Kinematics - Motivation

1. Dynamic systems often have several, bodies connected by gears, belts, and joints

2. Kinematics lets us relate the motion of connected components and generate expressions for their acceleration used for dynamics analysis

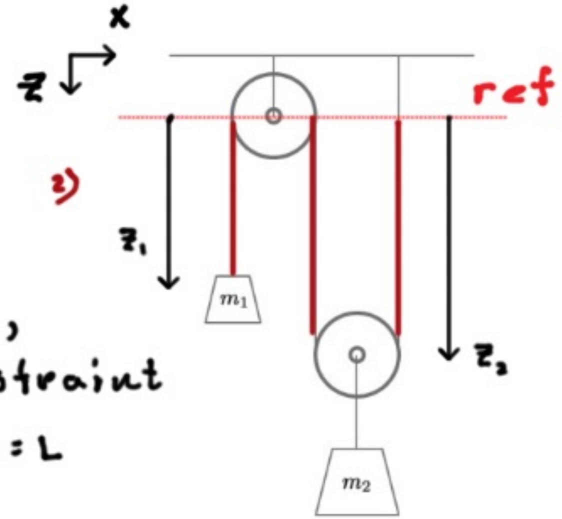
Skills

1. Given a description of a dynamic system
    a. Determine suitable variables for describing motion, constraint equations and degrees of freedom (DOF)
    b. Calculate expressions for key positions, velocities, and accelerations needed for dynamic equation generation
2. Develop constraint equations for specific types of systems containing:
    1. Pulleys
    2. Belts and gears
    3. Rack and pinions

Pulley Kinematic Analysis

1. Define an inertial coordinate frame
2. Define variables of motion, with respect to your frame, for all the bodies of interest
3. Create position constraint equations - one for each cord. Note: a cord's length is not necessarily constant, for example, if wound on a winch and drum
4. Differentiate the constraint equations to form velocity and acceleration constraints

1)



3) One cord, One constraint

$z_1 + 2z_2 = L$

4) $\dot{z}_1 + 2\dot{z}_2 = 0$

$\ddot{z}_1 + 2\ddot{z}_2 = 0$

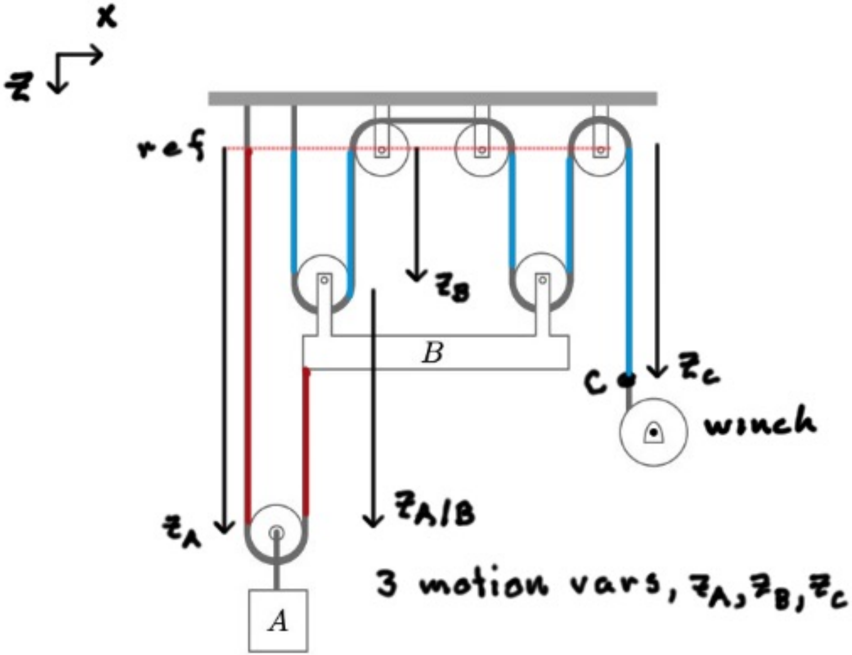
Example 1

Define motion variables and create suitable kinematic expressions for position, speed and acceleration.

- 2 cords, 2 constraint eqs.
- $z_A + z_{A/B} = L_1$, $z_{A/B} = z_A - z_B$

$$\boxed{2z_A - z_B = L_1}$$

- $\boxed{4z_B + z_C = L_2}$



Constraint Eqs: $2z_A - z_B = L_1$
$4z_B + z_C = L_2$

Degrees of Freedom and Motion Variables

1. For dynamic system analysis we need a set of **motion variables**, also known as **generalized coordinates**, that uniquely define the system's geometry.

2. Typically there is an infinite number of ways to define the motion variables. Even the number you use is not unique

3. Depending on the number of variables you use and how you define them, there may exist **constraint equations** that relate them

4. The number of degrees of freedom, DOF, is

   DOF = # of generalized coordinates - # of constraint equations

5. Returning to the pulleys

   DOF = # of moving objects - # of ropes

Ex 1: ?

Approach

1. Identify the objects that have mass - much like creating a FBD

2. Identify how each of the objects can move, e.g. translate only in x, translate in x and y and rotate, etc.

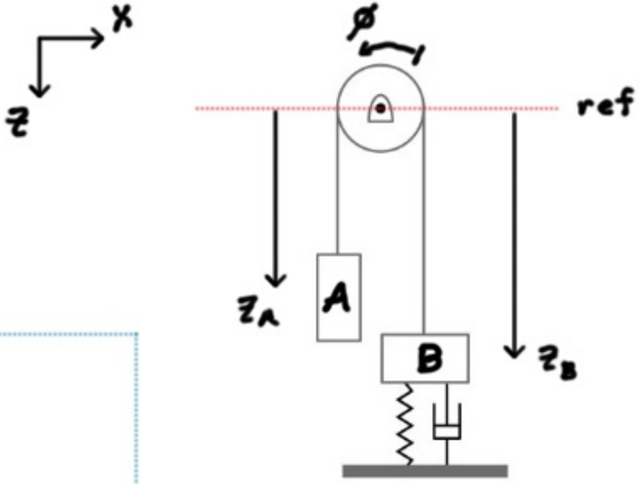
3. Define a set of motion variables (generalized coordinates) that uniquely describe the system's geometry, iterating on different choices of variables so that you get the fewest possible. **Think of it this way:** If some tells you the values of the motion variables, you should be able to sketch the system's unique configuration. If your set of motion variable selection passes this test, then it's correct

4. Identify any relationships between variables and develop their constraint equations, just like you did with pulleys

5. Calculate the number of DOFs

Example 2

Define motion variables and create suitable kinematic expressions if needed and calculate the number of DOFs. Assume that the cable doesn't stretch and it does not slip on the pulley. Also, assume both blocks and the pulley have significant mass.



Motion Vars: ϕ, z_A, z_B

Constraint Descriptions:
- No stretch
- No slip

Constraint Equations:
- $z_A + z_B = L$ & $r \cdot \Delta\phi = \Delta z_A$

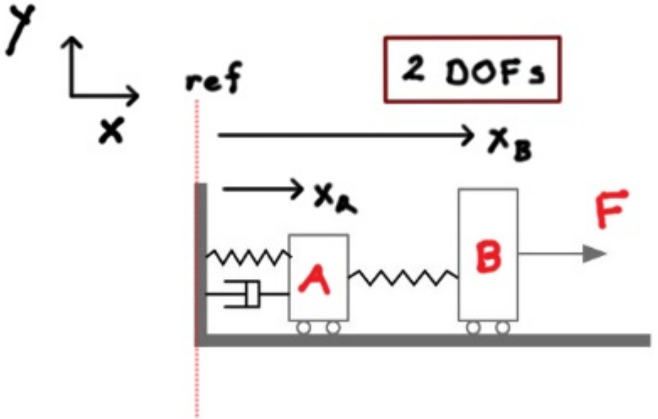
DOFs: $3 - 2 = 1$

Δ is disturbing. Maybe there are better motion var. choices.

$\boxed{\text{1 DOF}}$

Example 3

Define motion variables and create suitable kinematic expressions if needed and calculate the number of DOFs. Assume both blocks have significant mass.

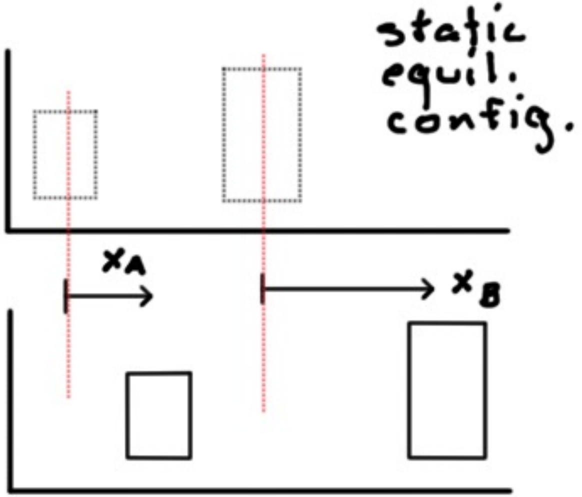


Motion Vars: x_A, x_B

Constraint Descriptions: None

Constraint Equations: None

DOFs: 2

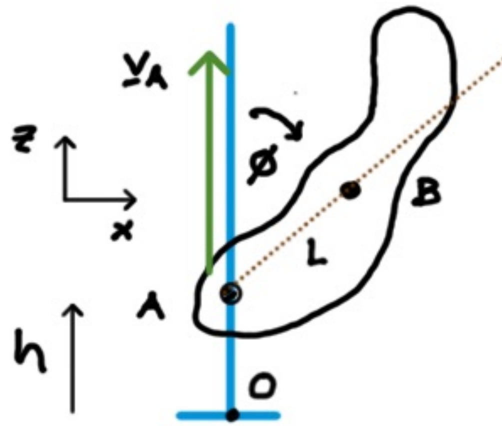


Better motion variable def'n:

Displacement relative to static equilibrium!

Velocity and Accelertion of a Fixed Point in a Rigid Body - Nonrotating Frame

Consider the potato whose angular velocity is ω with some reference point A and some fixed point in the potato B.



$$\mathbf{v}_B = \mathbf{v}_A + \omega \times \mathbf{p}_{B/A}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\omega} \times \mathbf{p}_{B/A} + \omega \times \left(\omega \times \mathbf{p}_{B/A}\right)$$

Terms:

$\underline{v}_A = \dot{h}\,\hat{z}$

$\underline{a}_A = \ddot{h}\,\hat{z}$

$\underline{\omega} = \dot{\phi}\,\hat{y}$

$\dot{\underline{\omega}} = \ddot{\phi}\,\hat{y}$

$\underline{p}_{B/A} = L(\sin\phi\,\hat{x} + \cos\phi\,\hat{z})$

$\underline{v}_B$, $\underline{a}_B$:

$\underline{v}_B = \dot{h}\,\hat{z} + L\dot{\phi}(-s\phi\,\hat{z} + c\phi\,\hat{x})$

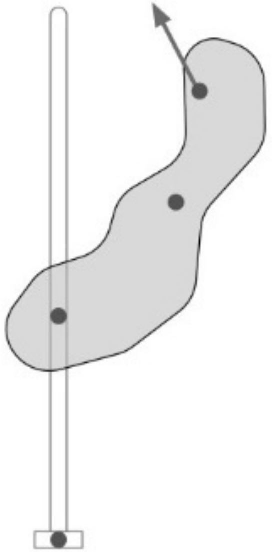
$\underline{a}_B = \ddot{h}\,\hat{z} + L\ddot{\phi}(-s\phi\,\hat{z} + c\phi\,\hat{x}) + \ldots$

$L\dot{\phi}^2(-s\phi\,\hat{x} - c\phi\,\hat{z})$

Note: $\underline{a}_B$ is deriv. of components of $\underline{v}_B$.

Example 4

Define suitable motion variables and create an expression for the acceleration of the potato's center of mass



Velocity and Accelertion of a Fixed Point in a Rigid Body - Rotating Frame

Consider the potato whose angular velocity is ω with some reference point A and some fixed point in the potato B.

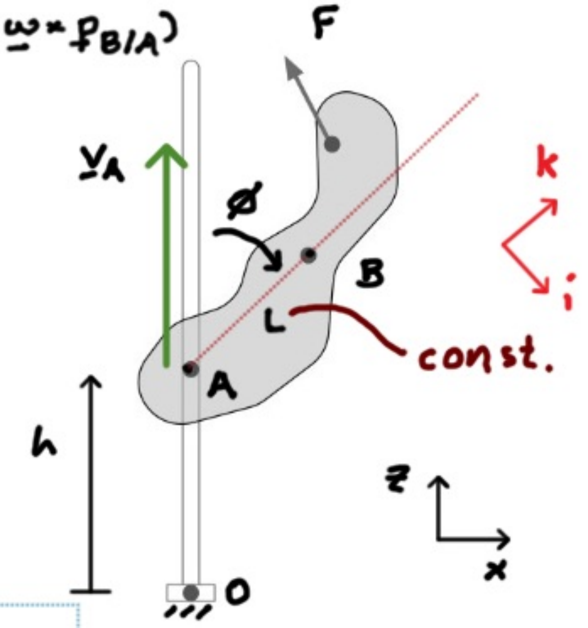
$$\mathbf{v}_B = \mathbf{v}_A + \omega \times \mathbf{p}_{B/A}$$

$\underline{\omega}$ is now the abs. ang. vel. of the rotating frame, a.k.a. body-fixed frame.

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\omega} \times \mathbf{p}_{B/A} + \omega \times \left(\omega \times \mathbf{p}_{B/A}\right)$$

Example 5 $\underline{v}_B = \underline{v}_A + \underline{\omega} \times \underline{r}_{B/A}, \quad \underline{a}_B = \underline{a}_A + \dot{\underline{\omega}} \times \underline{r}_{B/A} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{B/A})$

Define suitable motion variables and create an expression for the acceleration of the potato's center of mass



Terms:

$$\hat{z} = -s\phi\,\hat{i} + c\phi\,\hat{k}$$

$\underline{v}_A = \dot{h}\,\hat{z}$

$\underline{a}_A = \ddot{h}\,\hat{z}$

$\underline{\omega} = \dot{\phi}\,\hat{j}$

$\dot{\underline{\omega}} = \ddot{\phi}\,\hat{j}$

$\underline{r}_{B/A} = L\,\hat{k}$

$\underline{v}_B$, $\underline{a}_B$:

$$\underline{v}_B = \dot{h}\,\hat{z} + L\dot{\phi}\,\hat{i}$$

$$\underline{a}_B = \ddot{h}\,\hat{z} + L\ddot{\phi}\,\hat{i} - L\dot{\phi}^2\,\hat{k}$$

Note: core terms are simpler, as are cross products.

Velocity and Accelertion of a Moving Point in a Rigid Body - Rotating Frame

Consider the potato whose angular velocity is ω with some reference point A and some moving point in the potato B.

Vel of B as seen by observer sitting on the rotating frame at A.

$$\mathbf{v}_B = \mathbf{v}_A + \omega \times \mathbf{p}_{B/A} + (\dot{\mathbf{p}}_{B/A})_{rel}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\omega} \times \mathbf{p}_{B/A} + \omega \times \left(\omega \times \mathbf{p}_{B/A}\right) + (\ddot{\mathbf{p}}_{B/A})_{rel} + 2\omega \times (\dot{\mathbf{p}}_{B/A})_{rel}$$

Note: Special Case: Fixed pt. rotation, $\underline{v}_A = \underline{a}_A = 0$

$\underline{v}_B = (\dot{\underline{p}}_B)_{rel} + \underline{\omega} \times \underline{p}_B$

$\underline{a}_B = (\dot{\underline{v}}_B)_{rel} + \underline{\omega} \times \underline{v}_B$

1. Differentiate components of $\underline{p}_B, \underline{v}_B$
2. Form $\underline{v}_B, \underline{a}_B$

Example 6

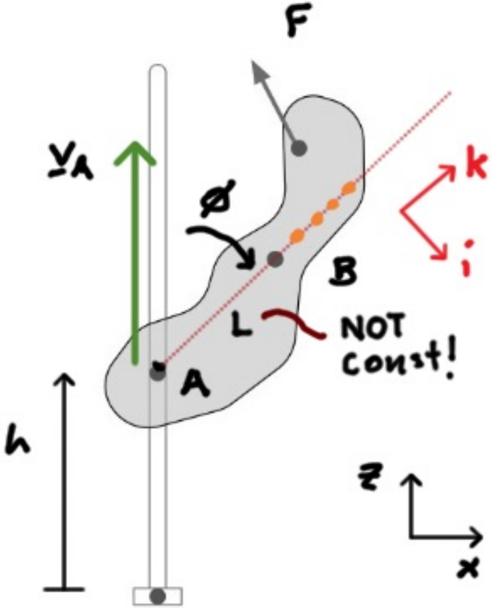
Define suitable motion variables and create an expression for the velocity of the ant crawling on the potato

Terms From p.12 And:

$$(\dot{\underline{p}}_{B/A})_{rel} = \dot{L}\,\hat{k}$$

$$(\ddot{\underline{p}}_{B/A})_{rel} = \ddot{L}\,\hat{k}$$

$$2\underline{\omega}\times(\dot{\underline{p}}_{B/A})_{rel} = 2\dot{L}\dot{\phi}\,\hat{i}$$



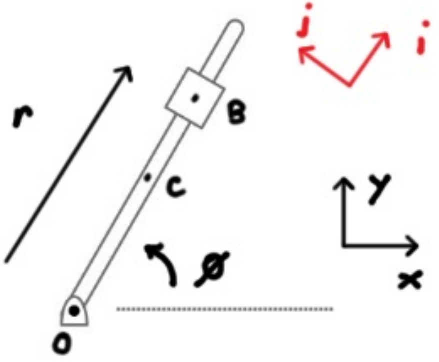
Kinematics Analysis Approach

1. Identify the location of the center of mass of each body
2. Attach a frame to each body. A frame may translate, rotate or both. If the frame rotates, define the frame’s absolute angular velocity and acceleration
3. Create postion vectors to each body’s center of mass using intermediate frames to make the expressions more compact
4. Apply the moving frame kinematics above, or develop them ‘organically’ as you form velocity and acceleration expressions
5. Note: For complex mechanisms, this can be difficult. In t his class we’ll keep things reasonable.

Example 7 - Rotation + Extension

The rod is free to rotate about the pin and collar slides freely on the rod. Define motion variables. How many DOFs does it have? Create an expression for the acceleration of the collar.

Variables: r, ϕ
Constraints: 0
DOFs: 2



Terms:

$$\underline{\omega} = \dot{\phi}\,\hat{k}$$

$$\underline{\rho}_B = r\,\hat{i}$$

$$\underline{v}_B = \dot{r}\,\hat{i} + r\dot{\phi}\,\hat{j}$$

$$\underline{a}_B = \ddot{r}\,\hat{i} + \dot{r}\dot{\phi}\,\hat{j} + r\ddot{\phi}\,\hat{j} + \dot{r}\dot{\phi}\,\hat{j} - r\dot{\phi}^2\,\hat{i}$$

$$= (\ddot{r} - r\dot{\phi}^2)\,\hat{i} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\hat{j}$$

Fixed Pt. Rotation

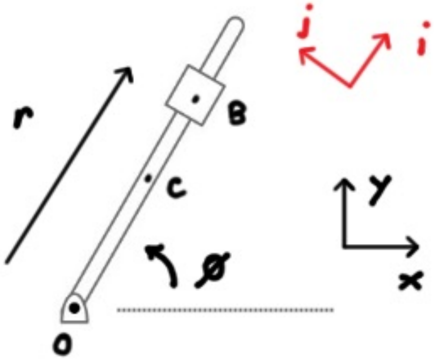
$$\underline{v}_B = (\dot{\underline{\rho}}_B)_{rel} + \underline{\omega} \times \underline{\rho}_B$$

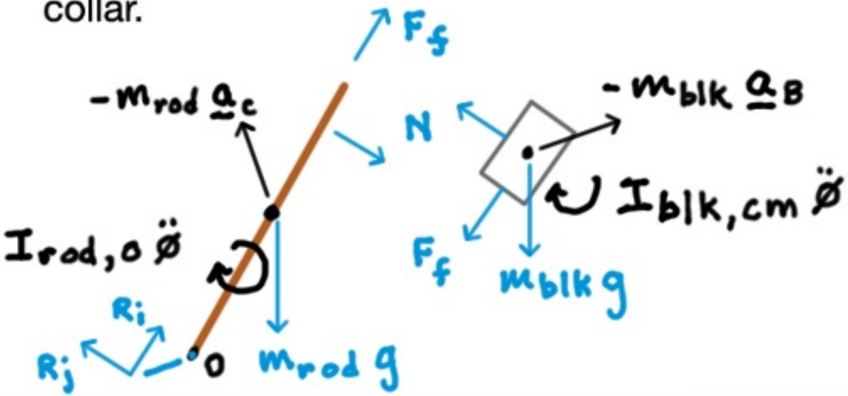
$$\underline{a}_B = (\dot{\underline{v}}_B)_{rel} + \underline{\omega} \times \underline{v}_B$$

Example 7 - Rotation + Extension - Dynamics

The rod is free to rotate about the pin and collar slides freely on the rod. Define motion variables. How many DOFs does it have? Create an expression for the acceleration of the collar.

$$\boxed{\begin{aligned} \Sigma \underline{F} - m\underline{a} &= 0 \\ \Sigma M - I\ddot{\phi} &= 0 \end{aligned}}$$





Unknowns: $\ddot{r}, \ddot{\phi}, N, F_f, R_i, R_j$

Equations: 6

$\therefore$ Solvable!

Rod:

$\Sigma \underline{F}_{rod} - m_{rod}\,\underline{a}_c = 0$ (2 eqs)

$\Sigma M_o - I_{rod,o}\,\ddot{\phi} = 0$ (1 eq)

Blk:

$\Sigma \underline{F}_{blk} - m_{blk}\,\underline{a}_B = 0$ (2 eqs)

$\cancel{\Sigma M_B - I_{blk,cm}\,\ddot{\phi} = 0 \text{ (1 eq)}}$

Friction: $F_f = \mu N$ (1 eq)

Instantaneous Center of Zero Velocity (Planar Motion)

1. At every instant of a body's general motion there exists an axis, orthogonal to the plane, where the motion of any point on the body can be expressed as a pure rotation about this axis.
2. The point of intersection of this axis and the plane is the instantaneous center of zero velocity, C. Note that the absolute acceleration of C is in general not zero.
3. To locate C you need two absolute velocity vectors of points moving with the body, and then:
    a. Case 1: The two vectors are not parallel. Draw lines perpendicular to the velocities from both base points. Their intersection is C
    b. Case 2: The two vectors are parallel. Draw two lines. One through both velocity vector base points and the other through their end points. The point of intersection is C

Find the mistake:

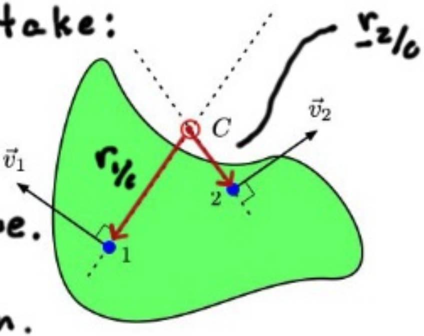
$\underline{v}_1 = \underline{\omega} \times \underline{r}_{1/c}$

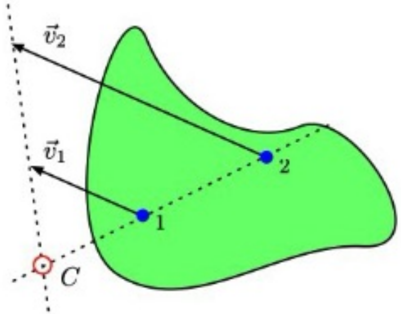
$\underline{v}_2 = \underline{\omega} \times \underline{r}_{2/c}$

Assume $\underline{v}_2$ is true.

$\underline{\omega}$ is ccw. ↺

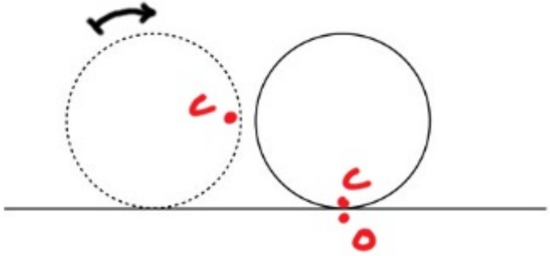
$\underline{v}_1$ is in opp. drxn.





Rolling

1. If there is NO sliding
    a. Translation and rotation are related through kinematics.
    b. Contact point has zero velocity
1. If there IS sliding, then the body has independent rotation and translation.
2. Analysis Approach: Velocity and Acceleration of a Fixed Point in a Rigid Body



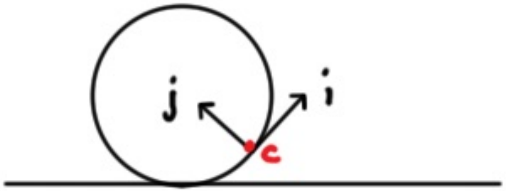
No Slip $\rightarrow V_c = 0$ (Deduction)

- $V_o = 0$
- $V_{c/o} = V_c - V_o$
- No Slip $\rightarrow V_{c/o} = 0$

- No Slip $\rightarrow V_c = 0$

No Slip $\not\rightarrow a_c = 0$?

Consider the body-fixed rotating frame at c.

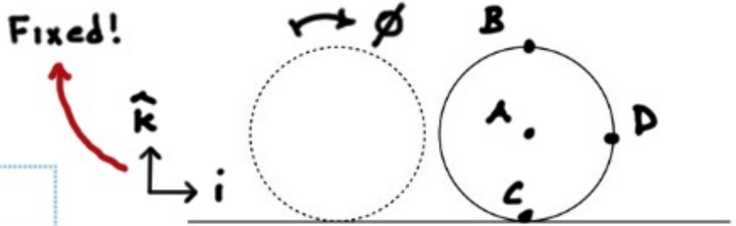


$\underline{a}_c$ must have an $\underline{\omega} \times \_\_$ term!

Example 8 - Rolling

Exploit $\underline{v}_c = \underline{0}$ to explore $\underline{v}$ & $\underline{a}$ of pts.

Examine the acceleration for points on the circumference of the non-skiddin wheel, incluiding its instantaneous zero velocity point where it contacts the road.



A) $\underline{v}_A = \underline{v}_c + \underline{\omega} \times \underline{\rho}_{A/c} = \boxed{0 + R\dot{\phi}\,\hat{i}}$

$\underline{a}_A = R\ddot{\phi}\,\hat{i}$

C) $\underline{a}_A = \underline{a}_c + \dot{\underline{\omega}} \times \underline{\rho}_{A/c} + \underline{\omega} \times (\underline{\omega} \times \underline{\rho}_{A/c})$ Solve for $\underline{a}_c$

$\underline{a}_c = R\ddot{\phi}\,\hat{i} - \dot{\underline{\omega}} \times \underline{\rho}_{A/c} - \underline{\omega} \times (\underline{\omega} \times \underline{\rho}_{A/c})$

$= \boxed{R\ddot{\phi}\,\hat{i} - R\ddot{\phi}\,\hat{i} - R\dot{\phi}^2\hat{k} = -R\dot{\phi}^2\hat{k}}$

$\underline{\rho}_{A/c} = R\hat{k}$

$\underline{\rho}_{B/c} = 2R\hat{k}$

$\underline{\rho}_{D/c} = R\hat{k} + R\hat{i}$

$\underline{\omega} = \dot{\phi}\,\hat{j}$

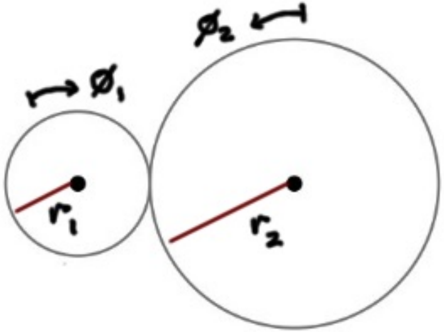
$\dot{\underline{\omega}} = \ddot{\phi}\,\hat{j}$

B) $\underline{v}_B = \underline{v}_c + \underline{\omega} \times \underline{\rho}_{B/c} = \boxed{0 + 2R\dot{\phi}\,\hat{i}}$

D) $\underline{v}_D = \underline{v}_c + \underline{\omega} \times \underline{\rho}_{D/c} = \boxed{0 + R\dot{\phi}(\hat{i} - \hat{k})}$

Example 9 - Gears and Belts (or Chains)

Develop the kinematics for this gear train, including an expression for its gear ratio. Also consider the power transfer from one gear to the other.



s (arc length)

Constraint: Equal s

$$s = r_1\phi_1 = r_2\phi_2$$
$$\phi_1 = {}^{r_2}/_{r_1} \cdot \phi_2$$

$$\boxed{\text{Gear Ratio: } N = \frac{r_2}{r_1},\ \phi_1 = N\phi_2}$$

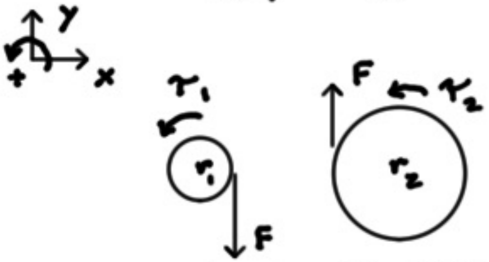
Note: $\dot{\phi}_1 = N\dot{\phi}_2$, $\ddot{\phi}_1 = N\ddot{\phi}_2$

E.g. If $N = 50$ & $\dot{\phi}_1 = 1000$ rpm then $\dot{\phi}_2 = 200$ rpm

Note: Torque effect is inverted!

$$\boxed{\tau_1 = \frac{1}{N}\tau_2}$$

Statics: Apply a τ_1, calc. τ_2 for equilibrium



$$\tau_1 - r_1 F = 0 \qquad \tau_2 - r_2 F = 0$$

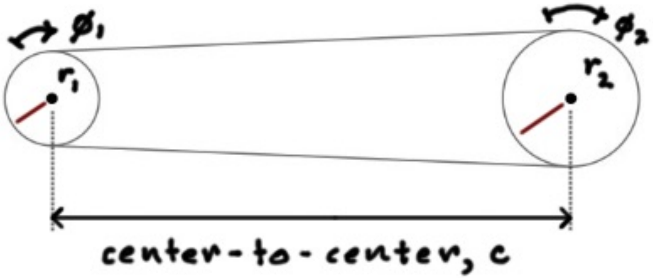
$$F = {}^{\tau_2}/_{r_2}$$

$$\tau_1 = \frac{r_1}{r_2}\tau_2$$

$$\boxed{\tau_1 = \frac{1}{N}\tau_2}$$

Example 10 - Gears and Belts (or Chains)

Develop the kinematics for this chain train system, including an expression for its gear ratio. Also consider the power transfer from one sprocket to the other.



s (arc length)

Constraint: Equal s

$s = r_1\phi_1 = r_2\phi_2$

$\phi_1 = \frac{r_2}{r_1}\phi_2$

$$\boxed{N = \frac{r_2}{r_1}, \quad \phi_1 = N\phi_2, \quad \tau_1 = \frac{1}{N}\tau_2}$$

Same as gear

Guideline: $c < 6(r_1 + r_2)$

Power

Princ. of Work & KE: $\Delta W = \Delta T$ where W is the work from ALL forces, internal & external.

Let $\Delta t \to 0$, $\dot{W} = \dot{T}$. The rate of work done on a sys. is the power associated with ALL forces.

consider:

$m\ddot{x} = F$ & $T = \frac{1}{2}mv^2$

$\dot{T} = mv\dot{v} = mv\ddot{x} = mv\cdot\frac{F}{m} = Fv$. Power always has the form: (force)(spd) or (Moment)(ang. vel.)

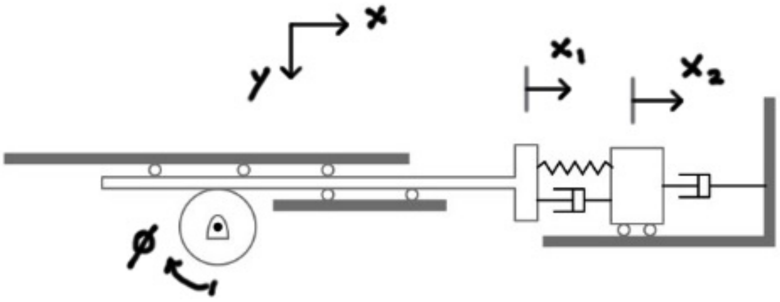
Example 11 - Rack and Pinion

Assign suitable variables of motion and develop any kinematic expressions for that relate them.

3 motion vars.

Constraint: $r\phi = x_1$

2 DOFs



Summary

1. Given a description of a dynamic system, and some time to practice… you can
    a. Determine suitable variables for describing motion, constraint equations and degrees of freedom (DOF)
    b. Calculate expressions for key positions, velocities, and accelerations needed for dynamic equation generation using either fixed or rotating frames
2. Perform kinematic analysis, such as computing expressions for acceleration, for systems containing:
    1. Pulleys
    2. Belts and gears
    3. Rack and pinions

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