1 Question 2-4

Given $f(n) = \sum_{n}^{i=1} i^k$ and $g(n) = n^{k+1}$, determine the relationship between $f(n)$ and $g(n)$.

2 Answer

We will first assume that $f(n) = O(g(n))$. This means that there exists a constant $c > 0$ and $n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

We first need to find an upper bound of the summation in $f(n)$. The upper bound is given by:n^{k+1}, this is because the largest term in the sum is n^k and there are n terms in the sum. Therefore, the sum is bounded by $n \cdot n^k = n^{k+1}$.

Then we will consider the inequality:

$$\sum_{i=1}^{n} i^k \leq c \cdot n^{k+1}$$

we can pick $c = 1$ and $n_0 = 1$ to satisfy the inequality.

Next, we will assume that $f(n) = \Omega(g(n))$. This means that there exists a constant $c > 0$ and $n_0 > 0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

We will consider the lower bound of the summation in $f(n)$. The lower bound is computed by integrating the function i^k from 1 to n. This is smaller if we divided the interval $[1, n]$ into n equal parts and calculate the area under the curve. The area under the curve is given by $\int_i^{i+1} x^k dx$ which is less than i^k. Therefore, the lower bound of the sum is given by

$$\int_1^n i^k = \frac{n^{k+1}}{k+1} - \frac{1}{k+1}$$

Then we will consider the inequality:

$$\sum_{i=1}^{n} i^k \geq \frac{n^{k+1}}{k+1} - \frac{1}{k+1} \geq c \cdot n^{k+1}$$
$$\frac{1}{k+1} - \frac{1}{n^{k+1}(k+1)} \geq c$$

Such a constant c exists, as shown by the limit of the right hand side of the inequality is $\frac{1}{k+1}$. Then, we can pick $c = \frac{1}{k+1} - \frac{1}{2^{k+1}(k+1)}$ and $n_0 = 2$ to satisfy the inequality.

In conclusion, since $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ we can conclude that $f(n) = \Theta(g(n))$ by choosing $c_1 = 1, c_2 = \frac{1}{k+1} - \frac{1}{2^{k+1}(k+1)}, n_0 = 2$.