1 Question 2-1

Given $f(n) = n^{1.01}$ and $g(n) = n\log(n)^2$, determine the relationship between $f(n)$ and $g(n)$.

2 Answer

We will first assume that $f(n) = O(g(n))$. This means that there exists a constant $c > 0$ and $n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

$$\begin{aligned} f(n) &\leq c \cdot g(n) \\ n^{1.01} &\leq c \cdot n\log(n)^2 \\ n^{0.01} &\leq c \cdot \log(n)^2 \\ \frac{n^{0.01}}{log(n)^2} &\leq c \end{aligned}$$

Then we will can find the limit of the right hand side of the inequality to be infinity, as n approaches infinity.

$$\begin{aligned} \lim_{n\to\infty} \frac{n^{0.01}}{log(n)^2} &= \lim_{n\to\infty} \frac{0.01n^{0.01}}{2\log(n)} && \text{(L'Hopital)} \\ &= \lim_{n\to\infty} \frac{0.0001n^{0.01}}{2} \\ &= \infty \end{aligned}$$

However, such a constant c does not exist, as the limit of the right hand side of the inequality is infinity. Therefore, it contradicts the assumption that $f(n) = O(g(n))$.

Next we will assume that $f(n) = \Omega(g(n))$. This means that there exists a constant $c > 0$ and $n_0 > 0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

$$\begin{aligned} f(n) &\geq c \cdot g(n) \\ n^{1.01} &\geq c \cdot n\log(n)^2 \\ n^{0.01} &\geq c \cdot \log(n)^2 \\ \frac{n^{0.01}}{log(n)^2} &\geq c \end{aligned}$$

Such a constant c exists, as shown by the limit of the right hand side of the inequality.

then we can pick $c = \frac{10^{10}}{1000^2 log(10)^2}$ and $n_0 = 10^{1000}$ to satisfy the inequality.

Therefore, $f(n) = \Omega(g(n))$. Since $f(n) = \Omega(g(n))$ and $f(n) \neq O(g(n))$ we can conclude that $f(n) \neq \Theta(g(n))$.