1 Question 1.3

$$6^{22345} \mod 7$$

2 Solution

Observe that there exists a pattern in the powers of 6 modulo 7:

$$\begin{aligned}
6^1 &\mod 7 = 6 \\
6^2 &\mod 7 = 1 \\
6^3 &\mod 7 = 6 \\
6^4 &\mod 7 = 1 \\
6^5 &\mod 7 = 6 \\
6^6 &\mod 7 = 1
\end{aligned}$$

Therefore, we suspect $6^{2k-1} \mod 7 = 6^{2\times 11173-1} \mod 7 = 6, \forall k \in \mathbb{Z}^+$. We can prove this with induction.

2.1 Proof by induction

Base case holds:

$$6^1 \mod 7 = 6^{6\times 3724+1} \mod 7 = 6$$

Inductive step: assume when $n = k$, the statement 6^{2k-1}holds, then consider $n = k + 1$

$$\begin{aligned}
6^{2(k+1)-1} \mod 7 &= 6^{2k+1} \mod 7 \\
&= 6^{2k-1} \times 6^2 \mod 7 \\
&= ((6^{2k-1} \mod 7) \times (6^2 \mod 7)) \mod 7 \\
&= 6 \times 1 \mod 7 \\
&= 6
\end{aligned}$$

So the statement holds for $n = k + 1 \forall k \in \mathbb{Z}^+$, and by the principle of mathematical induction, $6^{22345} \mod 7 = 6$.