CS370: Interpolation

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See figure 2.1

$$y = p(x)$$

We want to find a function p, such that the curve is 'nice' (where nice is piecewise polynomial or polynomial)

Given:
$(x_1, y_1), (x_2, y_2)..., (x_n, y_n)$ n points $x_1 < x_2 < ... < x_n$

Find a polynomial P(x) of degree $< n$
In general:

$$p(x) = c_1 + c_2 x + c_3 x^2 + ... + c_n x^{n-1}$$
$$p(x_1) = y_1$$
$$p(x_2) = y_2$$
$$...$$
$$p(x_n) = y_n$$

n unknowns, n equations (linear)

Example:
$(-1, 1), (1, 1), (2, 5), (4, 1)$
See figure 2.2

$$p(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$
$$p(-1) = c_1 - c_2 + c_3 - c_4 = 1$$
$$p(1) = c_1 + c_2 + c_3 + c_4 = 1$$
$$p(2) = c_1 + 2c_2 + 4c_3 + 8c_4 = 5$$
$$p(4) = c_1 + 4c_2 + 16c_3 + 64c_4 = 1$$

$$\left\{\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 4 & 16 & 64 & 1 \end{array}\right\} // \text{ Solve the matrix!!}$$

Now we are just writing out the solution...

$$\begin{aligned} p(x) &= c_1 + c_2 x + c_3 x^2 + c_4 x^3 \\ &= 1 + b_2(x-1) + b_3(x-1)^2 + b_4(x-1)^3 \\ &= L_1(x) + L_2(x) + 5L_3(x) + L_4(x) \end{aligned}$$

$$\begin{aligned} L_1(x) &= \frac{(x-1)(x-2)(x-4)}{-30} \\ L_2(x) &= \frac{(x+1)(x-2)(x-4)}{6} \\ L_3(x) &= \frac{(x+1)(x-1)(x-4)}{-6} \\ L_4(x) &= \frac{(x+1)(x-1)(x-2)}{30} \end{aligned}$$

I think we are writing it out this way so that we can easily plug in the vlues and get the correct points??

Question:

1. Does an interpolating polynomial always exist?
2. If (1) is true then is the answer always unique?

$$\begin{aligned} p(x) &= c_1 + c_2 x + ...c_n x^{n-1} \\ p(x_1) &= c_1 + c_2 x_1 + ...c_n x_1^{n-1} \\ p(x_2) &= c_1 + c_2 x_2 + ...c_n x_2^{n-1} \\ &... \\ p(x_n) &= c_1 + c_2 x_n + ...c_n x_n^{n-1} \end{aligned}$$

$$\begin{Bmatrix} 1 & x_1 & x_1^2 & ... & x_1^{n-1} \\ 1 & x_2 & x_2^2 & ... & x_2^{n-1} \\ ... & ... & ... & ... & ... \\ 1 & x_n & x_n^2 & ... & x_n^{n-1} \end{Bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{Bmatrix}$$

The first matrix is the vandermonde (V) matrix.
V is invertable, $V \times \overrightarrow{c} = \overrightarrow{y}$
det $V \neq 0$ and det $V = \pi(x_i - x_j) \neq 0$ for $i < j$

Remember what a determinate is, remember what invertible is, but we will never be asked to do it.

$p(x)$

$$p(x) = q_1(x)(x - x_1) + y_1$$
$$p(x) = q_2(x)(x - x_2) + y_2$$
$$...$$
$$p(x) = q_n(x)(x - x_n) + y_n$$

Lagrange Polynomial

$$(x_1, y_1), (x_2, y_2)...(x_n, y_n)$$
$$p(x) = y_1 L_1(x) + y_2 L_2(x) + ... + y_n L_n(x)$$

$L_i(x_i) = 1, L_i(x_j) = 0$ for $i \neq j$ and $deg(L_i) = n - 1$
Lets construct L_1 using the above

$$L_1(x) = \frac{(x - x_2)(x - x_3)...(x - x_n)}{(x_1 - x_2)(x_1 - x_3)...(x_1 - x_n)}$$

$$L_i(x) = \frac{(x - x_1)...(x - x_{i-1})(x - x_{i+1})...(x - x_n)}{(x_i - x_1)...(x_i - x_{i-1})...(x_i - x_n)}$$

$L_i(x_i) = 1$ and $L_j(x_j) = 0$ where $j \neq i$