WIKIPEDIA

Clebsch–Gordan coefficients

In physics, the **Clebsch–Gordan (CG) coefficients** are numbers that arise in angular momentum coupling in quantum mechanics. They appear as the expansion coefficients of total angular momentum eigenstates in an uncoupled tensor product basis. In more mathematical terms, the CG coefficients are used in representation theory, particularly of compact Lie groups, to perform the explicit direct sum decomposition of the tensor product of two irreducible representations (i.e., a reducible representation) into irreducible representations, in cases where the numbers and types of irreducible components are already known abstractly. The name derives from the German mathematicians Alfred Clebsch and Paul Gordan, who encountered an equivalent problem in invariant theory.

From a vector calculus perspective, the CG coefficients associated with the SO(3) group can be defined simply in terms of integrals of products of spherical harmonics and their complex conjugates. The addition of spins in quantum-mechanical terms can be read directly from this approach as spherical harmonics are eigenfunctions of total angular momentum and projection thereof onto an axis, and the integrals correspond to the Hilbert space inner product.[1] From the formal definition of angular momentum, recursion relations for the Clebsch–Gordan coefficients can be found. There also exist complicated explicit formulas for their direct calculation.[2]

The formulas below use Dirac's bra–ket notation and the Condon–Shortley phase convention[3] is adopted.

Contents

Angular momentum operators
Angular momentum states
Tensor product space
Formal definition of Clebsch–Gordan coefficients
Recursion relations
Explicit expression
Orthogonality relations
Special cases
Symmetry properties
Rules for phase factors
Relation to Wigner 3-j symbols
Relation to Wigner D-matrices
Relation to spherical harmonics
Other Properties
SU(N) Clebsch–Gordan coefficients
See also
Remarks
Notes
References
External links
Further reading

Angular momentum operators

Angular momentum operators are self-adjoint operators $\mathrm{j_x}$, $\mathrm{j_y}$, and $\mathrm{j_z}$ that satisfy the commutation relations

$$[\mathrm{j}_k, \mathrm{j}_l] \equiv \mathrm{j}_k \mathrm{j}_l - \mathrm{j}_l \mathrm{j}_k = i\hbar \varepsilon_{klm} \mathrm{j}_m \qquad k, l, m \in \{\mathrm{x}, \mathrm{y}, \mathrm{z}\},$$

where ε_{klm} is the Levi-Civita symbol. Together the three operators define a *vector operator*, a rank one Cartesian tensor operator,

$$\mathbf{j} = (\mathrm{j_x}, \mathrm{j_y}, \mathrm{j_z}).$$

It also known as a spherical vector, since it is also a spherical tensor operator. It is only for rank one that spherical tensor operators coincide with the Cartesian tensor operators.

By developing this concept further, one can define another operator $\mathbf{j}^2$ as the inner product of $\mathbf{j}$ with itself:

$$\mathbf{j}^2 = \mathrm{j_x}^2 + \mathrm{j_y}^2 + \mathrm{j_z}^2.$$

This is an example of a Casimir operator. It is diagonal and its eigenvalue characterizes the particular irreducible representation of the angular momentum algebra $\mathbf{so}(3) \cong \mathbf{su}(2)$. This is physically interpreted as the square of the total angular momentum of the states on which the representation acts.

One can also define *raising* (j_+) and *lowering* (j_-) operators, the so-called ladder operators,

$$\mathrm{j}_\pm = \mathrm{j_x} \pm i\mathrm{j_y}.$$

Angular momentum states

It can be shown from the above definitions that $\mathbf{j}^2$ commutes with $\mathrm{j_x}$, $\mathrm{j_y}$, and $\mathrm{j_z}$:

$$[\mathbf{j}^2, \mathrm{j}_k] = 0 \qquad k \in \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}.$$

When two Hermitian operators commute, a common set of eigenfunctions exists. Conventionally $\mathbf{j}^2$ and j_z are chosen. From the commutation relations the possible eigenvalues can be found. These states are denoted $|j\,m\rangle$ where j is the *angular momentum quantum number* and m is the *angular momentum projection* onto the z-axis. They satisfy the following eigenvalue equations:

$$\begin{aligned}\mathbf{j}^2|j\,m\rangle &= \hbar^2 j(j+1)|j\,m\rangle, && j\in\{0,\tfrac{1}{2},1,\tfrac{3}{2},\ldots\}\\ \mathrm{j}_z|j\,m\rangle &= \hbar m|j\,m\rangle, && m\in\{-j,-j+1,\ldots,j\}.\end{aligned}$$

The raising and lowering operators can be used to alter the value of m:

$$\mathrm{j}_\pm|j\,m\rangle = \hbar C_\pm(j,m)|j\,(m\pm 1)\rangle,$$

where the ladder coefficient is given by:

$$C_\pm(j,m) = \sqrt{j(j+1)-m(m\pm 1)} = \sqrt{(j\mp m)(j\pm m+1)}. \qquad (1)$$

In principle, one may also introduce a (possibly complex) phase factor in the definition of $C_\pm(j,m)$. The choice made in this article is in agreement with the Condon–Shortley phase convention. The angular momentum states are orthogonal (because their eigenvalues with respect to a Hermitian operator are distinct) and are assumed to be normalized:

$$\langle j\,m|j'\,m'\rangle = \delta_{j,j'}\delta_{m,m'}.$$

Here the italicized j and m denote integer or half-integer angular momentum quantum numbers of a particle or of a system. On the other hand, the roman j_x, j_y, j_z, j_+, j_-, and $\mathbf{j}^2$ denote operators. The δ symbols are Kronecker deltas.

Tensor product space

We now consider systems with two physically different angular momenta j_1 and j_2. Examples include the spin and the orbital angular momentum of a single electron, or the spins of two electrons, or the orbital angular momenta of two electrons. Mathematically, this means that the angular momentum operators act on a space V_1 of dimension $2j_1+1$ and also on a space V_2 of dimension $2j_2+1$. We are then going to define a family of "total angular momentum" operators acting on the tensor product space $V_1\otimes V_2$, which has dimension $(2j_1+1)(2j_2+1)$. The action of the total angular momentum operator on this space constitutes a representation of the su(2) Lie algebra, but a reducible one. The reduction of this reducible representation into irreducible pieces is the goal of Clebsch–Gordan theory.

Let V_1 be the $(2\,j_1+1)$-dimensional vector space spanned by the states

$$|j_1\,m_1\rangle, \qquad m_1\in\{-j_1,-j_1+1,\ldots,j_1\},$$

and V_2 the $(2\,j_2+1)$-dimensional vector space spanned by the states

$$|j_2\,m_2\rangle, \qquad m_2\in\{-j_2,-j_2+1,\ldots,j_2\}.$$

The tensor product of these spaces, $V_3\equiv V_1\otimes V_2$, has a $(2\,j_1+1)\,(2\,j_2+1)$-dimensional *uncoupled* basis

$$|j_1\,m_1\,j_2\,m_2\rangle \equiv |j_1\,m_1\rangle\otimes|j_2\,m_2\rangle, \quad m_1\in\{-j_1,-j_1+1,\ldots,j_1\}, \quad m_2\in\{-j_2,-j_2+1,\ldots,j_2\}.$$

Angular momentum operators are defined to act on states in V_3 in the following manner:

$$(\mathbf{j}\otimes 1)|j_1\,m_1\,j_2\,m_2\rangle \equiv \mathbf{j}|j_1\,m_1\rangle\otimes|j_2\,m_2\rangle$$

and

$$(1\otimes\mathbf{j})|j_1\,m_1\,j_2\,m_2\rangle \equiv |j_1\,m_1\rangle\otimes\mathbf{j}|j_2\,m_2\rangle,$$

where 1 denotes the identity operator.

The **total**[nb 1] **angular momentum** operators are defined by the coproduct (or tensor product) of the two representations acting on $V_1 \otimes V_2$,

$$\boxed{\mathbf{J}\equiv\mathbf{j}\otimes 1+1\otimes\mathbf{j}.}$$

The total angular momentum operators can be shown to *satisfy the very same commutation relations*,

$$[\mathrm{J}_k,\mathrm{J}_l] = i\hbar\varepsilon_{klm}\mathrm{J}_m,$$

where $k, l, m \in \{x, y, z\}$. Indeed, the preceding construction is the standard method[4] for constructing an action of a Lie algebra on a tensor product representation.

Hence, a set of *coupled* eigenstates exist for the total angular momentum operator as well,

$$\begin{aligned}\mathbf{J}^2|[j_1\,j_2]\,J\,M\rangle &= \hbar^2 J(J+1)|[j_1\,j_2]\,J\,M\rangle\\ \mathrm{J}_z|[j_1\,j_2]\,J\,M\rangle &= \hbar M|[j_1\,j_2]\,J\,M\rangle\end{aligned}$$

for $M\in\{-J, -J+1, \ldots, J\}$. Note that it is common to omit the $[j_1\,j_2]$ part.

The total angular momentum quantum number J must satisfy the triangular condition that

$$|j_1-j_2|\le J\le j_1+j_2,$$

such that the three nonnegative integer or half-integer values could correspond to the three sides of a triangle.[5]

The total number of total angular momentum eigenstates is necessarily equal to the dimension of V_3:

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} (2J+1) = (2j_1+1)(2j_2+1)\ .$$

As this computation suggests, the tensor product representation decomposes as the direct sum of one copy of each of the irreducible representations of dimension $2J+1$, where J ranges from $|j_1-j_2|$ to j_1+j_2 in increments of 1.[6] As an example, consider the tensor product of the three-dimensional representation corresponding to $j_1=1$ with the two-dimensional representation with $j_2=1/2$. The possible values of J are then $J=1/2$ and $J=3/2$. Thus, the six-dimensional tensor product representation decomposes as the direct sum of a two-dimensional representation and a four-dimensional representation.

The goal is now to describe the preceding decomposition explicitly, that is, to explicitly describe basis elements in the tensor product space for each of the component representations that arise.

The total angular momentum states form an orthonormal basis of V_3:

$$\langle J\,M|J'\,M'\rangle = \delta_{J,J'}\delta_{M,M'}\ .$$

These rules may be iterated to, e.g., combine n doublets (s=1/2) to obtain the Clebsch-Gordan decomposition series, (Catalan's triangle),

$$\mathbf{2}^{\otimes n} = \bigoplus_{k=0}^{\lfloor n/2\rfloor} \left(\frac{n+1-2k}{n+1}\binom{n+1}{k}\right)\ (\mathbf{n+1-2k})\ ,$$

where $\lfloor n/2\rfloor$ is the integer floor function; and the number preceding the boldface irreducible representation dimensionality ($2j$+1) label indicates multiplicity of that representation in the representation reduction.[7] For instance, from this formula, addition of three spin 1/2s yields a spin 3/2 and two spin 1/2s, $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = \mathbf{4}\oplus\mathbf{2}\oplus\mathbf{2}$.

Formal definition of Clebsch–Gordan coefficients

The coupled states can be expanded via the completeness relation (resolution of identity) in the uncoupled basis

$$|J\,M\rangle = \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} |j_1\,m_1\,j_2\,m_2\rangle\langle j_1\,m_1\,j_2\,m_2|J\,M\rangle \qquad (2)$$

The expansion coefficients

$$\langle j_1\,m_1\,j_2\,m_2|J\,M\rangle$$

are the *Clebsch–Gordan coefficients*. Note that some authors write them in a different order such as $\langle j_1\,j_2;\,m_1\,m_2|J\,M\rangle$.

Applying the operator

$$\mathrm{J_z} = \mathrm{j_z}\otimes 1 + 1\otimes\mathrm{j_z}$$

to both sides of the defining equation shows that the Clebsch–Gordan coefficients can only be nonzero when

$$M = m_1 + m_2\,.$$

Recursion relations

The recursion relations were discovered by physicist Giulio Racah from the Hebrew University of Jerusalem in 1941.

Applying the total angular momentum raising and lowering operators

$$\mathrm{J_\pm} = \mathrm{j_\pm}\otimes 1 + 1\otimes\mathrm{j_\pm}$$

to the left hand side of the defining equation gives

$$\begin{aligned}\mathrm{J_\pm}|[j_1\,j_2]\,J\,M\rangle &= \hbar C_\pm(J,M)|[j_1\,j_2]\,J\,(M\pm 1)\rangle\\ &= \hbar C_\pm(J,M)\sum_{m_1,m_2}|j_1\,m_1\,j_2\,m_2\rangle\langle j_1\,m_1\,j_2\,m_2|J\,(M\pm 1)\rangle\end{aligned}$$

Applying the same operators to the right hand side gives

$$\begin{aligned}&\mathrm{J_\pm}\sum_{m_1,m_2}|j_1\,m_1\,j_2\,m_2\rangle\langle j_1\,m_1\,j_2\,m_2|J\,M\rangle\\ &=\hbar\sum_{m_1,m_2}\Big(C_\pm(j_1,m_1)|j_1\,(m_1\pm 1)\,j_2\,m_2\rangle + C_\pm(j_2,m_2)|j_1\,m_1\,j_2\,(m_2\pm 1)\rangle\Big)\langle j_1\,m_1\,j_2\,m_2|J\,M\rangle\\ &=\hbar\sum_{m_1,m_2}|j_1\,m_1\,j_2\,m_2\rangle\Big(C_\pm(j_1,m_1\mp 1)\langle j_1\,(m_1\mp 1)\,j_2\,m_2|J\,M\rangle + C_\pm(j_2,m_2\mp 1)\langle j_1\,m_1\,j_2\,(m_2\mp 1)|J\,M\rangle\Big).\end{aligned}$$

where $C_\pm$ was defined in 1. Combining these results gives recursion relations for the Clebsch–Gordan coefficients:

$$C_\pm(J,M)\langle j_1\,m_1\,j_2\,m_2|J\,(M\pm 1)\rangle = C_\pm(j_1,m_1\mp 1)\langle j_1\,(m_1\mp 1)\,j_2\,m_2|J\,M\rangle + C_\pm(j_2,m_2\mp 1)\langle j_1\,m_1\,j_2\,(m_2\mp 1)|J\,M\rangle$$

.

Taking the upper sign with the condition that $M = J$ gives initial recursion relation:

$$0 = C_+(j_1, m_1 - 1)\langle j_1\,(m_1 - 1)\,j_2\,m_2|J\,J\rangle + C_+(j_2, m_2 - 1)\langle j_1\,m_1\,j_2\,(m_2 - 1)|J\,J\rangle.$$

In the Condon–Shortley phase convention, one adds the constraint that

$$\langle j_1\,j_1\,j_2\,(J - j_1)|J\,J\rangle > 0$$

(and is therefore also real).

The Clebsch–Gordan coefficients $\langle j_1\,m_1\,j_2\,m_2\,|\,J\,M\rangle$ can then be found from these recursion relations. The normalization is fixed by the requirement that the sum of the squares, which equivalent to the requirement that the norm of the state $|[j_1\,j_2]\,J\,J\rangle$ must be one.

The lower sign in the recursion relation can be used to find all the Clebsch–Gordan coefficients with $M = J - 1$. Repeated use of that equation gives all coefficients.

This procedure to find the Clebsch–Gordan coefficients shows that they are all real in the Condon–Shortley phase convention.

Explicit expression

Orthogonality relations

These are most clearly written down by introducing the alternative notation

$$\langle J\,M|j_1\,m_1\,j_2\,m_2\rangle \equiv \langle j_1\,m_1\,j_2\,m_2|J\,M\rangle$$

The first orthogonality relation is

$$\sum_{J=|j_1-j_2|}^{j_1+j_2}\sum_{M=-J}^{J}\langle j_1\,m_1\,j_2\,m_2|J\,M\rangle\langle J\,M|j_1\,m_1'\,j_2\,m_2'\rangle = \langle j_1\,m_1\,j_2\,m_2|j_1\,m_1'\,j_2\,m_2'\rangle = \delta_{m_1,m_1'}\delta_{m_2,m_2'}$$

(derived from the fact that $1 \equiv \sum_x |x\rangle\,\langle x|$) and the second one is

$$\sum_{m_1,m_2}\langle J\,M|j_1\,m_1\,j_2\,m_2\rangle\langle j_1\,m_1\,j_2\,m_2|J'\,M'\rangle = \langle J\,M|J'\,M'\rangle = \delta_{J,J'}\delta_{M,M'}.$$

Special cases

For $J = 0$ the Clebsch–Gordan coefficients are given by

$$\langle j_1\,m_1\,j_2\,m_2|0\,0\rangle = \delta_{j_1,j_2}\delta_{m_1,-m_2}\frac{(-1)^{j_1-m_1}}{\sqrt{2j_1+1}}.$$

For $J = j_1 + j_2$ and $M = J$ we have

$$\langle j_1\,j_1\,j_2\,j_2|(j_1 + j_2)\,(j_1 + j_2)\rangle = 1.$$

For $j_1 = j_2 = J\,/\,2$ and $m_1 = -m_2$ we have

$$\langle j_1\,m_1\,j_1\,(-m_1)|(2j_1)\,0\rangle = \frac{(2j_1)!^2}{(j_1 - m_1)!(j_1 + m_1)!\sqrt{(4j_1)!}}.$$

For $j_1 = j_2 = m_1 = -m_2$ we have

$$\langle j_1\,j_1\,j_1\,(-j_1)|J\,0\rangle = (2j_1)!\sqrt{\frac{2J+1}{(J+2j_1+1)!(2j_1-J)!}}.$$

For $j_2 = 1$, $m_2 = 0$ we have

$$\langle j_1\,m\,1\,0|(j_1 + 1)\,m\rangle = \sqrt{\frac{(j_1 - m + 1)(j_1 + m + 1)}{(2j_1 + 1)(j_1 + 1)}}$$

$$\langle j_1\,m\,1\,0|j_1\,m\rangle = \frac{m}{\sqrt{j_1(j_1 + 1)}}$$

$$\langle j_1\,m\,1\,0|(j_1 - 1)\,m\rangle = -\sqrt{\frac{(j_1 - m)(j_1 + m)}{j_1(2j_1 + 1)}}$$

For $j_2 = 1/2$ we have

$$\left\langle j_1\ \left(M-\frac{1}{2}\right)\ \frac{1}{2}\ \frac{1}{2}\middle|\left(j_1\pm\frac{1}{2}\right)\ M\right\rangle = \pm\sqrt{\frac{1}{2}\left(1\pm\frac{M}{j_1+\frac{1}{2}}\right)}$$

$$\left\langle j_1\ \left(M+\frac{1}{2}\right)\ \frac{1}{2}\ \left(-\frac{1}{2}\right)\middle|\left(j_1\pm\frac{1}{2}\right)\ M\right\rangle = \sqrt{\frac{1}{2}\left(1\mp\frac{M}{j_1+\frac{1}{2}}\right)}$$

Symmetry properties

$$\begin{aligned}
\langle j_1\, m_1\, j_2\, m_2 | J\, M\rangle &= (-1)^{j_1+j_2-J}\langle j_1\,(-m_1)\,j_2\,(-m_2)|J\,(-M)\rangle \\
&= (-1)^{j_1+j_2-J}\langle j_2\, m_2\, j_1\, m_1|J\, M\rangle \\
&= (-1)^{j_1-m_1}\sqrt{\frac{2J+1}{2j_2+1}}\langle j_1\, m_1\, J\,(-M)|j_2\,(-m_2)\rangle \\
&= (-1)^{j_2+m_2}\sqrt{\frac{2J+1}{2j_1+1}}\langle J\,(-M)\,j_2\, m_2|j_1\,(-m_1)\rangle \\
&= (-1)^{j_1-m_1}\sqrt{\frac{2J+1}{2j_2+1}}\langle J\, M\, j_1\,(-m_1)|j_2\, m_2\rangle \\
&= (-1)^{j_2+m_2}\sqrt{\frac{2J+1}{2j_1+1}}\langle j_2\,(-m_2)\,J\, M|j_1\, m_1\rangle
\end{aligned}$$

A convenient way to derive these relations is by converting the Clebsch–Gordan coefficients to Wigner 3-j symbols using **3**. The symmetry properties of Wigner 3-j symbols are much simpler.

Rules for phase factors

Care is needed when simplifying phase factors: a quantum number may be a half-integer rather than an integer, therefore $(-1)^{2k}$ is not necessarily 1 for a given quantum number k unless it can be proven to be an integer. Instead, it is replaced by the following weaker rule:

$$(-1)^{4k} = 1$$

for any angular-momentum-like quantum number k.

Nonetheless, a combination of j_i and m_i is always an integer, so the stronger rule applies for these combinations:

$$(-1)^{2(j_i-m_i)} = 1$$

This identity also holds if the sign of either j_i or m_i or both is reversed.

It is useful to observe that any phase factor for a given (j_i, m_i) pair can be reduced to the canonical form:

$$(-1)^{aj_i+b(j_i-m_i)}$$

where $a \in \{0, 1, 2, 3\}$ and $b \in \{0, 1\}$ (other conventions are possible too). Converting phase factors into this form makes it easy to tell whether two phase factors are equivalent. (Note that this form is only *locally* canonical: it fails to take into account the rules that govern combinations of (j_i, m_i) pairs such as the one described in the next paragraph.)

An additional rule holds for combinations of j_1, j_2, and j_3 that are related by a Clebsch-Gordan coefficient or Wigner 3-j symbol:

$$(-1)^{2(j_1+j_2+j_3)} = 1$$

This identity also holds if the sign of any j_i is reversed, or if any of them are substituted with an m_i instead.

Relation to Wigner 3-j symbols

Clebsch–Gordan coefficients are related to Wigner 3-j symbols which have more convenient symmetry relations.

$$\begin{aligned}
\langle j_1\, m_1\, j_2\, m_2|J\, M\rangle &= (-1)^{j_1-j_2+M}\sqrt{2J+1}\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M\end{pmatrix} \\
&= (-1)^{2j_2}(-1)^{J-M}\sqrt{2J+1}\begin{pmatrix} j_1 & J & j_2 \\ m_1 & -M & m_2\end{pmatrix}
\end{aligned} \quad (3)$$

The factor $(-1)^{2j_2}$ is due to the Condon–Shortley constraint that $\langle j_1\, j_1\, j_2\,(J-j_1)|J\, J\rangle > 0$, while $(-1)^{J-M}$ is due to the time-reversed nature of $|J\, M\rangle$.

Relation to Wigner D-matrices

$$\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta \, d\beta \int_0^{2\pi} d\gamma \, D^J_{M,K}(\alpha,\beta,\gamma)^* D^{j_1}_{m_1,k_1}(\alpha,\beta,\gamma) D^{j_2}_{m_2,k_2}(\alpha,\beta,\gamma)$$
$$= \frac{8\pi^2}{2J+1} \langle j_1 \, m_1 \, j_2 \, m_2 | J \, M \rangle \langle j_1 \, k_1 \, j_2 \, k_2 | J \, K \rangle$$

Relation to spherical harmonics

In the case where integers are involved, the coefficients can be related to integrals of spherical harmonics:

$$\int_{4\pi} Y_{\ell_1}^{m_1}{}^*(\Omega) Y_{\ell_2}^{m_2}{}^*(\Omega) Y_L^M(\Omega) \, d\Omega = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2L+1)}} \langle \ell_1 \, 0 \, \ell_2 \, 0 | L \, 0 \rangle \langle \ell_1 \, m_1 \, \ell_2 \, m_2 | L \, M \rangle$$

It follows from this and orthonormality of the spherical harmonics that CG coefficients are in fact the expansion coefficients of a product of two spherical harmonics in terms a single spherical harmonic:

$$Y_{\ell_1}^{m_1}(\Omega) Y_{\ell_2}^{m_2}(\Omega) = \sum_{L,M} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2L+1)}} \langle \ell_1 \, 0 \, \ell_2 \, 0 | L \, 0 \rangle \langle \ell_1 \, m_1 \, \ell_2 \, m_2 | L \, M \rangle Y_L^M(\Omega)$$

Other Properties

$$\sum_m (-1)^{j-m} \langle j \, m \, j \, (-m) | J \, 0 \rangle = \delta_{J,0} \sqrt{2j+1}$$

SU(N) Clebsch–Gordan coefficients

For arbitrary groups and their representations, Clebsch–Gordan coefficients are not known in general. However, algorithms to produce Clebsch–Gordan coefficients for the special unitary group are known.[8][9] In particular, SU(3) Clebsch-Gordan coefficients have been computed and tabulated because of their utility in characterizing hadronic decays, where a flavor-SU(3) symmetry exists that relates the up, down, and strange quarks.[10][11] A web interface for tabulating SU(N) Clebsch–Gordan coefficients (http://homepages.physik.uni-muenchen.de/~vondelft/Papers/ClebschGordan/) is readily available.

See also

- 3-j symbol
- 6-j symbol
- 9-j symbol
- Racah W-coefficient
- Spherical harmonics
- Spherical basis
- Tensor products of representations
- Associated Legendre polynomials
- Angular momentum
- Angular momentum coupling
- Total angular momentum quantum number
- Azimuthal quantum number
- Table of Clebsch–Gordan coefficients
- Wigner D-matrix
- Wigner–Eckart theorem
- Angular momentum diagrams (quantum mechanics)
- Clebsch–Gordan coefficient for SU(3)
- Littlewood-Richardson coefficient

Remarks

1. The word "total" is often overloaded to mean several different things. In this article, "total angular momentum" refers to a generic sum of two angular momentum operators $\mathbf{j}_1$ and $\mathbf{j}_2$. It is not to be confused with the other common use of the term "total angular momentum" that refers specifically to the sum of orbital angular momentum and spin.

Notes

1. Greiner & Müller 1994
2. Edmonds 1957
3. Condon & Shortley 1970
4. Hall 2015 Section 4.3.2
5. Merzbacher 1998
6. Hall 2015 Appendix C
7. Zachos, C K (1992). "Altering the Symmetry of Wavefunctions in Quantum Algebras and Supersymmetry". *Modern Physics Letters*. World Scientific Publishing. **A7**: 1595–1600. arXiv:hep-th/9203027 (https://arxiv.org/abs/hep-th/9203027). Bibcode:1992MPLA....7.1595Z (http://adsabs.harvard.edu/abs/1992MPLA....7.1595Z). doi:10.1142/S0217732392001270 (https://doi.org/10.1142%2FS0217732392001270).
8. Alex et al. 2011
9. Kaplan & Resnikoff 1967
10. de Swart 1963
11. Kaeding 1995

References

- Alex, A.; Kalus, M.; Huckleberry, A.; von Delft, J. (2011). "A numerical algorithm for the explicit calculation of SU(N) and SL(N,C) Clebsch–Gordan coefficients". *J. Math. Phys.* **82**: 023507. arXiv:1009.0437 (https://arxiv.org/abs/1009.0437). Bibcode:2011JMP....52b3507A (http://adsabs.harvard.edu/abs/2011JMP....52b3507A). doi:10.1063/1.3521562 (https://doi.org/10.1063%2F1.3521562).
- Condon, Edward U.; Shortley, G. H. (1970). "Ch. 3". *The Theory of Atomic Spectra*. Cambridge: Cambridge University Press. ISBN 0-521-09209-4.
- Edmonds, A. R. (1957). *Angular Momentum in Quantum Mechanics*. Princeton, New Jersey: Princeton University Press. ISBN 0-691-07912-9.
- Greiner, Walter; Müller, Berndt (1994). *Quantum Mechanics: Symmetries* (2nd ed.). Springer Verlag. ISBN 978-3540580805.
- Hall, Brian C. (2015), *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, Graduate Texts in Mathematics, **222** (2nd ed.), Springer, ISBN 978-3319134666.
- Kaplan, L. M.; Resnikoff, M. (1967). "Matrix products and explicit 3, 6, 9, and 12j coefficients of the regular representation of SU(n)". *J. Math. Phys.* **8**: 2194. Bibcode:1967JMP.....8.2194K (http://adsabs.harvard.edu/abs/1967JMP.....8.2194K). doi:10.1063/1.1705141 (https://doi.org/10.1063%2F1.1705141).

- Kaeding, Thomas (1995). "Tables of SU(3) isoscalar factors". *Atomic Data and Nuclear Data Tables*. **61**: 233–288. arXiv:nucl-th/9502037 (https://arxiv.org/abs/nucl-th/9502037) [nucl-th (https://arxiv.org/archive/nucl-th)]. Bibcode:1995ADNDT..61..233K (http://adsabs.harvard.edu/abs/1995ADNDT..61..233K). doi:10.1006/adnd.1995.1011 (https://doi.org/10.1006%2Fadnd.1995.1011).
- Merzbacher, Eugen (1998). *Quantum Mechanics* (3rd ed.). John Wiley. pp. 428–9. ISBN 0-471-88702-1.
- Albert Messiah (1966). *Quantum Mechanics* (Vols. I & II), English translation from French by G. M. Temmer. North Holland, John Wiley & Sons.
- de Swart, J. J. (1963). "The Octet model and its Clebsch-Gordan coefficients". *Rev. Mod. Phys*. **35**: 916. Bibcode:1963RvMP...35..916D (http://adsabs.harvard.edu/abs/1963RvMP...35..916D). doi:10.1103/RevModPhys.35.916 (https://doi.org/10.1103%2FRevModPhys.35.916).

External links

- PDF Table of Clebsch–Gordan Coefficients, Spherical Harmonics, and d-Functions (http://pdg.lbl.gov/2011/reviews/rpp2011-rev-clebsch-gordan-coefs.pdf)
- Clebsch–Gordan, 3-j and 6-j Coefficient Web Calculator (http://www.volya.net/index.php?id=vc)
- Downloadable Clebsch–Gordan Coefficient Calculator for Mac and Windows (http://phys.csuchico.edu/C-G/)
- Web interface for tabulating SU(N) Clebsch–Gordan coefficients (http://homepages.physik.uni-muenchen.de/~vondelft/Papers/ClebschGordan/)

Further reading

- *Quantum mechanics*, E. Zaarur, Y. Peleg, R. Pnini, Schaum's Easy Oulines Crash Course, McGraw Hill (USA), 2006, ISBN 978-007-145533-6
- *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles* (2nd Edition), R. Eisberg, R. Resnick, John Wiley & Sons, 1985, ISBN 978-0-471-87373-0
- *Quantum Mechanics*, E. Abers, Pearson Ed., Addison Wesley, Prentice Hall Inc, 2004, ISBN 978-0-13-146100-0
- *Physics of Atoms and Molecules*, B. H. Bransden, C. J. Joachain, Longman, 1983, ISBN 0-582-44401-2
- *The Cambridge Handbook of Physics Formulas*, G. Woan, Cambridge University Press, 2010, ISBN 978-0-521-57507-2.
- *Encyclopaedia of Physics* (2nd Edition), R. G. Lerner, G. L. Trigg, VHC publishers, 1991, ISBN (Verlagsgesellschaft) 3-527-26954-1, ISBN (VHC Inc.) 0-89573-752-3
- *McGraw Hill Encyclopaedia of Physics* (2nd Edition), C. B. Parker, 1994, ISBN 0-07-051400-3
- Biedenharn, L. C.; Louck, J. D. (1981). *Angular Momentum in Quantum Physics*. Reading, Massachusetts: Addison-Wesley. ISBN 0-201-13507-8.
- Brink, D. M.; Satchler, G. R. (1993). "Ch. 2". *Angular Momentum* (3rd ed.). Oxford: Clarendon Press. ISBN 0-19-851759-9.
- Messiah, Albert (1981). "Ch. XIII". *Quantum Mechanics (Volume II)*. New York: North Holland Publishing. ISBN 0-7204-0045-7.
- Zare, Richard N. (1988). "Ch. 2". *Angular Momentum*. New York: John Wiley & Sons. ISBN 0-471-85892-7.