Formal Languages

Recursively Enumerable Languages

Recursive Languages

Definition:

A language is **recursively enumerable** if some Turing machine accepts it

Let L be a recursively enumerable language

and M the Turing Machine that accepts it

For string w :

if $w \in L$ then M halts in a final state

if $w \notin L$ then M halts in a non-final state

or loops forever

Definition:

A language is **recursive**
if some Turing machine accepts it
and halts on any input string

In other words:

A language is recursive if there is
a membership algorithm for it

Let L be a recursive language

and M the Turing Machine that accepts it

For string w :

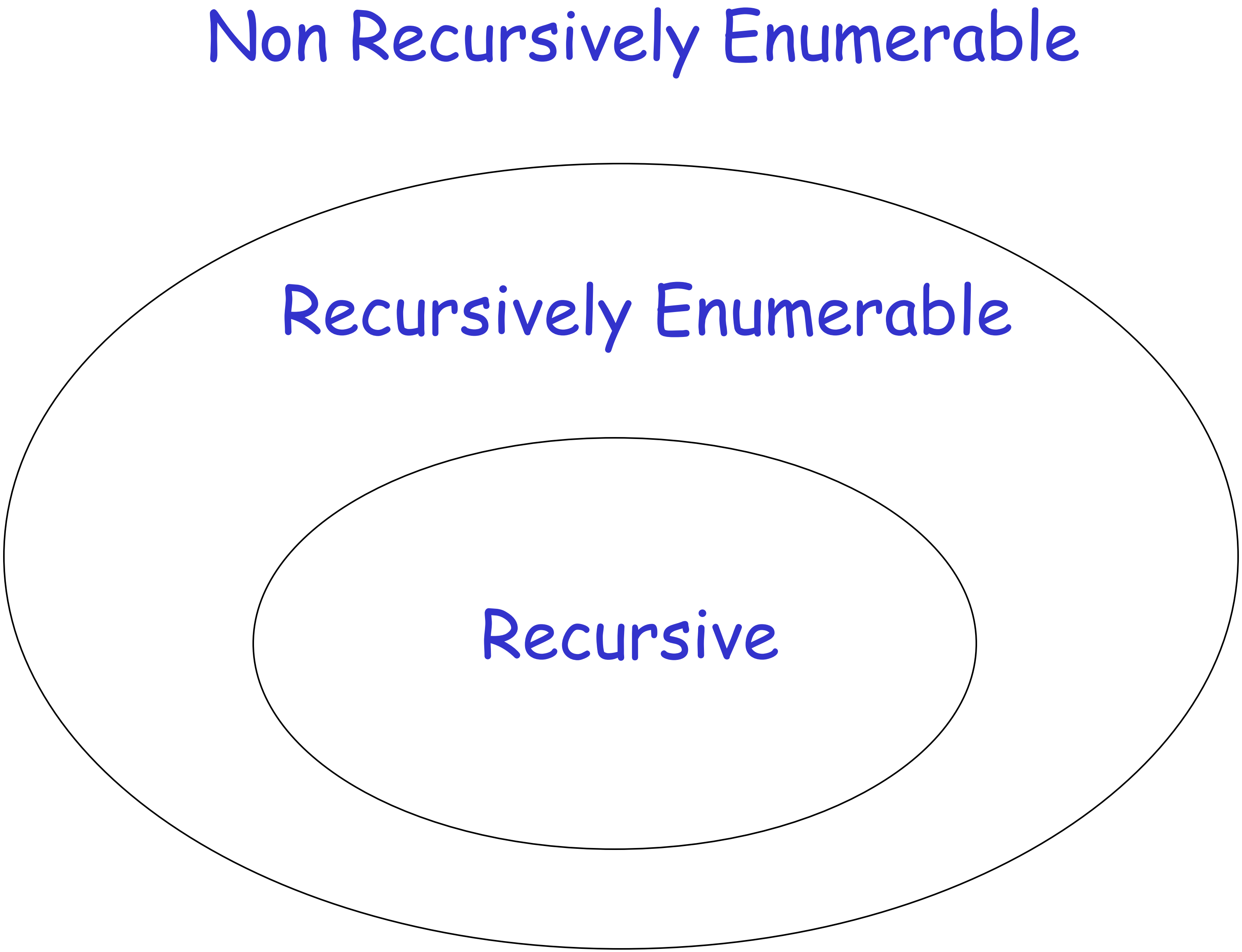
if $w \in L$ then M halts in a final state

if $w \notin L$ then M halts in a non-final state

We will prove:

1. There is a specific language which is not recursively enumerable (not accepted by any Turing Machine)

2. There is a specific language which is recursively enumerable but not recursive


Non Recursively Enumerable
Recursively Enumerable
Recursive

A Language which is not Recursively Enumerable

We want to find a language that is not Recursively Enumerable

This language is not accepted by any Turing Machine

Consider alphabet $\{a\}$

Strings: $a, \; aa, \; aaa, \; aaaa, \; \ldots$

$a^1 \quad a^2 \quad a^3 \quad a^4 \quad \ldots$

Consider Turing Machines
that accept languages over alphabet $\{a\}$

They are countable:

$$M_1,\ M_2,\ M_3,\ M_4,\ \ldots$$

Example language accepted by M_i

$$L(M_i) = \{aa, aaaa, aaaaaa\}$$

$$L(M_i) = \{a^2, a^4, a^6\}$$

Alternative representation

	a^1	a^2	a^3	a^4	a^5	a^6	a^7	$\cdots$
$L(M_i)$	0	1	0	1	0	1	0	$\cdots$

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

Consider the language

$$L = \{a^i : a^i \in L(M_i)\}$$

L consists of the 1's in the diagonal

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$L = \{a^3, a^4, \ldots\}$$

Consider the language $\overline{L}$

$$L = \{a^i : a^i \in L(M_i)\}$$

$$\overline{L} = \{a^i : a^i \notin L(M_i)\}$$

$\overline{L}$ consists of the 0's on the diagonal

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$\overline{L} = \{a^1, a^2, \ldots\}$$

Theorem:

Language $\overline{L}$ is not recursively enumerable

Proof:

Assume for contradiction that

$\overline{L}$ is recursively enumerable

There must exist some machine M_k that accepts $\overline{L}$

$$L(M_k) = \overline{L}$$

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

Question: $M_k = M_1$?

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$a^1 \in L(M_k)$$

Answer: $M_k \neq M_1$

$$a^1 \notin L(M_1)$$

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

Question: $M_k = M_2$?

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$a^2 \in L(M_k)$$

Answer: $M_k \neq M_2$

$$a^2 \notin L(M_2)$$

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

Question: $M_k = M_3$?

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$a^3 \notin L(M_k)$$

Answer: $M_k \neq M_3$

$$a^3 \in L(M_3)$$

Similarly: $M_k \neq M_i$ for any i

Because either:

$$a^i \in L(M_k) \qquad \text{or} \qquad a^i \notin L(M_k)$$

$$a^i \notin L(M_i) \qquad\qquad\quad a^i \in L(M_i)$$

Therefore, the machine M_k cannot exist

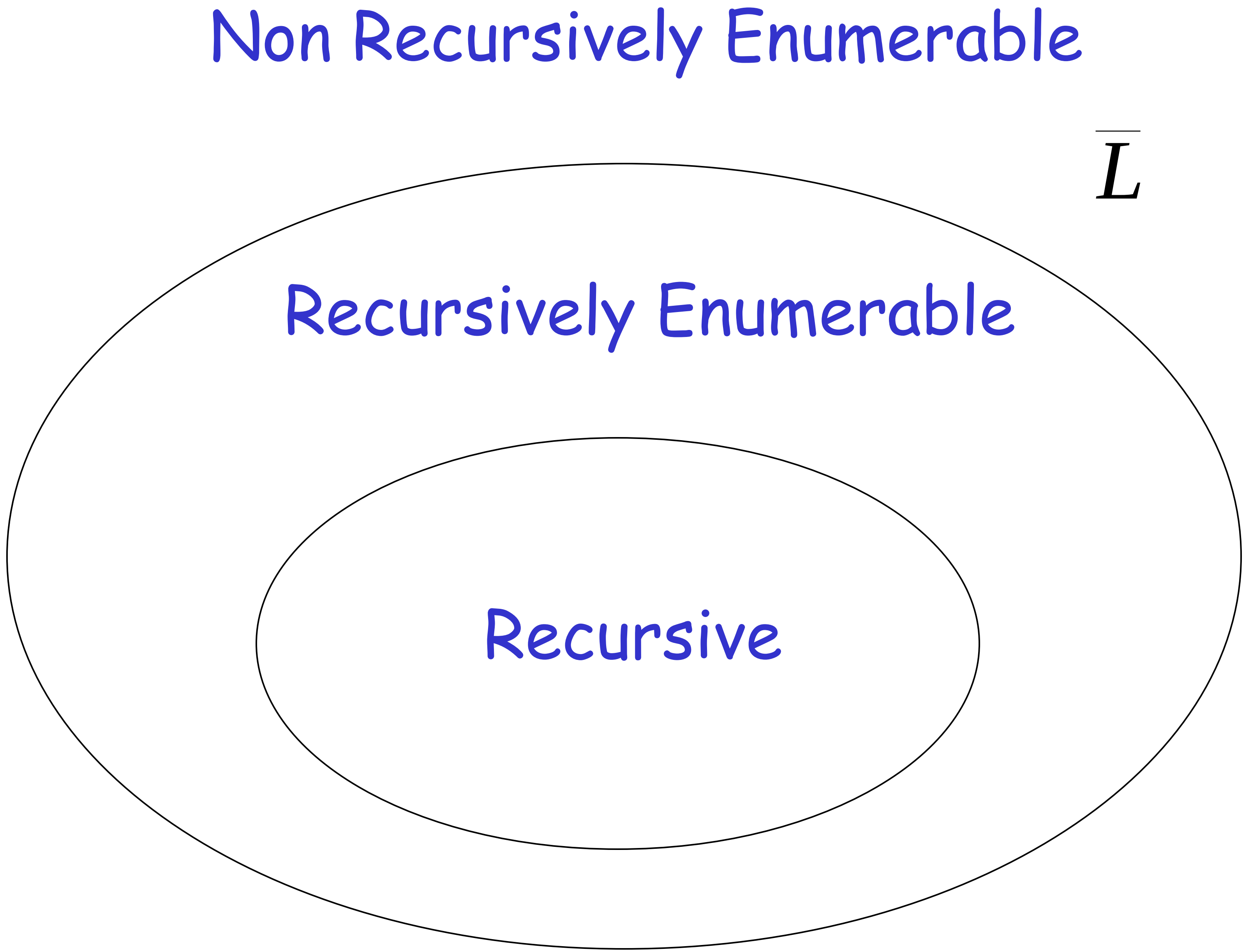
Therefore, the language $\overline{L}$ is not recursively enumerable

End of Proof

Observation:

There is no algorithm that describes $\overline{L}$

(otherwise $\overline{L}$ would be accepted by some Turing Machine)


Non Recursively Enumerable
$\overline{L}$
Recursively Enumerable
Recursive

A Language which is Recursively Enumerable and not Recursive

We want to find a language which

Is recursively enumerable

But not recursive

There is a Turing Machine that accepts the language

The machine doesn't halt on some input

We will prove that the language

$$L = \{a^i : a^i \in L(M_i)\}$$

Is recursively enumerable

but not recursive

	a^1	a^2	a^3	a^4	$\cdots$
$L(M_1)$	0	1	0	1	$\cdots$
$L(M_2)$	1	0	0	1	$\cdots$
$L(M_3)$	0	1	1	1	$\cdots$
$L(M_4)$	0	0	0	1	$\cdots$

$$L = \{a^3, a^4, \ldots\}$$

Theorem:

The language $L = \{a^i : a^i \in L(M_i)\}$

is recursively enumerable

Proof:

We will give a Turing Machine that accepts L

Turing Machine that accepts L

For any input string w

- Compute i, for which $w = a^i$
- Find Turing machine M_i

  (using an enumeration procedure for Turing Machines)
- Simulate M_i on input a^i
- If M_i accepts, then accept w

End of Proof

Observation:

Recursively enumerable

$$L = \{a^i : a^i \in L(M_i)\}$$

Not recursively enumerable

$$\overline{L} = \{a^i : a^i \notin L(M_i)\}$$

(Thus, also not recursive)

Theorem:

The language $L = \{a^i : a^i \in L(M_i)\}$

is not recursive

Proof:

Assume for contradiction that L is recursive

Then $\overline{L}$ is recursive:

Take the Turing Machine M that accepts L

M halts on any input:

If M accepts then reject

If M rejects then accept

Therefore:

$\overline{L}$ is recursive

But we know:

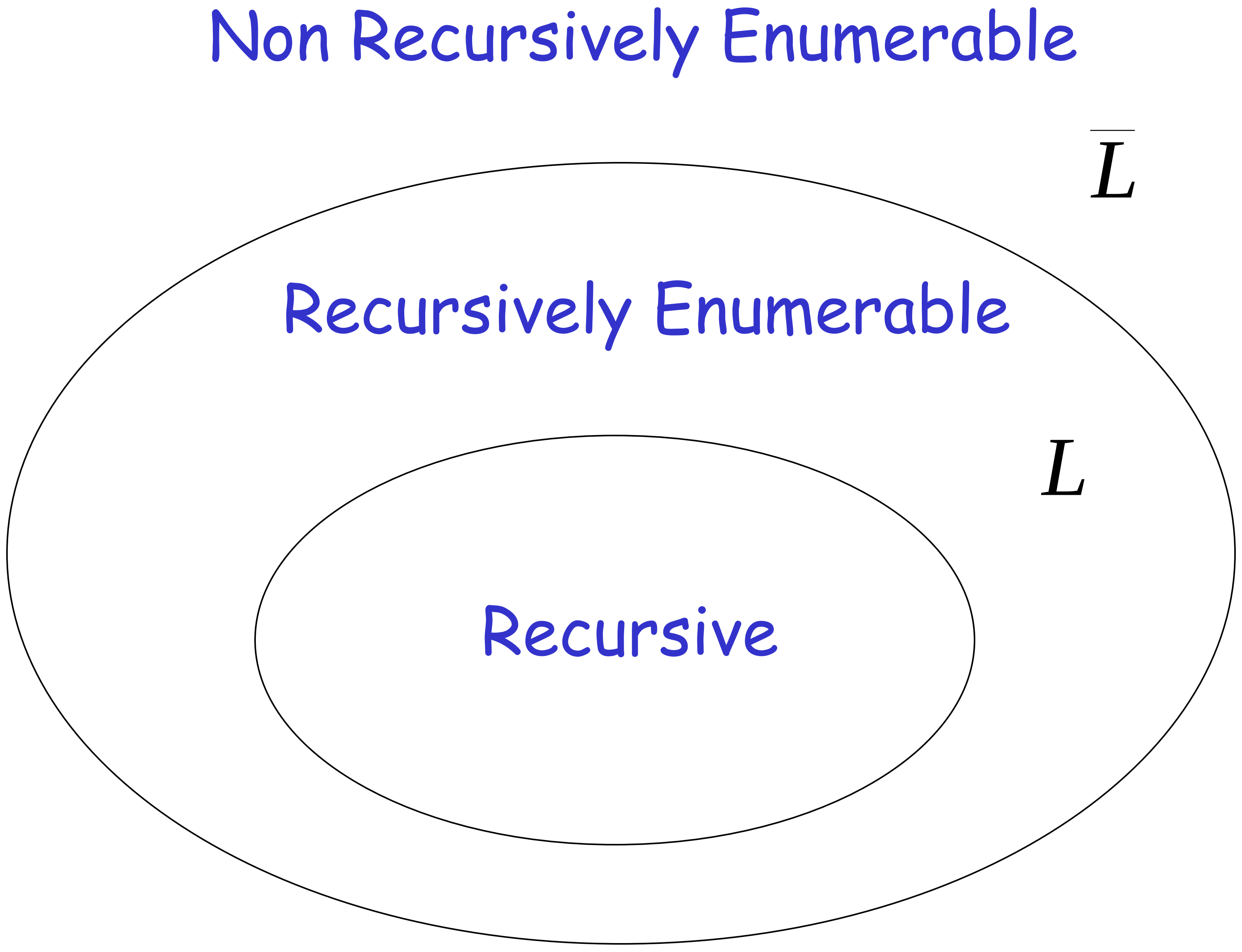
$\overline{L}$ is not recursively enumerable

thus, not recursive

CONTRADICTION!!!!

Therefore, L is not recursive

End of Proof


Non Recursively Enumerable
$\overline{L}$
Recursively Enumerable
L
Recursive

Recursive language?

Turing acceptable languages and Enumeration Procedures

We will prove:

(weak result)

- If a language is recursive then there is an enumeration procedure for it

(strong result)

- A language is recursively enumerable if and only if there is an enumeration procedure for it

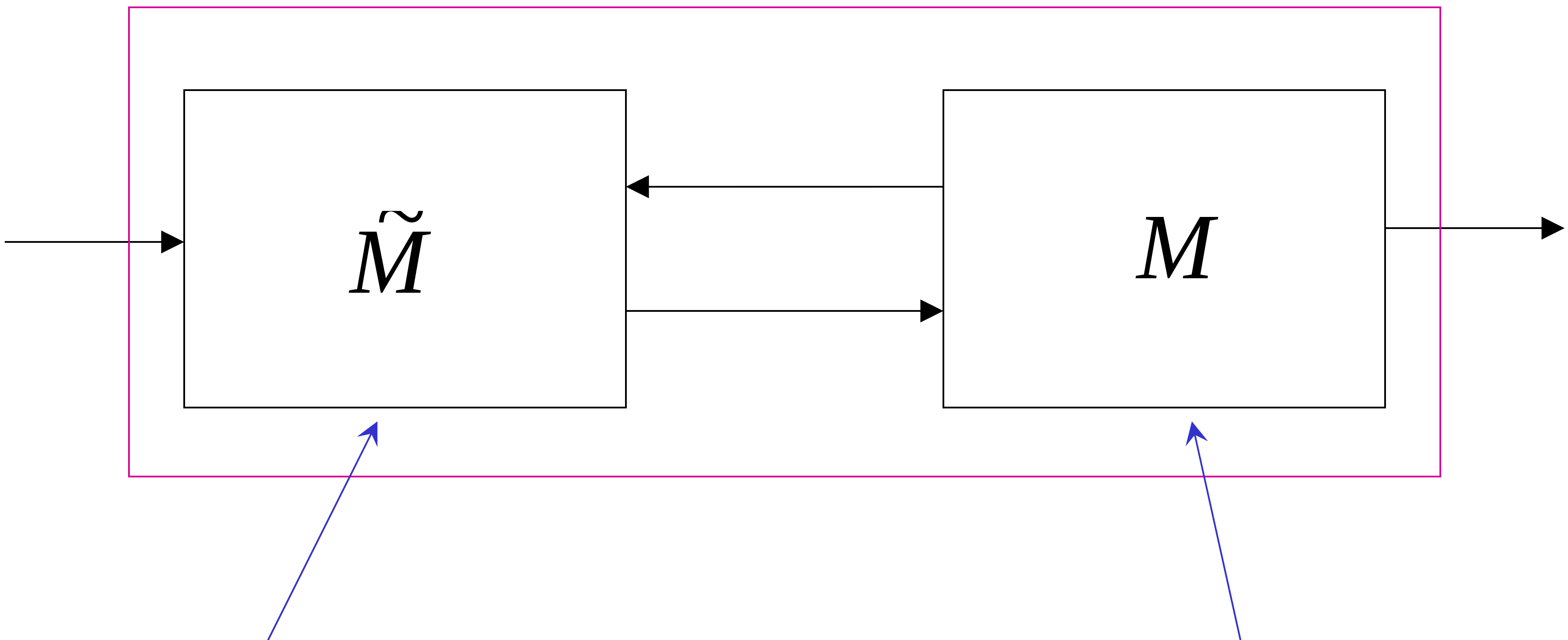
Theorem:

if a language L is recursive then there is an enumeration procedure for it

Proof?

Proof:

Enumeration Machine



Enumerates all strings of input alphabet

Accepts L

If the alphabet is $\{a,b\}$ then $\tilde{M}$ can enumerate strings as follows:

$$
\begin{array}{c}
a \\
b \\
aa \\
ab \\
ba \\
bb \\
aaa \\
aab \\
\ldots\ldots
\end{array}
$$

Enumeration procedure

Repeat:

$\tilde{M}$ generates a string w

M checks if $w \in L$

YES: print w to output

NO: ignore w

End of Proof

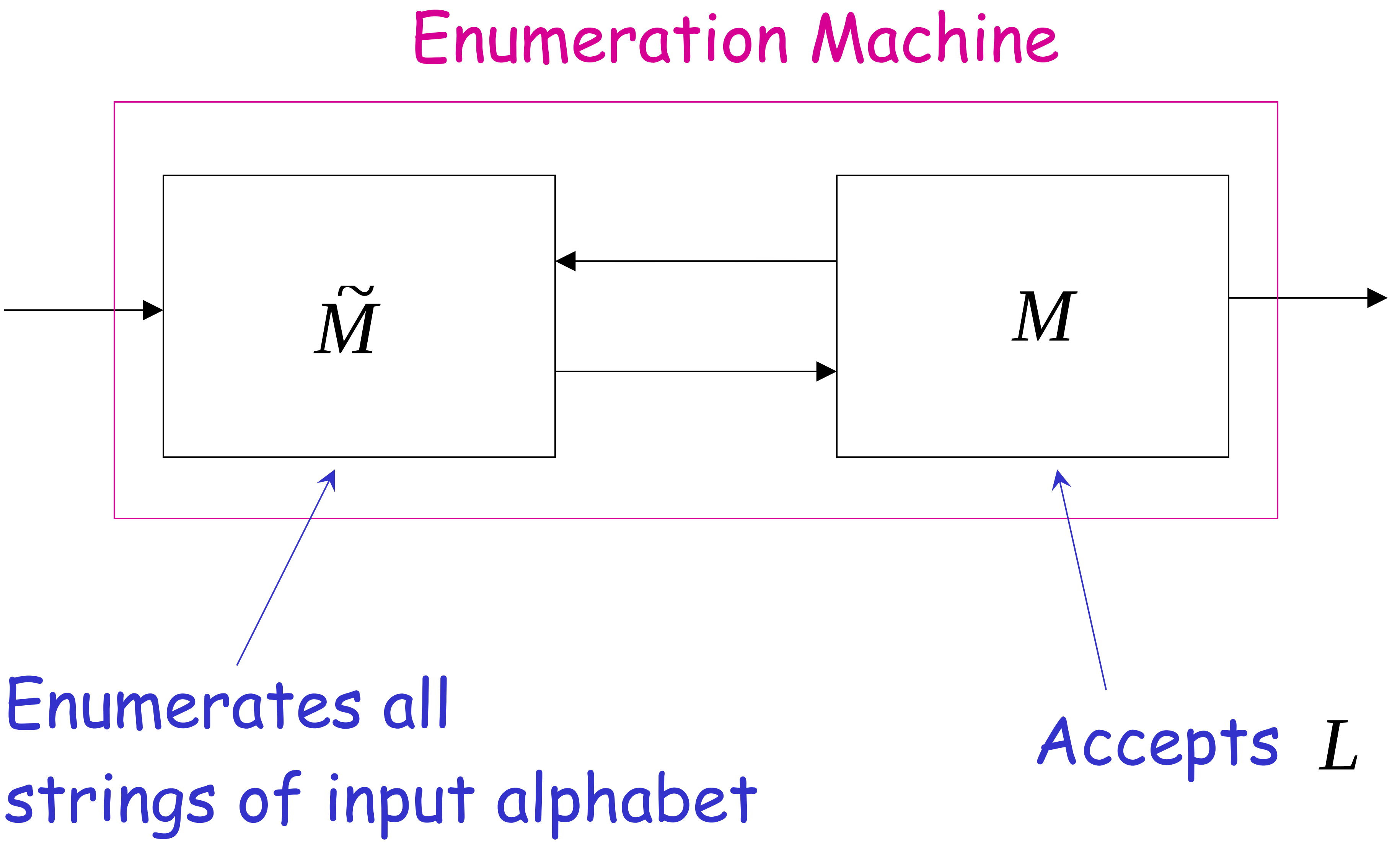
Example: $L = \{b, ab, bb, aaa, ...\}$

$\tilde{M}$	$L(M)$	Enumeration Output
a		
b	b	b
aa		
ab	ab	ab
ba		
bb	bb	bb
aaa	aaa	aaa
aab		
......	......	......

Theorem:

if language L is recursively enumerable then there is an enumeration procedure for it

Proof:



Enumeration Machine
$\tilde{M}$
M
Enumerates all strings of input alphabet
Accepts L

If the alphabet is $\{a, b\}$ then

$\tilde{M}$ can enumerate strings as follows:

$$a$$
$$b$$
$$aa$$
$$ab$$
$$ba$$
$$bb$$
$$aaa$$
$$aab$$

NAIVE APPROACH

Enumeration procedure

Repeat: $\tilde{M}$ generates a string w

M checks if $w \in L$

YES: print w to output

NO: ignore w

Problem?

NAIVE APPROACH

Enumeration procedure

Repeat: $\tilde{M}$ generates a string w

M checks if $w \in L$

YES: print w to output

NO: ignore w

Problem: If $w \notin L$
machine M may loop forever

BETTER APPROACH

$\tilde{M}$ Generates first string w_1

M executes first step on w_1

$\tilde{M}$ Generates second string w_2

M executes first step on w_2

second step on w_1

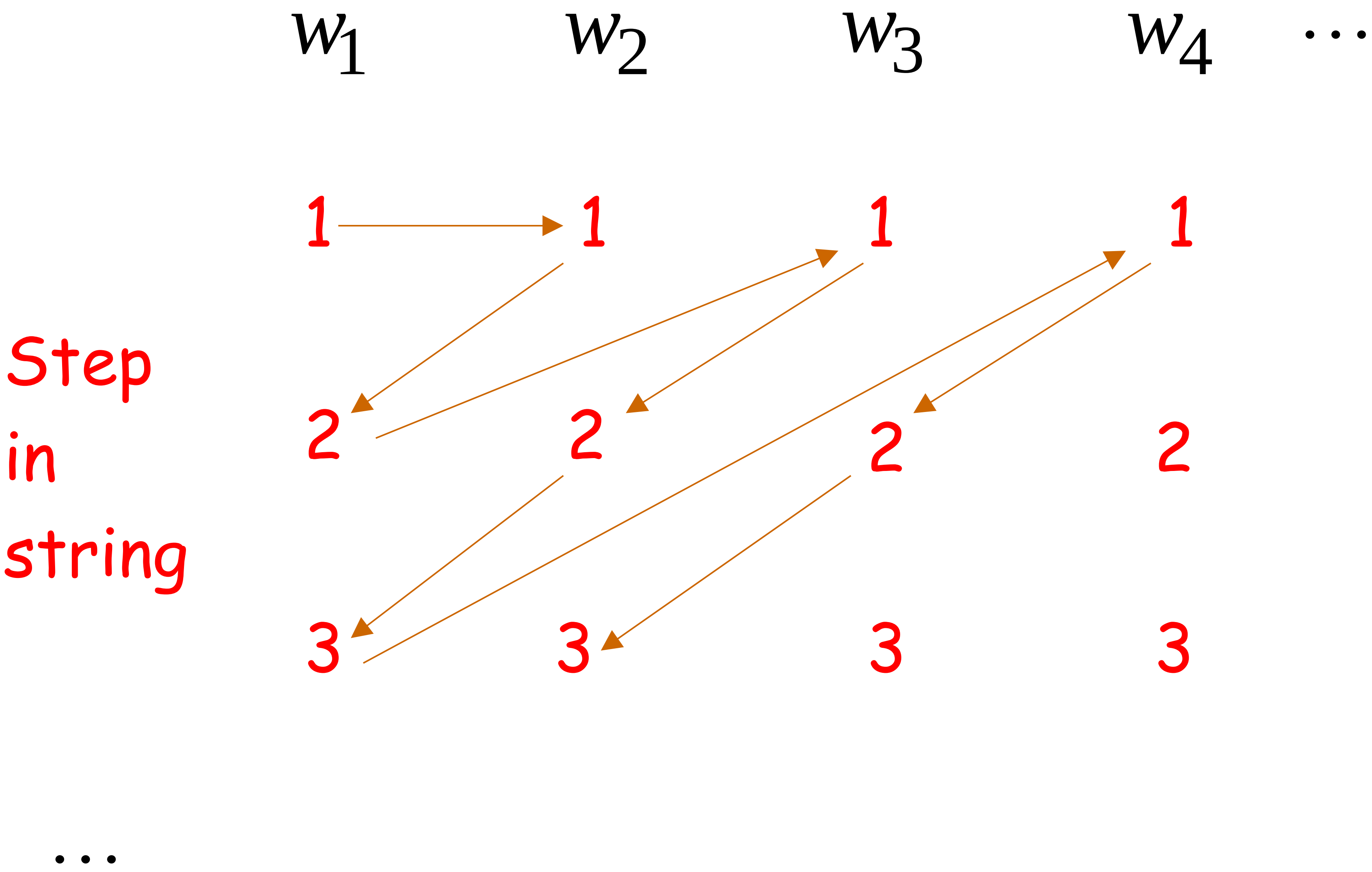
$\tilde{M}$ Generates third string w_3

M executes first step on w_3

second step on w_2

third step on w_1

And so on............



w_1 w_2 w_3 w_4 . . .
1 1 1 1
2 2 2 2
3 3 3 3
Step in string
. . .

If for any string w_i
machine M halts in a final state
then it prints w_i on the output

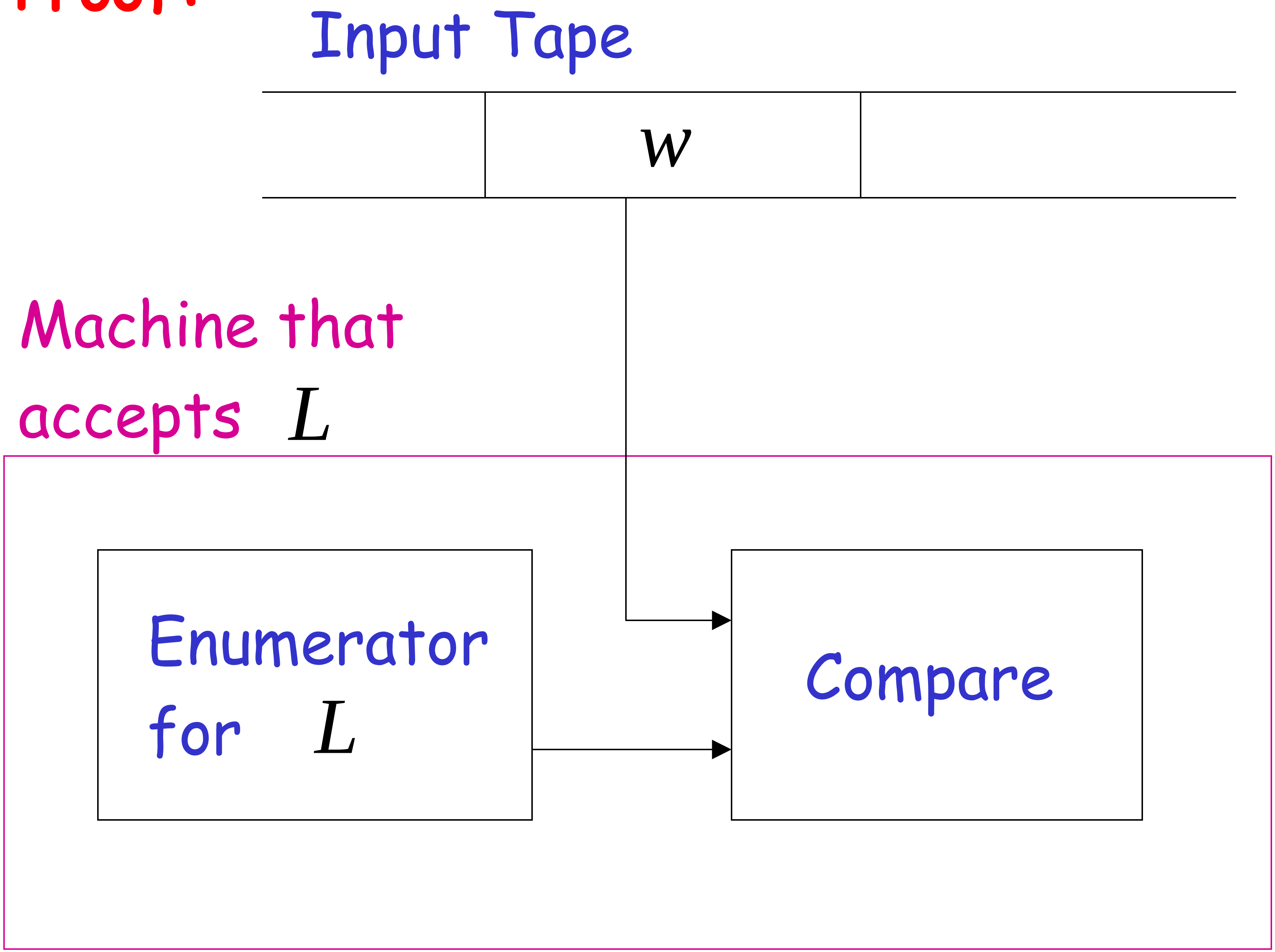
End of Proof

Theorem:

If for language L
there is an enumeration procedure
then L is recursively enumerable

Proof?

Proof:


Input Tape
w
Machine that accepts L
Enumerator for L
Compare

Turing machine that accepts L

For input string w

Repeat:

- Using the enumerator, generate the next string of L

- Compare generated string with w

  If same, accept and exit loop

End of Proof

We have proven:

A language is recursively enumerable
if and only if
there is an enumeration procedure for it