Logarithmic derivative

In mathematics, specifically in calculus and complex analysis, the **logarithmic derivative** of a function f is defined by the formula

$$\frac{f'}{f}$$

where f' is the derivative of f. Intuitively, this is the infinitesimal relative change in f; that is, the infinitesimal absolute change in f, namely f', scaled by the current value of f.

When f is a function $f(x)$ of a real variable x, and takes real, strictly positive values, this is equal to the derivative of $\ln(f)$, or the natural logarithm of f. This follows directly from the chain rule.

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Basic properties

Many properties of the real logarithm also apply to the logarithmic derivative, even when the function does *not* take values in the positive reals. For example, since the logarithm of a product is the sum of the logarithms of the factors, we have

$$(\log uv)' = (\log u + \log v)' = (\log u)' + (\log v)'.$$

So for positive-real-valued functions, the logarithmic derivative of a product is the sum of the logarithmic derivatives of the factors. But we can also use the Leibniz law for the derivative of a product to get

$$\frac{(uv)'}{uv} = \frac{u'v + uv'}{uv} = \frac{u'}{u} + \frac{v'}{v}.$$

Thus, it is true for *any* function that the logarithmic derivative of a product is the sum of the logarithmic derivatives of the factors (when they are defined).

A corollary to this is that the logarithmic derivative of the reciprocal of a function is the negation of the logarithmic derivative of the function:

$$\frac{(1/u)'}{1/u} = \frac{-u'/u^2}{1/u} = -\frac{u'}{u},$$

just as the logarithm of the reciprocal of a positive real number is the negation of the logarithm of the number.

More generally, the logarithmic derivative of a quotient is the difference of the logarithmic derivatives of the dividend and the divisor:

$$\frac{(u/v)'}{u/v} = \frac{(u'v - uv')/v^2}{u/v} = \frac{u'}{u} - \frac{v'}{v},$$

just as the logarithm of a quotient is the difference of the logarithms of the dividend and the divisor.

Generalising in another direction, the logarithmic derivative of a power (with constant real exponent) is the product of the exponent and the logarithmic derivative of the base:

$$\frac{(u^k)'}{u^k} = \frac{ku^{k-1}u'}{u^k} = k\frac{u'}{u},$$

just as the logarithm of a power is the product of the exponent and the logarithm of the base.

In summary, both derivatives and logarithms have a product rule, a reciprocal rule, a quotient rule, and a power rule (compare the list of logarithmic identities); each pair of rules is related through the logarithmic derivative.

Computing ordinary derivatives using logarithmic derivatives

Logarithmic derivatives can simplify the computation of derivatives requiring the product rule. The procedure is as follows: Suppose that $f(x) = u(x)v(x)$ and that we wish to compute $f'(x)$. Instead of computing it directly, we compute its logarithmic derivative. That is, we compute:

$$\frac{f'}{f} = \frac{u'}{u} + \frac{v'}{v}.$$

Multiplying through by f computes f':

$$f' = f\left(\frac{u'}{u} + \frac{v'}{v}\right).$$

This technique is most useful when f is a product of a large number of factors. This technique makes it possible to compute f' by computing the logarithmic derivative of each factor, summing, and multiplying by f.

Integrating factors

The logarithmic derivative idea is closely connected to the integrating factor method for first-order differential equations. In operator terms, write

$$D = \frac{d}{dx}$$

and let *M* denote the operator of multiplication by some given function $G(x)$. Then

$$M^{-1}DM$$

can be written (by the product rule) as

$$D + M^*$$

where M^* now denotes the multiplication operator by the logarithmic derivative

$$\frac{G'}{G}$$

In practice we are given an operator such as

$$D + F = L$$

and wish to solve equations

$$L(h) = f$$

for the function h, given f. This then reduces to solving

$$\frac{G'}{G} = F$$

which has as solution

$$exp\left(\int F\right)$$

with any indefinite integral of F.

Complex analysis

The formula as given can be applied more widely; for example if $f(z)$ is a meromorphic function, it makes sense at all complex values of z at which f has neither a zero nor a pole. Further, at a zero or a pole the logarithmic derivative behaves in a way that is easily analysed in terms of the particular case

$$z^n$$

with n an integer, $n \neq 0$. The logarithmic derivative is then

$$n/z,$$

and one can draw the general conclusion that for f meromorphic, the singularities of the logarithmic derivative of f are all *simple* poles, with residue n from a zero of order n, residue $-n$ from a pole of order n. See argument principle. This information is often exploited in contour integration.

In the field of Nevanlinna Theory, an important lemma states that the proximity function of a logarithmic derivative is small with respect to the Nevanlinna Characteristic of the original function, for instance $m(r, h'/h) = S(r, h) = o(T(r, h))$.

The multiplicative group

Behind the use of the logarithmic derivative lie two basic facts about GL_1, that is, the multiplicative group of [real numbers](#) or other [field](#). The [differential operator](#)

$$X \frac{d}{dX}$$

is [invariant](#) under 'translation' (replacing X by aX for a constant). And the [differential form](#)

$$dX/X$$

is likewise invariant. For functions F into GL_1, the formula

$$dF/F$$

is therefore a *[pullback](#)* of the invariant form.

Examples

- [Exponential growth](#) and [exponential decay](#) are processes with constant logarithmic derivative.
- In [mathematical finance](#), the [Greek](#) λ is the logarithmic derivative of derivative price with respect to underlying price.
- In [numerical analysis](#), the [condition number](#) is the infinitesimal relative change in the output for a relative change in the input, and is thus a ratio of logarithmic derivatives.

See also

- [Logarithmic differentiation](#)
- [Multiplicative calculus](#)

References