, $\mu = \frac{m}{m_c}$, $\lambda = \frac{l}{l_c}$, $\epsilon = \frac{x}{x_c}$ etc

4. rewrite the DE and IVP in terms of the new variables (using the chain rule). Note: the char schales will often have simple physical interpretations

Example: The Mixing Tank Problem
Consider a tank holding a mass m(t) of a chemical dissolved in a volume V of water. A solution with concentration C of the same chemical enters the tank at a rate f

The contents of the tank are mixed constantly, and the mixed solution exits the tank at $f\frac{L}{min}$. Find m(t)

We must have

$$\frac{dm}{dt} = (ratein) - (rateout)$$

$$= fc - f\frac{m}{V}$$

$$\frac{dm}{dt} = fc - \frac{fm}{v} \_|\_ m(0) = m_0$$

Consistency check?
$[\frac{dm}{dt}] = MT^{-1}$
$[fc] = [L^3T^{-1}] \times [ML^{-3}] = MT^{-1}$
$[\frac{fm}{v}] = \frac{[L^4T^{-1}][M]}{L^3} = MT^{-1}$

Starting from Jan 25th
From monday:

For the mixing tank problem,

$$\frac{dm}{dt} = fc - \frac{fm}{v}$$

$m(0) = m_0$
$m = mass$ of chamical in solution in tank,
$t = time$ in seconds
$f = flowrate$ ina and out in l/s
C =concenttration of inflowing solution in grams/litre
v = volume of tank in litres
(and $m_0 = initialmass$, $m(0)$)

To nondimensionalize this, list the variables/constants and dimensions
$[m] = M$, $[t] = T$, $[f] = L^3T^{-1}$, $[c] = ML^{-3}$, $[v] = L^3$

Find combinations of constants with dimensions of variabels, call these m_c, t_c
$m_c = cv$ and $t_c = \frac{v}{f}$

We now use these values of m and t as ratios/scales for our values

Define new variabels:

$$M = \frac{m}{m_c} = \frac{m}{cv}$$

$m = CV\mu$

$$\tau = \frac{t}{t_c} = \frac{t}{\frac{v}{f}} = \frac{ft}{v}$$

$t = \frac{V\tau}{f}$

Rewrite the DE in terms of μ, τ
$\frac{dm}{dt} = \frac{dm}{d\mu}\frac{d\mu}{d\tau}\frac{d\tau}{dt} = (cv)\frac{du}{d\tau}(\frac{f}{v}) = fc\frac{du}{d\tau}$

The original DE is therefore:

$$fc\frac{du}{d\tau} = fc - \frac{f}{v}(cv\mu)$$

That is:

$$\frac{du}{d\tau} = 1 - \mu$$

Initial conditiona? $m(0) = m_0 \to cv\mu(0) = m_0 \to \mu(0) = \frac{m_0}{cv}$

The solution is $\mu(\tau) = Ke^{-\tau} + 1$

so:

$$k = \frac{m_0}{cv} - 1$$

So:

$$\mu\tau = (\frac{m_0}{cv} - 1)e^{-\tau} + 1$$

In terms of mt?

$$\frac{m}{cv} = (\frac{m_0}{cv} - 1)e^{-\frac{ft}{v}} + 1$$

so:

$$m(t) = (m_0 - cv)e^{-\frac{ft}{v}} + cv$$

Example: The RC circuit

The IVP for the charge q(t) on a capacitor in an RC circuit is $\frac{dq}{dt} + \frac{q}{RC} = \frac{V(t)}{R}$, $q(0) = q_0$ usually $q(0) = 0$
Nondimensionalize? Assume $v(t) = v_0$
$[q] = Q$, $[t] = T$, $[R] = ML^2T^{-1}Q^{-2}$, $[C] = M^{-1}L^{-2}T^2Q^2$, $[V_0] = ML^2T^{-2}Q^{-1}$
$V_R = iR$, and $V_c = \frac{q}{C}$

q_c?? or t_c ??
Observe from the DE that $[RC] = T$, so use $t_c = RC$
Also, $[\frac{v_0}{R}] = \frac{Q}{T}$, so $[\frac{V_0}{R} \times t_c] = Q$ Use $q_c = \frac{V_0}{R}RC = V_0C$
Define $\tilde{q} = \frac{q}{q_c} = \frac{q}{v_0C}$
$\tau = \frac{t}{t_c} = \frac{t}{RC}$

$$\frac{dq}{dt} = \frac{dq}{d\tilde{q}}\frac{d\tilde{q}}{d\tau}\frac{d\tau}{dt} = (v_0C)\frac{d\tilde{q}}{d\tau}(\frac{1}{RC}) = \frac{V_0}{R}\frac{d\tilde{q}}{d\tau}$$

$\to$ our DE becomes $\frac{v_0}{R}\frac{d\tilde{q}}{d\tau} + \frac{V_0C\tilde{q}}{RC} = \frac{V_0}{R}$
ie $\frac{d\tilde{q}}{d\tau} + \tilde{q} = 1$ This is the same as the mixing tank DE!!

Interpretation of q_c? It's the charge on the capacitor when it is fully charged (so that the voltage matches the srouce voltage)
And t_c? Its the time required for $q(t)$ to reach $(1-e^{-1})V_0C$, ie $\approx 63\%$ of the maximum

Comment::: There may be more than one option for nondimensionaliation

Example:
Suppose $V(t) = V_0 \cos \omega t$

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0 \cos \omega t}{R}$$

using $q_C = V_0 C$ and $t_c = RC$, we got $\tilde{q} = \frac{q}{V_0 C}$ and $\tau = \frac{t}{RC}$
$\frac{d\tilde{q}}{d\tau} + \tilde{q} = \cos RC\omega\tau$
Alternatively, we coudl use $t_c = \frac{1}{\omega}$, and $\tau = \omega t$

This leads to $\frac{d\tilde{q}}{t\tau} + \frac{V_0}{R^2\omega C}\tilde{q} = \cos \tau$