# 1 Deriving posteriors

## Q 1.1

$$p(\mu|x) = p(x|\mu)p(\mu) \propto \text{Likelihood } \times \text{ prior} = p(x|\mu)p(\mu)$$

$$p(x|\mu) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2\right)$$

We set $\tau = \frac{1}{\sigma^2}$ to simplify the derivation

$$\left(\frac{1}{\sqrt{2\pi}}^N\right)\tau^{\frac{N}{2}} \exp\left(\frac{-\tau}{2}\cdot\sum_{i=1}^{N}(x_i-\mu)^2\right) \propto \exp\left(-\frac{\tau}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right)$$

For the prior we set $\tau_0 = \frac{1}{\sigma_0^2}$

$$p(\mu) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right) = \frac{1}{\sqrt{2\pi}}\sqrt{\tau_0} \exp\left(-\frac{\tau_0}{2}(\mu-\mu_0)^2\right) \propto \exp\left(-\frac{\tau_0}{2}(\mu-\mu_0)^2\right)$$

$$p(\mu|x) \propto \exp\left(-\frac{\tau_0}{2}(\mu-\mu_0)^2\right)\exp\left(-\frac{\tau}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right)$$

$$= \exp\left(\frac{-\tau}{2}\cdot\left(\sum_{i=1}^{N}x_i^2 - 2N\bar{x}\mu + N\mu^2\right) - \frac{\tau_0}{2}(\mu-\mu_0)^2\right)$$

$$= \exp\left(\frac{-\tau}{2}\cdot\left(2N\mu + N\mu^2\right) - \frac{\tau_0}{2}(\mu-\mu_0)^2\right)$$

$$= \exp\left(-\tau N\bar{x}\mu + \frac{\tau N\mu^2}{2} - \frac{\tau_0\mu^2}{2} + \tau_0\mu\mu_0 - \frac{\tau_0\mu_0^2}{2}\right) \propto \exp\left(\frac{\mu^2}{2}(\tau_0 - N\tau) + \mu(\tau N\bar{x} + \tau_0\mu_0)\right)$$

Completing the square gives us

$$\propto \exp\left(\frac{1}{2}(\tau_0 + N\tau)\left(\mu - \frac{\mu_0\tau_0 + N\bar{x}\tau}{\tau_0 + N\tau}\right)^2\right)$$

$$= \exp\left(\frac{-1}{2}\left(\frac{N}{\sigma^2} + \frac{1}{\sigma_0^2}\right)\left(\mu - \frac{\sigma_0^2 N}{\sigma^2 + N\sigma_0^2}\bar{x} + \frac{\sigma^2}{\sigma^2 + N\sigma_0^2}\mu_0\right)^2\right)$$

As we can see the posterior of the mean is now the form of a gaussian distrubution with

$$\mu' = \left(\mu - \frac{\sigma_0^2 N}{\sigma^2 + N\sigma_0^2}\bar{x} + \frac{\sigma^2}{\sigma^2 + N\sigma_0^2}\mu_0\right), \sigma'^2 = \left(\frac{N}{\sigma^2} + \frac{1}{\sigma_0^2}\right)^{-1}$$

$$\mu \sim \mathcal{N}\left(\frac{\sigma_0^2 N}{\sigma^2 + N\sigma_0^2}\bar{x} + \frac{\sigma^2}{\sigma^2 + N\sigma_0^2}\mu_0,\ \left(\frac{1}{\sigma_0^2} + \frac{N}{\sigma^2}\right)^{-1}\right)$$

*Q.E.D*

## Q 1.2

This time $\mu$ is fixed and we want to find $p(\sigma^2|X)$

$$p(\sigma^2|x) \propto p(x|\sigma^2)p(\sigma^2) \propto p(x|\sigma^2)p(\sigma^2)$$

The prior is:

$$p(\sigma^2) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} \exp\left(-\frac{\beta}{\sigma^2}\right)$$

The likelihood is:

$$p(x|\sigma^2) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2\right)$$

The posterior is

$$\begin{aligned} p(\sigma^2|x) \propto \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} \exp\left(-\frac{\beta}{\sigma^2}\right) \times \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2\right) \\ = \left(\frac{1}{\sigma^2}\right)^{\alpha+1+\frac{N}{2}} \exp\left(-\frac{1}{\sigma^2}\left(\beta + \frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right)\right) \\ \propto \left(\frac{1}{\sigma^2}\right)^{N/2+\alpha} \exp\left(-\frac{1}{\sigma^2}\left(\beta + \frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right)\right) \end{aligned}$$

We see that the posterior has the shape of the inverse gamma pdf with the right parameters

$$p(\sigma^2|x) \sim \text{InvGamma}\left(\frac{N}{2}+\alpha, \beta + \frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right) \quad Q.E.D$$

## Q1.3

In this excercise we treat both the mean and precision as unknown.
Posterior for $\mu$

$$p(\mu|X,\tau) \propto p(X|\mu,\tau^{-1})p(\mu|\mu_0,(n_0\tau^{-1})$$

$$p(\mu|X,\tau^{-1}) \propto \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}}\tau^{\frac{1}{2}}\frac{\sqrt{\tau n_0}}{\sqrt{2\pi}} \exp\left(\frac{-\tau}{2}(x_i-\mu)^2 + \frac{-\tau n_0}{2}(\mu-\mu_0)^2\right)$$

$$\propto \exp\left(\frac{-\tau}{2}\sum_{i=1}^{N}(x_i-\mu)^2 - \frac{\tau n_0}{2}(\mu-\mu_0)^2\right) = \exp\left(\frac{1}{2}(\tau n_0+N\tau)\left(\mu - \frac{\mu_0\tau n_0 + N\bar{x}\tau}{\tau n_0+N\tau}\right)^2\right)$$

$$p(\mu|X,\tau^{-1}) \propto N\left(\frac{N}{N+n_0}\bar{x} + \frac{n_0}{n_0+N}\mu_0, (\tau n_0+N\tau)^{-1}\right)$$

Moving on to the preciscion. We want to find out $p(\mu,\tau^{-1}|X)$. We can use a normal gamma prior and utilize our previous results

$$p(\mu,\tau^{-1}|X) \propto NormalGamma(\mu,\tau|\mu_0,(n_0\tau)^{-1},\alpha,\beta)p(X|\mu,\tau)$$

$$\propto Gaussian\left(\frac{N}{N+n_0}\bar{x} + \frac{n_0}{n_0+N}\mu_0, (\tau n_0+N\tau)^{-1}\right) Gamma\left(\frac{N}{2}+\alpha, \beta + \frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right) \cdot p(X|\mu,\tau)$$

$$\propto \tau^{\frac{N}{2}+\alpha} \exp\left(\frac{1}{2}(\tau n_0+N\tau)\left(\mu - \frac{\mu_0\tau n_0+N\bar{x}\tau}{\tau n_0+N\tau}\right)^2 - \tau\left(\beta + \frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right) + \frac{-\tau}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right)$$

$$\propto \tau^{\frac{N}{2}+\alpha} \exp\left(\frac{1}{2}(\tau n_0+N\tau)\left(\mu - \frac{\mu_0\tau n_0+N\bar{x}\tau}{\tau n_0+N\tau}\right)^2 - \tau\left(\beta + \frac{1}{2}\sum_{i=1}^{N}(x_i-\mu)^2\right)\right)$$

We take advantage of the fact that $\sum_{i=1}^{N}(x_i-\mu)^2 = \sum_{i=1}^{N}(x_i-\bar{x})^2 + N(\bar{x}-\mu)^2$

$$= \tau^{\frac{N}{2}+\alpha} \exp\left(\frac{1}{2}(\tau n_0+N\tau)\left(\mu - \frac{\mu_0\tau n_0+N\bar{x}\tau}{\tau n_0+N\tau}\right)^2 - \tau\left(\beta + \frac{1}{2}\sum_{i=1}^{N}(x_i-\bar{x})^2 + N(\bar{x}-\mu)^2\right)\right)$$

$$= \tau^{\frac{N}{2}+\alpha} \exp\left(\tau\left(\beta + \frac{1}{2}\sum_{i=1}^{N}(x_i-\bar{x})^2 + \frac{Nn_0}{2(N-n_0)}(\bar{x}-\mu_0)^2\right)\right)$$

$$= \mathrm{Ga}\left(\tau \mid \alpha + \frac{N}{2},\ \beta + \frac{1}{2}\sum_{i=1}^{N}(x_i-\bar{x})^2 + \frac{Nn_0}{2(N+n_0)}(\bar{x}-\mu_0)^2\right)$$

$$Q.E.D$$