

Intervals and Predictive Distributions

Professor Rodrigo Targino

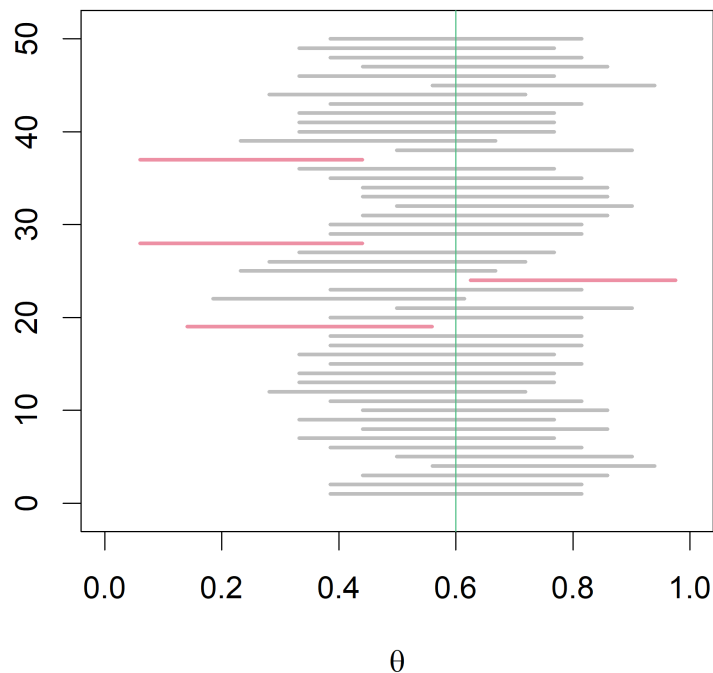
Announcements

- Reading: Chapters 8.1 (intervals) and 8.3 (posterior prediction) Bayes Rules
- Homework 2 due May 5, at 11:59pm

Reminder: Frequentist confidence interval

- Frequentist interval: $Pr(l(Y) < \theta < u(Y) \mid \theta) = 0.95$
 - Probability that the interval will cover the true value *before* the data are observed.
 - Interval is random since Y is random

Reminder: Frequentist confidence interval



We expect $0.05 \times 50 = 2.5$ will *not* cover the true parameter 0.6

Posterior Credible Intervals

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 - Probability that the interval will cover the true value *before* the data are observed.
 - Interval is random since Y is random
- **Bayesian Interval:** $Pr(l(y) < \theta < u(y) \mid Y = y) = 0.95$
 - Information about the the true value of θ *after* observeing $Y = y$.
 - θ is random (because we include a prior), y is observed so interval is non-random.

Posterior Credible Intervals (Quantile-based)

- The easiest way to obtain a confidence interval is to use the quantiles of the posterior distribution.

If we want $100 \times (1 - \alpha)\%$ interval, we find numbers $\theta_{\alpha/2}$ and $\theta_{1-\alpha/2}$ such that:

$$1. p(\theta < \theta_{\alpha/2} \mid Y = y) = \alpha/2$$

$$2. p(\theta > \theta_{1-\alpha/2} \mid Y = y) = \alpha/2$$

$$p(\theta \in [\theta_{\alpha/2}, \theta_{1-\alpha/2}] \mid Y = y) = 1 - \alpha$$

- Use quantile functions in R, e.g. `qbeta`, `qpois`, `qnorm` etc.

Example: interval for shooting skill in basketball

- The posterior distribution for Covington's shooting percentage is a

$$\text{Beta}(49 + 478, 50 + 873) = \text{Beta}(528, 924)$$

- For a 95% *credible* interval, $\alpha = 0.05$
 - Lower endpoint: `qbeta(0.025, 528, 924)`
 - Upper endpoint: `qbeta(0.975, 528, 924)`
 - $[\theta_{\alpha/2}, \theta_{1-\alpha/2}] = [0.34, 0.39]$

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 - $[\theta_{\alpha/2}, \theta_{1-\alpha/2}] = [0.34, 0.39]$
- Compared to frequentist *confidence* interval without prior information: $[0.39, 0.59]$
- End-of-season percentage was 0.37
- Credible intervals and confidence intervals have different meanings!

Highest Posterior Density (HPD) region

Definition: (HPD region) A $100 \times (1 - \alpha)$ HPD region consists of a subset of the parameter space, $R(y) \in \Theta$ such that

1. $\Pr(\theta \in R(y) | Y = y) = 1 - \alpha$

- The probability that θ is in the HPD region is $1 - \alpha$

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- The probability that θ is in the HPD region is $1 - \alpha$

2. If $\theta_a \in R(y)$, and $\theta_b \notin R(y)$, then $p(\theta_a | Y = y) > p(\theta_b | Y = y)$

- All points in an HPD region have a higher posterior density than points outside the region.

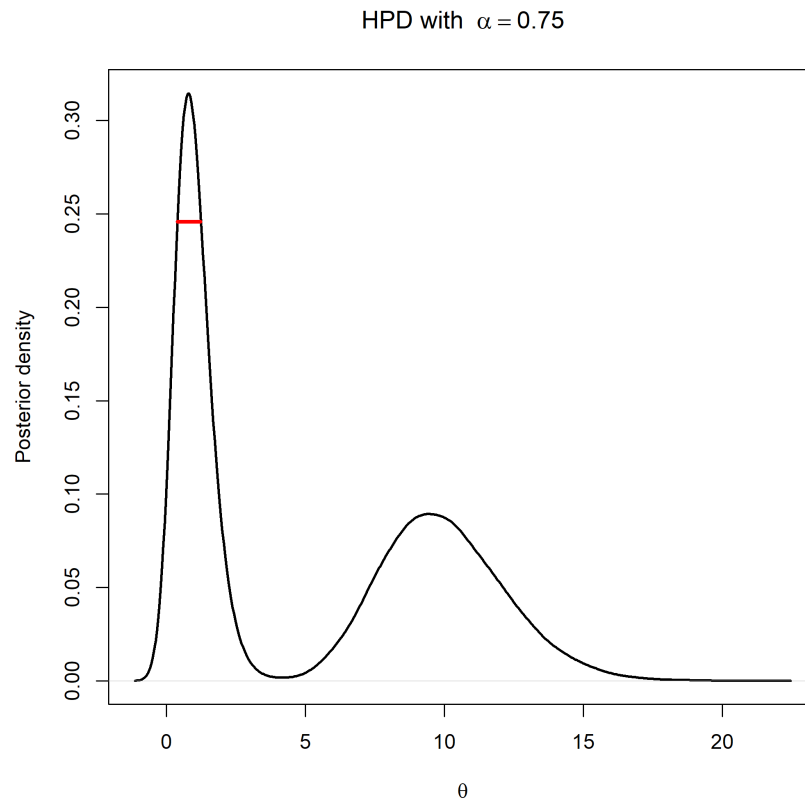
The HPD region can be discontinuous (hence "region")

Highest Posterior Density (HPD) region

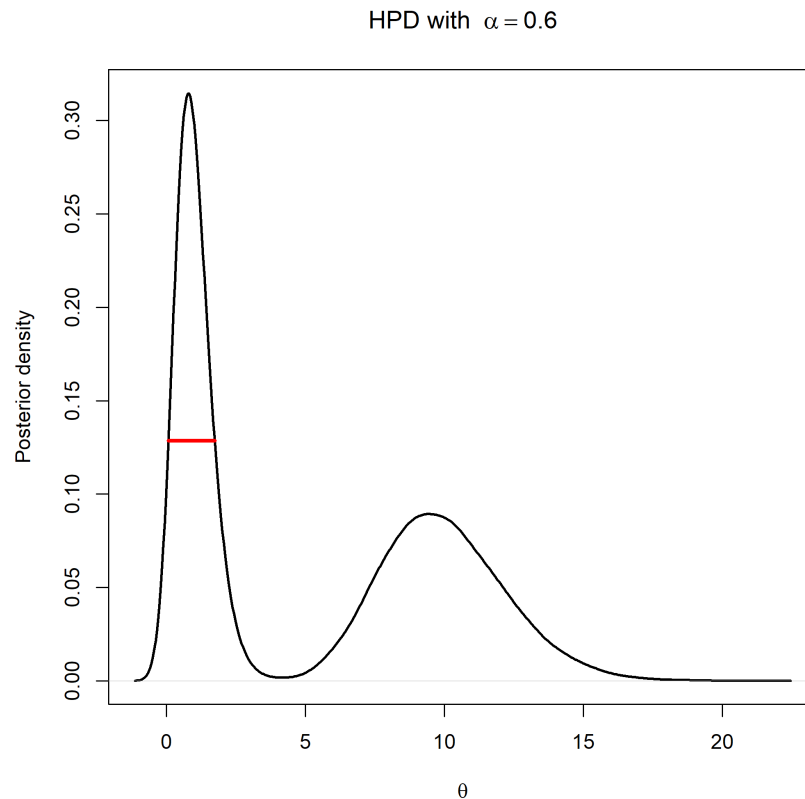
1. $p(\theta \in s(y) \mid Y = y) = 1 - \alpha$
2. If $\theta_a \in s(y)$, and $\theta_b \notin s(y)$, then $p(\theta_a \mid Y = y) > p(\theta_b \mid Y = y)$.
 - All points in an HPD region have a higher posterior density than points out- side the region.

The HPD region is the *smallest* region with probability $(1 - \alpha)\%$

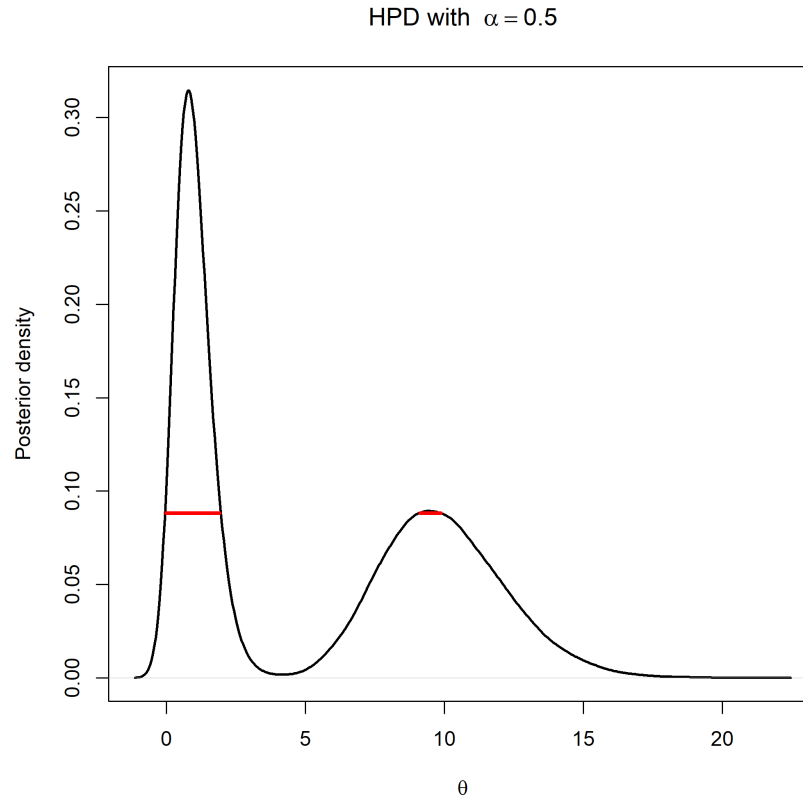
Highest Posterior Density (HPD) region



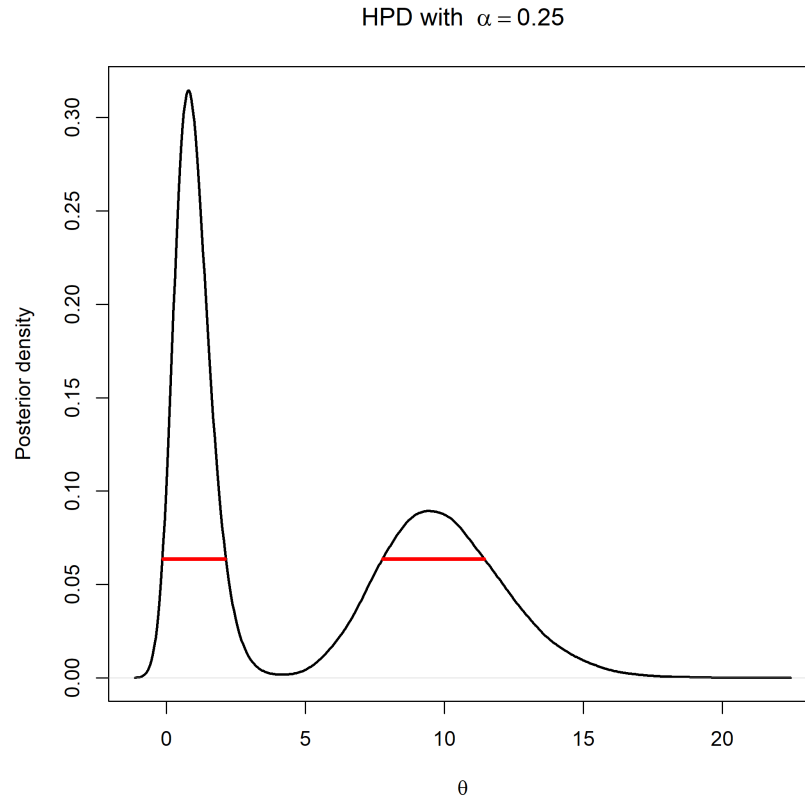
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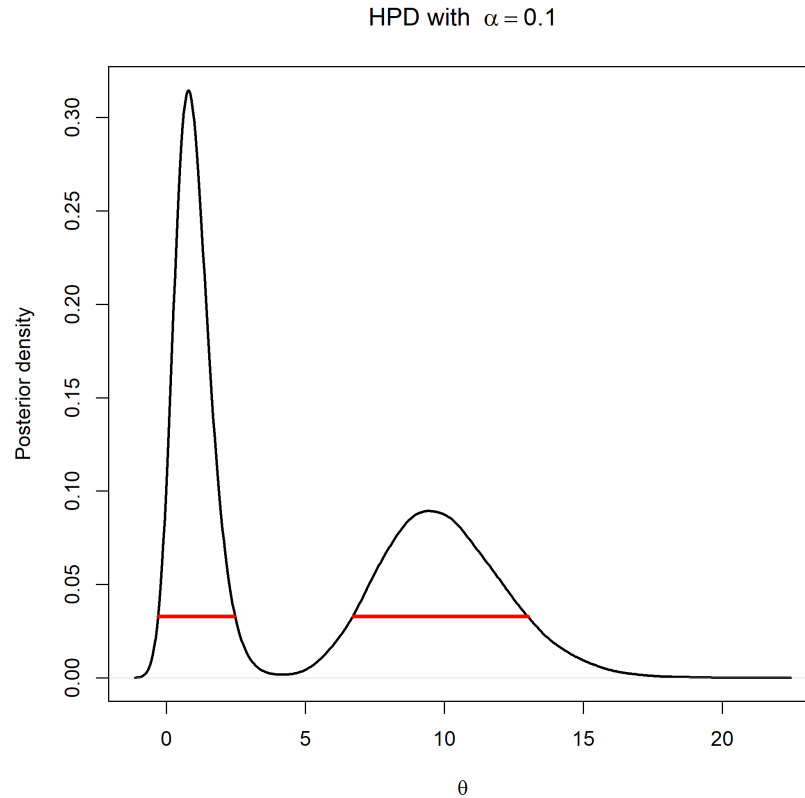
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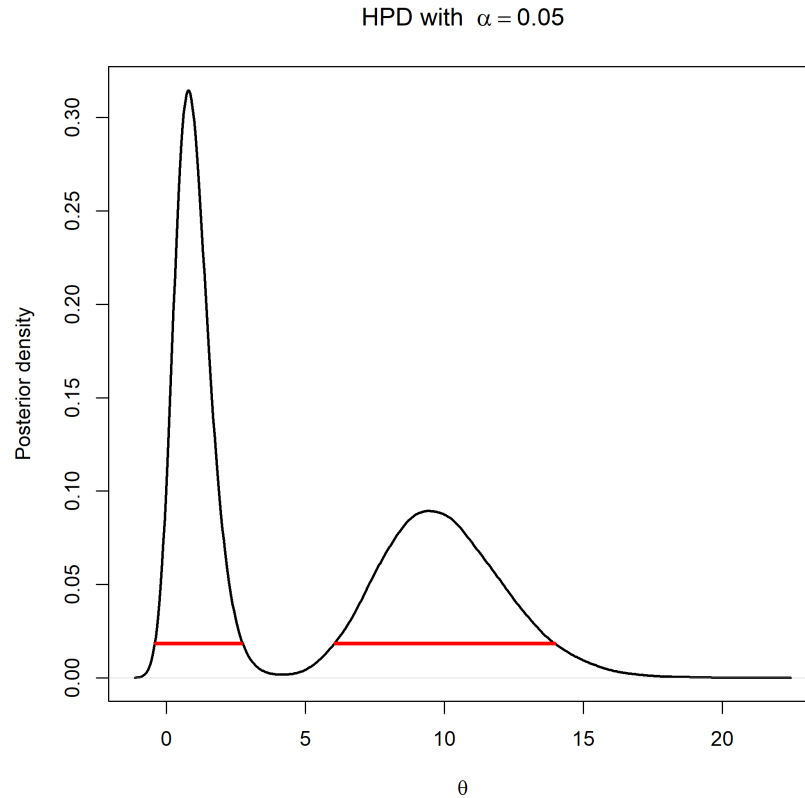
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Calibration: Frequentist Behavior of Bayesian Intervals

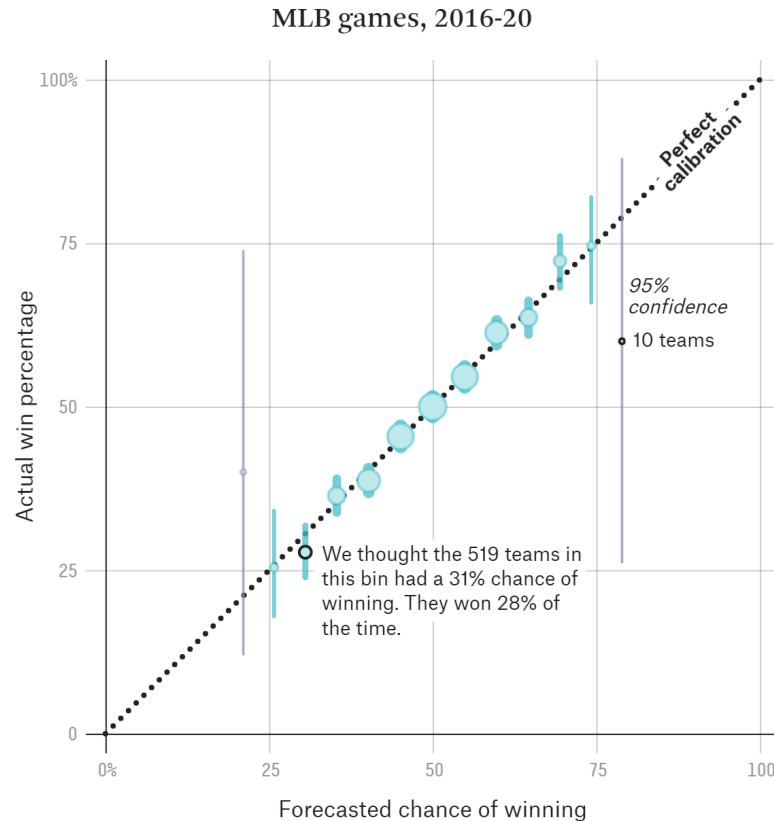
- A credible interval is calibrated if it has the right frequentist coverage
- Bayesian credible intervals usually won't have correct coverage
- If our prior was well-calibrated and the sampling model was correct, we'd have well-calibrated credible intervals
- Specifying *nearly* calibrated prior distributions is hard!

Calibration of MLB predictions

Indeed, single predictions are hard to judge on their own. So let's group every MLB game prediction (not just those from September 2018) into bins — for example, we'll throw every prediction that gave a team between a 37.5 percent and 42.5 percent chance of winning into the same “40 percent” group — and then plot the averages of each bin's forecasted chances of winning against their actual win percentage. If our forecast is **well-calibrated** — that is, if events happened roughly as often as we predicted over the long run — then all the bins on the calibration plot will be close to the 45-degree line; if our forecast was **poorly calibrated**, the bins will be further away.

source: <https://projects.fivethirtyeight.com/checking-our-work/>

Calibration of MLB predictions



source: <https://projects.fivethirtyeight.com/checking-our-work/>

Calibration of political predictions

Calibration for FiveThirtyEight “polls-plus” forecast

WIN PROBABILITY RANGE	NO. FORECASTS	EXPECTED NO. WINNERS	ACTUAL NO. WINNERS
95-100%	27	26 . 7	26
75-94%	15	13 . 1	14
50-74%	14	8 . 7	11
25-49%	13	4 . 8	3
5-24%	27	3 . 1	1
0-4%	88	0 . 8	1

source: <https://fivethirtyeight.com/features/when-we-say-70-percent-it-really-means-70-percent/>

The age guessing game*



*Bayesian edition

Interval Trivia

- I'm going to ask you ten questions about random facts
- For each, write down a 50% credible interval for *your* belief about the answer
- Goal:
 - Be well calibrated. 50% of your intervals should contain the true answer.

Percentage in California



CALIFORNIA REPUBLIC



What fraction of the US population is living in California?

Olympic Swimming Pool



How many gallons of water are there in an olympic swimming pool?

Household Income



What is the median household income in the US (dollars)?

Grooves on a quarter



How many grooves are there on the edge of quarter?

Gold in Fort Knox



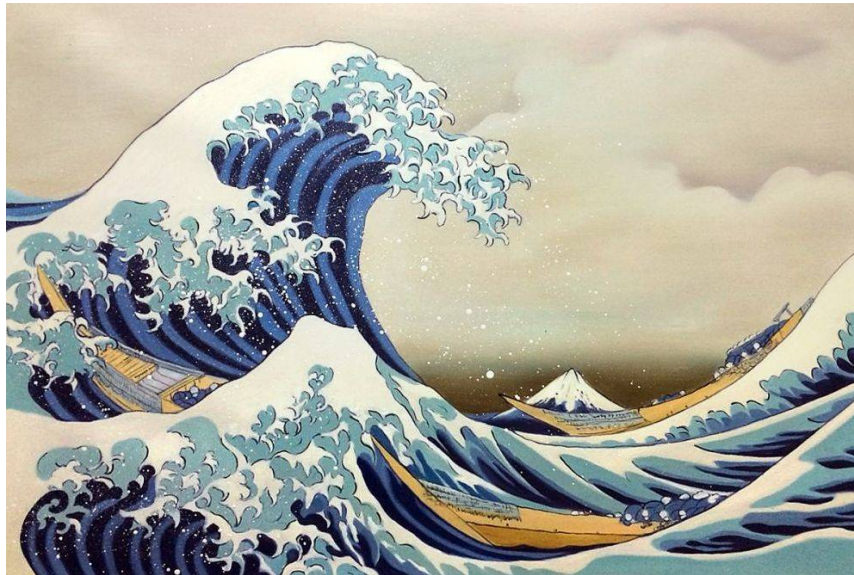
How many pounds of gold are there Fort Knox?

Population of Australia



What is the population of Australia (in millions)?

Tallest tsunami wave ever recorded



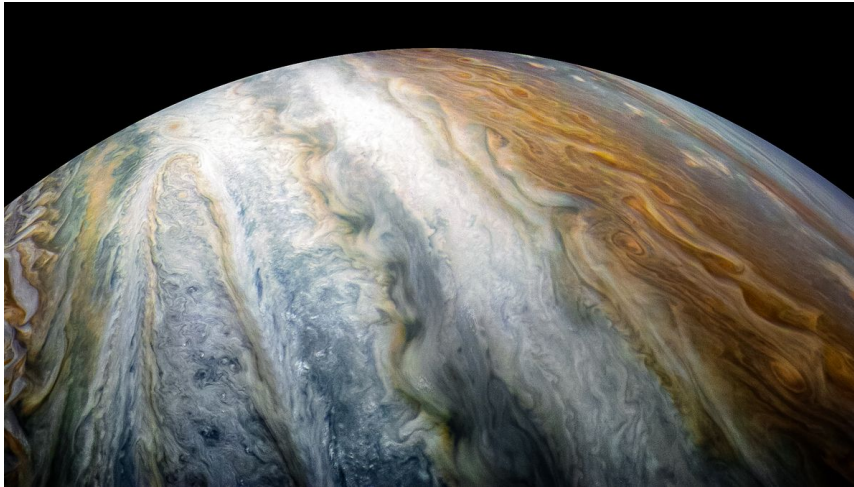
How tall was the tallest tsunami wave ever recorded (in feet)?

Disney world



In what year did Disneyland Park open to the public?

Jupiter



How many times larger in volume is Jupiter than Earth?

Netflix



In what year was Netflix founded?

All questions

N	Question	lower	upper
Q1	What fraction of the US population is living in California?	_____	_____
Q2	How many gallons of water are there in an olympic swimming pool?	_____	_____
Q3	What is the median household income in the US (dollars)?	_____	_____
Q4	How many grooves are there on the edge of quarter?	_____	_____
Q5	How many pounds of gold are there in Fort Knox?	_____	_____
Q6	What is the population of Australia (in millions)?	_____	_____
Q7	How tall was the tallest tsunami wave ever recorded (in feet)?	_____	_____
Q8	In what year did Disneyland open to the public?	_____	_____
Q9	How many times larger in volume, is Jupiter than Earth?	_____	_____
Q10	In what year was Netflix founded?	_____	_____

Answers

N	Question	Answers
Q1	What fraction of the US population is living in California?	12%
Q2	How many gallons of water are there in an olympic swimming pool?	660,000
Q3	What is the median household income in the US (dollars)?	\$70,784
Q4	How many grooves are there on the edge of quarter?	119
Q5	How many pounds of gold are there in Fort Knox?	9,206,250
Q6	What is the population of Australia (in millions)?	25.89 mil
Q7	How tall was the tallest tsunami wave ever recorded (in feet)?	1700 feet
Q8	In what year did Disneyland open to the public?	1955
Q9	How many times larger in volume, is Jupiter than Earth?	1,321.33
Q10	In what year was Netflix founded?	1997

References

N	References
Q1	https://www.census.gov/library/stories/state-by-state/california-population-change-between-census-decade.html
Q2	https://healingpicks.com/how-many-gallons-are-in-an-olympic-size-swimming-pool/
Q3	https://www.census.gov/library/visualizations/2022/comm/median-household-income.html
Q4	https://www.icpl.org/articles/why-does-quarter-have-119-ridges
Q5	https://www.usmint.gov/about/mint-tours-facilities/fort-knox
Q6	https://www.abs.gov.au/statistics/people/population
Q7	https://www.pbs.org/wgbh/nova/tsunami/once-nf.html
Q8	https://www.history.com/this-day-in-history/disneyland-opens
Q9	https://www.space.com/18392-how-big-is-jupiter.html
Q10	https://en.wikipedia.org/wiki/Netflix

Calibrated probability intervals

- Calibration is important but only part of the story!
- Want well calibrated but *small* intervals (big intervals tell us nothing)
- How you could have gotten a perfect score on the quiz:
 - For 5 of the answers select $[-1 \text{ Trillion}, +1 \text{ Trill}]$ (ensures it will cover)
 - For the other 5 answers select $[-0.01, + 0.01]$ (ensures it won't cover)

Calibrated probability intervals

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- Domain expertise helps us develop smaller prior distributions (calibration?)
 - Usually at the cost of calibration
 - My experience: people tend to be overconfident
 - Alternatives to domain expertise?

Subjective Bayesianism

- So far we have focused on defining priors using domain expertise
- "Subjective" Bayes
 - Essentially what we have discussed so far
 - Priors usually represent subjective judgements can't always be rigorously justified
- Alternative: "objective" Bayes

Objective Bayesianism

- Is there a way to define "objective" prior distributions?
 - Good default prior distributions for some problems?
 - "Non-informative" prior distributions?
- Also called "reference" or "default" priors

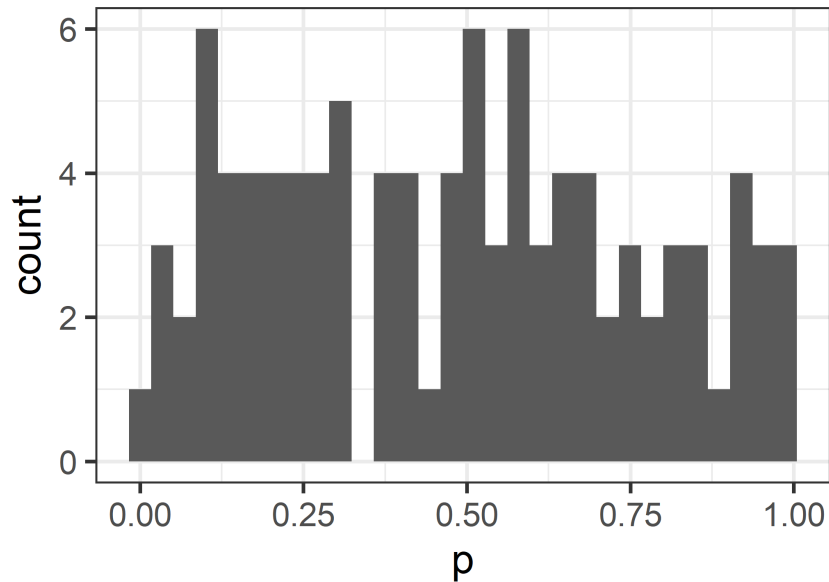
Objective Bayesianism

- Is there a way to define "objective" prior distributions?
 - Good default prior distributions for some problems?
 - "Non-informative" prior distributions?
- Also called "reference" or "default" priors
- Can we find prior distributions that lead to (approximately) correct frequentist calibration?
- Can we find prior distributions which minimize the amount of information contained in the distribution?
 - Principle of maximum entropy (MAXENT).

Difficulties with non-informative priors

Uniform distribution for p

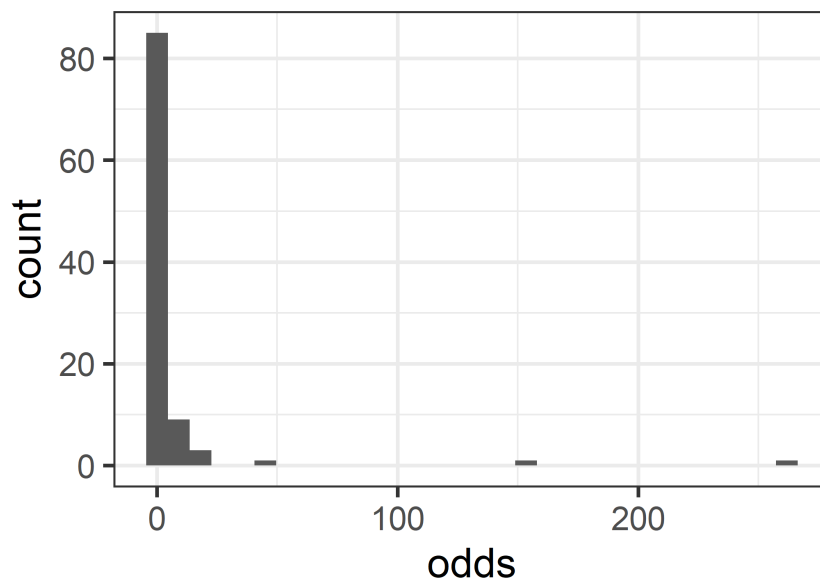
```
p <- runif(100)
tibble(p=p) %>% ggplot() +
  geom_histogram(aes(x=p), bins=30) +
  theme_bw(base_size=24)
```



Difficulties with non-informative priors

Implied distribution for odds = $p/(1-p)$

```
odds <- p/(1-p)
tibble(odds=odds) %>% ggplot() +
  geom_histogram(aes(x=odds)) +
  theme_bw(base_size=24)
```



Improper prior distributions

- For the Beta distribution we chose a uniform prior — e.g. $p(\theta) \propto \text{const}$. This was ok because
 - $\int_0^1 p(\theta) d\theta = \text{const} < \infty$
 - We say this prior distribution is *proper* because it is integrable
- For the Poisson distribution, try the same thing: $p(\lambda) \propto \text{const}$
 - $\int_0^\infty p(\lambda) d\lambda = \infty$
 - In this case we say $p(\lambda)$ is an *improper* prior

Improper prior distributions

- In the Poisson case, let us pretend the prior for λ is proper...

$$p(\lambda \mid y) \propto$$

Improper prior distributions

- Sometimes there is an absence of precise prior information
- The prior distribution does not have to be proper but the posterior does!
 - A proper distribution is one with an integrable density
 - If you use an improper prior distribution, you need to check that the posterior distribution is also proper

Posterior Predictive Distributions

Posterior predictive distribution

- An important feature of Bayesian inference is the existence of a predictive distribution for new observations.
 - Let \tilde{y} be a new (unseen) observation, and y_1, \dots, y_n the observed data.
 - The posterior predictive distribution is $p(\tilde{y} \mid y_1, \dots, y_n)$

Posterior predictive distribution

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 - Let \tilde{y} be a new (unseen) observation, and y_1, \dots, y_n the observed data.
 - The posterior predictive distribution is $p(\tilde{y} \mid y_1, \dots, y_n)$
- The predictive distribution does not depend on unknown parameters
- The predictive distribution only depends on observed data
- Asks: what is the probability distribution for new data given observations of old data?

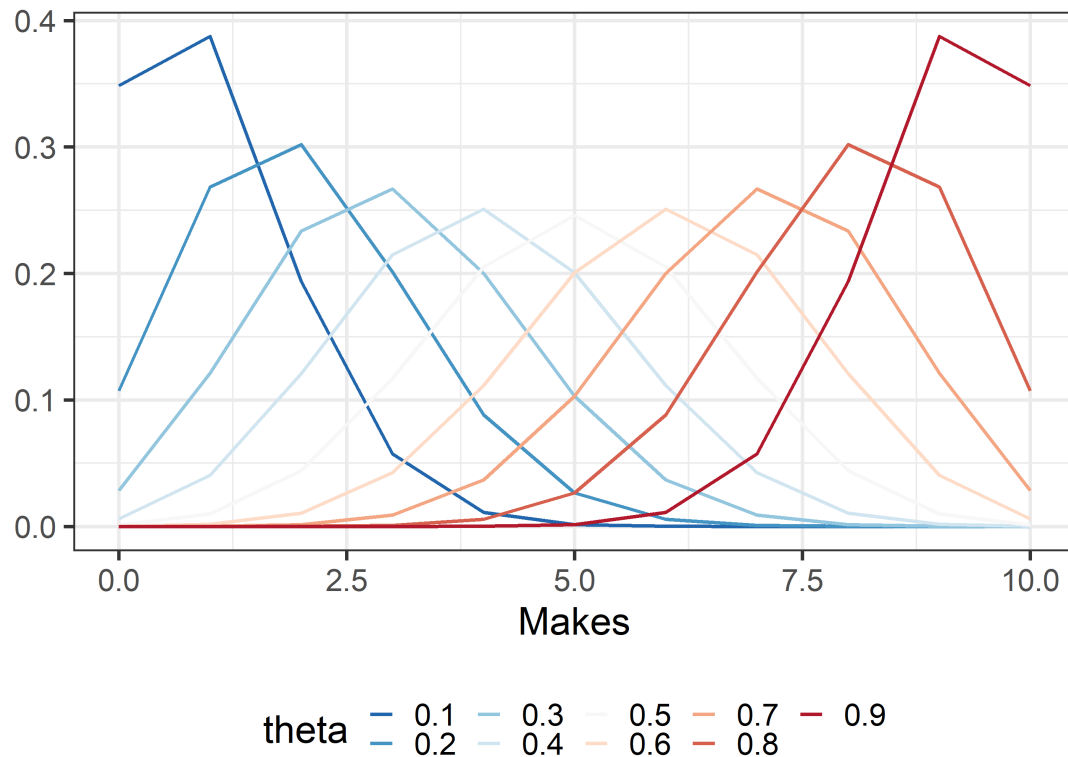
Another Basketball Example

- I take free throw shots and make 1 out of 2. How many do you think I will make if I take 10 more?
- If my true "skill" was 50%, then $\tilde{Y} \sim \text{Bin}(10, 0.50)$
- Is this the correct way to calculate the predictive distribution?

Posterior Prediction

If you know θ , then we know the distribution over future attempts:

$$\tilde{Y} \sim \text{Bin}(10, \theta)$$

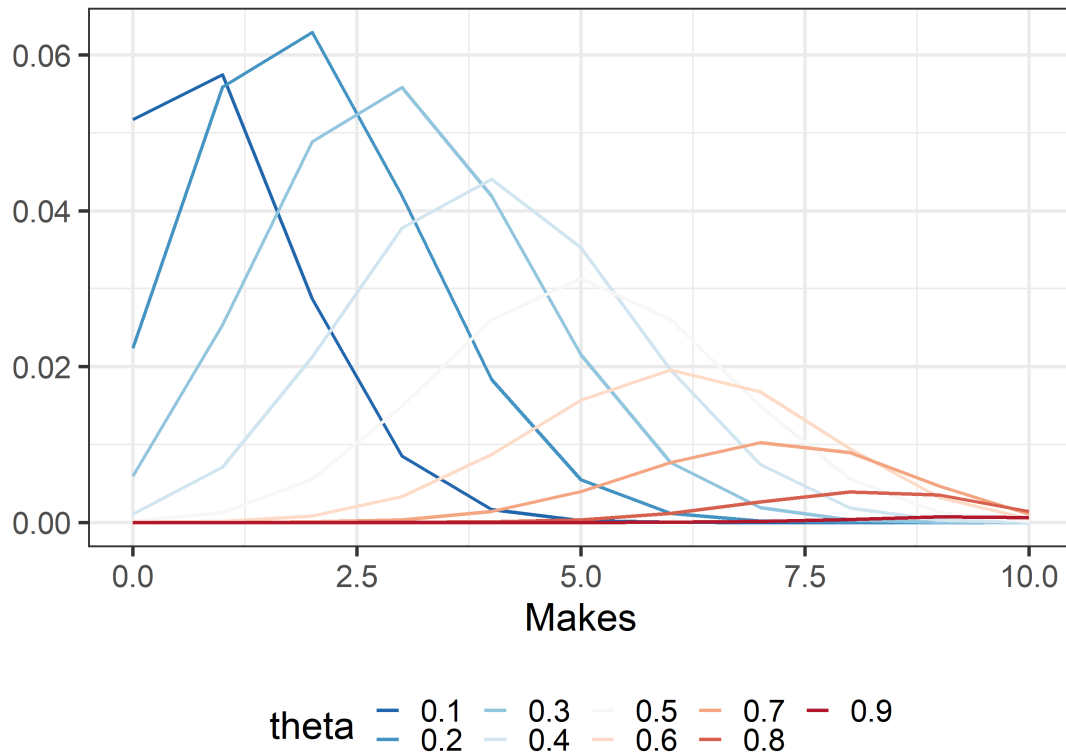


Posterior Prediction

- We already observed 1 make out of 2 tries.
- Assume a $\text{Beta}(1, 3)$ prior distribution
 - e.g. a priori you think I'm more likely to make 25% of my shots
- Then $p(\theta \mid Y = 1, n = 2)$ is a $\text{Beta}(2, 4)$
- Intuition: weight $\tilde{Y} \sim \text{Bin}(10, \theta)$ by $p(\theta \mid Y = 1, n = 2)$

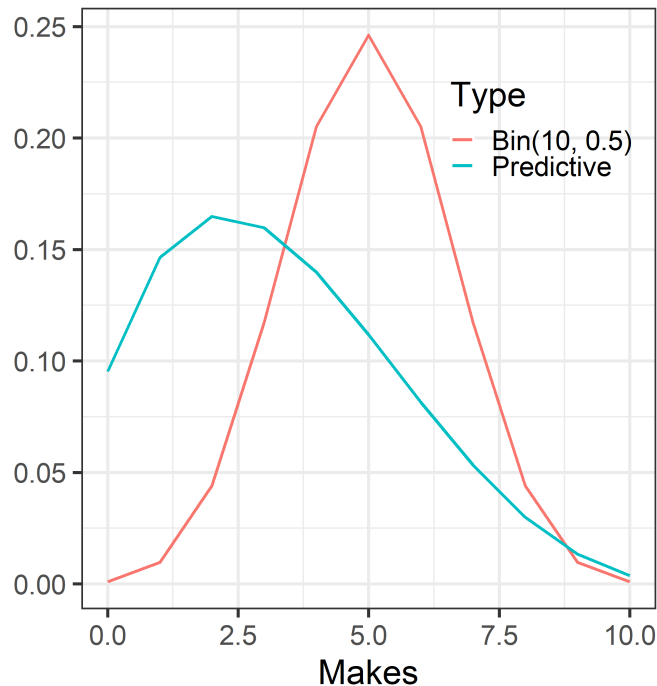
Posterior Prediction

If I take 10 more shots how many will I make?



Posterior Predictive Distribution

$$p(\theta) = \text{Beta}(1, 3), p(\theta | y) = \text{Beta}(2, 4)$$



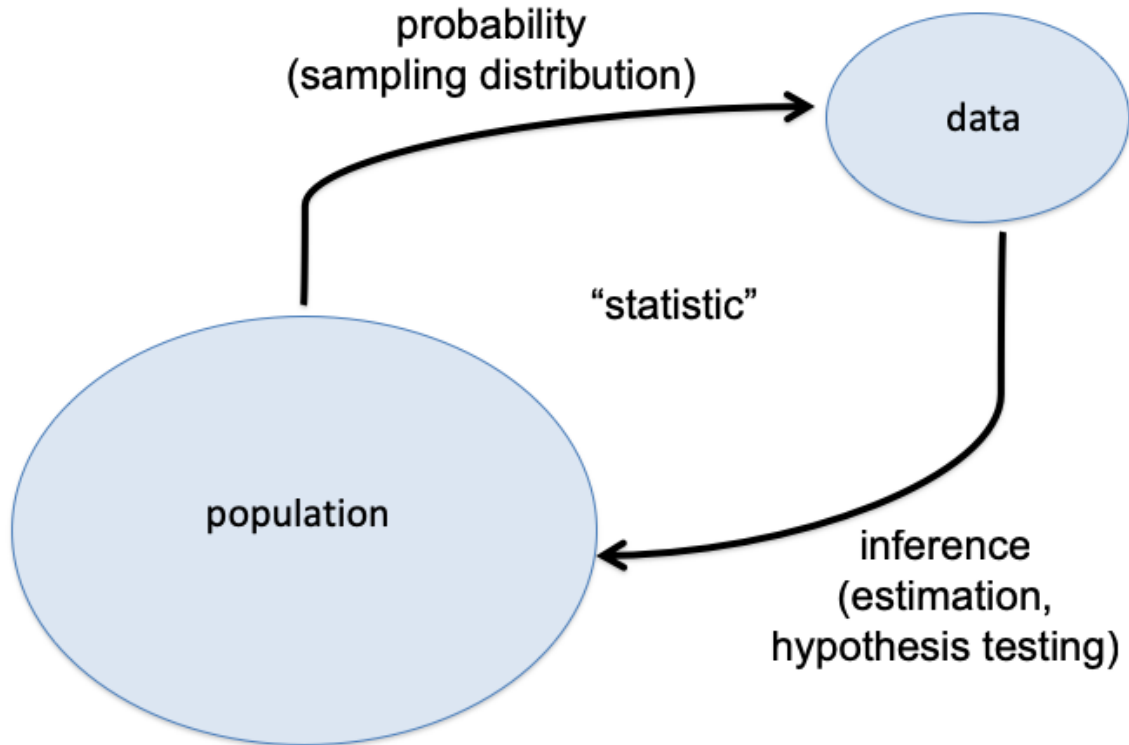
The predictive density, $p(\tilde{y} | y)$, answers the question "if I take 10 more shots how many will I make, given that I already made 1 of 2".

The posterior predictive distribution

$$\begin{aligned} p(\tilde{y} \mid y_1, \dots, y_n) &= \int p(\tilde{y}, \theta \mid y_1, \dots, y_n) d\theta \\ &= \int p(\tilde{y} \mid \theta) p(\theta \mid y_1, \dots, y_n) d\theta \end{aligned}$$

- The posterior predictive distribution describes our uncertainty about a new observation after seeing n observations
- It incorporates uncertainty due to the sampling in a model $p(\tilde{y} \mid \theta)$ *and* our posterior uncertainty about the data generating parameter, $p(\theta \mid y_1, \dots, y_n)$

Posterior Predictive Density



The prior predictive distribution

$$\begin{aligned} p(\tilde{y}) &= \int p(\tilde{y}, \theta) d\theta \\ &= \int p(\tilde{y} \mid \theta) p(\theta) d\theta \end{aligned}$$

- The prior predictive distribution describes our uncertainty about a new observation before seeing data

The prior predictive distribution

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- It incorporates uncertainty due to the sampling in a model $p(\tilde{y} \mid \theta)$ *and* our prior uncertainty about the data generating parameter, $p(\theta)$

Homework 1

- $\lambda \sim \text{Gamma}(\alpha, \beta)$
- $\tilde{Y} \sim \text{Pois}(\lambda)$
- $p(\tilde{y}) = \int p(\tilde{y} \mid \lambda)p(\lambda)d\lambda$ is a prior predictive distribution!
- "A Gamma-Poisson mixture is a Negative-Binomial Distribution"

Homework 1

$$\begin{aligned} p(\tilde{y}) &= \int p(\tilde{y} \mid \lambda) p(\lambda) d\lambda \\ &= \int \left(\frac{\lambda^{\tilde{y}}}{y!} e^{-\lambda} \right) \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{(\alpha-1)} e^{-\beta\lambda} \right) d\lambda \\ &= \frac{\beta^\alpha}{\Gamma(\alpha) y!} \int (\lambda^{(\alpha+y-1)} e^{-(\beta+1)\lambda}) d\lambda \end{aligned}$$

$\int (\lambda^{(\alpha+y-1)} e^{-(\beta+1)\lambda}) d\lambda$ looks like an unnormalized Gamma($\alpha + y, \beta + 1$)

Summary

- Bayesian credible intervals
 - Posterior probability that the value falls in the interval
 - Still strive for well-calibrated intervals (in the frequentist sense)
- Non-informative prior distributions
- Posterior predictive distributions
 - Estimated distribution for new data our uncertainty about the parameters