Gaussian Elimination as LU factorization

Solve $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ for x.

Form the augmented system and perform row operations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right] \xrightarrow[R_3 := R_3 - (1)R_1]{R_2 := R_2 - (1)R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 := R_3 - \left(-\frac{1}{3}\right)R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 0 & -5/3 & 10/3 \end{array}\right]$$

System is now upper triangular. Do back-substitution to find x.

$$-\frac{5}{3}x_3 = \frac{10}{3} \to x_3 = -2$$

$$-3x_2 + 1x_3 = 4 \to x_2 = -2$$

$$x_1 + x_2 + x_3 = 0 \to x_1 = 4$$

$$x = \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix}$$

The upper triangular matrix after row operations is the U in our LU factorization. But where is L? The multiplicative factors!

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1/3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -5/3 \end{bmatrix}$$

Factors

You can verify $A = LU$.

The diagonal entries of L are all 1's. We only need to store the <u>below</u> diagonal entries, i.e. the multiplicative factors.

To save space, store them back into "A", where we would otherwise put zeros.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{R_2 := R_2 - (1)R_1} \begin{bmatrix} 1 & 1 & 1 \\ \boxed{1} & -3 & 1 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{R_3 := R_3 - (1)R_1} \begin{bmatrix} 1 & 1 & 1 \\ \boxed{1} & -3 & 1 \\ \boxed{1} & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 := R_3 - (-\frac{1}{3})R_2} \begin{bmatrix} 1 & 1 & 1 \\ \boxed{1} & -3 & 1 \\ \boxed{1} & \boxed{-1/3} & -\frac{5}{3} \end{bmatrix}$$

↑
L components

Solving a different RHS for the same system

Now, solve $Ax=b$ for $b=\begin{bmatrix}6\\3\\2\end{bmatrix}$ without re-factorizing A.

We'll solve $Lz=b$, then $Ux=z$.

$Lz=b \rightarrow \begin{bmatrix}1&0&0\\1&1&0\\1&-1/3&1\end{bmatrix} z = \begin{bmatrix}6\\3\\2\end{bmatrix}$

$\therefore z_1 = 6$

$z_1 + z_2 = 3 \rightarrow z_2 = -3$

$z_1 - \frac{1}{3}z_2 + z_3 = 2$

$\rightarrow z_3 = -5$

so $z = \begin{bmatrix}6\\-3\\-5\end{bmatrix}$.

$Ux=z \rightarrow \begin{bmatrix}1&1&1\\0&-3&1\\0&0&-5/3\end{bmatrix} x = \begin{bmatrix}6\\-3\\-5\end{bmatrix} \rightarrow x = \begin{bmatrix}1\\2\\3\end{bmatrix}$