Formal Languages

Turing's Thesis

Turing's thesis:

Any computation carried out
by mechanical means
can be performed by a Turing Machine

(1930)

Computer Science Law:

A computation is mechanical
if and only if
it can be performed by a Turing Machine

There is no known model of computation
more powerful than Turing Machines

Definition of Algorithm:

An algorithm for function $f(w)$
is a
Turing Machine which computes $f(w)$

Algorithms are Turing Machines

When we say:

There exists an algorithm

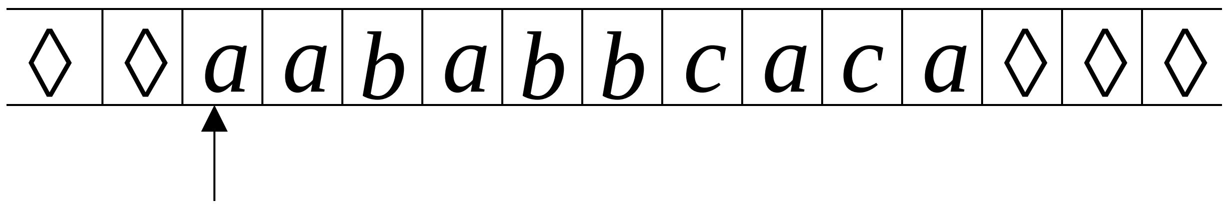
We mean:

There exists a Turing Machine
that executes the algorithm

Variations of the Turing Machine

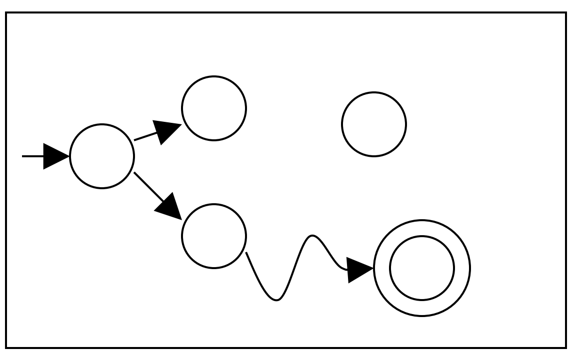
The Standard Model

Infinite Tape



Read-Write Head (Left or Right)

Control Unit



Deterministic

Variations of the Standard Model

Turing machines with:

- Stay-Option
- Semi-Infinite Tape
- Off-Line
- Multitape
- Multidimensional
- Nondeterministic

The variations form different Turing Machine **Classes**

We want to prove:

Each **Class** has the same power as the Standard Model

Same Power of two classes means:

The two classes of Turing machines accept the same languages

Same Power of two classes means:

For any machine M_1 of first class

there is a machine M_2 of second class

such that: $$L(M_1) = L(M_2)$$

And vice-versa

Simulation: a technique to prove same power

Simulate the machine of one class
with a machine of the other class

First Class

Original Machine

M_1

Second Class

Simulation Machine

M_2

M_1

Configurations in the Original Machine correspond to configurations in the Simulation Machine

$$\text{Original Machine:} \quad d_0 \succ d_1 \succ \cdots \succ d_n$$

$$\updownarrow \qquad \updownarrow \qquad \qquad \updownarrow$$

$$\text{Simulation Machine:} \quad d'_0 \overset{*}{\succ} d'_1 \overset{*}{\succ} \cdots \overset{*}{\succ} d'_n$$

Final Configuration

Original Machine: d_f

$\updownarrow$

Simulation Machine: d'_f

The Simulation Machine and the Original Machine accept the same language

Turing Machines with Stay-Option

The head can stay in the same position

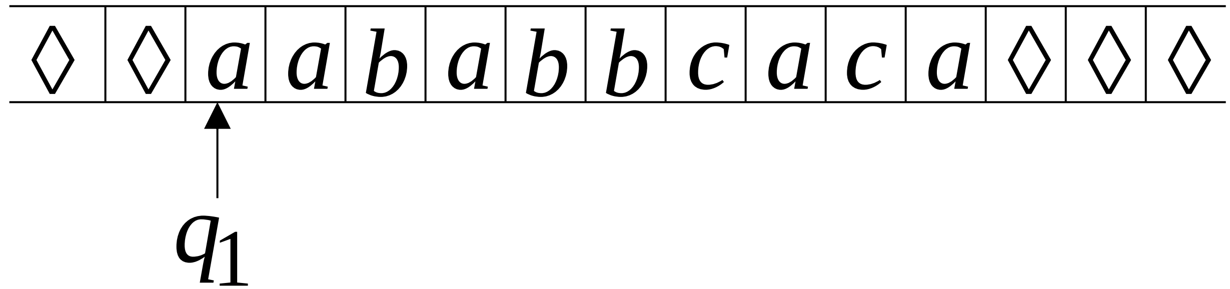
$\lozenge$	$\lozenge$	a	a	b	a	b	b	c	a	c	a	$\lozenge$	$\lozenge$	$\lozenge$

Left, Right, Stay

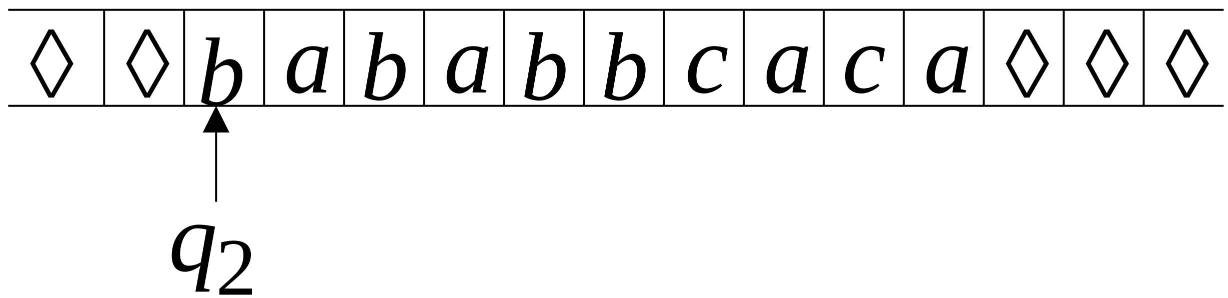
L,R,S: moves

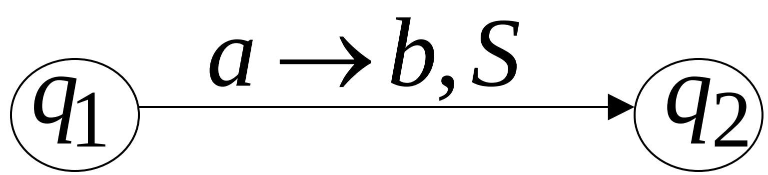
Example:

Time 1



Time 2





Theorem: Stay-Option Machines have the same power as Standard Turing machines

Proof?

Proof:

Part 1: Stay-Option Machines
are at least as powerful as
Standard machines

Proof: a Standard machine is also
a Stay-Option machine
(that never uses the S move)

Proof:

Part 2: Standard Machines are at least as powerful as Stay-Option machines

Proof: a standard machine can simulate a Stay-Option machine

Stay-Option Machine

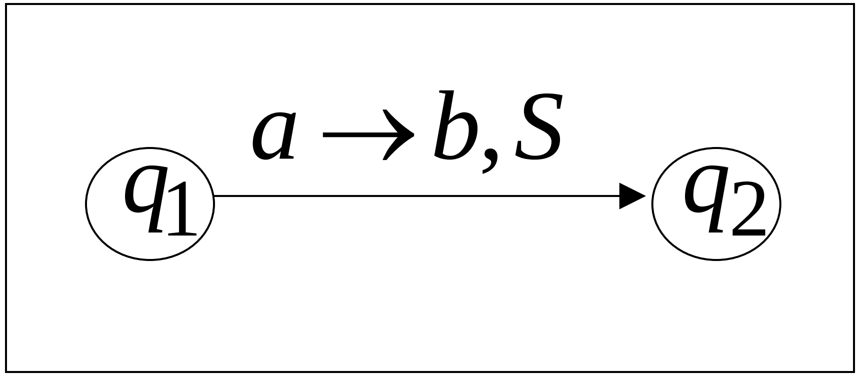
$$q_1 \xrightarrow{a \to b, L} q_2$$

Simulation in Standard Machine

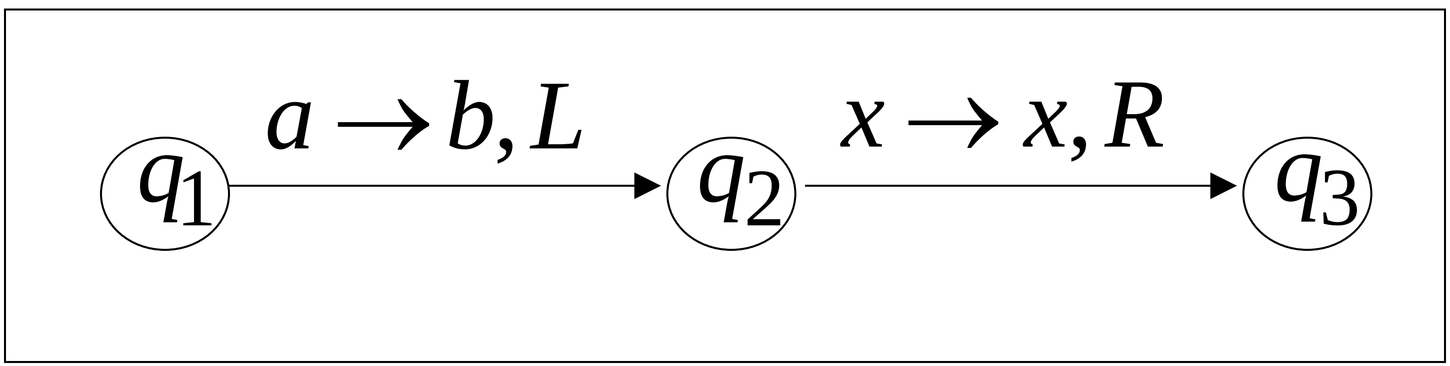
$$q_1 \xrightarrow{a \to b, L} q_2$$

Similar for Right moves

Stay-Option Machine



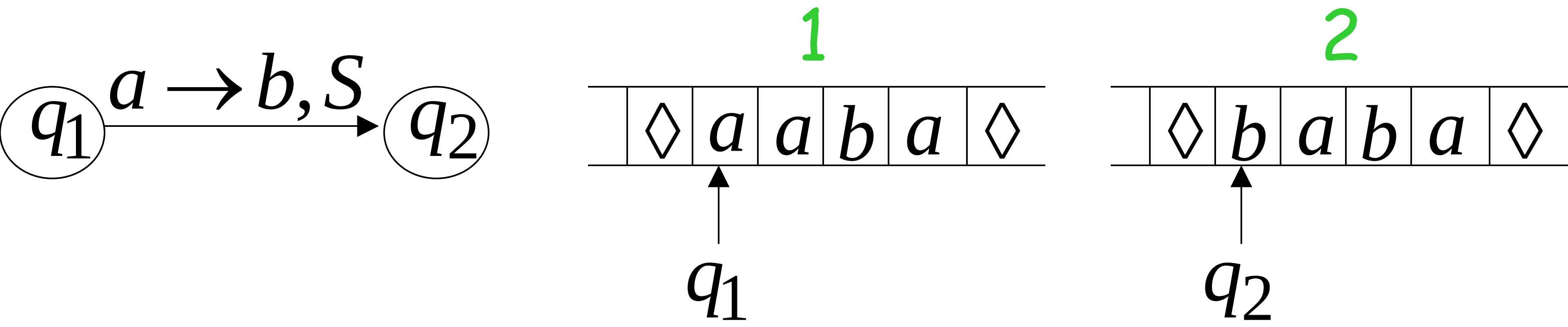
Simulation in Standard Machine



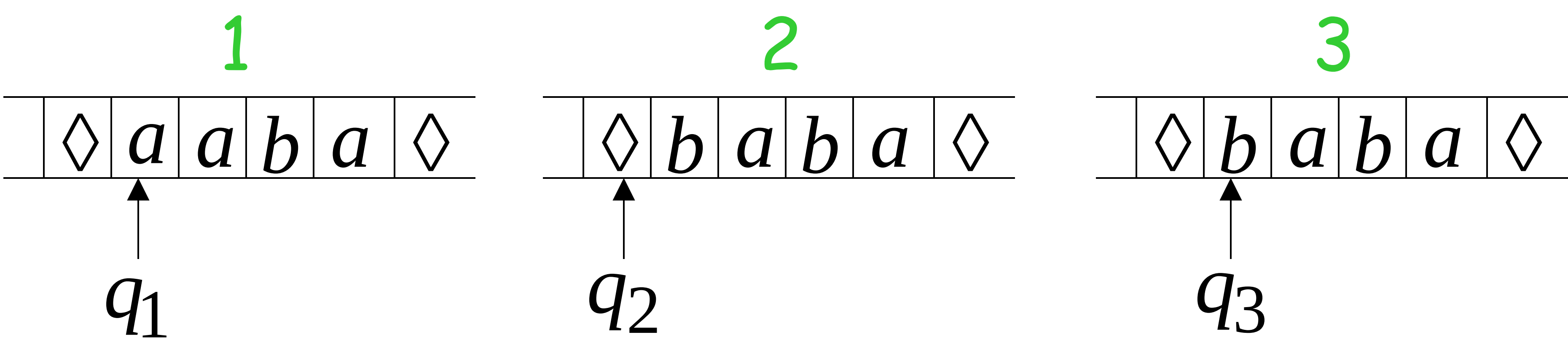
For every symbol x

Example

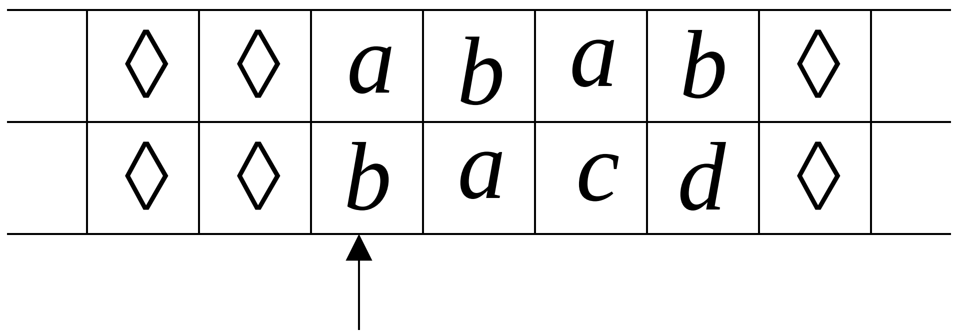
Stay-Option Machine:



Simulation in Standard Machine:



Standard Machine--Multiple Track Tape

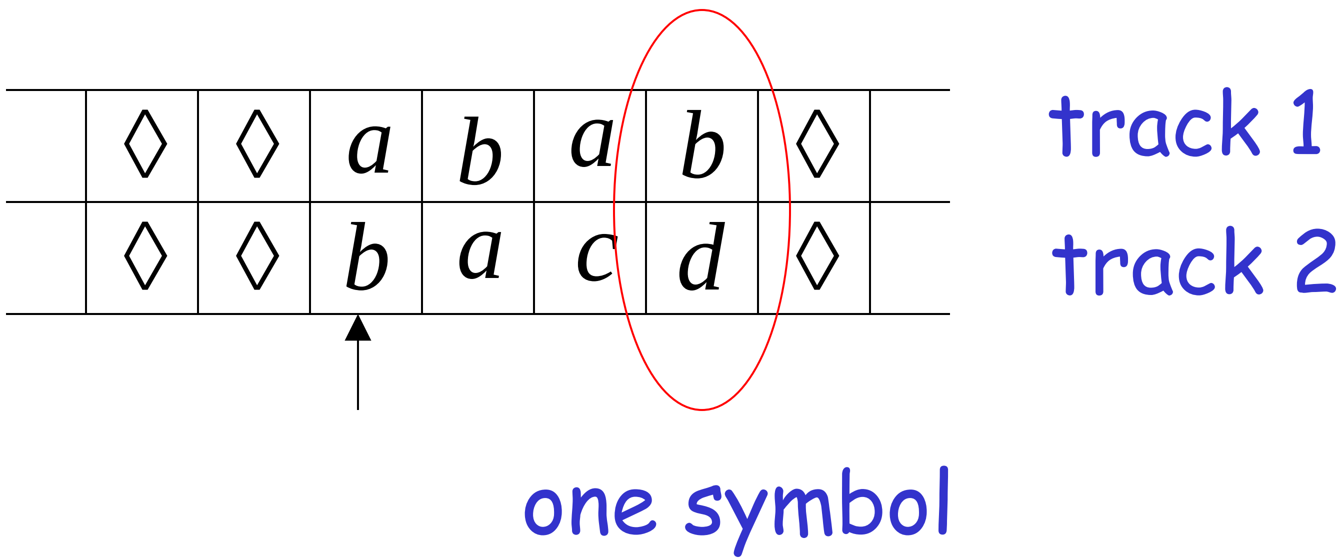


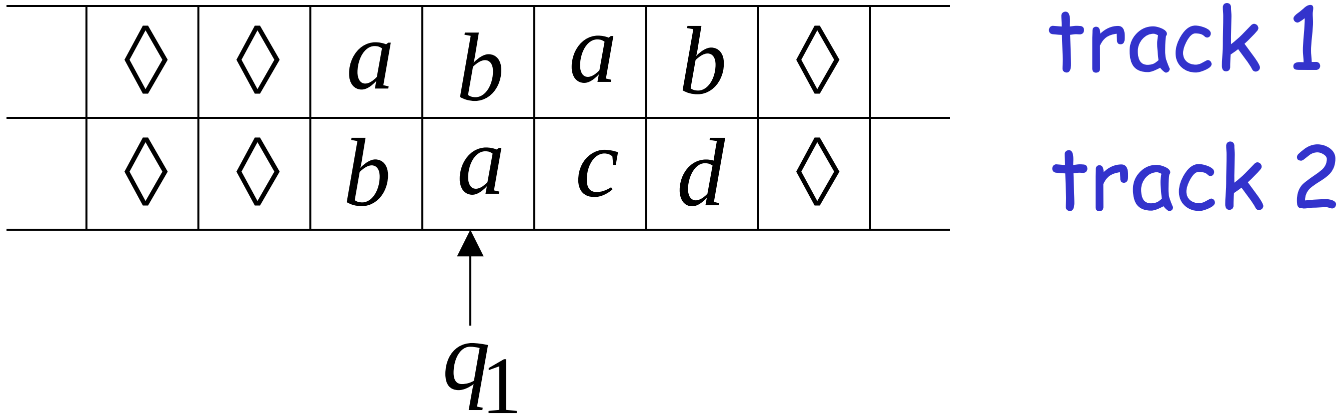
track 1

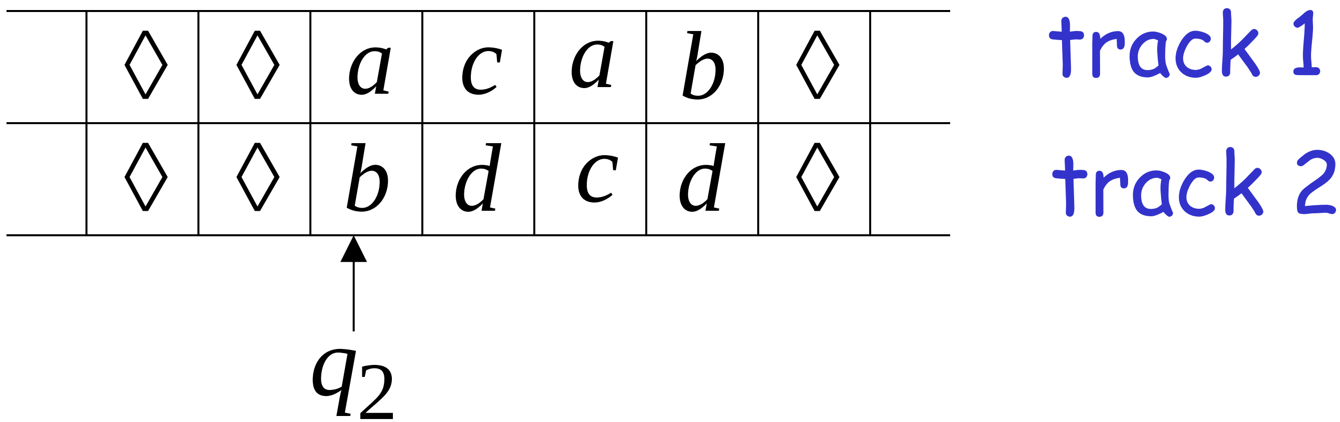
track 2

Proof of equivalence?

Standard Machine--Multiple Track Tape

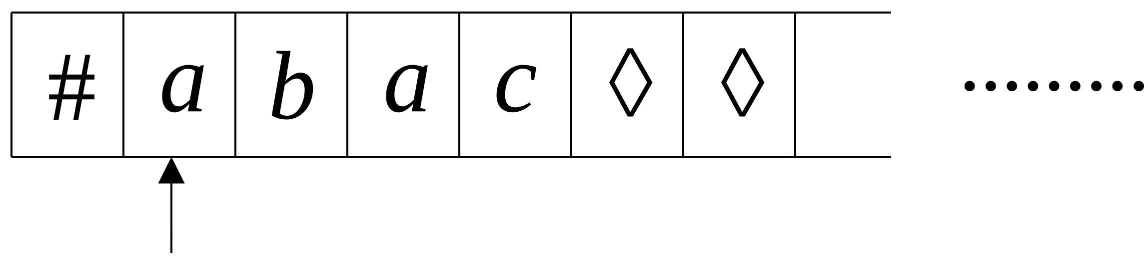






$$q_1 \xrightarrow{(b,a)\rightarrow(c,d),L} q_2$$

Semi-Infinite Tape



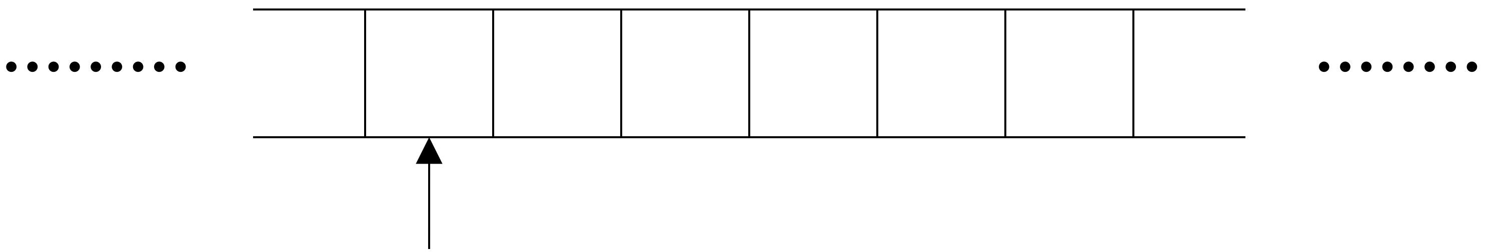
Proof of equivalence?

Standard Turing machines simulate
Semi-infinite tape machines:

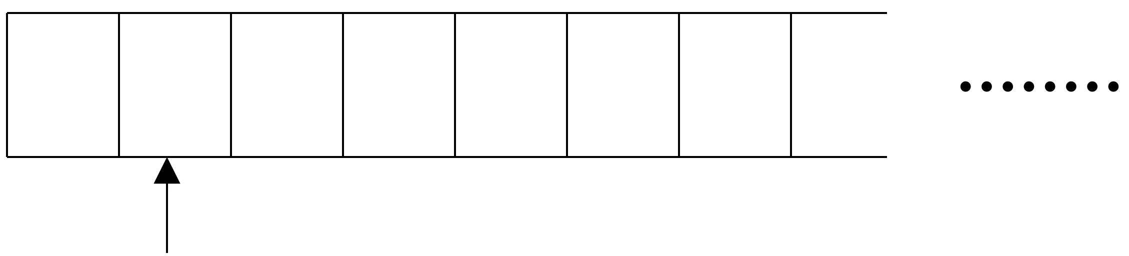
Trivial

Semi-infinite tape machines simulate Standard Turing machines:

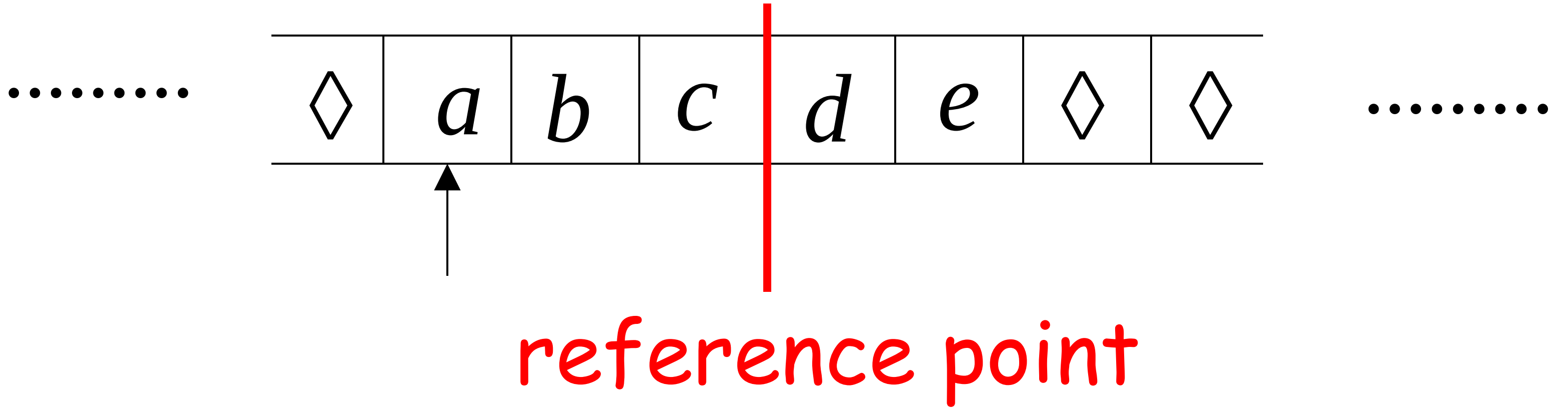
Standard machine



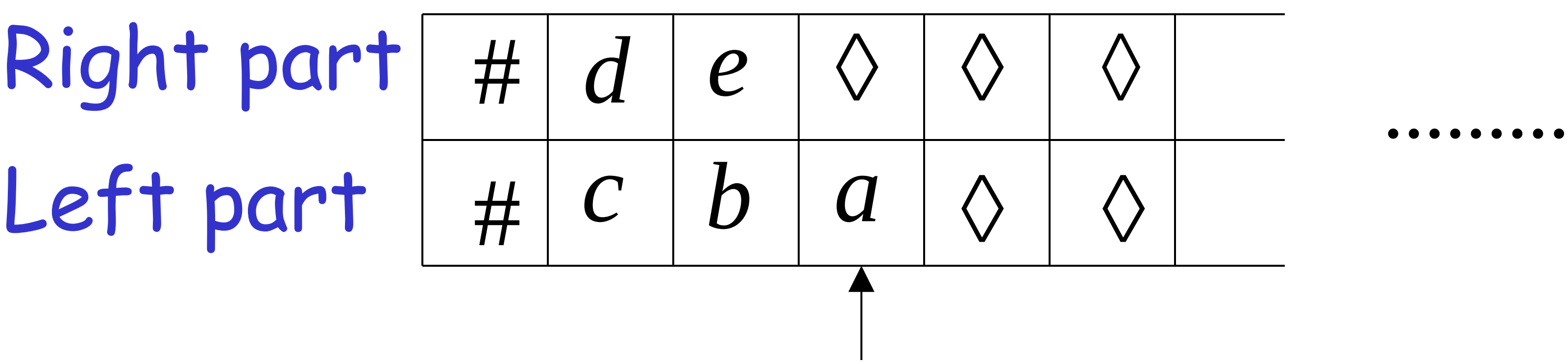
Semi-infinite tape machine



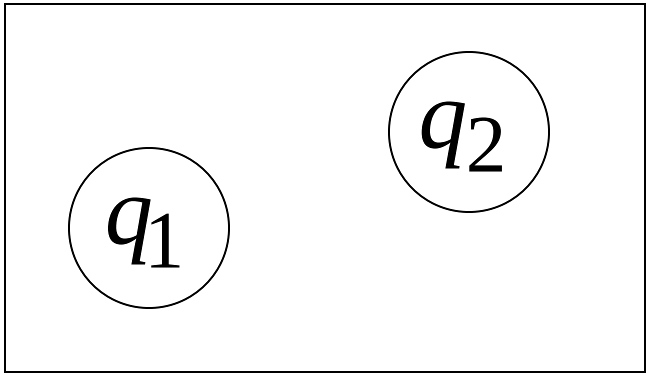
Standard machine



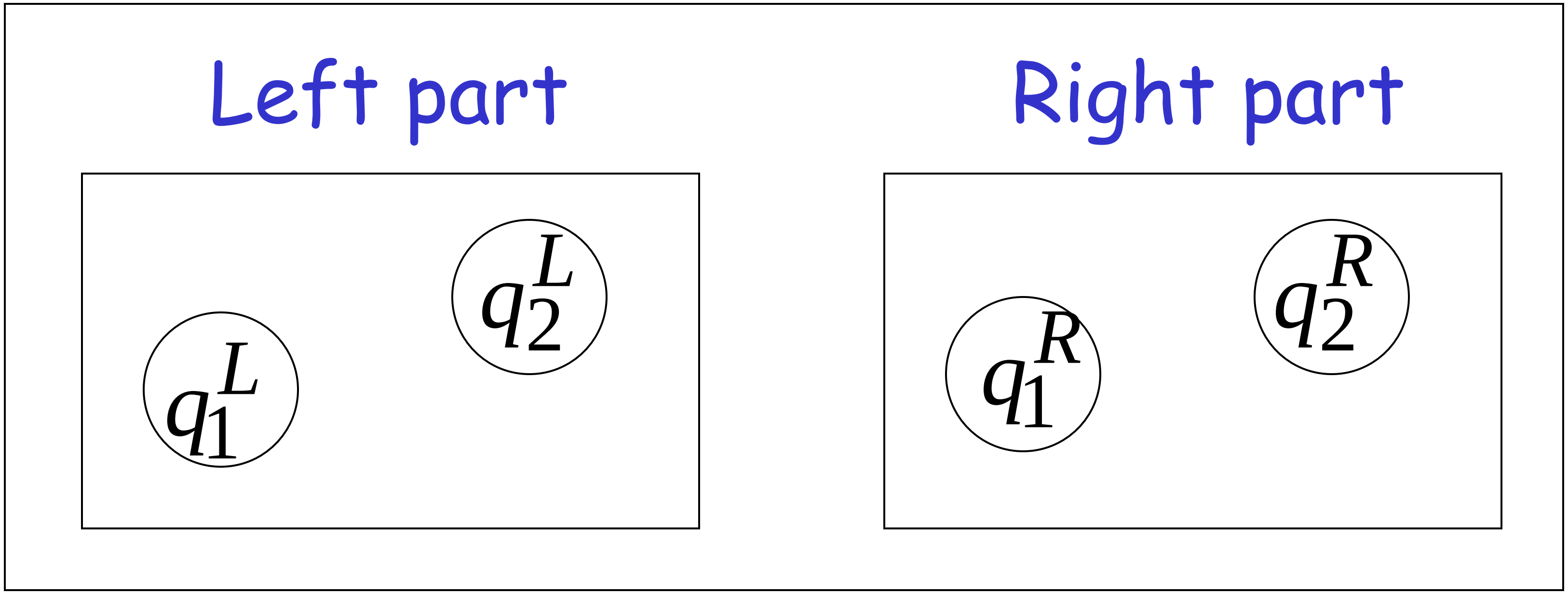
Semi-infinite tape machine with two tracks



Standard machine



Semi-infinite tape machine



Standard machine

$$q_1 \xrightarrow{a \to g, R} q_2$$

Semi-infinite tape machine

Right part

$$q_1^R \xrightarrow{(a, x) \to (g, x), R} q_2^R$$

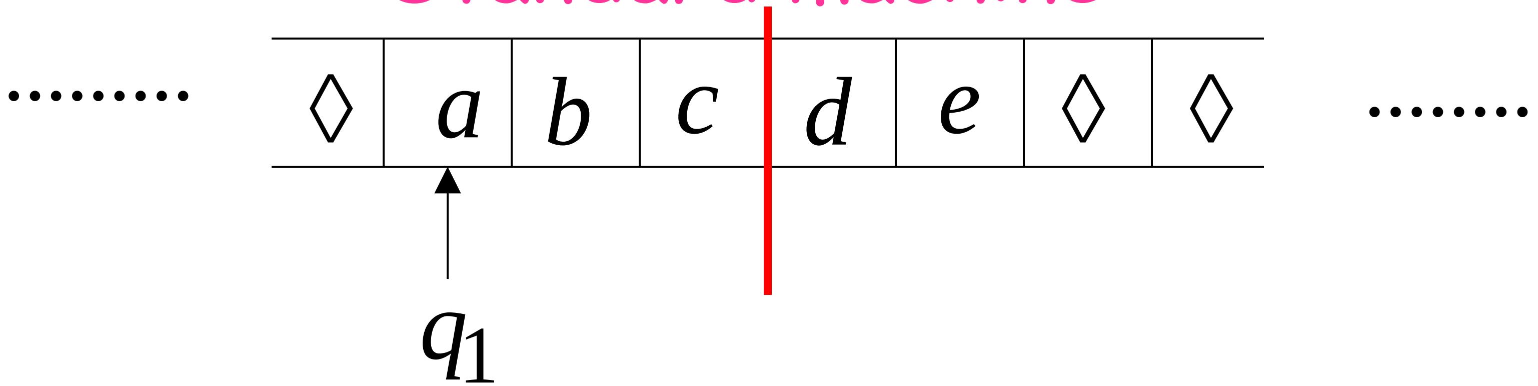
Left part

$$q_1^L \xrightarrow{(x, a) \to (x, g), L} q_2^L$$

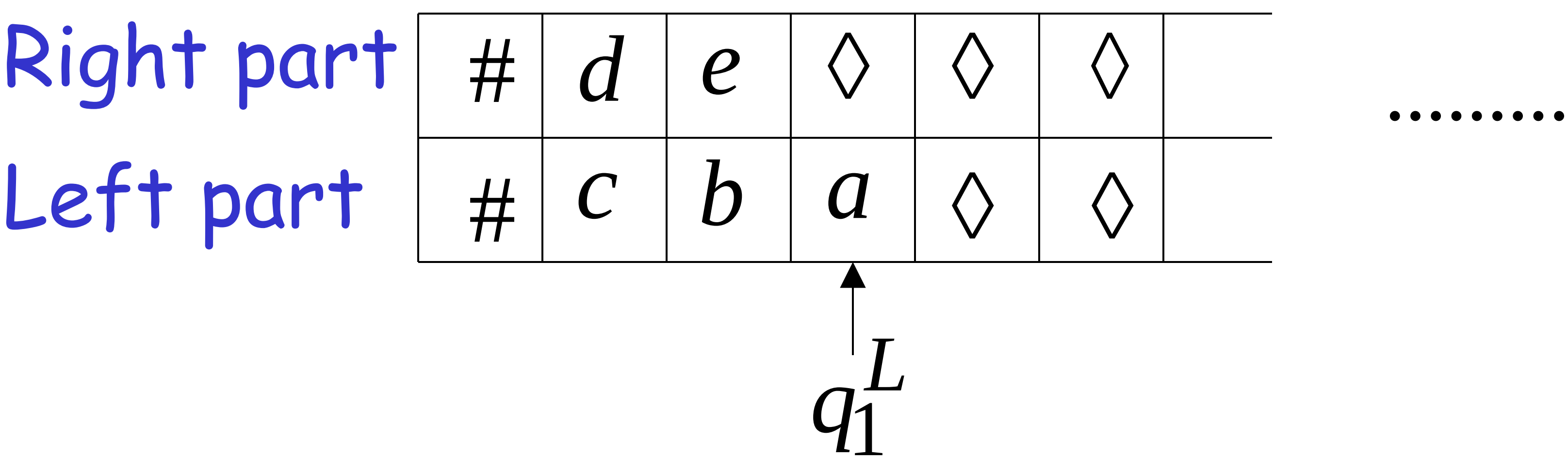
For all symbols x

Time 1

Standard machine

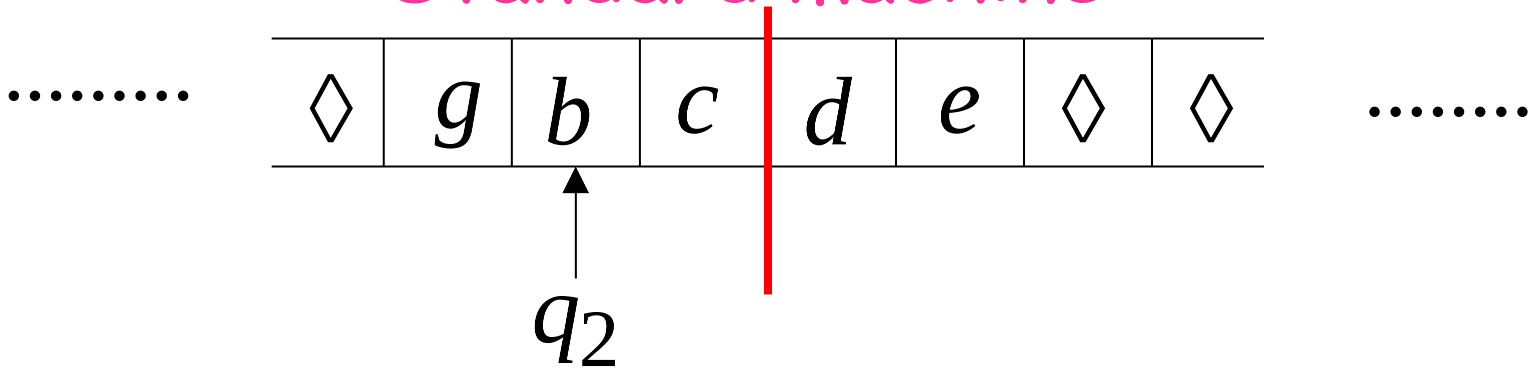


Semi-infinite tape machine

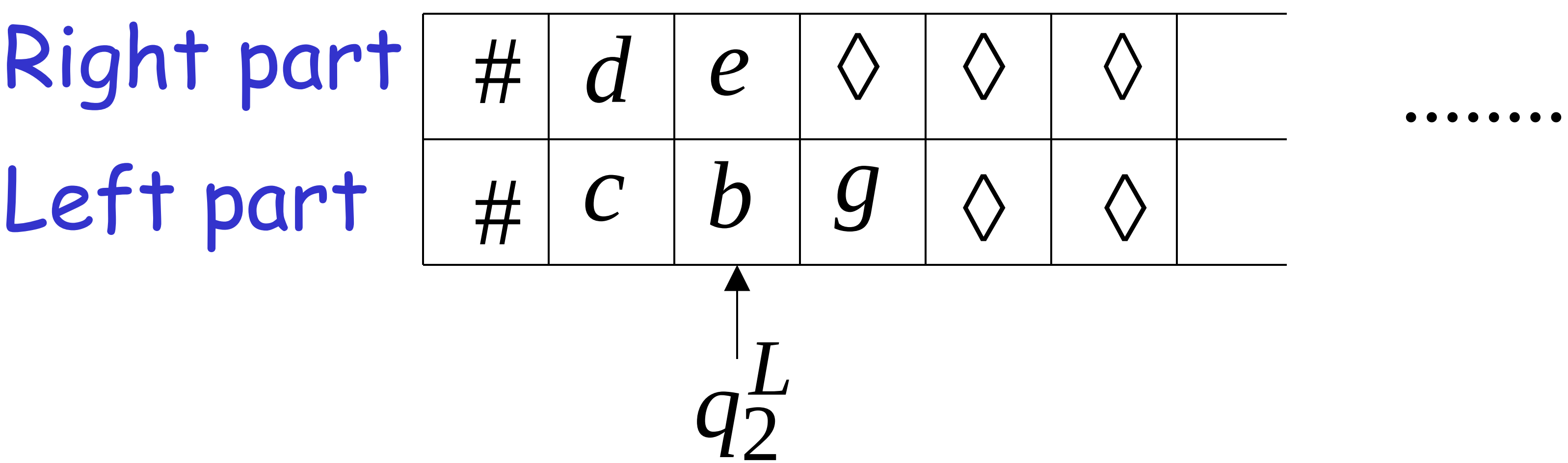


Time 2

Standard machine



Semi-infinite tape machine



At the border:

Semi-infinite tape machine

Right part

$$q_1^R \xrightarrow{(\#,\#) \to (\#,\#), R} q_1^L$$

Left part

$$q_1^L \xrightarrow{(\#,\#) \to (\#,\#), R} q_1^R$$

Semi-infinite tape machine

Time 1

Right part	#	d	e	◊	◊	◊		....
Left part	#	c	b	g	◊	◊		

$\uparrow$

q_1^L

Time 2

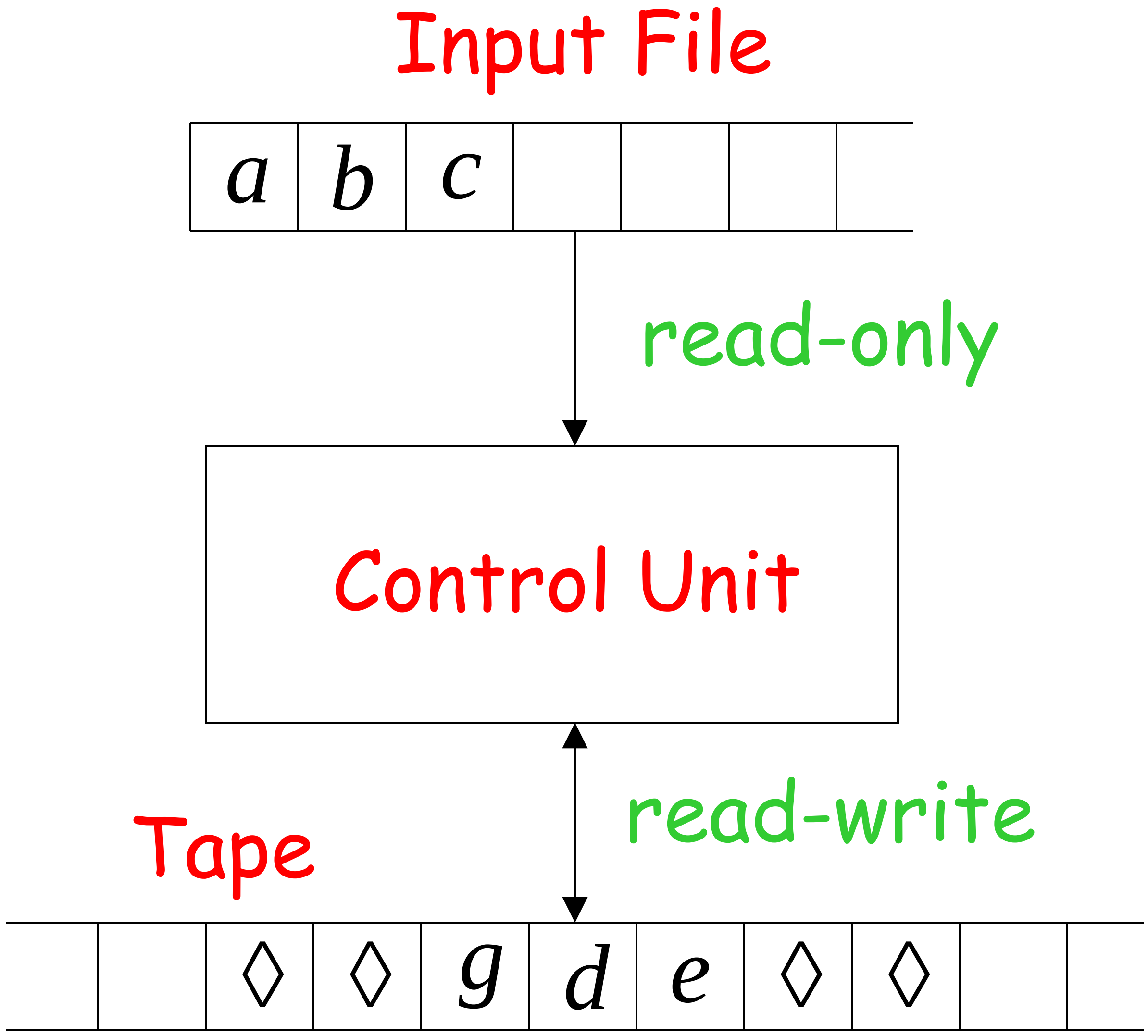
Right part	#	d	e	◊	◊	◊		....
Left part	#	c	b	g	◊	◊		

$\uparrow$

q_1^R

Theorem: Semi-infinite tape machines have the same power as Standard Turing machines

The Off-Line Machine



Proof of equivalence?

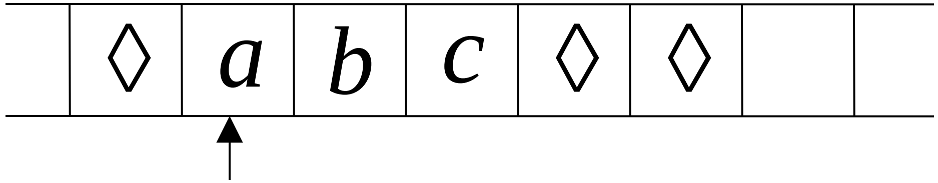
Off-line machines simulate Standard Turing Machines:

Off-line machine:

1. Copy input file to tape

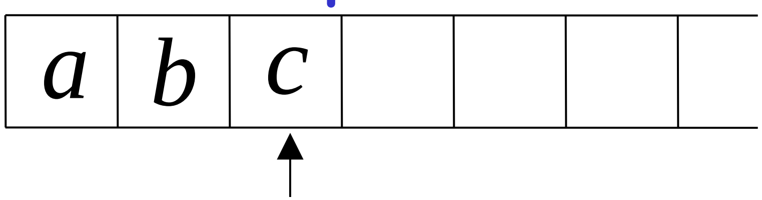
2. Continue computation as in Standard Turing machine

Standard machine

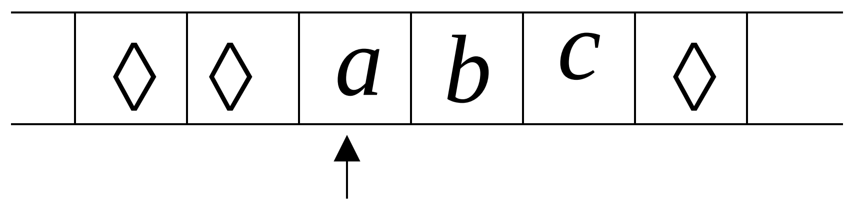


Off-line machine

Input File

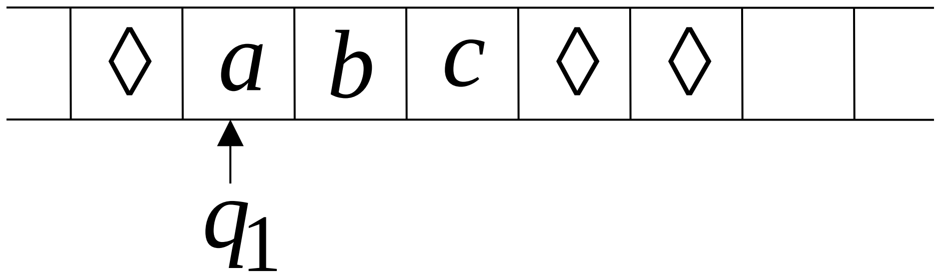


Tape



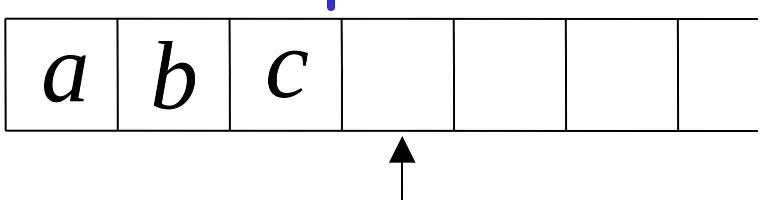
1. Copy input file to tape

Standard machine

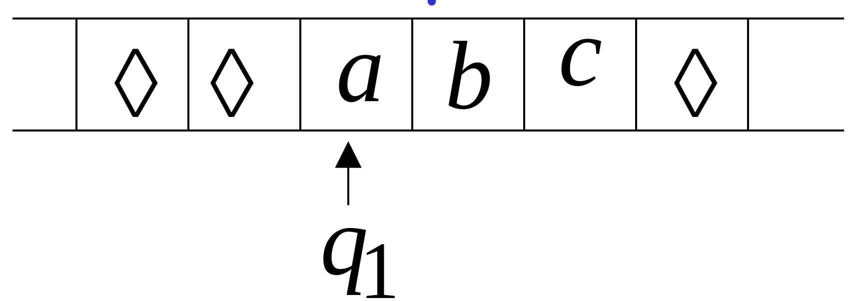


Off-line machine

Input File



Tape



2. Do computations as in Turing machine

Standard Turing machines simulate Off-line machines:

Use a Standard machine with four track tape
to keep track of
the Off-line input file and tape contents

Off-line Machine

Input File

a	b	c	d			

Tape

	$\Diamond$	$\Diamond$	e	f	g	$\Diamond$

Four track tape -- Standard Machine

	#	a	b	c	d			Input File
	#	0	0	1	0			head position
		e	f	g				Tape
		0	1	0				head position

Reference point

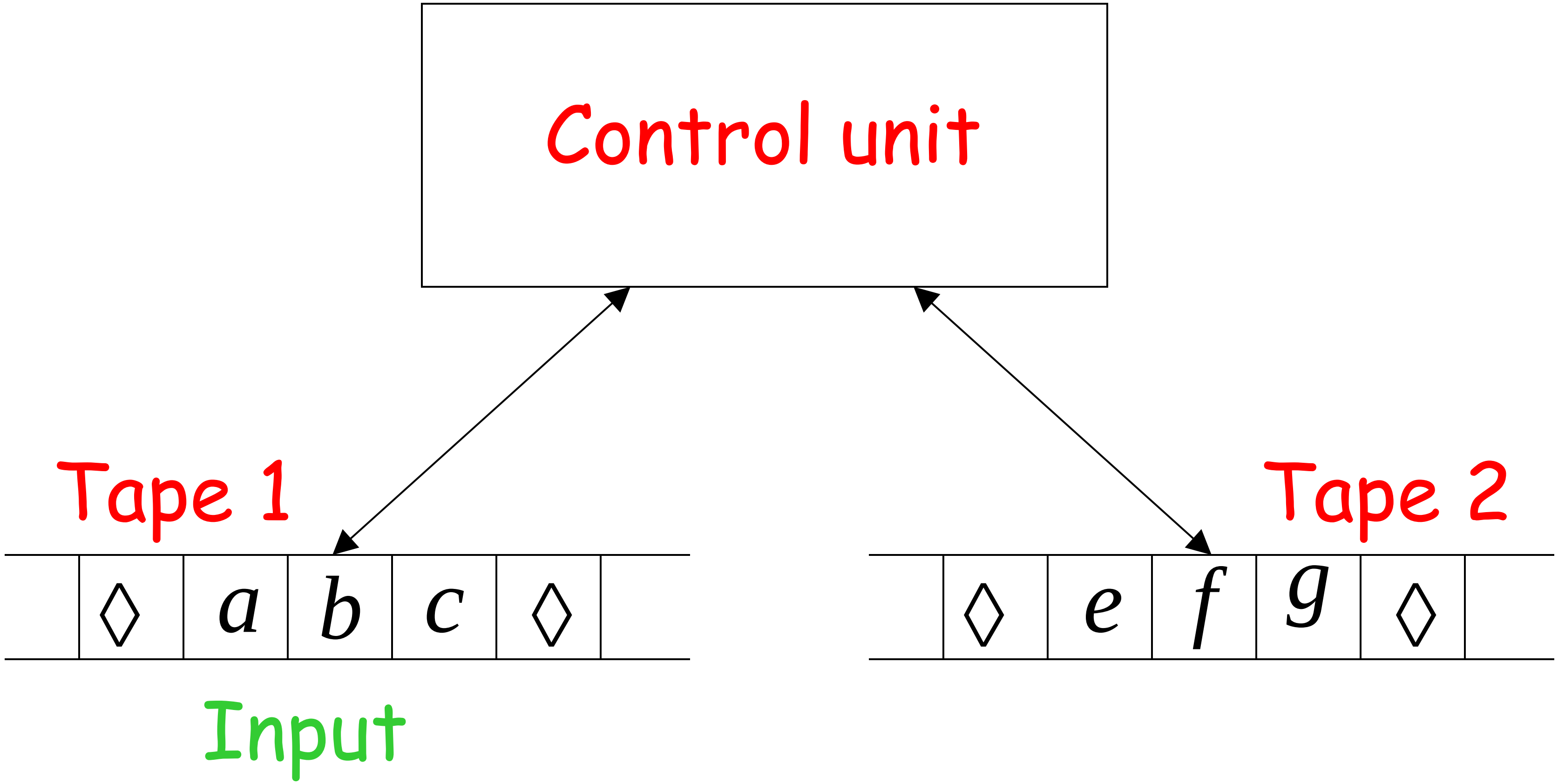
	#	a	b	c	d			Input File
	#	0	0	1	0			head position
		e	f	g				Tape
		0	1	0				head position

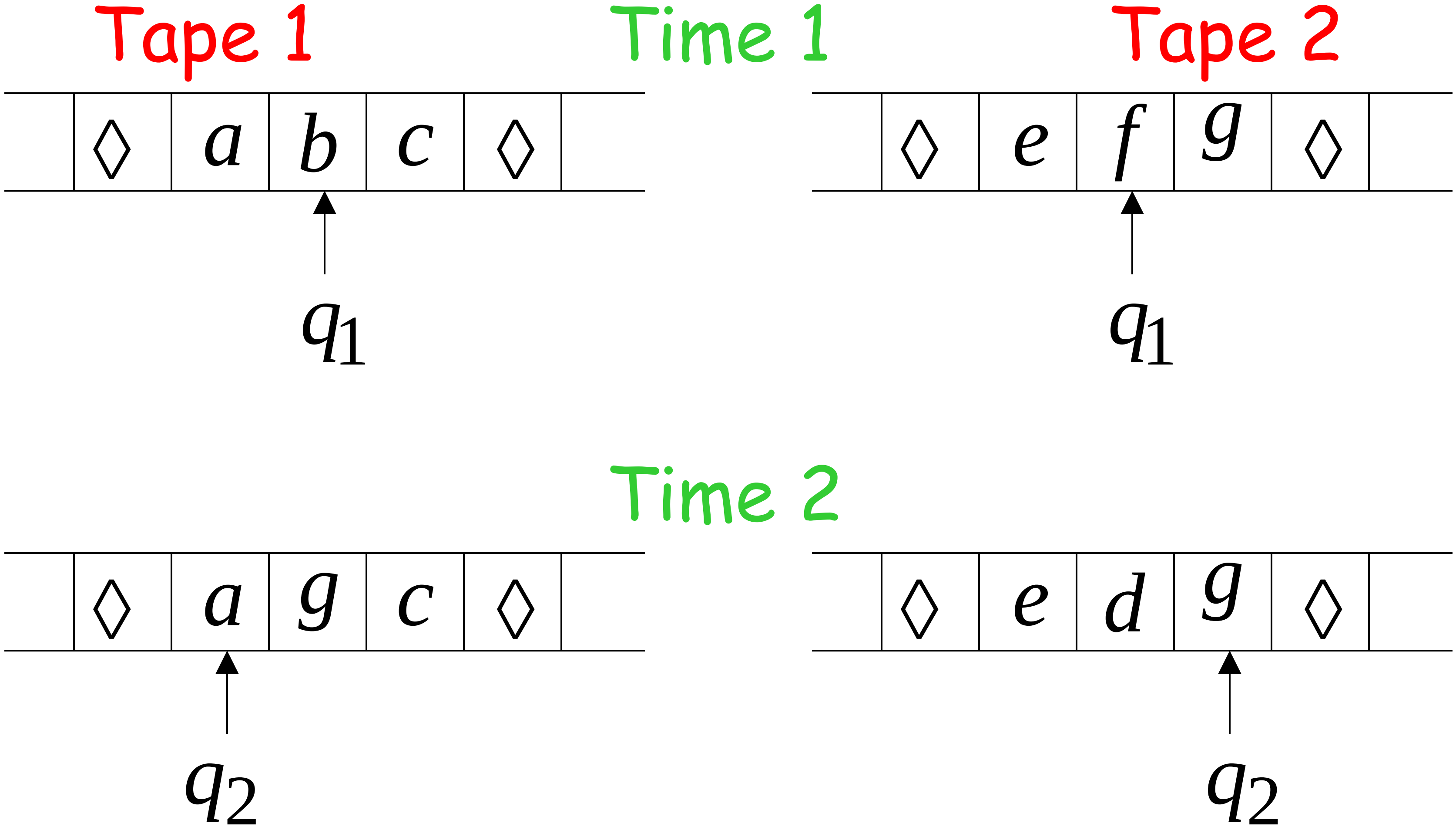
Repeat for each state transition:

- Return to reference point
- Find current input file symbol
- Find current tape symbol
- Make transition

Theorem: Off-line machines have the same power as Standard machines

Multitape Turing Machines





$$q_1 \xrightarrow{(b,f)\to(g,d),L,R} q_2$$

Proof of equivalence?

Multitape machines simulate Standard Machines:

Use just one tape

Standard machines simulate Multitape machines:

Standard machine:

- Use a multi-track tape

- A tape of the Multiple tape machine corresponds to a pair of tracks

Multitape Machine

Tape 1

	◊	a	b	c	◊	

Tape 2

	◊	e	f	g	h	◊

Standard machine with four track tape

		a	b	c				Tape 1
		0	1	0				head position
		e	f	g	h			Tape 2
		0	0	1	0			head position

Reference point

	#	a	b	c				Tape 1
	#	0	1	0				head position
	#	e	f	g	h			Tape 2
	#	0	0	1	0			head position

Repeat for each state transition:

- Return to reference point
- Find current symbol in Tape 1
- Find current symbol in Tape 2
- Make transition

Theorem: Multi-tape machines have the same power as Standard Turing Machines

Same power doesn't imply same speed:

Language $L = \{a^n b^n\}$

	Acceptance Time
Standard machine	n^2
Two-tape machine	n

Algorithms?

$$L = \{a^n b^n\}$$

Standard machine:

Go back and forth n^2 times

Two-tape machine:

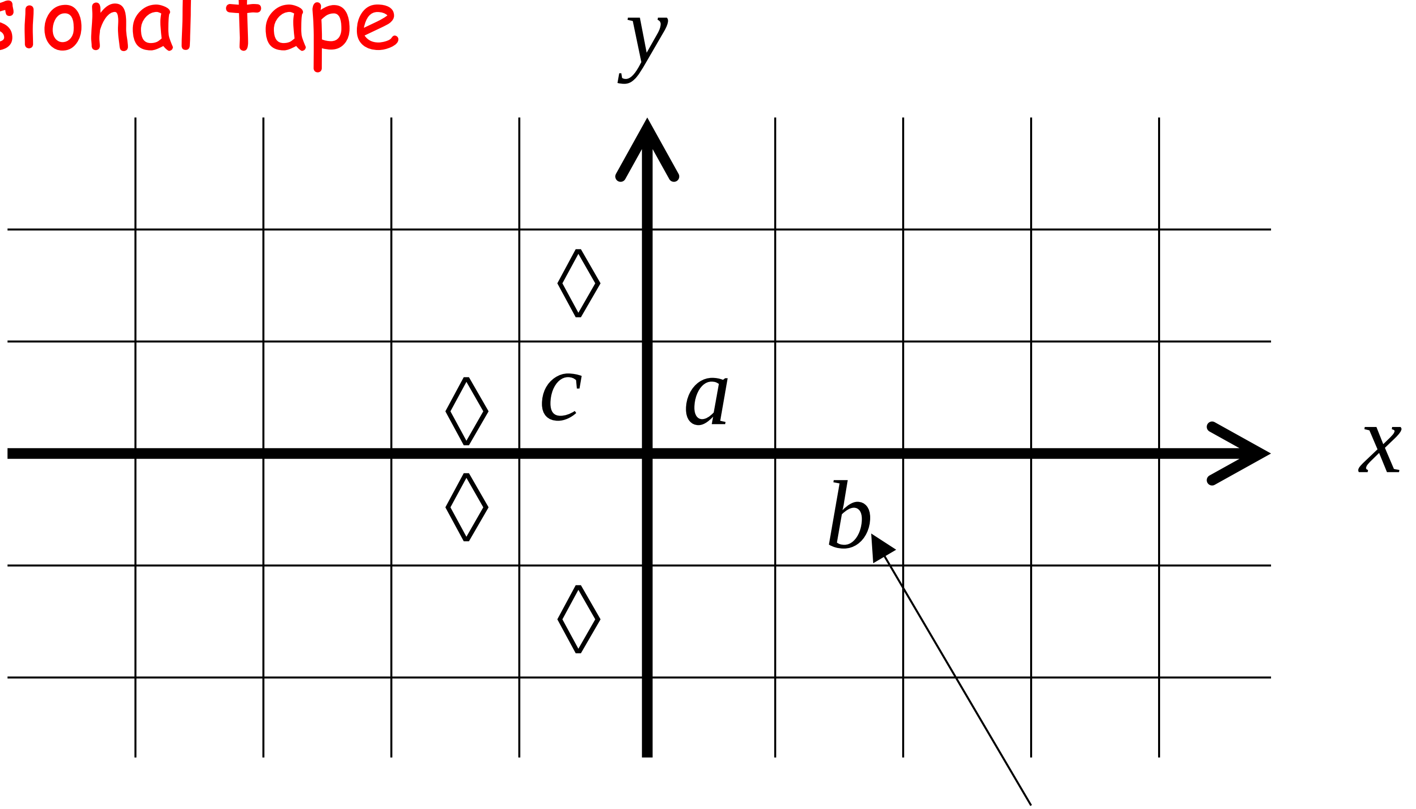
Copy b^n to tape 2 (n steps)

Leave a^n on tape 1 (n steps)

Compare tape 1 and tape 2 (n steps)

MultiDimensional Turing Machines

Two-dimensional tape



MOVES: L,R,U,D

U: up D: down

HEAD

Position: +2, -1

Proof of equivalence?

Multidimensional machines simulate Standard machines:

Use one dimension

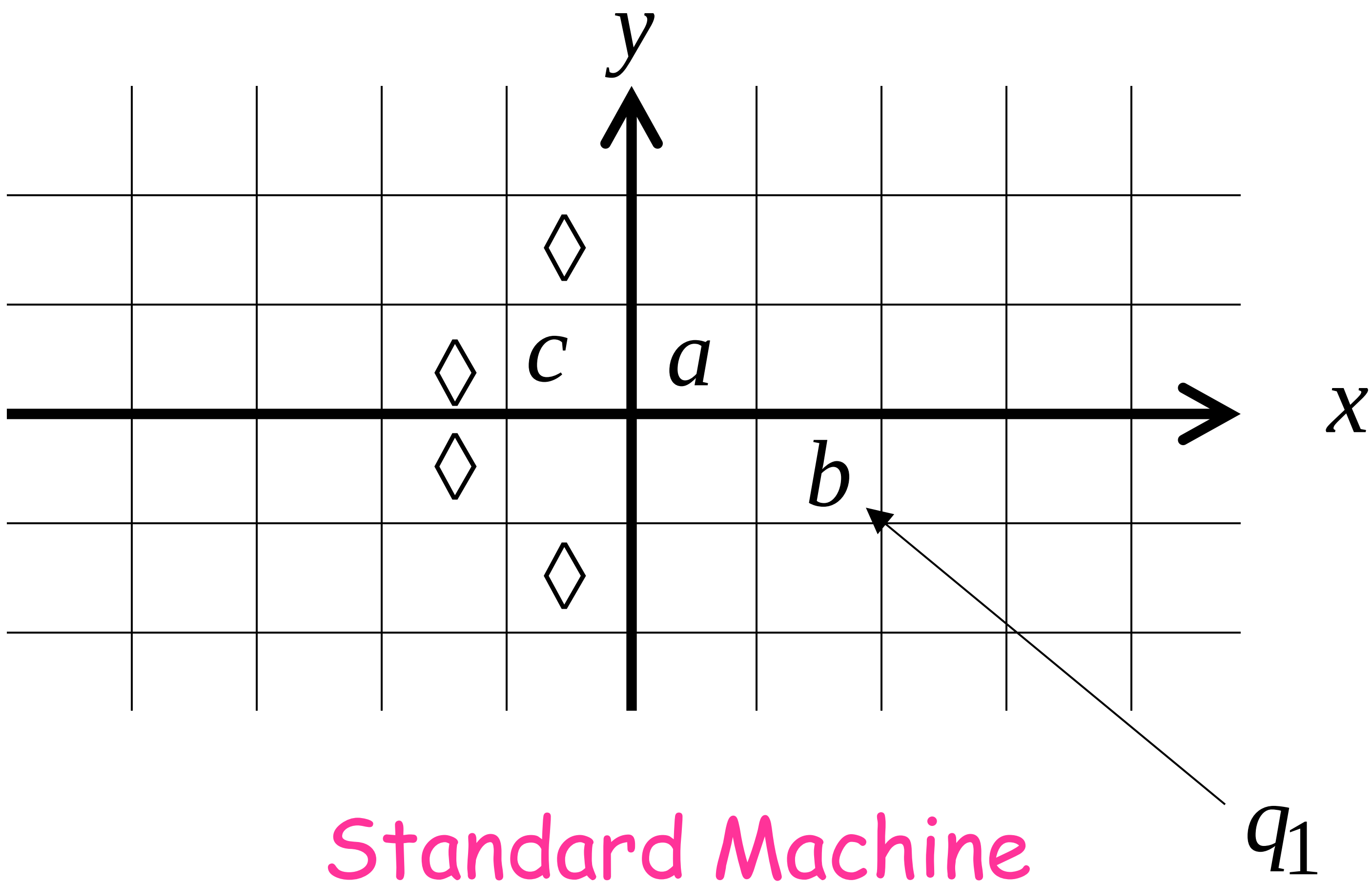
Standard machines simulate Multidimensional machines:

Standard machine:

- Use a two track tape

- Store symbols in track 1
- Store coordinates in track 2

Two-dimensional machine



Standard Machine

a				b					c	
1	#	1	#	2	#	−	1	#	−	1

q_1

symbols

coordinates

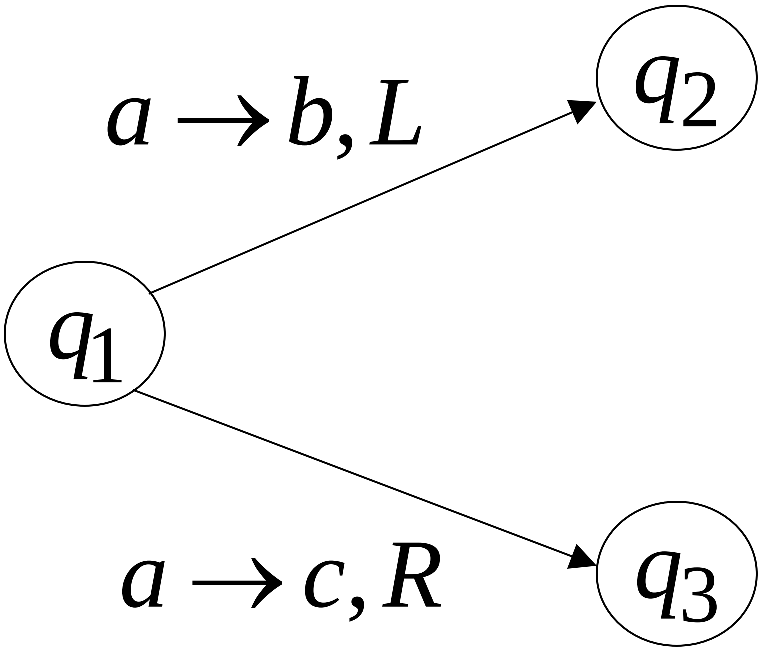
Standard machine:

Repeat for each transition

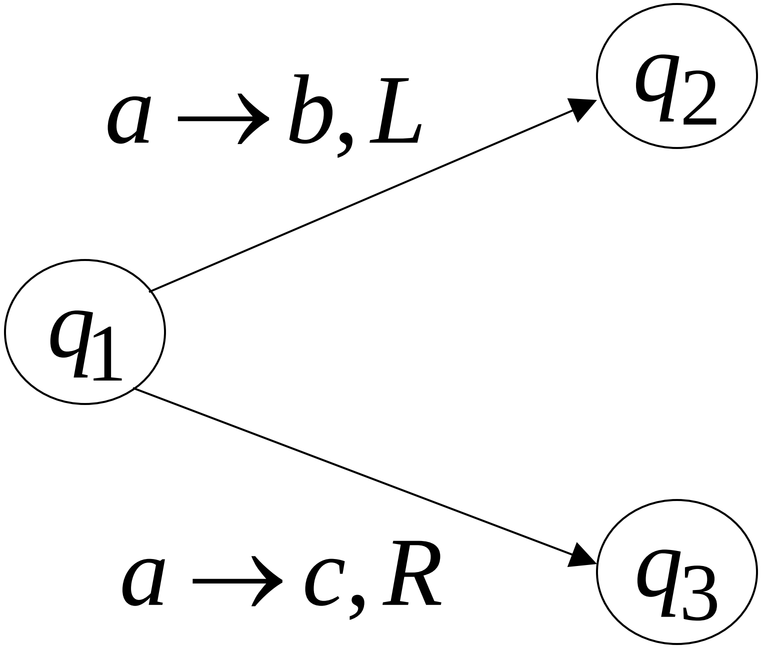
- Update current symbol
- Compute coordinates of next position
- Go to new position

Theorem: MultiDimensional Machines have the same power as Standard Turing Machines

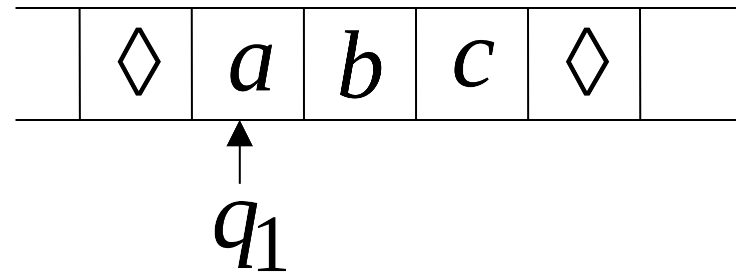
NonDeterministic Turing Machines



Non Deterministic Choice

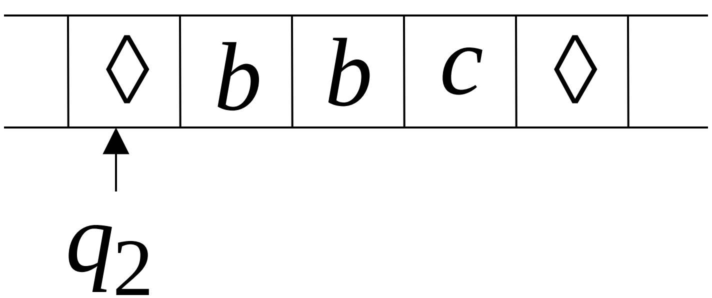


Time 0

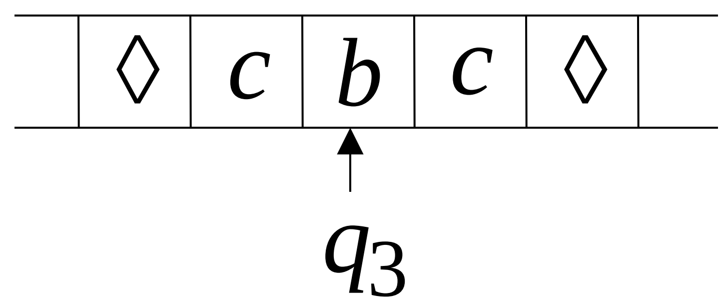


Time 1

Choice 1



Choice 2



Input string w is accepted if this is a possible computation:

$$q_0 w \overset{*}{\succ} x\, q_f\, y$$

Initial configuration

Final Configuration

Final state

Proof of equivalence?

NonDeterministic Machines simulate Standard (deterministic) Machines:

NonDeterministic Machines simulate Standard (deterministic) Machines:

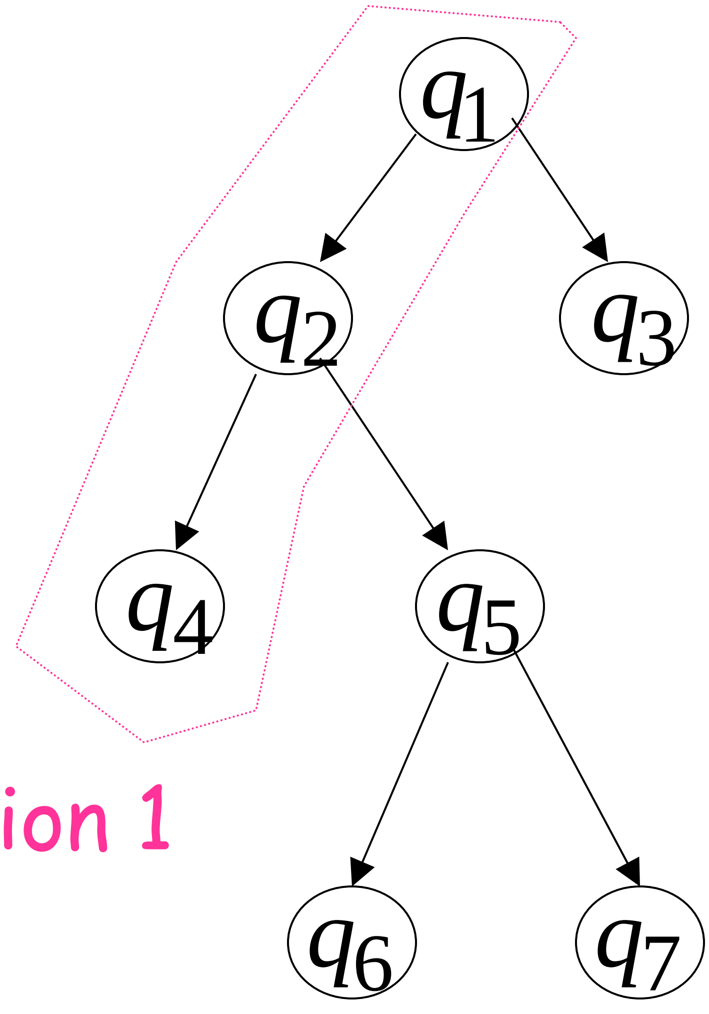
Every deterministic machine
is also a nondeterministic machine

Deterministic machines simulate NonDeterministic machines:

Deterministic machine:

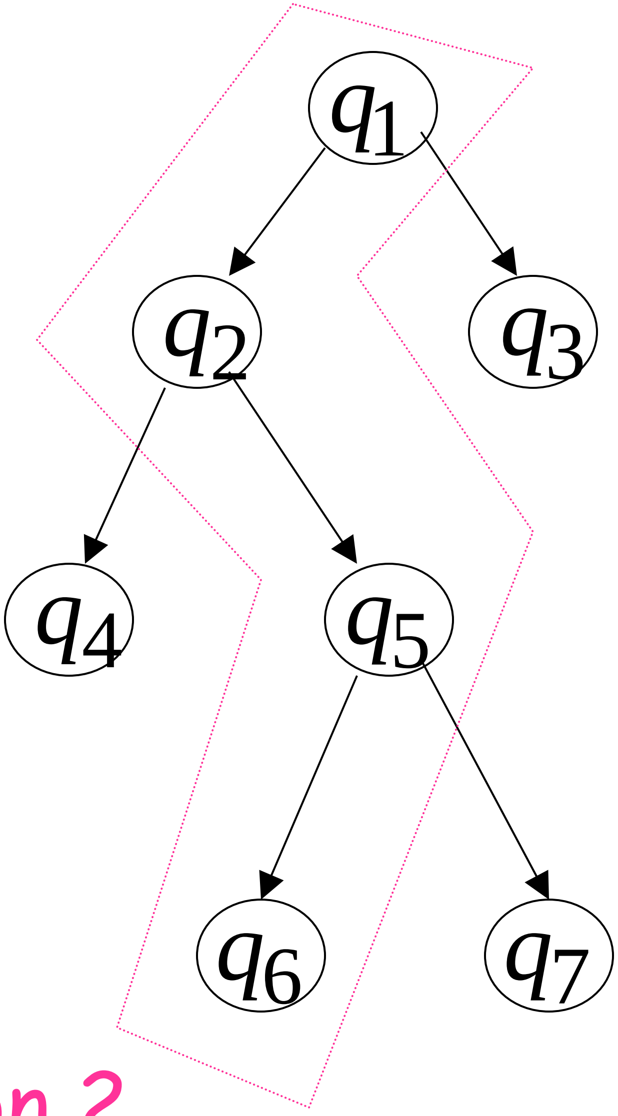
Keeps track of all possible computations

Non-Deterministic Choices



Computation 1

Non-Deterministic Choices



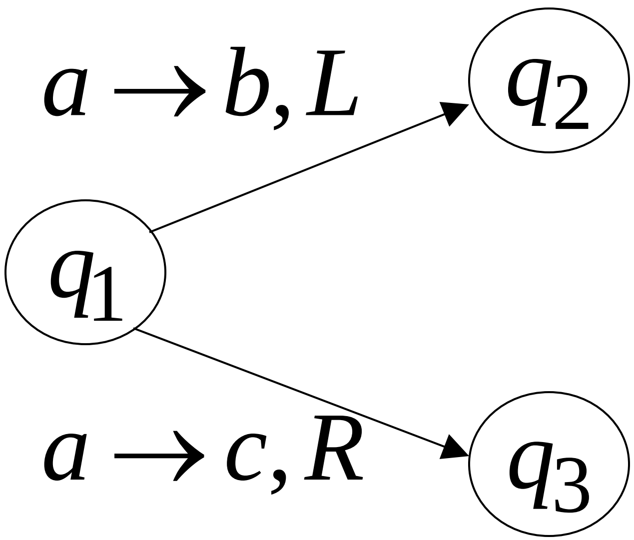
Computation 2

Simulation

Deterministic machine:

- Keeps track of all possible computations
- Stores computations in a two-dimensional tape

NonDeterministic machine



Time 0

	◊	a	b	c	◊	

q_1

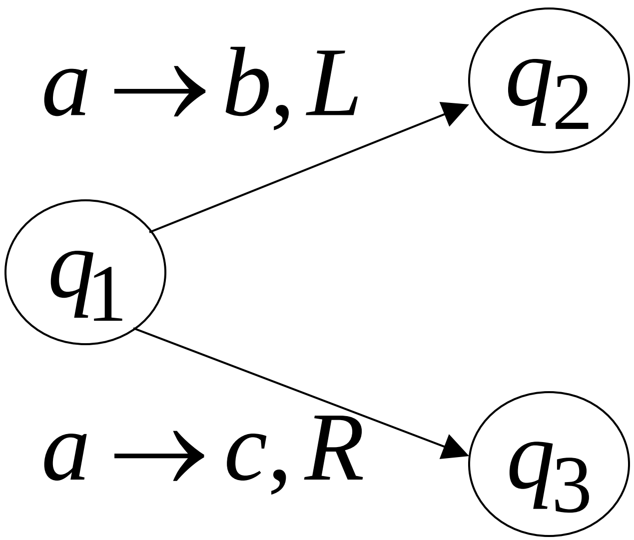
Deterministic machine

#	#	#	#	#	#
#	a	b	c	#	
#	q_1			#	
#	#	#	#	#	

Computation 1

NonDeterministic machine

Time 1



Choice 1

	◊	b	b	c	◊	
	q_2					

Choice 2

	◊	c	b	c	◊	
			q_3			

Deterministic machine

	#	#	#	#	#	#	
#		b	b	c	#		
#	q_2				#		
#		c	b	c	#		
#			q_3		#		

Computation 1

Computation 2

Repeat

- Execute a step in each computation:

- If there are two or more choices in current computation:
    1. Replicate configuration
    2. Change the state in the replica

Theorem: NonDeterministic Machines have the same power as Deterministic machines

Remark:

The simulation in the Deterministic machine takes exponential time of time needed by the NonDeterministic machine