Formal Languages
The Pumping Lemma for CFLs

Review: pumping lemma for regular languages

Take an **infinite** context-free language

Generates an infinite number
of different strings

Example:

$$S \rightarrow AB$$

$$A \rightarrow aBb$$

$$B \rightarrow Sb$$

$$B \rightarrow b$$

$$S \rightarrow AB$$

$$A \rightarrow aBb$$

$$B \rightarrow Sb$$

$$B \rightarrow b$$

In a derivation of a long string, variables are repeated

A derivation:

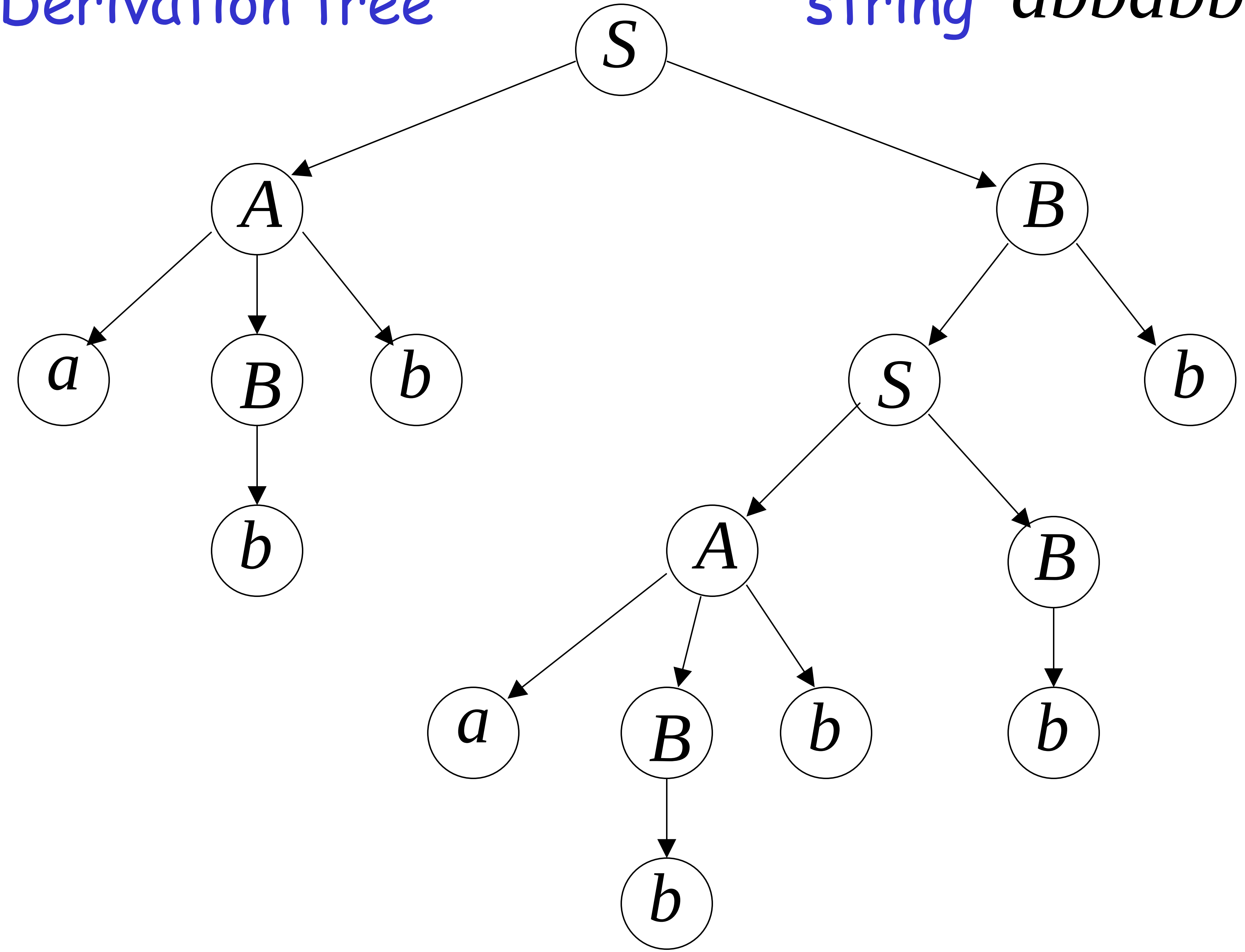
$$S \Rightarrow AB \Rightarrow aBbB \Rightarrow abb\underline{B}$$

$$\Rightarrow abbSb \Rightarrow abbABb \Rightarrow abbaBbBb \Rightarrow$$

$$\Rightarrow abbabb\underline{B}b \Rightarrow abbabbbb$$

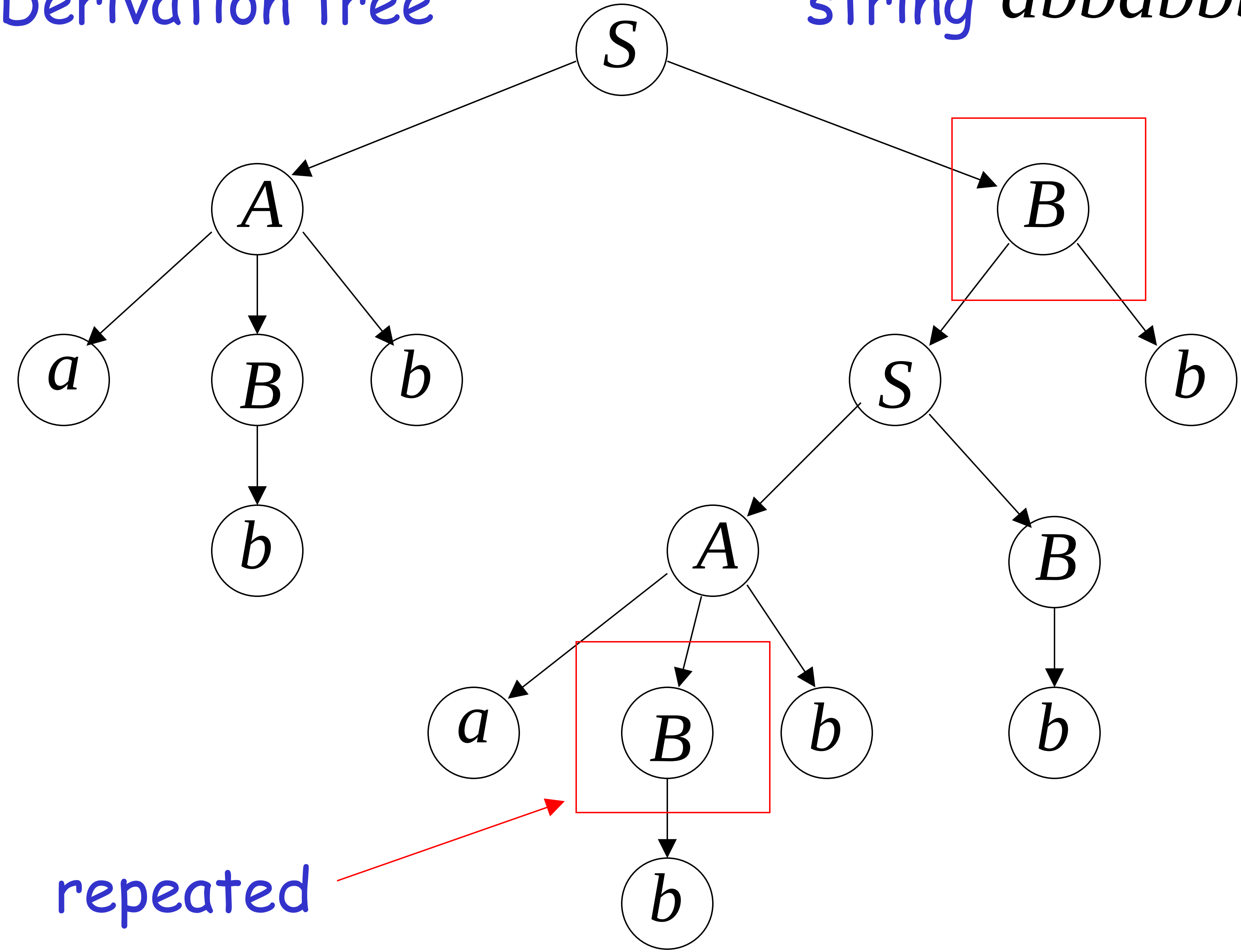
Derivation tree

string $abbabbbb$



Derivation tree

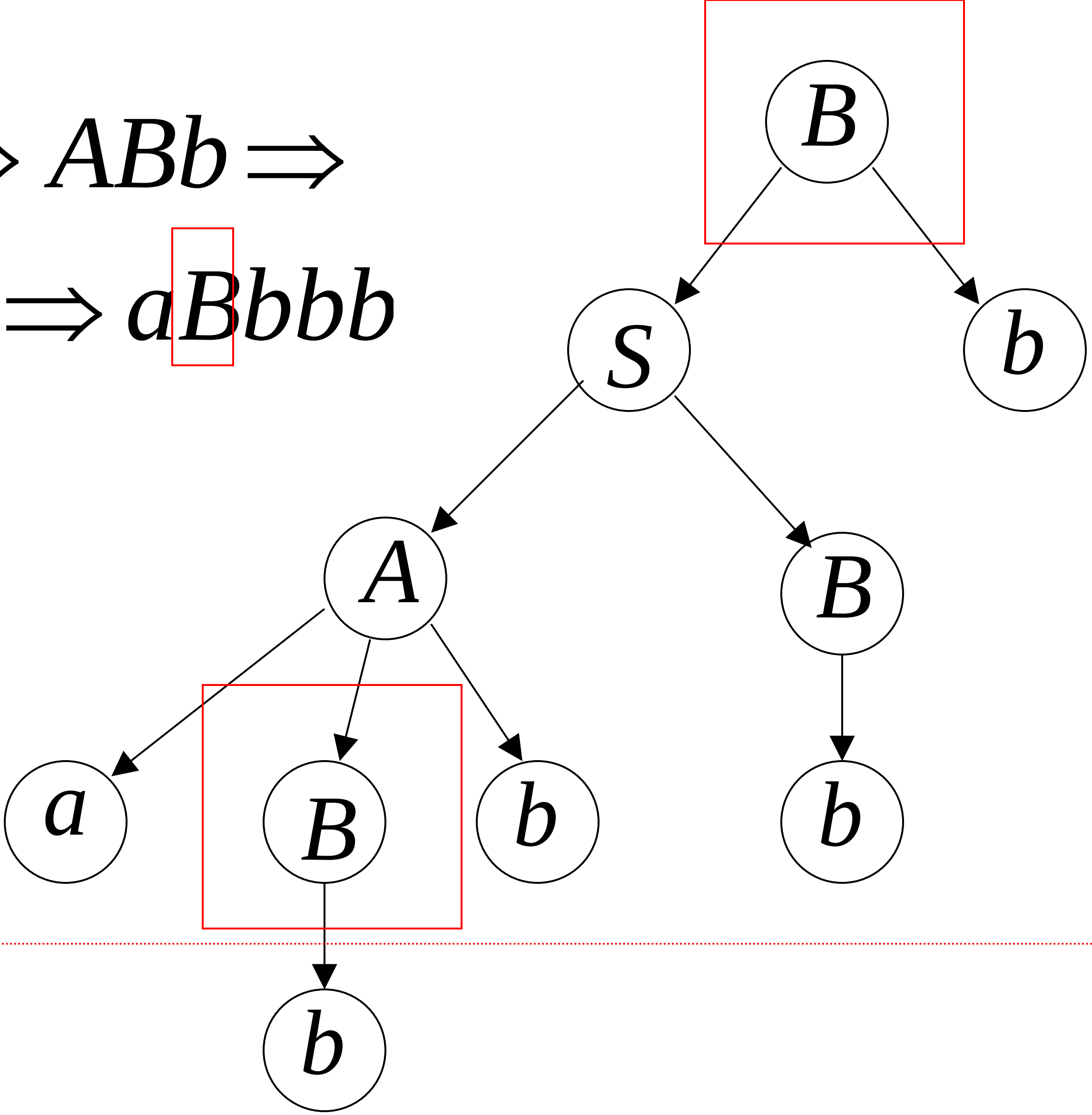
string $abbabbbb$



repeated

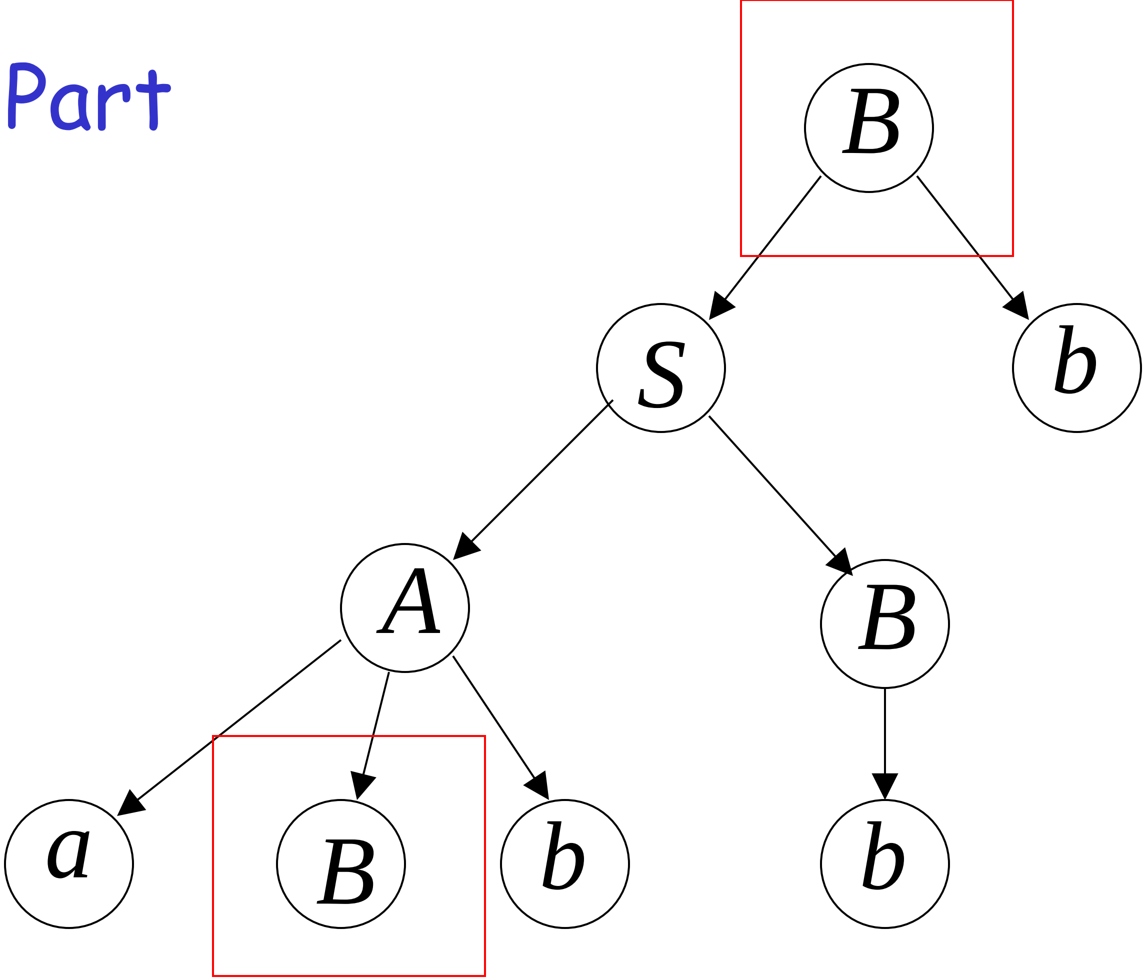
$$\boxed{B} \Rightarrow Sb \Rightarrow ABb \Rightarrow$$

$$\Rightarrow aBbBb \Rightarrow a\boxed{B}bbb$$



$$B \overset{*}{\Rightarrow} aBbbb \qquad B \Rightarrow b$$

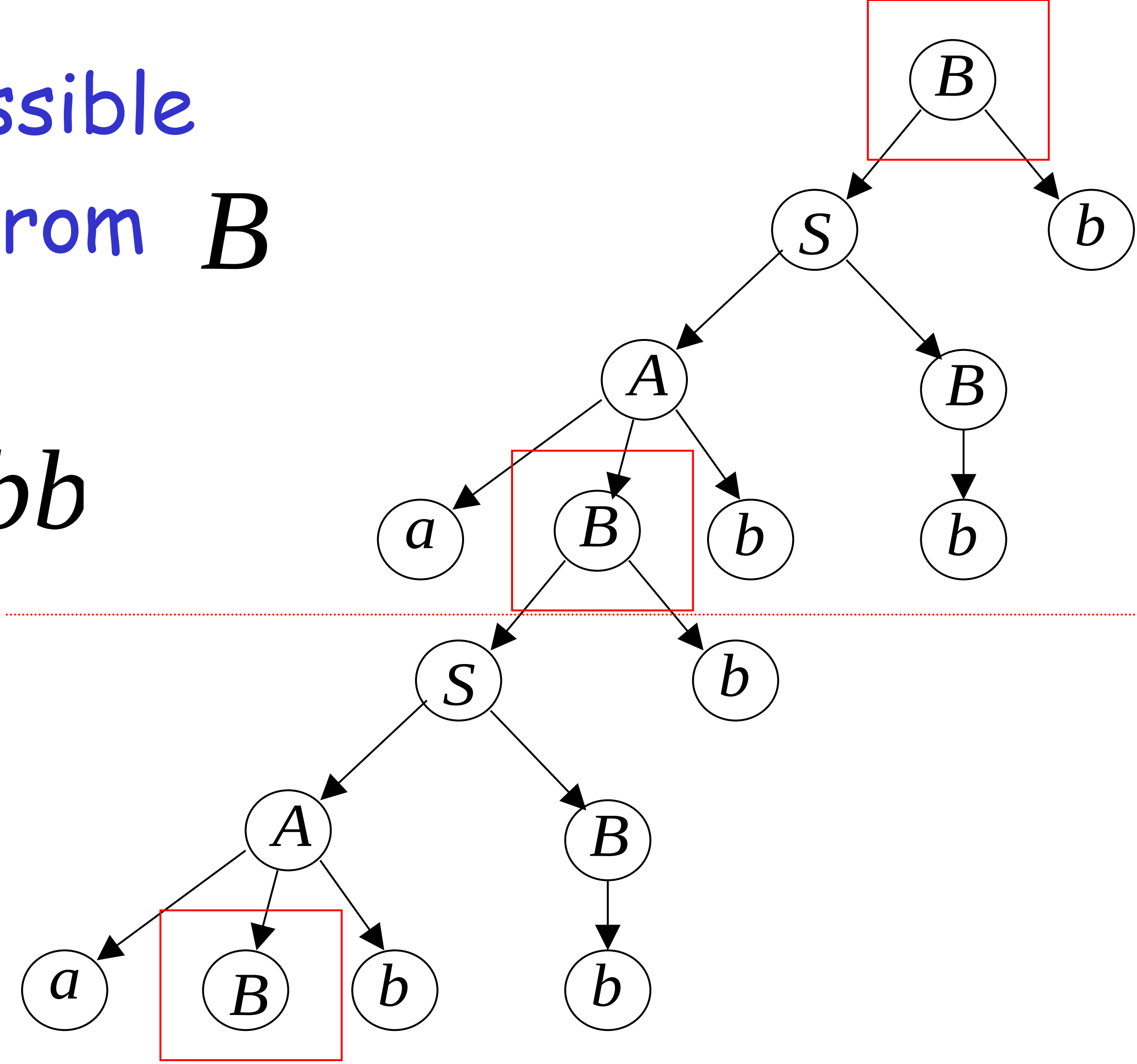
Repeated Part



$$B \overset{*}{\Rightarrow} aBbbb$$

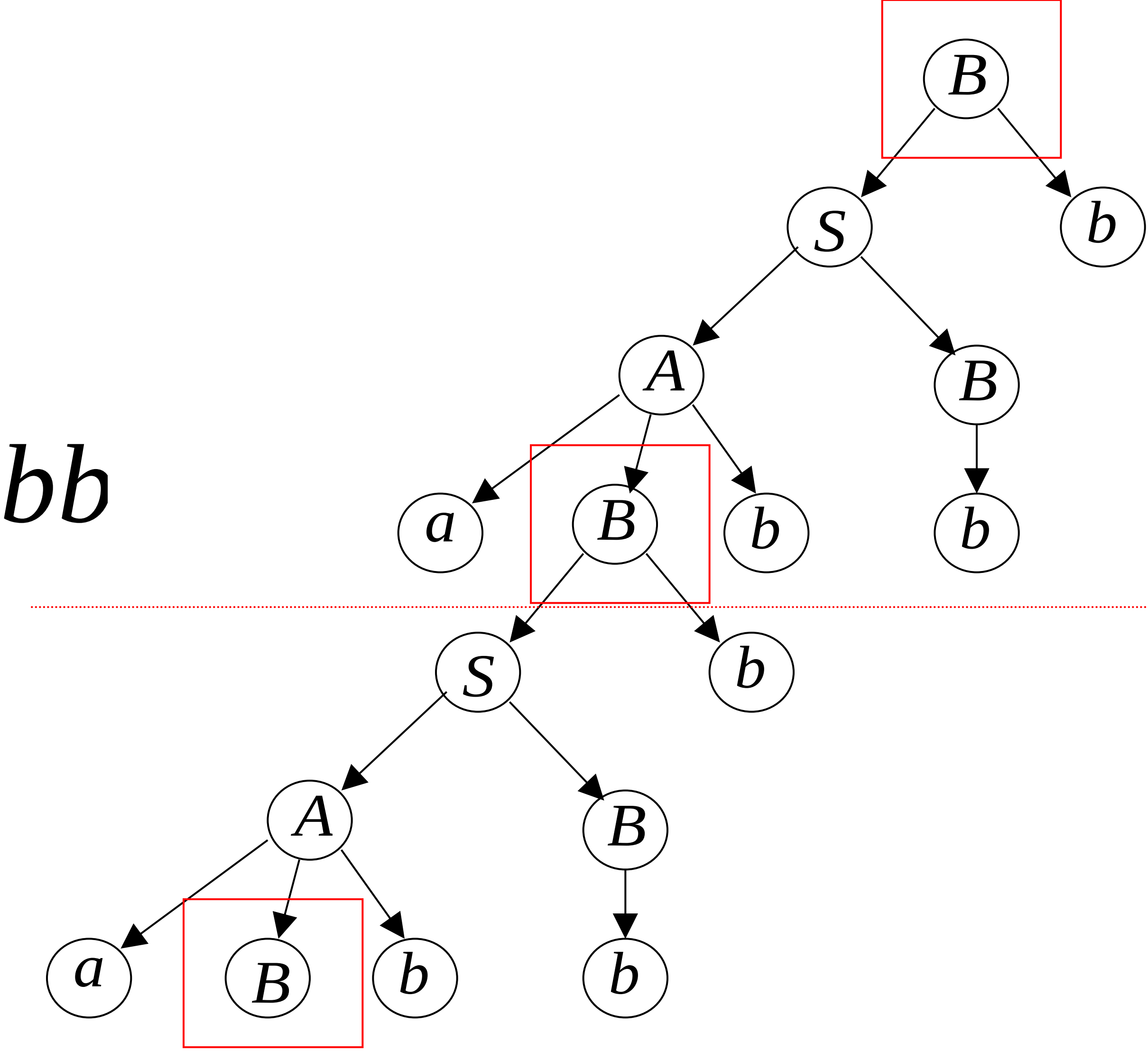
Another possible derivation from B

$$B \overset{*}{\Rightarrow} aBbbb$$



$$B \overset{*}{\Rightarrow} aBbbb \overset{*}{\Rightarrow} aaBbbbbbb$$

$$B \overset{*}{\Rightarrow} aBbbb$$



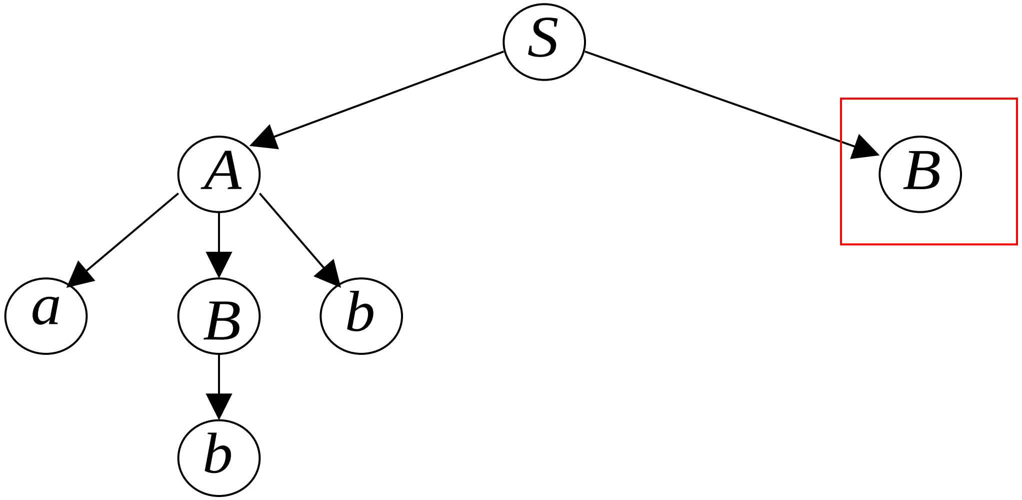
$$B \overset{*}{\Rightarrow} (a)B(bbb) \overset{*}{\Rightarrow} (a)^2 B(bbb)^2$$

A Derivation from S

$$S \overset{*}{\Rightarrow} abbBb$$

$$B \overset{*}{\Rightarrow} aBbbb$$

$$B \Rightarrow b$$

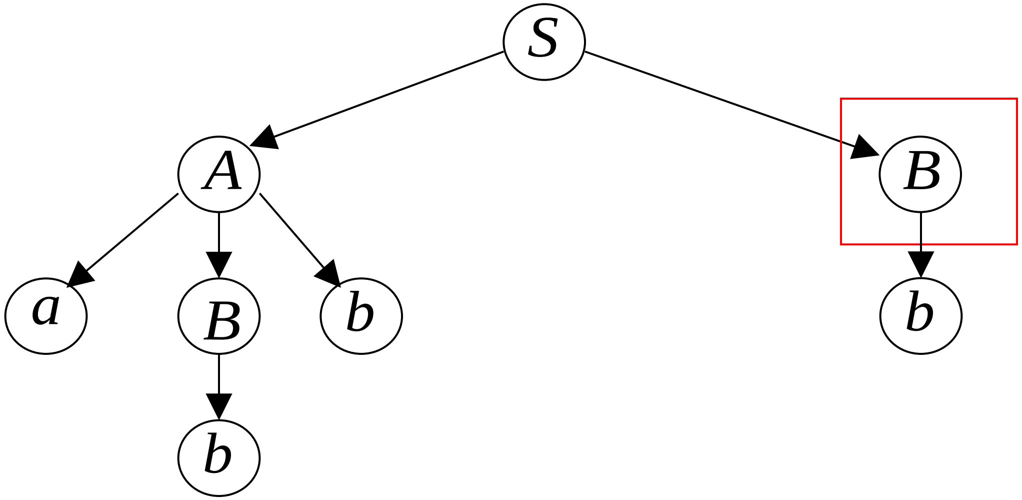


$$S \overset{*}{\Rightarrow} abbBb$$

$$S \overset{*}{\Rightarrow} abbBb$$

$$B \overset{*}{\Rightarrow} aBbbb$$

$$B \Rightarrow b$$



$$S \overset{*}{\Rightarrow} abbBb \Rightarrow abbbb \quad = abb(a)^0 b(bbb)^0$$

$$S \overset{*}{\Rightarrow} abbBb \qquad B \overset{*}{\Rightarrow} aBbbb \qquad B \Rightarrow b$$

$$\Downarrow$$

$$S \overset{*}{\Rightarrow} abb(a)^0 b(bbb)^0$$

$$\Downarrow$$

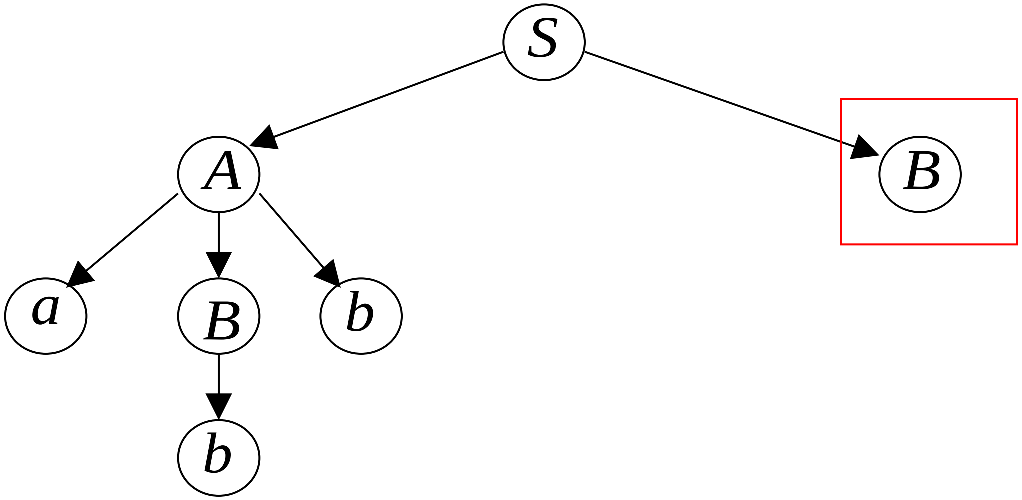
$$abb(a)^0 b(bbb)^0 \in L(G)$$

A Derivation from S

$$S \overset{*}{\Rightarrow} abbBb$$

$$B \overset{*}{\Rightarrow} aBbbb$$

$$B \Rightarrow b$$

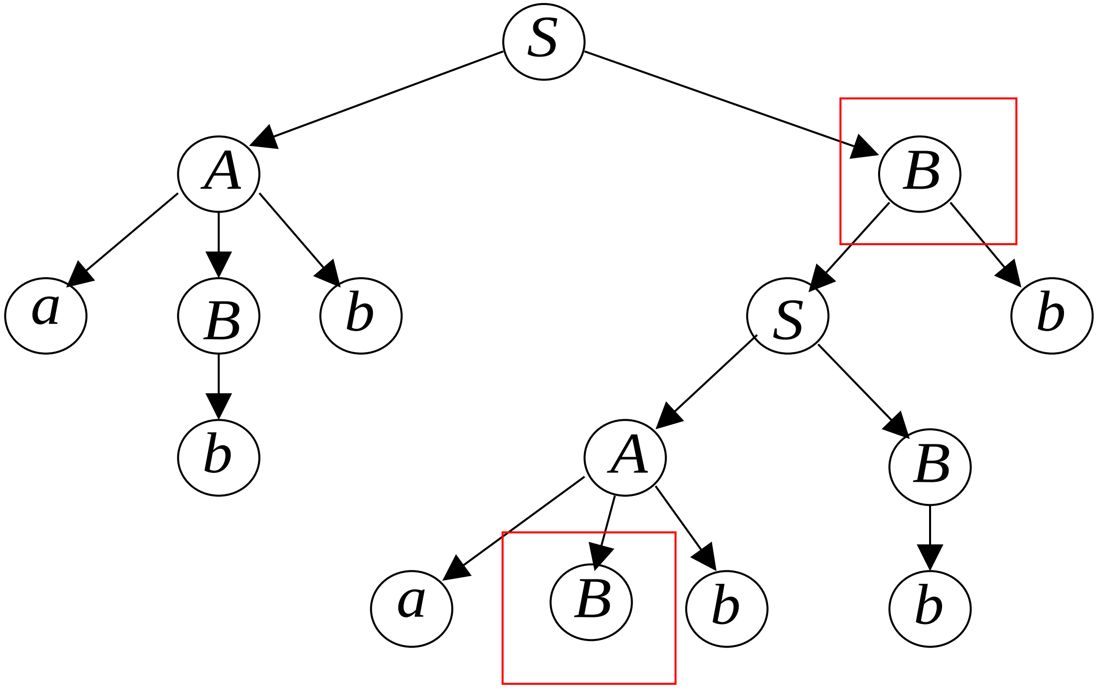


$$S \overset{*}{\Rightarrow} abbBb$$

$$S \overset{*}{\Rightarrow} abbBb$$

$$B \overset{*}{\Rightarrow} aBbbb$$

$$B \Rightarrow b$$

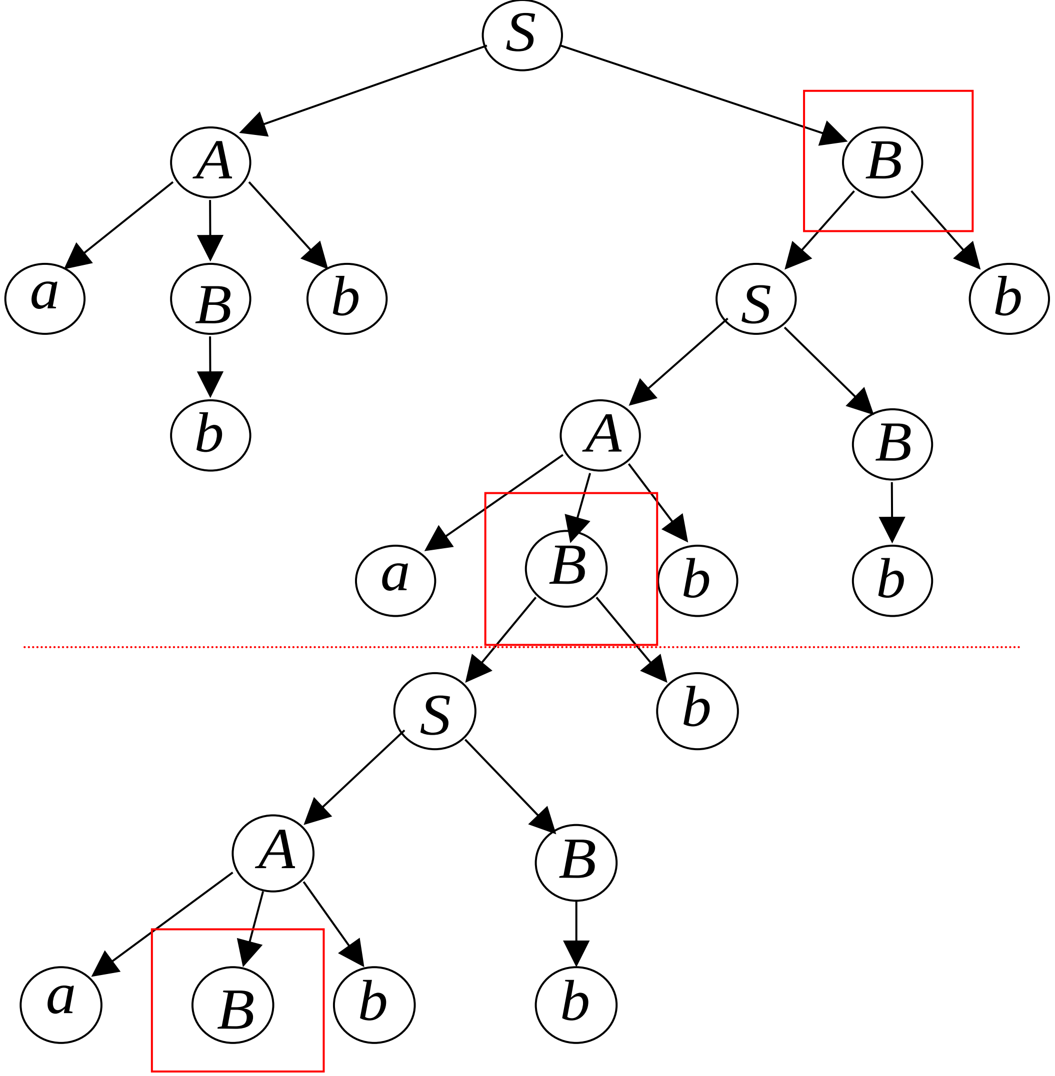


$$S \overset{*}{\Rightarrow} abbBb \overset{*}{\Rightarrow} abbaBbbb$$

$$S \overset{*}{\Rightarrow} abbBb$$

$$B \overset{*}{\Rightarrow} aBbbb$$

$$B \Rightarrow b$$

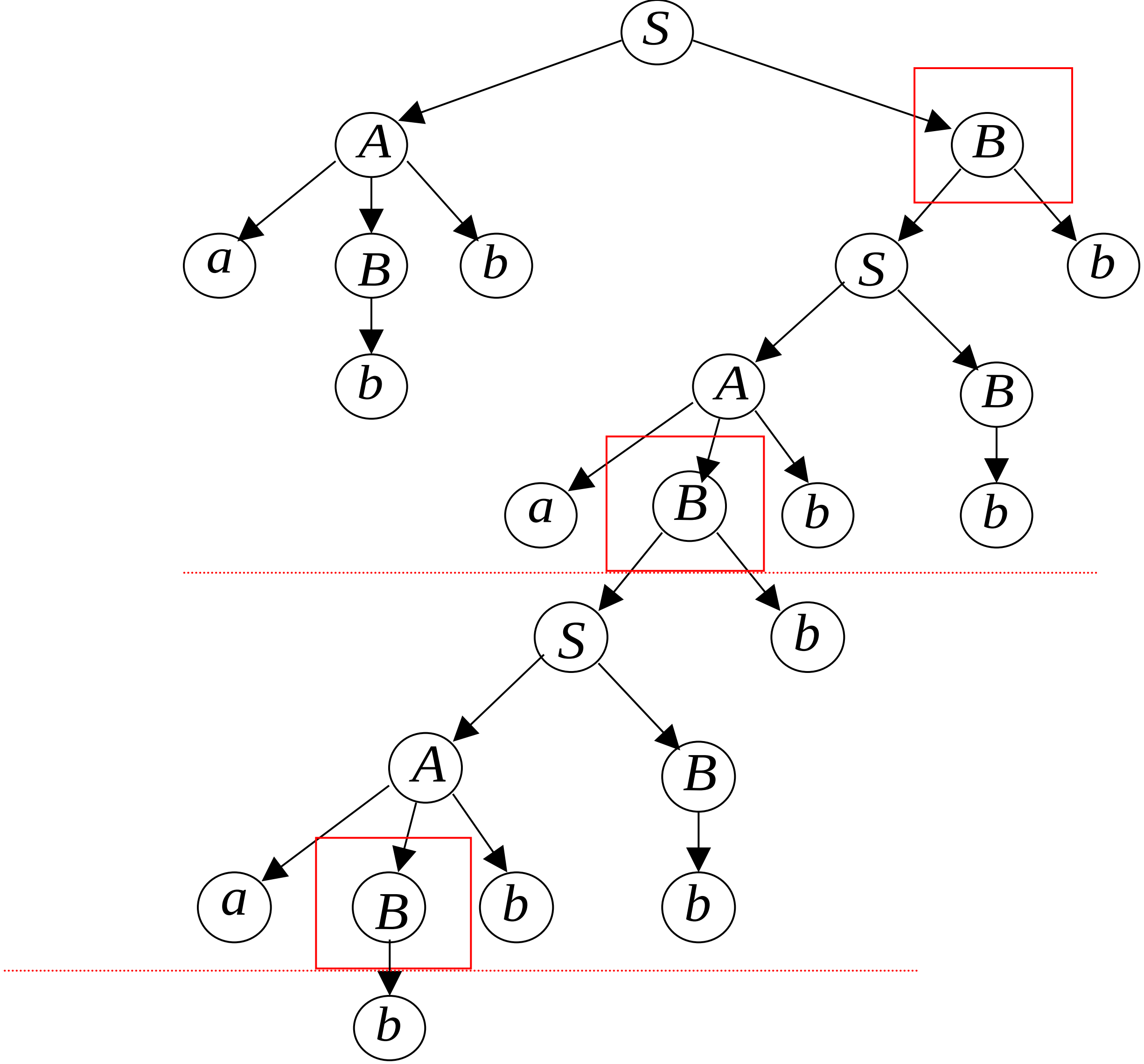


$$S \overset{*}{\Rightarrow} abb(a)B(bbb) \overset{*}{\Rightarrow} abb(a)^2 B(bbb)^2$$

$$S \overset{*}{\Rightarrow} abbBb$$

$$B \overset{*}{\Rightarrow} aBbbb$$

$$B \Rightarrow b$$



$$S \overset{*}{\Rightarrow} abb(a)^2 B(bbb)^2 \overset{*}{\Rightarrow} abb(a)^2 b(bbb)^2$$

$S \overset{*}{\Rightarrow} abbBb \qquad B \overset{*}{\Rightarrow} aBbbb \qquad B \Rightarrow b$

$$S \overset{*}{\Rightarrow} abb(a)^2 b(bbb)^2$$

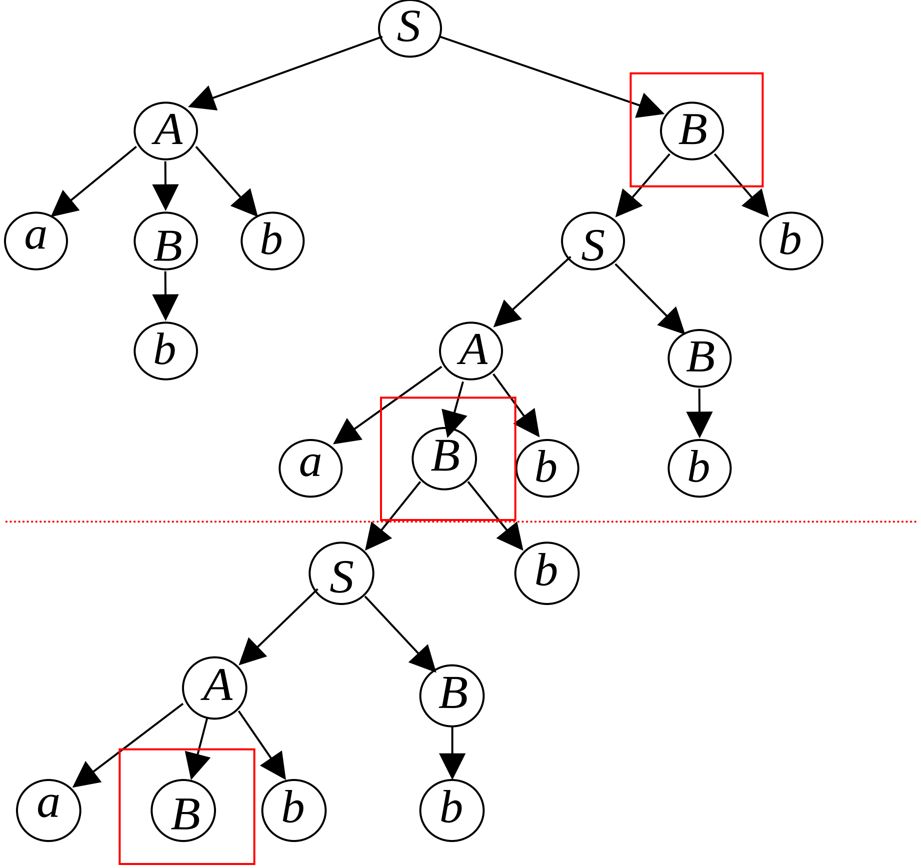
$$abb(a)^2 b(bbb)^2 \in L(G)$$

A Derivation from S

$$S \overset{*}{\Rightarrow} abbBb$$

$$B \overset{*}{\Rightarrow} aBbbb$$

$$B \Rightarrow b$$

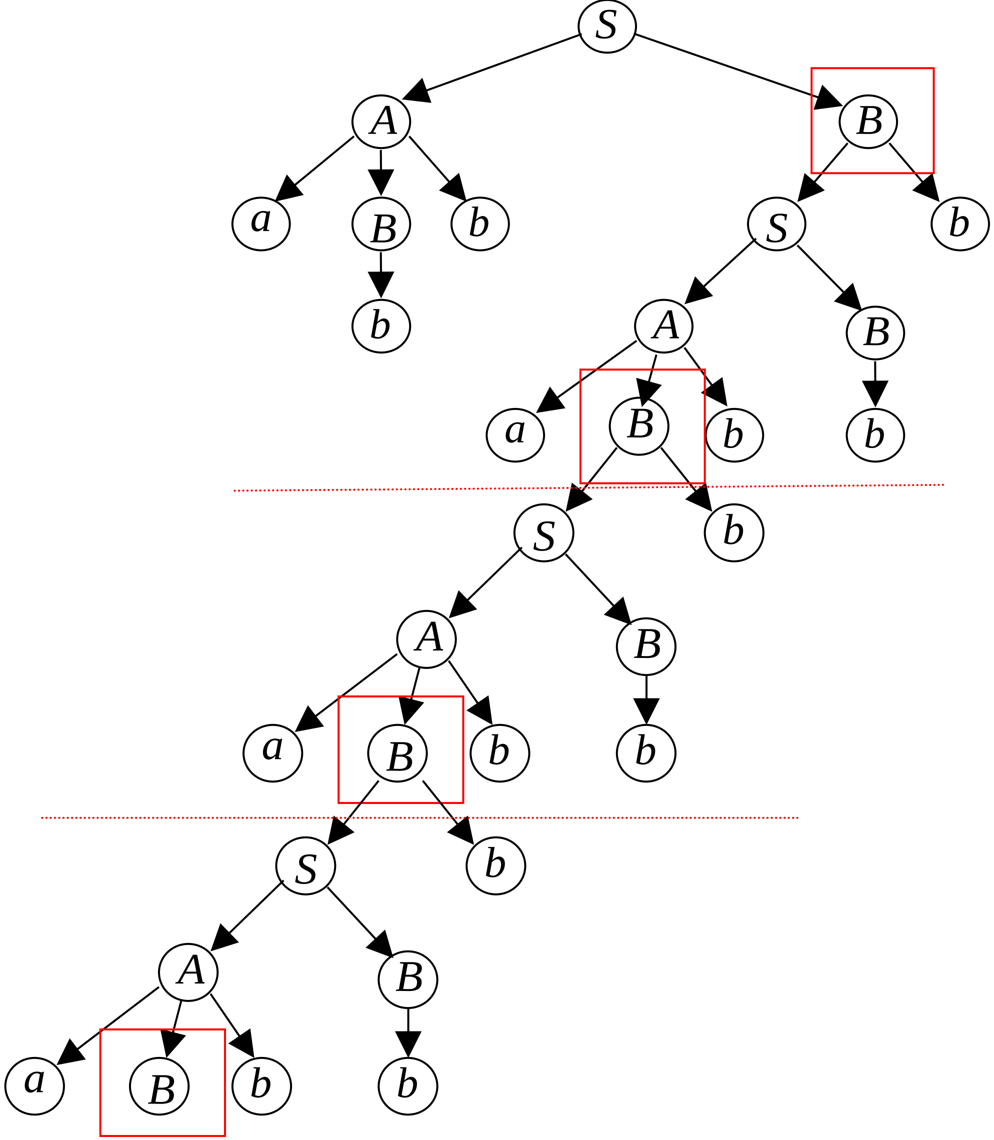


$$S \overset{*}{\Rightarrow} abb(a)^2 B(bbb)^2$$

$$S \overset{*}{\Rightarrow} abbBb$$

$$B \overset{*}{\Rightarrow} aBbbb$$

$$B \Rightarrow b$$

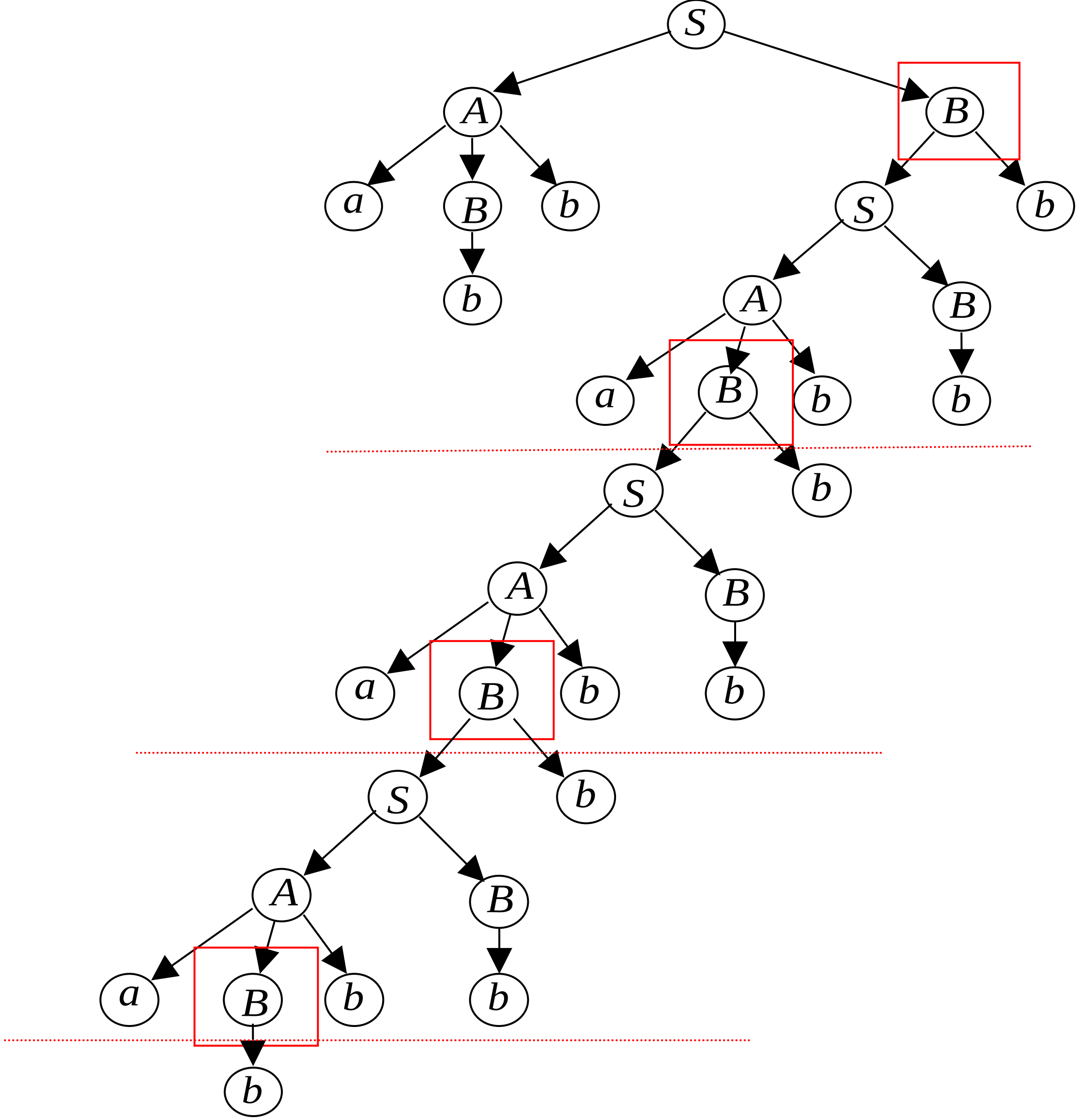


$$S \overset{*}{\Rightarrow} abb(a)^2 B(bbb)^2 \overset{*}{\Rightarrow} abb(a)^3 B(bbb)^3$$

$$S \overset{*}{\Rightarrow} abbBb$$

$$B \overset{*}{\Rightarrow} aBbbb$$

$$B \Rightarrow b$$



$$S \overset{*}{\Rightarrow} abb(a)^3B(bbb)^3 \Rightarrow abb(a)^3b(bbb)^3$$

$$S \overset{*}{\Rightarrow} abbBb \qquad B \overset{*}{\Rightarrow} aBbbb \qquad B \Rightarrow b$$

$$S \overset{*}{\Rightarrow} abb(a)^3 b(bbb)^3$$

$$abb(a)^3 b(bbb)^3 \in L(G)$$

In General:

$$S \overset{*}{\Rightarrow} abbBb \qquad B \overset{*}{\Rightarrow} aBbbb \qquad B \Rightarrow b$$

$$\Downarrow$$

$$S \overset{*}{\Rightarrow} abb(a)^i b(bbb)^i$$

$$\Downarrow$$

$$abb(a)^i b(bbb)^i \in L(G) \qquad i \geq 0$$

Consider now an infinite context-free language L

Let G be the grammar of $L-\{\lambda\}$

Take G so that it has no unit-productions

no λ-productions

Let p = (Number of productions) × (Largest right side of a production)

Let $m = p + 1$

Example $G:$

$$S \rightarrow AB$$

$$A \rightarrow aBb$$

$$B \rightarrow Sb$$

$$B \rightarrow b$$

$$p = 4 \times 3 = 12$$

$$m = p + 1 = 13$$

Take a string $w \in L(G)$

with length $|w| \geq m$

We will show:

in the derivation of w

a variable of G is repeated

$$S \overset{*}{\Rightarrow} w$$

$$v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow v_k \Rightarrow w$$

$$S = v_1$$

$$v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow v_k \Rightarrow w$$

$$|v_i| < |v_{i+1}| + f \longleftarrow$$

maximum right hand side of any production

$$\Downarrow$$

$$|w| < k \cdot f$$

$$\Downarrow$$

$$m \leq |w| \leq k \cdot f \quad \Longrightarrow \quad p < k \cdot f$$

$$v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow v_k \Rightarrow w$$

$$p < k \cdot f$$

$$\Downarrow$$

$$k > \frac{p}{f}$$

Number of productions in grammar

$$v_1 \Rightarrow v_2 \Rightarrow \cdots \Rightarrow v_k \Rightarrow w$$

$k >$ Number of productions in grammar

⇓

Some production must be repeated

$$v_1 \Rightarrow \cdots \Rightarrow a_1 A a_2 \Rightarrow \cdots \Rightarrow a_3 A a_4 \Rightarrow \cdots \Rightarrow w$$

Repeated variable

$$S \to r_1$$

$$A \to r_2$$

$$B \to r_2$$

$$\cdots$$

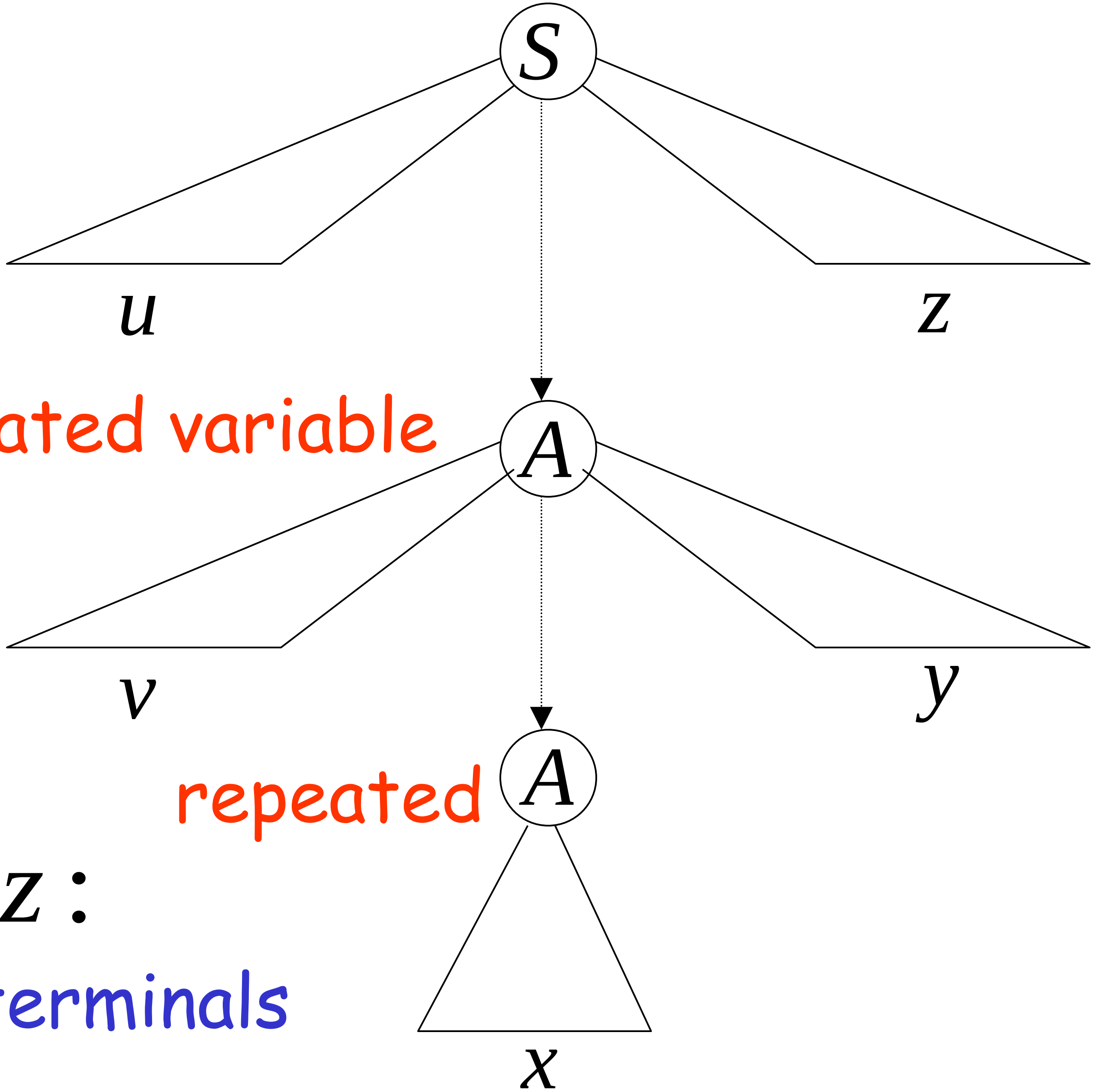
$w \in L(G)$ $\quad |w| \geq m$

Derivation of string w

$$S \Rightarrow \cdots \Rightarrow a_1 A a_2 \Rightarrow \cdots \Rightarrow a_3 A a_4 \Rightarrow \cdots \Rightarrow w$$

Some variable is repeated

Derivation tree of string w



Last repeated variable

repeated

$$w = uvxyz$$

u, v, x, y, z :

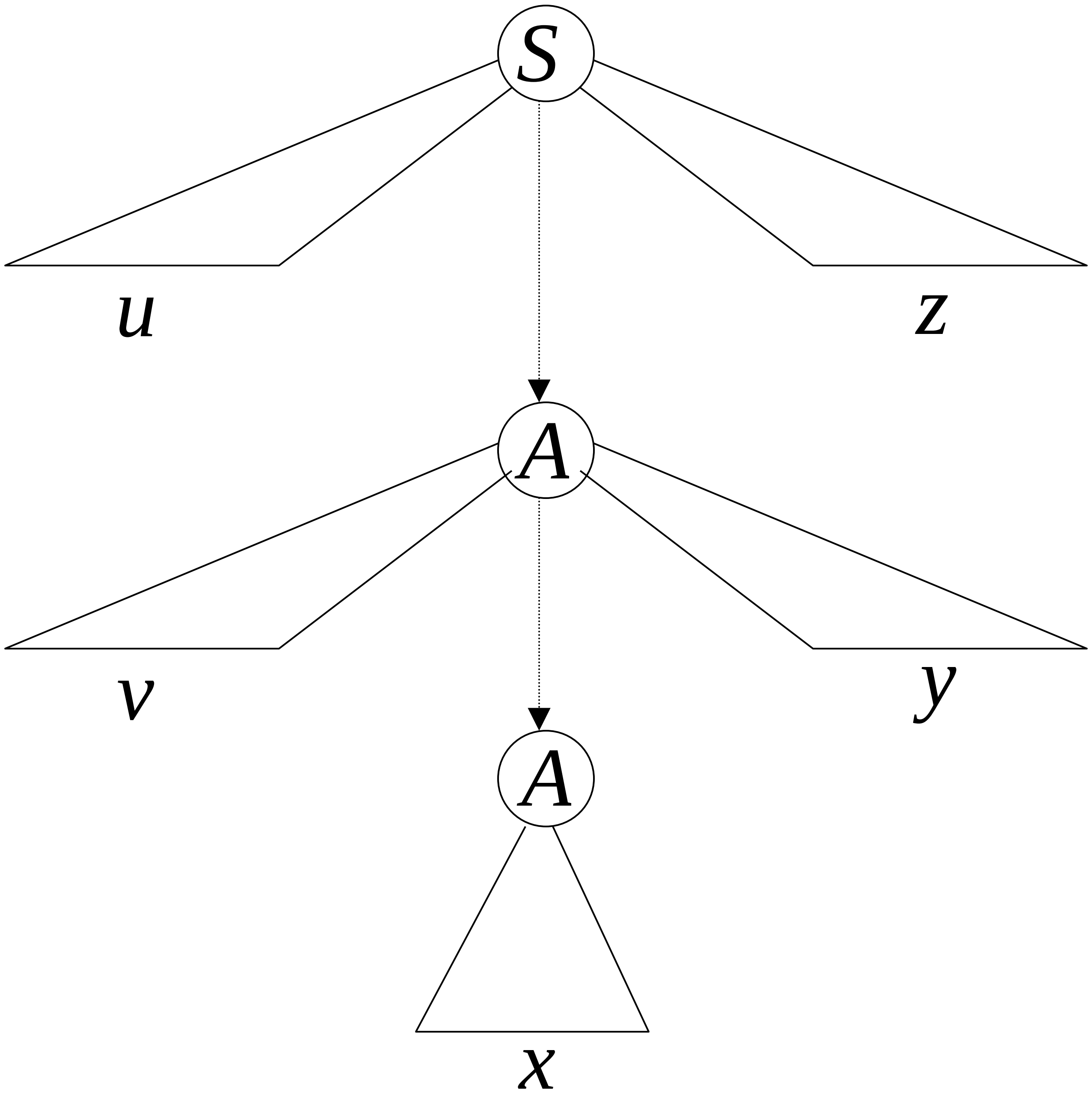
Strings of terminals

Possible derivations:

$$S \overset{*}{\Rightarrow} uAz$$

$$A \overset{*}{\Rightarrow} vAy$$

$$A \overset{*}{\Rightarrow} x$$



We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uxz$$

$$uv^0xy^0z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvxyz$$

The original $w = uv^1xy^1z$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow} uvvxyyz$$

$$uv^2xy^2z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \Rightarrow uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} uvvvxyyyz$$

$$uv^3xy^3z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} \ldots$$

$$\overset{*}{\Rightarrow} uvvv \cdots vAy \cdots yyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvv \cdots vxy \cdots yyyz$$

$$uv^i xy^i z$$

Therefore, any string of the form

$$uv^i xy^i z \qquad i \geq 0$$

is generated by the grammar G

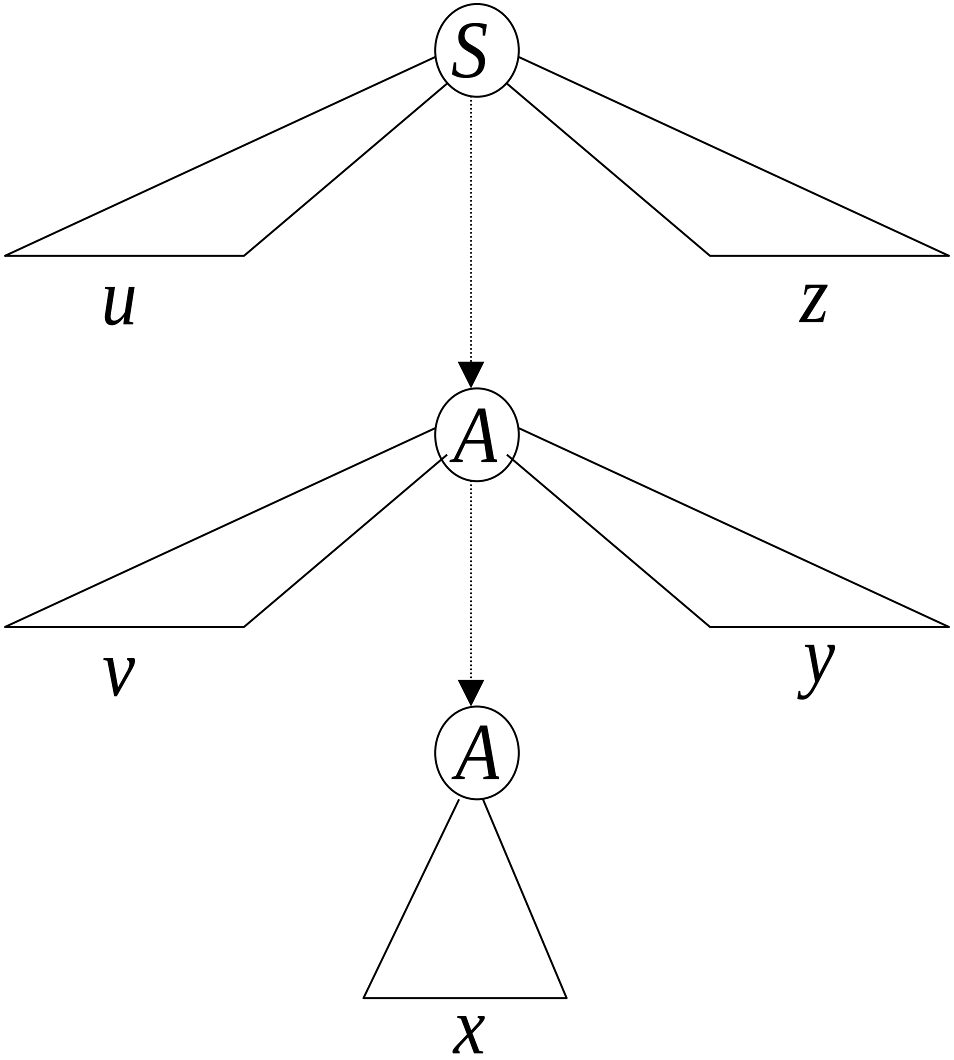
Therefore,

knowing that $uvxyz \in L(G)$

we also know that $uv^ixy^iz \in L(G)$

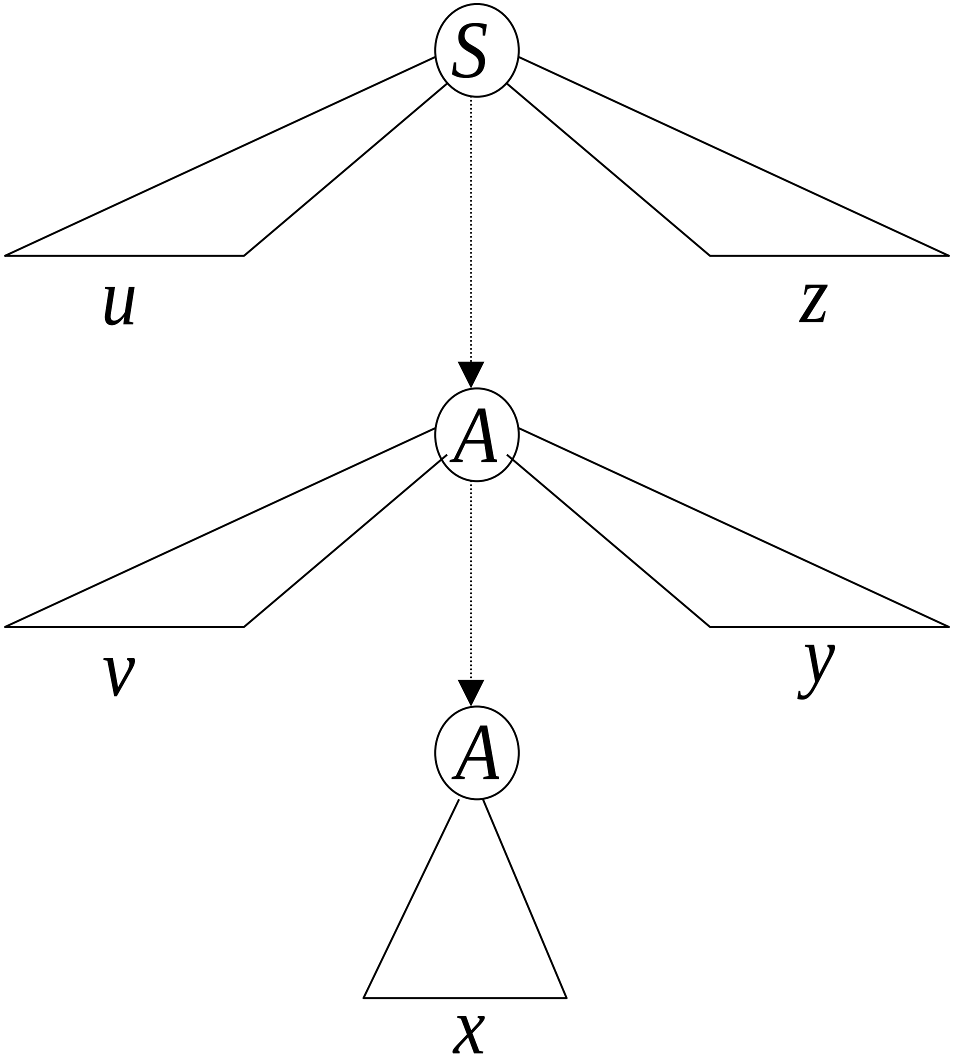
$$L(G) = L - \{\lambda\} \quad \Big\Downarrow$$

$$uv^ixy^iz \in L$$



Observation: $|vxy| \leq m$

Since A is the last repeated variable



Observation: $|vy| \geq 1$

Since there are no unit or λ-productions

The Pumping Lemma:

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

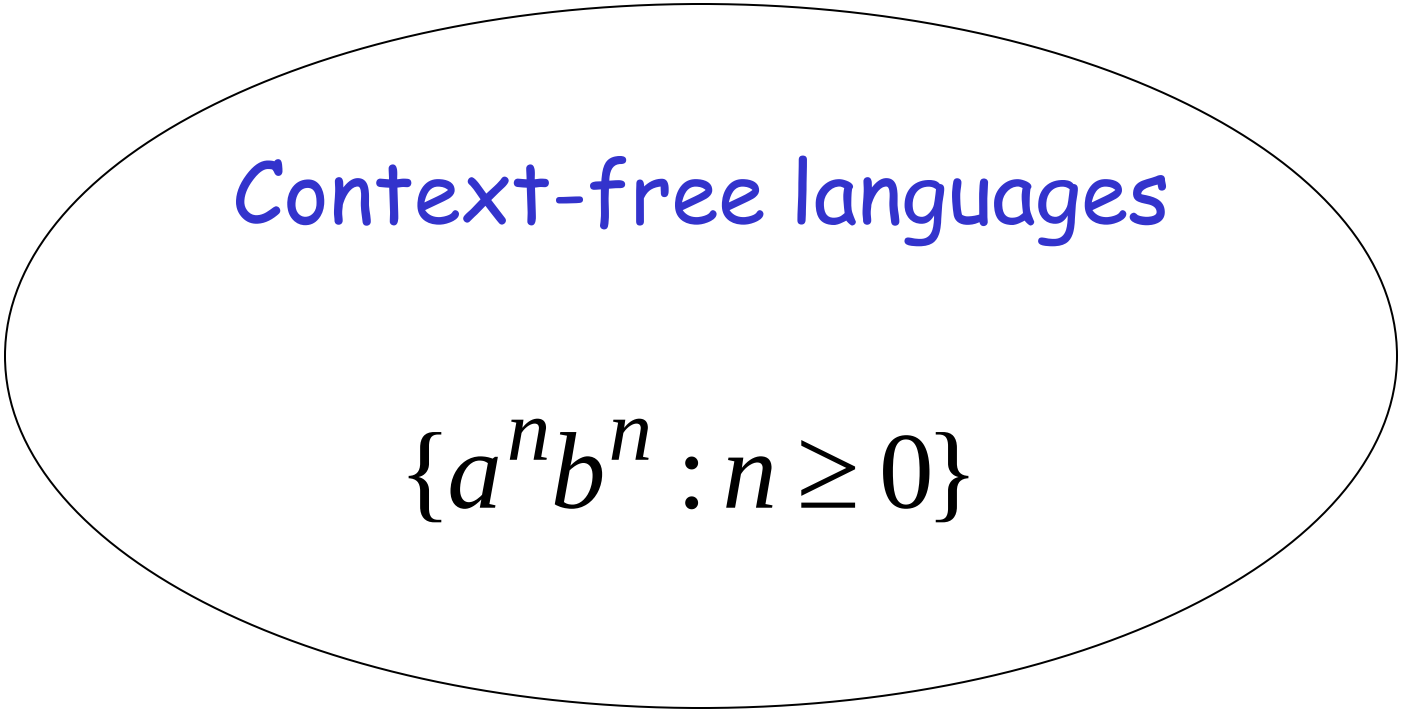
and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Applications of The Pumping Lemma

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$



Context-free languages

$$\{a^n b^n : n \geq 0\}$$

Theorem: The language

$$L = \{a^n b^n c^n : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^n b^n c^n : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^n b^n c^n : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string $w \in L$ with length $|w| \geq m$

We pick: $w = a^m b^m c^m$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

We can write: $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

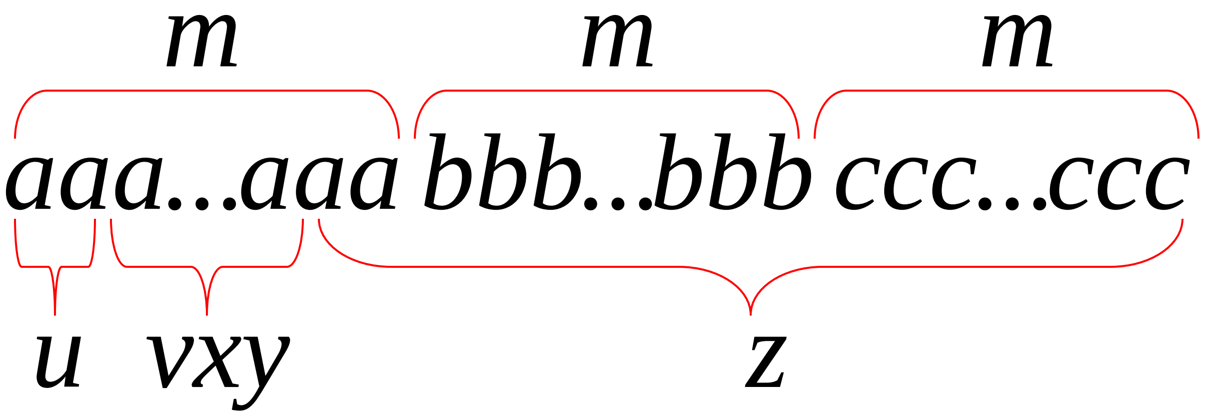
We examine **<u>all</u>** the possible locations of string vxy in w

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within a^m

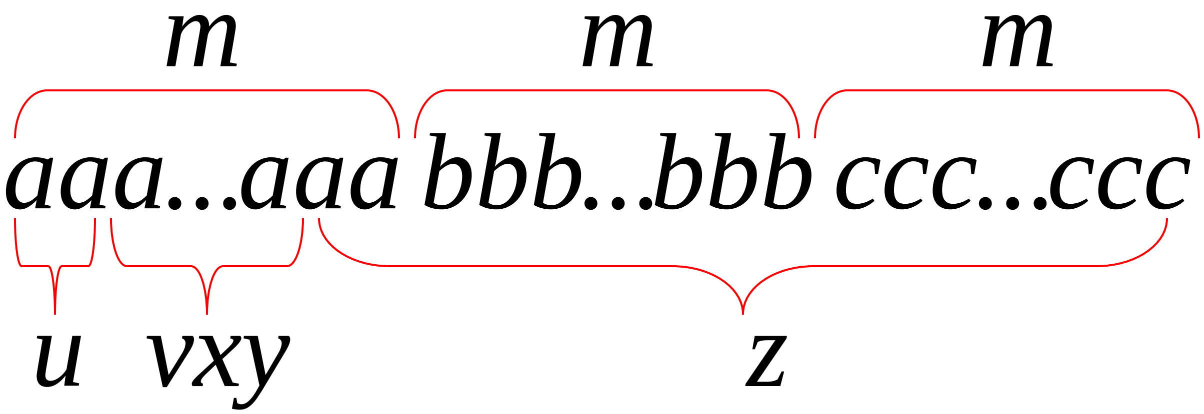


$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: v and y only contain a



$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: Repeating v and y

$$k \geq 1$$

$$\overbrace{\underbrace{a}_{u}\underbrace{aaaaa\ldots aaaaa}_{v^2xy^2}a}^{m+k}\underbrace{\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$$k \geq 1$$

$$\overbrace{aaaaaa\ldots aaaaaa}^{m+k}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{a}_{u}\underbrace{aaaaa\ldots aaaaa}_{v^2xy^2}\underbrace{a\,bbb\ldots bbb\,ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$$k \geq 1$$

However: $uv^2xy^2z = a^{m+k} b^m c^m \notin L$

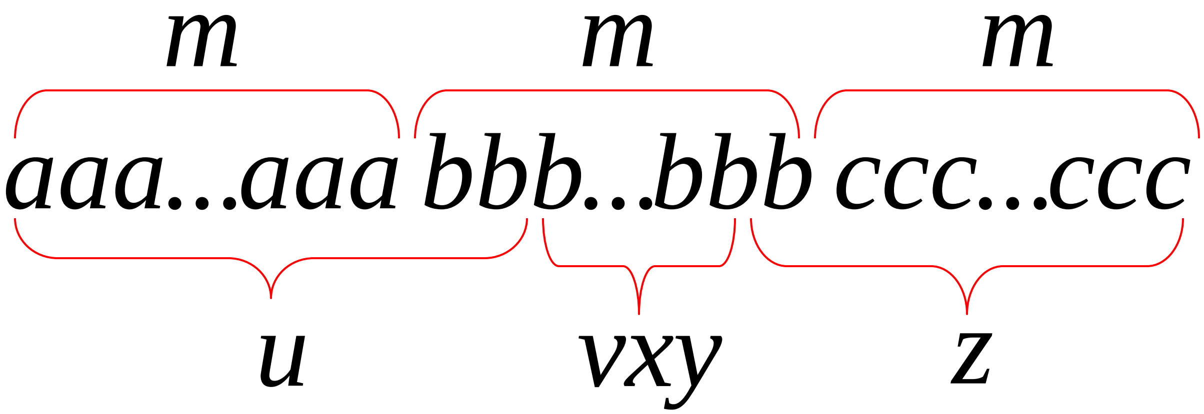
Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: vxy is within b^m

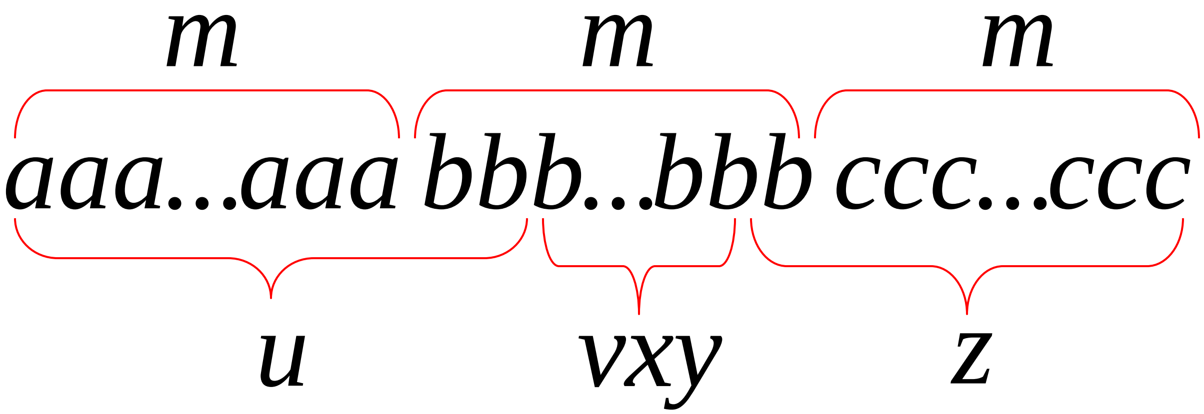


$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: Same analysis as in case 1



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: vxy is within c^m

$$\underbrace{\overbrace{aaa\ldots aaa}^{m}\;\overbrace{bbb\ldots bbb}^{m}}_{u}\;\overbrace{c\,c\,\underbrace{c\ldots c}_{vxy}\,\underbrace{cc}_{z}}^{m}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: Same analysis as in case 1

$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

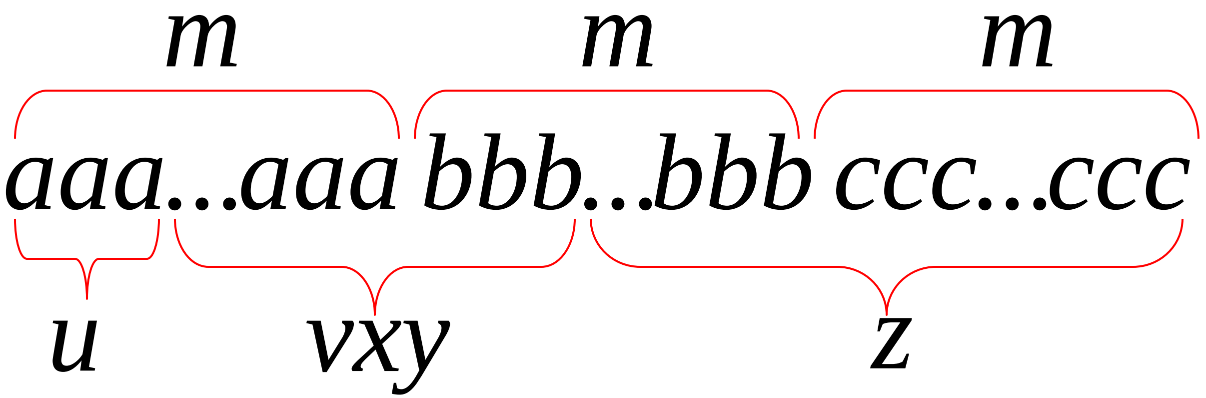
$$\underbrace{aaa\ldots aaa\,bbb\ldots bbb\,c}_{u}\underbrace{cc\ldots c}_{vxy}\underbrace{cc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: vxy overlaps a^m and b^m



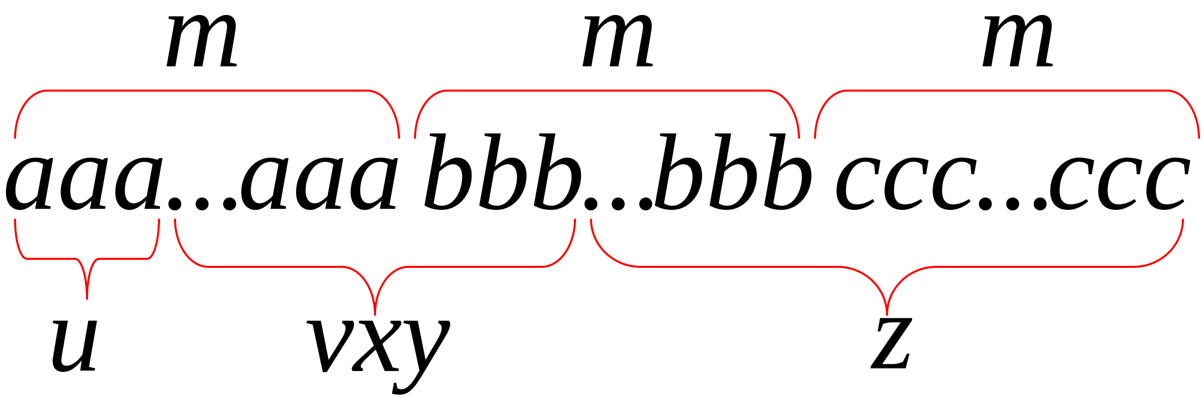
$$L=\{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Subcase 1: v contains only a

y contains only b



$$aaa...aaa\,bbb...bbb\,ccc...ccc$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Subcase 1: v contains only a

y contains only b

$$k_1 + k_2 \geq 1$$

$$\overbrace{aaa\ldots aaaaaaa}^{m+k_1}\overbrace{bbbbbbb\ldots bbb}^{m+k_2}\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaaaaaa\,bbbbbbb}_{v^2xy^2}\underbrace{\ldots bbb\,ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

$$\overbrace{aaa...aaaaaaa}^{m+k_1}\,\overbrace{bbbbbbbb...bbb}^{m+k_2}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaaaaaa\,bbbbbbbb}_{v^2xy^2}\underbrace{...bbb\,ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

However: $uv^2xy^2z = a^{m+k_1} b^{m+k_2} c^m \notin L$

Contradiction!!!

$$L=\{a^n b^n c^n : n \geq 0\}$$

$$w=a^m b^m c^m$$

$$w=uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Subcase 2: v contains a and b

y only contains b

$$\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaa\,bbb}_{vxy}\underbrace{...bbb\,ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

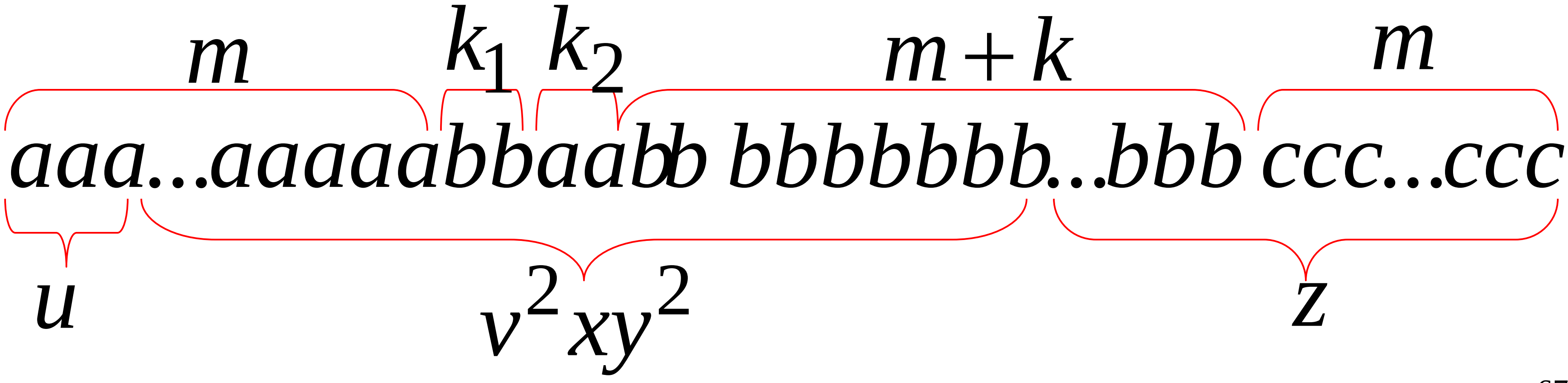
$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Subcase 2: v contains a and b

y only contains b

$$k_1 + k_2 + k \geq 1$$



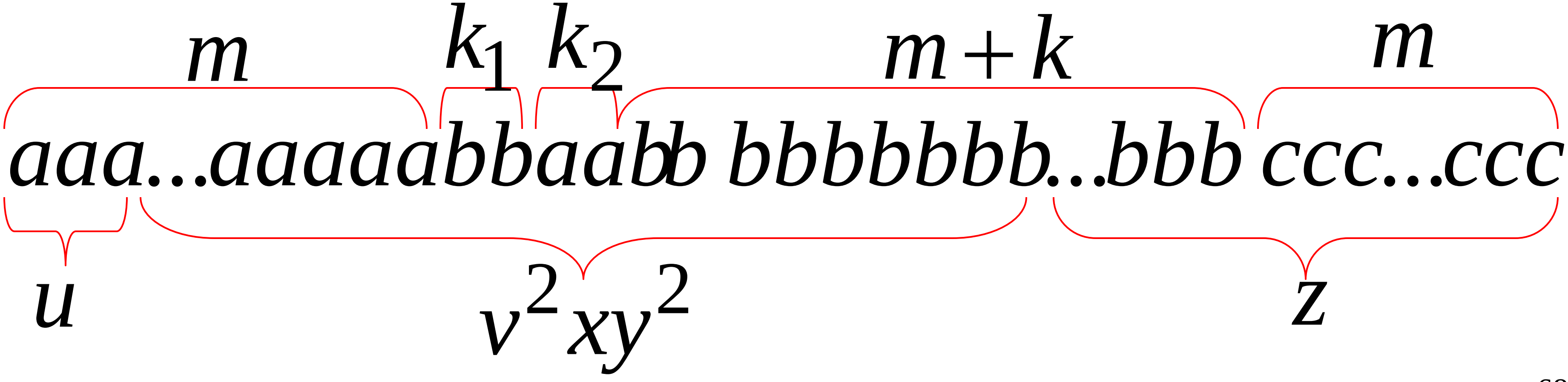
$$L=\{a^nb^nc^n : n\ge 0\}$$

$$w=a^mb^mc^m$$

$$w=uvxyz \qquad |vxy|\le m \qquad |vy|\ge 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z\in L$

$$k_1+k_2+k\ge 1$$



m k_1 k_2 $m+k$ m

$$aaa\ldots aaaaabbaabb\ bbbbbbbb\ldots bbb\ ccc\ldots ccc$$

u v^2xy^2 z

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

However: $k_1 + k_2 + k \geq 1$

$$uv^2xy^2z = a^m b^{k_1} a^{k_2} b^{m+k} c^m \notin L$$

Contradiction!!!

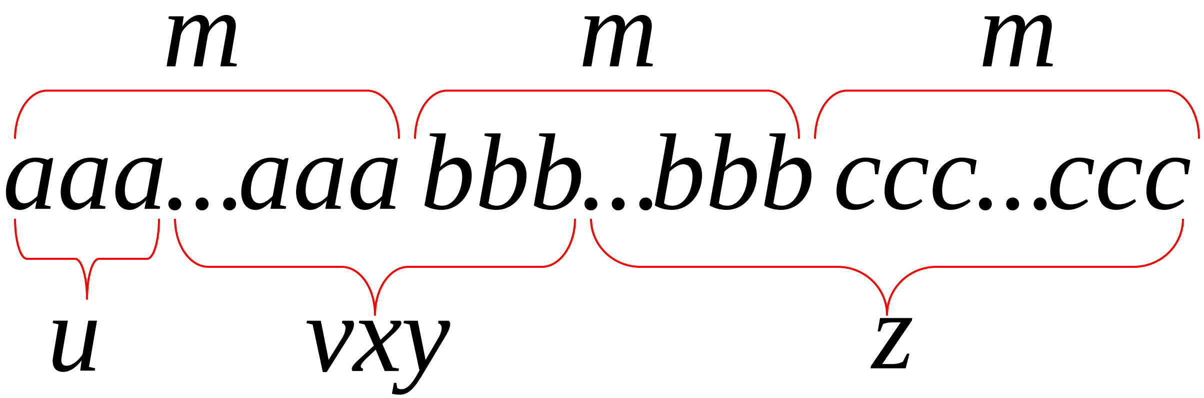
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Subcase 3: v only contains a

y contains a and b



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$w = uvxyz$ $\qquad |vxy| \leq m$ $\qquad |vy| \geq 1$

Case 4: Subcase 3: v only contains a

y contains a and b

Same analysis as for subcase 2

$$L=\{a^n b^n c^n : n \geq 0\}$$

$$w=a^m b^m c^m$$

$$w=uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 5: vxy overlaps b^m and c^m

$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

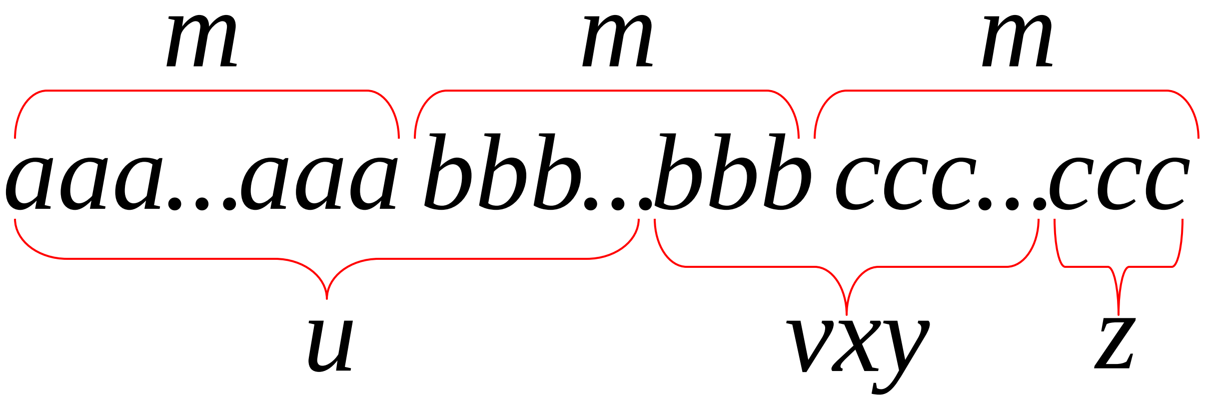
$$\underbrace{aaa\ldots aaa\,bbb\ldots}_{u}\underbrace{bbb\,ccc\ldots}_{vxy}\underbrace{ccc}_{z}$$

$$L=\{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 5: Same analysis as in case 4



There are no other cases to consider

(since $|vxy| \leq m$, string vxy cannot

overlap a^m, b^m and c^m at the same time)

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^n b^n c^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free