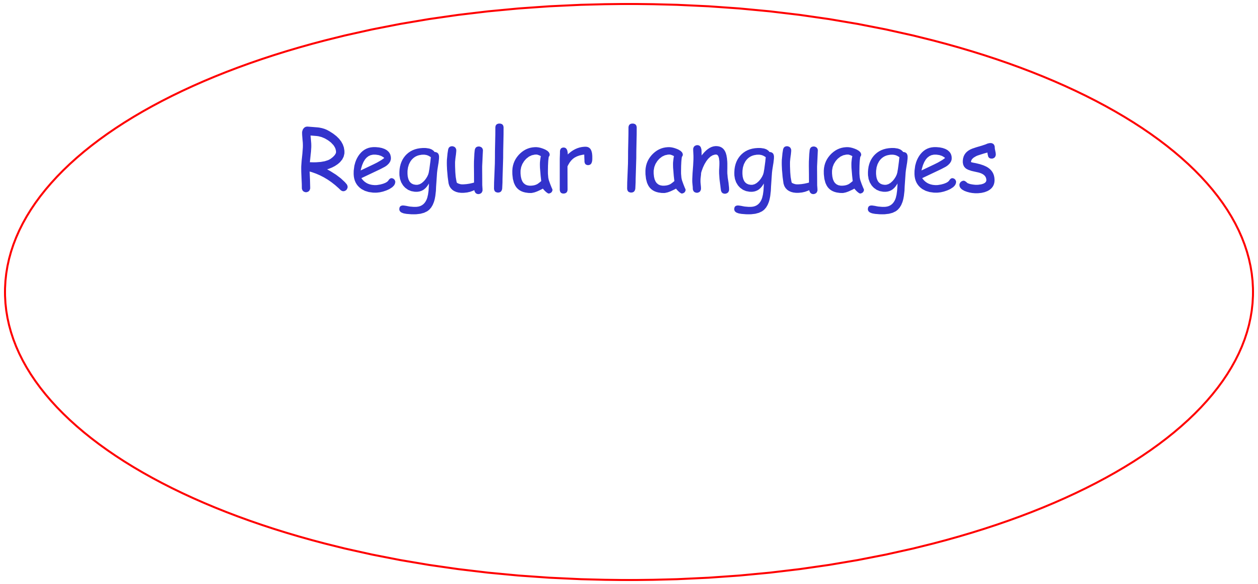
Formal Languages
The Pumping Lemma (2)

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

Non-regular languages

$$L=\{vv^R : v\in\Sigma^*\}$$



Regular languages

Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \qquad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m b^m a^m$

Write $a^m b^m b^m a^m = x\,y\,z$

From the Pumping Lemma

it must be that length $|x\,y| \leq m, \quad |y| \geq 1$

$$xyz = \rlap{$\overbrace{}^{m}$}\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{\ldots a\overbrace{b\ldots b}^{m}\overbrace{b\ldots b}^{m}\overbrace{a\ldots a}^{m}}_{z}$$

Thus: $y = a^k, \quad k \geq 1$

$$x\,y\,z = a^m b^m b^m a^m \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \in L$

$$x\,y\,z = a^m b^m b^m a^m \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^2\,z \;\in\; L$

$$xy^2z = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots a}_{y}\mathrlap{\underbrace{}_{z}}a\ldots a}^{m+k}\overbrace{b\ldots b}^{m}\overbrace{b\ldots b}^{m}\overbrace{a\ldots a}^{m} \in L$$

Thus: $a^{m+k} b^m b^m a^m \;\in L$

$$a^{m+k}b^mb^ma^m \in L \qquad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$

$$\Downarrow$$

$$a^{m+k}b^mb^ma^m \notin L$$

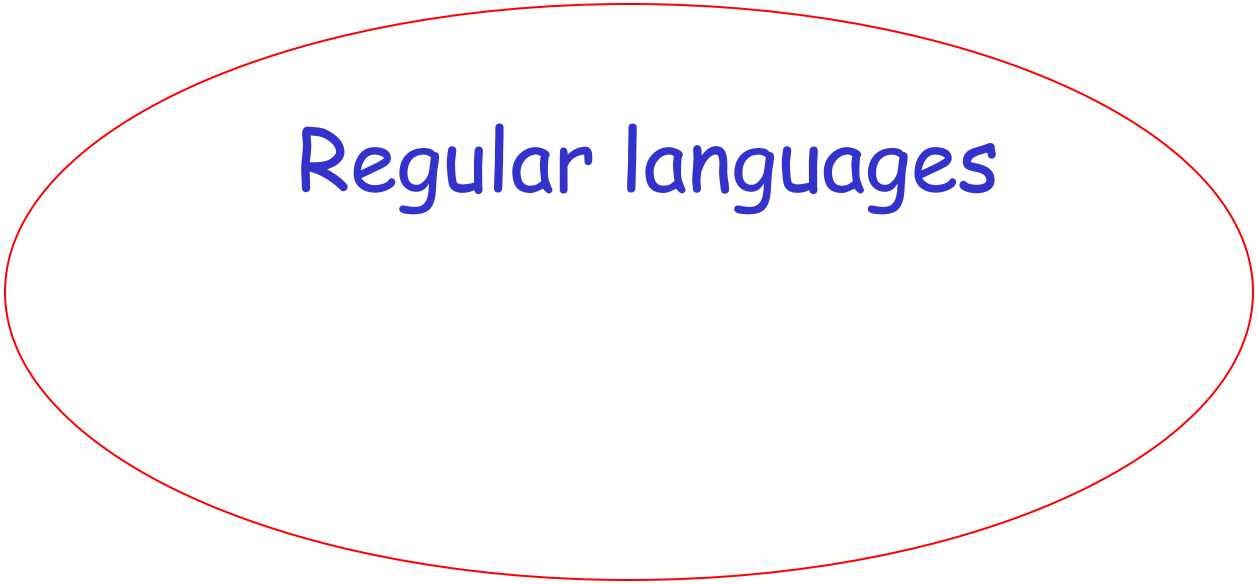
CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$



Regular languages

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Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m c^{2m}$

Write $a^m b^m c^{2m} = x\, y\, z$

From the Pumping Lemma
it must be that length $|x\,y| \leq m, \quad |y| \geq 1$

$$xyz = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\mathrlap{\underbrace{}_{z}}a\ldots a}^{m}\overbrace{b\ldots b}^{m}\overbrace{c\ldots cc\ldots c}^{2m}$$

Thus: $y = a^k, \quad k \geq 1$

$$x\,y\,z = a^m b^m c^{2m} \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^0\,z = xz \in L$

$$x\,y\,z = a^m b^m c^{2m} \qquad y = a^k, \quad k \geq 1$$

From the **Pumping Lemma:** $xz \in L$

$$xz = \mathrlap{\overbrace{}^{m-k}}\underbrace{a\ldots a}_{x}\underbrace{a\ldots a\overbrace{b\ldots b}^{m}\overbrace{c\ldots cc\ldots c}^{2m}}_{z} \in L$$

Thus: $a^{m-k} b^m c^{2m} \in L$

$$a^{m-k} b^{m} c^{2m} \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^l c^{n+l} : n, l \geq 0\}$

$$\Downarrow$$

$$a^{m-k} b^{m} c^{2m} \notin L$$

CONTRADICTION!!!

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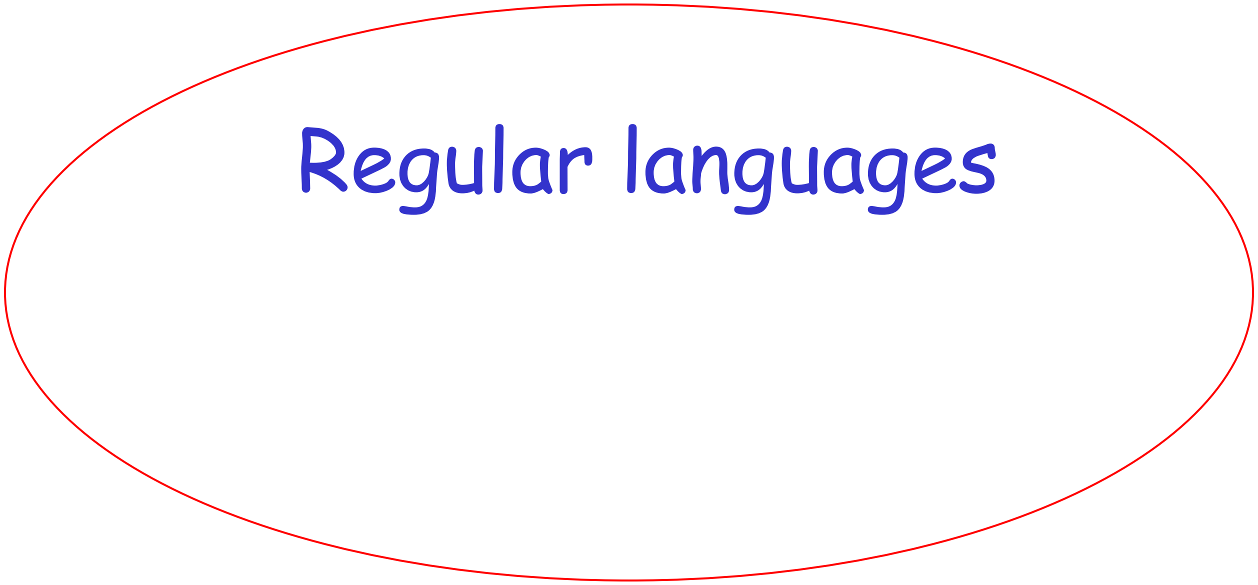
Non-regular languages

$$L = \{a^{n!} : \; n \geq 0\}$$



Regular languages

Theorem: The language $L = \{a^{n!} :\ n \geq 0\}$

is not regular

$$n! \ = \ 1 \cdot 2 \cdots (n-1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^{m!}$

Write $a^{m!} = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\ y| \leq m, \quad |y| \geq 1$

$$xyz = a^{m!} = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\rlap{$\underbrace{}_{z}$}a\ldots a}^{m}\overbrace{a\ldots aa\ldots a}^{m!-m}$$

Thus: $y = a^{k}, \quad 1 \leq k \leq m$

$x\,y\,z = a^{m!}$ $\qquad$ $y = a^k, \quad 1 \le k \le m$

From the Pumping Lemma: $\quad x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $\quad x\,y^2\,z \in L$

$x\,y\,z = a^{m!}$ $\qquad$ $y = a^{k}, \;\; 1 \le k \le m$

From the Pumping Lemma: $x\,y^{2}\,z \in L$

$$xy^{2}z = \rlap{$\overbrace{}^{m+k}\overbrace{}^{m!-m}$}\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\underbrace{a...aa...aa...aa...a}_{z} \in L$$

Thus: $a^{m!+k} \in L$

$$a^{m!+k} \in L \qquad 1 \le k \le m$$

Since: $L = \{a^{n!} : \ n \ge 0\}$

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There must exist p such that:

$$m! + k = p!$$

However: $m!+k \leq m!+m$ for $m>1$

$$\leq m!+m!$$

$$< m!m + m!$$

$$= m!(m+1)$$

$$= (m+1)!$$

$$\Downarrow$$

$$m!+k < (m+1)!$$

$$\Downarrow$$

$m!+k \neq p!$ for any p

$$a^{m!+k} \in L \qquad 1 \le k \le m$$

BUT: $L = \{a^{n!} : n \ge 0\}$

$$\Downarrow$$

$$a^{m!+k} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language