Hankel transform

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In mathematics, the **Hankel transform** expresses any given function *f*(*r*) as the weighted sum of an infinite number of Bessel functions of the first kind $J_\nu(kr)$. The Bessel functions in the sum are all of the same order ν, but differ in a scaling factor *k* along the *r*-axis. The necessary coefficient F_ν of each Bessel function in the sum, as a function of the scaling factor *k* constitutes the transformed function. The Hankel transform is an integral transform and was first developed by the mathematician Hermann Hankel. It is also known as the Fourier–Bessel transform. Just as the Fourier transform for an infinite interval is related to the Fourier series over a finite interval, so the Hankel transform over an infinite interval is related to the Fourier–Bessel series over a finite interval.

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Definition

The **Hankel transform** of order ν of a function *f*(*r*) is given by:

$$F_\nu(k) = \int_0^\infty f(r) J_\nu(kr)\, r\,\mathrm{d}r$$

where J_ν is the Bessel function of the first kind of order ν with $\nu \geq -\frac{1}{2}$. The inverse Hankel transform of $F_\nu(k)$ is defined as:

$$f(r) = \int_0^\infty F_\nu(k) J_\nu(kr) k\,\mathrm{d}k$$

which can be readily verified using the orthogonality relationship described below.

Domain of definition

Inverting a Hankel transform of a function *f*(*r*) is valid at every point at which *f*(*r*) is continuous provided that the function is defined in (0, ∞), is piecewise continuous and of bounded variation in every finite subinterval in (0, ∞), and

$$\int_0^\infty |f(r)|\, r^{\frac{1}{2}}\, \mathrm{d}r < \infty.$$

However, like the Fourier Transform, the domain can be extended by a density argument to include some functions whose above integral is not finite, for example $f(r) = (1+r)^{-3/2}$.

Alternative definition

An alternative definition says that the Hankel transform of *g*(*r*) is:[1]

$$h_\nu(k) = \int_0^\infty g(r) J_\nu(kr)\, \sqrt{kr}\, \mathrm{d}r$$

The two definitions are related:

$$\text{If } g(r) = f(r)\sqrt{r} \text{ then } h_\nu(k) = F_\nu(k)\sqrt{k}.$$

This means that, as with the previous definition, the Hankel transform defined this way is also its own inverse:

$$g(r) = \int_0^\infty h_\nu(k) J_\nu(kr)\, \sqrt{kr}\, \mathrm{d}k$$

The obvious domain now has the condition

$$\int_0^\infty |g(r)|\, \mathrm{d}r < \infty$$

but this can be extended. According to the reference given above, we can take the integral as the limit as the upper limit goes to infinity (an improper integral rather than a Lebesgue integral) and in this way the Hankel transform and its inverse work for all functions in $L^2(0, \infty)$.

Orthogonality

The Bessel functions form an orthogonal basis with respect to the weighting factor *r*:

$$\int_0^\infty J_\nu(kr) J_\nu(k'r) r\, \mathrm{d}r = \frac{\delta(k-k')}{k}, \qquad k, k' > 0.$$

The Plancherel theorem and Parseval's theorem

If *f*(*r*) and *g*(*r*) are such that their Hankel transforms $F_\nu(k)$ and $G_\nu(k)$ are well defined, then the Plancherel theorem states

$$\int_0^\infty f(r)g(r)r\,\mathrm{d}r = \int_0^\infty F_\nu(k)G_\nu(k)k\,\mathrm{d}k.$$

Parseval's theorem, which states:

$$\int_0^\infty |f(r)|^2 r\,\mathrm{d}r = \int_0^\infty |F_\nu(k)|^2 k\,\mathrm{d}k,$$

is a special case of the Plancherel theorem. These theorems can be proven using the orthogonality property.

Relation to other function transforms

Relation to the Fourier transform (circularly symmetric case)

The Hankel transform of order zero is essentially the 2-dimensional Fourier transform of a circularly symmetric function.

Consider a 2-dimensional function $f(\mathbf{r})$ of the radius vector $\mathbf{r}$. Its Fourier transform is:

$$F(\mathbf{k}) = \iint f(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}\mathbf{r}.$$

With no loss of generality, we can pick a polar coordinate system (r, θ) such that the $\mathbf{k}$ vector lies on the $\theta = 0$ axis(in K-space). The Fourier transform is now written in these polar coordinates as:

$$F(\mathbf{k}) = \int_{r=0}^\infty \int_{\theta=0}^{2\pi} f(r,\theta)e^{ikr\cos(\theta)}\,r\,\mathrm{d}\theta\,\mathrm{d}r$$

where θ is the angle between the $\mathbf{k}$ and $\mathbf{r}$ vectors. If the function f happens to be circularly symmetric, it will have no dependence on the angular variable θ and may be written $f(r)$. The integration over θ may be carried out, and the Fourier transform is now written:

$$F(\mathbf{k}) = F(k) = 2\pi \int_0^\infty f(r)J_0(kr)r\,\mathrm{d}r$$

which is just 2π times the zero-order Hankel transform of $f(r)$. For the reverse transform,

$$f(\mathbf{r}) = \frac{1}{(2\pi)^2}\iint F(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}\mathbf{k} = \frac{1}{2\pi}\int_0^\infty F(k)J_0(kr)k\,\mathrm{d}k$$

so $f(r)$ is $\frac{1}{2\pi}$ times the zero-order Hankel transform of $F(k)$.

Relation to the Fourier transform (radially symmetric case in n-dimensions)

For an n-dimensional Fourier transform,

$$F(\mathbf{k}) = \int f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^n\mathbf{r}$$

if the function f is radially symmetric, then:[2]

$$k^{\frac{n-2}{2}} F(k) = (2\pi)^{n/2} \int_0^\infty r^{\frac{n-2}{2}} f(r) J_{\frac{n-2}{2}}(kr) r \, dr$$

Relation to the Fourier transform (general 2D case)

To generalize: If f can be expanded in a multipole series,

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} f_m(r) e^{im\theta},$$

and if θ_k is the angle between the direction of $\mathbf{k}$ and the $\theta = 0$ axis,

$$\begin{aligned}
F(\mathbf{k}) &= \int_0^\infty r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, f(r,\theta) e^{ikr\cos(\theta-\theta_k)} \\
&= \sum_m \int_0^\infty r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, f_m(r) e^{im\theta} e^{ikr\cos(\theta-\theta_k)} \\
&= \sum_m e^{im\theta_k} \int_0^\infty r\,\mathrm{d}r\, f_m(r) \int_0^{2\pi} \mathrm{d}\varphi\, e^{im\varphi} e^{ikr\cos\varphi} \qquad\qquad \varphi = \theta - \theta_k \\
&= \sum_m e^{im\theta_k} \int_0^\infty r\,\mathrm{d}r\, f_m(r) 2\pi i^m J_m(kr) \\
&= 2\pi \sum_m i^m e^{im\theta_k} \int_0^\infty f_m(r) J_m(kr) r\,\mathrm{d}r. \\
&= 2\pi \sum_m i^m e^{im\theta_k} F_m(k)
\end{aligned}$$

where $F_m(k)$ is the m-th order Hankel transform of $f_m(r)$.

Functions inside a limited radius

Additionally, if f_m is sufficiently smooth near the origin and zero outside a radius R, it may be expanded into a Chebyshev series,

$$f_m(r) = r^m \sum_{t\geq 0} f_{mt} \left(1 - \left(\tfrac{r}{R}\right)^2\right)^t, \qquad 0 \leq r \leq R.$$

so that

$$\begin{aligned}
F(\mathbf{k}) &= 2\pi \sum_m i^m e^{im\theta_k} \sum_t f_{mt} \int_0^R r^m \left(1 - \left(\frac{r}{R}\right)^2\right)^t J_m(kr) r \, \mathrm{d}r && (*) \\
&= 2\pi \sum_m i^m e^{im\theta_k} R^{m+2} \sum_t f_{mt} \int_0^1 x^m (1 - x^2)^t J_m(kxR) x \, \mathrm{d}x && x = \frac{r}{R} \\
&= 2\pi \sum_m i^m e^{im\theta_k} R^{m+2} \sum_t f_{mt} \frac{t! 2^t}{(kR)^{1+t}} J_{m+t+1}(kR).
\end{aligned}$$

The above can be viewed as a more general case that is not as constrained as the previous case in the previous section. The numerically important aspect is that the expansion coefficients f_{mt} are accessible with Discrete Fourier transform techniques. Insertion into the previous formula yields

This is one flavor of fast Hankel transform techniques.

Relation to the Fourier and Abel transforms

The Hankel transform is one member of the FHA cycle of integral operators. In two dimensions, if we define A as the Abel transform operator, F as the Fourier transform operator and H as the zeroth order Hankel transform operator, then the special case of the projection-slice theorem for circularly symmetric functions states that:

$$FA = H.$$

In other words, applying the Abel transform to a 1-dimensional function and then applying the Fourier transform to that result is the same as applying the Hankel transform to that function. This concept can be extended to higher dimensions.

Some Hankel transform pairs

[3]

$f(r)$	$F_0(k)$
1	$\frac{\delta(k)}{k}$
$\frac{1}{r}$	$\frac{1}{k}$
r	$-\frac{1}{k^3}$
r^3	$\frac{9}{k^5}$
r^m	$\dfrac{2^{m+1}\Gamma\left(\frac{m}{2}+1\right)}{k^{m+2}\Gamma\left(-\frac{m}{2}\right)}, \qquad -2 < \Re(m) < -\frac{1}{2}$
$\dfrac{1}{\sqrt{r^2+z^2}}$	$\dfrac{e^{-k\lvert z\rvert}}{k}$
$\dfrac{1}{z^2+r^2}$	$K_0(kz) \qquad z \in \mathbf{C}$
$\frac{e^{iar}}{r}$	$\dfrac{i}{\sqrt{a^2-k^2}} \qquad a > 0, k < a.$
	$\dfrac{1}{\sqrt{k^2-a^2}} \qquad a > 0, k > a.$
$e^{-\frac{1}{2}a^2r^2}$	$\dfrac{1}{a^2}e^{-\frac{k^2}{2a^2}}$
$\dfrac{1}{r}J_0(lr)e^{-sr}$	$\dfrac{2}{\pi\sqrt{(k+l)^2+s^2}}K\left(\sqrt{\dfrac{4kl}{(k+l)^2+s^2}}\right)$
$-r^2f(r)$	$\dfrac{\mathrm{d}^2F_0}{\mathrm{d}k^2}+\dfrac{1}{k}\dfrac{\mathrm{d}F_0}{\mathrm{d}k}$

$f(r)$	$F_\nu(k)$
r^s	$\dfrac{2^{s+1}}{k^{s+2}}\dfrac{\Gamma\left(\frac{1}{2}(2+\nu+s)\right)}{\Gamma(\frac{1}{2}(\nu-s))}$
$r^{\nu-2s}\Gamma\left(s,r^2h\right)$	$\frac{1}{2}\left(\frac{k}{2}\right)^{2s-\nu-2}\gamma\left(1-s+\nu,\frac{k^2}{4h}\right)$
$e^{-r^2}r^\nu U\left(a,b,r^2\right)$	$\dfrac{\Gamma(2+\nu-b)}{2\Gamma(2+\nu-b+a)}\left(\frac{k}{2}\right)^\nu e^{-\frac{k^2}{4}}\,{}_1F_1\left(a,2+a-b+\nu,\frac{k^2}{4}\right)$
$r^nJ_\mu(lr)e^{-sr}$	Expressable in terms of elliptic integrals.[4]
$-r^2f(r)$	$\dfrac{\mathrm{d}^2F_\nu}{\mathrm{d}k^2}+\dfrac{1}{k}\dfrac{\mathrm{d}F_\nu}{\mathrm{d}k}-\dfrac{\nu^2}{k^2}F_\nu$

$K_n(z)$ is a modified Bessel function of the second kind. $K(z)$ is the complete elliptic integral of the first kind.

The expression

$$\frac{d^2 F_0}{dk^2} + \frac{1}{k} \frac{dF_0}{dk}$$

coincides with the expression for the Laplace operator in polar coordinates (k, θ) applied to a spherically symmetric function $F_0(k)$.

The Hankel transform of Zernike polynomials are essentially Bessel Functions (Noll 1976):

$$R_n^m(r) = (-1)^{\frac{n-m}{2}} \int_0^\infty J_{n+1}(k) J_m(kr)\, dk$$

for even $n - m \geq 0$.

Generalized Hankel transform for fan-beam geometry

The image reconstruction in polar coordinates with parallel beam CT projections can be done with Hankel transform. This theory has been generalized to fan-beam geometry. Generalized Bessel function is utilized. [5]

See also

- Fourier transform
- Integral transform
- Abel transform
- Fourier–Bessel series
- Neumann polynomial

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