Laplace Examples

Another giant of math, physics and engineering.

The math for this topic is covered in two .mlx files. Here we'll run through several examples

Laplace Transform Table

Laplace Transform Table

#	$f(t)$	$F(s)$	#	$f(t)$	$F(s)$
1	$\delta(t)$	1	11	$1-e^{-at}$	$\frac{a}{s(s+a)}$
2	u_s	$\frac{1}{s}$	12	$\frac{1}{a}(at-1+e^{-at})$	$\frac{a}{s^2(s+a)}$
3	t	$\frac{1}{s^2}$	13	$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$
4	t^2	$\frac{2!}{s^3}$	14	$(1-at)e^{-at}$	$\frac{s}{(s+a)^2}$
5	t^3	$\frac{3!}{s^4}$	15	$1-(1+at)e^{-at}$	$\frac{a^2}{s(s+a)^2}$
6	t^m	$\frac{m!}{s^{m+1}}$	16	$be^{-bt}-ae^{-at}$	$\frac{(b-a)s}{(s+a)(s+b)}$
7	e^{-at}	$\frac{1}{(s+a)}$	17	$\sin(bt)$	$\frac{b}{s^2+b^2}$
8	te^{-at}	$\frac{1}{(s+a)^2}$	18	$\cos(bt)$	$\frac{s}{s^2+b^2}$
9	$\frac{1}{2!}t^2e^{-at}$	$\frac{1}{(s+a)^3}$	19	$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$
10	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$	$\frac{1}{(s+a)^m}$	20	$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$

Laplace Properties Table

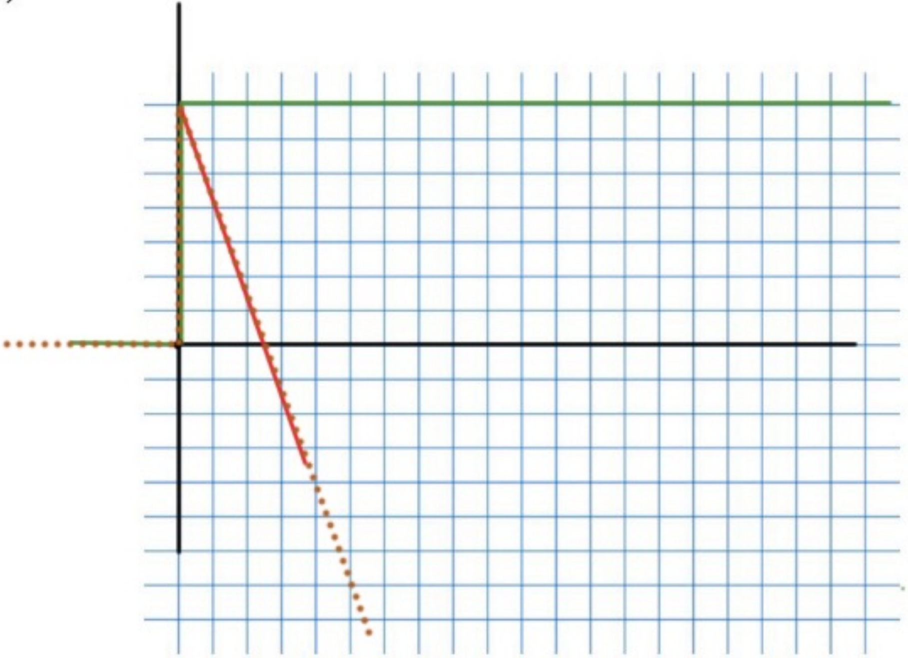
Laplace Transform Properties

$f(t)$	$F(s)$	Property
$\alpha f_1(t) + \beta f_2(t)$	$\alpha F_1(s) + \beta F_2(s)$	Superposition
$f(t-\lambda)$	$F(s)e^{-s\lambda}$	Time Delay ($\lambda \geq 0$)
$f(at)$	$\frac{1}{\lvert a\rvert}F(\frac{s}{a})$	Time Scaling
$e^{-at}f(t)$	$F(s+a)$	Shift in Frequency
$f^{(m)}(t)$	$s^m F(s) - s^{m-1}f(0)+$ $-s^{m-2}\dot{f}(0) - \cdots - f^{(m-1)}(0)$	Differentiation
$\int f(\zeta)d\zeta$	$\frac{1}{s}F(s)$	Integration
$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$	Convolution
$f(0)$	$\lim_{s\to\infty} sF(s)$	Initial Value Theorem
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$	Final Value Theorem
$tf(t)$	$-\frac{d}{ds}F(s)$	Multiplication by Time

Example 1

Sketch the function $g(t) = (7 - 3t)u_s(t)$ and create its LT.

$$G = \frac{7}{s} - \frac{3}{s^2} = \frac{7s - 3}{s^2}$$



Example 2

Find the LT of $y(t) = 4.2e^{-6t}\sin 7t$ $\qquad e^{-at}\sin(bt), \quad \dfrac{b}{(s+a)^2+b^2}$

$a=6, \; b=7$

$$Y = \frac{4.2 \cdot 7}{(s+6)^2+7^2} = \boxed{\frac{29.4}{s^2+12s+85}}$$

Example 3

Take the ILT of $V(s) = \frac{7}{s+5} - \frac{27}{s^2+9}$

$\sin(bt)$, $\frac{b}{s^2+b^2}$

$$V = \frac{7}{s+5} - \frac{9 \cdot 3}{s^2+3^2}$$

$$\boxed{v(t) = 7e^{-5t} - 9\sin(3t)}$$

Example 4

Take the LT of the DEQ model:

$$\ddot{y} + 2\dot{y} + 36y = \dot{u} + 5u, \;\; \dot{y}(0) = y(0) = 0$$

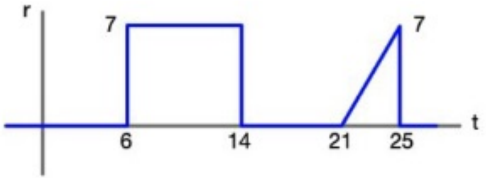
Then form the transfer function.

$$(s^2 + 2s + 36)Y = (s+5)U$$

$$\boxed{\frac{Y}{U} = \frac{s+5}{s^2 + 2s + 36}}$$

Example 5

Take the LT of the function sketched at right..



$$r = 7u_s(t-6) - 7u_s(t-14) + \frac{7}{4}(t-21)u_s(t-21) - \frac{7}{4}(t-25)u_s(t-25) - 7u_s(t-25)$$

$$R = \frac{7e^{-6s}}{s} - \frac{7e^{-14s}}{s} + \frac{7}{4}\left[\frac{e^{-21s}}{s^2} - \frac{e^{-25s}}{s^2}\right] - \frac{7}{s}e^{-25s}$$

$$\boxed{R = \frac{7}{s}\left(e^{-6s} - e^{-14s} - e^{-25s}\right) + \frac{7}{4s^2}\left(e^{-21s} - e^{-25s}\right)}$$

Example 6

Given the transfer function below, create its DEQ model

$$\frac{R}{U} = \frac{s}{s^2 + 5s + 36}$$

$$R \cdot (s^2 + 5s + 36) = U \cdot s$$

$$\boxed{\ddot{r} + 5\dot{r} + 36r = \dot{u}}$$

Example 7

Use the FVT to find the steady state value, y_{ss}, when u is a unit step. Reconstitute the DEQ model to check the answer using dc gain.

$$\frac{Y}{U} = \frac{72}{s^2 + 5s + 36}$$

$$Y = \frac{72}{s^2 + 5s + 36} \cdot U$$

$$U = \frac{1}{s}$$

$$Y = \frac{72}{s(s^2 + 5s + 36)}$$

$$Y_{ss} = \lim_{s \to 0} sY = 2$$

$$\ddot{y} + 5\dot{y} + 36y = 72u$$

$$2\zeta\omega_n = 5$$

$$\omega_n^2 = 36$$

$$K_{dc}\,\omega_n^2 = 72 \rightarrow K_{dc} = 2$$