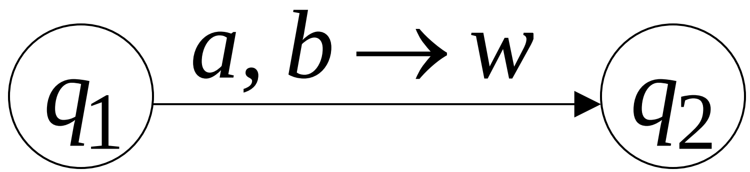
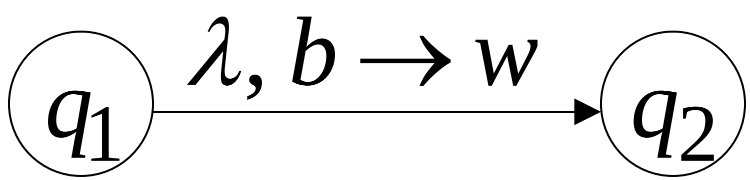
Formal Languages

Deterministic PDAs

Deterministic PDA: DPDA

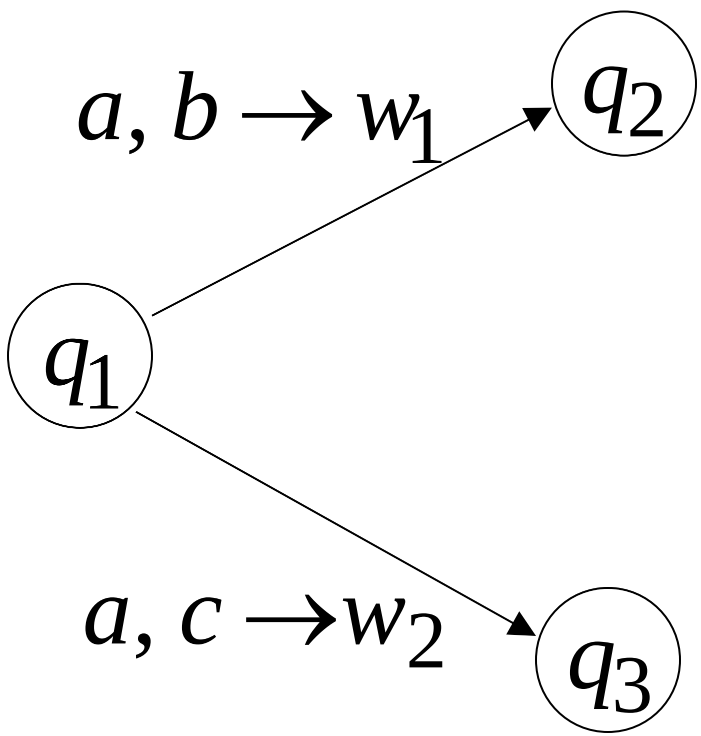
Allowed transitions:

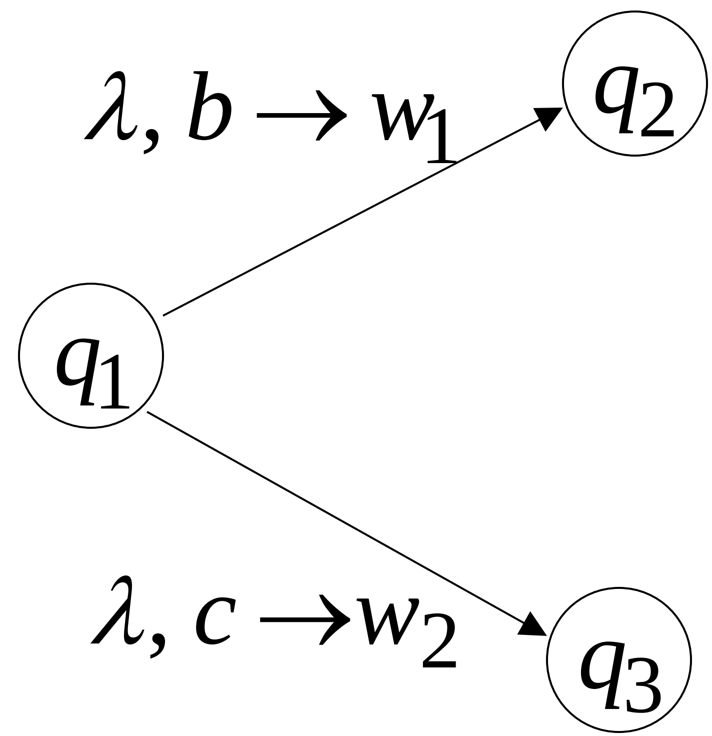




(deterministic choices)

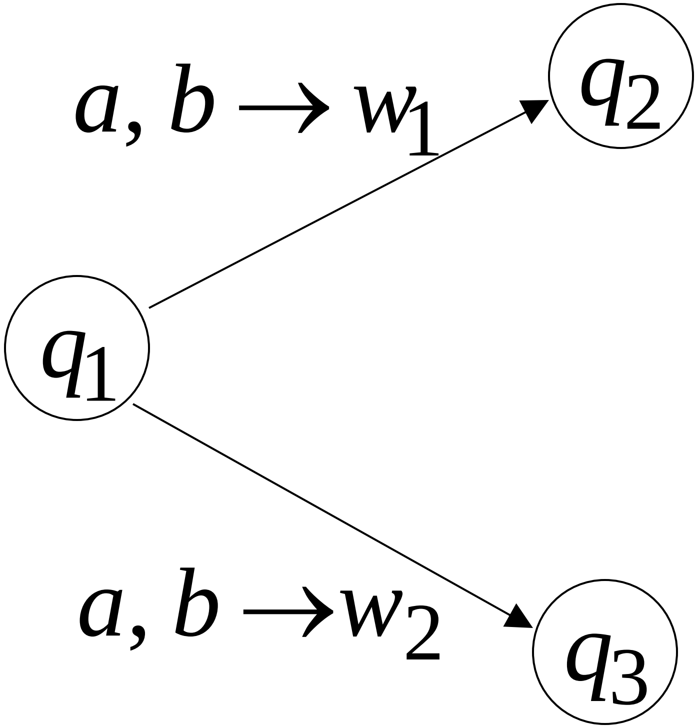
Allowed transitions:

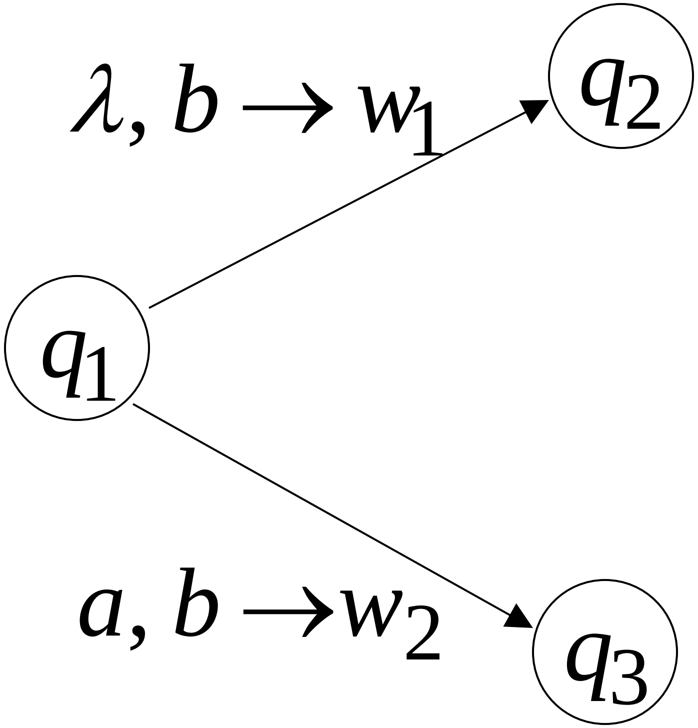




(deterministic choices)

Not allowed:

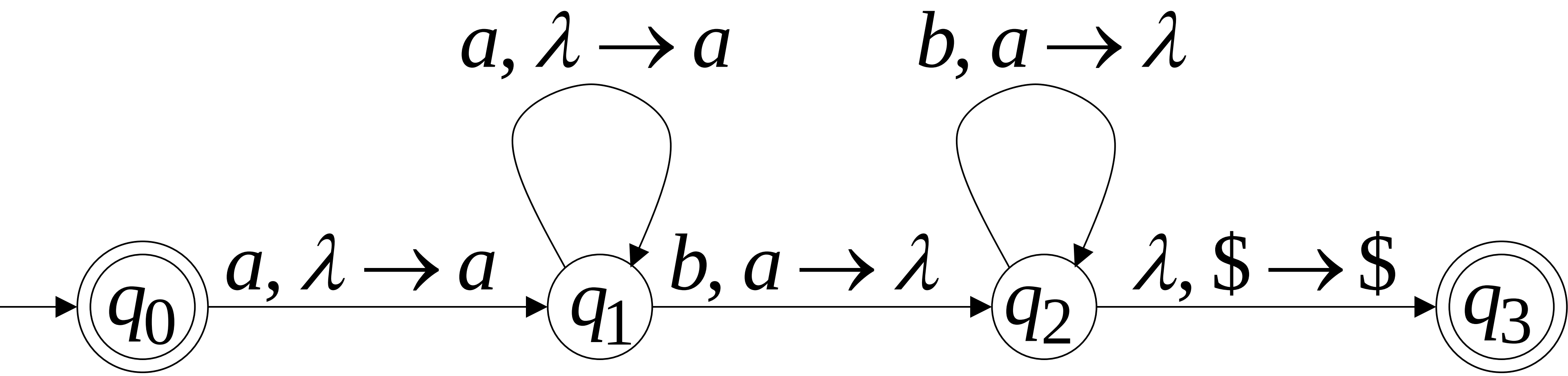




(non deterministic choices)

DPDA example

$$L(M) = \{a^n b^n : n \geq 0\}$$



Definition:

A language L is **deterministic context-free** if there exists some DPDA that accepts it

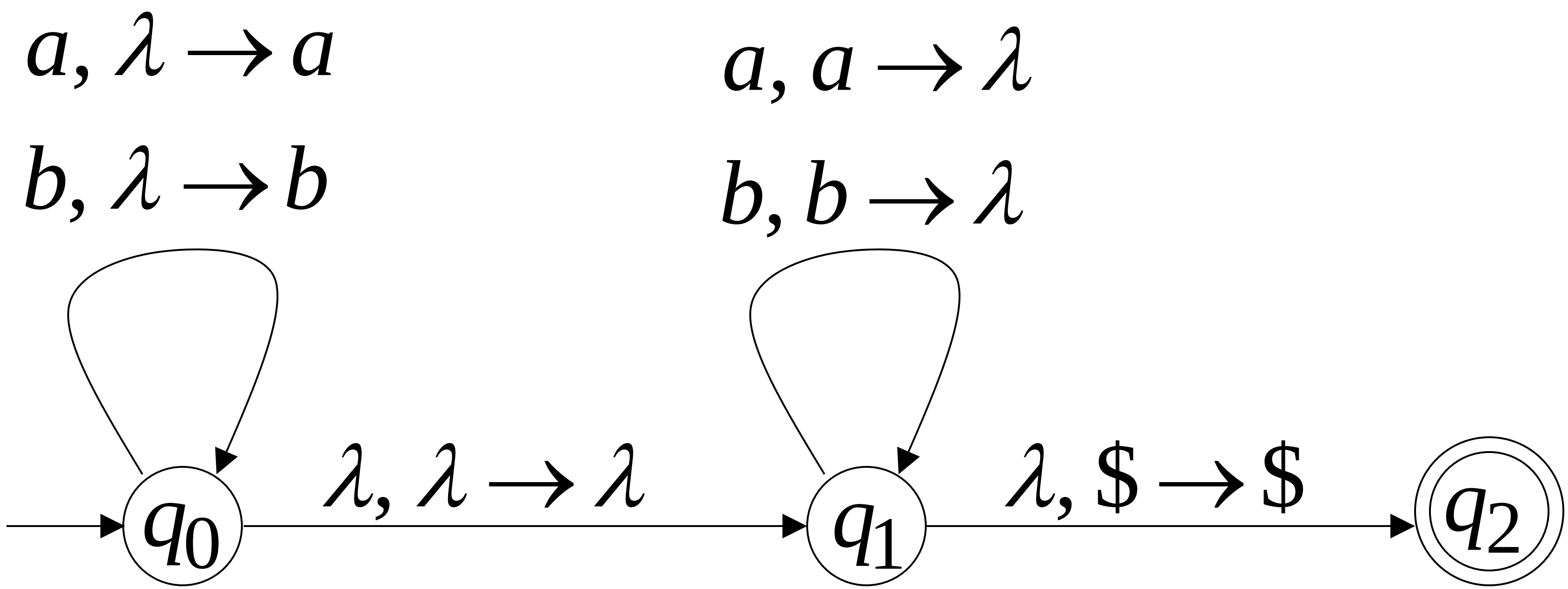
Example:

The language $L(M) = \{a^n b^n : n \geq 0\}$

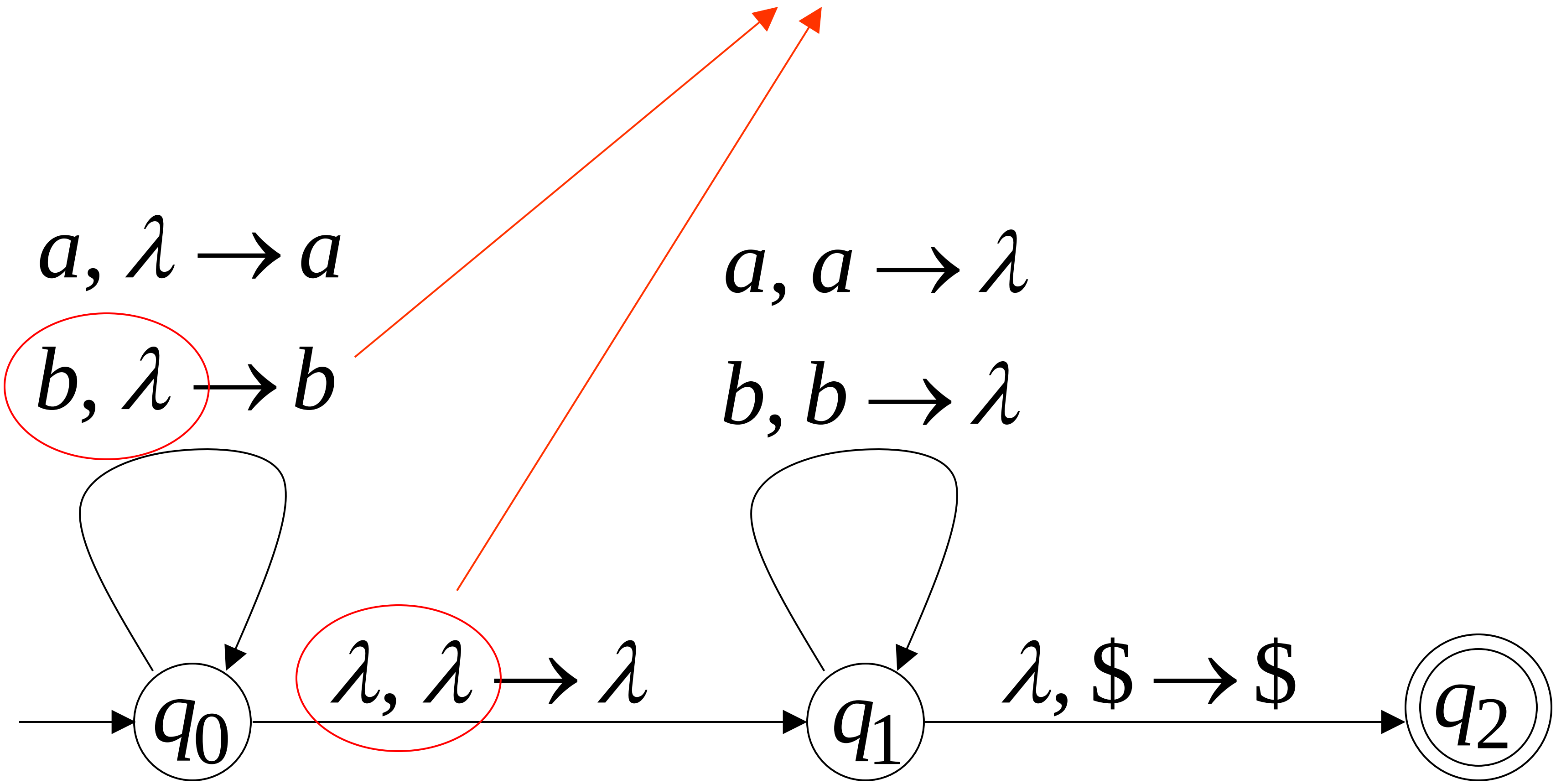
is **deterministic context-free**

Example of Non-DPDA (PDA)

$$L(M)=\{vv^R:\ v\in\{a,b\}^*\}$$



Not allowed in DPDAs



PDAs

Have More Power than

DPDAs

It holds that:

$$\left\{\begin{array}{c}\text{Deterministic}\\ \text{Context-Free}\\ \text{Languages}\\ \text{(DPDA)}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{Context-Free}\\ \text{Languages}\\ \text{PDAs}\end{array}\right\}$$

Since every DPDA is also a PDA

We will actually show:

$$L \notin \left\{ \begin{array}{c} \text{Deterministic} \\ \text{Context-Free} \\ \text{Languages} \\ \text{(DPDA)} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{Context-Free} \\ \text{Languages} \\ \text{(PDA)} \end{array} \right\} \ni L$$

We will show that there exists a context-free language L which is not accepted by any DPDA

The language is:

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\} \qquad n \geq 0$$

We will show:

- L is context-free
- L is **not** deterministic context-free

Proof?

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

Language L is context-free

Context-free grammar for L :

$$S \rightarrow S_1 \mid S_2 \qquad \{a^n b^n\} \cup \{a^n b^{2n}\}$$

$$S_1 \rightarrow a S_1 b \mid \lambda \qquad \{a^n b^n\}$$

$$S_2 \rightarrow a S_2 bb \mid \lambda \qquad \{a^n b^{2n}\}$$

Theorem:

The language $L = \{a^n b^n\} \cup \{a^n b^{2n}\}$

is **not** deterministic context-free

(there is **no** DPDA that accepts L)

Proof: Assume for contradiction that

$$L=\{a^n b^n\}\cup\{a^n b^{2n}\}$$

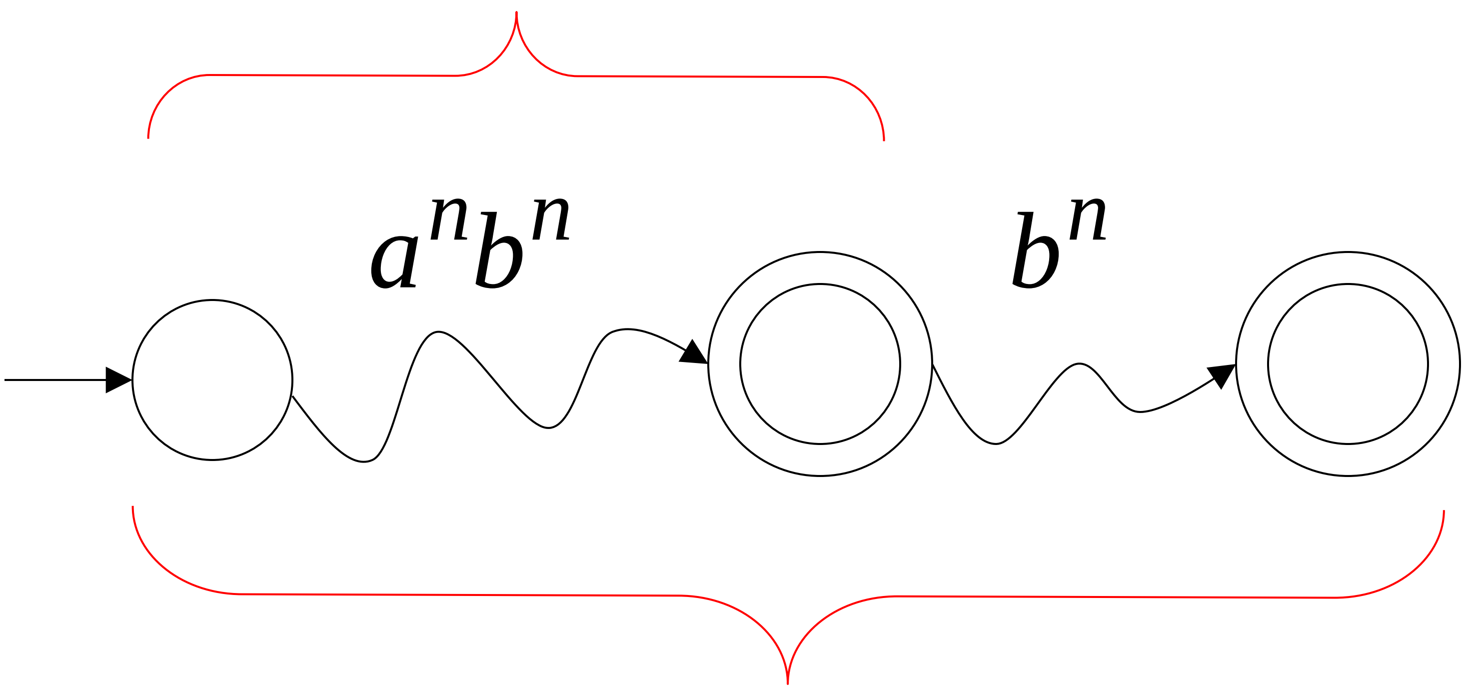
is deterministic context free

Therefore:

there is a DPDA M that accepts L

DPDA M with $L(M) = \{a^n b^n\} \cup \{a^n b^{2n}\}$

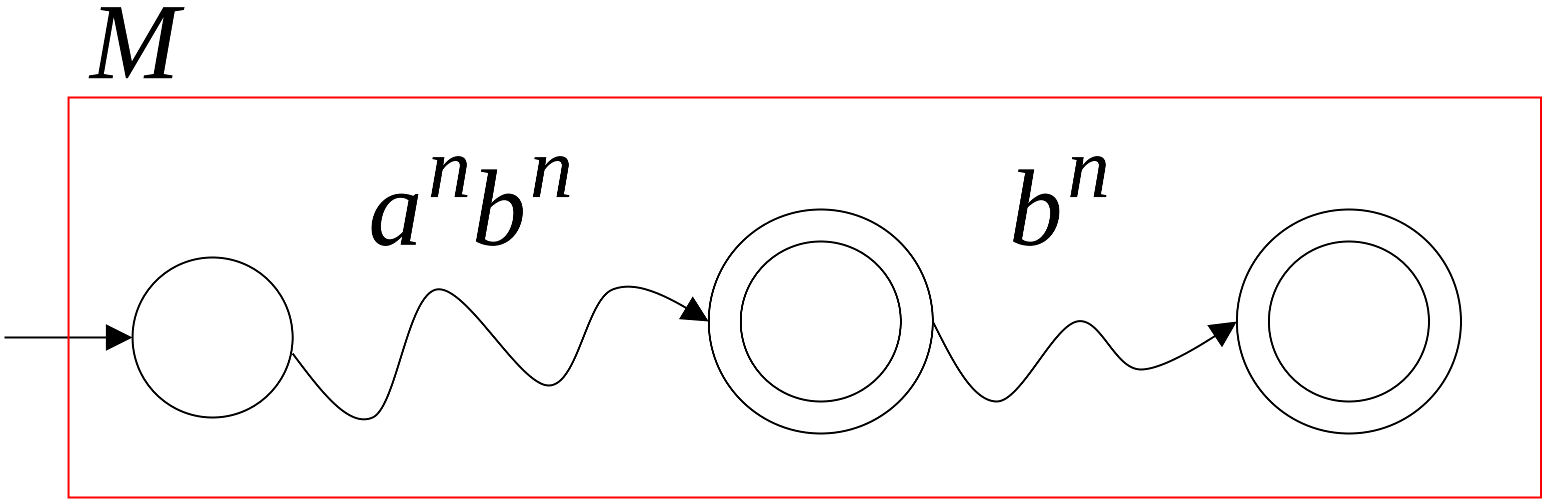
accepts $a^n b^n$



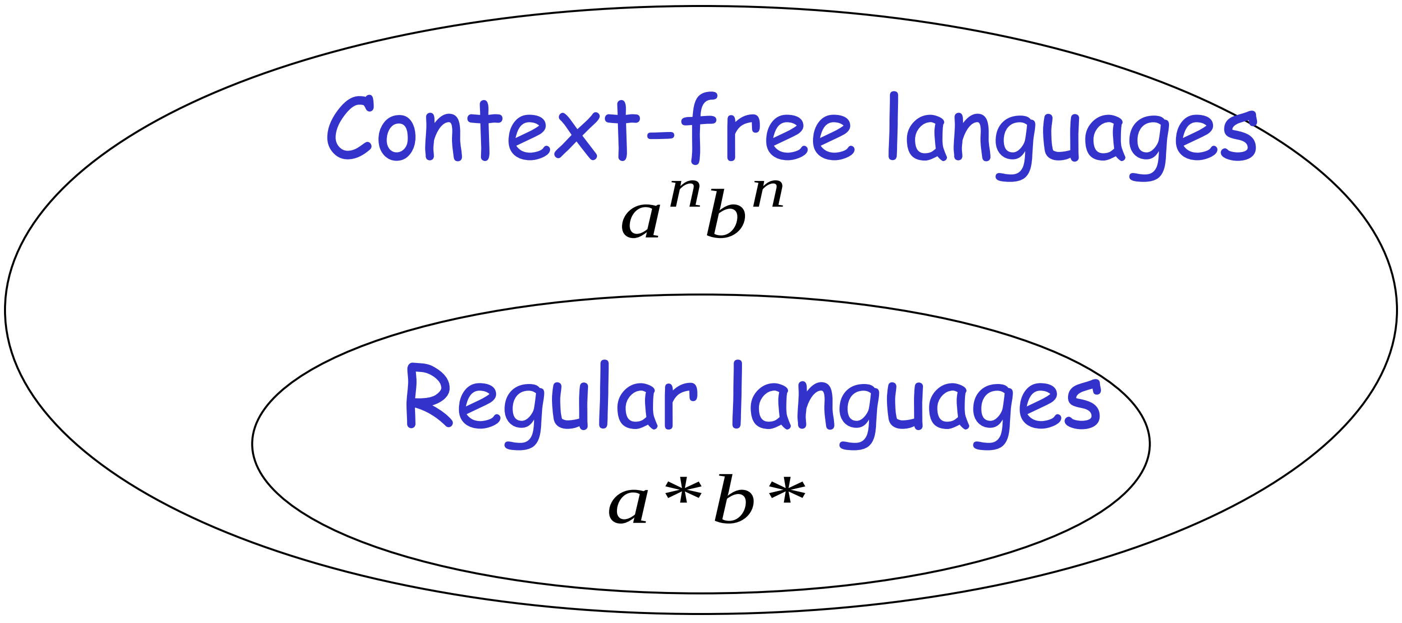
accepts $a^n b^{2n}$

DPDA M with $L(M)=\{a^n b^n\} \cup \{a^n b^{2n}\}$

Such a path exists due to determinism



Fact 1: The language $\{a^n b^n c^n\}$ is **not** context-free



(we will prove this at a later class using pumping lemma for context-free languages)

Fact 2: The language $L \cup \{a^n b^n c^n\}$
is not context-free

$$(L = \{a^n b^n\} \cup \{a^n b^{2n}\})$$

(we can prove this using pumping lemma
for context-free languages)

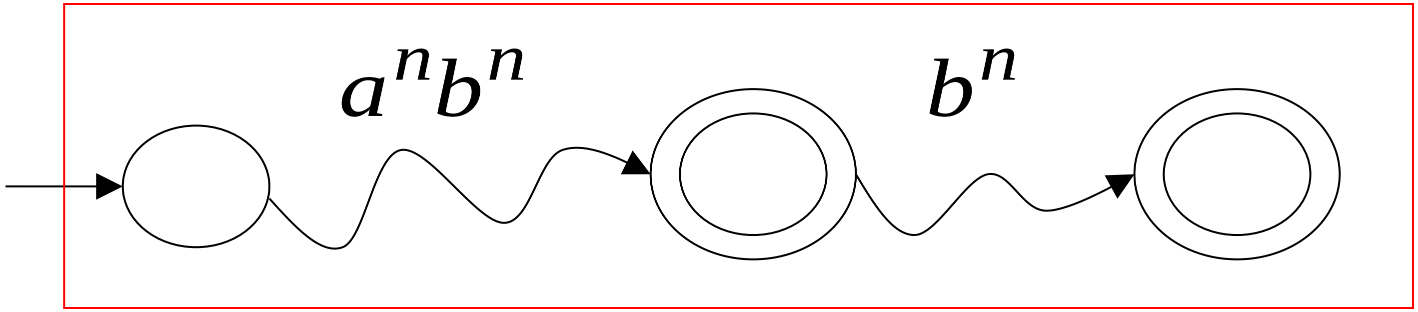
We will construct a PDA that accepts:

$$L \cup \{a^n b^n c^n\}$$

$$(L = \{a^n b^n\} \cup \{a^n b^{2n}\})$$

which is a contradiction!

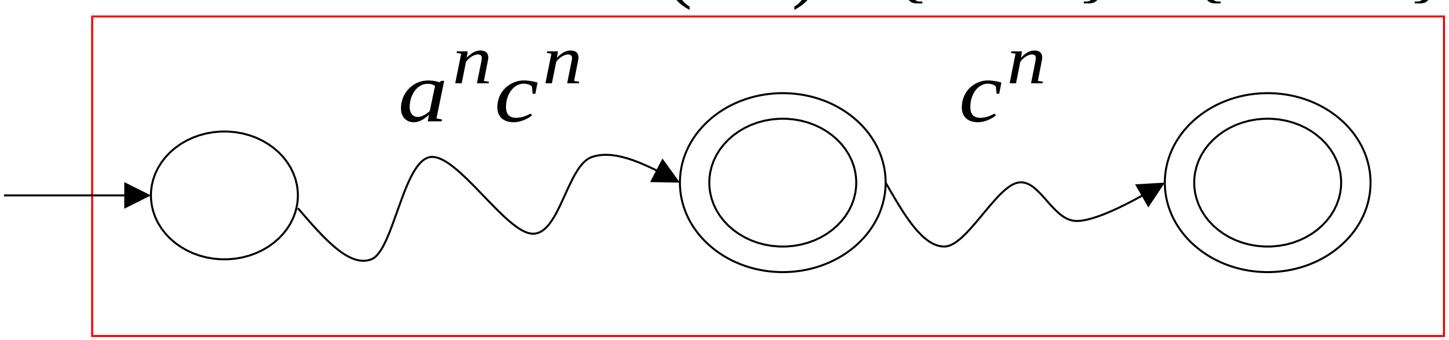
DPDA M $L(M)=\{a^nb^n\}\cup\{a^nb^{2n}\}$



Modify M

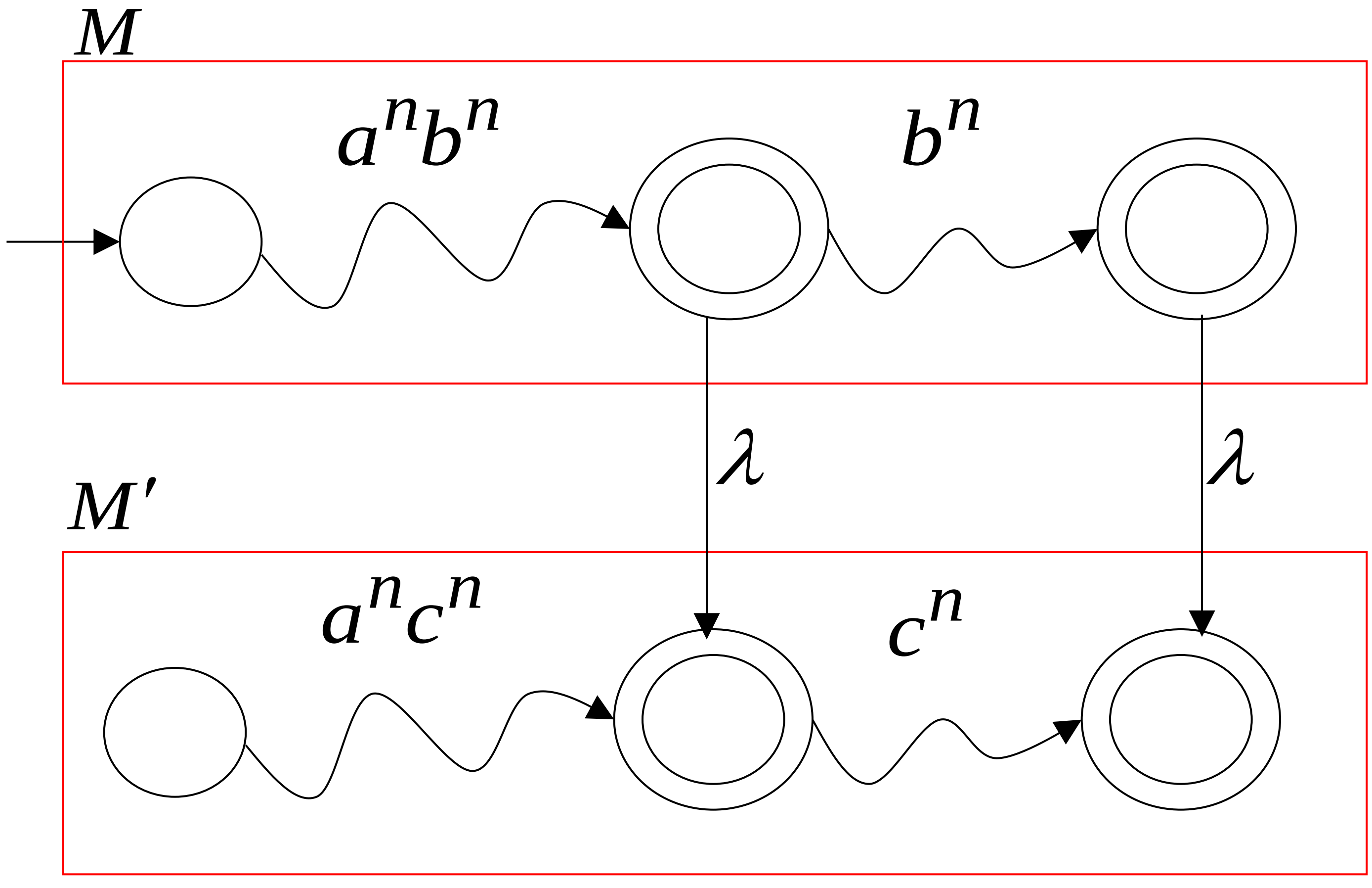
Replace b with c

DPDA M' $L(M')=\{a^nc^n\}\cup\{a^nc^{2n}\}$



A PDA that accepts $L \cup \{a^n b^n c^n\}$

Connect the final states of M
with the final states of M'



Since $L \cup \{a^n b^n c^n\}$ is accepted by a PDA

it is context-free

Contradiction!

(since $L \cup \{a^n b^n c^n\}$ is not context-free)

Therefore:

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

Is not deterministic context free

There is **no** DPDA that accepts it

End of Proof

Positive Properties of Context-Free languages

Union

Context-free languages
are closed under: **Union**

$$\left.\begin{array}{l} L_1 \text{ is context free} \\ \\ L_2 \text{ is context free} \end{array}\right\} \Longrightarrow \begin{array}{c} L_1 \cup L_2 \\ \text{is context-free} \end{array}$$

Proof?

Example

Language

Grammar

$$L_1 = \{a^n b^n\}$$

$$S_1 \to aS_1b \mid \lambda$$

$$L_2 = \{ww^R\}$$

$$S_2 \to aS_2a \mid bS_2b \mid \lambda$$

Union

$$L = \{a^n b^n\} \cup \{ww^R\}$$

$$S \to S_1 \mid S_2$$

In general:

For context-free languages $L_1, \; L_2$

with context-free grammars $G_1, \; G_2$

and start variables $S_1, \; S_2$

The grammar of the **union** $L_1 \cup L_2$

has new start variable S

and additional production $S \to S_1 \mid S_2$

Concatenation

Context-free languages are closed under: **Concatenation**

L_1 is context free

L_2 is context free

$\Rightarrow$ L_1L_2 is context-free

Proof?

Example

Language | **Grammar**

$$L_1 = \{a^n b^n\}$$

$$S_1 \rightarrow aS_1b \mid \lambda$$

$$L_2 = \{ww^R\}$$

$$S_2 \rightarrow aS_2a \mid bS_2b \mid \lambda$$

Concatenation

$$L = \{a^n b^n\}\{ww^R\}$$

$$S \rightarrow S_1 S_2$$

In general:

For context-free languages $L_1, \ L_2$

with context-free grammars $G_1, \ G_2$

and start variables $S_1, \ S_2$

The grammar of the **concatenation** $L_1 L_2$

has new start variable S

and additional production $S \to S_1 S_2$

Star Operation

Context-free languages are closed under: **Star-operation**

L is context free $\Rightarrow$ L^* is context-free

Proof?

Example

Language

$$L = \{a^n b^n\}$$

Grammar

$$S \rightarrow aSb \mid \lambda$$

Star Operation

$$L = \{a^n b^n\}^*$$

$$S_1 \rightarrow SS_1 \mid \lambda$$

In general:

For context-free language L

with context-free grammar G

and start variable S

The grammar of the **star operation** L^*

has new start variable S_1

and additional production $S_1 \to SS_1 \mid \lambda$

Negative Properties
of
Context-Free Languages

Intersection

Context-free languages
are **not** closed under: **intersection**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 \cap L_2$

not necessarily
context-free

Example

$$L_1 = \{a^n b^n c^m\}$$

Context-free:

$$S \rightarrow AC$$

$$A \rightarrow aAb \mid \lambda$$

$$C \rightarrow cC \mid \lambda$$

$$L_2 = \{a^n b^m c^m\}$$

Context-free:

$$S \rightarrow AB$$

$$A \rightarrow aA \mid \lambda$$

$$B \rightarrow bBc \mid \lambda$$

Intersection

$L_1 \cap L_2 = \{a^n b^n c^n\}$ **NOT** context-free

Complement

Context-free languages
are **not** closed under: **complement**

L is context free $\Longrightarrow$ $\overline{L}$ **not** necessarily context-free

Example

$L_1 = \{a^n b^n c^m\}$

Context-free:

$S \to AC$

$A \to aAb \mid \lambda$

$C \to cC \mid \lambda$

$L_2 = \{a^n b^m c^m\}$

Context-free:

$S \to AB$

$A \to aA \mid \lambda$

$B \to bBc \mid \lambda$

Complement

$$\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2 = \{a^n b^n c^n\}$$

NOT context-free

Intersection
of
Context-free languages
and
Regular Languages

The intersection of

a context-free language and

a regular language

is a context-free language

L_1 context free

L_2 regular

$\Rightarrow$ $L_1 \cap L_2$ context-free

Machine M_1

NPDA for L_1

context-free

Machine M_2

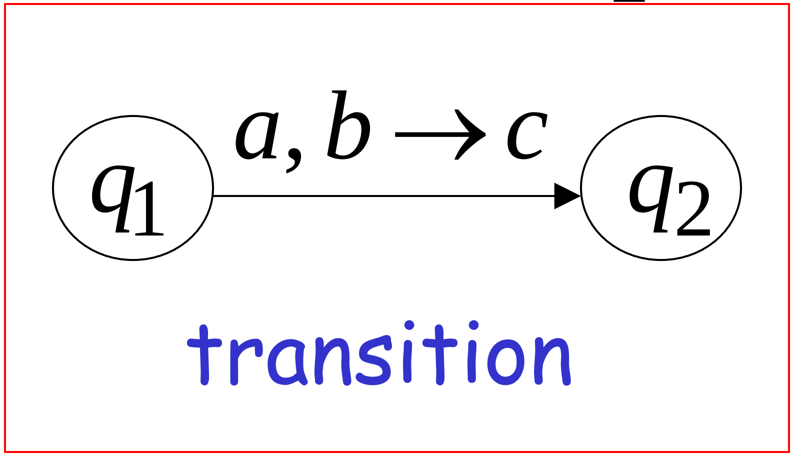
DFA for L_2

regular

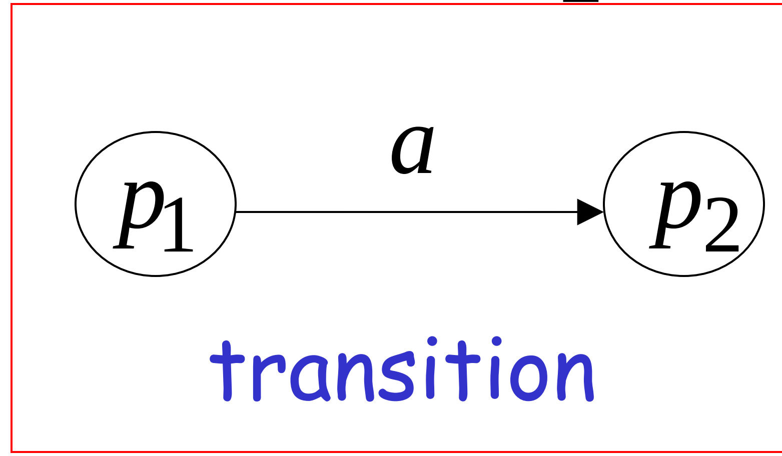
Construct a new NPDA machine M that accepts $L_1 \cap L_2$

M simulates in parallel M_1 and M_2

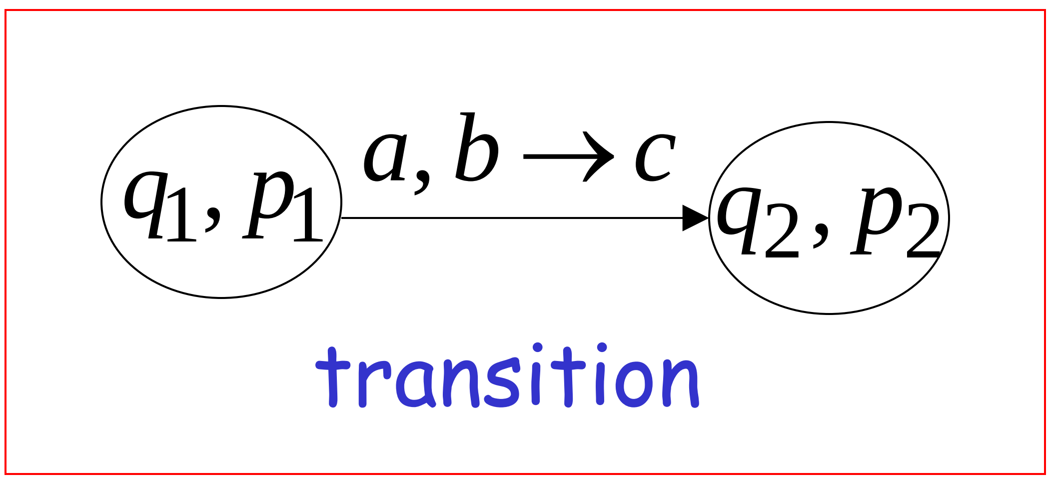
NPDA M_1



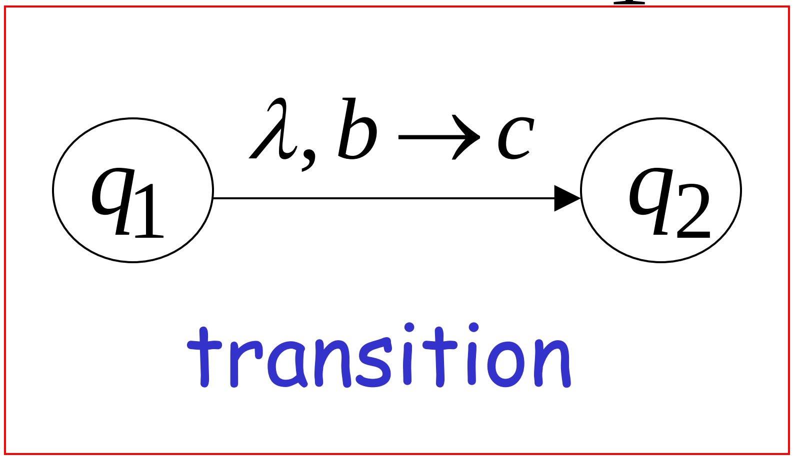
DFA M_2



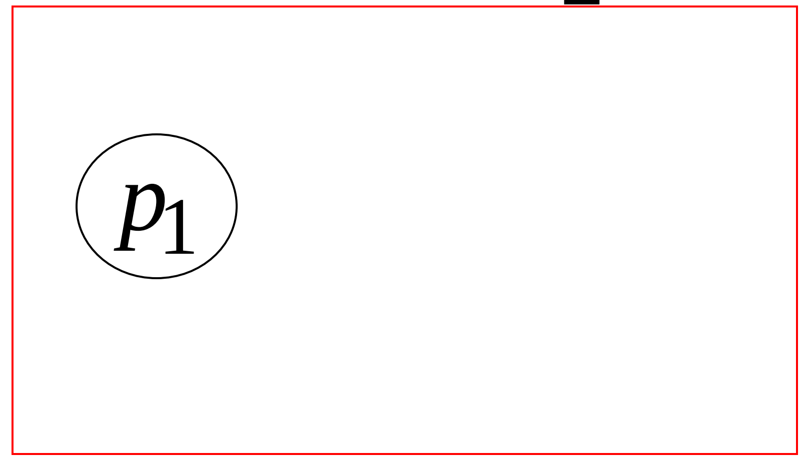
NPDA M



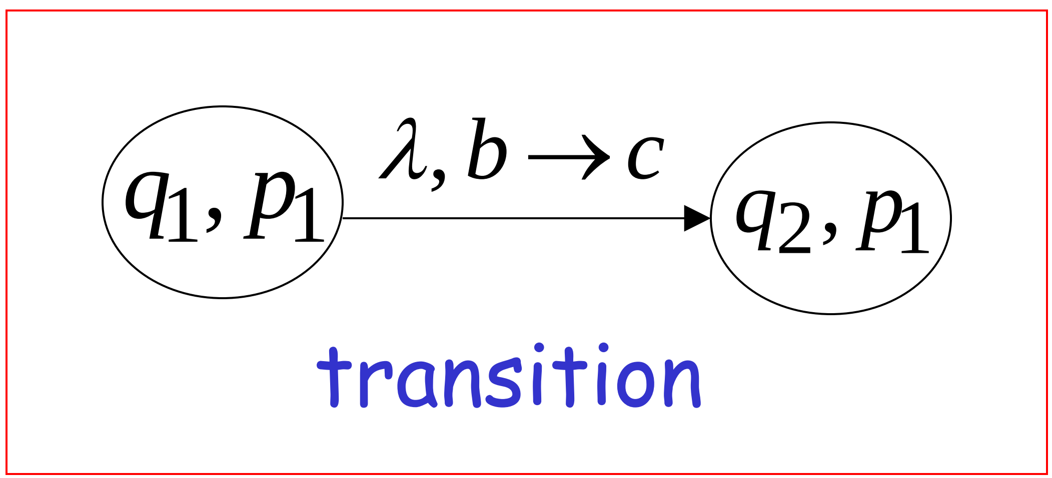
NPDA M_1



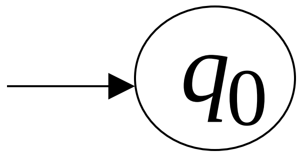
DFA M_2



NPDA M

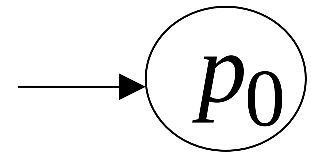


NPDA M_1



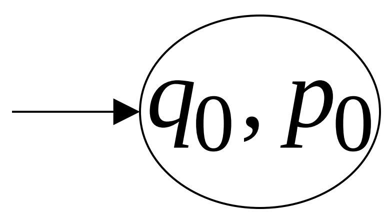
initial state

DFA M_2

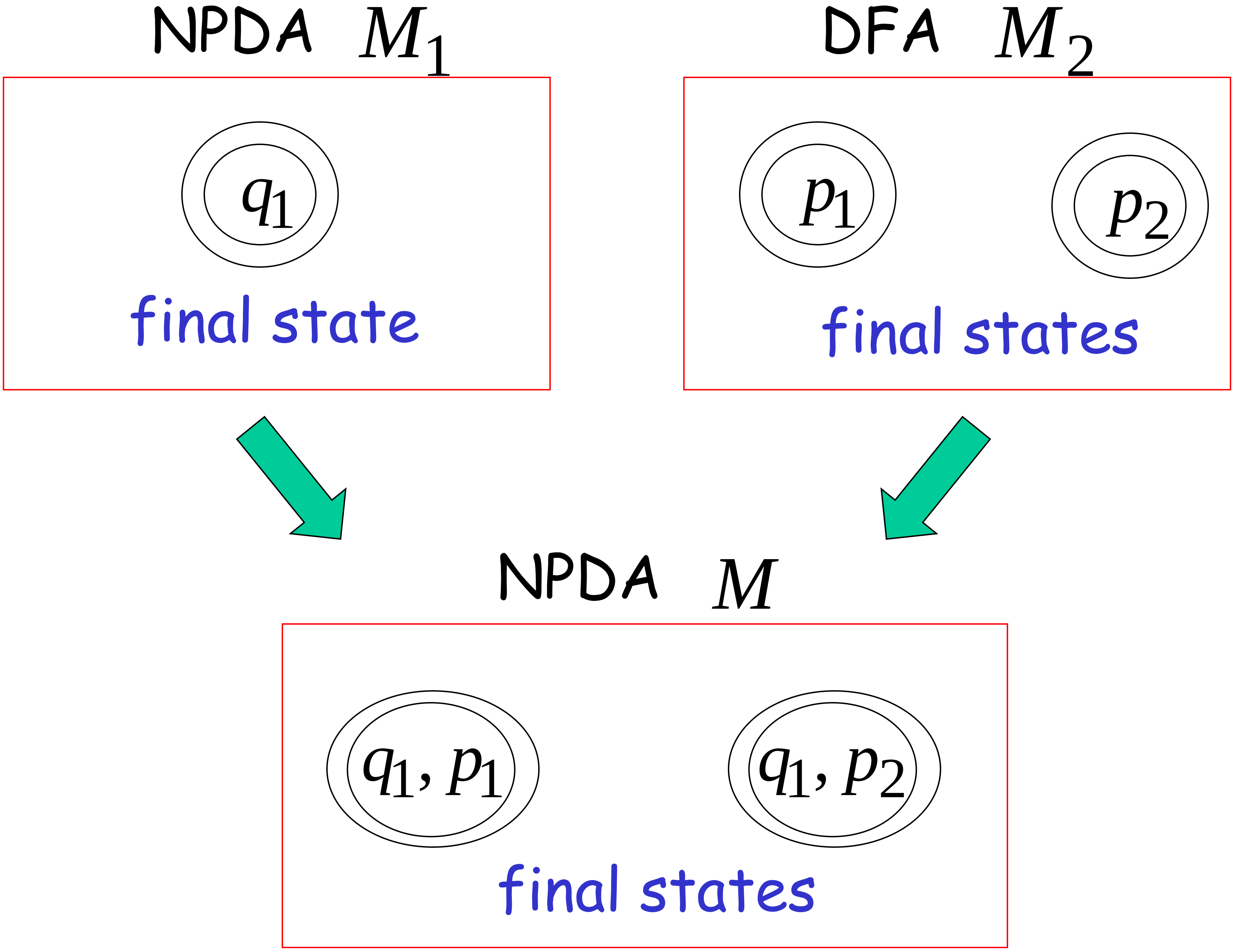


initial state

NPDA M



Initial state



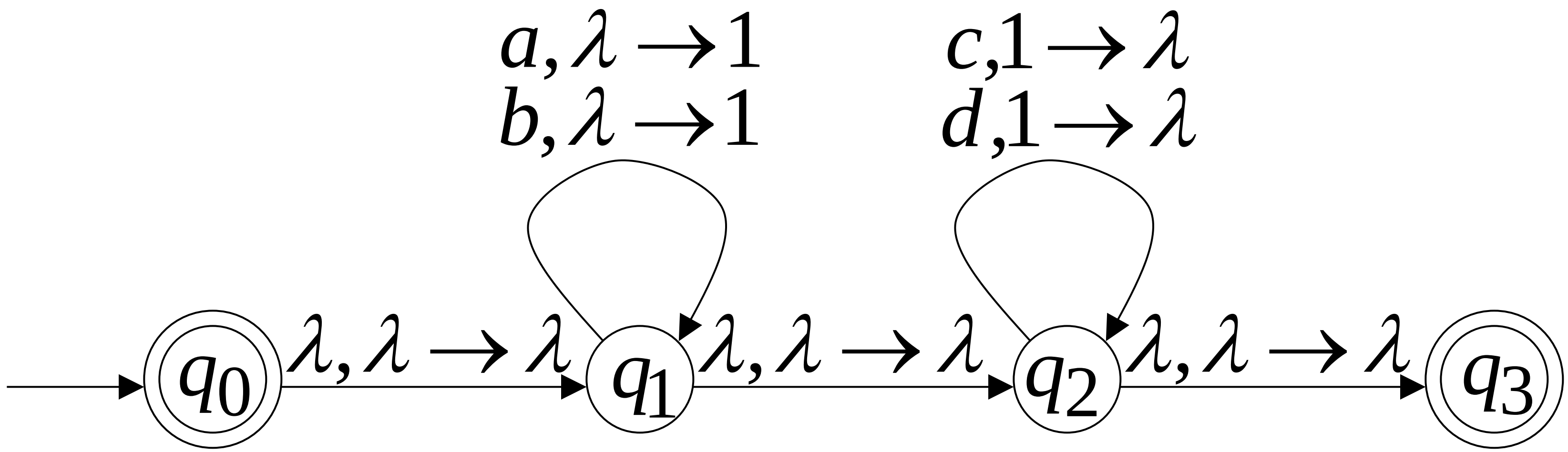
NPDA M_1
q_1
final state
DFA M_2
p_1
p_2
final states
NPDA M
q_1, p_1
q_1, p_2
final states

Example:

context-free

$$L_1 = \{w_1 w_2 : |w_1| = |w_2|, w_1 \in \{a,b\}^*, w_2 \in \{c,d\}^*\}$$

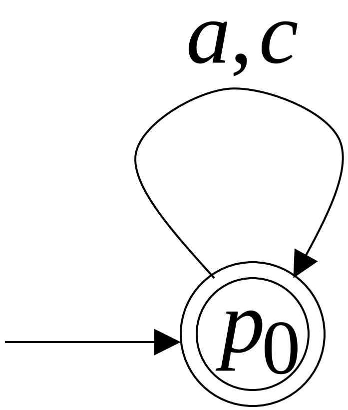
NPDA M_1



regular

$$L_2 = \{a, c\}^*$$

DFA M_2

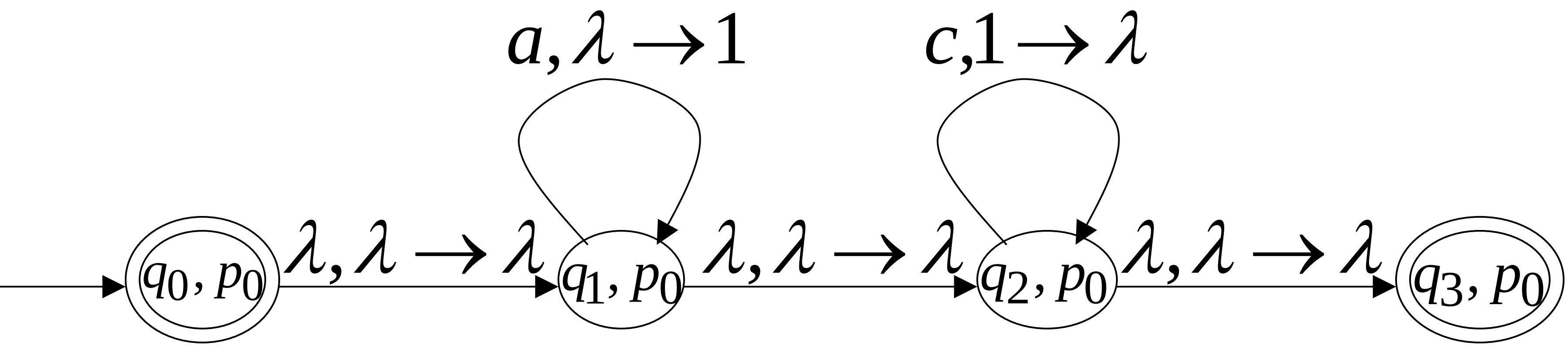


Intersection?

context-free

Automaton for: $L_1 \cap L_2 = \{a^n c^n : \ n \geq 0\}$

NPDA M



In General:

M simulates in parallel M_1 and M_2

M accepts string w if and only if

M_1 accepts string w and

M_2 accepts string w

$$L(M) = L(M_1) \cap L(M_2)$$

Therefore:

M is NPDA

$\Downarrow$

$L(M_1) \cap L(M_2)$ is context-free

$\Downarrow$

$L_1 \cap L_2$ is context-free

Applications
of
Regular Closure

The intersection of
a context-free language and
a regular language
is a context-free language

Regular Closure

L_1 context free

L_2 regular

$$\left.\begin{array}{l} L_1 \text{ context free} \\ L_2 \text{ regular} \end{array}\right\} \Rightarrow L_1 \cap L_2$$

context-free

An Application of Regular Closure

Prove that: $L = \{a^n b^n : \;\; n \neq 100, n \geq 0\}$

is context-free

We know:

$\{a^n b^n : n \geq 0\}$ is context-free

We also know:

$$L_1 = \{a^{100}b^{100}\}$$ is regular

$$\overline{L_1} = \{(a+b)^*\} - \{a^{100}b^{100}\}$$ is regular

$\{a^n b^n\}$ $\qquad$ $\overline{L_1} = \{(a+b)^*\} - \{a^{100} b^{100}\}$

context-free $\qquad$ regular

(regular closure) $\{a^n b^n\} \cap \overline{L_1}$ context-free

$$\{a^n b^n\} \cap \overline{L_1} = \{a^n b^n : \ n \neq 100, n \geq 0\} = L$$

is context-free

Another Application of Regular Closure

Prove that: $L = \{w: \quad n_a = n_b = n_c\}$

is **not** context-free

Proof?

If $L = \{w: \ n_a = n_b = n_c\}$ is context-free

(regular closure)

Then $L \cap \{a^*b^*c^*\} = \{a^n b^n c^n\}$

context-free regular context-free

Impossible!!!

Therefore, L is **not** context free

Decidable Properties of Context-Free Languages

Membership Question:

for context-free grammar G

find if string $w \in L(G)$

Membership Question:

for context-free grammar G
find if string $w \in L(G)$

Membership Algorithms: Parsers

- Exhaustive search parser
- **CYK** parsing algorithm

Empty Language Question:

for context-free grammar G

find if $L(G) = \varnothing$

Empty Language Question:

for context-free grammar G

find if $L(G)=\varnothing$

Algorithm:

1. Remove useless variables

2. Check if start variable S is useless

Infinite Language Question:

for context-free grammar G

find if $L(G)$ is infinite

Infinite Language Question:

for context-free grammar G

find if $L(G)$ is infinite

Algorithm:

1. Remove useless variables

2. Remove unit and λ productions

3. Create dependency graph for variables

4. If there is a loop in the dependency graph then the language is infinite

Example:

$$S \to AB$$

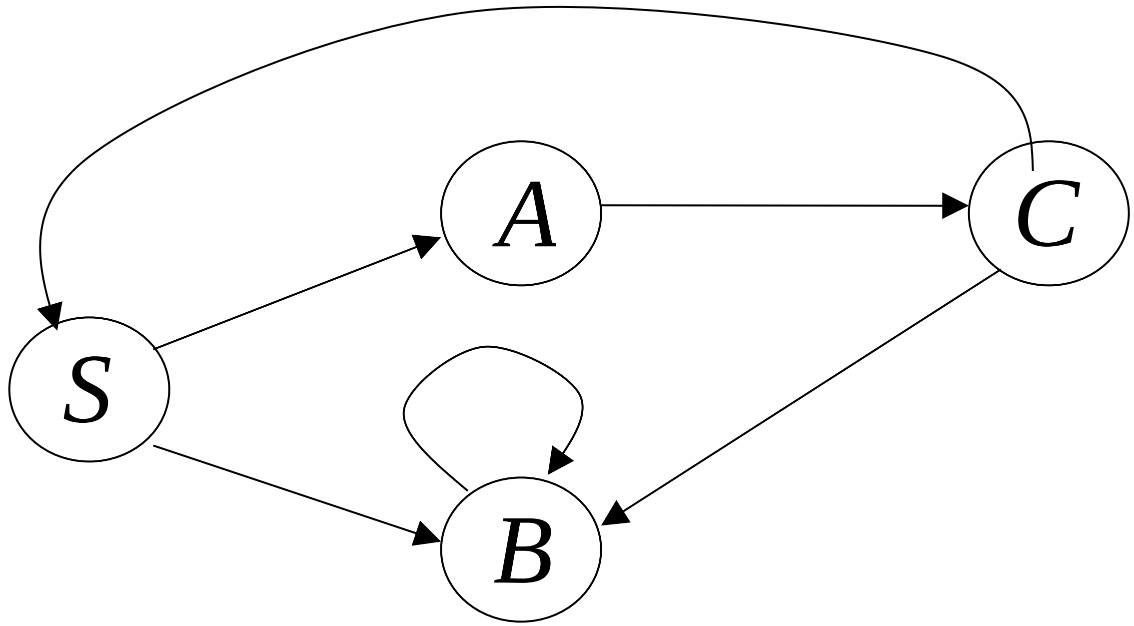
$$A \to aCb \mid a$$

$$B \to bB \mid bb$$

$$C \to cBS$$

Dependency graph

Infinite language



$$S \to AB$$

$$A \to aCb \mid a$$

$$B \to bB \mid bb$$

$$C \to cBS$$

$$S \Rightarrow AB \Rightarrow aCbB \Rightarrow acBSbB \Rightarrow acbbSbbb$$

$$S \overset{*}{\Rightarrow} acbbSbbb \overset{*}{\Rightarrow} (acbb)^2 S (bbb)^2$$

$$\overset{*}{\Rightarrow} (acbb)^i S (bbb)^i$$