

Formal Languages

More Applications of the Pumping Lemma for CFLs

The Pumping Lemma:

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

$$\{vv : v \in \{a,b\}^*\}$$

Context-free languages

$$\{a^n b^n : n \geq 0\}$$

$$\{ww^R : w \in \{a,b\}^*\}$$

Theorem: The language

$$L = \{vv : v \in \{a,b\}^*\}$$

is **not** context free

Proof: Use the Pumping Lemma
for context-free languages

$$L = \{vv : v \in \{a,b\}^*\}$$

Assume for contradiction that L
is context-free

Since L is context-free and infinite
we can apply the pumping lemma

$$L = \{vv : v \in \{a,b\}^*\}$$

Pumping Lemma gives a magic number m
such that:

Pick any string of L with length at least m

we pick: $a^m b^m a^m b^m \in L$

$$L = \{vv : v \in \{a,b\}^*\}$$

We can write: $a^m b^m a^m b^m = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

Pumping Lemma says:

$$uv^i xy^i z \in L \quad \text{for all } i \geq 0$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

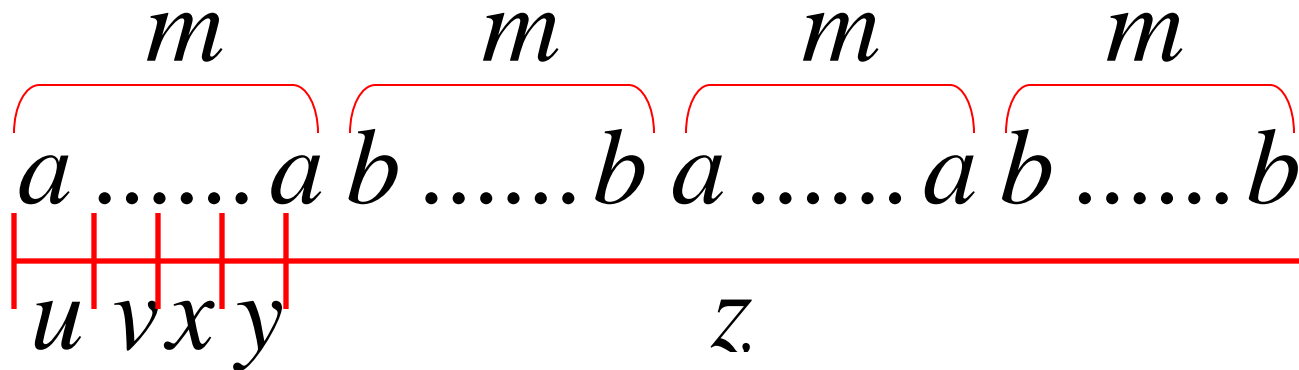
We examine all the possible locations
of string vxy in $a^m b^m a^m b^m$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

Case 1: vxy is within the first a^m

$$v = a^{k_1} \quad y = a^{k_2} \quad k_1 + k_2 \geq 1$$

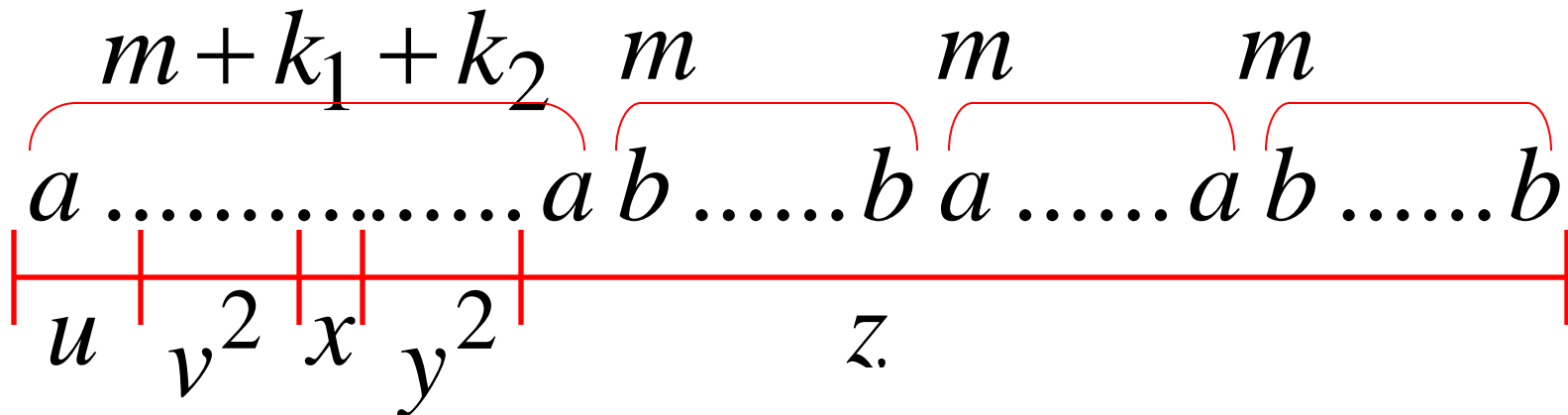


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$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

Case 1: vxy is within the first a^m

$$a^{m+k_1+k_2} b^m a^m b^m = uv^2 xy^2 z \notin L$$

$$k_1 + k_2 \geq 1$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

Case 1: vxy is within the first a^m

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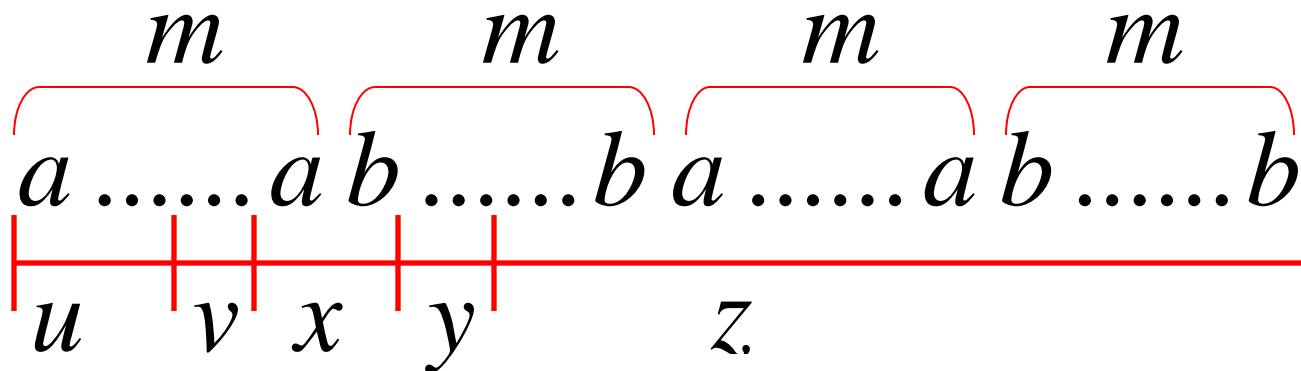
Contradiction!!!

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

Case 2: v is in the first a^m
 y is in the first b^m

$$v = a^{k_1} \quad y = b^{k_2} \quad k_1 + k_2 \geq 1$$

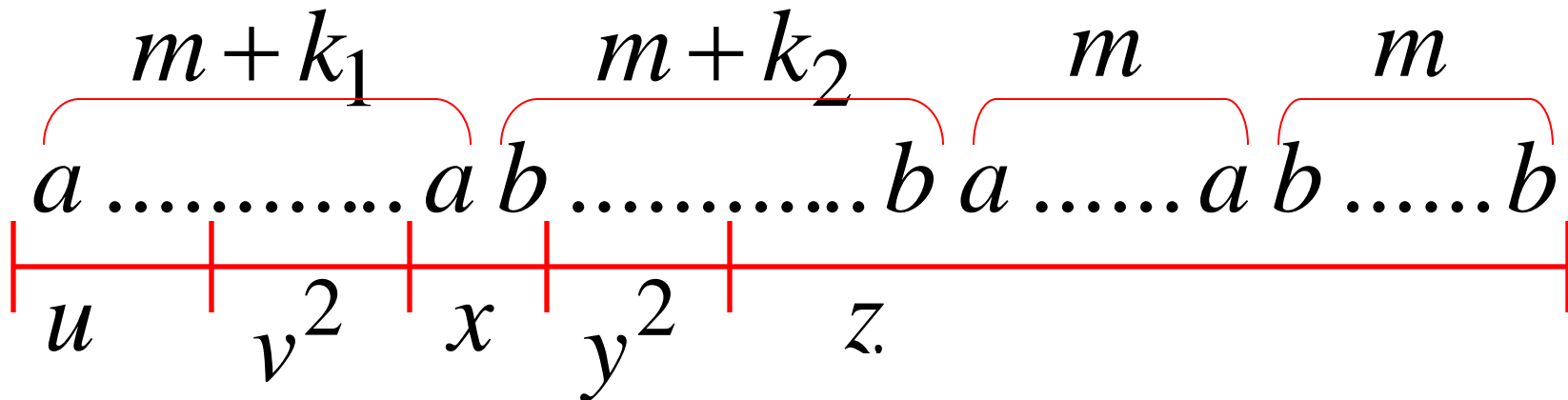


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$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

Case 2: v is in the first a^m
 y is in the first b^m

$$a^{m+k_1} b^{m+k_2} a^m b^m = uv^2 xy^2 z \notin L$$

$$k_1 + k_2 \geq 1$$

$$L = \{vv : v \in \{a,b\}^*\}$$

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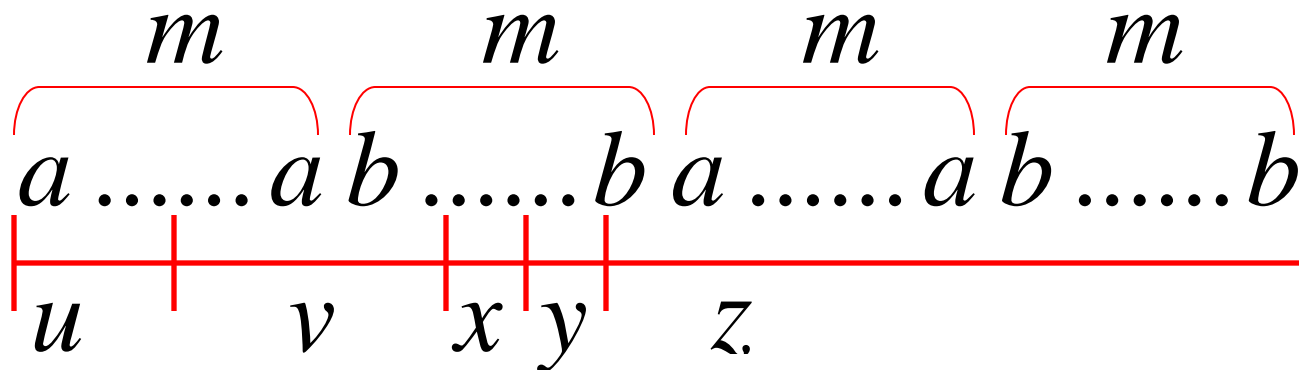
Contradiction!!!

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

Case 3: v overlaps the first $a^m b^m$
 y is in the first b^m

$$v = a^{k_1} b^{k_2} \quad y = b^{k_3} \quad k_1, k_2 \geq 1$$

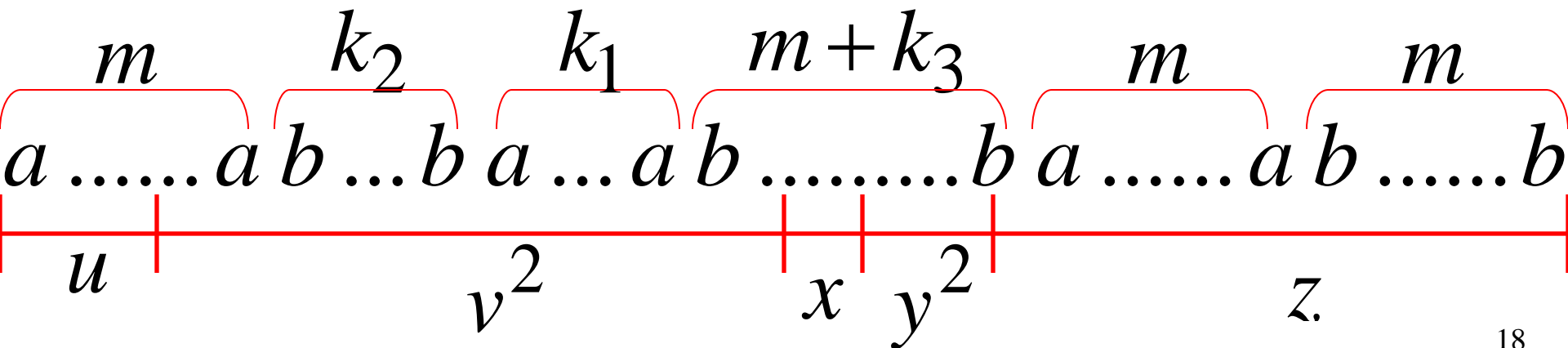


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$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

Case 3: v overlaps the first $a^m b^m$
 y is in the first b^m

$$a^m b^{k_2} a^{k_1} b^{m+k_3} a^m b^m = uv^2 xy^2 z \notin L$$

$$k_1, k_2 \geq 1$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

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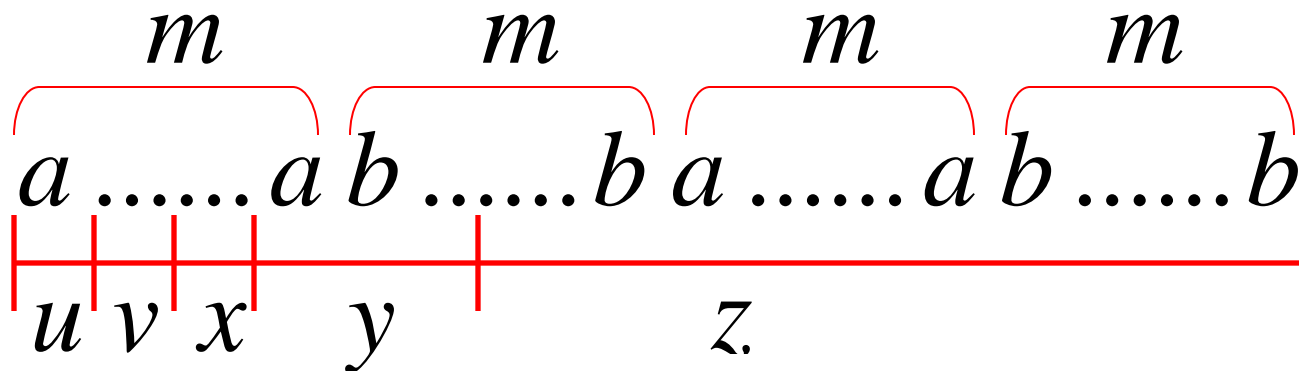
Contradiction!!!

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

Case 4: v in the first a^m
 y Overlaps the first $a^m b^m$

Analysis is similar to case 3



Other cases: vxy is within $a^m \boxed{b^m} a^m b^m$

or

$a^m b^m \boxed{a^m} b^m$

or

$a^m b^m a^m \boxed{b^m}$

Analysis is similar to case 1:

$\boxed{a^m} b^m a^m b^m$

More cases:

vxy overlaps

$a^m b^m a^m b^m$

or

$a^m b^m a^m b^m$

Analysis is similar to cases 2,3,4:

$a^m b^m a^m b^m$

There are no other cases to consider

Since $|vxy| \leq m$, it is impossible for vxy to overlap:

$$a^m b^m a^m b^m$$

or

$$a^m b^m a^m b^m$$

or

$$a^m b^m a^m b^m$$

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{vv : v \in \{a,b\}^*\}$$

is context-free must be wrong

Conclusion: L is not context-free

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

$$\{ww : w \in \{a,b\}^*\}$$

$$\{a^{n!} : n \geq 0\}$$

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Pumping Lemma gives a magic number m
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Pick any string of L with length at least m

we pick: $a^{m!} \in L$

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We can write: $a^{m!} = uvxyz$

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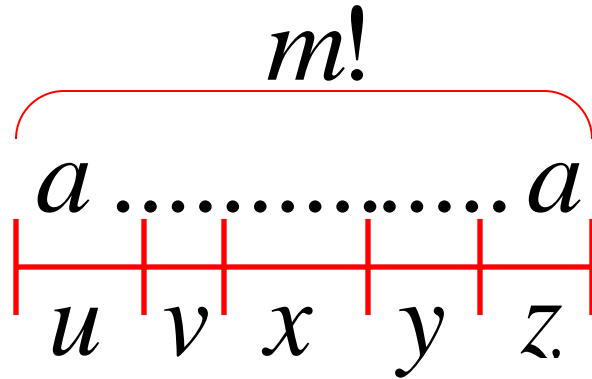
$$a^{m!} = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

We examine all the possible locations
of string vxy in $a^{m!}$

There is only one case to consider

$$L = \{a^{n!} : n \geq 0\}$$

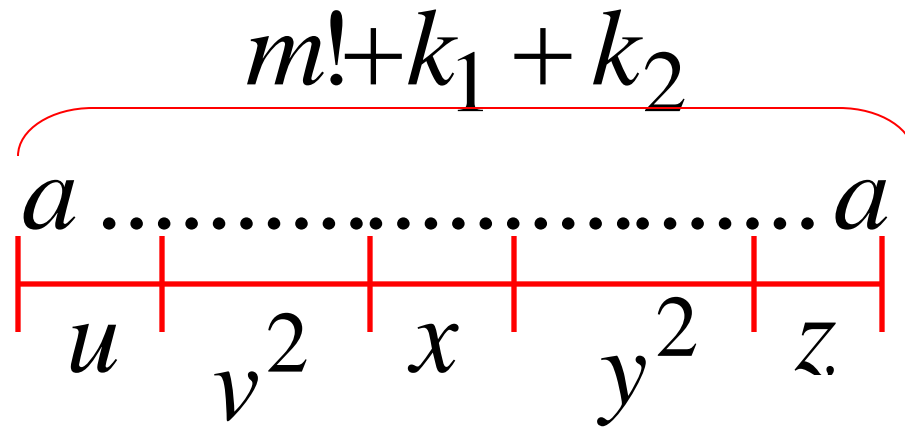
$$a^{m!} = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$



$$v = a^{k_1} \quad y = a^{k_2} \quad 1 \leq k_1 + k_2 \leq m$$

$$L = \{a^{n!} : n \geq 0\}$$

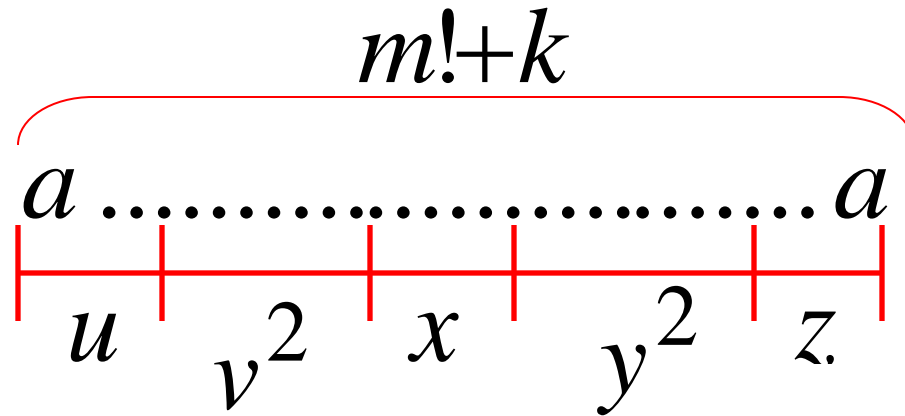
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$$k = k_1 + k_2$$

$$v = a^{k_1} \quad y = a^{k_2} \quad 1 \leq k \leq m$$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

$$a^{m!+k} = uv^2xy^2z$$

$$1 \leq k \leq m$$

Since $1 \leq k \leq m$, for $m \geq 2$ we have:

$$m! + k \leq m! + m$$

$$< m! + m!m$$

$$= m!(1 + m)$$

$$= (m + 1)!$$



$$m! < m! + k < (m + 1)!$$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

$$m! < m! + k < (m+1)!$$



$$a^{m!+k} = uv^2xy^2z \notin L$$

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$$a^{m!} = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

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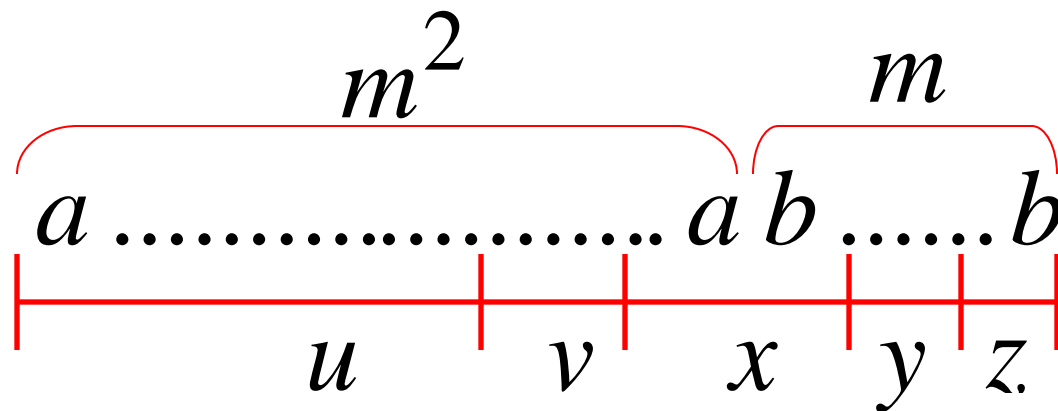
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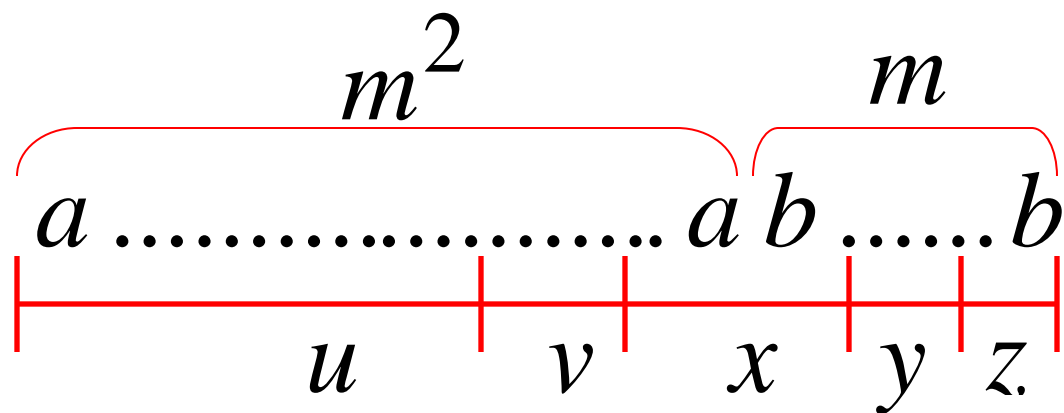
Most complicated case: v is in a^{m^2}
 y is in b^m



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$$a^{m^2} b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

$$v = a^{k_1} \quad y = b^{k_2} \quad 1 \leq k_1 + k_2 \leq m$$

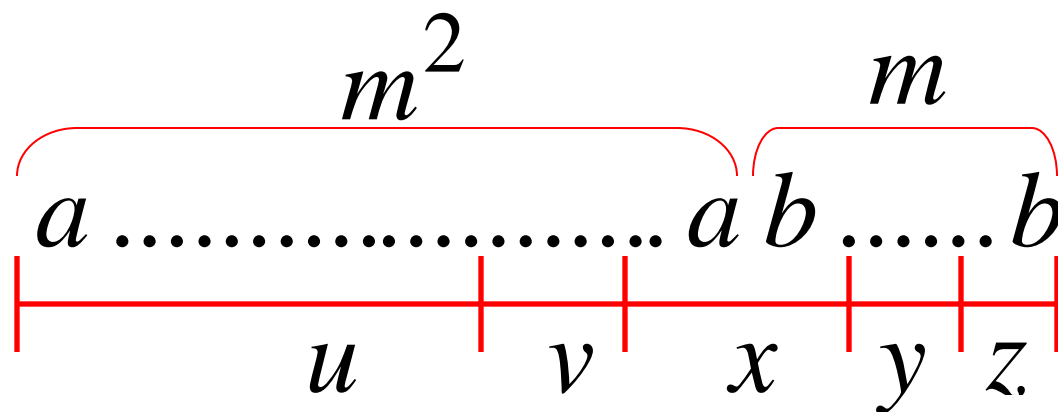


$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \quad y = b^{k_2} \quad 1 \leq k_1 + k_2 \leq m$$

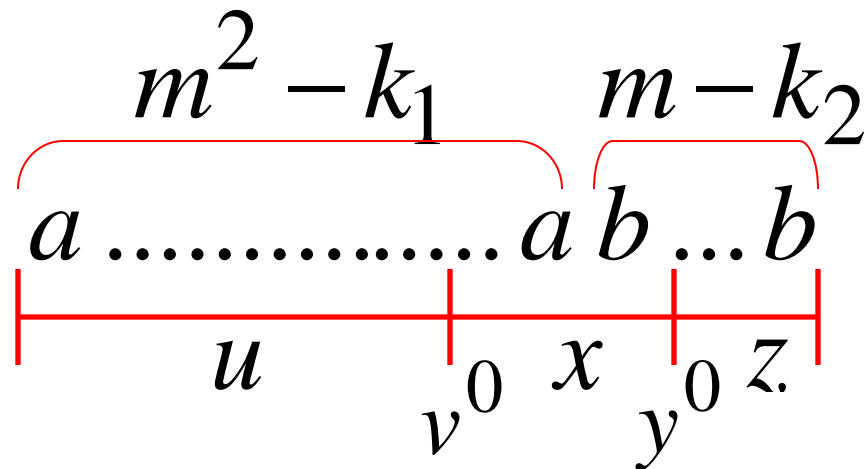


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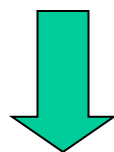
$$v = a^{k_1} \quad y = b^{k_2} \quad 1 \leq k_1 + k_2 \leq m$$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z$$

$$k_1 \neq 0 \text{ and } k_2 \neq 0 \quad 1 \leq k_1 + k_2 \leq m$$



$$\begin{aligned} (m - k_2)^2 &\leq (m - 1)^2 \\ &= m^2 - 2m + 1 \\ &< m^2 - k_1 \end{aligned}$$

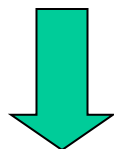


$$m^2 - k_1 \neq (m - k_2)^2$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

$$m^2 - k_1 \neq (m - k_2)^2$$



$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z \notin L$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

However, from Pumping Lemma: $uv^0xy^0z \in L$

$$a^{m^2-k_1} b^{m-k_2} = uv^0xy^0z \notin L$$

Contradiction!!!

When we examine the rest of the cases
we also obtain a contradiction

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^{n^2}b^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free