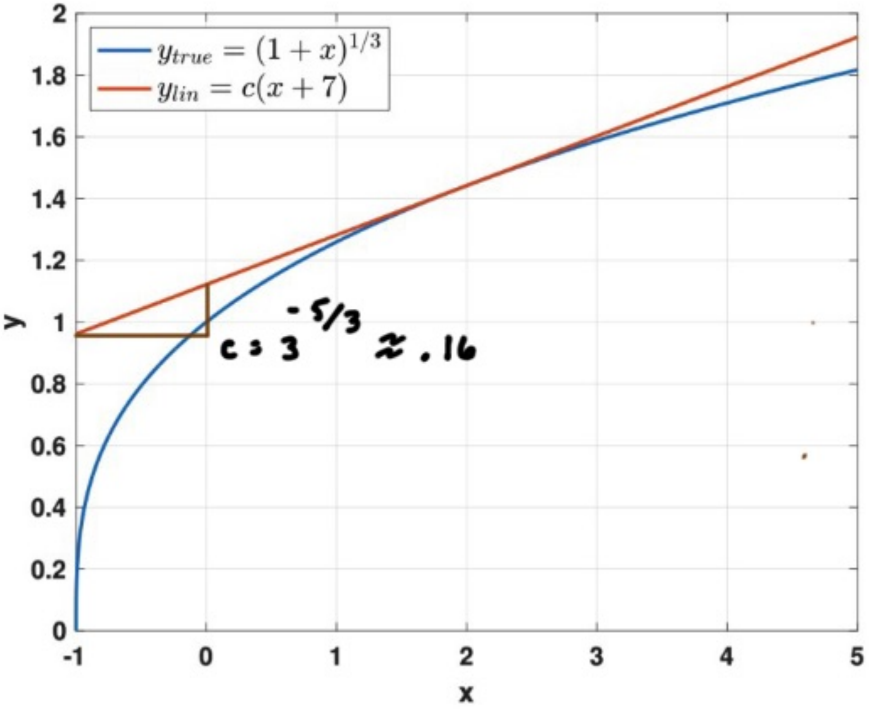
Linearization - Motivation

1. Nearly all dynamic systems are nonlinear

2. If the **component models** of a system are **linear**, then the system is linear, and it's easier to analyze and control

3. Engineers often design components of a system to operate in a **linear regime**

4. We need to develop skills/tools to recognize **linear vs. nonlinear** systems and to be able to **create a linear DEQ model from a nonlinear** one when appropriate…



Skills

1. Terminology:
    a. Equilibrium configuration and equations of equilibrium
    b. Taylor series
    c. High order terms
    d. Linear operating regime
2. Determine if a system is linear or nonlinear by analyzing its step response at two different equilibrium configurations
3. Linearize a function of one variable about an equilibrium configuration, with and without MATLAB
4. Given a nonlinear DEQ model with an input
    1. … and given the desired equilibrium configuration, determine the corresponding constant input
    2. … or given a constant input, determine the equilibrium configuration
    3. Create an approximate, linear, DEQ model

Linear Operating Regime

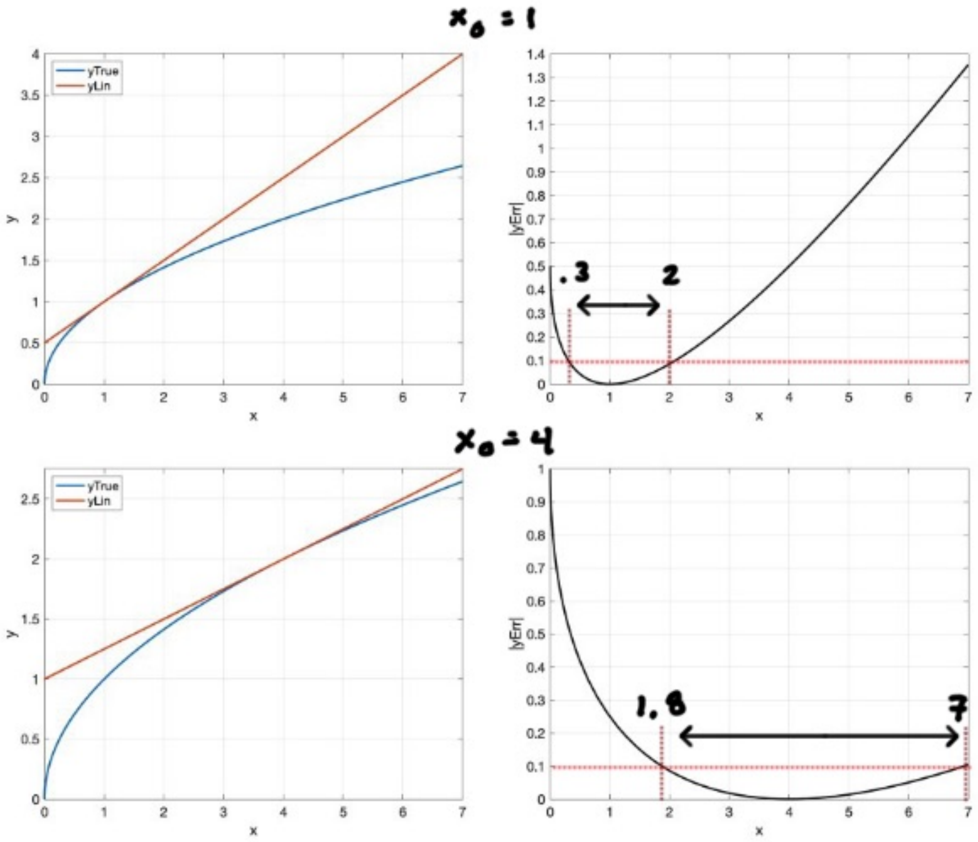
Consider the function: $y = x^{1/2}$

- Depends on the application
- Depends on the linearization point (configuration)

Metrics

$$y_{err} = y_{true} - y_{lin}$$

$$\% y_{err} = \frac{100(y_{true} - y_{lin})}{y_{true}}$$



Linearizing a Function of One Variable

Motivation - Given: x_0 and $f(x_0)$

Estimate: $f(x_0 + \delta x)$

for any δx in some linear operating regime.

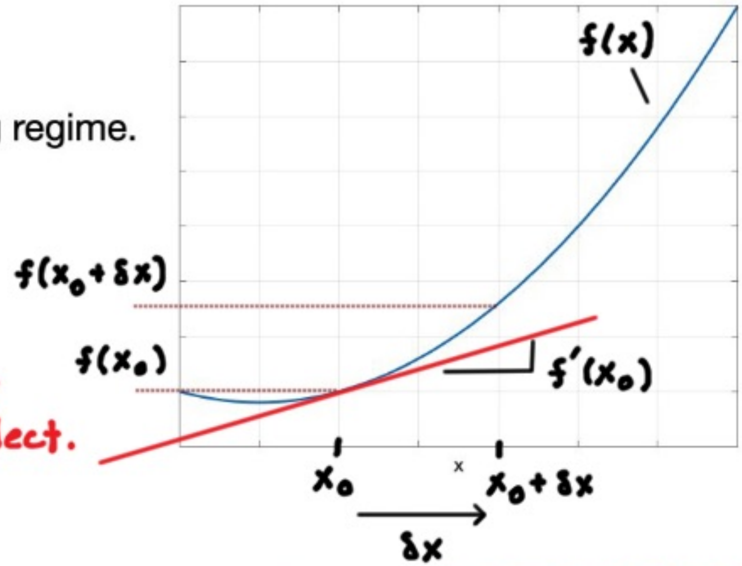
Taylor Series -

$f(x) = f(x_0) + (x - x_0) f'(x_0) + \cdots$ (1)

Higher order terms, $(x-x_0)^2, (x-x_0)^3 \ldots$

If $(x - x_0) << 1$, e.g. $< .1$, then neglect.

$f(x) \approx f(x_0) + (x - x_0) f'(x_0) + \cdots$ (2)



Define: $x = x_0 + \delta x$ or $x - x_0 = \delta x$ Sub. into (2)

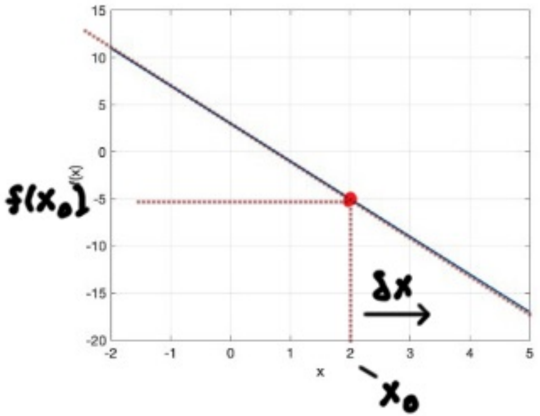
$$\boxed{f(x_0 + \delta x) \approx f(x_0) + \delta x \cdot f'(x_0)} \quad (3)$$

Note: (3) is a fn of δx, so, we can name it $f(\delta x)$.

Example 1

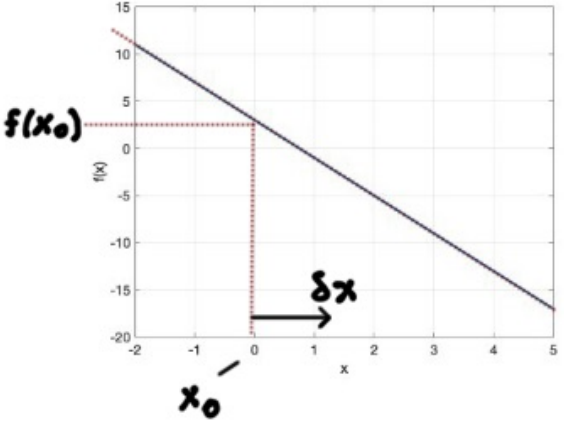
$a)\ f(x) = 3 - 4x,\ x_0 = 2$

$$f(x_0 + \delta x) = f(x_0) + \delta x \cdot f'(x_0)$$
$$= -5 - 4\delta x$$



$b)\ f(x) = 3 - 4x,\ x_0 = 0$

$$f(x_0 + \delta x) = 3 - 4x$$



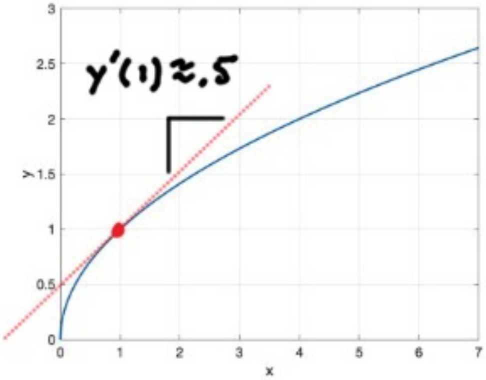
Example 2

$y(x) = \sqrt{x},\ x_0 = 1$ (linearize me ...)

$$y(x_0 + \delta x) \approx y(x_0) + y'(x_0) \cdot \delta x$$

$$\approx 1 + \frac{1}{2} x^{-1/2}\Big|_{x=1} \cdot \delta x$$

$$\boxed{\approx 1 + \frac{1}{2}\delta x}$$



Example 3

$a)\quad y = \sin(ax)$

$b)\quad y = \cos(ax)$

$c)\quad y = x^n$

$d)\quad y = e^{ax}$

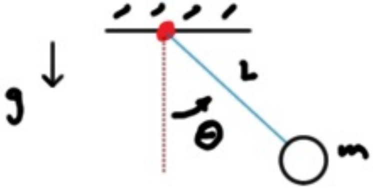
$e)\quad y = x\sin(2x)$

Equilibrium Configurations

1. An equilibrium configuration is a set of values of dependent variables and inputs such that…
2. If the system is placed in this configuration and then released, there is no motion - it stays there forever
3. Nonlinear dynamic systems can have multiple equilibrium configurations
4. We can **make** equilibrium configurations by setting the input to some value
5. We'll denote equilibrium configurations using a zero subscript, e.g. (θ_0, τ_0)

Large Angle Pendulum

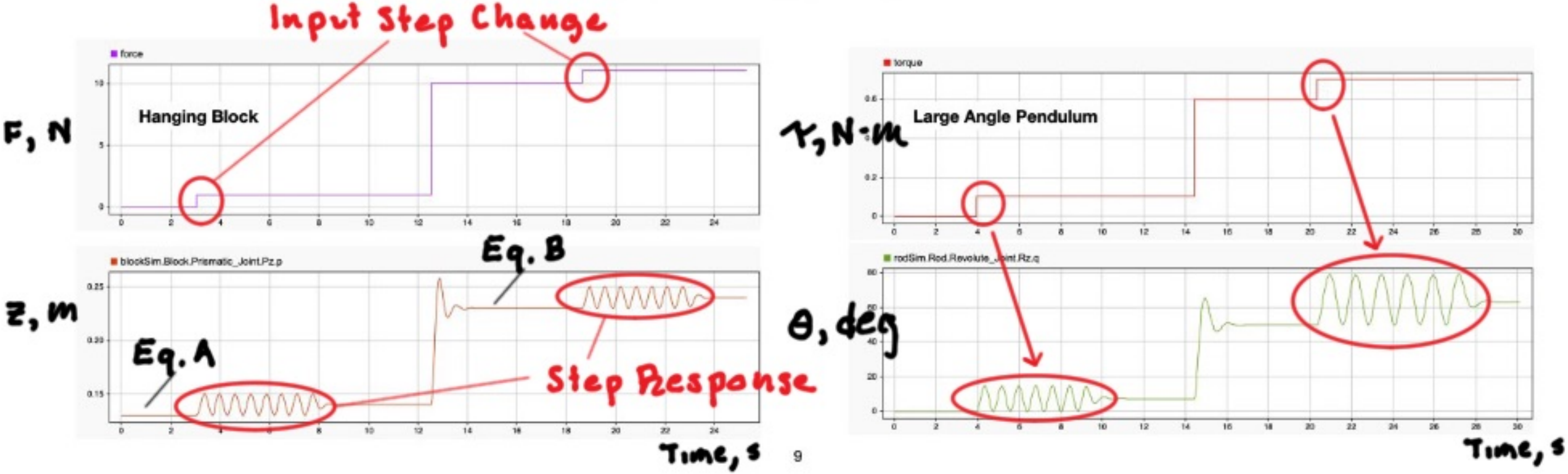
$$mL^2\ddot{\theta} + mgL\sin(\theta) = \tau$$



$\theta_0 = 0$ & $180°$

Equilibrium Configurations, Nonlinear Systems and Data

1. The response of a linear system to small input changes is invariant to its equilibrium configuration
2. The response of a nonlinear system to small input changes depends on its equilibrium configuration
3. Two cases
    1. Large angle pendulum (nonlinear)
    2. Block attached to the ceiling by a spring (linear)



Calculating Equilibrium (With Example)

Consider the large angle pendulum DEQ model: $mL^2\ddot{\theta} + mgL\sin(\theta) = \tau$

1. Set all d/dt terms to zero AND replace dependent variables and inputs with 0 subscript versions

$mgL\sin(\theta_0) = \tau_0$ Equation of Equilibrium

2. Solve for the dependent variable equilibrium configuration for some value of the input, or in general

$$\theta_0 = \sin^{-1}\left(\frac{\tau_0}{mgL}\right)$$

Note: if $m = 1$ kg, $g = 10\ m/s^2$, $L = 2$ m, $\tau_0 = 5$ N-m then

$$\theta_0 = \sin^{-1}(5/20) \approx 14.5°$$

Example 4 - Calculating Equilibrium

Given the DEQ model below, find the equations of equilibrium and an equilibrium configuration such that $x_{1,0} = 4$

$$\ddot{x}_1 + \sqrt{x_1} - x_2 = 0 \qquad \ddot{x}_2 - \dot{x}_1 + 3\cos(x_2) - x_1 = u$$

$$\boxed{x_{1,0}^{1/2} - x_{2,0} = 0 \qquad 3\cos(x_{2,0}) - x_{1,0} = u_0}$$

Eqs. of Equil.

$$2 - x_{2,0} = 0$$

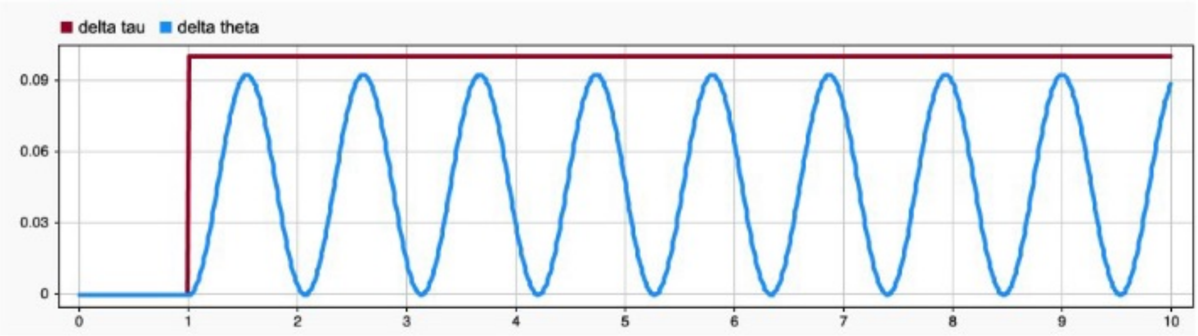
$$x_{2,0} = 2 \qquad 3\cos(2) - 4 = u_0$$

$$\boxed{\text{Equil. Config.: } x_{1,0} = 4,\ x_{2,0} = 2,\ u_0 = 3\cos(2) - 4 \approx -5.25}$$

DEQ Linearization

- Starting with a nonlinear DEQ we'll generate a linear DEQ that is valid only when motion is in the linear operating regime
- The linearized DEQ model uses **new dependent variables and new input variables** that describe how small changes in the input generate small changes in the dependent variables
- Here's an example of a large angle pendulum DEQ linearized at $\theta_0 = 30°$ and its step response

$$mL^2\ddot{\theta} + mgL\sin\theta = \tau \;\rightarrow mL^2\,\ddot{\delta\theta} + \frac{\sqrt{3}}{2}mgL\,\delta\theta = \delta\tau$$



DEQ Linearization - Procedure

1. Define an equilibrium configuration, e.g. x_0, y_0, u_0 establishing the system's **equations of equilibrium**

2. Define variables of small motion about equilibrium, e.g. $x = x_0 + \delta x, y = y_0 + \delta y, u = u_0 + \delta u$ and **substitute them into the nonlinear differential equations**

3. Identify the original DEQ's nonlinear terms and linearize them about the equilibrium configuration

4. Use the **equations of equilibrium** to cancel terms in the linearized differential equations

5. The resulting differential equation model should have δ quantities as dependent variables without ANY **standalone** equilibrium quantities, e.g. x_0, u_0, etc.

Example 6 - DEQ Linearization

Linearize the DEQ model below about the equilibrium configuration found in Example 5.

$$\ddot{x}_1 + \sqrt{x_1} - x_2 = 0 \qquad \ddot{x}_2 - \dot{x}_1 + 3\cos(x_2) - x_1 = u$$

1. Define an equilibrium configuration, e.g. x_0, y_0, u_0 establishing the system's **equations of equilibrium**

$$x_{1,0}^{1/2} - x_{2,0} = 0 \qquad 3\cos(x_{2,0}) - x_{1,0} = u_0$$

From Ex 5...

$$\boxed{\text{Equil. Config.: } x_{1,0} = 4,\ x_{2,0} = 2,\ u_0 = 3\cos(2) - 4 \approx -5.25}$$

Example 6 - DEQ Linearization (continued)

$$\ddot{x}_1 + \sqrt{x_1} - x_2 = 0$$
$$\ddot{x}_2 - \dot{x}_1 + 3\cos(x_2) - x_1 = u$$

2. Define variables of small motion about equilibrium, e.g. $x = x_0 + \delta x, y = y_0 + \delta y, u = u_0 + \delta u$ and **substitute them into the nonlinear differential equations**

3. Identify the original DEQ's nonlinear terms and linearize them about the equilibrium configuration

$$x_{1,0}^{1/2} - x_{2,0} = 0$$
$$3\cos(x_{2,0}) - x_{1,0} = u_0$$

$$\delta\ddot{x}_1 + (x_{1,0} + \delta x_1)^{1/2} - (x_{2,0} + \delta x_2) = 0$$
$$\delta\ddot{x}_2 - \delta\dot{x}_1 + 3\cos(x_{2,0} + \delta x_2) - (x_{1,0} + \delta x_1) = u_0 + \delta u$$

$$(x_{1,0} + \delta x_1)^{1/2} \approx x_{1,0}^{1/2} + \frac{1}{2} x_1^{-1/2}\Big|_{x_1 = x_{1,0}} \cdot \delta x_1 = x_{1,0}^{1/2} + \frac{1}{2} x_{1,0}^{-1/2} \cdot \delta x_1$$

$$\cos(x_{2,0} + \delta x_2) \approx \cos(x_{2,0}) - \sin(x_{2,0}) \cdot \delta x_2$$

Example 6 - DEQ Linearization (continued)

$$\ddot{x}_1 + \sqrt{x_1} - x_2 = 0$$
$$\ddot{x}_2 - \dot{x}_1 + 3\cos(x_2) - x_1 = u$$

2. Define variables of small motion about equilibrium, e.g. $x = x_0 + \delta x, y = y_0 + \delta y, u = u_0 + \delta u$ and **substitute them into the nonlinear differential equations**

3. Identify the original DEQ's nonlinear terms and linearize them about the equilibrium configuration

$$\delta\ddot{x}_1 + \underline{x_{1,0}^{1/2} + \frac{1}{2} x_{1,0}^{-1/2} \cdot \delta x_1} - (x_{2,0} + \delta x_2) = 0$$

$$\delta\ddot{x}_2 - \delta\dot{x}_1 + \underline{3[\cos(x_{2,0}) - \sin(x_{2,0}) \cdot \delta x_2]} - (x_{1,0} + \delta x_1) = u_0 + \delta u$$

Example 6 - DEQ Linearization (continued)

$$\ddot{x}_1 + \sqrt{x_1} - x_2 = 0$$
$$\ddot{x}_2 - \dot{x}_1 + 3\cos(x_2) - x_1 = u$$

4. Use the **equations of equilibrium** to cancel terms in the linearized differential equations

$x_{1,0} = 4,\ x_{2,0} = 2,\ u_0 = 3\cos(2) - 4 \approx -5.25$

5. The resulting differential equation model should have δ quantities as dependent variables without ANY **standalone** equilibrium quantities, e.g. x_0, u_0, etc.

$$x_{1,0}^{1/2} - x_{2,0} = 0$$
$$3\cos(x_{2,0}) - x_{1,0} = u_0$$

$$\delta\ddot{x}_1 + \underline{x_{1,0}^{1/2} + \frac{1}{2}x_{1,0}^{-1/2} \cdot \delta x_1} - (x_{2,0} + \delta x_2) = 0$$

$$\delta\ddot{x}_2 - \delta\dot{x}_1 + 3[\underline{\cos(x_{2,0}) - \sin(x_{2,0}) \cdot \delta x_2}] - (x_{1,0} + \delta x_1) = u_0 + \delta u$$

$$\delta\ddot{x}_1 + \frac{1}{2}x_{1,0}^{-1/2} \cdot \delta x_1 - \delta x_2 = 0$$

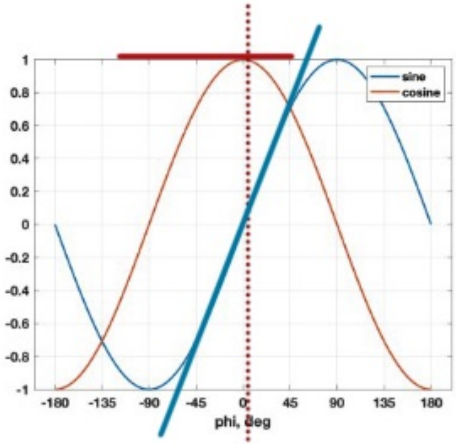
or

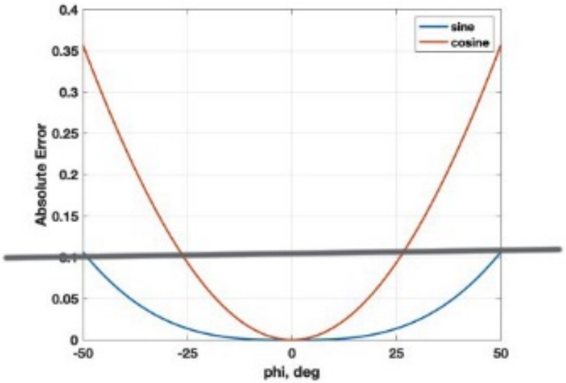
$$\boxed{\begin{aligned} &\delta\ddot{x}_1 + \frac{1}{4}\delta x_1 - \delta x_2 = 0 \\ &\delta\ddot{x}_2 - \delta\dot{x}_1 - 3\sin(2)\,\delta x_2 - \delta x_1 = \delta u \end{aligned}}$$

Final Note

1. The $\sin \phi$ and $\cos \phi$ functions appear often in nonlinear DEQ models

2. If the equilibrium configuration is $\phi_0 = 0$, then they linearize to ϕ and 1 respectively but their linear operating regimes are quite different.





Summary

- Given a system of nonlinear DEQs you can (with more practice…)
  a. Create its equations of equilibrium and find an equilibrium configuration
  b. Create a linear model

- Given step responses at two different equilibrium points, determine if the system is linear or nonlinear

- Linearize nonlinear functions