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Vector spherical harmonics

In mathematics, **vector spherical harmonics (VSH)** are an extension of the scalar spherical harmonics for use with vector fields. The components of the VSH are complex-valued functions expressed in the spherical coordinate basis vectors.

Contents

Definition

Several conventions have been used to define the VSH.[1][2][3][4][5] We follow that of Barrera *et al.*. Given a scalar spherical harmonic $Y_{\ell m}(\theta, \varphi)$, we define three VSH:

- $\mathbf{Y}_{lm} = Y_{lm}\hat{\mathbf{r}}$
- $\mathbf{\Psi}_{lm} = r\nabla Y_{lm}$
- $\mathbf{\Phi}_{lm} = \mathbf{r}\times\nabla Y_{lm}$

with $\hat{\mathbf{r}}$ being the unit vector along the radial direction in spherical coordinates and $\mathbf{r}$ the vector along the radial direction with the same norm as the radius: $(r, 0, 0)$. The radial factors are included to guarantee that the dimensions of the VSH are the same as those of the ordinary spherical harmonics and that the VSH do not depend on the radial spherical coordinate.

The interest of these new vector fields is to separate the radial dependence from the angular one when using spherical coordinates, so that a vector field admits a multipole expansion

$$\mathbf{E} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(E^r_{lm}(r)\mathbf{Y}_{lm} + E^{(1)}_{lm}(r)\mathbf{\Psi}_{lm} + E^{(2)}_{lm}(r)\mathbf{\Phi}_{lm}\right)$$

The labels on the components reflect that E^r_{lm} is the radial component of the vector field, while $E^{(1)}_{lm}$ and $E^{(2)}_{lm}$ are transverse components.

Main Properties

Symmetry

Like the scalar spherical harmonics, the VSH satisfy

$$\mathbf{Y}_{l,-m} = (-1)^m\mathbf{Y}^*_{lm} \qquad \mathbf{\Psi}_{l,-m} = (-1)^m\mathbf{\Psi}^*_{lm} \qquad \mathbf{\Phi}_{l,-m} = (-1)^m\mathbf{\Phi}^*_{lm}$$

which cuts the number of independent functions roughly in half. The star * indicates complex conjugate.

Orthogonality

The VSH are orthogonal in the usual three-dimensional way at a point $\mathbf{r}$

$$\mathbf{Y}_{lm}\cdot\mathbf{\Psi}_{lm} = 0 \qquad \mathbf{Y}_{lm}\cdot\mathbf{\Phi}_{lm} = 0 \qquad \mathbf{\Psi}_{lm}\cdot\mathbf{\Phi}_{lm} = 0$$

They are also orthogonal in the Hilbert space

$$\int \mathbf{Y}_{lm} \cdot \mathbf{Y}^*_{l'm'}\, \mathrm{d}\Omega = \delta_{ll'} \delta_{mm'}$$

$$\int \mathbf{\Psi}_{lm} \cdot \mathbf{\Psi}^*_{l'm'}\, \mathrm{d}\Omega = l(l+1)\delta_{ll'} \delta_{mm'}$$

$$\int \mathbf{\Phi}_{lm} \cdot \mathbf{\Phi}^*_{l'm'}\, \mathrm{d}\Omega = l(l+1)\delta_{ll'} \delta_{mm'}$$

$$\int \mathbf{Y}_{lm} \cdot \mathbf{\Psi}^*_{l'm'}\, \mathrm{d}\Omega = 0$$

$$\int \mathbf{Y}_{lm} \cdot \mathbf{\Phi}^*_{l'm'}\, \mathrm{d}\Omega = 0$$

$$\int \mathbf{\Psi}_{lm} \cdot \mathbf{\Phi}^*_{l'm'}\, \mathrm{d}\Omega = 0$$

An additional result at a single point $\mathbf{r}$ (not reported in Barrera et al, 1985) is (for all l, m, l', m')

$$\mathbf{Y}_{lm} \cdot \mathbf{\Psi}_{l'm'} = 0 \qquad \mathbf{Y}_{lm} \cdot \mathbf{\Phi}_{l'm'} = 0$$

Vector multipole moments

The orthogonality relations allow one to compute the spherical multipole moments of a vector field as

$$E^r_{lm} = \int \mathbf{E} \cdot \mathbf{Y}^*_{lm}\, \mathrm{d}\Omega$$

$$E^{(1)}_{lm} = \frac{1}{l(l+1)} \int \mathbf{E} \cdot \mathbf{\Psi}^*_{lm}\, \mathrm{d}\Omega$$

$$E^{(2)}_{lm} = \frac{1}{l(l+1)} \int \mathbf{E} \cdot \mathbf{\Phi}^*_{lm}\, \mathrm{d}\Omega$$

The gradient of a scalar field

Given the multipole expansion of a scalar field

$$\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{lm}(r) Y_{lm}(\theta, \phi)$$

we can express its gradient in terms of the VSH as

$$\nabla\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{\mathrm{d}\phi_{lm}}{\mathrm{d}r} \mathbf{Y}_{lm} + \frac{\phi_{lm}}{r} \mathbf{\Psi}_{lm} \right)$$

Divergence

For any multipole field we have

$$\nabla \cdot (f(r)\mathbf{Y}_{lm}) = \left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{2}{r} f \right) Y_{lm}$$

$$\nabla \cdot (f(r)\mathbf{\Psi}_{lm}) = -\frac{l(l+1)}{r} f Y_{lm}$$

$$\nabla \cdot (f(r)\mathbf{\Phi}_{lm}) = 0$$

By superposition we obtain the divergence of any vector field

$$\nabla \cdot \mathbf{E} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{\mathrm{d}E^r_{lm}}{\mathrm{d}r} + \frac{2}{r} E^r_{lm} - \frac{l(l+1)}{r} E^{(1)}_{lm} \right) Y_{lm}$$

we see that the component on $\mathbf{\Phi}_{\ell m}$ is always solenoidal.

Curl

For any multipole field we have

$$\nabla \times (f(r)\mathbf{Y}_{lm}) = -\frac{1}{r} f \mathbf{\Phi}_{lm}$$

$$\nabla \times (f(r)\mathbf{\Psi}_{lm}) = \left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{1}{r} f \right) \mathbf{\Phi}_{lm}$$

$$\nabla \times (f(r)\mathbf{\Phi}_{lm}) = -\frac{l(l+1)}{r} f \mathbf{Y}_{lm} - \left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{1}{r} f \right) \mathbf{\Psi}_{lm}$$

By superposition we obtain the curl of any vector field

$$\nabla \times \mathbf{E} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(-\frac{l(l+1)}{r} E^{(2)}_{lm} \mathbf{Y}_{lm} - \left(\frac{\mathrm{d}E^{(2)}_{lm}}{\mathrm{d}r} + \frac{1}{r} E^{(2)}_{lm} \right) \mathbf{\Psi}_{lm} + \left(-\frac{1}{r} E^r_{lm} + \frac{\mathrm{d}E^{(1)}_{lm}}{\mathrm{d}r} + \frac{1}{r} E^{(1)}_{lm} \right) \mathbf{\Phi}_{lm} \right)$$

Examples

First vector spherical harmonics

- $l = 0$
  - $\mathbf{Y}_{00} = \sqrt{\frac{1}{4\pi}} \hat{\mathbf{r}}$
  - $\mathbf{\Psi}_{00} = \mathbf{0}$
  - $\mathbf{\Phi}_{00} = \mathbf{0}$
- $l = 1$
  - $\mathbf{Y}_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \, \hat{\mathbf{r}}$
  - $\mathbf{Y}_{11} = -\sqrt{\frac{3}{8\pi}} \mathrm{e}^{\mathrm{i}\varphi} \sin\theta \, \hat{\mathbf{r}}$
  - $\mathbf{\Psi}_{10} = -\sqrt{\frac{3}{4\pi}} \sin\theta \, \hat{\theta}$
  - $\mathbf{\Psi}_{11} = -\sqrt{\frac{3}{8\pi}} \mathrm{e}^{\mathrm{i}\varphi} \left(\cos\theta \, \hat{\theta} + \mathrm{i} \, \hat{\varphi} \right)$
  - $\mathbf{\Phi}_{10} = -\sqrt{\frac{3}{4\pi}} \sin\theta \, \hat{\varphi}$
  - $\mathbf{\Phi}_{11} = \sqrt{\frac{3}{8\pi}} \mathrm{e}^{\mathrm{i}\varphi} \left(\mathrm{i} \, \hat{\theta} - \cos\theta \, \hat{\varphi} \right)$
- $l = 2$
  - $\mathbf{Y}_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}} \left(3\cos^2\theta - 1\right) \hat{\mathbf{r}}$
  - $\mathbf{Y}_{21} = -\sqrt{\frac{15}{8\pi}} \, \sin\theta \, \cos\theta \, e^{i\varphi} \, \hat{\mathbf{r}}$
  - $\mathbf{Y}_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \, \sin^2\theta \, e^{2i\varphi} \, \hat{\mathbf{r}}$

- $\mathbf{\Psi}_{20} = -\frac{3}{2}\sqrt{\frac{5}{\pi}}\,\sin\theta\,\cos\theta\,\hat{\theta}$
- $\mathbf{\Psi}_{21} = -\sqrt{\frac{15}{8\pi}}\,e^{i\varphi}\,\left(\cos 2\theta\,\hat{\theta} + \mathrm{i}\cos\theta\,\hat{\varphi}\right)$
- $\mathbf{\Psi}_{22} = \sqrt{\frac{15}{8\pi}}\,\sin\theta\,e^{2i\varphi}\,\left(\cos\theta\,\hat{\theta} + \mathrm{i}\,\hat{\varphi}\right)$

- $\mathbf{\Phi}_{20} = -\frac{3}{2}\sqrt{\frac{5}{\pi}}\,\sin\theta\,\cos\theta\,\hat{\phi}$
- $\mathbf{\Phi}_{21} = \sqrt{\frac{15}{8\pi}}\,e^{i\varphi}\,\left(\mathrm{i}\cos\theta\,\hat{\theta} - \cos 2\theta\,\hat{\varphi}\right)$
- $\mathbf{\Phi}_{22} = \sqrt{\frac{15}{8\pi}}\,\sin\theta\,e^{2i\varphi}\,\left(-\mathrm{i}\,\hat{\theta} + \cos\theta\,\hat{\varphi}\right)$

The expression for negative values of m are obtained applying the symmetry relations.

Applications

Electrodynamics

The VSH are especially useful in the study of multipole radiation fields. For instance, a magnetic multipole is due to an oscillating current with angular frequency ω and complex amplitude

$$\hat{\mathbf{J}} = J(r)\mathbf{\Phi}_{lm}$$

and the corresponding electric and magnetic fields can be written as

$$\hat{\mathbf{E}} = E(r)\mathbf{\Phi}_{lm}$$

$$\hat{\mathbf{B}} = B^r(r)\mathbf{Y}_{lm} + B^{(1)}(r)\mathbf{\Psi}_{lm}$$

Substituting into Maxwell equations, Gauss' law is automatically satisfied

$$\nabla\cdot\hat{\mathbf{E}} = 0$$

while Faraday's law decouples in

$$\nabla\times\hat{\mathbf{E}} = -\mathrm{i}\omega\hat{\mathbf{B}} \qquad \Rightarrow \qquad \begin{cases} \dfrac{l(l+1)}{r}E = \mathrm{i}\omega B^r \\[2ex] \dfrac{\mathrm{d}E}{\mathrm{d}r} + \dfrac{E}{r} = \mathrm{i}\omega B^{(1)} \end{cases}$$

Gauss' law for the magnetic field implies

$$\nabla\cdot\hat{\mathbf{B}} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}B^r}{\mathrm{d}r} + \frac{2}{r}B^r - \frac{l(l+1)}{r}B^{(1)} = 0$$

and Ampère-Maxwell's equation gives

$$\nabla\times\hat{\mathbf{B}} = \mu_0\hat{\mathbf{J}} + \mathrm{i}\mu_0\varepsilon_0\omega\hat{\mathbf{E}} \quad \Rightarrow \quad -\frac{B^r}{r} + \frac{\mathrm{d}B^{(1)}}{\mathrm{d}r} + \frac{B^{(1)}}{r} = \mu_0 J + \mathrm{i}\omega\mu_0\varepsilon_0 E$$

In this way, the partial differential equations have been transformed into a set of ordinary differential equations.

Fluid dynamics

In the calculation of the Stokes' law for the drag that a viscous fluid exerts on a small spherical particle, the velocity distribution obeys Navier-Stokes equations neglecting inertia, i.e.

$$\nabla\cdot\mathbf{v} = 0$$

$$\mathbf{0} = -\nabla p + \eta\nabla^2\mathbf{v}$$

with the boundary conditions

$$\mathbf{v} = \mathbf{0} \quad (r = a)$$

$$\mathbf{v} = -\mathbf{U}_0 \quad (r \to \infty)$$

being **U** the relative velocity of the particle to the fluid far from the particle. In spherical coordinates this velocity at infinity can be written as

$$\mathbf{U}_0 = U_0 \left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right) = U_0 \left(\mathbf{Y}_{10} + \mathbf{\Psi}_{10}\right)$$

The last expression suggest an expansion on spherical harmonics for the liquid velocity and the pressure

$$p = p(r) Y_{10}$$

$$\mathbf{v} = v^r(r)\mathbf{Y}_{10} + v^{(1)}(r)\mathbf{\Psi}_{10}$$

Substitution in the Navier-Stokes equations produces a set of ordinary differential equations for the coefficients.

See also

- Spherical harmonics
- Spin spherical harmonics
- Spin-weighted spherical harmonics
- Electromagnetic radiation
- Spherical basis

References

1. R.G. Barrera, G.A. Estévez and J. Giraldo, *Vector spherical harmonics and their application to magnetostatics*, Eur. J. Phys. **6** 287-294 (1985)
2. B. Carrascal, G.A. Estevez, P. Lee and V. Lorenzo *Vector spherical harmonics and their application to classical electrodynamics*, Eur. J. Phys., **12**, 184-191 (1991)
3. E. L. Hill, *The theory of Vector Spherical Harmonics*, Am. J. Phys. **22**, 211-214 (1954)
4. E. J. Weinberg, *Monopole vector spherical harmonics*, Phys. Rev. D. **49**, 1086-1092 (1994)
5. P.M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part II*, New York: McGraw-Hill, 1898-1901 (1953)

External links

- *Vector Spherical Harmonics* at Eric Weisstein's Mathworld (http://mathworld.wolfram.com/VectorSphericalHarmonic.html)