Problem 1

- (1) D
- (2) 2
- (3) $k = 5$
- (4) $f(n) = O(g(n))$
- (5) $S = 1536$
- (6) Number of swap is 8
- (7) $[1, 2, 7, 8, 9, 6, 3, 4, 5]$

(8) Height of the node is the longest path or number of edges from the node to the children. Denote the height as h(N) where N is the node.

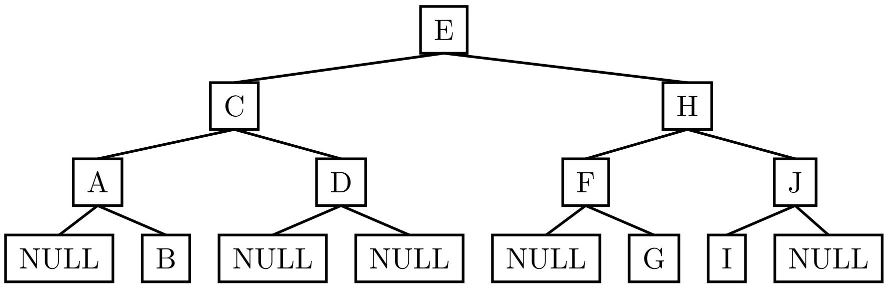
$$h(N) = \begin{cases} 0 \text{ if N is a leaf node} \\ 1 + \max(h(\text{N's left child node}), h(\text{N's right child node})) \end{cases}$$

Depth of the node is the longest path or number of edges from the root to the node. Denote the depth as d(N) where N is the node.

$$d(N) = \begin{cases} 0 \text{ if N is the root node} \\ 1 + d(\text{N's parent node}) \end{cases}$$

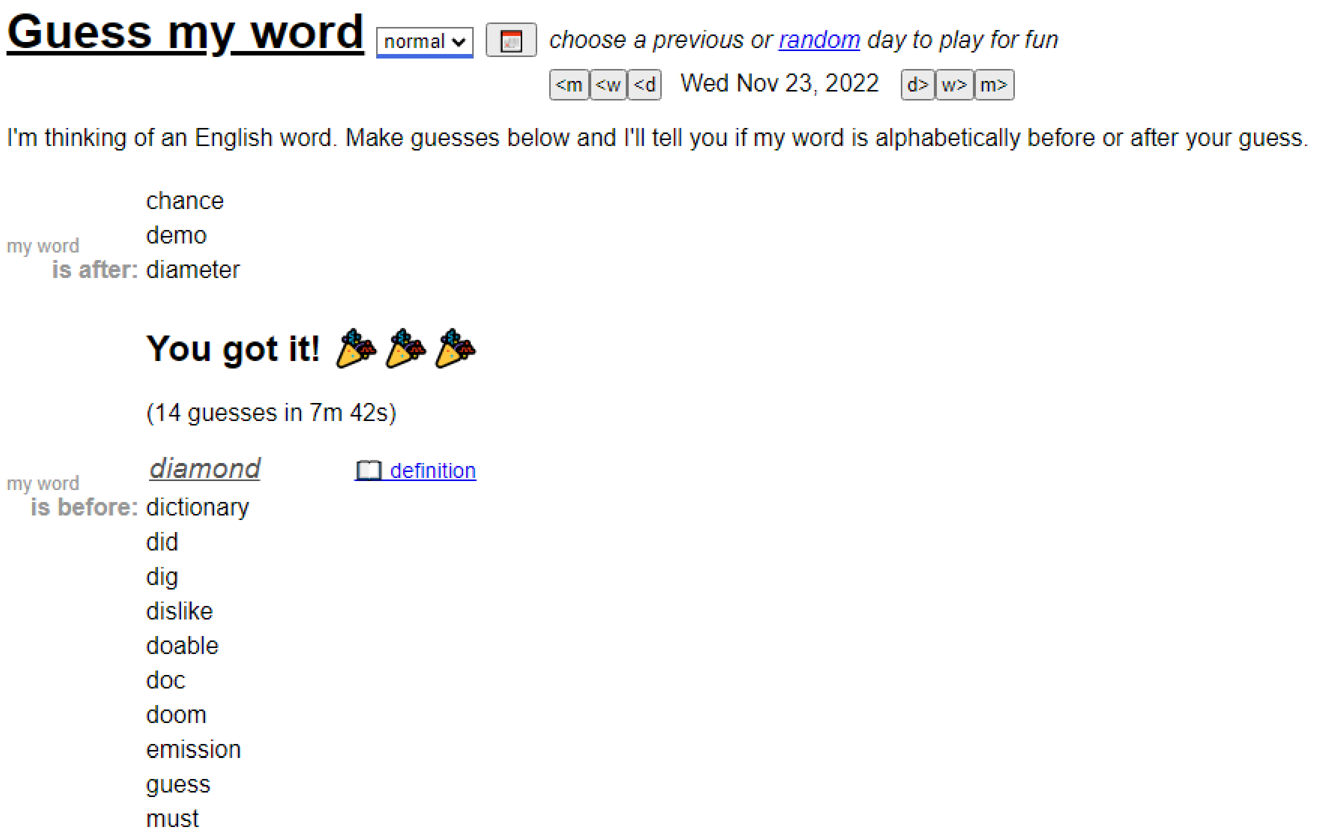
(9)
- Preorder traversal:[a b d e c f h l m q i g j n o k p]
- Inorder traversal: [d b e a l h q m f i c n j o g k p]
- Postorder traversal:[d e b l q m h i f n o j p k g c a]

(10)



Problem 2

(a)


Guess my word normal choose a previous or random day to play for fun

<m <w <d Wed Nov 23, 2022 d> w> m>

I'm thinking of an English word. Make guesses below and I'll tell you if my word is alphabetically before or after your guess.

chance
demo
my word is after: diameter

You got it! 🎉🎉🎉

(14 guesses in 7m 42s)

diamond definition
my word is before: dictionary
did
dig
dislike
doable
doc
doom
emission
guess
must

Figure 1: Solution (2-a)

(b)

Initilaize an offset to 1, denoting the depth of letter for comparison. For example, we the depth is 1 for castle and cattle, then we're comparing the 2nd letter of the two words.
Approximate the offset letter of the middle word in the dictionary, alphabetically, it is m-started word, the the ascii code is $\frac{\text{int('z')}-\text{int('a')}}{2}$ and compare it with the target word. If response says "before, then the target word must be in the first half of the dictionary. Otherwise, the target word must be in the second half of the dictionary.
Recursively repeat: Shrink the find range by half based on the previous step, approximate the offset letter of the middle word in the new range, and compare it with the target word. If the target word is less than the middle word, then the target word must be in the first half of the new range. Otherwise, the target word must be in the second half of the new range. If the range is reduced to 0 but the word is not found, then increase the offset by 1 and repeat the process.
This is an simplified version of search, actual search should consider the distribution of the words in the dictionary. For example, m may not be the middle word in the dictionary. and there's much less word contains letters like xyz than abc.

(c)

$T(n) = T\left(\frac{n}{2}\right) + O(1)$
Every time, the middle word is being compared and cost O(1) time. and the range is reduced by half every time by picking the middle value.
Master theorem has the following form: $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
where $a = 1, b = 2, f(n) = O(1)$

- Consider case 2: $f(n) = \Theta\left(n^{\log_b a} \log^k n\right)$ for some $k \geq 0$, then $T(n) = \Theta\left(n^{\log_b a} \log^{k+1} n\right)$

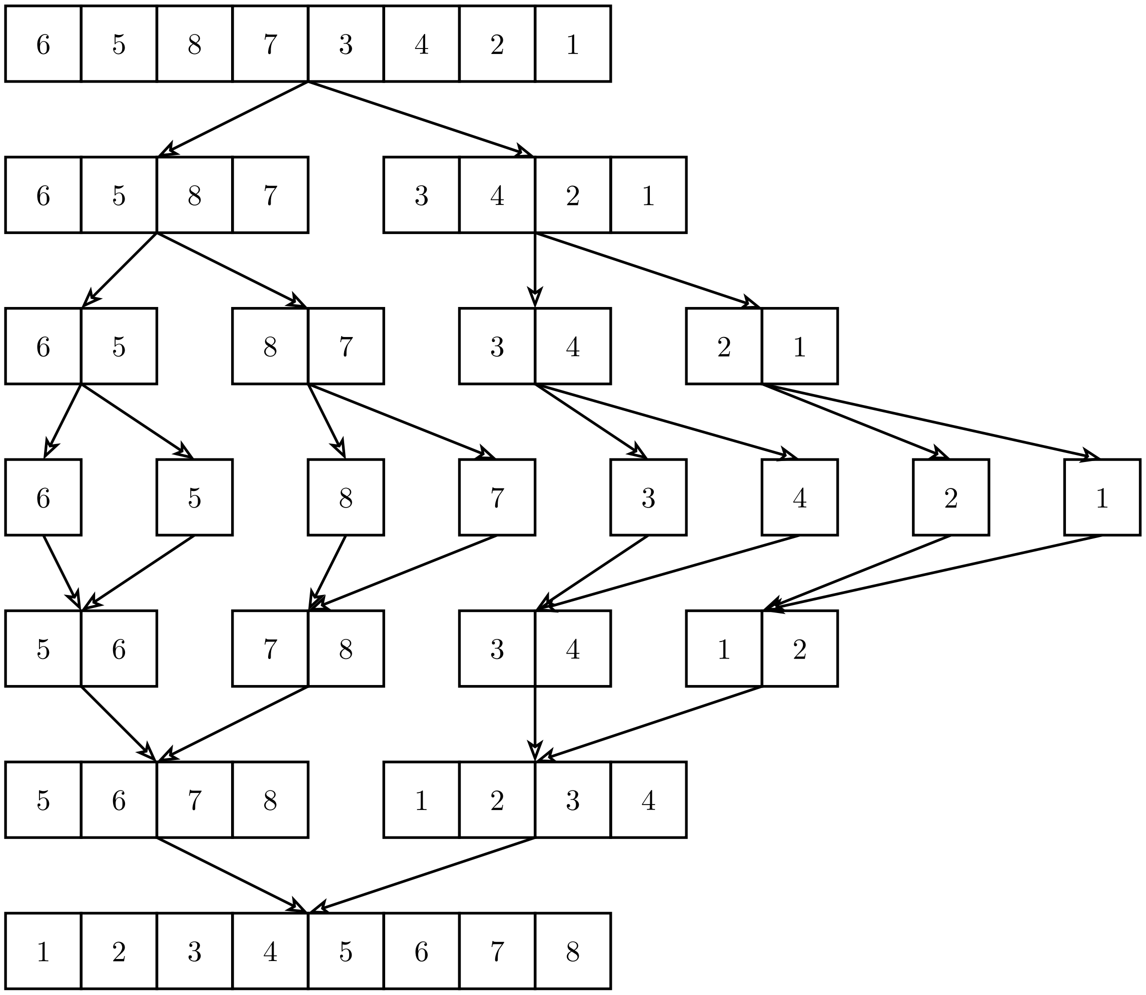
here, $\log_b a = 0$ and we let $k = 0$ we have $O(1) = \Theta(n^0) = \Theta(1)$ therefore $T(n) = \Theta(\log(n))$

(d)

$\log(267751) \approx 18$
Assume you cannot quit the game half way The maximum number of comparisons is 18, which means it costs at maximum 18 dollars to find the word. So to profit 1 bucks, we can only do 14 comparison, which minimize the range to $2^4 = 16$ and and the expectation is $E(\text{profit}) = \frac{1}{16} * 1 + \frac{15}{16} * (-4) = -2.75$ if we do 13 comparison the expectation is even lower.

Problem 3

(a)



- 4 Comparison: 6 and 5, 8 and 7, 3 and 4, 2 and 1
- 4 comparison: 5 and 7, 7 and 6, 3 and 1, 3 and 2,
- 4 comparison: 5 and 1, 5 and 2, 5 and 3, 5 and 4

Total 12 comparison

(b)

$M = 2M\left(\frac{n}{2}\right) + \frac{n}{2}$
So merge sort is a divide and conquer algorithm, it divides the array into two halves and recursively compare and merge the two halves. The division step therefore, results in 2 subproblems of size n/2, which gives $2M\left(\frac{n}{2}\right)$ to merge from.
The merge step is the conquer step, it compare and merges the two sorted halves into a single sorted array. And the comparison takes up from $n-1$ (alternating subarray) to $\frac{n}{2}$ (all elements from one is smaller than the other). Since we only consider the minimum, then the number of comparison is $\frac{n}{2}$
Therefore, we have $M(n) = 2M\left(\frac{n}{2}\right) + \frac{n}{2}$

(c)

We use the substitution method to solve the recurrence.

$$\begin{aligned}
M(n) &= 2\frac{\frac{n}{2}\log\left(\frac{n}{2}\right)}{2} + \frac{n}{2} \\
M(n) &= \frac{n}{2}\log\frac{n}{2} + \frac{n}{2} \\
M(n) &= \frac{n}{2}(\log n - \log 2) + \frac{n}{2} \\
M(n) &= \frac{n}{2}\log n - \frac{n}{2} + \frac{n}{2} \\
M(n) &= \frac{n}{2}\log n
\end{aligned}$$

Therefore, by substitution method, we have $M(n) = \frac{n\log n}{2}$

(d)

The merge sort divide the given array 3 times and at the bottom level made 4 comparison no matter the order of the array. The variation of the comparison number occurs in the 2nd merge and top level merge. consider 2 cases for the 2nd merge, either $3 * 2 = 6$ comparison is made for each pair of subarray or $2 * 2 = 4$ comparison is made for each pair of subarray. If former case is true, there's no way to have $12 - 4 - 6 = 2$ comparison at the top level. Therefore, the 2nd merge must have 4 comparison for each pair of subarray and the top level must have 4 comparison consequentially
to have 4 comparison at the top level, one array must be greater than the other for all its value. the probabily is $\frac{2}{8C4} = \frac{1}{35}$ where C is the combination operator
To have 4 comparison at 2nd level, the one array must be greater than the other for all its value. $\frac{2}{4C2} = \frac{1}{3}$
Therefore, the probability of having 4 comparison at the top level and 2nd level is $\frac{1}{35} * \frac{1}{3} * \frac{1}{3} = \frac{1}{315}$

Problem 4

(a)

In seleciton sort, we find the minimum element in the unsorted array and swap it with the first element. Then we find the minimum element in the remaining unsorted array and swap it with the second element. We repeat this process until the array is sorted. If the input is sorted, the algorithm still needs to compare every element with the minimum element in the unsorted array. Therefore, The number of comparison is $n + (n-1) + (n-2) + ... + 1 = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$
By definition of big O, we need to find c and n_0 such that $\frac{n^2}{2} + \frac{n}{2} \leq cn$ for all $n \geq n_0$
Note that $\frac{n^2}{2} + \frac{n}{2} \leq \frac{n^2}{2} + \frac{n^2}{2} \leq n^2$
Let $c = 1$

$$\frac{n^2}{2} + \frac{n}{2} \leq n^2$$
$$\frac{n}{2} \leq \frac{n^2}{2}$$
$$1 \leq n$$

so $n_0 = 1$
Therefore, $\frac{n^2}{2} + \frac{n}{2} = O(n)$
For Insertion sort, we start from the second element and compare it with the first element, if it is smaller, we swap them. Then we compare the third element with the second element and swap if necessary. We repeat this process until the array is sorted. However, for this input, the inner loop will not be executed since the 2nd element is never smaller than the first element. Therefore, the number of comparison is $n - 1$
By definition of big O, we need to find c and n_0 such that $n - 1 \leq cn$ for all $n \geq n_0$ note that $n - 1 \leq n$
Therefore, $n - 1 = O(n)$
So O(n) is smaller than O(n^2), therfore the insertion sort is better

(b)

In heap sort, we first build a max heap from the array, then we swap the first element with the last element and heapify the array. We repeat this process until the array is sorted.
The heap building process can be expressed

$$\begin{aligned}\sum_{i=\lfloor\log n\rfloor}^{0} 2^i(\lfloor\log n\rfloor - i) &\leq \sum_{i=\lfloor\log n\rfloor}^{0} \frac{n}{2^h}h \\ &\leq n\sum_{i=\lfloor\log n\rfloor}^{0} \frac{h}{2^h} \\ &\leq n\sum_{i=0}^{\infty} \frac{h}{2^h} \\ &= 2n\end{aligned}$$

By letting $c = 2$ and $n_0 = 1$, we have $2n = O(n)$ Therefore, the heap building process is $O(n)$
then we have the extraction process, which consist of n extraction and heapify process. The heapify process is $O(\log n)$, so the extraction process is $O(n\log n)$
Therefore, the total time complexity of heap sort is $O(n) + O(n\log n) = O(n\log n)$
Qucik sort consist of partition and sort process. The partition process will select a pivot and partition the array into two halves, then recursively partition the two halves. In this case, the partition will always select the last element as the pivot, and the partition process will always result in a partition of 1 and n-1. Therefore, the partition process makes k-1 comparison. where k is the kth partition. The total number of comparison is thus $n + (n-1) + (n-2) + ... + 1$ and we have proved in the previous question that this is $O(n^2)$
Therefore, the total time complexity of quick sort is $O(n^2)$ and heap sort is better than quick sort

(c)

The bubble sort does not depends on the input, it will always compare every element with the next element and swap if necessary. Therefore, the number of comparison is $\frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2}$
By definition of big O, we need to find c and n_0 such that $\frac{n^2}{2} - \frac{n}{2} \leq cn$ for all $n \geq n_0$
since $\frac{n^2}{2} - \frac{n}{2} < \frac{n^2}{2} + \frac{n}{2} \leq c^n$ which we have proved in the previous question, we can let $c = 1$ and $n_0 = 1$
Therefore, $\frac{n^2}{2} - \frac{n}{2} = O(n^2)$
In bucket sort, we first divide the array into buckets, then sort each bucket and concatenate them. Since n is an integer, We initialize n empty buckets, this takes O(n) time. Then we insert element to its bucket and sort each bucket using insertion sort, which takes O(n) time since each bucket have the same element. Finally, we concatenate the buckets, which takes O(n) time. Therefore, the total time complexity is $O(n)$
Therefore, the bucket sort is better than bubble sort for this input

(d)

In Counting sort with the given input, we Initilaize the number of occurance from 0-9 to zero. This takes 10 operations. Then we iterating through the array and count the occurance of each element, which takes n operations. Then we iterate through the occurance array and sum the previous element, which takes n operations Finally, we iterate through the array and place the element in the correct position, which takes n+10 operations. Therefore, the total time complexity is $3n + 20 = O(n)$
let $c = 4$

$$3n + 20 \leq 4n$$
$$n \geq 20$$

Therefore, $n_0 = 20$
In merge sort, as we shown in the previous question, the best time complexity is $O(n \log n)$
The worst time complexity of counting sort is still $O(n \log n)$ since we can express the number of comparison as $M(n) = 2M\left(\frac{n}{2}\right) + n - 1$
Using Master theorem, we have $a = 2, b = 2, f(n) = n - 1$

- Consider case 2: $f(n) = \Theta\left(n^{\log_b a} \log^k n\right)$ for some $k \geq 0$, then $T(n) = \Theta\left(n^{\log_b a} \log^{k+1} n\right)$

here, $\log_b a = 1$ and we let $k = 0$ we have
$O(n - 1) = \Theta(n^1 \log^0 n) = \Theta(n)$ therefore $T(n) = \Theta(n \log n)$
Therefore, the counting sort is better than merge sort for this input

Problem 5

(a)

BFS traverse the graph level by level, starting from the root node, then each immediate neighbor before moving the to the neighbors's neighbor. It utilizes a queue (in which elements lastly inserted is popped out last(LILO)), by enqueuing the starting node and dequeuing the node and enqueuing its neighbors, then repeat. The time complexity of BFS is $O(V + E)$ where V is the number of vertices and E is the number of edges. It also finds the shortest path in an unweighted graph.
DFS traverse the graph by going as deep as possible along each branch before backtracking. It utilizes a stack (in which element lastly inserted is poped first (LIFO)), by pushing the starting node and popping the node and pushing its neighbors, then repeat. The time complexity of DFS is $O(V + E)$ where V is the number of vertices and E is the number of edges. It is not guaranteed to find the shortest path in an unweighted graph, but it can be used to find the connected components of a graph.
For the graph above, we have

- BFS Traversal:[a b c d e f g h i j k l m n o p q]
- DFS Traversal: [a b d e c f h l m q i g j n o k p]

(b)

Initilaize a path array set the starting node as the root node
Step 2:

- iterate through the starting node's children
- Append the node to the path array
- Recursively call the function with the child node as the starting node
- Append the node to the path array

psudo code:

```
global path
function DFS(node):
  path.append(node)
  left = node.left
  right = node.right
  if left:
    path.append((node, left))
    DFS(left, path)
    path.append((left, node))
  if right:
    path.append((node, right))
    DFS(right, path)
    path.append((right, node))

  return path

DFS(root)
```

This is the algorithm to traverse each edge exactly twice in different directions using DFS,

- the algorithm will recursively gets to the leaf node and then backtracks to the parent node, then to the next child node if it exists.
- Each edge is visited in the forward direction when the call stack is being filled. And the edge is visited in the backward direction when the function is being popped out of the call stack.

- Therefore, each edge is visited exactly twice in different directions.

Since the graph is a binary tree and vertices is visited twice only, the time complexity is $O(n)$

(c)

By definition of distance between two vertices, it is the number of edges in the shortest path between the two vertices. Therefore, the distance between two vertices is the number of edges in the shortest path between the two vertices.
Note that the futhest pairs of vertices is always between 2 leaves. We can prove this by contradiction, if the furthest pair is not between 2 leaves, for example between a leaf and a non-leaf node, then the distance between the two vertices is the sum of the distance between the leaf and the shared node and the distance between the shared node and the non-leaf node, which is less than the distance between the child leaves of the non-leaf node and the leaf. Therefore, the furthest pair of vertices is always between 2 leaves.
Pairs (g, d), (g, e), (g, n) ,(g, o),(g,p) which all have 7 edges between them. any other pairs of vertices have less than 7 edges between them.

(d)

Step 1: Initilaize a diameter variable to 0
Step 2:

- Define a recursive function to find the height of the node
- If the node is null, return 0
- Recursively call the function with the left child node and right child node to get the height
- Update the diameter to the maximum of the sum of the height of the left child node and the height of the right child node and the current diameter and add 1

Step 3: Call the function with the root node and get the diameter
psudo code:

```
diameter = 0
def height(node):
  if node is None:
    return 0
  left = height(node.left)
  right = height(node.right)
  diameter = max(diameter, left + right)
  return max(left, right) + 1

height(root)
```

As we proves in the previous question, the furthest pair of vertices is always between 2 leaves. Therefore, the diameter of the tree is the distance between the two leaves. Note that each two leave always have a shared node, therefore the distance is the sum of the heights from the shared node to corresponding leaves. By finding the maximum of the sum of the heights of the left and right child nodes, we can find the diameter of the tree.
The recursive approach ensures that maximum distance on every subtree is calculated. And since the each node is visited once, the time complexity is $O(n)$