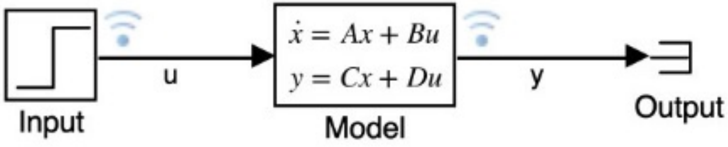
State Space - Motivation

1. **Standard approach** for representing linear, DEQ models

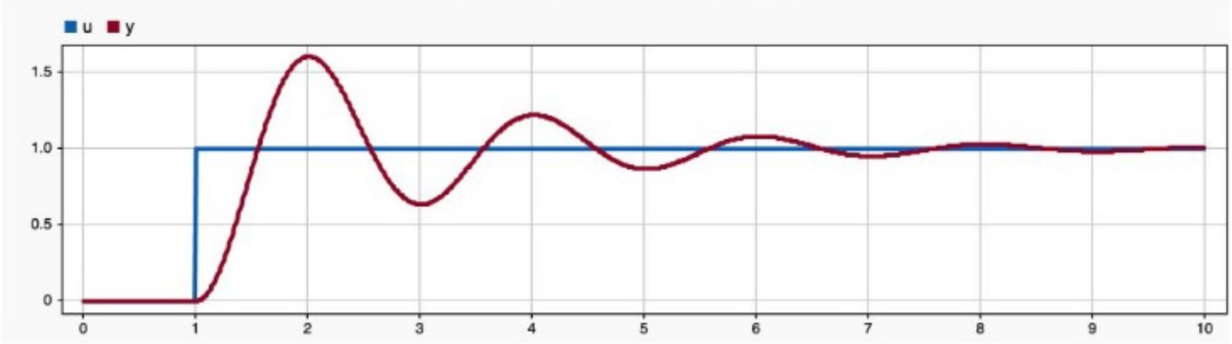
2. Permits a vast range of **analysis** and modern control system design techniques

3. **The gateway for numerical simulation**

Simulink Model



Step Response (Data Inspector)



Skills

1. Terminology:
    a. State equation
    b. Output equation
    c. State variables
    d. Eigenvalues
2. Given a set of linear differential equations:
    1. Create a state space representation
    2. Create a state space object in MATLAB, a step response and check its eigenvalues
3. Given two state space representations, be able to determine if they ***might*** represent the same dynamic system by checking eigenvalues

Introduction

1. A set of first order differential equations, and output equations, written in matrix form

$\underset{\sim}{x} : n \times 1$

$\underset{\approx}{A} : n \times n$

$\underset{\sim}{x} : \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$

state derivs.

state matrix

state vector

input vector

$$\dot{\mathbf{x}} = A\,\mathbf{x} + B\,\mathbf{u}$$ State Equ

$\underset{\sim}{u} : m \times 1$

$\underset{\approx}{B} : n \times m$

$\underset{\sim}{u} : \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{Bmatrix}$

$\underset{\sim}{y} : p \times 1$

$\underset{\approx}{C} : p \times n$

$\underset{\sim}{y} : \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{Bmatrix}$

output vector

$$\mathbf{y} = C\,\mathbf{x} + D\,\mathbf{u}$$ Output Equ

2. Defined by its 4 matrices, A, B, C, and D

3. Given a set of differential equations

   a. There are an infinite number of valid state space representations, A, B, C, and D matrices

   b. **All** of its state space representations have identical properties, such as the **eigenvalues of their A matrices**.

Procedure

1. Identify the system's inputs and outputs, (u and y)
2. Determine the systems order (n)
    1. Order: the sum of the highest derivative for each dependent variable
    2. This is the number of states you'll need.
3. Define the states - no unique, correct definition. We'll use one that works nearly all the time
4. Express the outputs (y) in terms of the states and the inputs (x and u)
5. Express the state derivatives (x dot) in terms of the states and inputs (x and u)
6. Fill the matrices (A, B, C and D)
7. Muscle Memory: Check the eigenvalues of A

Example 1

Given the DEQ model below, create a **state space representation,** where u is the input and h is the output. Use MATLAB to create its step response and check its eigenvalues using a **MATLAB state space object**.

$$\ddot{h} + 4\dot{h} - 3v = 0$$

$$\dot{v} - 7\dot{h} + 17v = 3u$$

order, n = 3

Output Eqn: $\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}$

$y_1 = h = x_1$ So, $\underline{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\underline{\underline{C}}} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \overbrace{\begin{bmatrix} 0 \end{bmatrix}}^{\underline{\underline{D}}} u$

$x_1 = h \quad \dot{x}_1 = \dot{h} = x_2$

$x_2 = \dot{h} \quad \dot{x}_2 = \ddot{h} = -4\dot{h} + 3v = -4x_2 + 3x_3$

$x_3 = v \quad \dot{x}_3 = \dot{v} = 7\dot{h} - 17v + 3u = 7x_2 - 17x_3 + 3u$

State Equ: $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$

$$\dot{\underline{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ 0 & 7 & -17 \end{bmatrix}}_{\underline{\underline{A}}} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}}_{\underline{\underline{B}}} u$$

Example 1 - continued

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$$\ddot{h} + 4\dot{h} - 3v = 0$$

$$\dot{v} - 7\dot{h} + 17v = 3u$$

Example 2

Given the linear, time-invariant DEQ model below, create a **state space representation,** where τ is the input and the outputs are $y_1 = \phi$ and $y_2 = \dot{\phi} - 3\dot{\theta}$ Use MATLAB to create its step response and check its eigenvalues using a **MATLAB state space object**.

$$\dddot{\theta} + 4\ddot{\theta} + 3\dot{\theta} + 2\theta - \dot{\phi} = 3\tau$$

$$\ddot{\phi} - \dot{\theta} + 3\dot{\phi} + 20\phi = 2\tau$$

order: $n = 3 + 2 = 5$

$x_1 = \theta \qquad \dot{x}_1 = \dot{\theta} = x_2$

$x_2 = \dot{\theta} \qquad \dot{x}_2 = \ddot{\theta} = x_3$

$x_3 = \ddot{\theta} \qquad \dot{x}_3 = \dddot{\theta} = -4x_3 - 3x_2 - 2x_1 + x_5 + 3\tau$

$x_4 = \phi \qquad \dot{x}_4 = \dot{\phi} = x_5$

$x_5 = \dot{\phi} \qquad \dot{x}_5 = \ddot{\phi} = x_2 - 3x_5 - 20x_4 + 2\tau$

State Equ: $\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$

$$\underline{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & -3 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -20 & -3 \end{bmatrix}}_{\underline{\underline{A}}} \underline{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 2 \end{bmatrix}}_{\underline{\underline{B}}} u$$

Output Eqn: $\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}$

$y_1 = \phi = x_4$

$y_2 = \dot{\phi} - 3\dot{\theta} = x_5 - 3x_2$

$$\underline{y} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 0 & 1 \end{bmatrix}}_{\underline{\underline{C}}} \underline{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{\underline{D}}} \tau$$

Example 2 - continued

Given the linear, time-invariant DEQ model below, create a **state space representation,** where τ is the input and the outputs are $y_1 = \phi$ and $y_2 = \dot{\phi} - 3\dot{\theta}$ Use MATLAB to create its step response and check its eigenvalues using a **MATLAB state space object**.

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$$\dddot{\theta} + 4\ddot{\theta} + 3\dot{\theta} + 2\theta - \dot{\phi} = 3\tau$$

$$\ddot{\phi} - \dot{\theta} + 3\dot{\phi} + 20\phi = 2\tau$$

Example 3

Create a **state space representation** for the system below where u is the input and w is the output.

order: $n = 2$

$$\ddot{w} + 2\dot{w} + 5w = 2u + 3\dot{u}$$

$x_1 = w \quad \dot{x}_1 = x_2$

$x_2 = \dot{w} \quad \dot{x}_2 = \ddot{w} = -2\dot{w} - 5w + 2u + 3\dot{u}$

$= -2x_2 - 5x_1 + 2u + ??$

Example 3 - continued

Create a **state space representation** for the system below where u is the input and w is the output.

$$\ddot{w} + 2\dot{w} + 5w = 2u + 3\dot{u}$$

$y = x_1$, $\underline{y} = [1 \quad 0]\,\underline{x} + [0]\,u$

$\underset{=}{C}$ $\underset{=}{D}$

$x_1 = w$ $\quad \dot{x}_1 = \dot{w} = x_2 + 3u$

$x_2 = \dot{w} - 3u$ $\quad \dot{x}_2 = \ddot{w} - 3\dot{u} = -2\dot{w} - 5w + 2u$

$-2(x_2 + 3u) - 5x_1 + 2u$

$-2x_2 - 5x_1 - 4u$

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 3 \\ -4 \end{bmatrix} u$$

$\underset{=}{A}$ $\underset{=}{B}$

Equivalent State Space Models and Eigenvalues

- A dynamic system can be represented using an infinite number of equivalent state space models
- The reason why two equivalent state space models have different A, B, C or D matrices is caused by how their state variables were defined
- The A matrices of any two equivalent state space models will have the same eigenvalues
- Consider: $\ddot{w} + 4\dot{w} + 5w = u$ and two different state variable definitions

$x_1 = w \quad \dot{x}_1 = x_2$

$x_2 = \dot{w} \quad \dot{x}_2 = \ddot{w} = -4x_2 - 5x_1 + u$

$$\underline{A}_1 = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}$$

$x_1 = w - \dot{w} \quad \dot{x}_1 = \dot{w} - \ddot{w} =$

$x_2 - (-4x_2 - 5(x_1 + x_2) + u)$

$= 10x_2 + 5x_1 - u$

$x_2 = \dot{w} \quad \dot{x}_2 = \ddot{w} = -4\dot{w} - 5w + u$

$= -4x_2 - 5(x_1 + x_2) + u$

$= -5x_1 - 9x_2 + u$

$$\underline{A}_2 = \begin{bmatrix} 5 & 0 \\ -5 & -9 \end{bmatrix}$$

Example 4

Create two A matrices for the dynamic system: $\ddot{w} + 4\dot{w} + 5w = u$ where

a. $x_1 = w, \quad x_2 = \dot{w}$

b. $x_1 = w - \dot{w}, \; x_2 = \dot{w}$

and check their eigenvalues to be sure they are identical

$$\underline{A}_1 = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \qquad \underline{A}_2 = \begin{bmatrix} 5 & 0 \\ -5 & -9 \end{bmatrix}$$

Summary

- Given a system of linear DEQs you can (with more practice…)
    a. Create a state space representation
    b. Create a MATLAB state space object and so some analysis (step response, check eigenvalues, maybe more)

- Given two state space representations determine if they **might be** for the same dynamic system.

Summary

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