Inverse Laplace

Laplace Transforms are one of the best ways (in my opinion) to solve linear DEQs given some strange input function.

Our focus is to use ILT to understand the **character of the a DEQ model response** to just about any input.

With practice you'll be able to look at a Laplace Domain function and see the time-domain response.

In these notes we'll work just with the ILT, then apply it to find the response of a DEQ in the next set

Skills

1. Given any $Y(s)$ calculate $y(t)$
2. Given a $Y(s)$ be able to classify it as being strictly proper, proper or improper. As long as it is NOT improper, be able to decompose it into partial fractions

Terminology

1. Rational Function
2. Polynomial Order
3. Multiplicity
4. Simple (in the context of roots)
5. Rational Function Flavors - strictly proper, proper, improper

Laplace Transform Table

Laplace Transform Table

#	$f(t)$	$F(s)$	#	$f(t)$	$F(s)$
1	$\delta(t)$	1	11	$1-e^{-at}$	$\frac{a}{s(s+a)}$
2	u_s	$\frac{1}{s}$	12	$\frac{1}{a}(at-1+e^{-at})$	$\frac{a}{s^2(s+a)}$
3	t	$\frac{1}{s^2}$	13	$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$
4	t^2	$\frac{2!}{s^3}$	14	$(1-at)e^{-at}$	$\frac{s}{(s+a)^2}$
5	t^3	$\frac{3!}{s^4}$	15	$1-(1+at)e^{-at}$	$\frac{a^2}{s(s+a)^2}$
6	t^m	$\frac{m!}{s^{m+1}}$	16	$be^{-bt}-ae^{-at}$	$\frac{(b-a)s}{(s+a)(s+b)}$
7	e^{-at}	$\frac{1}{(s+a)}$	17	$\sin(bt)$	$\frac{b}{s^2+b^2}$
8	te^{-at}	$\frac{1}{(s+a)^2}$	18	$\cos(bt)$	$\frac{s}{s^2+b^2}$
9	$\frac{1}{2!}t^2e^{-at}$	$\frac{1}{(s+a)^3}$	19	$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$
10	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$	$\frac{1}{(s+a)^m}$	20	$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$

Laplace Properties Table

Laplace Transform Properties

$f(t)$	$F(s)$	Property
$\alpha f_1(t) + \beta f_2(t)$	$\alpha F_1(s) + \beta F_2(s)$	Superposition
$f(t-\lambda)$	$F(s)e^{-s\lambda}$	Time Delay ($\lambda \geq 0$)
$f(at)$	$\frac{1}{\lvert a\rvert}F(\frac{s}{a})$	Time Scaling
$e^{-at}f(t)$	$F(s+a)$	Shift in Frequency
$f^{(m)}(t)$	$s^m F(s) - s^{m-1}f(0) + -s^{m-2}\dot{f}(0) - \cdots - f^{(m-1)}(0)$	Differentiation
$\int f(\zeta)d\zeta$	$\frac{1}{s}F(s)$	Integration
$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$	Convolution
$f(0)$	$\lim_{s\to\infty} sF(s)$	Initial Value Theorem
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$	Final Value Theorem
$tf(t)$	$-\frac{d}{ds}F(s)$	Multiplication by Time

Big Picture

Given a DEQ with initial conditions and an input function, write out the form of its solution, a.k.a. its response, e.g. $\ddot{y} + 3\dot{y} + 2y = u$ with zero initial conditions and $u = 4e^{-3t}$.

Steps

1. Take the LT of the DEQ and solve for its dependent variable, e.g. $Y(s) = \dfrac{4}{s^3 + 6s^2 + 11s + 6}$, If $Y(s)$ is in the LT Table, then write down $y(t)$ and you're done! Unlikely…
2. Expand $Y(s)$ into partial fractions such that each term IS in the LT Table. This introduces unspecified coefficients, e.g. $Y(s) = \dfrac{c_1}{s+1} + \dfrac{c_2}{s+2} + \dfrac{c_3}{s+3}$
3. Write out the form of $y(t)$ using the LT Table, e.g. $y(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}$. We now know the character of the response!
4. If you are very ambitious, solve for the coefficients.

Background - Polynomials and Rational Functions

Polynomial Roots - Given a polynomial $F(s)$, all the solutions to the equation $F(s) = 0$. The roots can be real or complex. Complex roots always appear as complex conjugate pairs: $s_{1,2} = -4 \pm 2j$

Root Multiplicity - The number of times each root of $F(s) = 0$ appears. If a root appears just once, then it is called simple.

Rational Function - a function that can be written with a numerator and denominator, both being polynomials, e.g.

$$X(s) = \frac{s+3}{s^2 + 4s + 100}$$

Strictly Proper Rational Function - The order of the denominator is **greater than** the order of the numerator

Proper Rational Function - The order of the denominator and numerator are **equal**

Improper Rational Function - the order of the denominator is **less than** the order of the numerator.

Example 1

Find and classify the roots of the denominator of: $Q(s) = \dfrac{s^2 + 3s + 7}{s^2(s+4)(s^2+2s+3)}$

Roots: $s = 0$, $m = 2$

$s = -4$, simple, real

$s^2 + 2s + 3 = 0$, $s = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{2} = -1 \pm \sqrt{2}\, j$, complex

MATLAB (numeric)

```
>> D = conv( [1 4 0 0 ],[1 2 3] )

>> roots(D)
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Partial Fraction Expansion Steps

1. The PFE terms, form and number, are determined from the denominator
2. The number of PFE terms is the same as the order of the denominator
3. Each unique root contributes n terms to the PFE where n is the multiplicity of the root
4. Complex roots contribute two terms to the PFE with their form motivated by your LT table - elements 19 and 20. NOTE: All we need to know are the values of a and b. Recall completing the square…

It's best to look at a complete, nasty example: $V(s) = \dfrac{s^3 + 6s + 14}{s^2(s+3)^3(s+7)(s^2+2s+10)}$

$$V(s) = \frac{c_1}{s^2} + \frac{c_2}{s} + \frac{c_3}{(s+3)^3} + \frac{c_4}{(s+3)^2} + \frac{c_5}{s+3} + \cdots$$

$$+ \frac{c_6}{s+7} + ? + \frac{3 \cdot c_7}{s^2+2s+10} + \frac{(s+1)c_8}{s^2+2s+10}$$

$s^2+2s+10=0$
complex roots, so
2 terms: #19, #20.
Need a & b.
Completed Square
$(s+1)^2+3^2$
$a=1,\ b=3$

Partial Fraction Expansion - Proper Rational Functions

1. PFE works great for **strictly proper** rational functions
2. If the rational function is **proper**, you must do a long division first. You'll get a constant plus a strictly proper rational function, which you can do PFE on. When this happens, you'll always have an impulse function in the resulting time-domain function.
3. At the end of the notes there is one example, we'll work with **strictly proper** rational functions in this course, but always look before you do the PFE.

Example 2

Find the form of the ILT for the rational function below

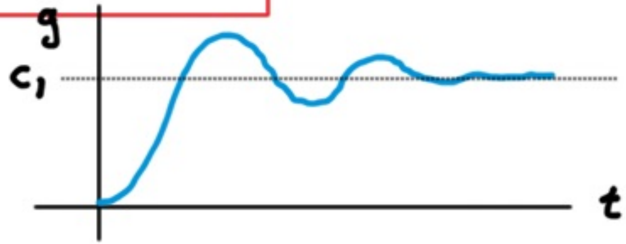
$$G(s) = \frac{s+3}{s(s^2+14s+58)}$$

complex roots: #19, #20

complete the sqr to find a & b. $(s+7)^2 + 3^2 \therefore a=7, b=3$

$$G(s) = \frac{c_1}{s} + \frac{3c_2}{s^2+14s+58} + \frac{c_3(s+7)}{s^2+14s+58}$$

$$\boxed{g(t) = c_1 u_s + c_2 e^{-7t} \sin 3t + c_3 e^{-7t} \cos 3t}$$



Solving for the Coefficients

We aren't going to do this often, but it's important to see it at least once. Here are the steps using an example

1. Given the rational function $Y(s) = \dfrac{s-4}{(s+2)(s+5)}$

2. Form its PFE and equate both forms of $Y(s)$: $\dfrac{s-4}{(s+2)(s+5)} = \dfrac{c_1}{s+2} + \dfrac{c_2}{s+5}$

3. Recombine the right side to get a common denominator

$$\frac{s-4}{(s+2)(s+5)} = \frac{c_1}{s+2} + \frac{c_2}{s+5} = \frac{c_1(s+5) + c_2(s+2)}{(s+2)(s+5)}$$

4. Equate the numerators and note that s is a variable and can take on ANY value. So the only way to balance the numertors is if the coefficients of all s terms are 0

$$s - 4 = (c_1 + c_2)s + (5c_2 + 2c_2)$$

or

$$c_1 + c_2 = 1 \text{ and } 5c_1 + 2c_2 = -4, \text{ or } c_1 = -2 \text{ and } c_2 = 3.$$

Example 3

H is strictly proper

Find the form of the ILT of the function below and calculate the coefficients

$$H(s) = \frac{6s^2 + 40s + 28}{s(s+4)(s+7)} \qquad H(s) = \frac{c_1}{s} + \frac{c_2}{s+4} + \frac{c_3}{s+7} \qquad \boxed{h(t) = c_1 u_s + c_2 e^{-4t} + c_3 e^{-7t}}$$

common Denom.

$$\frac{c_1}{s} + \frac{c_2}{s+4} + \frac{c_3}{s+7} = \frac{c_1(s+4)(s+7) + c_2 s(s+7) + c_3 s(s+4)}{s(s+4)(s+7)}$$

$$6s^2 + 40s + 28 = c_1(s^2 + 11s + 28) + c_2(s^2 + 7s) + c_3(s^2 + 4s)$$

$$= (c_1 + c_2 + c_3)s^2 + (11c_1 + 7c_2 + 4c_3)s + 28c_1$$

3eq, 3unk

$$\left.\begin{aligned} 6 &= c_1 + c_2 + c_3 \\ 40 &= 11c_1 + 7c_2 + 4c_3 \\ 28 &= 28c_1 \end{aligned}\right\} \boxed{\begin{aligned} c_1 &= 1 \\ c_2 &= 2 \\ c_3 &= 3 \end{aligned}}$$

Example 4

Y is proper: long divide ...

Find the form of the ILT of the function below

$$Y(s) = \frac{s^2 + 4s + 2}{s(s+4)}$$

$$\begin{array}{r} 1 \\ s^2+4s \overline{)\, s^2+4s+2} \\ -(s^2+4s) \\ \hline 2 \end{array}$$

$$, \quad Y(s) = 1 + \frac{2}{s(s+4)}$$

strictly proper

$$\frac{2}{s(s+4)} = \frac{c_1}{s} + \frac{c_2}{s+4}$$

$$\boxed{y(t) = \delta(t) + c_1 u_s + c_2 e^{-4t}}$$

Example 5

z is strictly proper

Find the form of the ILT for the function below

$$Z(s) = \frac{s-1}{(s+2)(s^2+2s+10)}$$

complex factor, complete sqr, a & b for #19 & #20.

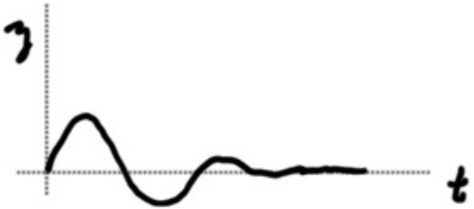
$(s+1)^2+3^2$, $a=1$, $b=3$!

$$Z = \frac{c_1}{s+2} + \frac{3c_2}{s^2+2s+10} + \frac{c_3(s+1)}{s^2+2s+10}$$

$$\boxed{z(t) = c_1 e^{-2t} + c_2 e^{-t} \sin 3t + c_3 e^{-t} \cos 3t}$$

$c_1 + c_3 = 0 \ldots$
subs cmd

MATLAB



$z(0) = c_1 + c_3$
$z(\infty) = 0$

Decays & Oscillates

Summary

With a little more practice you can…

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