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3-j symbol

In quantum mechanics, the **Wigner 3-j symbols**, also called 3-*jm* symbols, are an alternative to Clebsch–Gordan coefficients for the purpose of adding angular momenta.[1] While the two approaches address exactly the same physical problem, the 3-*j* symbols do so more symmetrically, and thus have greater and simpler symmetry properties than the Clebsch-Gordan coefficients.

Contents

Mathematical relation to Clebsch-Gordan coefficients

The 3-*j* symbols are given in terms of the Clebsch-Gordan coefficients by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1\, m_1\, j_2\, m_2 | j_3\, (-m_3) \rangle.$$

The j's and m's are angular momentum quantum numbers, i.e., every j (and every corresponding m) is either a nonnegative integer or half-odd-integer. The exponent of the sign factor is always an integer, so it remains the same when transposed to the left hand side, and the inverse relation follows upon making the substitution $m_3 \to -m_3$:

$$\langle j_1\, m_1\, j_2\, m_2 | j_3\, m_3 \rangle = (-1)^{-j_1+j_2-m_3} \sqrt{2j_3+1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}.$$

Definitional relation to Clebsch-Gordan coefficients

The C-G coefficients are defined so as to express the addition of two angular momenta in terms of a third:

$$|j_3\, m_3\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1\, m_1\, j_2\, m_2 | j_3\, m_3 \rangle |j_1\, m_1\, j_2\, m_2\rangle.$$

The 3-*j* symbols, on the other hand, are the coefficients with which three angular momenta must be added so that the resultant is zero:

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = |0\,0\rangle.$$

Here, $|0\,0\rangle$ is the zero angular momentum state ($j = m = 0$). It is apparent that the 3-*j* symbol treats all three angular momenta involved in the addition problem on an equal footing, and is therefore more symmetrical than the C-G coefficient.

Since the state $|0\,0\rangle$ is unchanged by rotation, one also says that the contraction of the product of three rotational states with a 3-*j* symbol is invariant under rotations.

Selection rules

The Wigner 3-*j* symbol is zero unless all these conditions are satisfied:

$$\begin{aligned} & m_i \in \{-j_i, -j_i+1, -j_i+2, \ldots, j_i\}, \quad (i = 1, 2, 3). \\ & m_1 + m_2 + m_3 = 0 \\ & |j_1 - j_2| \le j_3 \le j_1 + j_2 \\ & (j_1 + j_2 + j_3) \text{ is an integer (and, moreover, an even integer if } m_1 = m_2 = m_3 = 0) \end{aligned}$$

Symmetry properties

A 3-*j* symbol is invariant under an even permutation of its columns:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}.$$

An odd permutation of the columns gives a phase factor:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}.$$

Changing the sign of the m quantum numbers (time-reversal) also gives a phase:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

The 3-*j* symbols also have so-called Regge symmetries, which are not due to permutations or time-reversal.[2] These symmetries are,

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \begin{pmatrix} j_1 & \frac{j_2+j_3-m_1}{2} & \frac{j_2+j_3+m_1}{2} \\ j_3-j_2 & \frac{j_2-j_3-m_1}{2}-m_3 & \frac{j_2-j_3+m_1}{2}+m_3 \end{pmatrix}. \\ \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= (-1)^{j_1+j_2+j_3} \begin{pmatrix} \frac{j_2+j_3+m_1}{2} & \frac{j_1+j_3+m_2}{2} & \frac{j_1+j_2+m_3}{2} \\ j_1-\frac{j_2+j_3-m_1}{2} & j_2-\frac{j_1+j_3-m_2}{2} & j_3-\frac{j_1+j_2-m_3}{2} \end{pmatrix}. \end{aligned}$$

With the Regge symmetries, the 3-*j* symbol has a total of 72 symmetries. These are best displayed by the definition of a Regge symbol which is a one-to-one correspondence between it and a 3-*j* symbol and assumes the properties of a semi-magic square[3]

$$R = \begin{array}{|ccc|} \hline -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \\ \hline \end{array}$$

whereby the 72 symmetries now correspond to 3! row and 3! column interchanges plus a transposition of the matrix. These facts can be used to devise an effective storage scheme.[4]

Orthogonality relations

A system of two angular momenta with magnitudes j_1 and j_2, say, can be described either in terms of the uncoupled basis states (labeled by the quantum numbers m_1 and m_2), or the coupled basis states (labeled by j_3 and m_3). The 3-j symbols constitute a unitary transformation between these two bases, and this unitarity implies the orthogonality relations,

$$(2j_3+1)\sum_{m_1 m_2}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3'\\ m_1 & m_2 & m_3'\end{pmatrix} = \delta_{j_3,j_3'}\delta_{m_3,m_3'}\begin{Bmatrix} j_1 & j_2 & j_3\end{Bmatrix}.$$

$$\sum_{j_3 m_3}(2j_3+1)\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1' & m_2' & m_3\end{pmatrix} = \delta_{m_1,m_1'}\delta_{m_2,m_2'}.$$

The *triangular delta* $\{j_1 \; j_2 \; j_3\}$ is equal to 1 when the triad (j_1, j_2, j_3) satisfies the triangle conditions, and zero otherwise. The triangular delta itself is sometimes confusingly called[5] a "3-j symbol" (without the "m") in analogy to 6-j and 9-j symbols, all of which are irreducible summations of 3-jm symbols where no m variables remain.

Relation to spherical harmonics

The 3-*jm* symbols give the integral of the products of three spherical harmonics

$$\int Y_{l_1m_1}(\theta,\varphi)Y_{l_2m_2}(\theta,\varphi)Y_{l_3m_3}(\theta,\varphi)\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$=\sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix}$$

with l_1, l_2 and l_3 integers.

Relation to integrals of spin-weighted spherical harmonics

Similar relations exist for the spin-weighted spherical harmonics if $s_1+s_2+s_3=0$:

$$\int d\hat{\mathbf{n}}\,{}_{s_1}Y_{j_1m_1}(\hat{\mathbf{n}})\,{}_{s_2}Y_{j_2m_2}(\hat{\mathbf{n}})\,{}_{s_3}Y_{j_3m_3}(\hat{\mathbf{n}})$$
$$=\sqrt{\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{4\pi}}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3\\ -s_1 & -s_2 & -s_3\end{pmatrix}$$

Recursion relations

$$-\sqrt{(l_3\mp s_3)(l_3\pm s_3+1)}\begin{pmatrix} l_1 & l_2 & l_3\\ s_1 & s_2 & s_3\pm 1\end{pmatrix}$$
$$=\sqrt{(l_1\mp s_1)(l_1\pm s_1+1)}\begin{pmatrix} l_1 & l_2 & l_3\\ s_1\pm 1 & s_2 & s_3\end{pmatrix}+\sqrt{(l_2\mp s_2)(l_2\pm s_2+1)}\begin{pmatrix} l_1 & l_2 & l_3\\ s_1 & s_2\pm 1 & s_3\end{pmatrix}$$

Asymptotic expressions

For $l_1 \ll l_2, l_3$ a non-zero 3-j symbol has

$$\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix}\approx(-1)^{l_3+m_3}\frac{d^{l_1}_{m_1,l_3-l_2}(\theta)}{\sqrt{2l_3+1}}$$

where $\cos(\theta)=-2m_3/(2l_3+1)$ and d^l_{mn} is a Wigner function. Generally a better approximation obeying the Regge symmetry is given by

$$\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix}\approx(-1)^{l_3+m_3}\frac{d^{l_1}_{m_1,l_3-l_2}(\theta)}{\sqrt{l_2+l_3+1}}$$

where $\cos(\theta) = (m_2 - m_3)/(l_2 + l_3 + 1)$.

Metric tensor

The following quantity acts as a metric tensor in angular momentum theory and is also known as a *Wigner 1-jm symbol*,[1]

$$\begin{pmatrix} & j & \\ m & & m' \end{pmatrix} := \sqrt{2j+1}\begin{pmatrix} j & 0 & j \\ m & 0 & m' \end{pmatrix} = (-1)^{j-m'}\delta_{m,-m'}$$

It can be used to perform time-reversal on angular momenta.

Other properties

$$\sum_m (-1)^{j-m}\begin{pmatrix} j & j & J \\ m & -m & 0 \end{pmatrix} = \sqrt{2j+1}\,\delta_{J,0}$$

$$\frac{1}{2}\int_{-1}^{1} P_{l_1}(x)P_{l_2}(x)P_l(x)\,dx = \begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2$$

Relation to Racah V-coefficients

Wigner 3-j symbols are related to Racah V-coefficients[6] by a simple phase:

$$V(j_1 j_2 j_3; m_1 m_2 m_3) = (-1)^{j_1-j_2-j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

See also

- Clebsch–Gordan coefficients
- Spherical harmonics
- 6-j symbol
- 9-j symbol

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External links

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- 369j-symbol calculator at the Plasma Laboratory of Weizmann Institute of Science (http://plasma-gate.weizmann.ac.il/369j.html) (Numerical)
- Frederik J Simons: Matlab software archive, the code THREEJ.M (http://geoweb.princeton.edu/people/simons/software.html)
- Sage (mathematics software) (http://www.sagemath.org/) Gives exact answer for any value of j, m
- Johansson, H.T.; Forssén, C. "(WIGXJPF)" (http://fy.chalmers.se/subatom/wigxjpf/). (accurate; C, fortran, python)
- Johansson, H.T. "(FASTWIGXJ)" (http://fy.chalmers.se/subatom/fastwigxj/). (fast lookup, accurate; C, fortran)