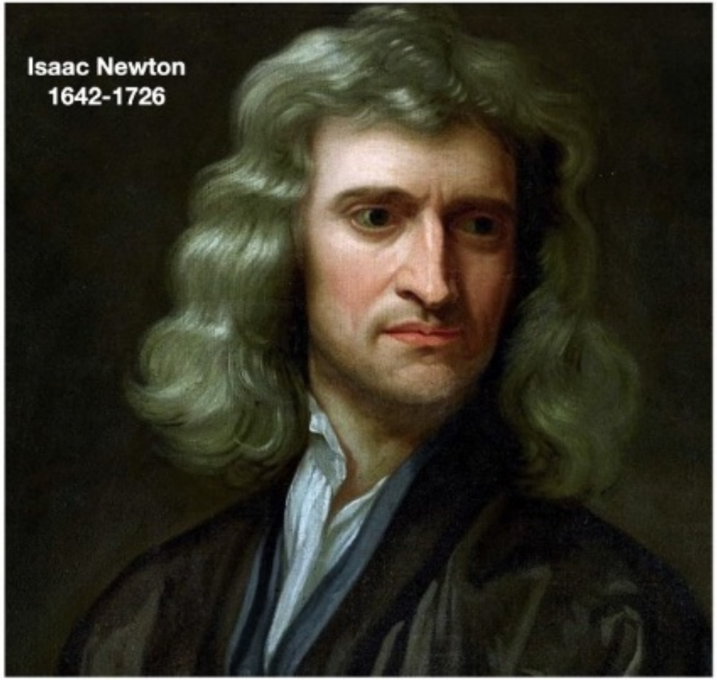
Solving DEQs

Given a DEQ model you have many ways to analyze and solve it.

We'll finish off how to use Laplace to solve DEQs and then summarize all the items in your DEQ toolbox and how might use them depending on the situation.

$$\frac{dy}{dx} = f(x, y)$$

$$\dot{y} = -4y + u$$



Skills

1. Solve DEQs using Laplace Transforms, either by hand or in concert with MATLAB
2. Given a DEQ model pick an approach for how to analyze it

Laplace Transform Table

Laplace Transform Table

#	$f(t)$	$F(s)$	#	$f(t)$	$F(s)$
1	$\delta(t)$	1	11	$1-e^{-at}$	$\frac{a}{s(s+a)}$
2	u_s	$\frac{1}{s}$	12	$\frac{1}{a}(at-1+e^{-at})$	$\frac{a}{s^2(s+a)}$
3	t	$\frac{1}{s^2}$	13	$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$
4	t^2	$\frac{2!}{s^3}$	14	$(1-at)e^{-at}$	$\frac{s}{(s+a)^2}$
5	t^3	$\frac{3!}{s^4}$	15	$1-(1+at)e^{-at}$	$\frac{a^2}{s(s+a)^2}$
6	t^m	$\frac{m!}{s^{m+1}}$	16	$be^{-bt}-ae^{-at}$	$\frac{(b-a)s}{(s+a)(s+b)}$
7	e^{-at}	$\frac{1}{(s+a)}$	17	$\sin{(bt)}$	$\frac{b}{s^2+b^2}$
8	te^{-at}	$\frac{1}{(s+a)^2}$	18	$\cos{(bt)}$	$\frac{s}{s^2+b^2}$
9	$\frac{1}{2!}t^2e^{-at}$	$\frac{1}{(s+a)^3}$	19	$e^{-at}\cos{(bt)}$	$\frac{s+a}{(s+a)^2+b^2}$
10	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$	$\frac{1}{(s+a)^m}$	20	$e^{-at}\sin{(bt)}$	$\frac{b}{(s+a)^2+b^2}$

Laplace Properties Table

Laplace Transform Properties

$f(t)$	$F(s)$	Property
$\alpha f_1(t) + \beta f_2(t)$	$\alpha F_1(s) + \beta F_2(s)$	Superposition
$f(t-\lambda)$	$F(s)e^{-s\lambda}$	Time Delay ($\lambda \geq 0$)
$f(at)$	$\frac{1}{\lvert a\rvert}F(\frac{s}{a})$	Time Scaling
$e^{-at}f(t)$	$F(s+a)$	Shift in Frequency
$f^{(m)}(t)$	$s^m F(s) - s^{m-1}f(0) +$ $-s^{m-2}\dot{f}(0) - \cdots - f^{(m-1)}(0)$	Differentiation
$\int f(\zeta)d\zeta$	$\frac{1}{s}F(s)$	Integration
$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$	Convolution
$f(0)$	$\lim_{s\to\infty} sF(s)$	Initial Value Theorem
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$	Final Value Theorem
$tf(t)$	$-\frac{d}{ds}F(s)$	Multiplication by Time

Big Picture

Given a DEQ with initial conditions and an input function, write out the form of its solution, a.k.a. its response, e.g. $\ddot{y} + 3\dot{y} + 2y = u$ with zero initial conditions and $u = 4e^{-3t}$.

Steps

1. Take the LT of the DEQ and solve for its dependent variable, e.g. $Y(s) = \dfrac{4}{s^3 + 6s^2 + 11s + 6}$, If $Y(s)$ is in the LT Table, then write down $y(t)$ and you're done! Unlikely…
2. Expand $Y(s)$ into partial fractions such that each term IS in the LT Table. This introduces unspecified coefficients, e.g. $Y(s) = \dfrac{c_1}{s+1} + \dfrac{c_2}{s+2} + \dfrac{c_3}{s+3}$
3. Write out the form of $y(t)$ using the LT Table, e.g. $y(t) = c_1e^{-t} + c_2e^{-2t} + c_3e^{-3t}$. We now know the character of the response!
4. If you are very ambitious, solve for the coefficients.

Example 1

Solve the DEQ model: $2\dot{g} + g = t\,u_s, \quad g(0) = 4$

Note: This is a 1st order DEQ with a 2 second time constant. It's input is a ramp, so, it's likely that the response will also be ramp-like.

Approach: Obtain the form of the solution, then turn to MATLAB to get the coefficients and a plot.

Laplace ①

$$2(sG - g(0)) + G = \frac{1}{s^2}$$

$$G(2s+1) = 2(4) + \frac{1}{s^2} = \frac{8s^2+1}{s^2}$$

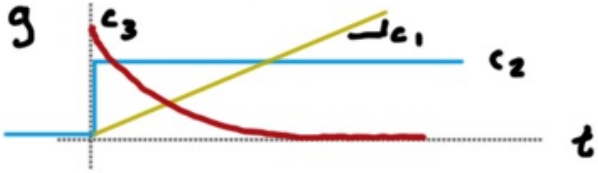
$$G = \frac{8s^2+1}{s^2(2s+1)} = \frac{4s^2+1/2}{s^2(s+1/2)}$$

PFE ②

$$G = \frac{c_1}{s^2} + \frac{c_2}{s} + \frac{c_3}{s+1/2}$$

ILT ③

$$g = c_1 t + c_2 u_s + c_3 e^{-1/2 t}$$



MATLAB ④

```
syms s t
G=(4*s^2 + 1/2)/s^2/(s+1/2)
g=ilaplace(G,s,t)
```

Also... `fplot(g,[0,10])`

Example 1 - continued

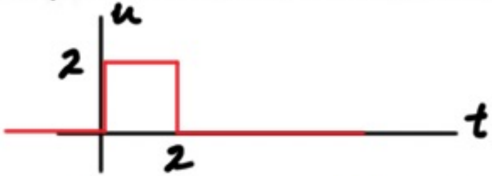
Solve the DEQ model: $2\dot{g} + g = t\,u_s, \quad g(0) = 4$

Example 2

Obtain the pulse response for the system whose transfer function is: $\dfrac{Y}{U} = \dfrac{1}{s+3}$. The pulse input is shown below.

Note: This is a first order system. It may be instructive to write out its DEQ model, though not needed to solve the problem. Also, transfer functions assume all initial conditions are zero.

Approach: Since a pulse is the sum of two steps, use the superposition, find the solution to a step, then tack on the solution to the other step, delayed in time… We can also use MATLAB



$Y \cdot (s+3) = U \cdot 1$, $\dot{y} + 3y = u$

$\tau = 1/3$ sec, $K_{dc} = 1$

Laplace u ①

$u = 2u_s - 2u_s(t-2)$

$U = \dfrac{2}{s} - \dfrac{2e^{-2s}}{s}$

Superposition Plan ②

Let $u = u_1 + u_2$, $u_1 = 2u_s$, $u_2 = -2u_s(t-2)$

$\therefore y = y_1 + y_2$ Also, if y_1 is known, $y_2 = -y_1$ & delayed 2 sec. $\therefore$ Find the step response!

Example 2 - continued

Obtain the pulse response for the system whose transfer function is: $\dfrac{Y}{U} = \dfrac{1}{s+3}$. The pulse input is shown below.

step response

$$Y_1 = \frac{1}{s+3} U = \frac{2}{s(s+3)}$$

$$Y_1 = \frac{c_1}{s} + \frac{c_2}{s+3}$$

$$y_1 = c_1 + c_2 e^{-3t}$$

so,

$$y_2 = -\left(c_1 + c_2 e^{-3(t-2)}\right) u_s(t-2)$$

MATLAB

```
syms s t
Y1 = 2/s/(s+3)
y1 = ilaplace(Y1, s, t)
```

$$y1 = \frac{2}{3} - \frac{2}{3} e^{-3t}$$

c_1 c_2

$$y = y_1 + y_2$$

More MATLAB ...

```
Y2 = -Y1 * exp(-2*s) ...
```

Example 2 - continued

Obtain the pulse response for the system whose transfer function is: $\dfrac{Y}{U} = \dfrac{1}{s+3}$. The pulse input is shown below.

Example 3

Find the solution to the DEQ: $\ddot{x} + 2\dot{x} + 17x = 17e^{-2t}u_s, \quad x(0) = 0, \dot{x}(0) = 3$

Note: This is a 2nd order DEQ model with $\omega_n = \sqrt{17}$ and $\zeta = 1/\sqrt{17}$, so its underdamped. The dc gain is up for interpreation. We could say the left side is $17u$ with $u = e^{-2t}u_s$ and then $k_{dc} = 1$. But this doesn't matter too much for what we are doing. Also, as $t \to \infty$ the input (right side) goes to 0. It seems reasonable to then expect $x_{ss} = 0$.

Approach: Use Laplace to get the form, then use MATLAB for the details and a plot.

LT ①

$$s^2 X - s\overset{\times}{x(0)} - \overset{3}{\dot{x}(0)} + 2(sX - \overset{\times}{x(0)}) + 17X = \frac{17}{s+2}$$

Factor ②

$$X \cdot (s^2 + 2s + 17) = 3 + \frac{17}{s+2} = \frac{3s+23}{s+2}$$

Solve X ③

$$X = \frac{3s+23}{(s+2)(s^2+2s+17)}$$

Example 3 - continued

Find the solution to the DEQ: $\ddot{x} + 2\dot{x} + 17x = 17e^{-2t}u_s$, $x(0) = 0$, $\dot{x}(0) = 3$

$$X = \frac{3s+23}{(s+2)(s^2+2s+17)}$$

PFE 4

$s^2+2s+17 = (s+1)^2 + 4^2$, $a=1$, $b=4$
for #19, #20.

$$X = \frac{c_1}{s+2} + \frac{4c_2}{(\quad)} + \frac{c_3(s+1)}{(\quad)}$$

ILT 5

$$\boxed{x = c_1e^{-2t} + c_2e^{-t}\sin 4t + c_3e^{-t}\cos 4t}$$

Note: $x(0) = c_1 + c_3 = 0$
$x(\infty) = 0$, $X_{ss} = 0$ ✓

MATLAB: $c_1 = 1$
$c_2 = 1$
$c_3 = -1$

Example 3 - continued

Find the solution to the DEQ: $\ddot{x} + 2\dot{x} + 17x = 17e^{-2t}u_s, \quad x(0) = 0, \dot{x}(0) = 3$

Example 3 - continued

Find the solution to the DEQ: $\ddot{x} + 2\dot{x} + 17x = 17e^{-2t}u_s, \quad x(0) = 0, \dot{x}(0) = 3$

Summary

Given a DEQ model, or even a transfer function, you have many ways to approach analysis

1. If its 1st or 2nd order, extract response parameters to get a general sense of the motion
2. Regardless of the order, extract the dc gain to understand its steady state configuration
3. Represent it in state space and simulate it in MATLAB/Simulink
4. Use Laplace Transforms to create the form of the solution, and if you want to go further, solve for the coefficients by hand or with MATLAB. At that point, you should plot it.