The Principle of Superposition

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The principle of superposition

This is the defining characteristic of linear systems:
If y_1 is a solution to $y' + k(x)y = f_1(x)$
and y_2 is a solution to $y' + k(x)y = f_2(x)$
then $y_1 + y_2$ is a solution to $y' + k(x)y = f_1(x) + f_2(x)$

Proof:
IF y_1 and y_2 are as described thenL

$$\begin{aligned}(y_1 + y_2)' + k(x)(y_1 + y_2) &= y_1' + k(x)y_1 + y_2' + k(x)y_2 \\ &= f_1(x) + f_2(x)\end{aligned}$$

We are used to using x as input and finding y. With differentials we have a differn way to look at it. We can look at the function on the right as the input. We can decide the function f(x) on the righthand side which then determines what y becomes.

A special case:
If y_h is a solution to $y' + k(x)y = 0$
and y_p is a solution to $y' + k(x)y = f(x)$
then $y_h + y_p$ is also a solution to $y' + k(x)y = f(x)$

Note: we call the two equations homogeneous and inhomogeneous, respectively.

Now consider two other observations:

- If k(x) is a constant, then we find y_h by inspection

$$\frac{dy}{dx} + ky = 0 \rightarrow y_h = Ce^{-kx}$$

- The general solution to a first order equation needs one constant of integration

These suggest another method for solving linear equations (usdful if k(x) is constant)

<u>TO SOLVE:</u> $\frac{dy}{dx} + ky = f(x)$

1. Find y_h (by inspection): $y = Ce^{-kx}$
2. Find y_p any particular solution to the inhomogeneous DE
3. The general solution to the full DE will be $y = y_h + y_p$

How do we find y_p?
- we can often guess its form!

The method of undetermined Coefficients

<u>Example 1</u>

$$\frac{dy}{dx} = 2y = e^{3x}$$

The solution to the homogeneous equation $y' = 2y = 0$ is $y_h = Ce^{-2x}$
For y_p we guess that $y_p = Ae^{3x}$, for some $A \epsilon R$
Plug this into the DE: $y_p' = 3Ae^{3x}$
So $y_p' + 2y_p = e^{3x} \rightarrow 3Ae^{3x} + 2Ae^{3x} = e^{3x}$
Our guess works if $A = \frac{1}{5}$
Therefore $y_p = \frac{1}{5}e^{3x}$ is a solution and the general solution is $y = y_h + y_p = Ce^{-2x} + \frac{1}{5}e^{3x}$

<u>Example2:</u>

$$\frac{dy}{dx} = y - x^2$$

$$\frac{dy}{dx} - y = -x^2$$

We have $y_h = Ce^x$
We guess $y_p = Ax^2 + Bx + C$
$\rightarrow y_p' = 2Ax + B$

$\rightarrow$ The DE gives $(2Ax + B) - (Ax^2 + Bx + C) = -x^2$
ie $-Ax^2 + (2A - B)x + (B - C) = -x^2$
$-A = -1$ and $2A - B = 0$ and $B - C = 0$
$A = 1$ and $B = 2$ and $C = 2$
$y = y_h + y_p = Ce^x + x^2 + 2x + 2$

Continuing on Jan 16th

summary:

Forcing Term	Trial Function
ae^{kx}	Ae^{kx}
$a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ where $a_n \neq 0$	$A_n x^n + A_{n-1}x^{n-1} + ... + A_0$
$a\cos(kx) + b\sin(kx)$	$A\cos(kx) + B\sin(kx)$
$x^n e^{kx}$	$(A_n x^n + A_{n-1}x^{n-1} + ... + A_0)e^{kx}$
$x^n(a\cos(kx) + b\sin(kx))$ (one of a or b = 0)	$(A_n x^n + A_{n-1}x^{n-1} + ... + A_0)(C\cos(kx) + \sin(kx))$
$e^{ax}\cos(bx)$	$e^{ax}(A\cos(bx) + B\sin(bx))$

One problem exists, Example:

$$\frac{dy}{dx} + 2y = e^{-2x}$$

If we try $y_p = Ae^{-2x}$ we get $y_p' = -2Ae^{-2x}$, and so $y_p' + 2y' = e^{-2x} \rightarrow -2Ae^{-2x} + 2Ae^{-2x} = e^{-2x} \rightarrow 0 = 1$

Whoops the above is wrong, oh no! What happened? The homogeneous solution is $y_h = Ce^{-2x}$ so this cannot solve the inhomogeneous problem!

What else might work?
Try $y = Axe^{-2x}$
$\rightarrow y' = Ae^{-2x} - 2Axe^{-2x}$
The DE becomes $(Ae^{-2x} = 2Axe^{-2x}) + 2Axe^{-2x} = e^{-2x}$
This works if $A = 1$,
$\rightarrow y = Ce^{-2x} + xe^{-2x}$

* IF our usual trial function matches the homogeneous solution, we'll need to multiply the trial function by x
We prove that the solution works by plugging it into the function

Example (more problems with the above)

$$\frac{dy}{dx} + 2y = xe^{-2x}$$

Normally for xe^{-2x}, we'd guess that $y_p = (Ax + B)e^{-2x}$
This will fail. (Try it????)
Try instead: $y = (Ax^2 + Bx)e^{-2x}$

$$y' + 2y = xe^{-2x}$$
$$\to (2Ax + B)e^{-2x} - 2(Ax^2 + Bx)e^{-2x} + 2(Ax^2 + Bx)e^{-2x}$$
$$= xe^{-2x}$$
$$\iff (2Ax + B)e^{-2x} = xe^{-2x}$$
$$\iff (2Ax + B) = x$$
$$\iff A = \frac{1}{2}B = 0$$

So $y = Ce^{-2x} + \frac{1}{2}xe^{-2x}$

Guessing your work is a standard method - and can be faster than finding the Integration constant. This is an endorsed method and should be used.