Markov matrices satisfy $|\lambda_i| \leq 1$

First show $|\lambda_i| \leq 1$ for Q^T.

Let λ and $\vec{x}$ be an eigenvalue/vector pair for Q^T.

So $Q^T\vec{x} = \lambda\vec{x}$.

Let k be the index of the largest magnitude entry of $\vec{X}$.

So $|X_j| \leq |X_k|$ for all j.

We have $(Q^T\vec{X})_k = \sum_{j=1}^{n} Q_{jk} X_j = \lambda X_k$.

Next, take absolute values...

$$|\lambda X_k| = |\lambda||X_k| = \left|\sum_{j=1}^{n} Q_{jk} X_j\right|$$

Use Δ inequality and that Q's entries are non-negative.

$$\leq \sum_{j=1}^{n} Q_{jk} |X_j|$$

$$\leq \sum_{j=1}^{n} Q_{jk} |X_k| \quad \text{since } |X_j| \leq |X_k|.$$

$$\leq |X_k| \cancel{\sum_{j=1}^{n} Q_{jk}}^{\,1} \quad \text{since col sums of } Q \text{ are } 1.$$

So $|\lambda||X_k| \leq |X_k|$

$\therefore$ $|\lambda| \leq 1$ for Q^T, and also for Q, since they have the same eigenvalues.

Page Rank Convergence

If M is a positive Markov matrix, Page Rank converges to a unique vector p^∞ for initial probability vector p^0.

Let $\vec{x}_\ell$ be the corresponding eigenvector for λ_ℓ, for all ℓ.
Assume p^0 is a linear combination of eigenvectors $\vec{x}_\ell$.

$$p^0 = \sum_\ell c_\ell \vec{x}_\ell \quad \text{for scalars } c_\ell.$$

Assume eigenvalues are in sorted order, so $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots$
Then $\vec{x}_1$ corresponds to $\lambda_1 = 1$.

Page Rank computes

$$(M)^k p^0 = (M)^k \sum_{\ell=1}^{R} c_\ell \vec{x}_\ell = \sum_{\ell=1}^{R} (M)^k c_\ell \vec{x}_\ell = \sum_{\ell=1}^{R} \lambda_\ell^k c_\ell \vec{x}_\ell$$

$$= c_1 \vec{x}_1 + \sum_{\ell=2}^{R} \lambda_\ell^k c_\ell \vec{x}_\ell$$

Th'm 7.8 said $|\lambda_\ell| < 1$ for $\ell > 1$, since $\lambda_1 = 1$ was unique.
Hence $\lim_{k \to \infty} \lambda_\ell^k = 0$ for $\ell > 1$.

Therefore $p^\infty = \lim_{k \to \infty} (M)^k p^0 = c_1 \vec{x}_1$.

Other eigenvector components of p^0 are scaled towards 0.

Uniqueness?

If we start with a different vector $q^0 = \sum b_\ell \vec{x}_\ell$,
we find $q^\infty = b_1 \vec{x}_1$.

Since q^∞ and p^∞ are both probability vectors, both sum to 1.

Then $\sum_{i=1}^{R} b_1 x_1(i) = 1 = \sum_{i=1}^{R} c_1 x_1(i)$

$$b_1 \sum_{i=1}^{R} x_1(i) = c_1 \sum_{i=1}^{R} x_1(i)$$

$$b_1 = c_1$$

$\therefore$ $p^\infty = q^\infty$, and so PageRank converges to a unique vector.