Proof - step 2

Converting

PDAs
to
Context-Free Grammars

$$\left\{\begin{array}{c}\text{Context-Free}\\ \text{Languages}\\ \text{(Grammars)}\end{array}\right\} \supseteq \left\{\begin{array}{c}\text{Languages}\\ \text{Accepted by}\\ \text{PDAs}\end{array}\right\}$$

Convert any PDA M to a context-free grammar G with: $L(G) = L(M)$

We will convert PDA M to

a context-free grammar G such that:

G simulates computations of M with leftmost derivations

Some Necessary Modifications

If necessary, modify the PDA so that:

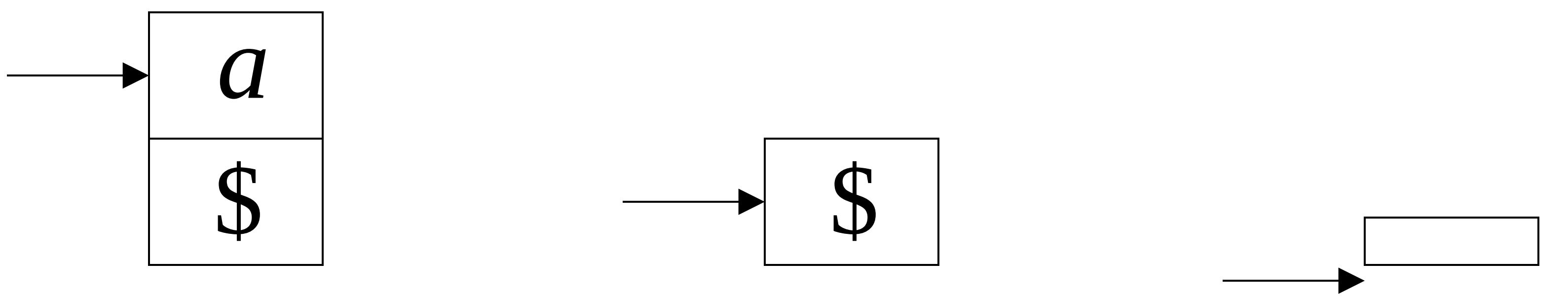
1. The stack is never empty during computation

2. It has a single accept state
   and empties the stack when it accepts a string

3. Has transitions without popping λ

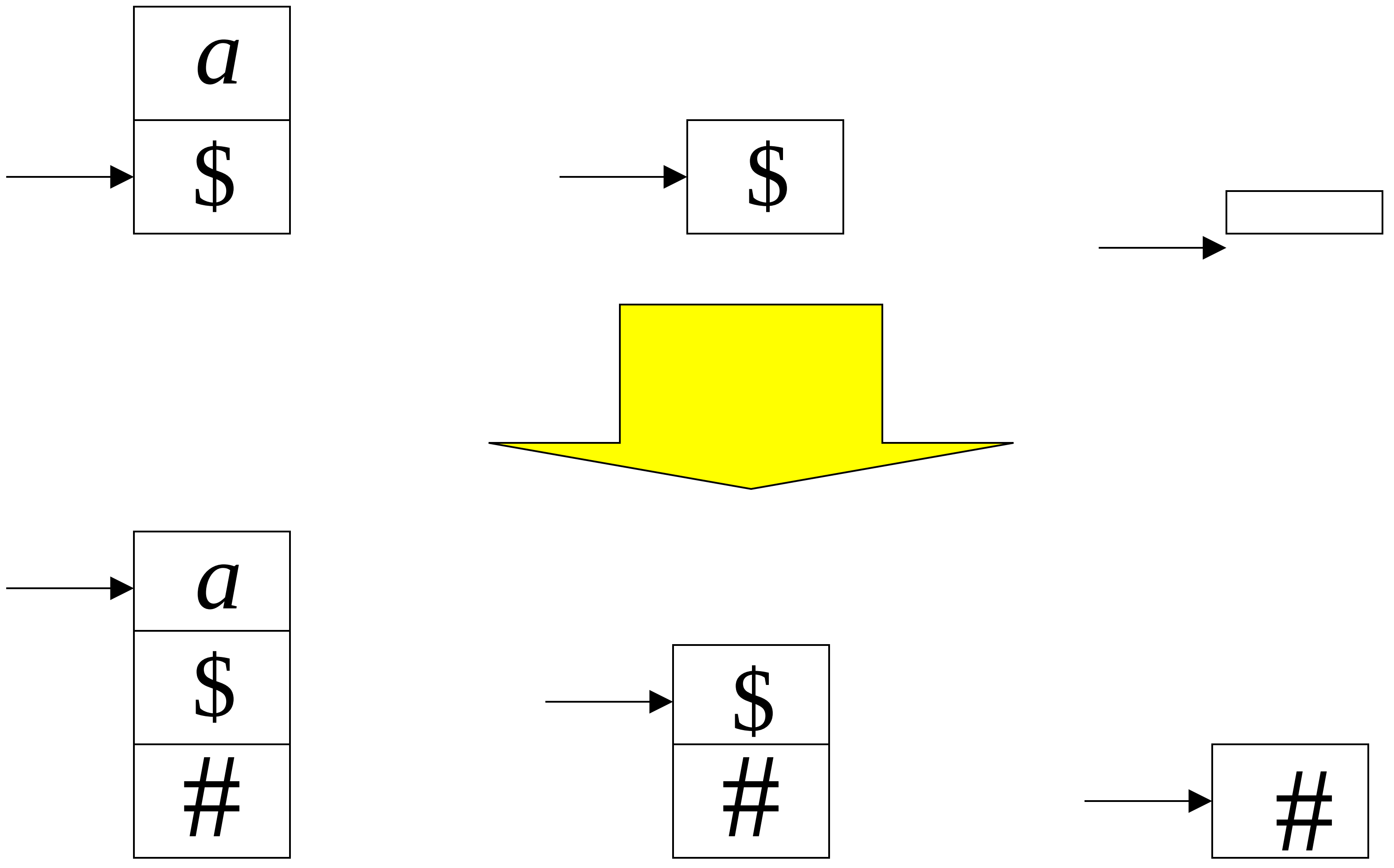
1. Modify the PDA so that the stack is never empty during computation

Stack

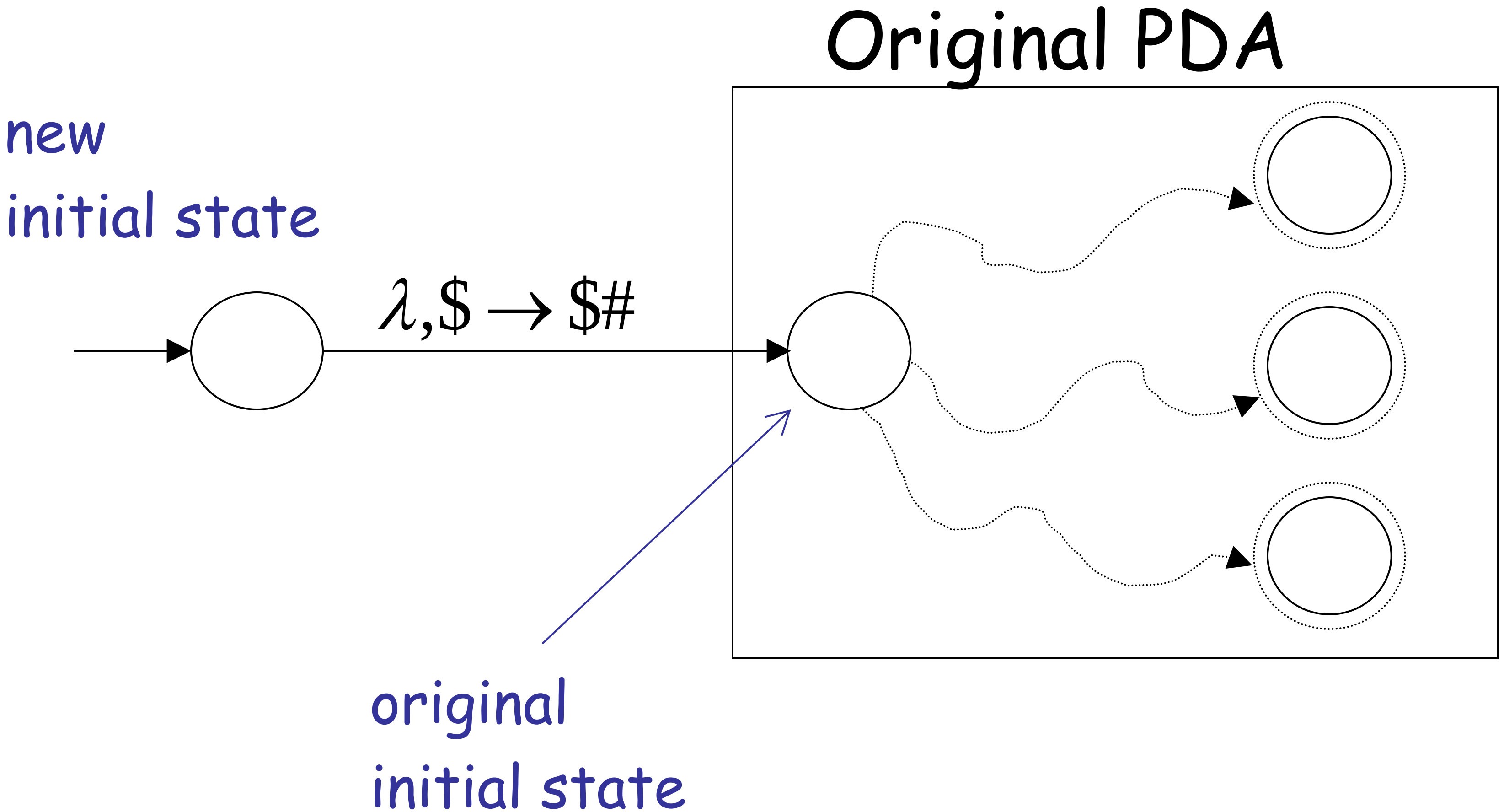


OK OK NOT OK

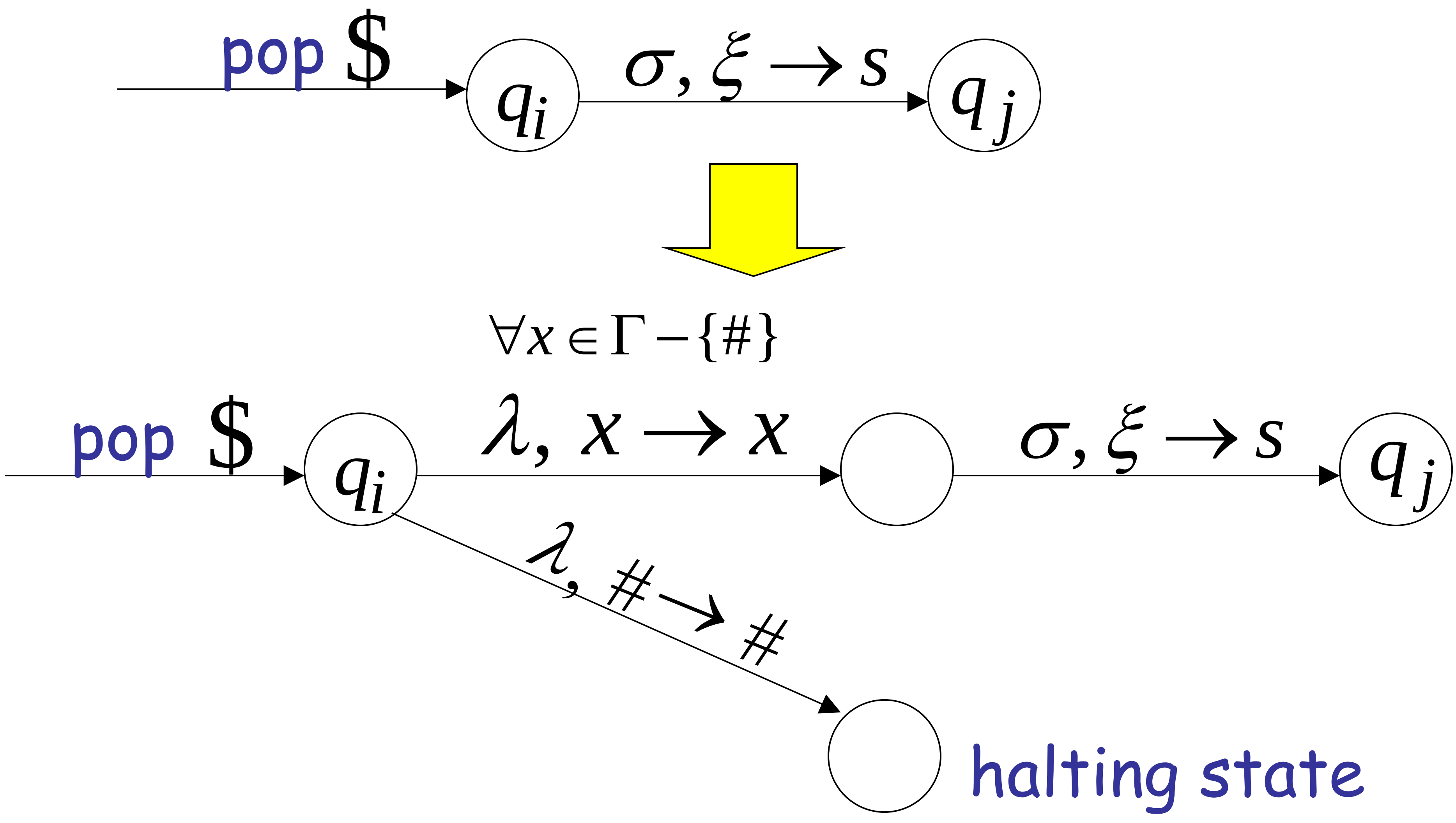
Introduce the new symbol $\#$ to mark the bottom of the stack



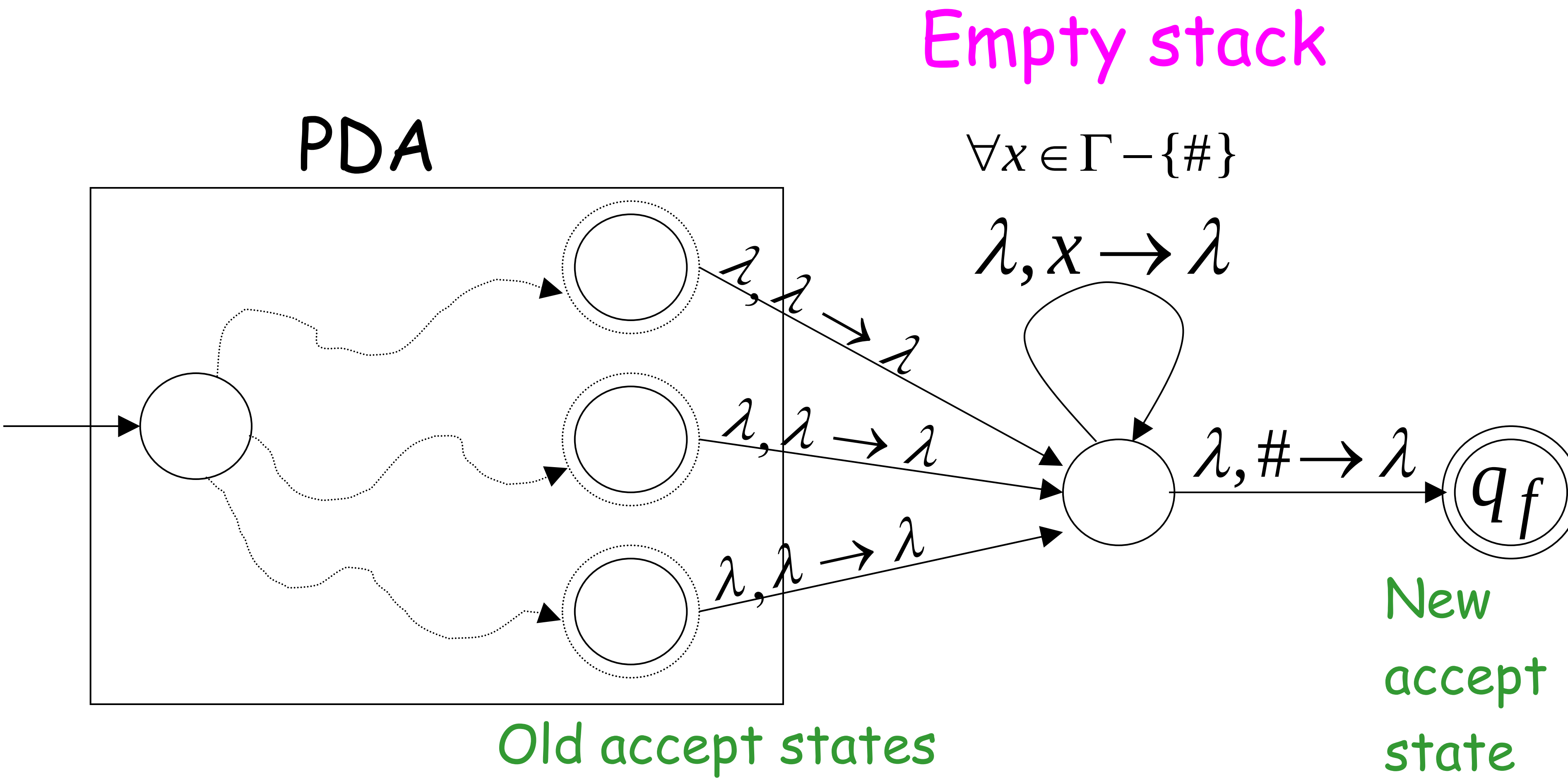
At the beginning insert # into the stack



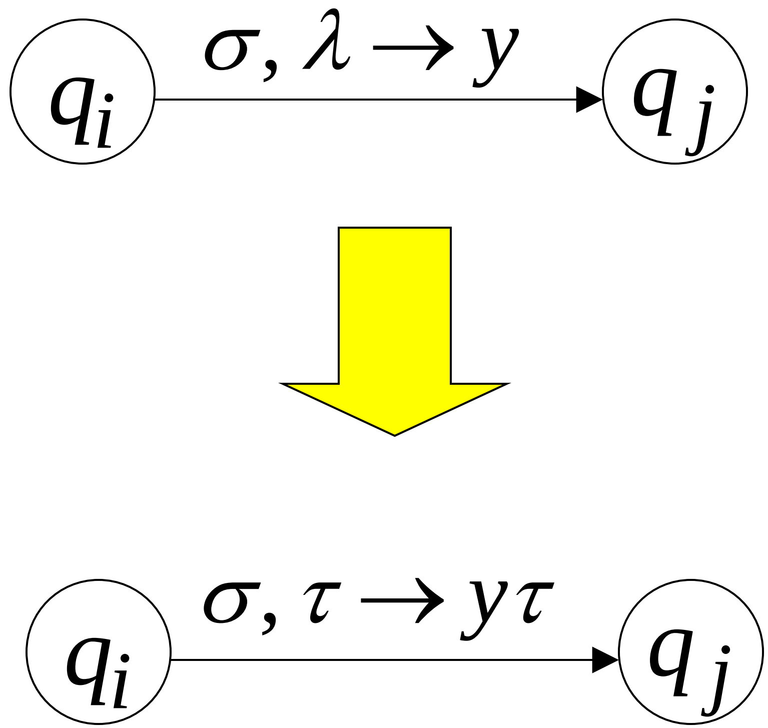
Convert all transitions so that after popping $\$$ the automaton halts



2. Modify the PDA so that at end it empties stack and has a unique accept state



3. Modify the PDA so that it has no transitions popping λ :



$$\forall \tau \in \Gamma - \{\#\}$$

Example of a PDA in correct form:

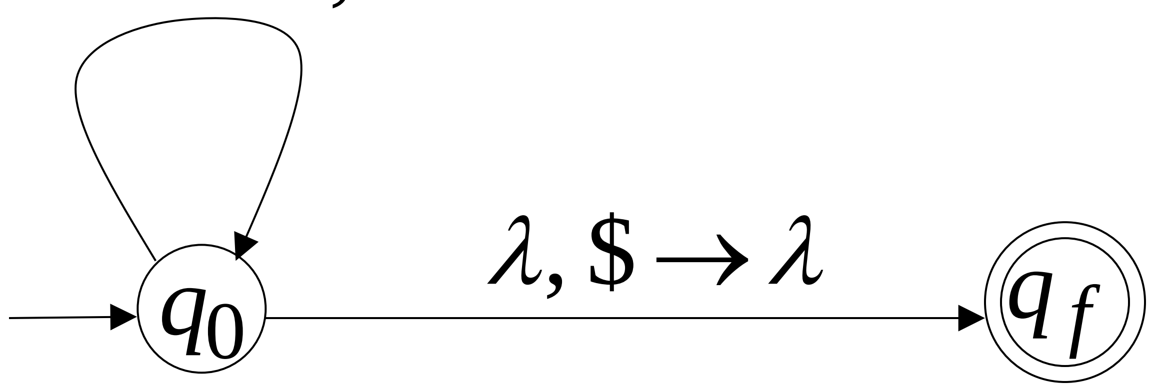
(modifications are not necessary)

$$L(M) = \{w \in \{a,b\}^* : \quad n_a(w) = n_b(w)\}$$

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



Grammar Construction

In grammar G:

Variables: A PDA stack symbols

Terminals: a PDA input symbols

Start Variable: $\$$ or $\#$ Stack bottom symbol

PDA transition

$$q \xrightarrow{a,\ A \to \lambda} q_1$$

Grammar production

$$A \to a$$

PDA transition

$$q \xrightarrow{a,\ A \rightarrow B_1 B_2 \cdots B_m} q_1$$

Grammar production

$$A \rightarrow a B_1 B_2 \cdots B_m$$

Grammar leftmost derivation

PDA computation

S	$\succ (q_1, \sigma_1 \cdots \sigma_k \sigma_{k+1} \cdots \sigma_n, \$)$
$\Rightarrow \cdots$	$\succ \cdots$
$\Rightarrow \sigma_1 \cdots \sigma_k X_1 \cdots X_m$	$\succ (q_1, \sigma_{k+1} \cdots \sigma_n, X_1 \cdots X_m)$
$\Rightarrow \cdots$	$\succ \cdots$
$\Rightarrow \sigma_1 \cdots \sigma_k \sigma_{k+1} \cdots \sigma_n$	$\succ (q_2, \lambda, \lambda)$

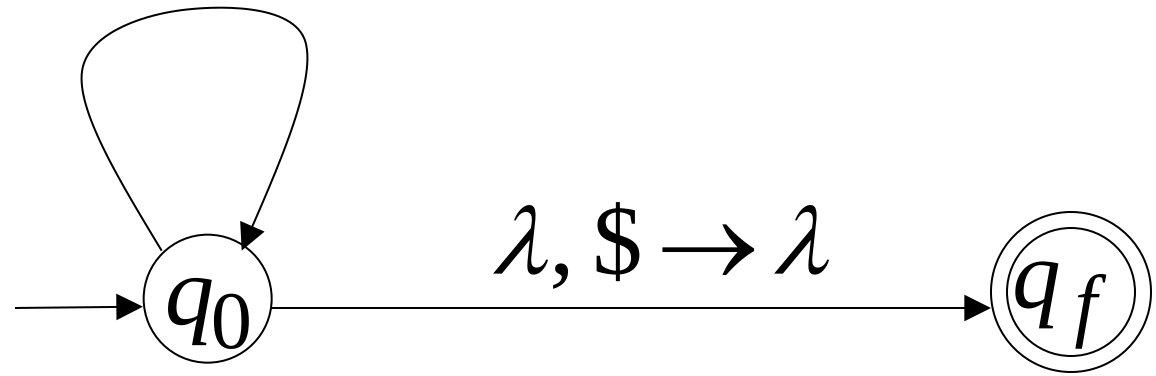
Leftmost variable

Example PDA:

$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$



Grammar:

$\$ \rightarrow a0\$ \qquad \$ \rightarrow b1\$$

$0 \rightarrow a00 \qquad 1 \rightarrow b11$

$1 \rightarrow a \qquad 0 \rightarrow b \qquad \$ \rightarrow \lambda$

Grammar Leftmost derivation:

$$
\begin{aligned}
&\$ \\
\Rightarrow\ & a0\$ \\
\Rightarrow\ & ab\$ \\
\Rightarrow\ & abb1\$ \\
\Rightarrow\ & abba\$ \\
\Rightarrow\ & abba
\end{aligned}
$$

PDA Computation:

$$
\begin{aligned}
&(q_0, abba, \$) \\
\succ\ & (q_0, bba, 0\$) \\
\succ\ & (q_0, ba, \$) \\
\succ\ & (q_0, a, 1\$) \\
\succ\ & (q_0, \lambda, \$) \\
\succ\ & (q_f, \lambda, \lambda)
\end{aligned}
$$

Derivation: $\$$

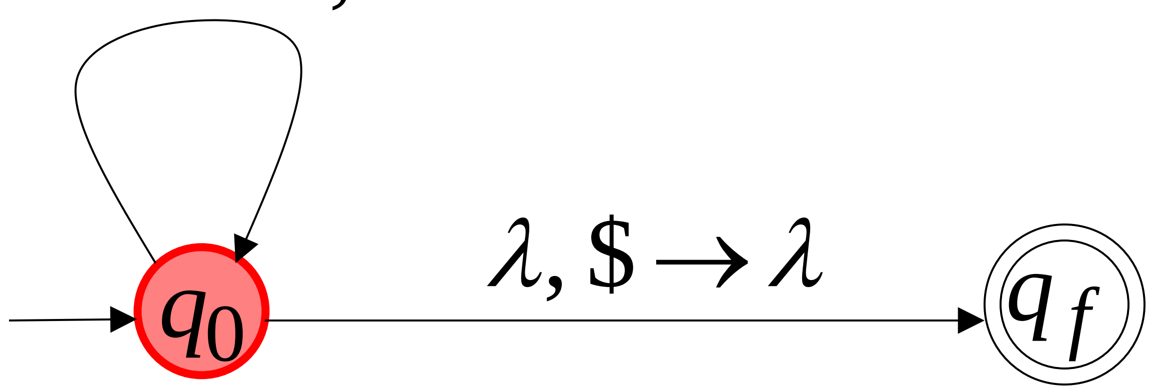
a	b	b	a

Time 0

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$



Derivation: $\$ \Longrightarrow a0\$$

a	b	b	a

Time 1

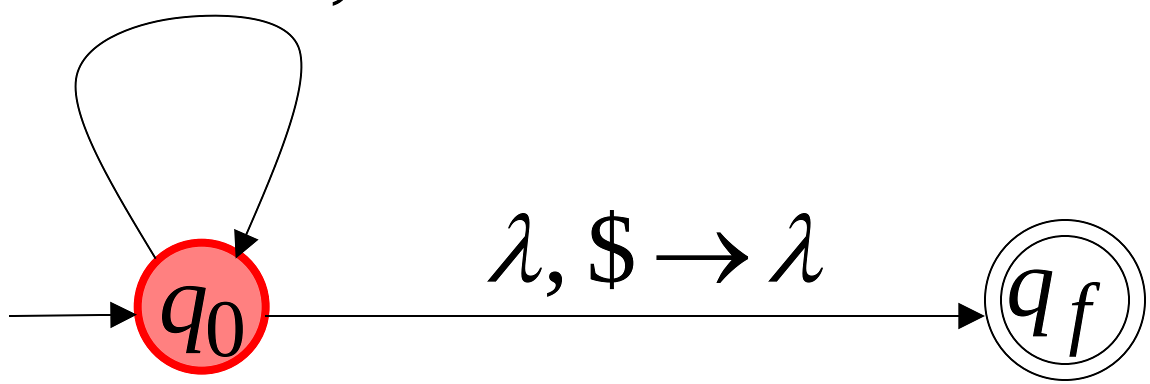
$a, \$ \rightarrow 0\$$ $\quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad b, 0 \rightarrow \lambda$

0
$

Stack



Derivation: $\$ \Rightarrow a0\$ \Rightarrow ab\$$

a	b	b	a

Time 2

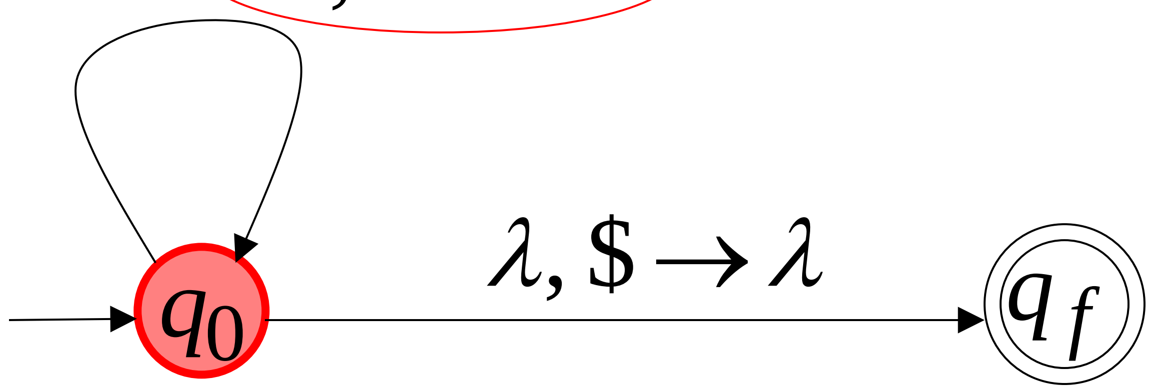
$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

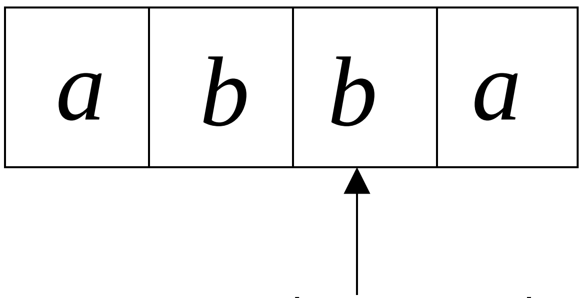
$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$

$\$$

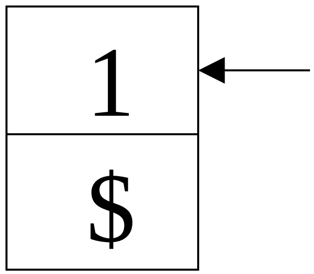
Stack



Derivation: $\$ \Rightarrow a0\$ \Rightarrow ab\$ \Rightarrow abb1\$$



Time 3

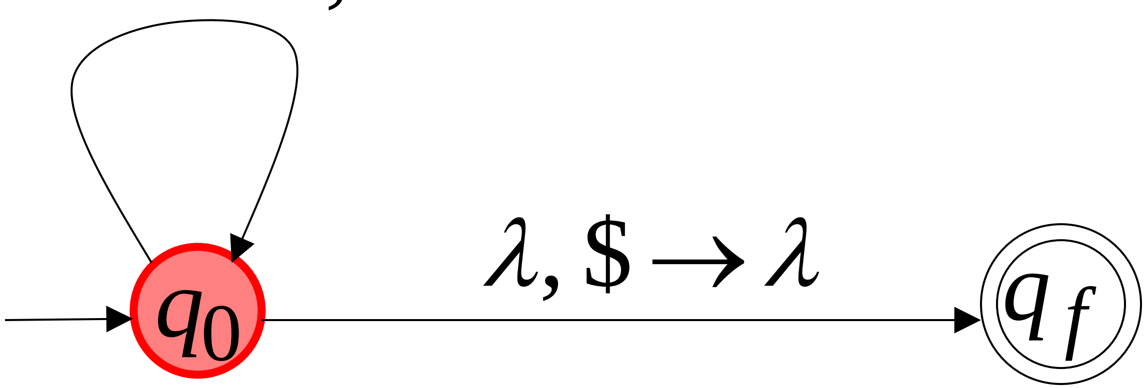


Stack

$a, \$ \rightarrow 0\$$ $\quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad b, 1 \rightarrow 11$

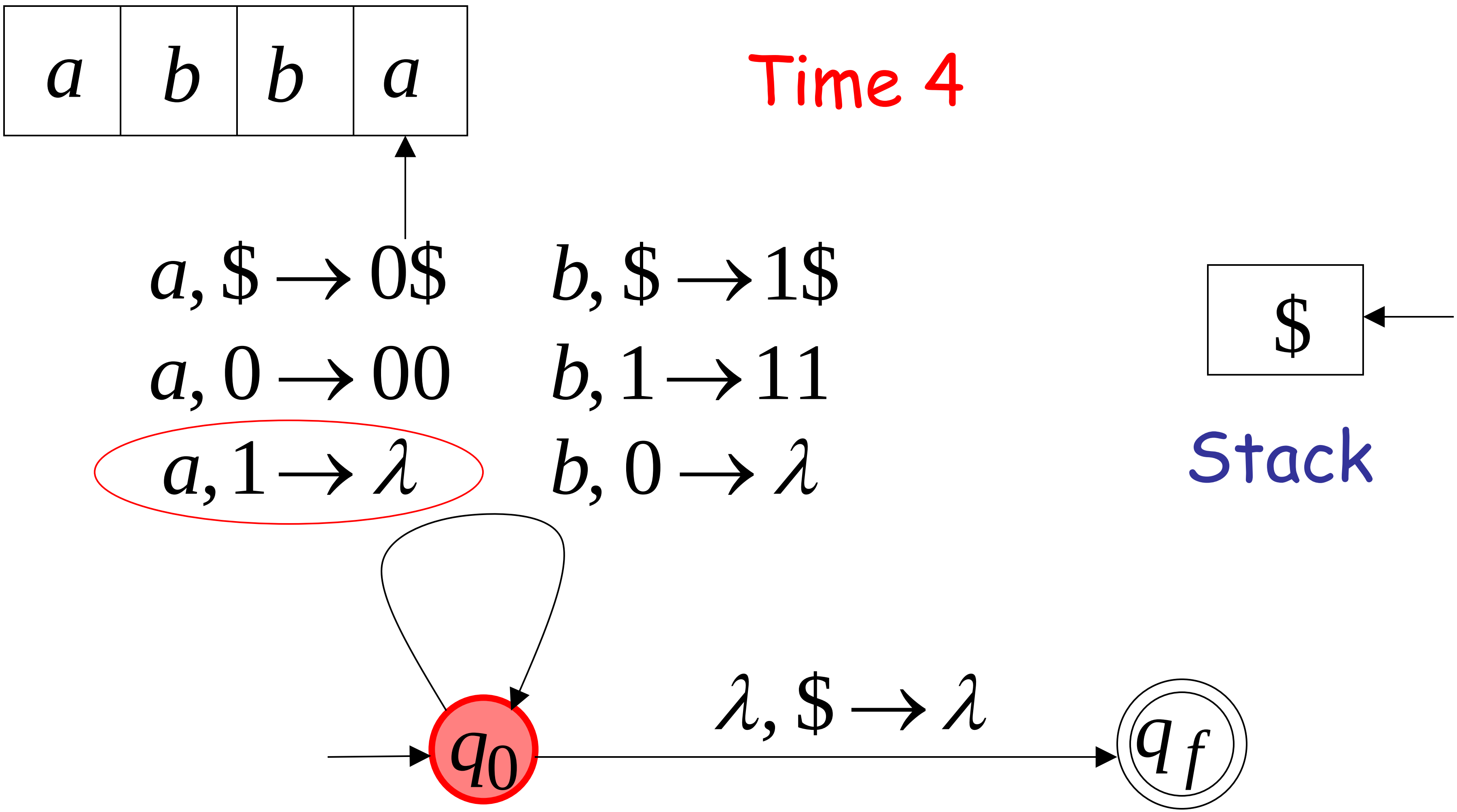
$a, 1 \rightarrow \lambda$ $\quad b, 0 \rightarrow \lambda$



Derivation: $\$ \Rightarrow a0\$ \Rightarrow ab\$ \Rightarrow abb1\$$

$abba\$$

Time 4



Derivation: $\$ \Rightarrow a0\$ \Rightarrow ab\$ \Rightarrow abb1\$$

$abba\$ \Rightarrow abba$

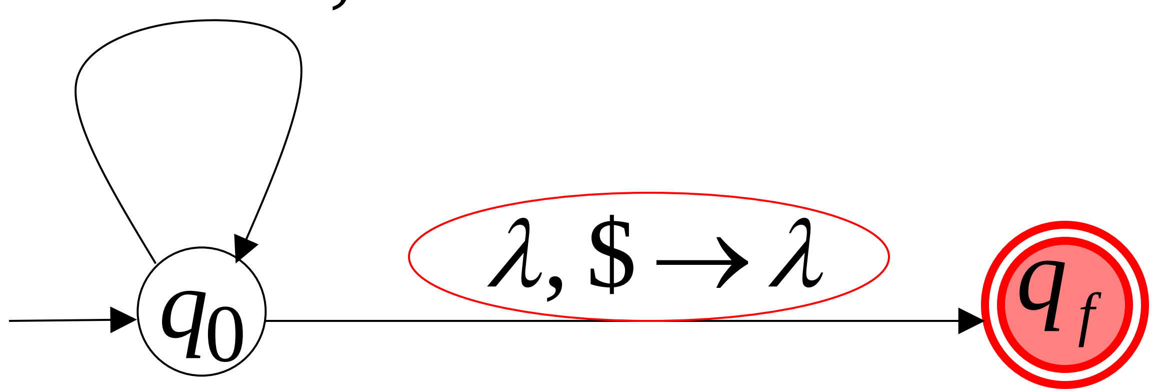
a	b	b	a

Time 5

$a, \$ \rightarrow 0\$$ $\quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad b, 0 \rightarrow \lambda$

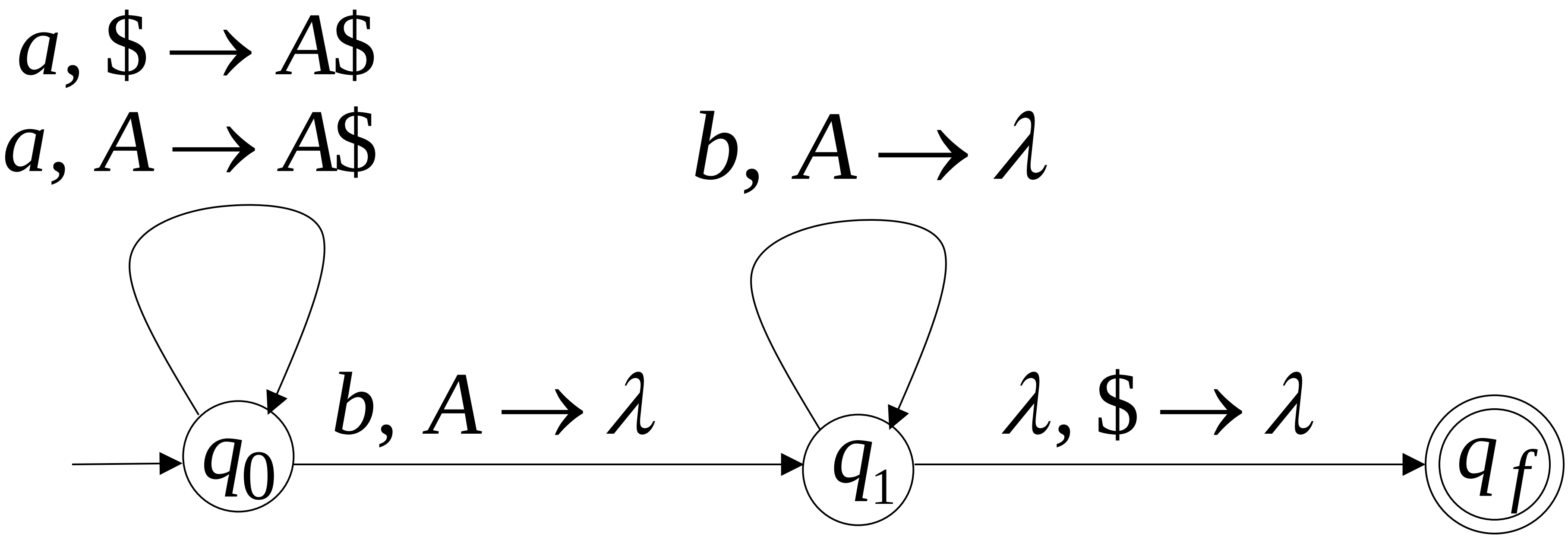


empty

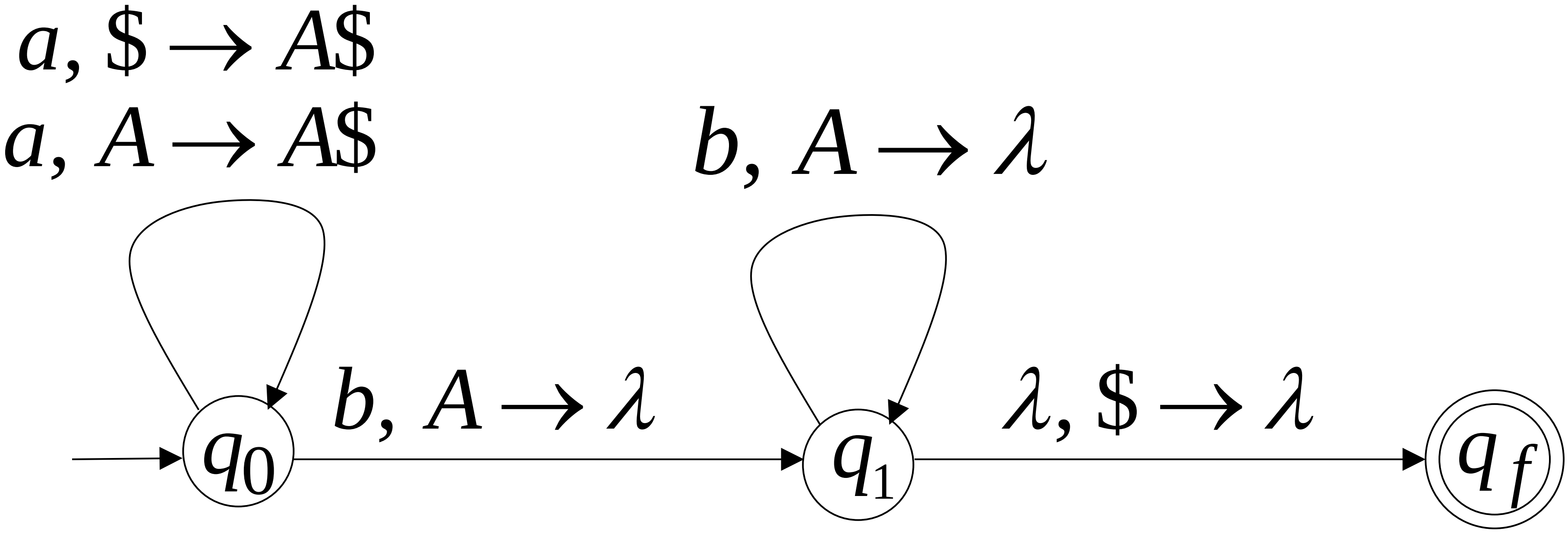
Stack

Exercise□

However, this grammar conversion does not work for all PDAs:



$$L(M) = \{a^n b^n : n \geq 1\}$$



Grammar:

$$\$ \rightarrow aA\$ \qquad A \rightarrow aA\$$$

$$\$ \rightarrow \lambda \qquad A \rightarrow b$$

Bad Derivation:

$$S \Rightarrow aA\$ \Rightarrow aaA\$ \Rightarrow aab\$ \Rightarrow aab \notin L(M)$$

Grammar:

$$\$ \rightarrow aA\$ \qquad A \rightarrow aA\$$$

$$\$ \rightarrow \lambda \qquad A \rightarrow b$$

What went wrong?

The Correct Grammar Construction

In grammar G:

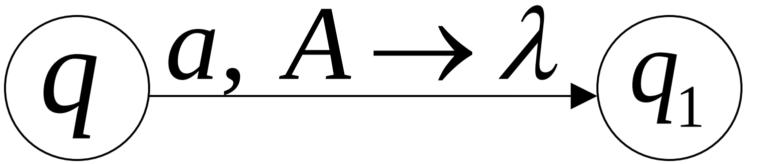
PDA stack symbol

Variables: $(q_i A q_j)$

PDA states

Terminals: Input symbols of PDA

PDA transition



Grammar production

$$(qAq_1) \to a$$

PDA transition

$$q \xrightarrow{a,\ A \to B_1 B_2 \cdots B_m} q_1$$

Grammar production

$$(qAq_{m+1}) \to a(q_1 B_1 q_2)(q_2 B_2 q_3)\cdots(q_m B_m q_{m+1})$$

For all possible states $q_2, \ldots, q_{m+1}$ in PDA

Stack bottom symbol

$\$ \text{ or } \#$

Start Variable: $(q_o Z q_f)$

Start state

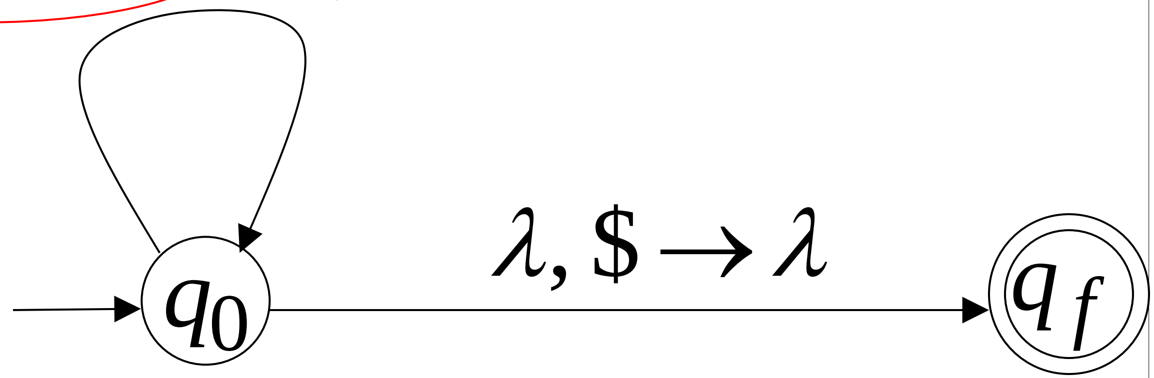
accept state

Example:

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



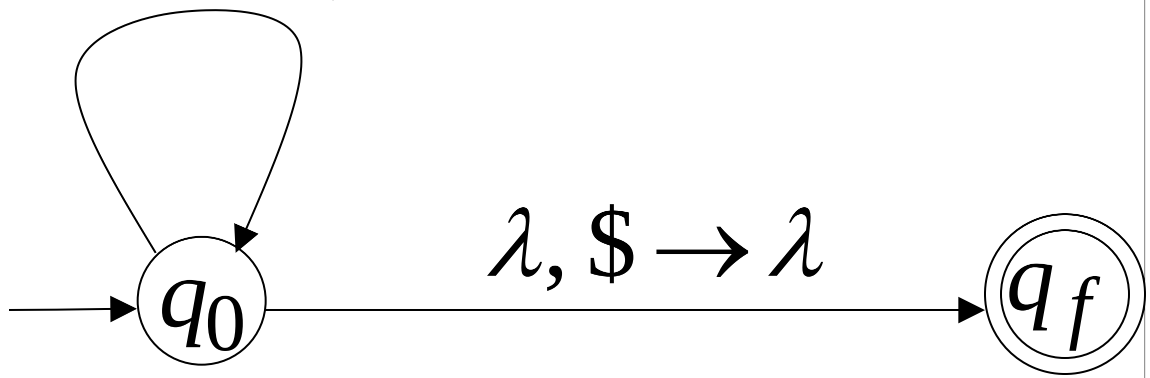
Grammar production: $(q_0 1 q_0) \rightarrow a$

Example:

$$a, \$ \to 0\$ \qquad b, \$ \to 1\$$$

$$a, 0 \to 00 \qquad b, 1 \to 11$$

$$a, 1 \to \lambda \qquad b, 0 \to \lambda$$



Grammar productions:

$$(q_0 \$ q_0) \to b(q_0 1 q_0)(q_0 \$ q_0) \mid b(q_0 1 q_f)(q_f \$ q_0)$$

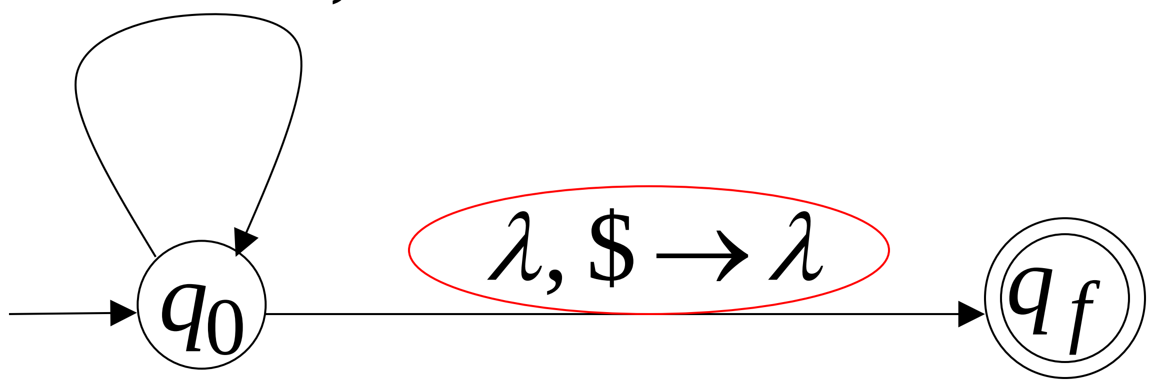
$$(q_0 \$ q_f) \to b(q_0 1 q_0)(q_0 \$ q_f) \mid b(q_0 1 q_f)(q_f \$ q_f)$$

Example:

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



Grammar production: $(q_0 \$ q_f) \rightarrow \lambda$

Resulting Grammar: $(q_0 \$ q_f)$: start variable

$$(q_0\$q_0) \rightarrow b(q_0 1 q_0)(q_0\$q_0) \mid b(q_0 1 q_f)(q_f\$q_0)$$

$$(q_0\$q_f) \rightarrow b(q_0 1 q_0)(q_0\$q_f) \mid b(q_0 1 q_f)(q_f\$q_f)$$

$$(q_0 1 q_0) \rightarrow b(q_0 1 q_0)(q_0 1 q_0) \mid b(q_0 1 q_f)(q_f 1 q_0)$$

$$(q_0 1 q_f) \rightarrow b(q_0 1 q_0)(q_0 1 q_f) \mid b(q_0 1 q_f)(q_f 1 q_f)$$

$$(q_0\$q_0) \rightarrow a(q_0 0 q_0)(q_0\$q_0) \mid a(q_0 0 q_f)(q_f\$q_0)$$

$$(q_0\$q_f) \rightarrow a(q_0 0 q_0)(q_0\$q_f) \mid a(q_0 0 q_f)(q_f\$q_f)$$

$$(q_0 0 q_0) \rightarrow a(q_0 0 q_0)(q_0 0 q_0) \mid a(q_0 0 q_f)(q_f 0 q_0)$$

$$(q_0 0 q_f) \rightarrow a(q_0 0 q_0)(q_0 0 q_f) \mid a(q_0 0 q_f)(q_f 0 q_f)$$

$$(q_0 1 q_0) \rightarrow a$$

$$(q_0 0 q_0) \rightarrow b$$

$$(q_0 \$ q_f) \rightarrow \lambda$$

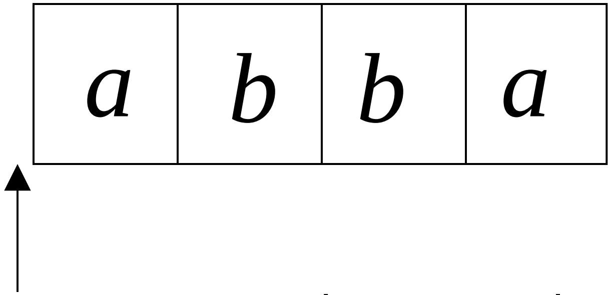
Grammar
Leftmost
derivation

PDA
computation

$$(q_0\$q_f)$$
$$\Rightarrow a(q_0 0 q_0)(q_0\$q_f)$$
$$\Rightarrow ab(q_0\$q_f)$$
$$\Rightarrow abb(q_0 1 q_0)(q_0\$q_f)$$
$$\Rightarrow abba(q_0\$q_f)$$
$$\Rightarrow abba$$

$$(q_0, abba, \$)$$
$$\succ (q_0, bba, 0\$)$$
$$\succ (q_0, ba, \$)$$
$$\succ (q_0, a, 1\$)$$
$$\succ (q_0, \lambda, \$)$$
$$\succ (q_f, \lambda, \lambda)$$

Derivation: $(q_0 \$ q_f)$

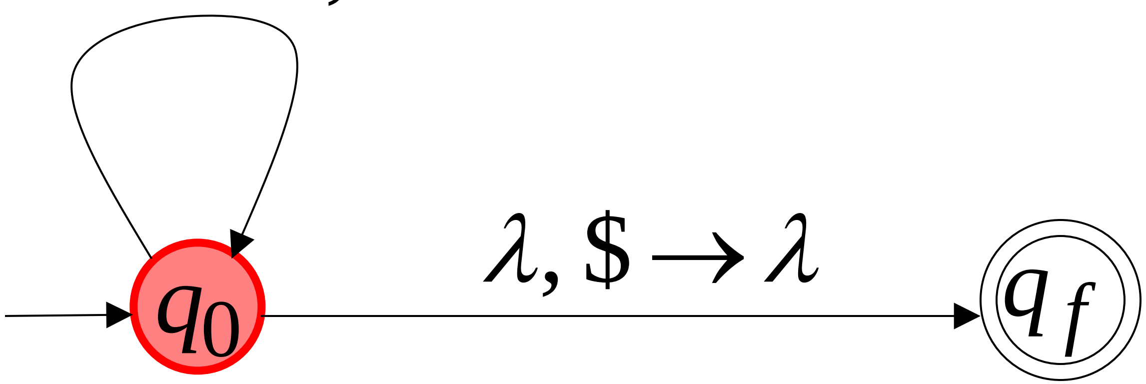


Time 0

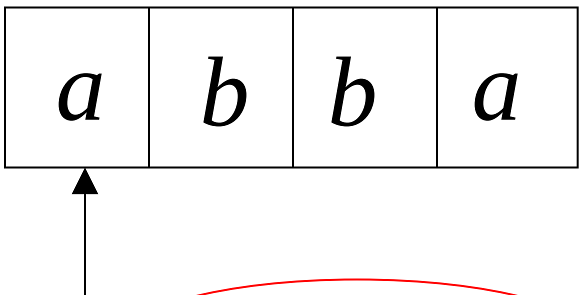
$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



Derivation: $(q_0\$q_f) \Longrightarrow a(q_0 0 q_0)(q_0\$q_f)$

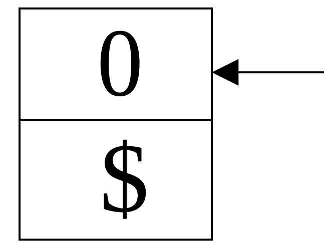


Time 1

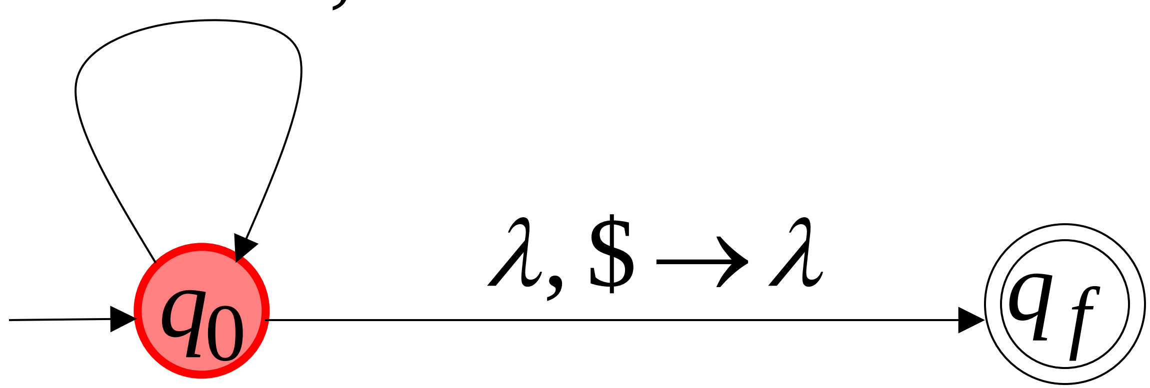
$a, \$ \rightarrow 0\$$ $\quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad b, 1 \rightarrow 11$

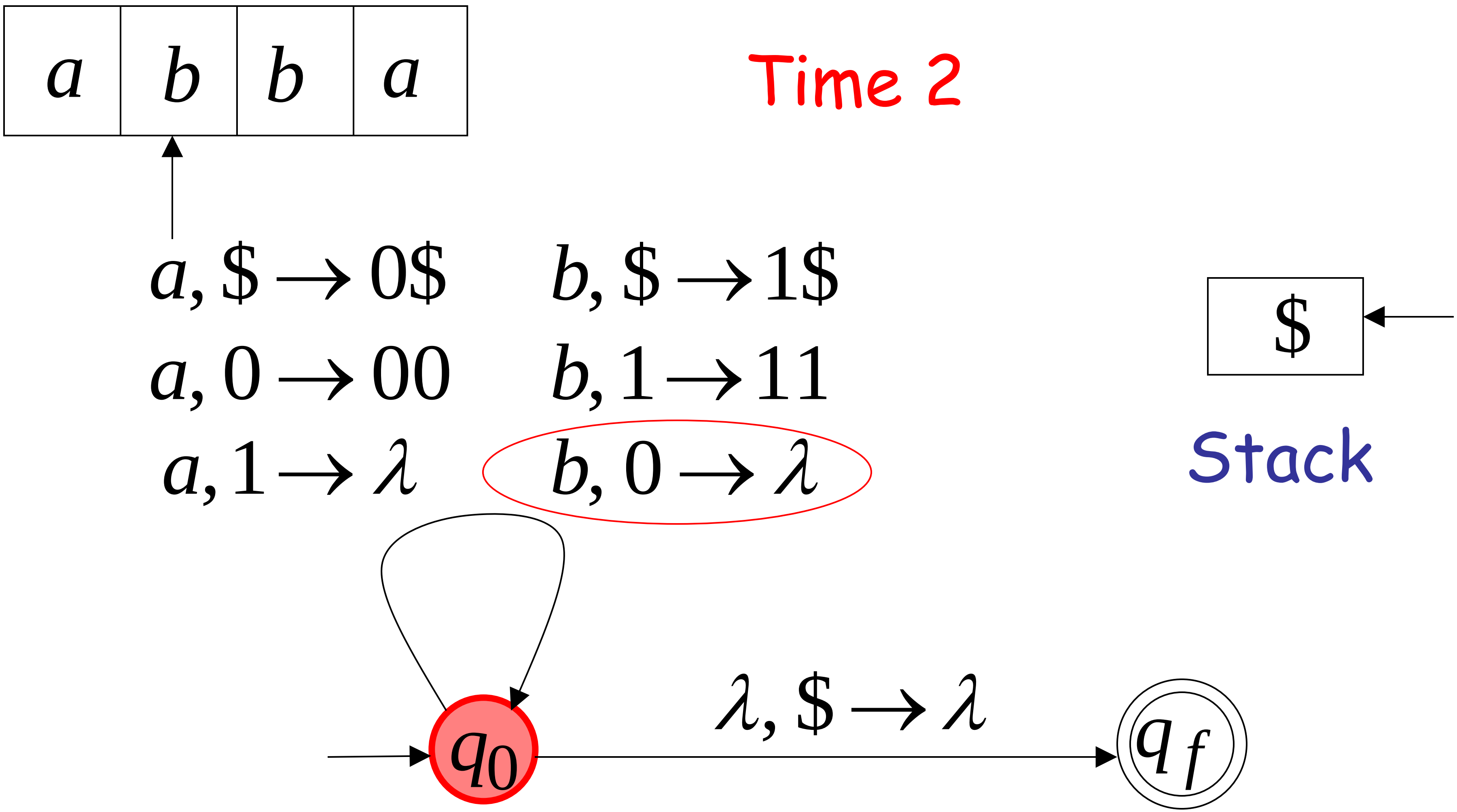
$a, 1 \rightarrow \lambda$ $\quad b, 0 \rightarrow \lambda$



Stack



Derivation: $(q_0\$q_f) \Longrightarrow a(q_0 0 q_0)(q_0\$q_f) \Longrightarrow ab(q_0\$q_f)$



Derivation: $(q_0\$q_f) \Rightarrow a(q_0 0 q_0)(q_0\$q_f) \Rightarrow ab(q_0\$q_f)$

$\Rightarrow abb(q_0 1 q_0)(q_0\$q_f)$

---

a	b	b	a

Time 3

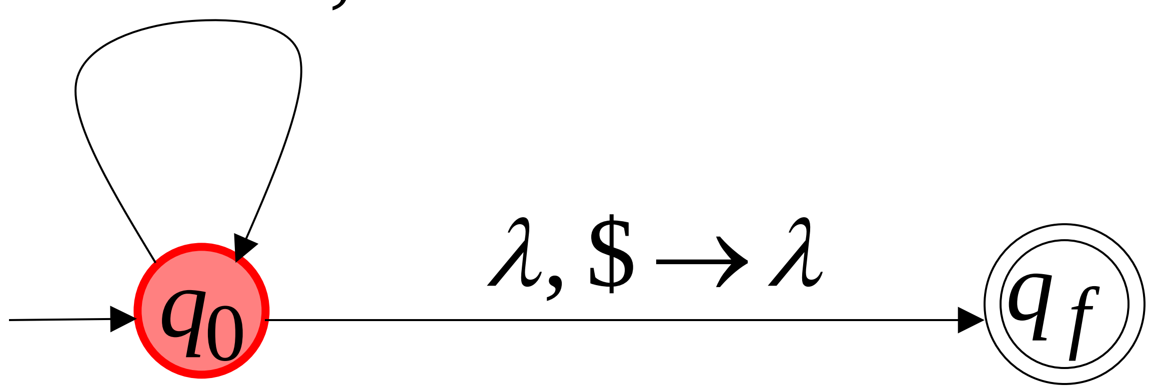
$a,\$ \rightarrow 0\$$ $\quad$ $b,\$ \rightarrow 1\$$

$a,0 \rightarrow 00$ $\quad$ $b,1 \rightarrow 11$

$a,1 \rightarrow \lambda$ $\quad$ $b,0 \rightarrow \lambda$

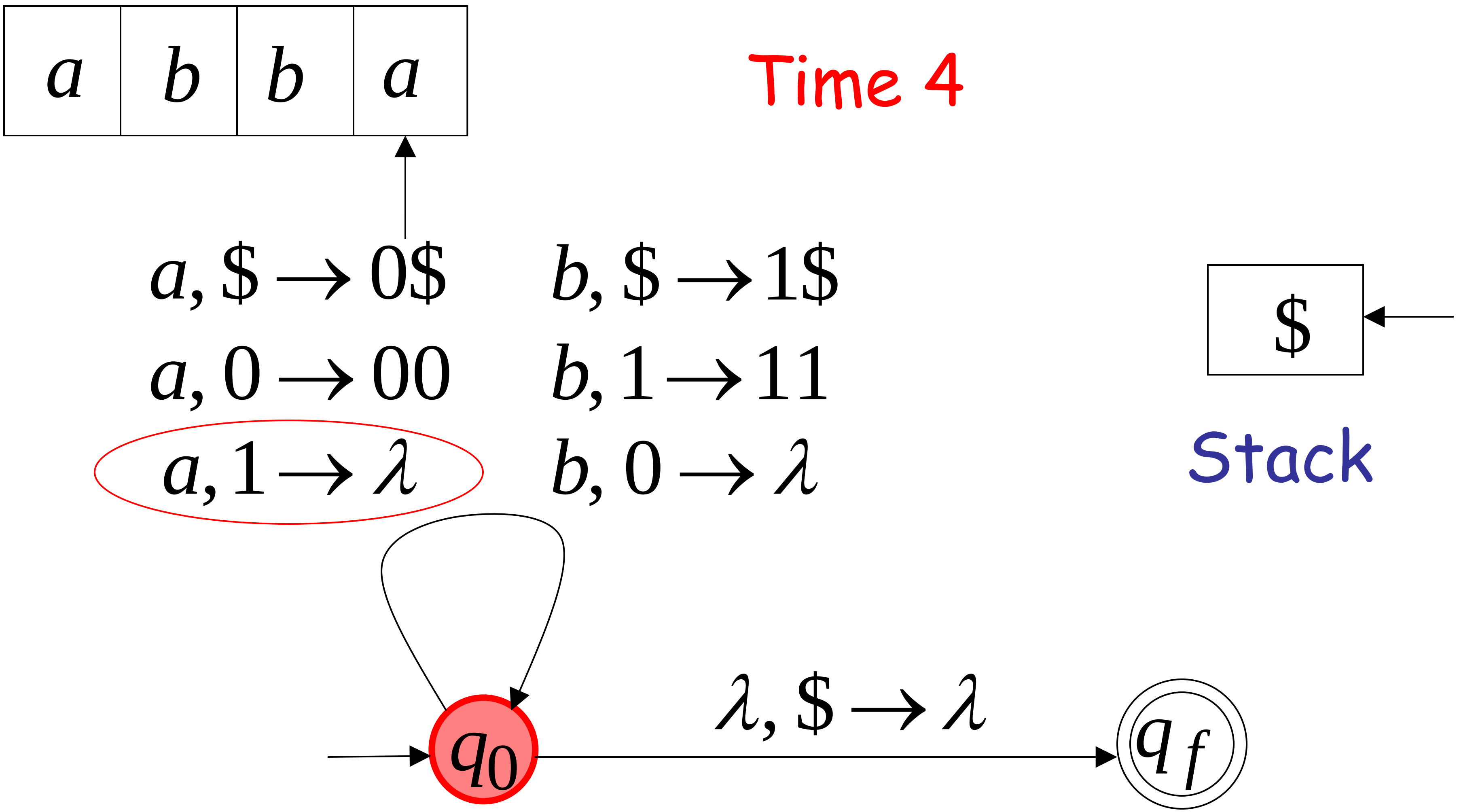
1
$

Stack



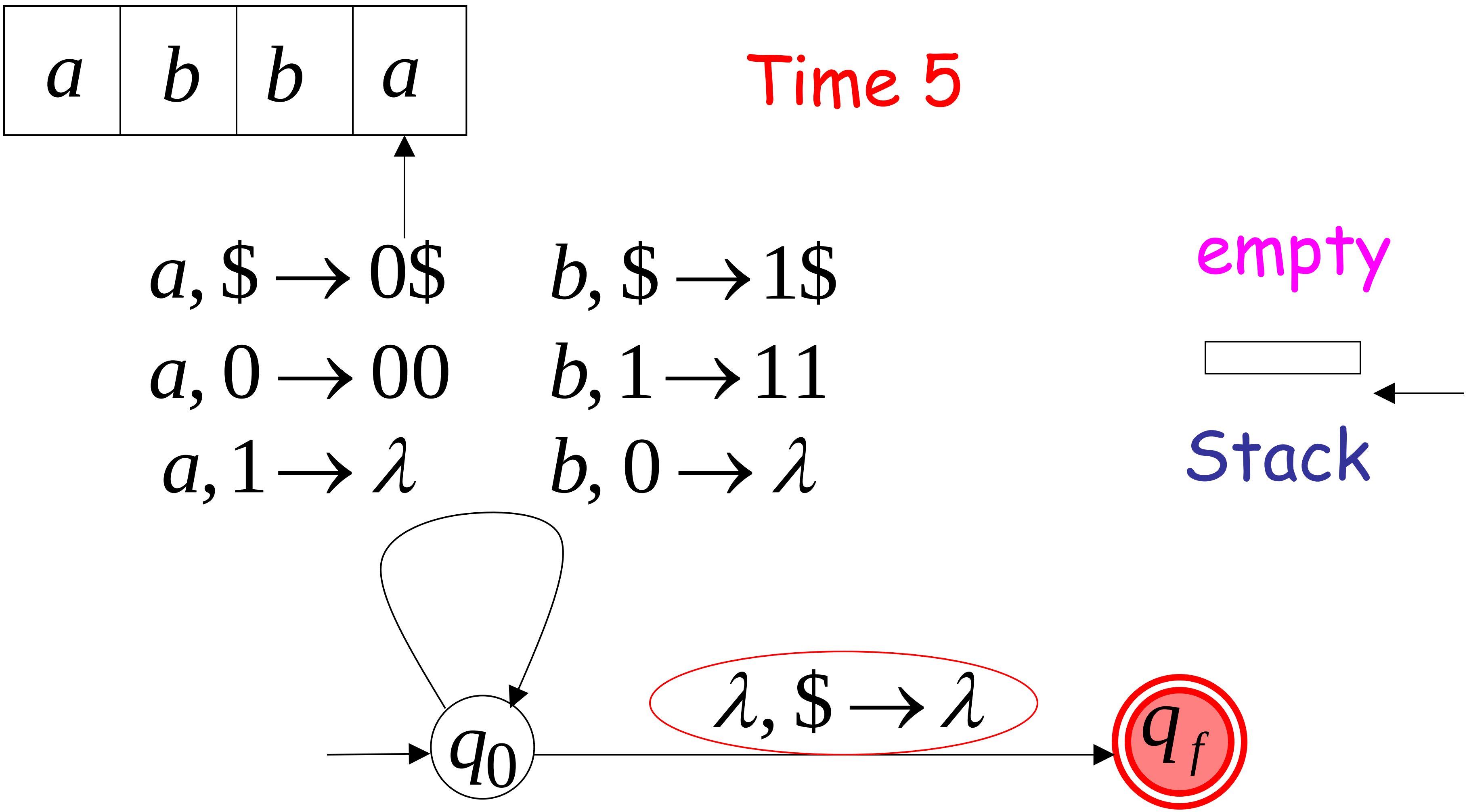
Derivation: $(q_0\$q_f) \Rightarrow a(q_0 0 q_0)(q_0\$q_f) \Rightarrow ab(q_0\$q_f)$

$\Rightarrow abb(q_0 1 q_0)(q_0\$q_f) \Rightarrow abba(q_0\$q_f)$



Derivation: $(q_0\$q_f) \Rightarrow a(q_0 0 q_0)(q_0\$q_f) \Rightarrow ab(q_0\$q_f)$

$\Rightarrow abb(q_0 1 q_0)(q_0\$q_f) \Rightarrow abba(q_0\$q_f) \Rightarrow abba$



In general:

Grammar

$$(q_i A q_j) \overset{*}{\Rightarrow} wB$$

If and Only if

PDA

$$(q_i, w, A) \overset{*}{\succ} (q_j, \lambda, B)$$

Thus:

Grammar generates w

$$(q_0\$q_f) \stackrel{*}{\Rightarrow} w$$

If and Only if

PDA accepts w

$$(q_0, w, \$) \stackrel{*}{\succ} (q_f, \lambda, \lambda)$$

Therefore:

For any PDA
there is a context-free grammar
that accepts the same language

$$\left\{\begin{array}{c}\text{Context-Free}\\\text{Languages}\\\text{(Grammars)}\end{array}\right\} \supseteq \left\{\begin{array}{c}\text{Languages}\\\text{Accepted by}\\\text{PDAs}\end{array}\right\}$$