Conjugate Symmetry

If f_n are all real, then $F_k = \overline{F_{N-k}}$.

We'll use 3 facts:

① $W^{N-k} = e^{\frac{2\pi i(N-k)}{N}} = \overbrace{e^{\frac{2\pi i N}{N}}}^{1} e^{\frac{-2\pi i k}{N}} = W^{-k}$

② Recalling that $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$, we observe that $\overline{e^{i\theta}} = e^{-i\theta}$. $\therefore$ also $\overline{W^j} = W^{-j}$.

③ For real x, $x = \bar{x}$, since its imaginary part is 0.

Now $\overline{F_{N-k}} = \overline{\frac{1}{N}\sum_{n=0}^{N-1} f_n W^{-n(N-k)}}$ ← conjugate

$$= \frac{1}{N}\sum_{n=0}^{N-1} \overline{f_n}\, \overline{W^{-n(N-k)}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} f_n\, \overline{W^{-n(N-k)}} \quad \text{by ③ since } f_n \text{ are real.}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} f_n\, \overline{W^{-n(-k)}} \quad \text{by ①}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} f_n W^{-nk} \quad \text{by ②}$$

$$= F_k \quad \text{by definition}$$