One Parameter Models

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Announcements

- Reading: Chapter 2 and 3, Bayes Rules

Bayesian Inference

- In frequentist inference, θ is treated as a fixed unknown constant
- In Bayesian inference, θ is treated as a random variable
- Need to specify a model for the joint distribution $p(y, \theta) = p(y \mid \theta)p(\theta)$

Setup

- The *sample space* $\mathcal{Y}$ is the set of all possible datasets. We observe one dataset y from which we hope to learn about the world.
    - Y is a random variable, y is a realization of that random variable
- The *parameter space* Θ is the set of all possible parameter values θ
    - θ encodes the population characteristics that we want to learn about!

Bayesian Inference in a Nutshell

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2. Our *sampling model* $p(y \mid \theta)$ describes our belief about what data we are likely to observe when the true population parameter is θ.

3. Once we actually observe data, y, we update our beliefs about θ by computing *the posterior distribution* $p(\theta \mid y)$. We do this with Bayes' rule!

Bayes' Rule

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- $P(A \mid B)$ is the conditional probability of A given B
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- $P(A)$ and $P(B)$ are called the marginal probability of A and B (unconditional)

Bayes' Rule for Bayesian Statistics

$$P(\theta \mid y) = \frac{P(y \mid \theta)P(\theta)}{P(y)}$$

- $P(\theta \mid y)$ is the posterior distribution
- $L(\theta) \propto P(y \mid \theta)$ is the likelihood
- $P(\theta)$ is the prior distribution
- $P(y) = \int_{\Theta} p(y \mid \tilde{\theta})p(\tilde{\theta})d\tilde{\theta}$ is the model evidence

Computing the Posterior Distribution

$$\begin{aligned} P(\theta \mid y) &= \frac{P(y \mid \theta)P(\theta)}{P(y)} \\ &\propto P(y \mid \theta)P(\theta) \\ &\propto L(\theta)P(\theta) \end{aligned}$$

- Start with a subjective belief (prior)
- Update it with evidence from data (likelihood)
- Summarize what you learn (posterior)

The posterior is proportional to the likelihood times the prior!

Bayesian vs Frequentist

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  - Estimators are random (due to sampling variability)
  - Asks: what would I expect to see if I repeated the experiment?"
- In Bayesian inference, unknown parameters are random variables.
  - Need to specify a prior distribution for θ (not easy)
  - Asks: "what do I *believe* are plausible values for the unknown parameters given the data?"
  - Who cares what might have happened, focus on what *did* happen by conditioning on observed data.

Example: estimating shooting skill in basketball

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- At that time, his three point field goal percentage, 0.49, was the best in the league and would have ranked in the top ten all time
- How can we estimate his true shooting skill?
    - Think of "true shooting skill" as the fraction he would make if he took infinitely many shots

Example: estimating shooting skill in basketball

- Assume every shot is independent (reasonable) and identically distributed (less reasonble?)
- Let $Y \sim \text{Bin}(n, \theta)$ where θ corresponds to his true skill
- Frequentist inference tells us that the maximum likelihood estimate is simply $\frac{y}{n} = 49/100 = 0.49$
- What would our estimates be if we use Bayesian inference?
    - What properties do we want for our prior distribution?

Cromwell's Rule

The use of priors placing a probability of 0 or 1 on events should be avoided except where those events are excluded by logical impossibility.

If a prior places probabilities of 0 or 1 on an event, then no amount of data can update that prior.

> I beseech [beg] you, in the bowels of Christ, think it possible that you may be mistaken.
>
> --- Oliver Cromwell

Cromwell's Rule

> Leave a little probability for the moon being made of green cheese; it can be as small as 1 in a million, but have it there since otherwise an army of astronauts returning with samples of the said cheese will leave you unmoved.
>
> --- Dennis Lindley (1991)

If $p(\theta = a) = 0$ for a value of a, then the posterior distribution is always zero, regardless of what the data says

$$p(\theta = a|y) \propto p(y|\theta = a)p(\theta = a) = 0$$

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- What would our estimates be if we use Bayesian inference?
  - If our prior reflects "complete ignorance" about basketball?
  - What if we want to incorporate prior domain knowledge?

The Binomial Model

- The uniform prior: $p(\theta) = \text{Unif}(0,1) = \mathbf{1}\{\theta \in [0,1]\}$
    - A "non-informative" prior
- Posterior: $p(\theta \mid y) \propto \underbrace{\theta^y (1-\theta)^{n-y}}_{\text{likelihood}} \times \underbrace{\mathbf{1}\{\theta \in [0,1]\}}_{\text{prior}}$
- The above posterior density is is a density over θ.

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- The above posterior density is is a density over θ.
- $p(\theta \mid y) \sim \mathrm{Beta}(y+1, n-y+1) = \frac{\Gamma(n)}{\Gamma(n-y)\Gamma(y)} \theta^y (1-\theta)^{n-y}$

Example: estimating shooting skill in basketball

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## Warning: Using `size` aesthetic for lines was deprecated in ggplot2 3.4.
## ℹ Please use `linewidth` instead.
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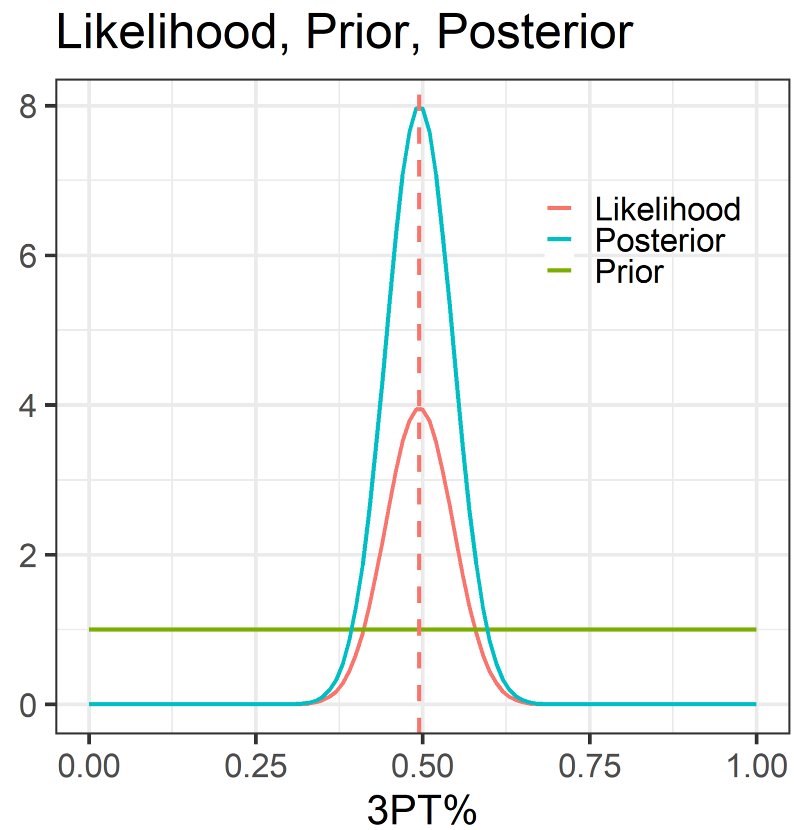

Posterior is proportional to the likelihood

Summarizing Posterior Results

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  - $E[\theta \mid y] = \int_\Theta \theta p(\theta \mid y) d\theta$ (the posterior mean)

  - $\arg\max p(\theta \mid y)$ (*maximum a posteriori* estimate)

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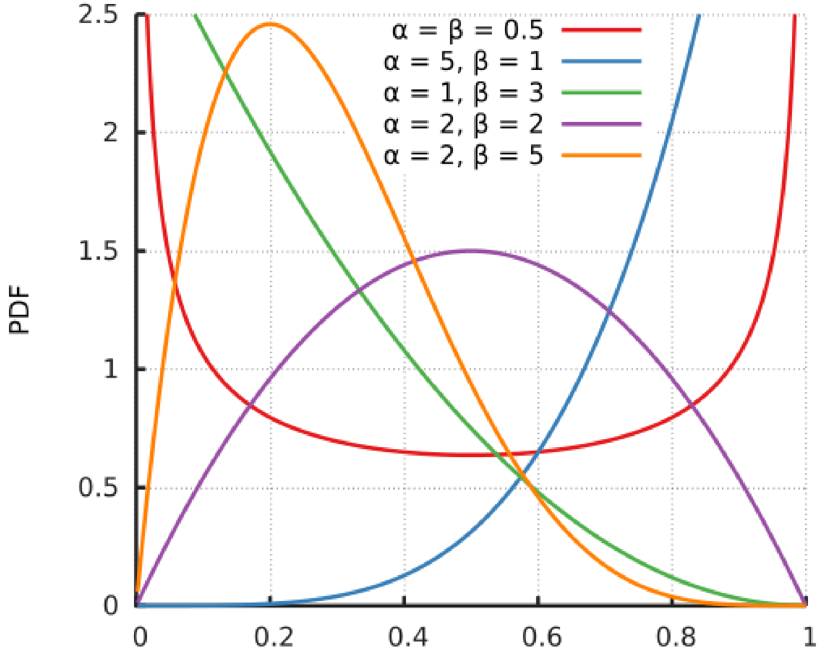
- Posterior credible intervals: for any region $R(y)$ of the parameter space compute the probability that θ is in that region: $p(\theta \in R(y))$

Beta Distributions



$$\mathrm{Beta}(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

Summarizing Posterior Results

- $\text{Beta}(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$
- The mean of a $\text{Beta}(\alpha, \beta)$ distribution r.v. $\frac{\alpha}{\alpha+\beta}$
- The mode of a $\text{Beta}(\alpha, \beta)$ distributed r.v. is $\frac{\alpha-1}{\alpha+\beta-2}$
- The variance of a $\text{Beta}(\alpha, \beta)$ r.v. is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
- In R: `dbeta`, `rbeta`, `pbeta`, `qbeta`

Informative prior distributions

- At that time, his three point field goal percentage, 0.49, was the best in the league and would have ranked in the top ten all time
- It seems very unlikely that this level of skill would continue for an entire season of play.
- A uniform prior distribution doesn't reflect our known beliefs. We need to choose a more *informative* prior distribution

Informative prior distributions

- When $p(\theta) \sim U(0,1)$ then the posterior was a Beta distribution
- Remember: the binomial likelihood is $L(\theta) \propto \theta^y (1-\theta)^{n-y}$
- Choose a prior with a similar looking form: $p(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$

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- Choose a prior with a similar looking form: $p(\theta) \propto \theta^{\alpha-1}(1 - \theta)^{\beta-1}$
- Then $p(\theta \mid y) \propto \theta^{y+\alpha-1}(1 - \theta)^{n-y+\beta-1}$ is a $\text{Beta}(y + \alpha, n - y + \beta)$
- For the binomial model, a beta prior distribution implies a beta posterior distribution!
- The family of Beta distributions is called a **conjugate prior** distribution for the binomial likelihood.

Conjugate Prior Distributions

Definition: A class of prior distributions, $\mathcal{P}$ for θ is called *conjugate* for a sampling model $p(Y|\theta)$ if $p(\theta) \in \mathcal{P} \implies p(\theta|y) \in \mathcal{P}$

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- Conjugate priors are very convenient because they make calculations easy
- The parameters for conjugate prior distribution have nice interpretations
- Note: convenience is not correctness. Best to choose prior distributions that reflect your true knowledge / experience, not convenience. We'll return to this later.

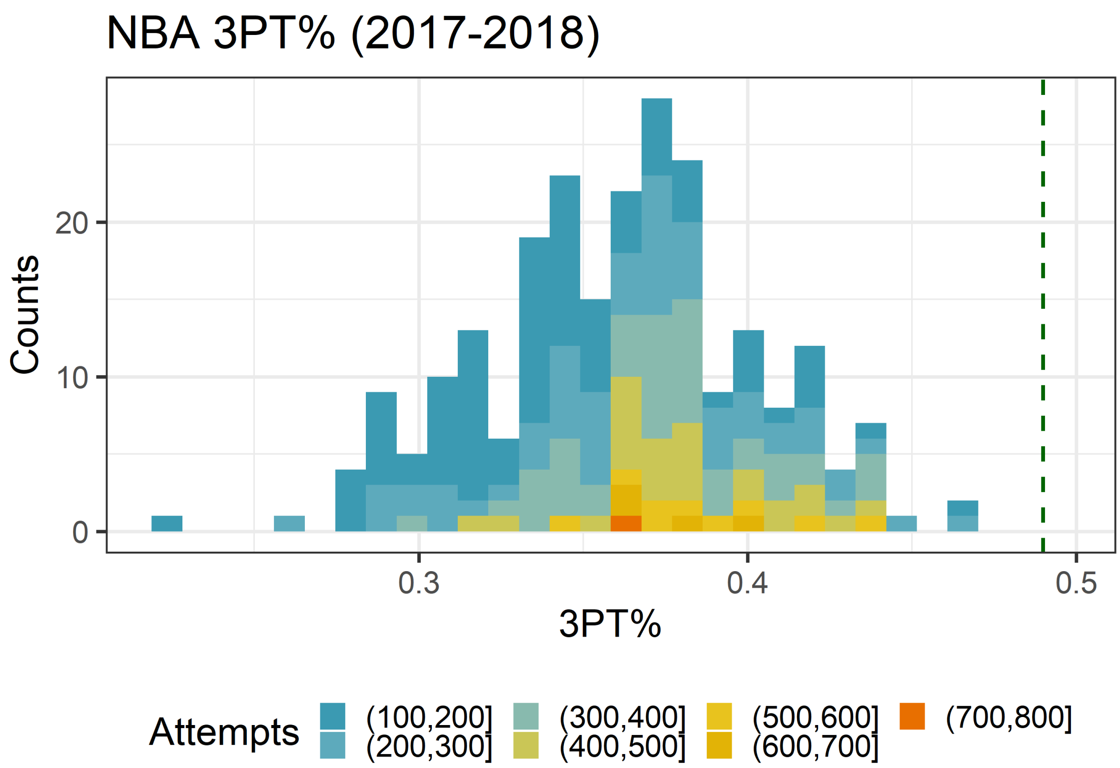
Pseudo-Counts Interpretation

- Observe y successes, $n - y$ failures
- If $p(\theta) \sim \text{Beta}(\alpha, \beta)$ then $p(\theta \mid y) = \text{Beta}(y + \alpha, n - y + \beta)$
- What is $E[\theta \mid y]$?

Example: estimating shooting skill in basketball

- On November 18, 2017, an NBA basketball player, Robert Covington, had made 49 out of 100 three point shot attempts.
- At that time, his three point field goal percentage, 0.49, was the best in the league and would have ranked in the 10 ten all time
- Prior knowledge tells us it is unlikely this will continue!
- How can we use Bayesian inference to better estimate his true skill?

Three point shooting in 2017-2018



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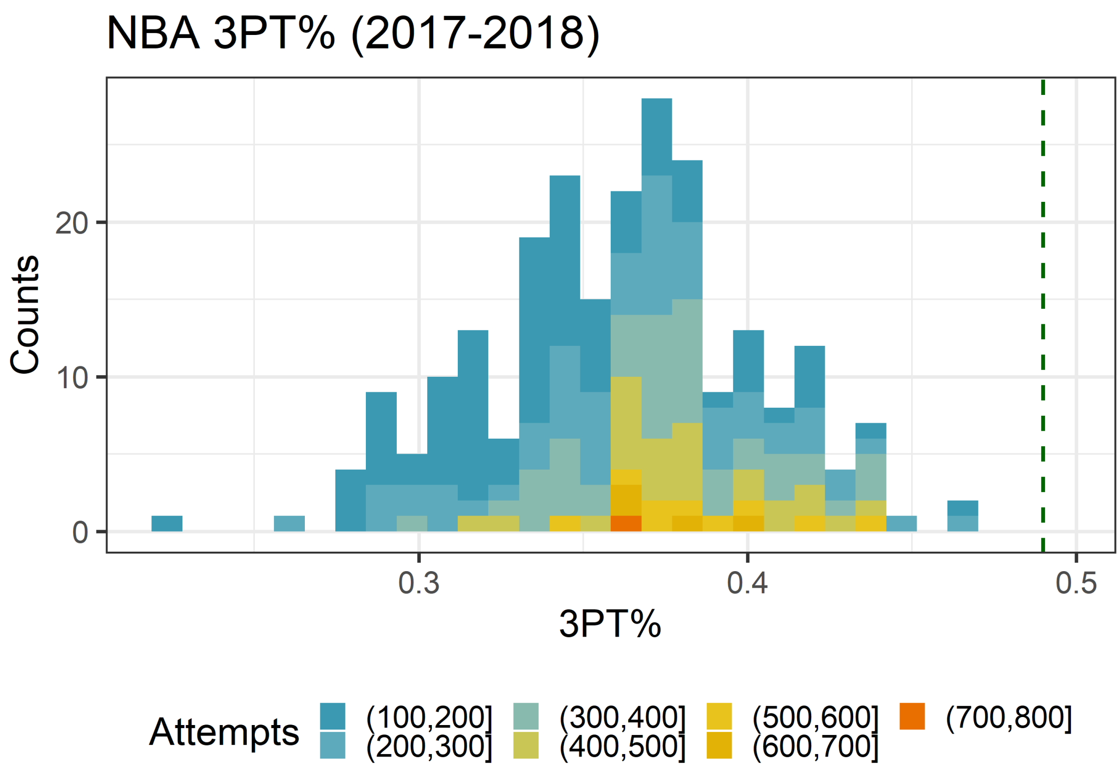


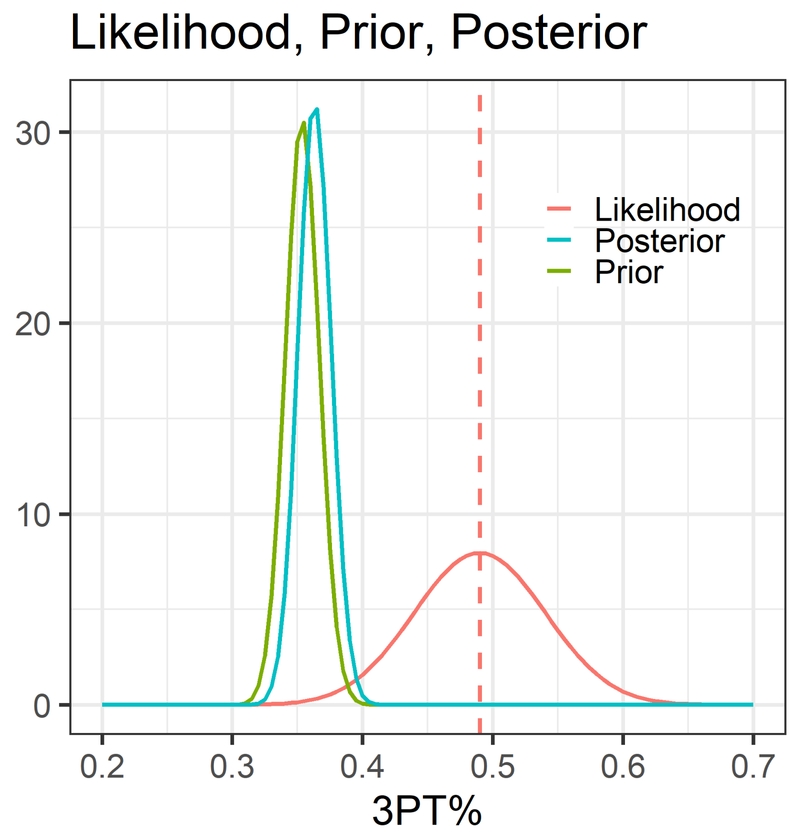
Regression Toward the Mean

What is a reasonable model?

- If we believe that his skill doesn't change much year to year, use past data to inform prior
- In his first 4 seasons combined Robert Covington made a total of 478 out of 1351 three point shots (0.35%, just below average).
- Choose a Beta(478, 873) prior (pseudo-count interpretation)

R. Covington 2017-2018 estimates

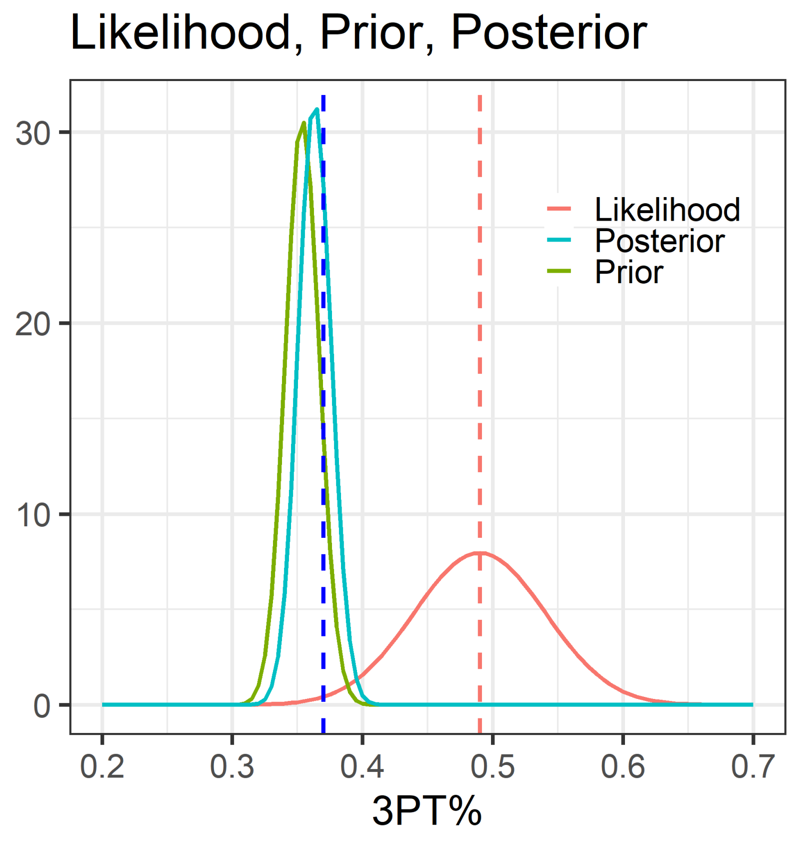
After 100 shots Robert Covington's 3PT% was 0.49



MLE = 0.49, posterior mean = 0.36

How did we do?

Robert Covington's end of season 3PT% was 0.37



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The Poisson Distribution

- A useful model for count data
- Events occur independently at some rate λ
- Mean = variance = λ.
- Example applications:
    - Epidemiology (disease incidence)
    - Astronomy (e.g. the number of meteorites entering the solar system each year)
    - The number of patients entering the emergency room
    - The number of times a neuron in the brain "fires"

Poisson model

Assume $Y_1, \ldots, Y_n$ are n i.i.d. observations from a $\text{Pois}(\lambda)$

Poisson model with exposure

- Often times we include an "exposure" term in the Poisson model:

$$p(y_i \mid \nu_i \lambda) = (\nu_i \lambda)^y e^{\nu_i \lambda} / y_i!$$

- How many cars do we expect to pass an intersection in one hour? How many in two hours?
  - If we model the distribution as Poisson, we expect twice as many in two hours as in one hours.
- Homework: exposure is the length of the chapter

Poisson model example

- In a particular county 3 people out of a population of 100,000 died of asthma
- Assume a Poisson sampling model with rate λ (units are rate of deaths per 100,000 people)
- How do we specify a prior distribution for λ?
- How would our Bayesian estimate for λ differ?

Conjugate Prior for the Poisson Distribution

Assume n i.i.d observations of a Poisson(λ)

$$\begin{aligned} p(\lambda \mid y_1, \ldots y_n) &\propto L(\lambda) \times p(\lambda) \\ &\propto \lambda^{\sum y_i} e^{-n\lambda} \times p(\lambda) \end{aligned}$$

- A prior distribution for λ should have support on $\mathbb{R}^+$, the positive real line
- Bayesian definition of sufficiency: $p(\lambda \mid s, y_1, \ldots y_n) = p(\lambda \mid s)$
  - For the Poisson, $\sum y_i$ is sufficient
- Can we find a density of the form $p(\lambda) \propto \lambda^{k_1} e^{k_2 \lambda}$?

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- Can we find a density of the form $p(\lambda) \propto \lambda^{k_1} e^{k_2 \lambda}$?
- Gamma$(a, b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}$

The Gamma distribution

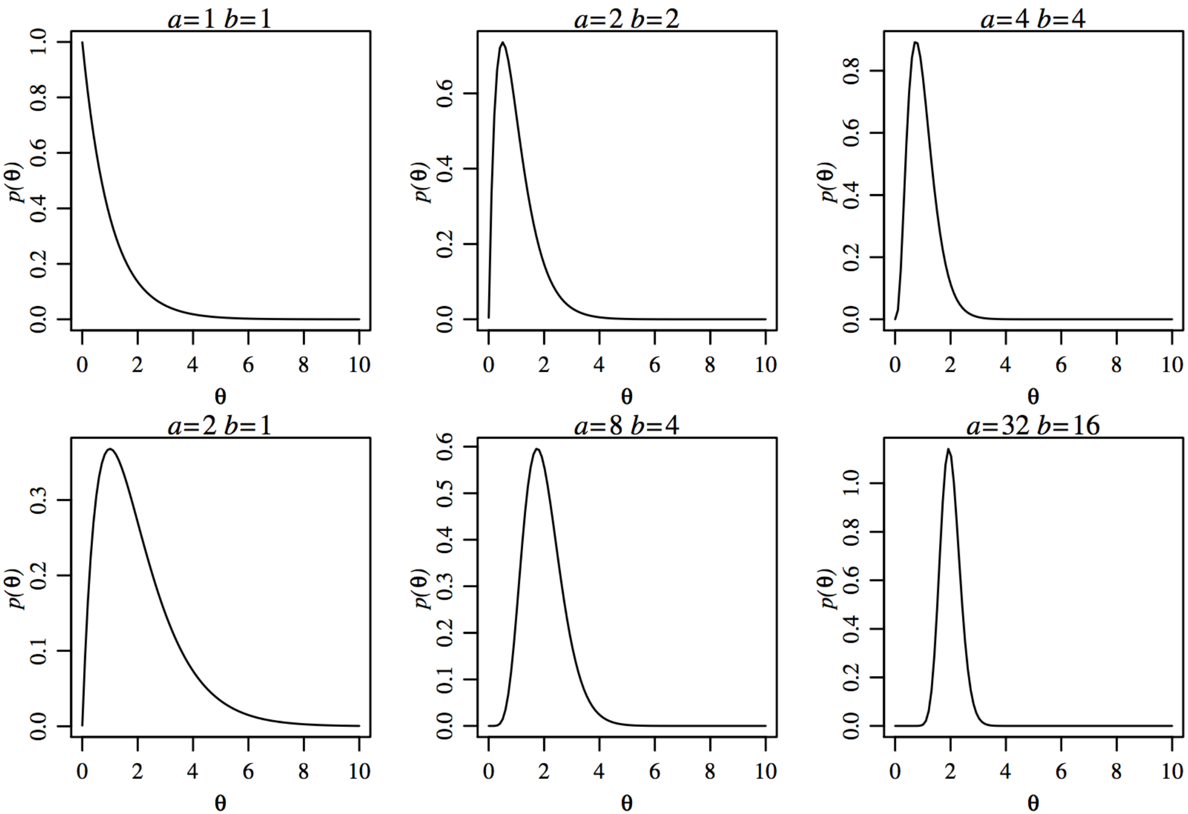


Fig. 3.8. Gamma densities.

The Gamma distribution

Useful properties of the Gamma distribution:

- $E[\lambda] = a/b$
- $\mathrm{Var}[\lambda] = a/b^2$
- $\mathrm{mode}[\lambda] = (a-1)/b$ if $a > 1$, 0 otherwise
- In R: `dgamma`, `rgamma`, `pgamma`, `qgamma`

The posterior in the Poisson-Gamma model

Assume one observation with $y_i \sim \text{Pois}(\lambda \nu_i)$ where ν_i is the exposure

$$
\begin{aligned}
p(\lambda \mid y_i) &\propto L(\lambda) \times p(\lambda) \\
&\propto (\lambda \nu_i)^{y_i} e^{-\lambda \nu_i} \times \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} \\
&\propto (\lambda)^{y_i + a - 1} e^{-(b+\nu_i)\lambda}
\end{aligned}
$$

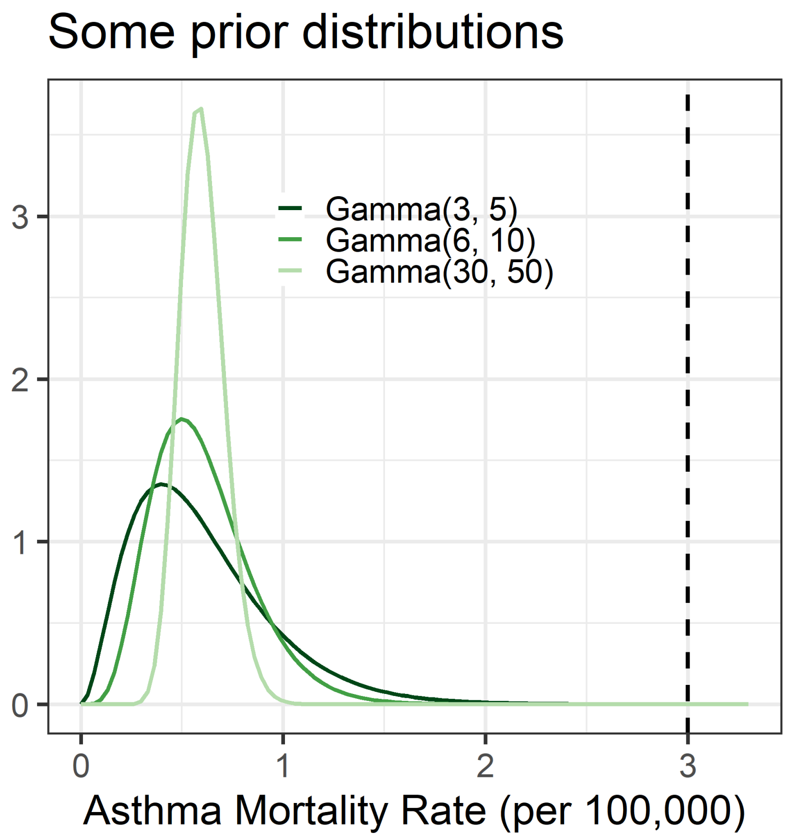
$$
p(\lambda \mid y, a, b) = \text{Gamma}(y_i + a, b + \nu_i)
$$

What is the posterior distribution for n observations, y_1, ... y_n, with exposures ν_1 ... ν_n?

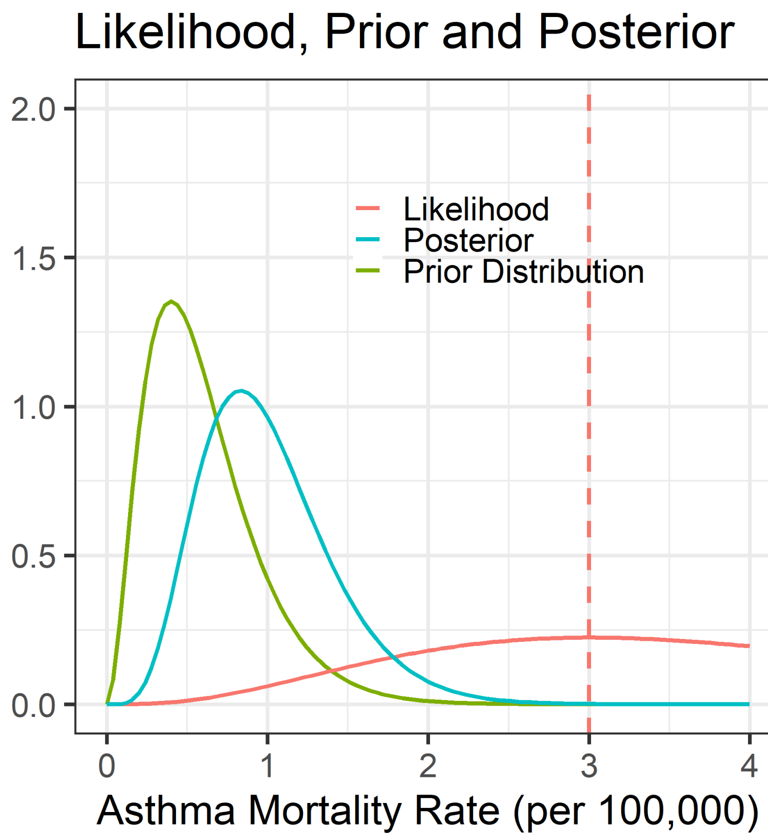
Poisson model example

- In a particular county 3 people out of a population of 100,000 died of asthma
- Assume a Poisson sampling model with rate λ
    - Units are rate of deaths per 100,000 people/year
- Experts know that typical rates of asthma mortality in the US are closer to 0.6 per 100,000
- Let's choose a Gamma distribution with a mean of 0.6 and appropriate uncertainty.

Possible Gamma prior distributions



Asthma Mortality



Using Gamma(3, 5) prior distribution

The posterior mean

$$
\begin{aligned}
E[\lambda \mid y_1, \ldots y_n] &= \frac{a + \sum y_i}{b + n} \\
&= \frac{b}{b+n}\frac{a}{b} + \frac{n}{b+n}\frac{\sum y_i}{n} \\
&= (1-w)\frac{a}{b} + w\hat{\lambda}_{\text{MLE}}
\end{aligned}
$$

- $w \to 1$ as $n \to \infty$ (data dominates prior)
- b can be interpreted as the number of *prior* observations
    - Analogous to n or total prior exposure
- a can be interpreted as the sum of the counts from prior total exposure of b
    - Analogous to $\sum_i y_i$

Asthma Mortality

- Suppose that nine additional years of data are obtained for the city
- The mortality rate of 3 per 100,000 is maintained: we find $y = 30$ deaths over 10 years.
- How has the posterior distribution changed?

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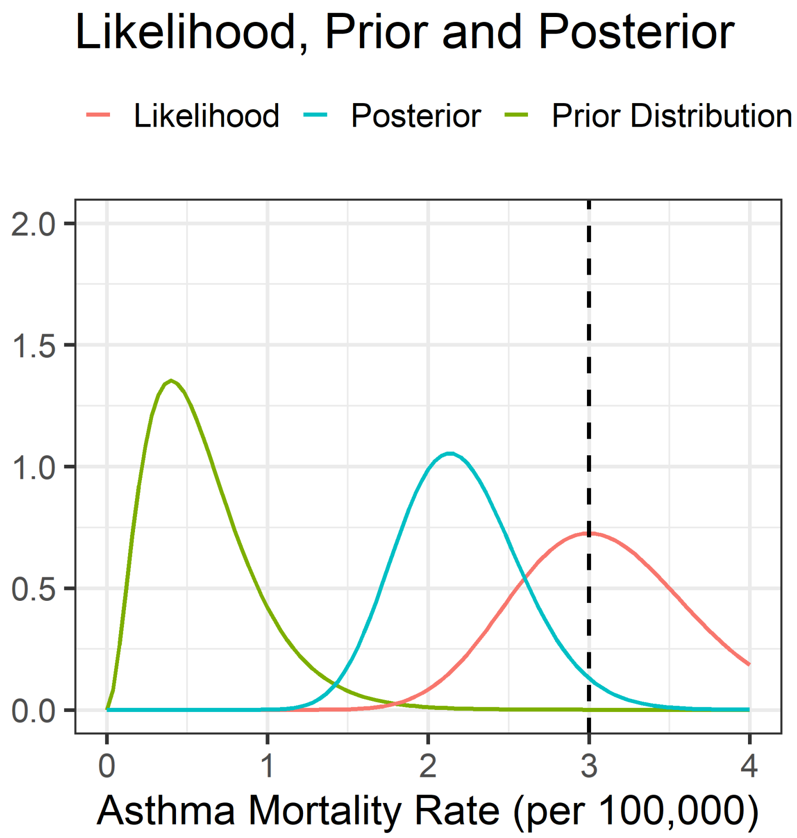
- The mortality rate of 3 per 100,000 is maintained: we find $y = 30$ deaths over 10 years.
- How has the posterior distribution changed?
- Two related approaches: use "all at once approach" or assume "Bayesian updating"

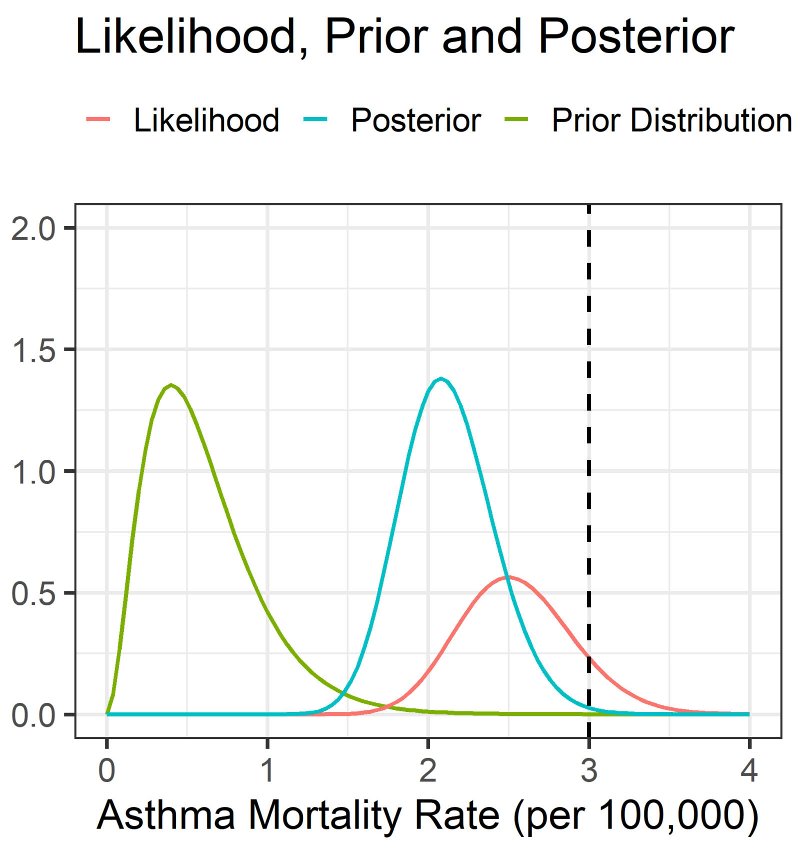
Asthma Mortality ("All At Once" Approach)



Using Gamma(3, 5) prior distribution

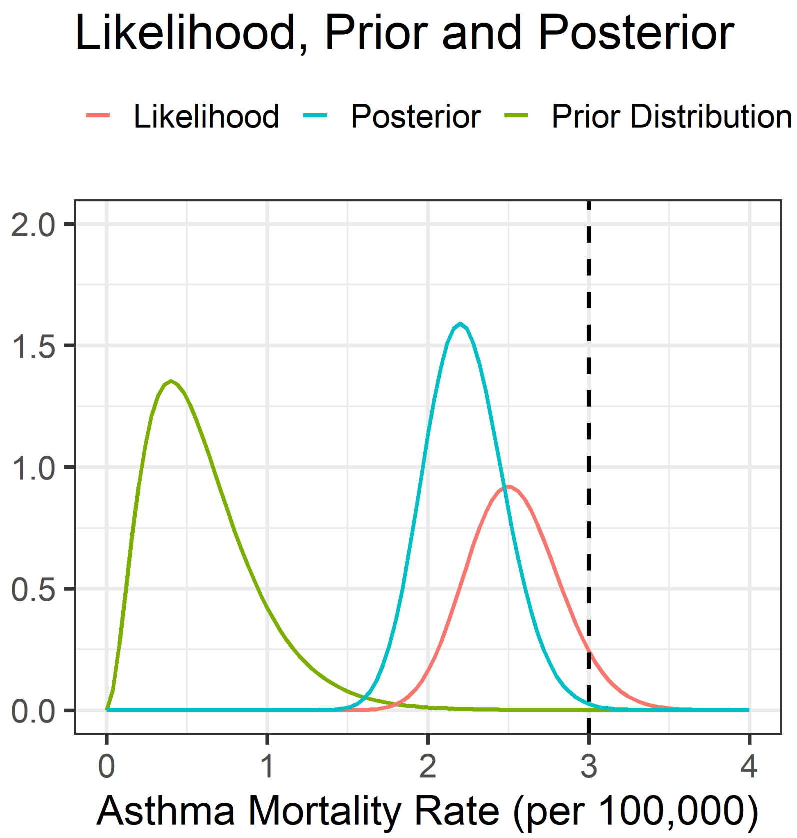
Asthma Mortality ("All At Once" Approach)

After 20 years we've see 50 deaths...



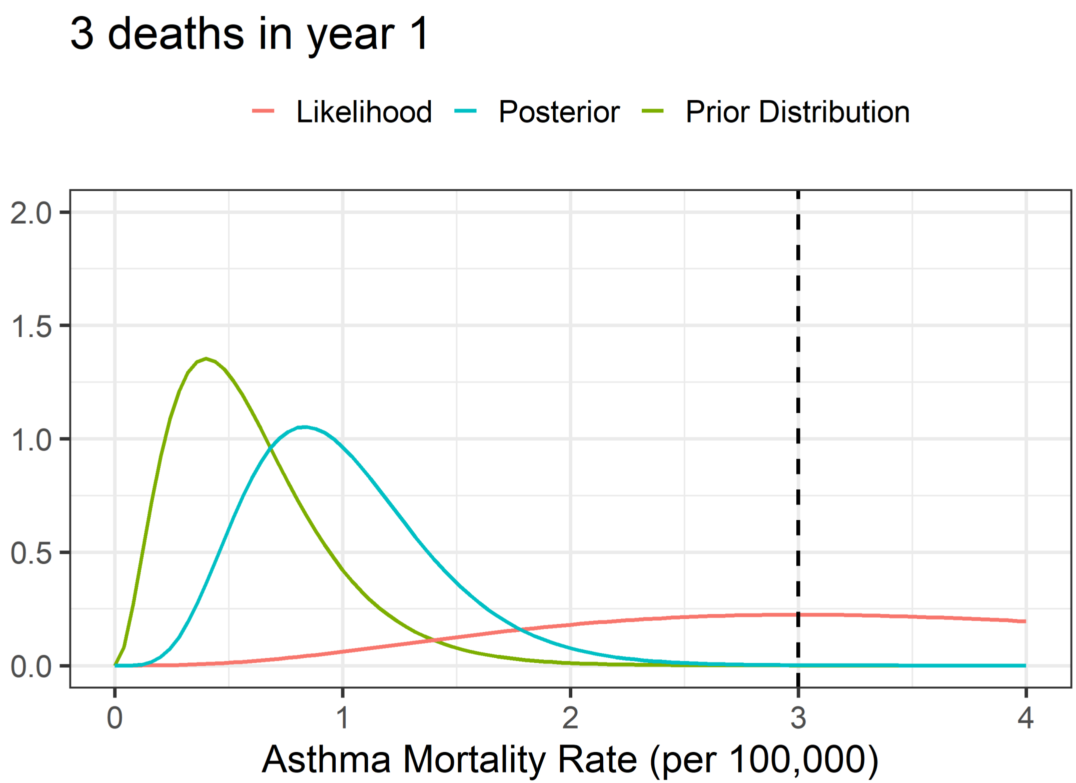
Asthma Mortality ("All At Once" Approach)

After 30 years we've see 75 deaths...



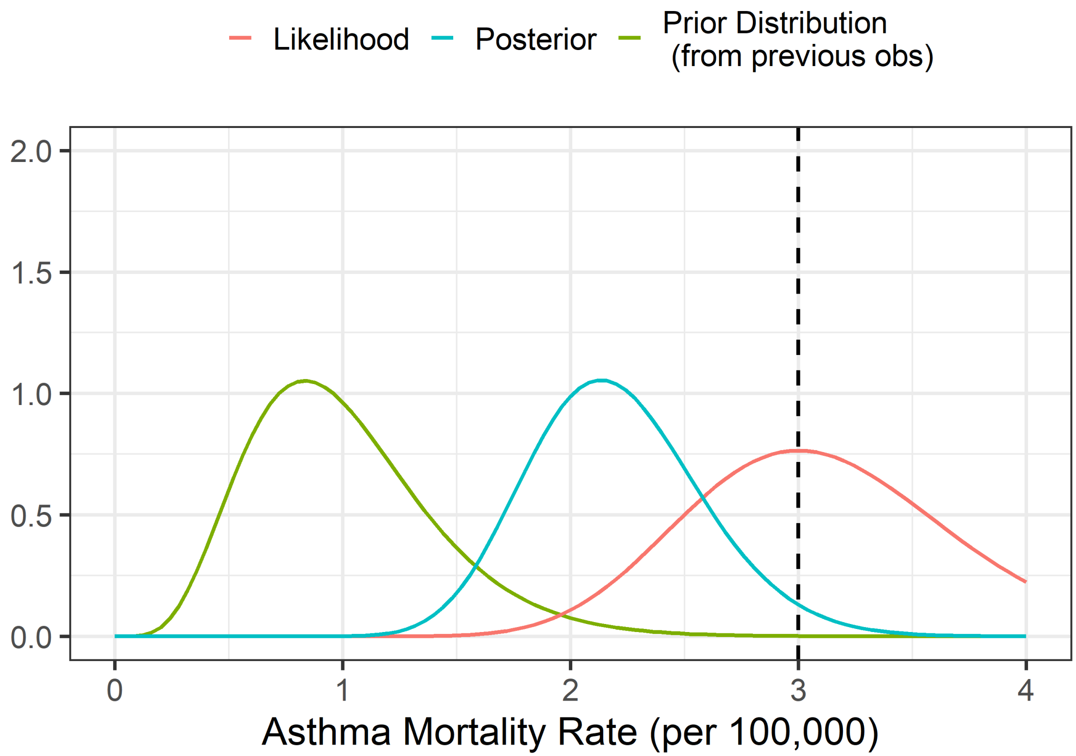
Asthma Mortality (Updating)

Perspective of continous "updating" of the posterior distribution



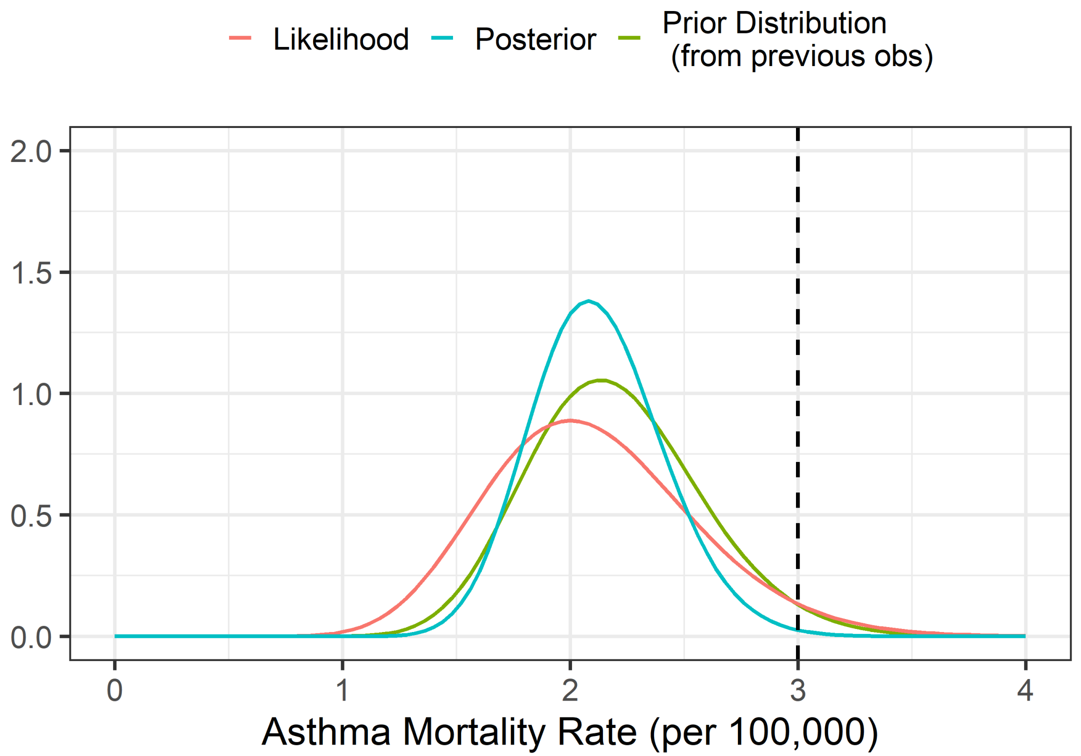
Asthma Mortality (Continuous Updating)

Prior mean, previous data (3+3)/(5+1) = 1
New data: 27 deaths in 9 more years, 27/9 = 3



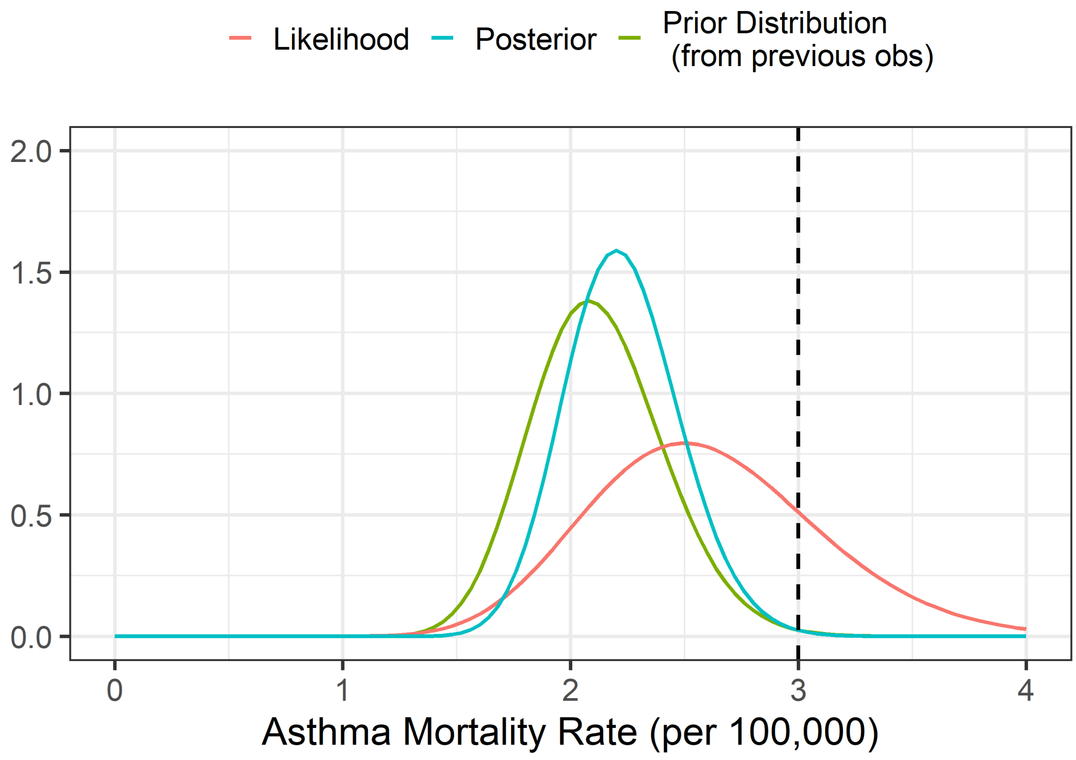
Asthma Mortality (Continuous Updating)

New prior" mean 33/15 = 2.2
New data, 20 deaths in 10 more years 20/10 = 2



Asthma Mortality

New prior" mean 53/25 = 2.12
New data, 25 deaths in 10 more years 25/10 = 2.5



Summary

- The Beta distribution
  - Conjugate prior for Binomial likelihood
- The Gamma distribution
  - Conjugate prior for the Poisson likelihood
- Pseudo-counts interpretations of conjugate prior distributions