Fundamental Subspaces and Their Relations to SVD

Given an operator A: $H_1 \to H_2$, one defines fundamental subspaces as follows:

1. R(A) = $\{$y $\in H_2 : \exists\, x \in H_1 \,\text{such that}\, y = Ax\}$ known as the Range of A.
2. N(A) = $\{x \in H_1 :\ Ax = 0\}$ known as the Nullspace of A.
3. R(A*) = $\{y \in H_1 : \exists\, x \in H_2 \,\text{such that}\, y = A^*x\}$
4. N(A*) = $\{x \in H_2 :\ A^*x = 0\}$

Some key relationships amongst these 4 fundamental subspaces include:

1. H_1 = R(A*) $\oplus$ N(A)
2. H_2 = R(A) $\oplus$ N(A*)
3. dim(H_1) = dim(R(A*)) + dim(N(A))
4. dim(H_2) = dim(R(A)) + dim(N(A*))

The important takeaway is on how these four spaces can be identified via SVD. If the operator A is a complex $m \times n$ matrix A, i.e. $A \in \mathbb{C}^{m \times n}$, with rank(A) = p where $p \leq \min(m,\, n)$, then the SVD of A is given by

$$A = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 \Sigma V_1^T$$

where $\Sigma = \text{diag}(\sigma_1, ..., \sigma_p)$ and U, V are unitary matrices (transpose of the matrix is the inverse of itself). Furthermore, the SVD of A identifies the fundamental subspaces as follows:

1. R(A) = span(U_1) $\subseteq \mathbb{C}^m$
2. N(A*) = span(U_2) $\subseteq \mathbb{C}^m$
3. R(A*) = span(V_1) $\subseteq \mathbb{C}^n$
4. N(A) = span(V_2) $\subseteq \mathbb{C}^n$