1 Question 1.1

$$3^{1500} \mod 11$$

2 Solution

$$\begin{aligned}
3 &\mod 11 = 3 \\
3^2 &\mod 11 = 9 \\
3^3 &\mod 11 = 5 \\
3^4 &\mod 11 = 4 \\
3^5 &\mod 11 = 1 \\
3^6 &\mod 11 = 3 \\
3^7 &\mod 11 = 9 \\
3^8 &\mod 11 = 5 \\
3^9 &\mod 11 = 4 \\
3^{10} &\mod 11 = 1
\end{aligned}$$

Therefore, we suspect $3^{1500} \mod 11 = 3^{5\times 500} \mod 11 = 1$. we can prove this with induction.

2.1 Proof by induction

To prove

$$3^{5n} \mod 11 = 1$$

Base case holds:

$$3^{1500} \mod 11 = 3^{10\times 150} \mod 11 = 1$$

Inductive step: assume when n = k, the statement holds, then consider $n = k+1$

$$\begin{aligned}
3^{5(k+1)} \mod 11 &= 3^{5k+5} \mod 11 \\
&= 3^{5k} \times 3^5 \mod 11 \\
&= (3^{5k} \mod 11 \times 3^5 \mod 11) \mod 11 \\
&= 1 \times 1 \mod 11 \\
&= 1
\end{aligned}$$

So the statement holds for $n = k + 1 \forall k \in \mathbb{Z}^+$, and by the principle of mathematical induction, $3^{5n} \mod 11 = 1$.

2.2 Alternative Solution

We can see that

$$\begin{aligned}
3^{1500} \mod 11 &= 3^{4+8+16+64+128+256+1024} \mod 11 \\
&= 3^4 \times 3^8 \times 3^{16} \times 3^{64} \times 3^{128} \times 3^{256} \times 3^{1024} \mod 11 \\
&= (3^4 \mod 11 \times 3^8 \mod 11 \times \cdots \times 3^{1024} \mod 11) \mod 11 \\
&= 1
\end{aligned}$$