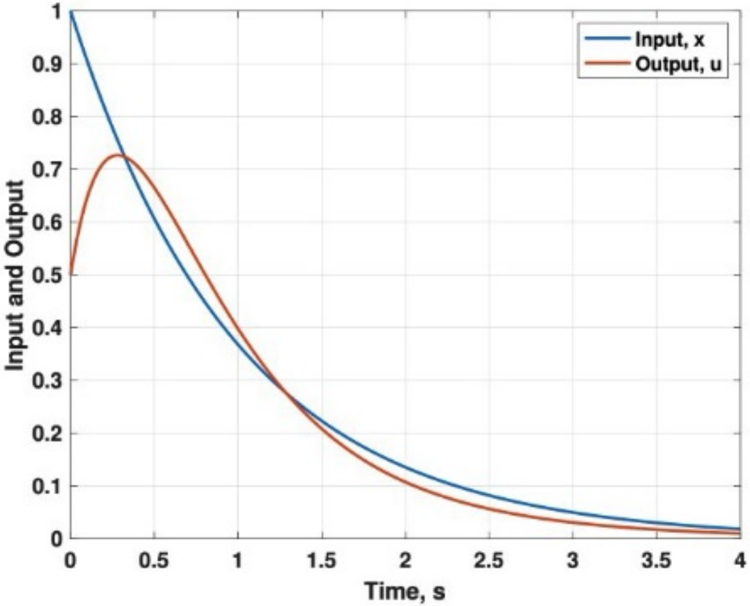
Differential Equation Review - Motivation

1. This course focuses on motion analysis grounded in **differential equations**. There's a little fluids and thermal system analysis too.

2. We need to all be on the same page with **terminology**

3. We will solve the occasional DEQ, but please rest easy! The course will mostly focus on other techniques to glean information on how a dynamic system behaves when poked

$$\ddot{x} + 5\dot{x} + 6x = e^{-t}, \;\; x(0) = .5, \dot{x}(0) = 2$$



Skills

1. Terminology:
    a. dependent and independent variables
    b. coefficients - constants and functions
    c. system order
    d. decoupled vs coupled systems of DEQs
    e. initial conditions
2. Know what it means to “solve” a system of DEQs
3. Be able to classify a system of DEQs as being
    a. linear or nonlinear
    b. time-invariant or time-varying
4. Use MATLAB to solve a differential equation and plot the solution vs time

Dependent/Independent Variables, Initial Conditions and Inputs

Consider the DEQ: $\ddot{x}(t) + 5\dot{x}(t) + 6x(t) = e^{-t}, \;\; x(0) = .5, \dot{x}(0) = 2$

1. A DEQ is simply an equation created from some unknown function, x(t), and its derivatives. The unknown function is the **dependent variable** and the quantity it depends on, t, is the **independent variable**.

2. The number of **initial conditions (ICs)** is equal to the highest derivative of the dependent variable

3. In this course the independent variable will always be time, t, and it's rarely shown. Also, if the **ICs** are not given, they are likely zero, but always ask…

$$\ddot{x} + 5\dot{x} + 6x = e^{-t}$$

4. The "standalone" exponential function of time on the right is called an **input**. It's better to write this as a variable so we can make it anything, e.g., replace the exponential with a sine wave, etc.

$$\ddot{x} + 5\dot{x} + 6x = u$$

5. The quantities that multiply the dependent variables and their derivatives are called **coefficients**. They may be functions of t.

Multiple DEQs, Coupling and Order

1. Often we need more than one DEQ to model a dynamic system. The DEQs below are called **decoupled** since each equation contains only one "family" of **dependent variables**. If the ICs are zero and u is not zero, then x will 'move' but phi will stay 0.

$$\ddot{x} + 5\dot{x} + 6x = u$$
$$\dddot{\phi} + 2\dot{\phi} + 7\phi = 0$$

2. Here's a **system of coupled DEQs**. If all the ICs are zero, but u is not, x will 'move' due to u, and phi will 'move' due to x.

$$\ddot{x} + 5\dot{x} + 6x = u$$
$$\dddot{\phi} + 2\dot{\phi} + 7\phi - \dot{x} = 0$$

3. The **order** of a dynamic system is the sum of the highest derivatives of all the **dependent variables**. Note: this is also the number of **ICs** the system should have.

Example

Consider the **system of differential equations** below used to model the swinging of a ship crane's payload

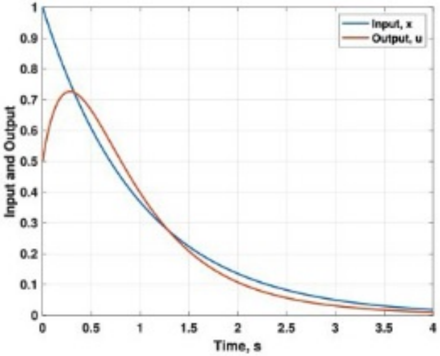
$$\ddot{\beta} + \dot{\beta} + (\beta - \phi) = u_1$$
$$\ddot{\phi} + 2\dot{\phi} + 2\phi - \dot{x} = 0$$
$$\dot{x} + 5x + \beta = u_2$$

- List the dependent variables
- What variable(s) likely represent the inputs?
- What is the system's order?
- Is the set of DEQs coupled or decoupled
- If the ICs are NOT assumed to be zero, how many are needed, and which dependent variables (and derivatives) should they span?

DEQ Solutions - Single Equation Model

Consider the DEQ: $\ddot{x} + 5\dot{x} + 6x = e^{-t},$ * $x(0) = .5,\ \dot{x}(0) = 2$

Its solution is the function x(t) that satisfies the DEQ AND the ICs: $x = \frac{1}{2}\left(e^{-t} + 5e^{-2t} - 5e^{-3t}\right)$



ICs: $\dot{x} = \frac{1}{2}(-e^{-t} - 10e^{-2t} + 15e^{-3t})$ $\quad \dot{x}(0) = \frac{1}{2}(-1-10+15) = 2$

$x(0) = \frac{1}{2}(1+5-5) = \frac{1}{2}$

DEQ: $\ddot{x} = \frac{1}{2}(e^{-t} + 20e^{-2t} - 45e^{-3t})$ Substitute *

$$\frac{1}{2}(e^{-t} + 20e^{-2t} - 45e^{-3t}) + \frac{5}{2}(-e^{-t} - 10e^{-2t} + 15e^{-3t}) + \cdots$$

$$+ \frac{6}{2}(e^{-t} + 5e^{-2t} - 5e^{-3t}) \stackrel{?}{=} e^{-t}$$

$$e^{-t}\underbrace{\left(\frac{1}{2} - \frac{5}{2} + \frac{6}{2}\right)}_{1} + e^{-2t}\underbrace{(10 - 25 + 15)}_{0} + e^{-3t}\underbrace{\left(\frac{-45}{2} + \frac{75}{2} - \frac{30}{2}\right)}_{0} \stackrel{?}{=} e^{-t}$$

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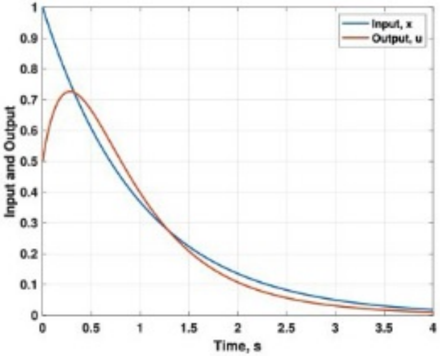

Time Invariant vs Time Varying DEQs

If ANY of the **coefficients** are functions of time, then the DEQ (or system of DEQs) is called **time-varying**. Otherwise, it's **time-invariant**.

Examples

$\ddot{x} + 5\dot{x} + 6x = e^{-t}$

$\ddot{y} + \dot{y} + (2 + \sin(t))\, y = u$

$\ddot{y} + \dot{y} + \sin(y)y = u$

Your body is an example of a time-varying dynamic system. At 21 it recovers from injuries and stress (inputs) more quickly than someone… a bit older….

Linear vs Nonlinear DEQs

If ALL of a system's DEQs have the form where the **coefficients** are either constants or functions of time, then it's a **linear system**. Otherwise it's a **nonlinear system**.

Examples

$\ddot{y} + \dot{y} + \sin(y)y = u$

$\dddot{w} + e^{-2w} + w = \sin t$

$\dddot{w} + e^{-2t}\dot{w} + w = \sin t$

Linear systems are MUCH easier to analyze and use in modular designs. Although all dynamic systems are nonlinear we often operate them with limited motion such that they can be approximated as linear.

Solving Differential Equations in MATLAB - Background

- MATLAB has two main types of variables: (1) numeric and (2) symbolic

```
>> y = 5; % numeric variable, double
```

```
>> syms u % symbolic variable
```

- We can also make symbolic functions. The statement below will create two things: (1) the symbolic variable t and (2) the symbolic function y(t)

```
>> syms x(t) % symbolic function
```

- Finally, we can make symbolic functions that are derivatives of other symbolic functions

```
>> dxdt1 = diff(x,t,1); % first derivative
```

```
>> dxdt2 = diff(x,t,2); % second derivative
```

- And here's how to create the DEQ: $\ddot{x} + 5\dot{x} + 6x = e^{-t}$

```
>> deq = dxdt2 + 5*dxdt1 + 6*x == exp(-t)
```

Solving Differential Equations in MATLAB

Solve and plot the DEQ: $\ddot{x} + 5\dot{x} + 6x = e^{-t}, \;\; x(0) = .5, \, \dot{x}(0) = 2$

- Use the variable deq from the previous page, define some ICs and use the dsolve command to find x(t)

```
>> x0 = x(0) == 1/2;
>> dxdt0 = dxdt(0) == 2;
>> xSol = dsvole( deq, [x0,dxdt0] )
```

dsolve

- For a quick plot, use fplot

```
>> fplot(xSol, [0,4]);
```

- For nicer plots, evaluate the solution at time points of interest to you, convert to numeric and plot

```
>> myt = 0 : .01 : 4;
>> plot(myt, double(xSol(myt)) );
```

Solving Differential Equations in MATLAB - Script

Solve and plot the DEQ: $\ddot{x} + 5\dot{x} + 6x = e^{-t},\ \ x(0) = .5,\ \dot{x}(0) = 2$

```
% Clean up
clearvars;
close('all');

syms x(t) % symbolic function

% Create symbolic fns for 1st and 2nd derivs and the input
dxdt1 = diff(x,t,1);
dxdt2 = diff(x,t,2);
u(t)  = exp(-t);

% Form the DEQ
deq = dxdt2 + 5*dxdt1 + 6*x == u;

% Create ICs
x0    = x(0)     == 1/2;
dxdt0 = dxdt1(0) == 2;

% Solve the DEQ
xSol(t) = dsolve(deq,[x0,dxdt0]);

% Here's a quick plot
figure(1);fplot(xSol,[0,4]);

% Here's a plot using time points of interest
myt = 0:.01:4;
figure(2);
plot(myt,double(xSol(myt)));
```

Summary

- Given a system of DEQs you can (with more practice…)
  a. Identify its dependent variables and inputs
  b. Classify it in two ways: (1) linear vs nonlinear and (2) time-invariant vs time-varying
- Given a system of DEQs and a possible solution, check if the solution is correct.
- From a plot of a DEQ solution verify it matches the given ICs
- Solve a DEQ using MATLAB and plot it using two different techniques

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