Transfer Functions

We've been picking around the edges of this topic for a few days now. It's time to take a deeper dive.

Transfer functions are yet another valid way to represent a dynamic system model

One of the best model forms for control system analysis and design

Skills

1. Given a DEQ model, find the transfer functions between its inputs and outputs
2. Cacluate the poles and zeros of a transfer function
3. Apply the FVT properly to a transfer function model
4. Use MATLAB to create transfer function objects and use them for analysis
5. Given a transfer function model, determine if it’s stable or unstable

Terminology

1. Poles
2. Zeros
3. Complex Plane
4. Pole-Zero Diagram
5. Stability
6. Characteristic Equation
7. Step Response and Impulse Response

Transfer Function Facts

1. A Laplace Domain model of a dynamic system where **by definition all initial conditions are zero**
2. The functional representation of the **OUTPUT / INPUT** of a dynamic system
3. The Laplace Transform of a dynamic system's **impulse response**
4. We often give the transfer function a new letter designation, e.g. $G(s)$
5. A transfer function's
    a. poles are the roots of its denominator (NOTE: they are equal to the eigenvalues of the A matrix)
    b. zeros are the roots of its numerator
    c. characteristic equation is the denominator polynomial, usually denoted, $F(s)$. Since we are often interested in pole locations, we almost always will analyze the equation: $F(s) = 0$
    d. order is defined as the order of its characteristic equation and is the order of the dynamic system
6. We can represent a transfer function graphically by plotting its poles and zeros in the complex plane. This is called a pole-zero diagram. NOTE: This representation loses constant information. The complex plane is divided into 3 regions: left half plane (LHP), right half plane (RHP) and the imaginary axis.

DEQ Model To Tranfer Function

1. Know which variables are the input and the output, e.g. y, u
2. LT the DEQ model equations assuming all initial conditions are zero
3. Substitute as needed to form ONE equation that relates the output and input, e.g. $(s^2 + s + 10)Y = (s + 3)U$
4. Create the transfer function by solving for the OUTPUT / INPUT, e.g. $\frac{Y}{U} = G = \frac{s + 3}{s^2 + s + 10}$

Example 1

Find the transfer function model from the DEQ model below. Create its pole-zero diagram.
Compare its poles to the eigenvalues of its state matrix A (K = 3)

$$\dddot{y} + 3\ddot{y} + k\dot{y} + 7y = 2\dot{u} + 5u$$

$$(s^3 + 3s^2 + ks + 7)Y = (2s + 5)U$$

$$\frac{Y}{U} = \frac{2s + 5}{s^3 + 3s^2 + 3s + 7}$$

MATLAB (numeric)

```
G = tf([2,5],[1,3,3,7])
pole(G)
zero(G)
pzmap(G)
gs = ss(G)
eig(gs)
step(G)
damp(G)
```

Example 1 - continued

$$\dddot{y} + 3\ddot{y} + k\dot{y} + 7y = 2\dot{u} + 5u$$

Example 2

Find the transfer function between the output, w, and the input, v for the DEQ model below and make a pole-zero diagram. What is its characteristic equation? Compare its poles to the eigenvalues of its state matrix, A.

$$3\ddot{y} + 6\dot{y} + 27y - 6w = 9v$$

$$\dot{w} - y + 4w = 0$$

Find $G = W/V$
Eliminate: Y

$$\ddot{y} + 2\dot{y} + 9y - 2w = 3v$$

$$(s^2+2s+9)Y - 2W = 3V \quad 1.$$

$$(s+4)W = Y \quad 2.$$

Sub 2. into 1.

$$(s^2+2s+9)(s+4)W - 2W = 3V$$

```
>> conv([1,2,9],[1,4])
```

$$(s^3+6s^2+17s+36-2)W = 3V$$

$$\boxed{\frac{W}{V} = \frac{3}{s^3+6s^2+17s+34}}$$

$$F(s) = s^3+6s^2+17s+34$$

```
>> H = tf(3,[1,6,17,34])
>> pole(H)
>> pzmap(H)
>> hs = ss(H)
>> eig(hs.A)
```

Example 2 - continued

$$3\ddot{y} + 6\dot{y} + 27y - 6w = 9v$$

$$\dot{w} - y + 4w = 0$$

Example 3

Given the transfer function model below, find its DEQ model

$$G(s) = \frac{68}{(s+1)(s^2+6s+34)}$$

$$\frac{Y}{U} = \frac{68}{(s+1)(s^2+6s+34)}$$

$$(s+1)(s^2+6s+34)\cdot Y = 68\,U$$

```
>> conv([1,1],[1,6,34])
```

$$(s^3+7s^2+40s+34)Y = 68\,U$$

$$\boxed{\dddot{y} + 7\ddot{y} + 40\dot{y} + 34y = 68u}$$

Response Types

1. Early in the class several response types were introduced, e.g. step and impulse. Recall
   (a) **Step Response:** Set all initial conditions to zero, give the system a unit step input and record the output
   (b) **Impulse Response:** Set all initial conditions to zero, give the system a unit impulse input and record the output
2. Any response is easily created if you have the system's transfer function since it relates OUTPUT / INPUT. Do some algebra, apply the input of interest and solve for the output. Viola!

Example 4

Calculate the system's impulse, step and ramp response given its transfer function

$$\frac{Y}{U} = G = \frac{3}{(s+2)(s+3)}$$

$u = \delta$, $U = 1$, $Y = \frac{3}{(s+2)(s+3)}$, $Y = \frac{c_1}{s+2} + \frac{c_2}{s+3}$, $y = c_1 e^{-2t} + c_2 e^{-3t}$

$u = u_s$, $U = \frac{1}{s}$, $Y = \frac{3}{s(s+2)(s+3)}$, $Y = \frac{c_1}{s} + \frac{c_2}{s+2} + \frac{c_3}{s+3}$, $y = c_1 u_s + c_2 e^{-2t} + c_3 e^{-3t}$

$u = t \cdot u_s$, $U = \frac{1}{s^2}$, $Y = \frac{3}{s^2(s+2)(s+3)}$, $Y = \frac{c_1}{s^2} + \frac{c_2}{s} + \frac{c_3}{s+2} + \frac{c_4}{s+3}$, $y = c_1 t + c_2 + c_3 e^{-2t} + c_4 e^{-3t}$

Transfer Function and Impulse Response

Earlier it was stated that s system's transfer function is the Laplace Tranform of the system's impulse response Let's look at that more formally

1. Consider the transfer function, $\frac{Y}{U} = G$

2. Rearrange, and solve for the output, $Y = G \cdot U$

3. Let the input be an impulse and viola!, $Y_{impulse} = G$

The implication is profound and wonderful. It means we can create a transfer function experimentally! Whack the dynamics system and measure its impulse response. Take the LT of the signal and you now have the systems transfer function model!

FVT Redux

Now that we have some new terminology, we can state the FVT correctly, knowing when it does and does not apply. Recall, the FVT only makes sense if the response HAS a final value. There are two situations where it will not reach steady state: (1) it oscillates forever or (2) the response grows without bound, that is, the dynamic system is unstable.

FVT: Given some retional function Y, where all the poles of sY are in the LHP, its final value exists and is:

$$y_{ss} = \lim_{s \to 0} sY$$

Example 5

Given the transfer function of a dc motor driven rack and pinion system where the input the motor voltage and the output is the cart position, calculate the system's steady state speed for (1) an impulse input and (2) a step input. What would be the transfer fuction that relates cart speed to input voltage?

$$H(s) = \frac{210}{s(s+10)}$$

1) $\frac{X}{V} = \frac{210}{s(s+10)}$ $V = 1$, $X = \frac{210}{s(s+10)}$, check: $sX = \frac{210}{s+10}$ All poles in LHP

$x_{ss} = \lim_{s \to 0} sX = \frac{210}{10} = 21$ cm

2) $V = \frac{1}{s}$, $X = \frac{210}{s^2(s+10)}$, check: $sX = \frac{210}{s(s+10)}$ Pole on im. axis FVT does NOT apply.

Transfer Functions and Multiple Inputs

Sometimes we have systems with more than one input or more than one output. We can form transfer functions for each output and each input. For example, if we have 1 output, Y, and 2 inputs, U, W we would manipulate the system to create the form

$$Y = G_{yu} \cdot U + G_{yw} \cdot W, \quad (1)$$

we can extract two transfer functions as

$$G_{yu} = \frac{Y}{U}, \quad G_{yw} = \frac{Y}{W}$$

The interpretation goes like this. IF $W = 0$ and $U \neq 0$ then G_{yu} can be used for analysis. Conversely, IF $W \neq 0$, $U = 0$, then G_{yw} is the transfer function you need to analyze the system. If both inputs are active, no worries, we have superposition and we can use Eq. 1.

Example 6

Given the DEQ model below where τ is an applied torque input and w is an external wind disturbance input find the transfer function that relates the disturbance to the wind turbine angle θ.

Find θ / w

$$\ddot{\theta} + 9\theta - \alpha = \tau$$
$$\dot{\alpha} + 13\alpha = w$$

(1) $(s^2+9)\theta - \alpha = \tau$, (2) $(s+13)\alpha = w$

Solve (1) for α, sub into (2)

$\alpha = (s^2+9)\theta - \tau \longrightarrow (s+13)[(s^2+9)\theta - \tau] = w$

$(s+13)(s^2+9)\theta = (s+13)\tau + w$

$$\theta = \frac{1}{s^2+9}\tau + \boxed{\frac{1}{(s+13)(s^2+9)}} w$$

Transfer Function Algebra

Sometimes we have dynamic systems with internal variables that we aren't interested in. If we can form transfer functions in terms of these variables, then they are easy to remove.

Example 7

Given the block diagram of the forward path of a control system, create the transfer function from the error, E, to the plant output Y

Introduction to Stability

1. Stability is a property of a dynamic system.
2. If a dynamic system is stable then its output will be bounded for ANY bounded input, e.g. a step.
3. A dynamic system is stable if and only if ALL its poles are in the left half of the complex plane (LHP). If it has poles on the imaginary axis, but none in the RHP, it is UNSTABLE but it will sometimes be called marginally stable.
4. We'll develop some nice methods for stability analysis soon. Here's an easy one. If any of the coefficients of the characteristic equation are negative, the system will have a pole(s) outside the LHP and is thus unstable.
5. Feedback control can easily make a stable system unstable…

Example 8

Evaluate the stability of the systems whose transfer functions are given below

1. $G = \dfrac{s-1}{(s+2)(s+3)}$
2. $G = \dfrac{5}{s^2-2s+3}$
3. $G = \dfrac{s+2}{s^3+6s^2+11s+70}$

MATLAB

We'll use MATLAB's transfer function tools extensively for the rest of the semester.

The first step is to create a transfer function object, e.g.

$$G = \frac{s+34}{(s+1)(s^2+s+17)}$$

```
>> G = tf( [1 34], conv([1 1],[1 1 17]) )
```

Summary

With a little more practice, you can

1. Create transfer functions from DEQ models
2. Use the FVT to calculate a system's steady state value, when applicable
3. Determine system stability from its poles
4. Create a MATLAB transfer function object