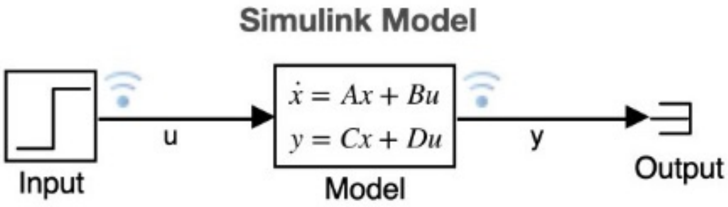
Simulink and Integration - Motivation

1. Simulation is used extensively in engineering analysis and design

2. Simulations always produce results - trust but verify…

3. Very fast way to gain insight into how your dynamic system responds to different situations (inputs and ics)

4. Simulation of a dynamic system is the numerical solution of its DEQ model - so it's important to know something about how they work - numerical integration



Skills

1. Terminology:
    a. Integration method and order
    b. Step size
    c. Difference equation
    d. Euler, trapezoidal and RK integration methods
2. Given a state space model
    a. Create a simulink simulation, with suitably chosen step size and integration method, and do some analysis with it
    b. Numerically solve it for any input and initial condition

Assumptions and Notation

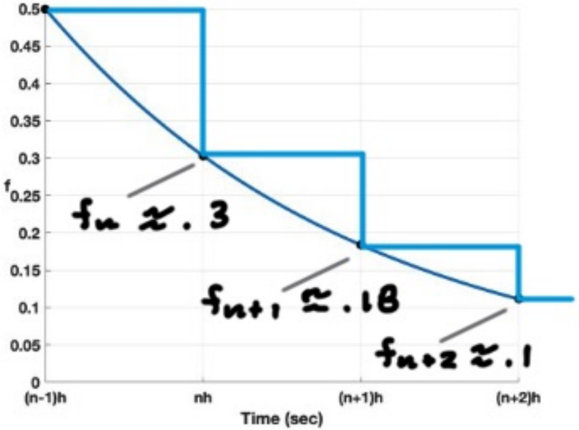
1. The simulation will update at a fixed time interval denoted h. Everything in it is sampled at this rate

2. The sample number will be denoted n - yes, we also use this for the order of a dynamic system. Sorry about that…

3. With these two concepts we can discretize continuous functions, for example, $u(t) = \sin 3t$ will only exist at $n = 0, 1, \ldots$ or

$n=0,\ u(0)=0$
$n=1,\ u(h)=\sin(h)$
$n=2,\ u(2h)=\sin(2h)$

4. More generally, the sampled values of the function $f(t)$ are



End Game

Given some dynamic system DEQ model, we will pick a **numerical integration method**, then solve the DEQ model numerically - that is, we'll simulate it.

For example, the DEQ

$$\dot{x}(t) = -4x(t) + u(t), x(0) = x_0$$

can be solved (simulated) by the **difference equation**

$$x_{n+1} = (1 - 4h)x_n + hu_n$$

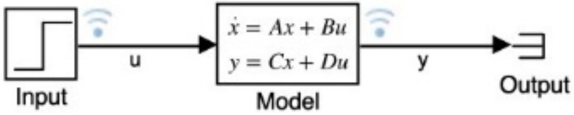
Let $u(t) = \sin(t)$, $h = .1$, $1 - 4h = .6$, $x_0 = 1$

$x_1 = .6x_0 + .1 \sin(0) = .6$

$x_2 = .6x_1 + .1 \sin(.1) = .36 + .1 \sin(.1) \approx .37$

$x_3 = .6x_2 + .1 \sin(.2) \approx .24$

and in Simulink



Introduction to Numerical Integration and Euler's Method

Consider the first order, linear DEQ model of a dynamic system

$$\dot{x} = f(t)$$

Consider 1 panel

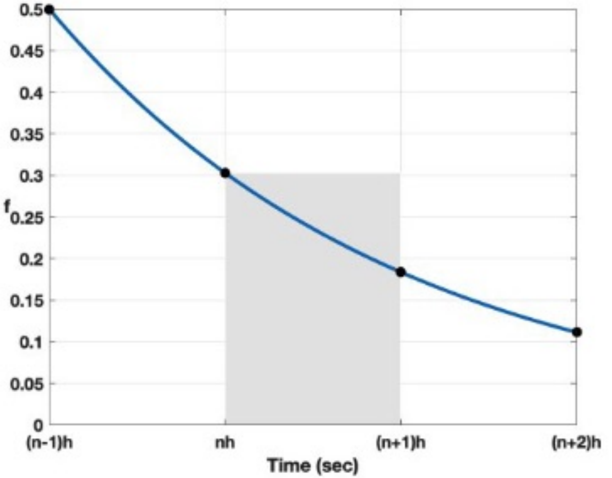
Let's solve it by integrating both sides.

$$\int_{nh}^{(n+1)h} \frac{dx}{dt}\, dt = \int_{nh}^{(n+1)h} f(t)\, dt$$

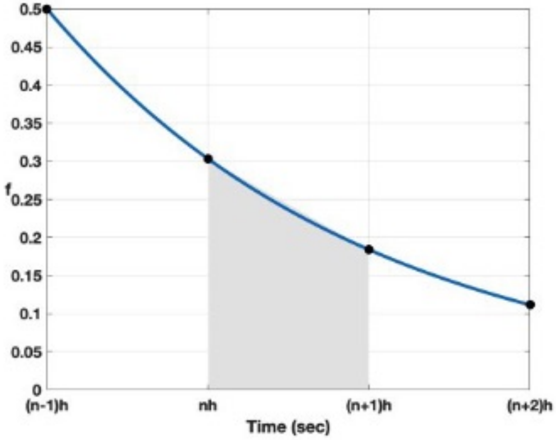
$$\int_{x_n}^{x_{n+1}} dx \approx h f_n$$

$$x_{n+1} - x_n \approx h f_n \quad \text{or} \quad \boxed{x_{n+1} \approx x_n + h f_n}$$

Euler: One f evaluation and errors are proportional to h.



Some Other Methods



Trapezoidal: Two $f(x)$ evaluations (2nd order method), errors are proportional to h^2.

Runge-Kutta: Many flavors, 2nd order, 3rd order, 4th order, etc. In Simulink ode2, ode3, ode4, … Errors are proportional to h^2, h^3, h^4 etc.

Extend To Any Dynamic System

- Replace the scalar $\dot{x} = f(t)$ with the system of equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$
- For example, Euler integration becomes

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\,\mathbf{f_n}(\mathbf{x}_n, \mathbf{u}_n, nh)$$

- While there are many other considerations, stability for one, you'll need to find a balance between the number of function evaluations and the time step to achieve the accuracy you need.

Time Step Selection - Linear Systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) = A\ \mathbf{x} + B\ \mathbf{u}$$

1. You need the integration method to be “faster” than the characteristic response of the dynamic system
2. Find the magnitudes of the eigenvalues of A - works for complex eigenvalues too

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>> abs(eig(A))
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3. Pick out the largest one as λ_{max}
4. Set your time step such that $h\lambda_{max} \leq 0.001$. This will give painfully small values of h for Euler - a first order integration method. You can increase h, roughly, by factors of 2 for each increase in integration method order as long as $h\lambda_{max} < .1$.
5. Always test your selection by decreasing h by a factor of 10, rerun, and make sure the results are very small. Small depends on the application, but differences that are less than 1×10^{-6} is often sufficient.

Example 1 - Time Step and Integration Method Selection

Given the state space model below, calculate a suitable integration time step for both Euler and RK-4. Create a Simulink model and compare step responses for both integration methods.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -20 & -3 & 1 & 1 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = 0$$

>> eig(A) $\longrightarrow$ $-1.5 \pm 4.2i, -7, -2$

>> abs(eig(A)) $\longrightarrow$ $4.47, 4.47, 7, 2$

$\lambda_{max} = 7$

Euler: $h = .001/7 \approx .00014$ sec

RK-4: $h = .00014\,(8) \approx .001$ This can be incr. a bit...

Summary

Given a system of linear DEQs you can (with more practice…)

a. Create a Simulink simulation
b. Pick a suitable integration method and step size
c. Explore its simulated step response