Lecture 11

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March 3rd, 2017

Missed a couple of classes...

Working with Piecewise-Defined Functions

To extend our method to IVPs with piecewise-defined functions, we need a simple, standard way of expressing them. We will use this:

Definition: The Heaviside function (Aka the Unit Step function) is defined as:

$H(t) = 0$ when $t < 0$ and $H(t) = 1$ when $t \geq 0$

Examples:

eg $e^{-t}H(t) = 0$ for $t < 0$ and e^{-t} for $t \geq 0$

We can shift this effect:
$(4 - t^2)H(t-1) = 0$ for $t < 1$ and $4 - t^2$ for $t \geq 1$

We can use H(t) to express almost any piecewise-defined fuction

ie: Sketch the graph of $f(t) = t^2 + (t - t^2)H(t) + (1-t)H(t-1)$
Solution:
For $t < 0$ $f(t) = t^2 + 0 + 0 = t^2$
For $0 \leq t < 1$ $f(t) = t^2 + (t - t^2) + 0 = t$
For $t \geq 1$ $f(t) = t^2 + (t - t^2) + (1 - t) = 1$

We'll need to do this in reverse:
eg. Write $f(t) = |t|$ in terms of $H(t)$
$|t| = -t$ for $t < 0$ and t for $T \geq 0$
Then: $f(t) = -t + 2tH(t)$
Comment: There are other ways to do this, however our standard strucutre will (always) be:

$$f(t) = \underline{...} + \underline{...}H(t-a) + \underline{...}H(t-b) + \underline{...}H(t-c) + ...$$

Exxample: Rewrite:
$f(t) = 2t + 4$ for $t < -1$ and 3 for $t\epsilon[-1, 1)$ and $3t$ for $t\epsilon[1, \infty)$

Solution:
We want $f(t) = \_ + \_H(t + 1) + \_H(t - 1)$
Filling in the blanks:

$$f(t) = (2t + 4) + (-2t - 1)H(t + 1) + (3t - 3)H(t - 1)$$

In general:
$f(t) = f1$ if $t < a$, $f2$ if $t\epsilon[a, b)$ $f3$ if $t \geq b$

$$f(t) = f1 + (f2 - f1)H(t - a) + (f3 - f2)H(t - b)$$

There are a couple of tricks which may be useful:

- Consider 1 - H(t-a) this is 1 for $t < a$ and 0 for $t \geq a$ (an off switch)
- Consider $H(t - a) - H(t - b)$ where $a < b = 0$ for $t < a$ and 1 for $a \leq t \leq b$ and 0 for $t \geq b$

The Second Shift Theorem

What is $L\{H(t)\}$? It's $L\{H(t)\} = \int_0^\infty H(t)e^{-st}dt$
$= \int_0^\infty e^{-st}dt = L1 = \frac{1}{S^2}$ for $s > 0$

More generally: $L\{f(t)H(t)\} = L\{f(t)\} = F(s)$
From now on we'll write our fomulas as:

$$\begin{array}{c|c} L\{H(t)\} = \frac{1}{S} & L^{-}1\{\frac{1}{S}\} = H(t) \\ L\{e^{at}H(t)\} = \frac{1}{S-a} & L^{-1}\{\frac{1}{S-a}\} = e^{at}H(t) \\ \text{etc} & \text{etc} \end{array}$$

What happens to H(t-a)?
$L\{H(t - a)\} = \int_0^\infty H(t - a)e^{-st}dt$
$= \int_0^a 0 * e^{-st}dt + \int_a^\infty 1 * e^{-st}dt$
$= 0 - \frac{e^{-st}}{S}\_a^\infty = \frac{e^{-as}}{S}$

And now the big question, what about $F(t)H(t-a))$

Proof

$L\{f(t-a)H(t-a)\} = e^{-as}F(s)$
$\therefore L\{f(t-a)H(t-a)\} = \int_0^\infty e^{-st}f(t-a)H(t-a)dt$
We let $\tau = t-a$ and $d\tau = dt$
$= \int_{-a}^\infty e^{-s(\tau+a)}f(\tau)H(\tau)d\tau$
$= \int_{-a}^0 0d\tau + e^{-as}\int_0^\infty e^{-s\tau}f(\tau)d\tau = e^{-as}F(s)$

$L\{H(t)\} = \frac{1}{S}$	$L^{-1}\{\frac{1}{S}\} = H(t)$
$L\{e^{at}H(t)\} = \frac{1}{S-a}$	$L^{-1}\{\frac{1}{S-a}\} = e^{at}H(t)$
$L\{t^nH(t)\} = \frac{n!}{S^{n+1}}$	$L^{-1}\{\frac{n!}{S^{n-1}}\} = t^nH(t)$
$L\{cosatH(t)\} = \frac{S}{S^2+a^2}$	$L^{-1}\{\frac{S}{S^2+a^2}\} = cosatH(t)$
$L\{sinatH(t)\} = \frac{a}{S^2+a^2}$	$L^{-1}\{\frac{a}{S^2+a^2}\} = sinatH(t)$
$L\{f(t-a)H(t-a)\} = e^{-as}F(s)$	$L^{-1}\{e^{-as}F(s)\} = f(t-a)H(t-a)$
$L\{e^{at}f(t)\} = F(s-a)$	$L^{-1}\{F(s-a)\} = e^{at}f(t)$

Using the 2nd shift theorem

Example1)
Let f(t) = t for $t\epsilon[0,2)$ and 0 for $t \geq 2$
Find F(s)
Solution:
Write $f(t) = t[H(t)-H(t-2)] = tH(t)-tH(t-2) = tH(t)-(t-2+2)H(t-2)$
$= t * H(t) - (t-2)H(t-2) - 2H(t-2)$
and now we can see that:
$F(s) = \frac{1}{S^2} - \frac{e^{-2s}}{S^2} - \frac{e^{-2s}}{S}$

2)
Find G(s) if g(t) = $\frac{1}{3}t$ for $t\epsilon[0,3)$ and $2-\frac{1}{3}t$ for $t\epsilon[3,6)$ and 0 for $t \geq 6$

Solution:

$$g(t) = \frac{1}{3}t[H(t)-H(t-3)] + (2-\frac{1}{3}t)[H(t-3)-H(t-6)]$$
$$= \frac{1}{3}tH(t) + (2-\frac{2}{3}t)H(t-3) + (\frac{1}{3}t-2)H(t-6)$$

$$= \frac{1}{3}tH(t) - \frac{2}{3}(t-3)H(t-3) + \frac{1}{3}(t-6)H(t-6)$$

$$\rightarrow G(s) = \frac{1}{3S^2} - \frac{2}{3S^2}e^{-3s} + \frac{1}{3S^2}e^{-6s}$$

3)
Evaluate $L^{-1}\{\frac{e^{-S}}{S^2+1}\}$
Solution:
Since $L^{-1}\{\frac{1}{S^2+1}\} = sintH(t)$
We have $L^{-1}\{\frac{e^{-s}}{S^2+1} = sin(t-1)H(t-1)\}$

4)
Find $L^{-1}\{\frac{e^{-3S}}{(S+1)^4}\}$
Solution:
$L^{-1}\{\frac{1}{(S+1)^4}\} = e^{-t}L^{-1}\{\frac{1}{S^4}\}$
$= \frac{1}{6}e^{-t}L^{-1}\{\frac{6}{S^4}\}$
$= \frac{1}{6}t^3e^{-t}H(t)$
$\rightarrow L^{-1}\{\frac{e^{-3S}}{(S+1)^4}\} = \frac{1}{6}(t-3)^3e^{-(t-3)}H(t-3)$

5)
Solve the IVP $\frac{dy}{dt} + 3y = f(t)$, y(0) = 0
where $f(t) = 3$ for $t\epsilon[0,10)$ and 6 for $t\epsilon[10,20)$ and 3 for $t\epsilon[20,30)$ and 0 for $t\epsilon[30,\infty)$

Solution:

$$y' + 3y = 3H(t) + 3H(t-10) - 3H(t-20) - 3H(t-30)$$

$$\rightarrow sY + 3Y = \frac{3}{S} + \frac{3}{S}e^{-10S} - \frac{3}{S}e^{-20S} - \frac{3}{S}e^{-30S}$$

$$\rightarrow Y(s) = \frac{3}{S(S+3)}[1 + e^{-10s} - e^{-20S} - e^{-30S}]$$

$$Y(s) = [\frac{1}{S} - \frac{1}{S+3}](1 + e^{-10S} - e^{-20S} - e^{30S})$$

$\rightarrow y(t) = (1-e^{-3t})H(t) + (1-e^{-3(t-10)})H(t-10) - (1-e^{-3(t-20)})H(t-20) - (1-e^{-3(t-30)})H(t-30)$

In piecewise-defined form:
$y(t) =$

$1-e^{-3t}$ for $t\epsilon[0,10)$
$2-(1+e^{30})e^{-3t}$ for $t\epsilon[10.20)$
$1-(1+e^{30}-e^{60})e^{-3t}$ for $t\epsilon[20,30)$
$-(1+e^{30}-e^{60}-e^{90})e^{-3t}$ for $t\epsilon[30,\infty)$

6)
Solve the IVP $x''+3x'+2x=f(t)$ x(0) = 1, x'(0) = -1
where $f(t) == sin(t)H(t-\frac{\pi}{2})$
Solution:
Note that $f(t)=cos(t-\frac{\pi}{2})H(t-\frac{\pi}{2})$
So $(S^2X(s)-sx(0)-x'(0))-3(sX(s)-x(0))+2X(s)$
$=\frac{S}{S^2+1}e^{\frac{-\pi}{2}S}$
$\rightarrow (S^2+3S+2)X(s)=s+2+\frac{S}{S^2+1}e^{\frac{-\pi}{2}S}$

$$\rightarrow x(s)=\frac{S+2}{(s+1)(s-2)}+\frac{se^{\frac{-\pi S}{2}}}{(S^2+1)(s+1)(s+2)}$$

Example:
Solve hte IVP $x''+3x'+2x=f(t)$ $x(0)=1$, $x'(0)=-1$
Where $f(t)=sintH(t-\frac{\pi}{2})$

Solution: First $sin(t)=cos(t-\frac{\pi}{2})$ so $f(t)=cos(t-\frac{\pi}{2})H(t-\frac{\pi}{2})$

Applying the transofmration gives:

$$S^2X-sx(0)-x'(0)+3sx-3x(0)+2x=\frac{S}{S^2+1}e^{\frac{\pi}{2}S}$$

$$\rightarrow x(s)=\frac{s+2}{(s+1)(s+2)}+\frac{se^{-\frac{\pi}{2}s}}{(s+1)(s+2)(s^2+1)}$$

$$=\frac{1}{s+1}+e^{-\frac{\pi}{2}s}[\frac{\frac{-1}{2}}{s+1}+\frac{\frac{2}{5}}{s+2}+\frac{\frac{1}{10}s+\frac{3}{10}}{s^2+1}]$$

This gives us:

$$x(t)=e^{-t}H(t)+H(t-\frac{\pi}{2})+[\frac{-1}{2}e^{-t}+\frac{2}{5}e^{\pi}e^{-2t}+\frac{1}{10}sin(t)-\frac{3}{10}cos(t)]$$

Systems of First-Order Linear Equations

In some applications (mixing tanks, chemical reactions etc) we find that he behaviour of multiple quantities are related to each other. In a relatively simple case, we might have:

1. $\frac{dx}{dt} = ax + by + f_1(t)$
2. $\frac{dy}{dt} = cx + dy + f_2(t)$

Since these are linear, it is possible to write the system in vector form.
Letting $\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$
we have: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$
More concisely, $\overrightarrow{x}' = A\overrightarrow{x} + \overrightarrow{f}(t)$

Eg:
If $\frac{dx}{dt} = 2x - 3y + t$
and $\frac{dy}{dt} = x + 4y$
Then we would want to write:
$\overrightarrow{x}' = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \overrightarrow{x} + \begin{bmatrix} t \\ 0 \end{bmatrix}$