MATHS ASSIGNMENT 3

Q1. In a bombing action, there is 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completly. How many bombbs are required to be dropped to give a 99% chance or better of destroying the target?

Let p be the probability that a bomb strikes the target.

$\therefore p = \frac{1}{2}, \ q = \frac{1}{2}$

Let X denote the no. of bombs that hit the target out of total n bombs.

$X=x$	0	1	2	......	n
$P(x)$	${}^nC_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^n$ $\left(\frac{1}{2}\right)^n$	${}^nC_1\left(\frac{1}{2}\right)^n$	${}^nC_2\left(\frac{1}{2}\right)^n$		$\left(\frac{1}{2}\right)^n$

The probability that k bombs out of n hit the target is given by

$$p(k) = {}^nC_k\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = {}^nC_k\left(\frac{1}{2}\right)^n$$

MENT 3

e is 50%
strike the
needed to
. How many
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that hit
mbs.

n

$\left(\frac{1}{2}\right)^n$

bs out of

$k\left(\frac{1}{2}\right)^n$

2^n

$P\{2 \le X \le n\} \ge 0.99$

$$\sum_{k=2}^{n} {}^nC_k \left(\frac{1}{2}\right)^n \ge 0.99$$

$$1 - \left[{}^nC_0 \left(\frac{1}{2}\right)^n + {}^nC_1 \left(\frac{1}{2}\right)^n\right] \ge 0.99$$

$$1 - \left[\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n\right] \ge 0.99$$

$$1 - \left(\frac{1}{2}\right)^n (1+n) \ge 0.99$$

$$\frac{(1+n)}{2^n} \le 0.01$$

For $n=1$ $1 \le 0.01$ ×

For $n=2$ $\frac{3}{4} \le 0.01$ ×

For $n=3$ $\frac{1}{2} \le$ ×

For $n=4$ $\frac{5}{16} \le$ ×

For $n=5$ $3/16$ ×

For $n=6$ $7/64$ ×

For $n=7$ $1/16$ ×

For $n=8$ $9/256$ ×

For $n=9$ $5/256$ ×

For $n=10$ $11/1024$

For $\boxed{n=11}$ $\frac{3}{512} \le 0.01$ ✓

2. A sortie of 20 aero-planes is sent on an operational fight. The chance that an aeroplane fails to return is 5%. Find the probability that

i) one plane does not return

ii) at the most 5 planes does not return

iii) what is the most probable no. of returns

$$p = \frac{5}{100}, \quad q = \frac{95}{100}$$

a) $^{20}C_1 \left(\frac{5}{100}\right)^1 \left(\frac{95}{100}\right)^{19}$

b) $^{20}C_0 \left(\frac{5}{100}\right)^0 \left(\frac{95}{100}\right)^{20} + {}^{20}C_1 \left(\frac{5}{100}\right)^1 \left(\frac{95}{100}\right)^{19}$
$+ {}^{20}C_2 \left(\frac{5}{100}\right)^2 \left(\frac{95}{100}\right)^{18} + {}^{20}C_3 \left(\frac{5}{100}\right)^3 \left(\frac{95}{100}\right)^{17}$
$+ {}^{20}C_4 \left(\frac{5}{100}\right)^4 \left(\frac{95}{100}\right)^{16} + {}^{20}C_5 \left(\frac{5}{100}\right)^5 \left(\frac{95}{100}\right)^{15}$

c) $^nC_K p^K q^{n-K} \quad n = 20, \; p = \frac{5}{100}, \; q = \frac{95}{100}$

$K = 0 \Rightarrow {}^{20}C_0 \left(\frac{5}{100}\right)^0 \left(\frac{95}{100}\right)^{20} = 0.3584$

$K = 1 \Rightarrow {}^{20}C_1 \left(\frac{5}{100}\right)^1 \left(\frac{95}{100}\right)^{19} = 0.3773$

$K = 2 \Rightarrow {}^{20}C_2 \left(\frac{5}{100}\right)^2 \left(\frac{95}{100}\right)^{18} = 0.1886$

3

→ Max probability at $k=1$

$\therefore$ Ans $= 20 - k$
$= 20 - 1$
$= 19$

3 A car hire firm has 2 cars which it hires out day by day. The no. of demands for a car on each day is distributed as a Poisson variate with mean 1.5. Calculate the proportion of days on which i) neither car is used ii) some demand is refused.

$\lambda = 1.5$
x = no. of demands per day / no. of days

$P(x) = \dfrac{\lambda^x e^{-\lambda}}{x!} = \dfrac{1.5^x e^{-1.5}}{x!}$

The proportion of days in which there are x demands for days is directly proportional to x demands for the car.

i) $P(X=0) = \dfrac{(1.5)^0 e^{-1.5}}{0!} = e^{-1.5}$

ii) $P(X>2) = 1 - [P(0) + P(1) + P(2)]$
$= 1 - \left[\dfrac{(1.5)^0 e^{-1.5}}{0!} + \dfrac{(1.5)^1 e^{-1.5}}{1!} + \dfrac{(1.5)^2 e^{-1.5}}{2!}\right]$
$= 1 - e^{-1.5}[1 + 1.5 + 1.25]$
$= 0.1911$

4. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and variance of the distribution.

$P\{X < 45\} = 0.31$
$P\{X > 64\} = 0.08$

$P\{X < 45\} = 0.31$

$P\left\{Z < \frac{45-\mu}{\sigma}\right\} = 0.31$

$\phi\left(\frac{45-\mu}{\sigma}\right) = 0.31$

$1 - \phi\left(\frac{\mu-45}{\sigma}\right) = 0.31$

$\phi\left(\frac{\mu-45}{\sigma}\right) = 0.69$

$\frac{\mu-45}{\sigma} = 0.5$

$\mu - 45 = 0.5\sigma$

$\mu - 0.5 = 45$

the items
4. find
ribution.

$P\{X > 64\} = 0.08$

$P\{X < 64\} = 1 - 0.08 = 0.92$

$P\left\{z < \frac{64-\mu}{\sigma}\right\} = 0.92$

$\Phi\left(\frac{64-\mu}{\sigma}\right) = 0.92$

$\frac{64-\mu}{\sigma} = 1.41$

$64 - \mu = 1.41\sigma$

$\cancel{64 - \mu} \quad \mu + 1.41\sigma = 64$

$\mu = 49.97 \simeq 50$

$\sigma = 9.94 \simeq 10$

5. A fair coin is tossed 500 times. Find the probability that the number of heads will not differ from 250 by
i) more than 10
ii) more than 30

$$\mu = 500 \times \frac{1}{2} = 250 = np$$

$$V(x) = \sigma^2 = npq = 500 \times \frac{1}{2} \times \frac{1}{2} = 125$$

$$\sigma = 5\sqrt{5}$$

$$\cancel{\therefore f(x) = \frac{1}{5\sqrt{5}}}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

$$f(x) = \frac{1}{5\sqrt{5}\sqrt{2\pi}} e^{-\frac{(x-250)^2}{2\times(5\sqrt{5})^2}}$$

$$P\{240 \le X \le 260\}$$

$$= P\left\{\frac{240-250}{5\sqrt{5}} \le Z \le \frac{260-250}{5\sqrt{5}}\right\}$$

$$= P\left\{\frac{-\cancel{10}^{\,2}}{\cancel{5}\sqrt{5}} \le Z \le \frac{\cancel{10}^{\,2}}{\cancel{5}\sqrt{5}}\right\}$$

$$= P\{-0.8944 \le Z \le 0.8944\}$$

Find
heads

$= 125$

$< x < \infty$

$\frac{260-250}{5\sqrt{5}}$?

42

$= 2\Phi(0.89) - 1$

$= 2 \times 0.7995 - 1$

$= 0.599$

6. The increase in sales per day in a shop is exponentially distributed with Rs 800 as the average. If sales tax is at the rate of 6%, find the probability that the increase in sales tax will exceed Rs 30 per day?

Mean = 80

X = increase in sales / day

$f(x) = \alpha e^{-\alpha x}$

$\alpha = \frac{1}{800}$

sales tax = 6%

X = ? such that sales tax = 30

$X \times \frac{6}{100} = 30$

$X = \frac{3000}{6} = 500$

$P(X \geq 500) = \int_{500}^{\infty} \frac{1}{800} e^{-\frac{x}{800}} dx$

$= \left[\frac{e^{-\frac{x}{800}}}{-1/800} \right]_{500}^{\infty} \times \frac{1}{800}$

$= \left[-e^{-\frac{x}{800}} \right]_{500}^{\infty}$

$= e^{-5/8} \simeq 0.535$

7. Obtain the mean and variance of Gamma Distribution.

$$f(x) = \frac{\alpha}{\Gamma r} (\alpha x)^{r-1} e^{-\alpha x} \quad , x > 0$$

$$\text{Mean} = \int_{-\infty}^{\infty} x f(x)$$

$$= \int_{0}^{\infty} x \cdot \frac{\alpha}{\Gamma r} (\alpha x)^{r-1} e^{-\alpha x} \, dx$$

Let $\alpha x = t$

$\alpha \, dx = dt \Rightarrow dx = \frac{dt}{\alpha}$

$$= \frac{\alpha}{\Gamma r} \int_{0}^{\infty} \frac{t}{\alpha} t^{r-1} e^{-t} \frac{dt}{\alpha}$$

$$= \frac{\alpha}{\Gamma r} \frac{1}{\alpha^2} \int_{0}^{\infty} e^{-t} t^{r} \, dt$$

$$= \frac{1}{\alpha \Gamma r} \times \Gamma(r+1) = \frac{r \Gamma r}{\alpha \Gamma r}$$

$$\boxed{E(X) = \frac{r}{\alpha}}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \int_0^\infty x^2 \frac{\alpha}{\Gamma r} (\alpha x)^{r-1} e^{-\alpha x} dx - \left(\frac{r}{\alpha}\right)^2$$

$$= \frac{\alpha}{\Gamma r} \int_0^\infty x^2 (\alpha x)^{r-1} e^{-\alpha x} dx - \left(\frac{r}{\alpha}\right)^2$$

Let $\alpha x = t \Rightarrow x = \frac{t}{\alpha}$

$\alpha\, dx = dt \Rightarrow dx = \frac{dt}{\alpha}$

$$= \frac{\alpha}{\Gamma r} \int_0^\infty \left(\frac{t}{\alpha}\right)^2 t^{r-1} e^{-t} \frac{dt}{\alpha} - \left(\frac{r}{\alpha}\right)^2$$

$$= \frac{\alpha}{\Gamma r} \times \frac{1}{\alpha^3} \int_0^\infty e^{-t} t^{r+1} dt - \left(\frac{r}{\alpha}\right)^2$$

$$= \frac{1}{\alpha^2 \Gamma r} \Gamma(r+2) - \left(\frac{r}{\alpha}\right)^2$$

$$= \frac{1}{\alpha^2 \Gamma r} (r+1) \Gamma(r+1) - \left(\frac{r}{\alpha}\right)^2$$

$$= \frac{1}{\alpha^2 \cancel{\Gamma r}} \times (r+1) \times r \times \cancel{\Gamma r} - \left(\frac{r}{\alpha}\right)^2$$

$$= \frac{r^2 + r}{\alpha^2} - \frac{r^2}{\alpha^2} = \frac{r}{\alpha^2}$$

$-\left(\frac{r}{\alpha}\right)^2$

$-\left(\frac{r}{\alpha}\right)^2$

$-\left(\frac{r}{\alpha}\right)^2$

$-\left(\frac{r}{\alpha}\right)^2$

$)^2$

$\frac{r}{\alpha})^2$

$\frac{r}{\alpha^2}$

8. The daily consumption of milk in a city in excess of 20,000 gallons is approximately distributed as a Gamma distribution with mean 20,000 and variance $2(10,000)^2$. The city has a daily stock of 30,000 gallons. What is the probability that the stock is insufficient in a particular day?

Let the daily ~~co~~ consumption of milk by X.

Let $Y = X - 20,000$ represent the daily consumption in excess of 20,000.

$$f(y) = \frac{\alpha}{\Gamma r} (\alpha y)^{r-1} e^{-\alpha y}$$

$$E(Y) = \frac{r}{\alpha} = 20,000$$

$$r = 20000\,\alpha$$

$$V(Y) = \frac{r}{\alpha^2} = 2 \times (10,000)^2$$

$$r = 2 \times (10,000)^2 \alpha^2$$

$$\alpha = \frac{1}{10,000}$$

$$r = 20,000 \times \frac{1}{10,000} = 2$$

$$f(y) = \frac{\frac{1}{10,000}}{\Gamma 2} \left(\frac{y}{10,000}\right)^{2-1} e^{-\frac{y}{10,000}}$$

$$= \frac{1}{10,000} \left(\frac{y}{10,000}\right) e^{-\frac{y}{10,000}}$$

$$P\{X > 30,000\} = P\{Y > 10,000\}$$

$$= \frac{1}{10,000} \int_{10,000}^{\infty} \frac{y}{10,000} e^{-\frac{y}{10,000}} dy$$

$$\frac{y}{10,000} = t$$

$$dy = 10,000\, dt$$

$$= \frac{1}{\cancel{10,000}} \int_{1}^{\infty} t\, e^{-t} \times \cancel{10,000}\, dt$$

$$= \left[-t\, e^{-t}\right]_{1}^{\infty} + \int_{1}^{\infty} e^{-t} dt$$

$$= e^{-1} + \left[-e^{-t}\right]_{1}^{\infty}$$

$$= \frac{1}{e} + \frac{1}{e}$$

$$= \frac{2}{e}$$

9. Suppose that the life lengths of 2 electronic devices D_1 and D_2 have distributions $N(40, 36)$ and $N(45, 9)$ respectively. If the electronic device is to be used for 45 hours period, which device is to be preferred? If it is to be used for a 48 hours period, which device is to be preferred?

$D_1 \rightarrow N(40, 36)$
$D_2 \rightarrow N(45, 9)$

D_1

$P\{X \geq 45\} = 1 - P\{X < 45\}$

$= 1 - P\left\{Z < \frac{45-\mu}{\sigma}\right\}$

$= 1 - P\left\{Z < \frac{5}{6}\right\}$

$= 1 - P\{Z < 0.8333\}$

$= 1 - \phi(0.8333)$

$= 1 - 0.7967$

$= 0.2033$

$$P\{X \geq 48\} = 1 - P\{X < 48\}$$

$$= 1 - P\left\{Z < \frac{48-\mu}{\sigma}\right\}$$

$$= 1 - P\{Z < 1.333\}$$

$$\doteq 1 - \phi(1.333)$$

$$= 1 - 0.9082$$

$$= 0.0918$$

D_2

$$P\{X \geq 45\} = 1 - P\{X < 45\}$$

$$= 1 - P\left\{Z < \frac{45-\mu}{\sigma}\right\}$$

$$= 1 - P\{Z < 0\}$$

$$= 1 - \phi(0)$$

$$= 1 - 0.500$$

$$= 0.5$$

$$P\{X \geq 48\} = 1 - P\{X < 48\}$$

$$= 1 - P\left\{Z < \frac{48-\mu}{\sigma}\right\}$$

$$= 1 - P\{Z < 1\}$$

$$= 1 - \Phi(1)$$

$$= 1 - 0.8413$$

$$= 0.1587$$

a) D_2 $\quad \because \overset{D_2}{P\{X \geq 45\}} > \overset{D_1}{P\{X \geq 45\}}$

b) D_2 $\quad \because \overset{D_2}{P\{X \geq 48\}} > \overset{D_1}{P\{X \geq 48\}}$

10. If X has distribution $N(\mu, \sigma^2)$ and if $Y = (X-\mu)/\sigma$, then prove that Y has distribution $N(0,1)$.

$$Y = \frac{X-\mu}{\sigma}$$

$$E(Y) = \sum_i y_i P(y_i)$$

$$= \sum_i \left(\frac{X-\mu}{\sigma}\right) \times P(x_i)$$

$$= \sum \frac{X P(x_i) - \mu \sum P(x_i)}{\sigma}$$

$$= \frac{\mu - \mu}{\sigma}$$

$$E(Y) = 0$$

$$V(Y) = \sum_i y_i^2 P(y_i) - \sum_i y_i P(y_i)$$

$$= \sum_i \left(\frac{X-\mu}{\sigma}\right)^2 P(x_i) - 0$$

$$= \sum \left(\frac{X^2 + \mu^2 - 2X\mu}{\sigma^2}\right) P(x_i)$$

$$= \frac{\sum X^2 P(x_i) + \mu^2 \sum P(x_i) - 2\mu \sum X P(x_i)}{\sigma^2}$$

$$= \frac{\sum X^2 P(x_i) - \mu^2}{\sigma^2}$$

$$= \frac{E(X^2) - [E(X)]^2}{\sigma^2}$$

$$= \frac{\sigma^2}{\sigma^2}$$

$$= 1$$