1 Question 2-3

Given $f(n) = log(n)^{log(n)}$ and $g(n) = 2^{(log(n))^2}$, determine the relationship between $f(n)$ and $g(n)$.

2 Answer

We will first assume that $f(n) = O(g(n))$. This means that there exists a constant $c > 0$ and $n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

$$\begin{aligned}
f(n) &\leq c \cdot g(n) \\
log(n)^{log(n)} &\leq c \cdot 2^{(log(n))^2} \\
log(n) \cdot log(log(n)) &\leq c \cdot 2^{(log(n))^2} \\
log(n) \cdot log(log(n)) &\leq log(c) \cdot (log(n))^2 \\
\frac{log(log(n))}{log(n)} &\leq log(c)
\end{aligned}$$

Then we will can find the limit of the right hand side of the inequality to be 0, as n approaches infinity.

$$\begin{aligned}
\lim_{n\to\infty} \frac{log(log(n))}{log(n)} &= \lim_{n\to\infty} \frac{\frac{1}{n \cdot log(n)}}{\frac{1}{n}} && \text{(L'Hopital)} \\
&= \lim_{n\to\infty} \frac{1}{log(n)} \\
&= 0
\end{aligned}$$

Therefore, such a constant c exists, and $f(n) = O(g(n))$. we can pick $c = 2^{\frac{log(3 \cdot log(10))}{3 \cdot log(10)}}$ and $n_0 = 10^3$ to satisfy the inequality.

Now we will assume that $f(n) = \Omega(g(n))$. This means that there exists a constant $c > 0$ and $n_0 > 0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

$$\begin{aligned}
f(n) &\geq c \cdot g(n) \\
\frac{log(log(n))}{log(n)} &\geq log(c)
\end{aligned}$$

Then we will can find the limit of the right hand side of the inequality to be infinity, as n approaches infinity. Therefore, such a constant c does not exist, and $f(n) \neq \Omega(g(n))$.

Since $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$ we can conclude that $f(n) \neq \Theta(g(n))$.