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# Butterfly FFT Example

$$f = \begin{bmatrix}4\\3\\2\\1\\4\\3\\2\\1\end{bmatrix} \xrightarrow{N=8} \begin{bmatrix}4\\3\\2\\1\\0\\0\\0\\0\end{bmatrix} \xrightarrow{N=4} \begin{bmatrix}3\\2\\1\\-i\\0\\0\\0\\0\end{bmatrix} \xrightarrow{N=2} \begin{bmatrix}2.5\\0.5\\0.5-0.5i\\0.5+0.5i\\0\\0\\0\\0\end{bmatrix}$$

(Still out of order!)

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Stage 1, $N=8$, $W = e^{\frac{2\pi i}{8}} = e^{\pi i/4}$

$g$: $\frac{1}{2}\begin{bmatrix}4\\3\\2\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}4\\3\\2\\1\end{bmatrix} = \begin{bmatrix}4\\3\\2\\1\end{bmatrix} = g$

$h$: $h_0 = \frac{1}{2}(4-4)W^0 = 0$ $\quad h_2 = \frac{1}{2}(2-2)W^{-2} = 0$

$h_1 = \frac{1}{2}(3-3)W^{-1} = 0$ $\quad h_3 = \frac{1}{2}(1-1)W^{-3} = 0$

$\therefore h = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

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Stage 2, $N=4$, $W = e^{\pi i/2}$

Top vector: $g_0 = \frac{1}{2}(4+2) = 3$ $\quad g_1 = \frac{1}{2}(3+1) = 2$ $\quad g = \begin{bmatrix}3\\2\end{bmatrix}$

$h_0 = \frac{1}{2}(4-2)W^0 = 1$ $\quad h_1 = \frac{1}{2}(3-1)W^{-1}$ $\quad h = \begin{bmatrix}1\\-i\end{bmatrix}$

$= 1\cdot(-i) = -i$

Bottom vector: all zeros again

Stage 3, $N=2$, $W = e^{-\pi i}$

Top pair: $g_0 = \frac{1}{2}(3+2) = 2.5$ $\quad h_0 = \frac{1}{2}(3-2)W^0 = 0.5$

Second pair: $g_0 = \frac{1}{2}(1-i) = 0.5-0.5i$ $\quad h_0 = \frac{1}{2}(1+i) = 0.5+0.5i$

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"Unscramble" step:

| Original indices | | Bit-Reversed indices | |
|---|---|---|---|
| 0 | 000 | 000 | 0 |
| 1 | 001 | 100 | 4 |
| 2 | 010 | 010 | 2 |
| 3 | 011 | 110 | 6 |
| 4 | 100 | 001 | 1 |
| 5 | 101 | 101 | 5 |
| 6 | 110 | 011 | 3 |
| 7 | 111 | 111 | 7 |

So...

$$\begin{bmatrix} 2.5 \\ 0.5 \\ 0.5-0.5i \\ 0.5+0.5i \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow F = \begin{bmatrix} 2.5 \\ 0 \\ 0.5-0.5i \\ 0 \\ 0.5 \\ 0 \\ 0.5+0.5i \\ 0 \end{bmatrix}$$

↑ Final answer

Notice: data was real so DFT shows conjugate symmetry

# Another FFT example

$$f = \begin{bmatrix} 2+i \\ 3 \\ -2 \\ -2i \end{bmatrix} \xrightarrow{N=4} \begin{bmatrix} i/2 \\ 3/2 - i \\ 2+i/2 \\ 1-3i/2 \end{bmatrix} \xrightarrow{N=2} \begin{bmatrix} 3/4 - i/4 \\ -3/4 + 3i/4 \\ 3/2 - i/2 \\ 1/2 + i \end{bmatrix} \xrightarrow{\text{Reorder}} \begin{bmatrix} 3/4 - i/4 \\ 3/2 - i/2 \\ -3/4 + 3i/4 \\ 1/2 + i \end{bmatrix}$$

## Stage 1, $N=4$, $W = e^{\pi i/2}$

$g_0 = \frac{1}{2}(2+i-2) = i/2$ $\quad g_1 = \frac{1}{2}(3-2i) = 3/2 - i$

$h_0 = \frac{1}{2}(2+i+2)\cdot W^0 = 2 + i/2$ $\quad h_1 = \frac{1}{2}(3+2i)W^{-1} = (3/2+i)(-i)$

$= 1 - 3/2\, i$

## Stage 2, $N=2$, $W = e^{\pi i}$

Top: $g_0 = \frac{1}{2}(i/2 + 3/2 - i) = 3/4 - i/4$

$h_0 = \frac{1}{2}(i/2 - 3/2 + i) = -3/4 + 3i/4$

Bottom: $g_0 = \frac{1}{2}(2 + i/2 + 1 - 3i/2) = 3/2 - i/2$

$h_0 = \frac{1}{2}(2 + i/2 - 1 + 3i/2) = 1/2 + i$

## Unscramble:

| | | | |
|---|---|---|---|
| 0 | 00 | 00 | 0 |
| 1 | 01 | 10 | 2 |
| 2 | 10 | 01 | 1 |
| 3 | 11 | 11 | 3 |