Formal Languages

Decidability

Consider problems with answer YES or NO

Examples:

- Does Machine M have three states ?

- Is string w a binary number?

- Does DFA M accept any input?

A problem is decidable if some Turing machine decides (solves) the problem

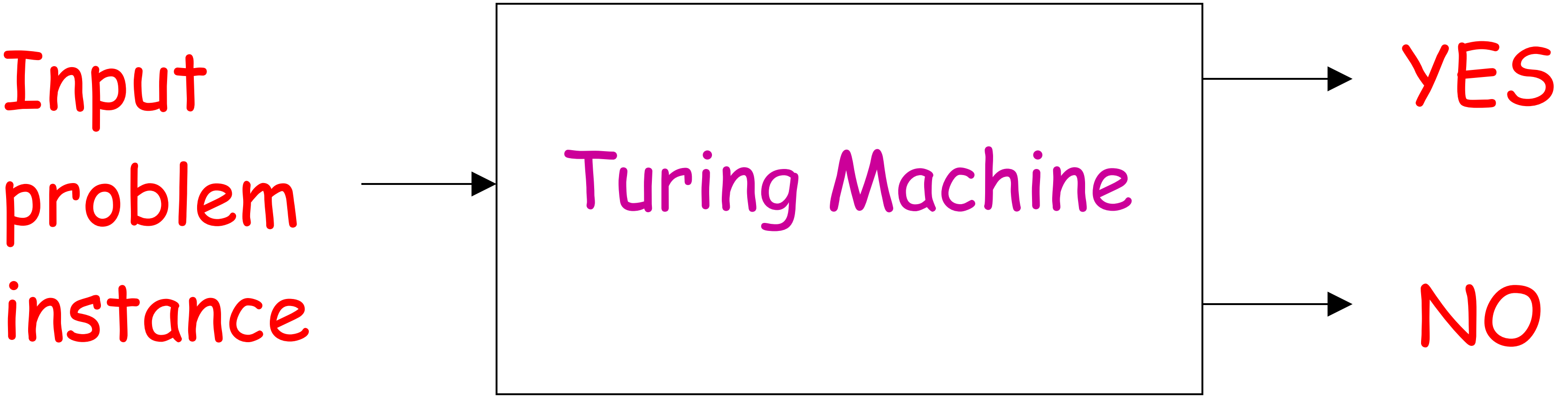
Decidable problems:

- Does Machine M have three states ?

- Is string w a binary number?

- Does DFA M accept any input?

The Turing machine that decides (solves) a problem answers YES or NO for each instance of the problem



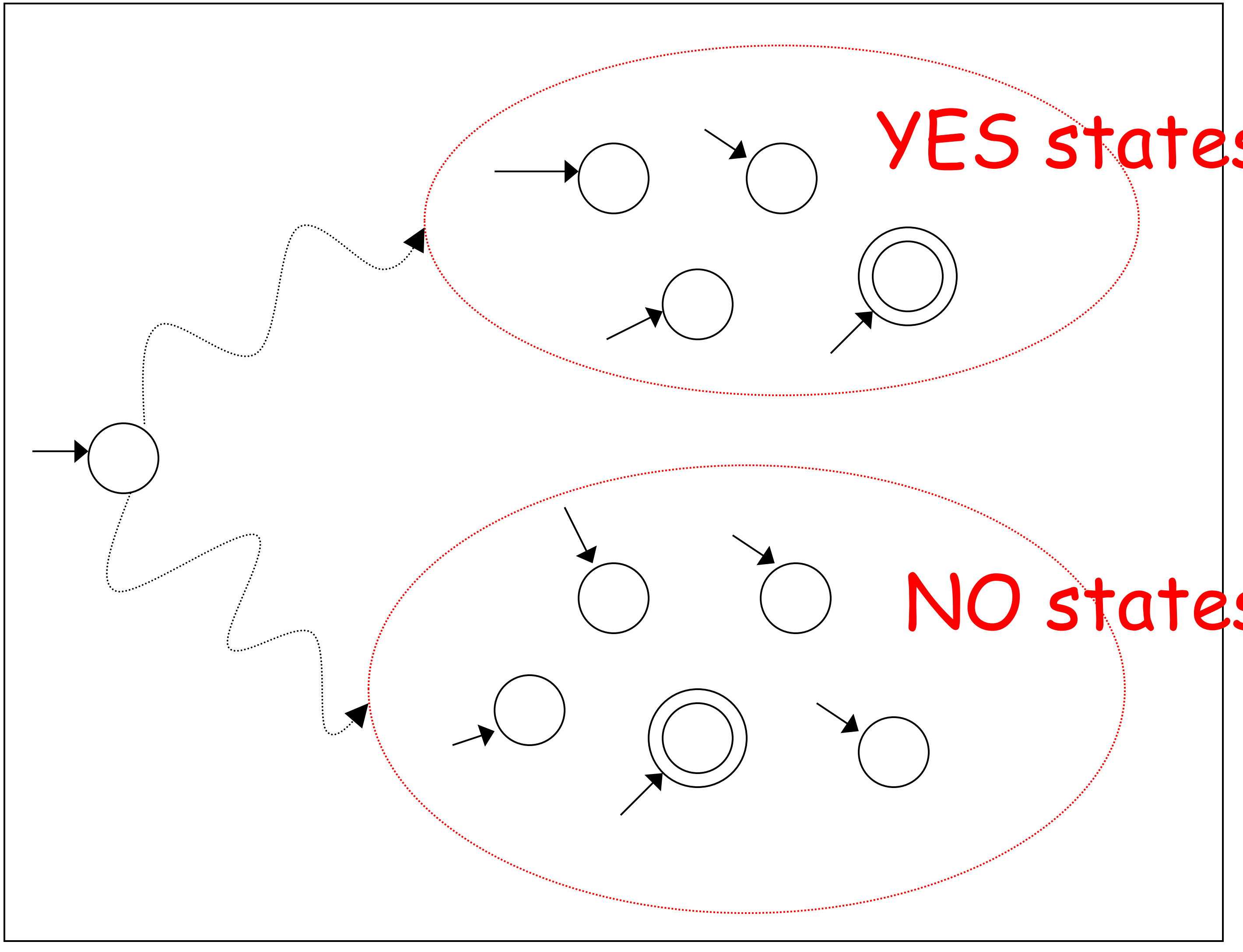
The machine that decides (solves) a problem:

- If the answer is YES
  then halts in a yes state

- If the answer is NO
  then halts in a no state

These states do not have to be final states

Turing Machine that decides a problem



YES and NO states are halting states

Difference between
Recursive Languages and Decidable problems

For decidable problems:

The YES states do not have to be final states

Some problems are undecidable:

which means:
there is no Turing Machine that
solves all instances of the problem

A simple undecidable problem:

The membership problem

The Membership Problem

Input:
- Turing Machine M
- String w

Question: Does M accept w ?

$$w \in L(M)?$$

Theorem:

The membership problem is undecidable

(there are M and w for which we cannot decide whether $w \in L(M)$)

Proof?

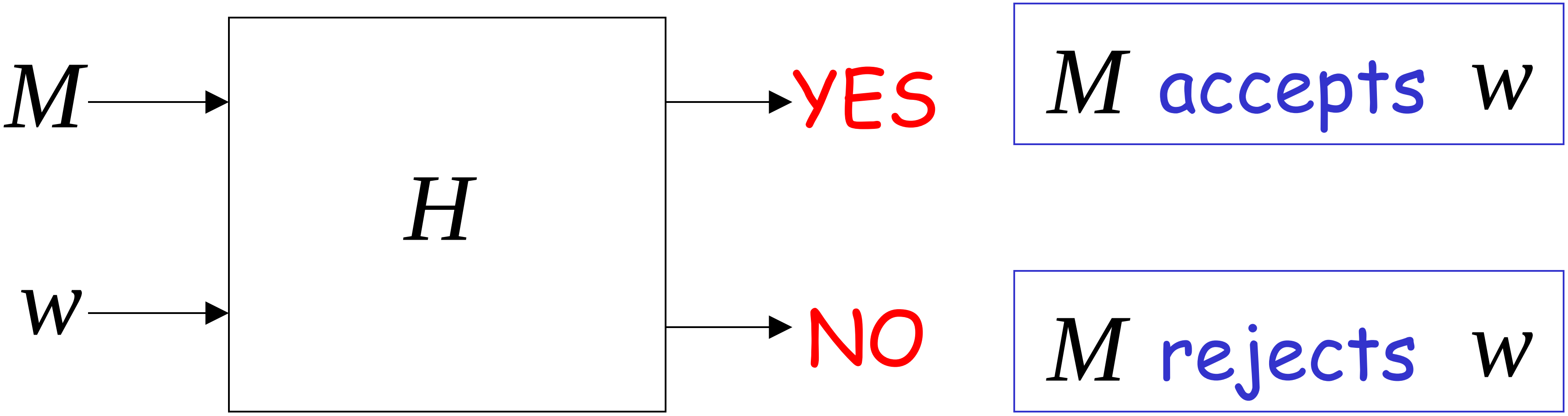
Theorem:

The membership problem is undecidable

(there are M and w for which we cannot decide whether $w \in L(M)$)

Proof: Assume for contradiction that the membership problem is decidable

Thus, there exists a Turing Machine H that solves the membership problem



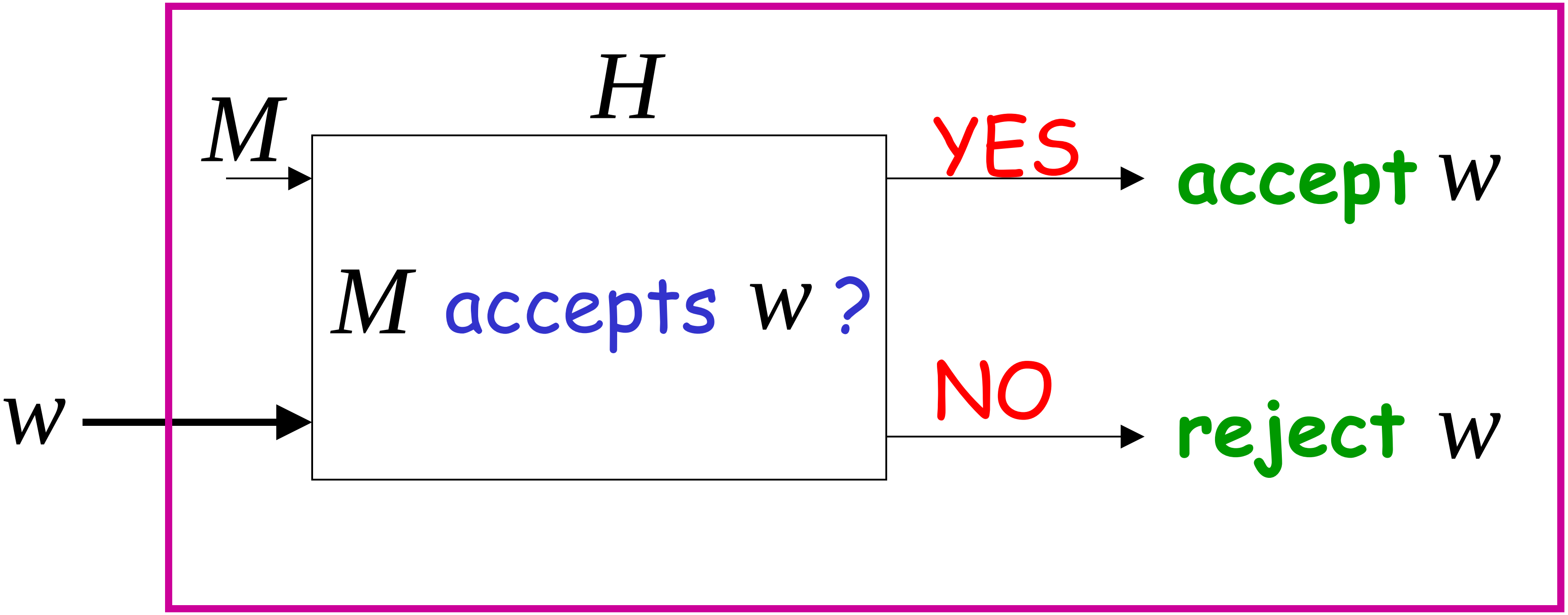
Let L be a recursively enumerable language

Let M be the Turing Machine that accepts L

We will prove that L is also recursive:

we will describe a Turing machine that accepts L and halts on any input

Turing Machine that accepts L
and halts on any input



Therefore, L is recursive

Since L is chosen arbitrarily, every recursively enumerable language is also recursive

But there are recursively enumerable languages which are not recursive

Contradiction!!!!

Therefore, the membership problem is undecidable

END OF PROOF

Another famous undecidable problem:

The halting problem

The Halting Problem

Input:
- Turing Machine M
- String w

Question: Does M halt on input w ?

Theorem:

The halting problem is undecidable

(there are M and w for which we cannot decide whether M halts on input w)

Proof?

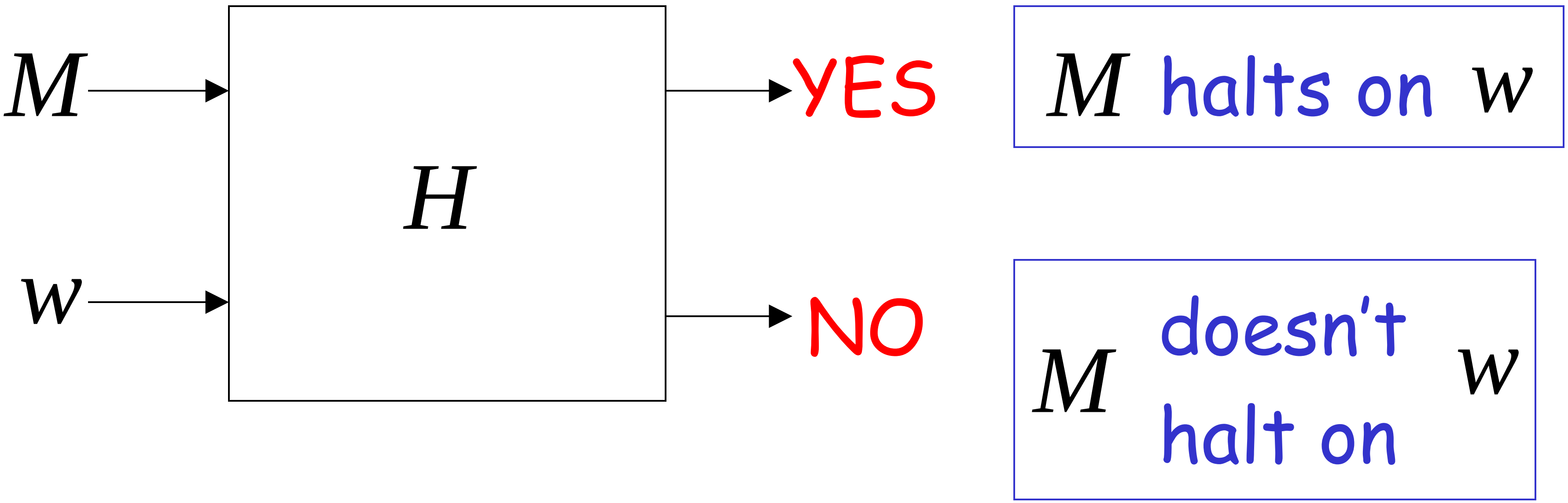
Theorem:

The halting problem is undecidable

(there are M and w for which we cannot decide whether M halts on input w)

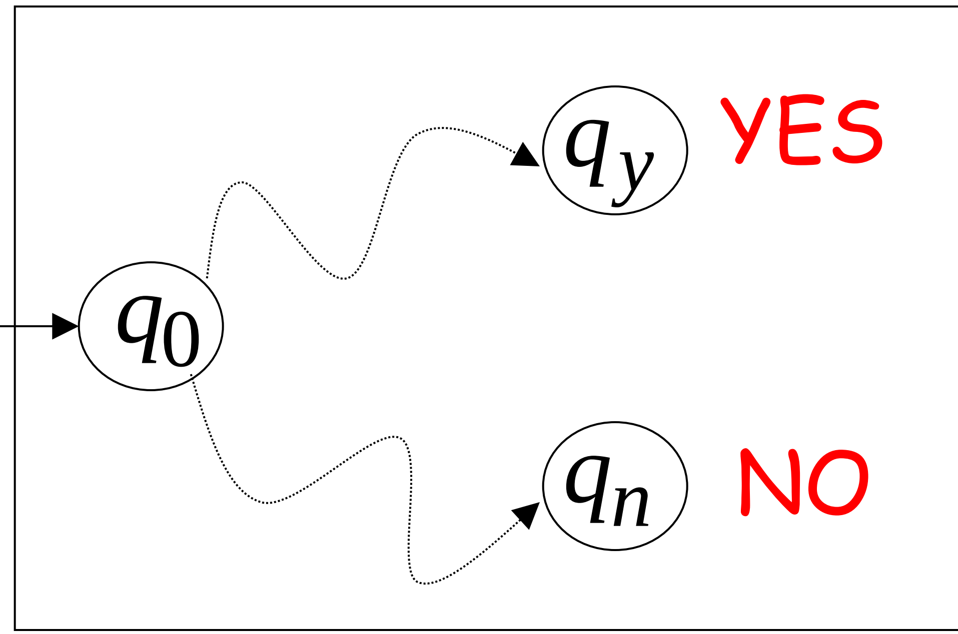
Proof: Assume for contradiction that the halting problem is decidable

Thus, there exists Turing Machine H that solves the halting problem



Construction of H

H



Input:
initial tape contents

$w_M \; w$

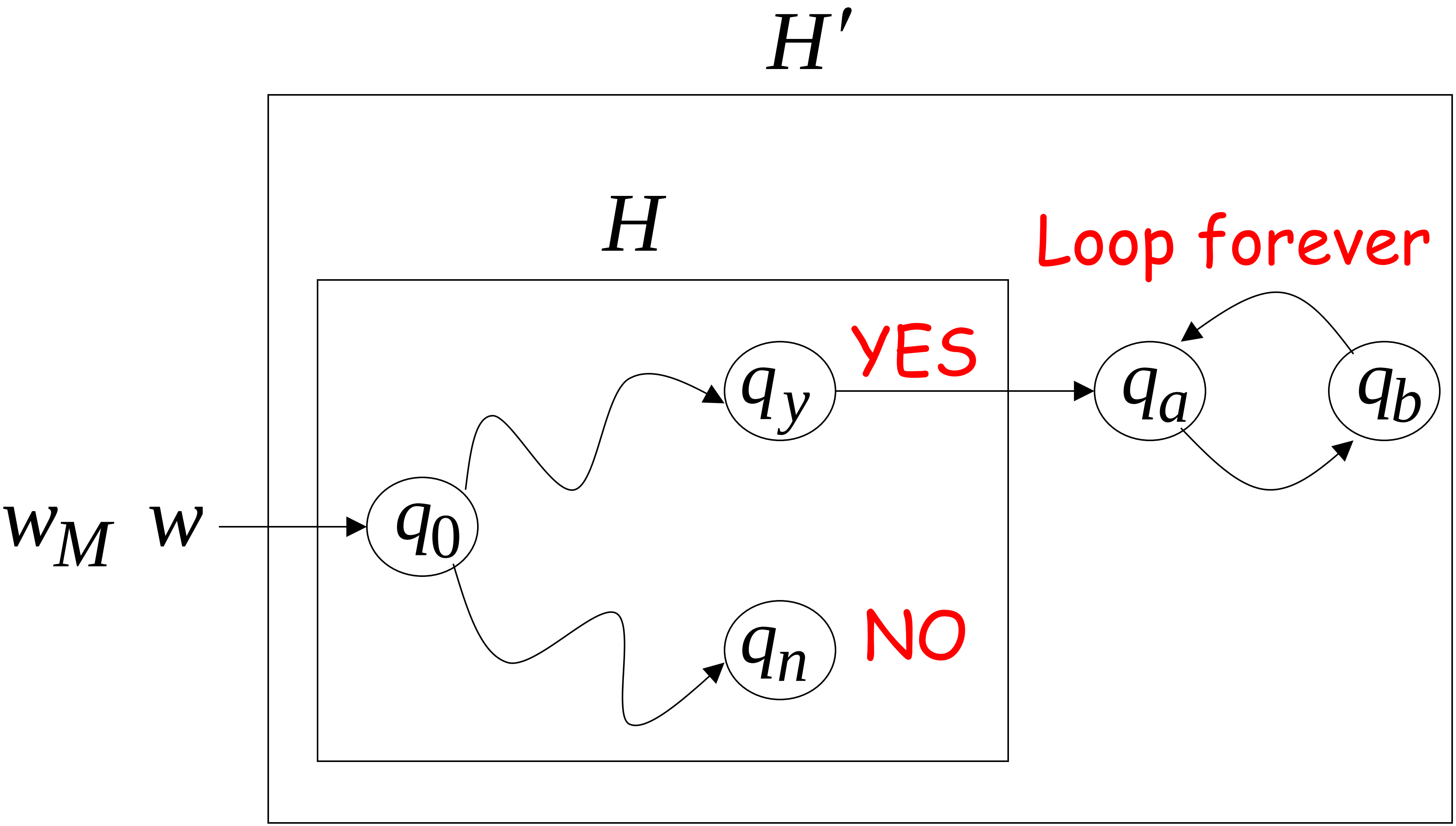
Encoding of M

String w

Construct machine H' :

If H returns YES then loop forever

If H returns NO then halt


H'
H
Loop forever
YES
NO
q_y
q_a
q_b
q_0
q_n
w_M w

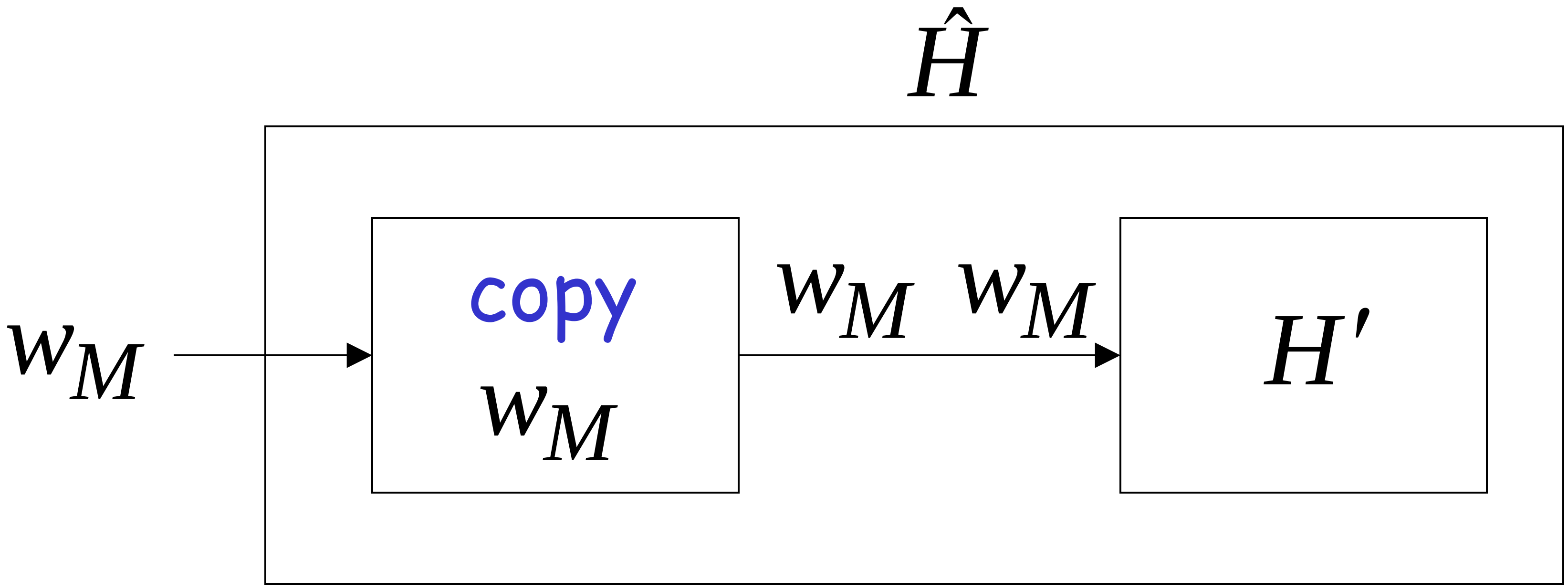
Construct machine $\hat{H}$:

Input: w_M (machine M)

If M halts on input w_M

Then loop forever

Else halt


$\hat{H}$
w_M
copy
w_M
w_M
w_M
H'

Run machine $\hat{H}$ on itself:

Input: $w_{\hat{H}}$ (machine $\hat{H}$)

If $\hat{H}$ halts on input $w_{\hat{H}}$

<u>**Then**</u> it will loop forever

<u>**Else**</u> it will halt

$\hat{H}$ on input $w_{\hat{H}}$:

If $\hat{H}$ halts then it loops forever

If $\hat{H}$ doesn't halt then it halts

Contradiction !!!!!

Therefore, we have a contradiction

The halting problem is undecidable

END OF PROOF

Another proof of the same theorem:

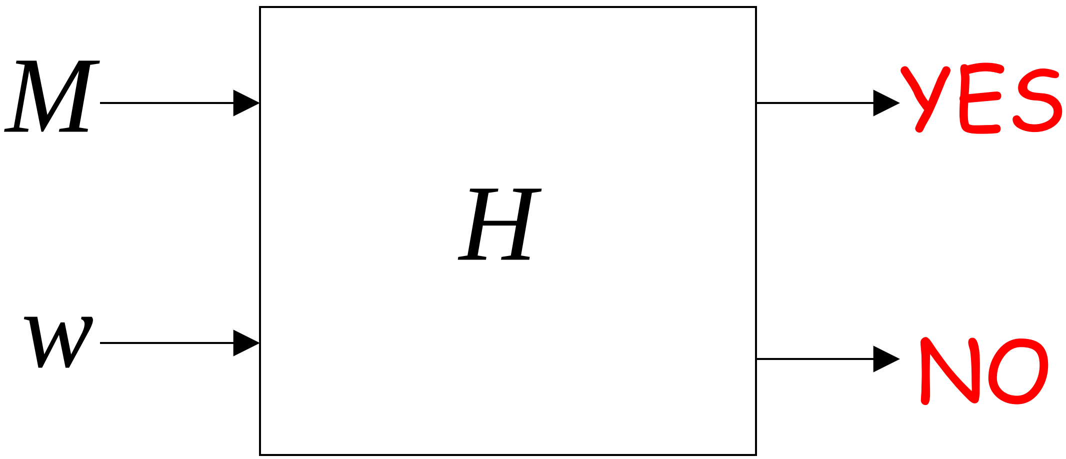
If the halting problem were decidable then every recursively enumerable language would be recursive

Theorem:

The halting problem is undecidable

Proof: Assume for contradiction that the halting problem is decidable

There exists Turing Machine H
that solves the halting problem



M halts on w

M doesn't halt on w

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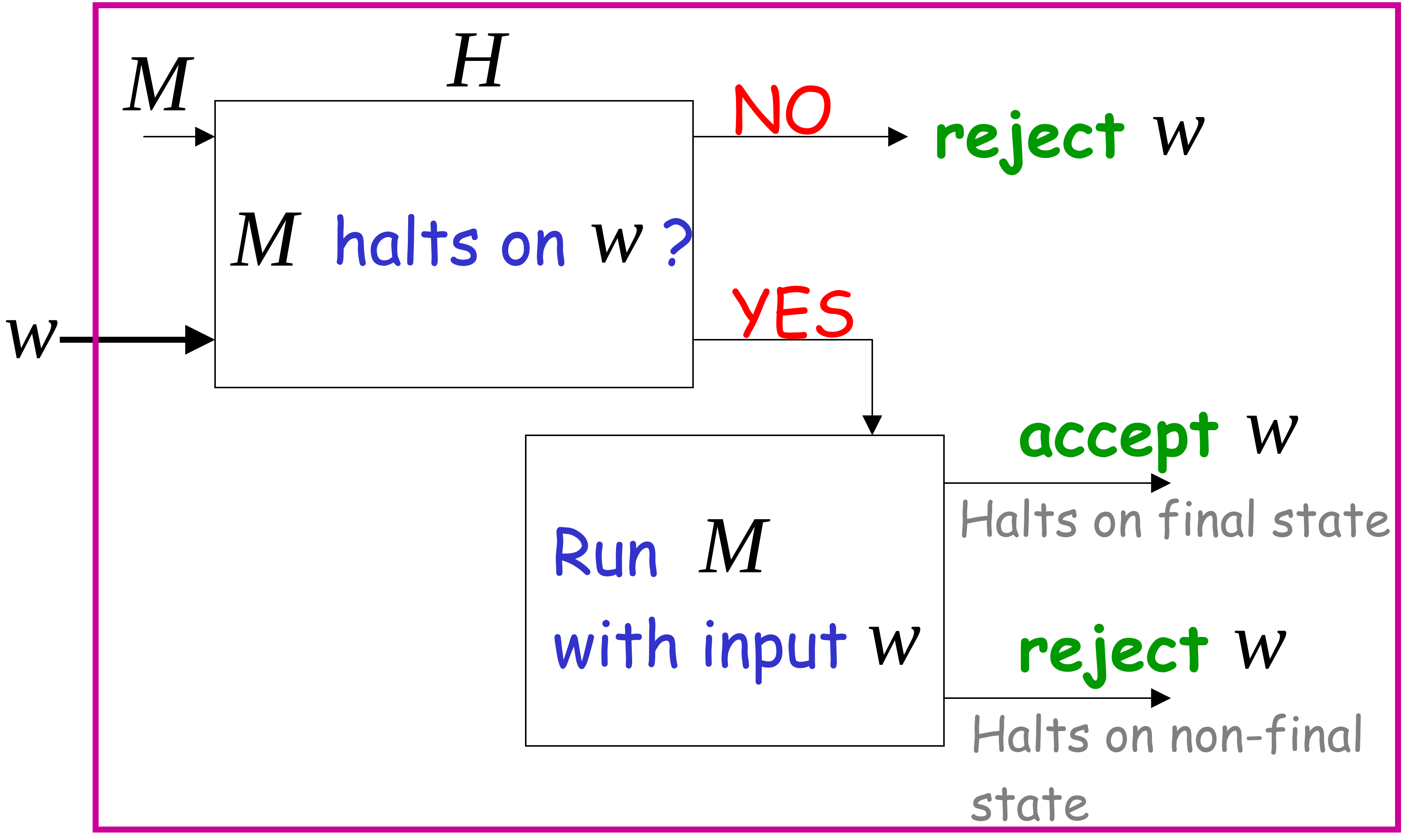
Turing Machine that accepts L and halts on any input



Therefore L is recursive

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But there are recursively enumerable languages which are not recursive

Contradiction!!!!

Therefore, the halting problem is undecidable

END OF PROOF