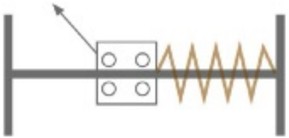
1st And 2nd Order DEQs - Motivation

1. A huge range of dynamic systems can be approximately modeled by 1st or 2nd order DEQs

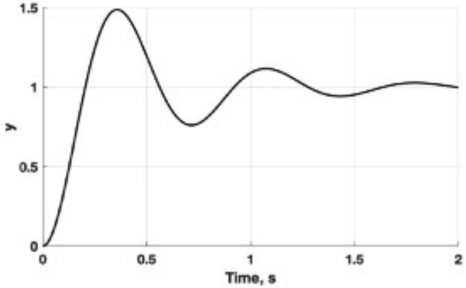
2. These models are uniquely described by 2 and 3 parameters respectively. If you know their values you have complete understanding of how the system responds to ANY input

3. Being able to (1) extract them and (2) set them is a fundamental skill. With it, you can readily analyze and design a large class of dynamic systems



$$\ddot{y} + 4\dot{y} + 81y = 50u$$

$$\omega_n = 9\,r/s \qquad \zeta = \frac{2}{9} \qquad k_{dc} = \frac{50}{81}$$



Skills

1. Terminology and Notation
    a. Frequency and Period
    b. Transient and Steady State
    c. DC Gain
    d. Damping ratio and natural frequency

2. Create a 1st order DEQ model from step response data

3. Given a 1st or 2nd order DEQ model, extract the DC gain, time constant (1st order systems) and both damping ratio and natural frequency (2nd order systems)

4. Classify a 2nd order system according to its damping ratio value

5. Use MATLAB to extract the damping ratio and natural frequency from a state space model

Example 0 - Frequency Warm-Up

What is the frequency, in both r/s and Hz, and the period for the signal below.

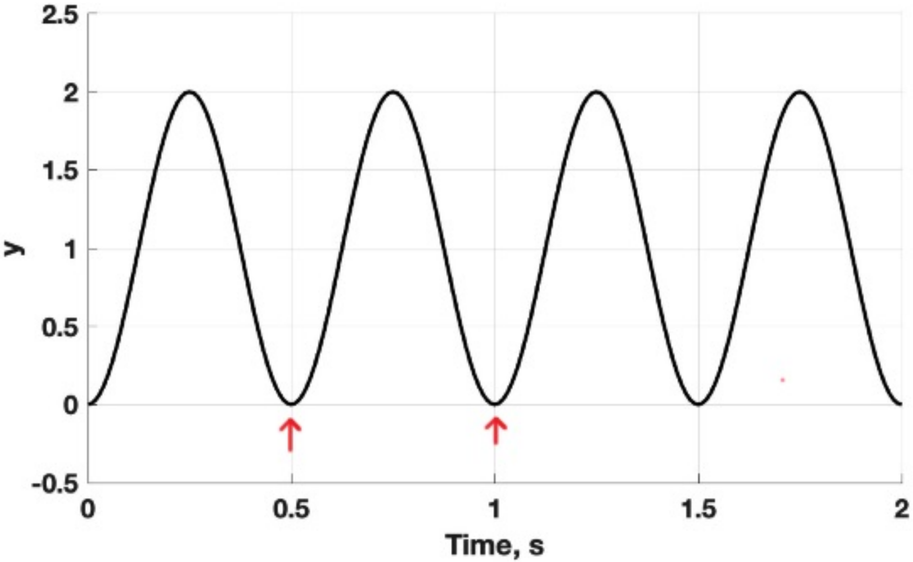
$T \approx .5$ sec

$f = 1/T = 2$ Hz

$\omega = 2\pi \cdot f$

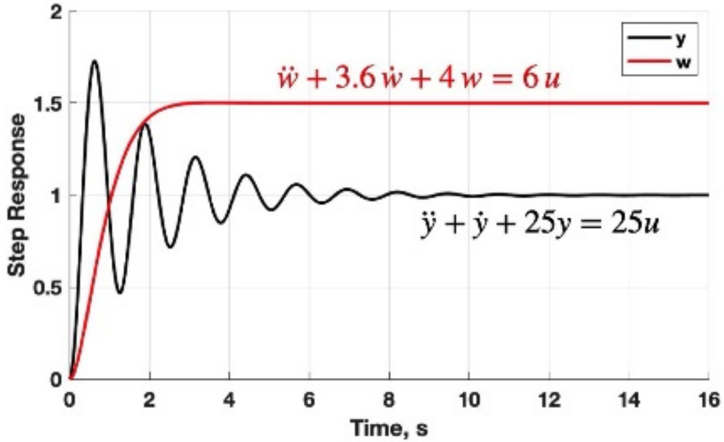
$\frac{rad}{cyc} \cdot \frac{cyc}{sec} = \frac{rad}{sec}$

$= 4\pi$ r/s ≈ 12.6 r/s



Terminology - Transient, Steady State and DC Gain

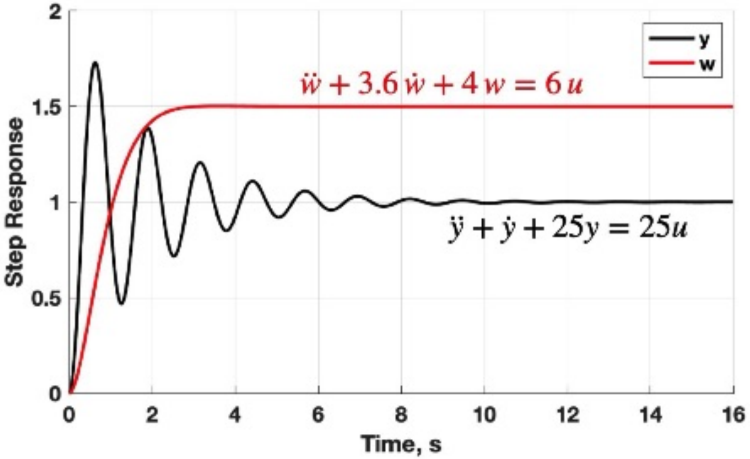
1. **Step Response** - ICs = 0, Input = Unit Step
2. Two pieces
   a. **Transient**
   b. **Steady state**
3. **Steady state output**
   a. Easy to see from step response data
   b. Easy to calculate from DEQs
4. The time of transition from transient to steady state…
5. **DC Gain**, k_{dc} - The steady state value of a system's **step response**.
6. To calculate DC Gain:
   a. Set the input to one, e.g. $u = 1$
   b. Set derivatives to 0, e.g. $\dot{y} = \ddot{y} = \ldots = 0$
   c. Solve what's left



Note: Same procedure as finding an equilibrium point, but for the special case of a step input.

Example 1

Calculate the DC Gain for the two DEQ models at right.



Why DC Gain?

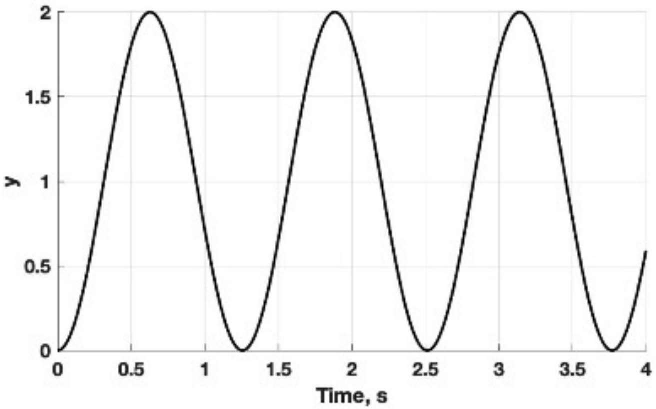
The steady state output, y_{ss}, for ANY input u that has a steady state value of u_{ss}, is equal to

$$y_{ss} = k_{dc}\, u_{ss}$$

NOTE: Some dynamic systems don't have a steady state value, and thus they don't have a DC Gain.

$$\ddot{y} + 25y = u$$

Applying the procedure gives $y_{ss} = 1/25$. But, this is nonsense



Example 2

What is the steady state output, w_{ss}, for

$$\dddot{w} + 2\ddot{w} + 3\dot{w} + 4w = u$$

when $u = 3(1 - e^{-2t})$?

$\boxed{w_{ss} = .75}$

First Order Linear Dynamic Systems

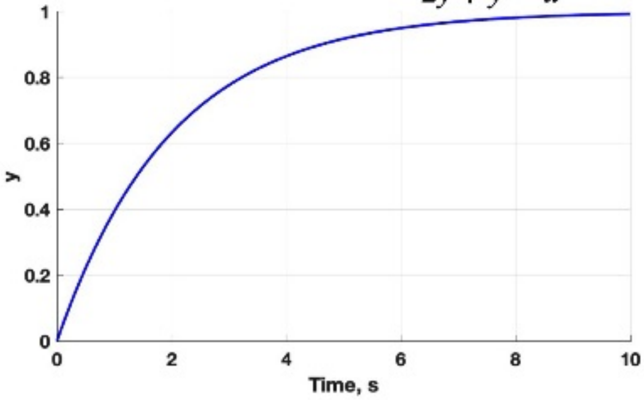
These occur in thermal, fluid, chemical and electrical systems. Sensor and actuator "dynamics" are also often modeled using first order DEQs. Every linear, time-invariant, first order system is uniquely defined by two parameters: τ and k_{dc} and has the form:

$$\tau \dot{y}(t) + y(t) = k_{dc}\, u(t)$$

Time Constant, τ: The time it takes the step response to get to 63% of its steady state value.

$2\dot{y} + y = u$

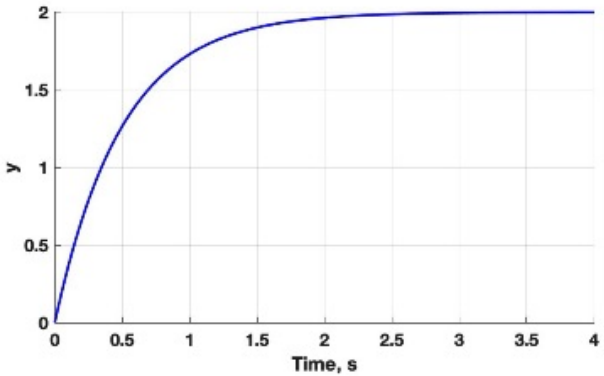
Inspection...
$\tau = 2$ sec



Example 3

The experimentally obtained unit step response is shown below. Calculate the system's DC Gain and time constant, then estimate its DEQ model.

$$\tau \dot{y} + y = K_{dc} u$$



$$\boxed{\text{The model: } .5\dot{y} + y = 2u}$$

Bonus: why .63 at $t = \tau$?

$y = K_{dc}(1 - e^{-1/\tau\, t})$

$y(\tau) = K_{dc}(1 - e^{-1})$

$\approx .63\, K_{dc}$

Example 4

For each of the first order systems below, calculate their time constants and DC Gains.

$$\tau \dot{y} + y = K_{dc} u$$

1. $4\dot{T} + T = 5u$

2. $\dot{y} + 5y = 10u$

3. $4\dot{w} + 2w = 6u$

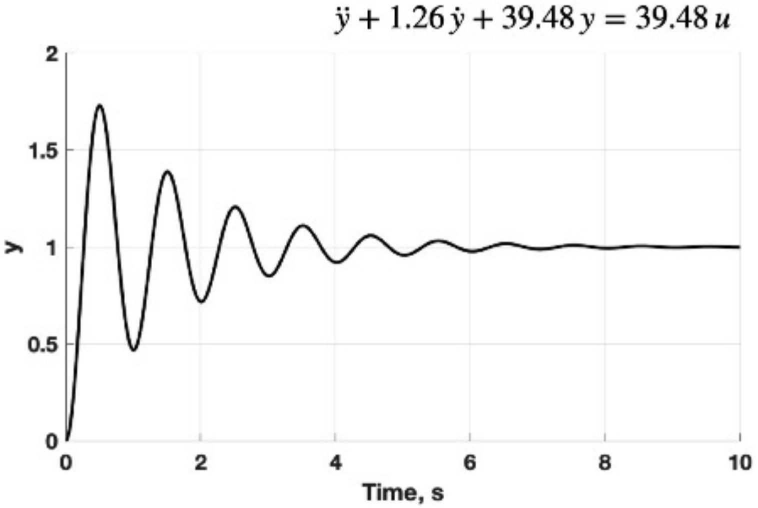
Second Order Linear Dynamic Systems

These occur in nearly all engineering disciplines. Every linear, time-invariant, second order system is uniquely defined by three parameters: ω_n, ζ, and k_{dc} and has the form:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = k_{dc}\omega_n^2\, u(t)$$

Damping Ratio, ζ, n.d.: An indicator of how quickly the step response **oscillation** decays.

Natural Frequency, ω_n, rad/s: The frequency of the step response IF $\zeta = 0$.



Second Order Linear Dynamic Systems - Damping Ratio

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K_{dc}\,\omega_n^2 u$$

The effect of ζ on the step response is:

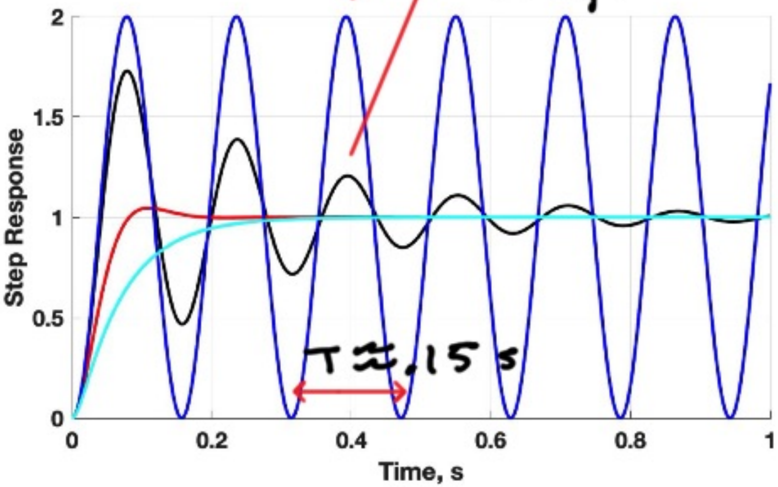
a. $\zeta < 0$: The step response blows up (unstable)

b. $\zeta = 0$: The step response oscillates forever (undamped)

c. $0 < \zeta < 1$: The step response has some oscillation. Very little oscillation for large ζ and lots of oscillation of small ζ (~~undamped~~) underdamped

d. $\zeta \geq 1$: The step response has no oscillation. It looks much like a first order step response except for some subtle differences at $t = 0$ (overdamped)

≈ equal freq



All of these systems have $\omega_n = 40$, but ζ is 0, 0.1, 0.7 and 1.5

$T \approx .15\,s \rightarrow \omega \approx \frac{2\pi}{T} \approx 42$ r/s

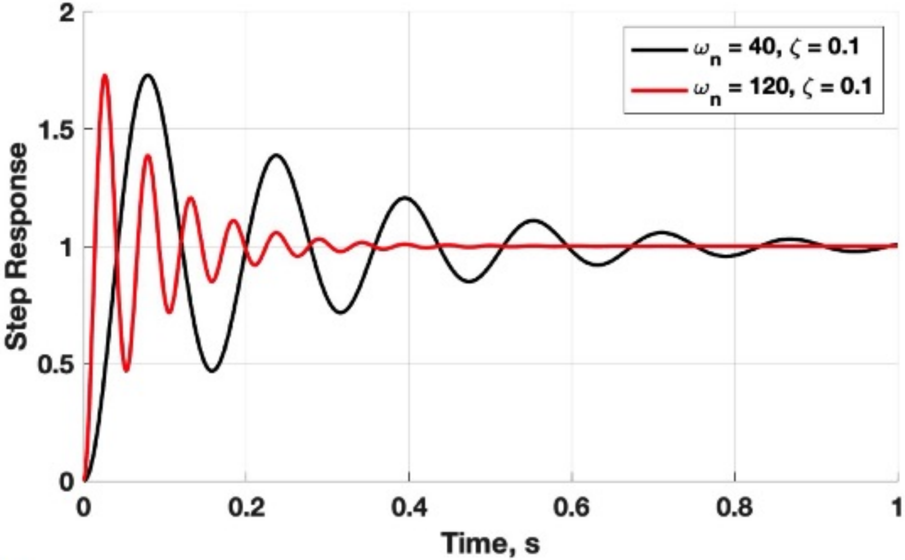
Second Order Linear Dynamic Systems - Natural Frequency

In a relative sense, large ω_n means higher frequency oscillation, or a “faster” response.

$\omega_n = 120$ r/s case

decays faster, yet,

$\zeta = .1$ same for both



$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K_{dc}\,\omega_n^2 u$$

Example 5

$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K_{dc}\,\omega_n^2 u$

For each of the second order systems below, calculate their damping ratios, natural frequencies and DC gains, then classify the response in terms of its damping and make a rough step response sketch.

1. $\ddot{T} + 2\dot{T} + 9T = 18u$

2. $\ddot{y} + 25y = 12.5u$

3. $\ddot{w} + 18\dot{w} + 36w = 6u$

MATLAB

The MATLAB command for extracting the damping ratio and natural frequency from a state space model is

```
>> damp(myss)
```

where `myss` is a state space object. Note: You'll need to pick through the results for what you want.

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = k_{dc}\omega_n^2 u$$

$$x_1 = y,\ x_2 = \dot{y}$$

$$\longrightarrow$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ k_{dc}\omega_n^2 \end{bmatrix}$$

$$C = [1 \quad 0] \qquad D = 0$$

Summary

You can (with a little more practice)

1. Create a first order DEQ model from its step response data
2. Decompose both first and second order system DEQs into their defining parameters $(k_{dc}, \tau, \zeta, \omega_n)$
3. Classify the damping of second order systems (overdamped, underdamped, undamped)
4. Calculate the steady state value of a dynamic system for any steady state input, u_{ss}, using its DC Gain
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