Levi-Civita symbol

In mathematics, particularly in linear algebra, tensor analysis, and differential geometry, the **Levi-Civita symbol** represents a collection of numbers; defined from the sign of a permutation of the natural numbers 1, 2, …, n, for some positive integer n. It is named after the Italian mathematician and physicist Tullio Levi-Civita. Other names include the **permutation symbol**, **antisymmetric symbol**, or **alternating symbol**, which refer to its antisymmetric property and definition in terms of permutations.

The standard letters to denote the Levi-Civita symbol are the Greek lower case epsilon ε or ϵ, or less commonly the Latin lower case e. Index notation allows one to display permutations in a way compatible with tensor analysis:

$$\varepsilon_{i_1 i_2 \ldots i_n}$$

where *each* index i_1, i_2, …, i_n takes values 1, 2, …, n. There are n^n indexed values of $\varepsilon_{i_1 i_2 \ldots i_n}$, which can be arranged into an n-dimensional array. The key defining property of the symbol is *total antisymmetry* in all the indices. When any two indices are interchanged, equal or not, the symbol is negated:

$$\varepsilon_{\ldots i_p \ldots i_q \ldots} = -\varepsilon_{\ldots i_q \ldots i_p \ldots}.$$

If any two indices are equal, the symbol is zero. When all indices are unequal, we have:

$$\varepsilon_{i_1 i_2 \ldots i_n} = (-1)^p \varepsilon_{12 \ldots n},$$

where p (called the parity of the permutation) is the number of pairwise interchanges of indices necessary to unscramble i_1, i_2, …, i_n into the order 1, 2, …, n, and the factor $(-1)^p$ is called the sign or signature of the permutation. The value $\varepsilon_{12 \ldots n}$ must be defined, else the particular values of the symbol for all permutations are indeterminate. Most authors choose $\varepsilon_{12 \ldots n} = +1$, which means the Levi-Civita symbol equals the sign of a permutation when the indices are all unequal. This choice is used throughout this article.

The term "n-dimensional Levi-Civita symbol" refers to the fact that the number of indices on the symbol n matches the dimensionality of the vector space in question, which may be Euclidean or non-Euclidean, e.g. $\mathbb{R}^3$ or Minkowski space. The values of the Levi-Civita symbol are independent of any metric tensor and coordinate system. Also, the specific term "symbol" emphasizes that it is not a tensor because of how it transforms between coordinate systems; however it can be interpreted as a tensor density.

The Levi-Civita symbol allows the determinant of a square matrix, and the cross product of two vectors in three-dimensional Euclidean space, to be expressed in index notation.

Contents

Definition

The Levi-Civita symbol is most often used in three and four dimensions, and to some extent in two dimensions, so these are given here before defining the general case.

Two dimensions

In two dimensions, Levi-Civita symbol is defined by:

$$\varepsilon_{ij} = \begin{cases} +1 & \text{if } (i,j) = (1,2) \\ -1 & \text{if } (i,j) = (2,1) \\ 0 & \text{if } i = j \end{cases}$$

The values can be arranged into a 2 × 2 antisymmetric matrix:

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Use of the two-dimensional symbol is relatively uncommon, although in certain specialized topics like supersymmetry[1] and twistor theory[2] it appears in the context of 2-spinors. The three- and higher-dimensional Levi-Civita symbols are used more commonly.
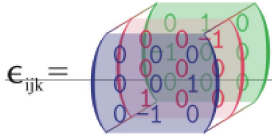
Three dimensions

In three dimensions, the Levi-Civita symbol is defined by:[3]

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),\ (2,3,1),\ \text{or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1),\ (1,3,2),\ \text{or } (2,1,3), \\ 0 & \text{if } i = j,\ \text{or } j = k,\ \text{or } k = i \end{cases}$$

That is, ε_{ijk} is 1 if (i, j, k) is an even permutation of (1, 2, 3), −1 if it is an odd permutation, and 0 if any index is repeated. In three dimensions only, the cyclic permutations of (1, 2, 3) are all even permutations, similarly the anticyclic permutations are all odd permutations. This means in 3d it is sufficient to take cyclic or anticyclic permutations of (1, 2, 3) and easily obtain all the even or odd permutations.

Analogous to 2-dimensional matrices, the values of the 3-dimensional Levi-Civita symbol can be arranged into a 3 × 3 × 3 array:

where i is the depth (blue: $i = 1$; red: $i = 2$; green: $i = 3$), j is the row and k is the column.

Some examples:

$$\varepsilon_{132} = -\varepsilon_{123} = -1$$
$$\varepsilon_{312} = -\varepsilon_{213} = -(-\varepsilon_{123}) = 1$$
$$\varepsilon_{231} = -\varepsilon_{132} = -(-\varepsilon_{123}) = 1$$
$$\varepsilon_{232} = -\varepsilon_{232} = 0$$

For the indices (i, j, k) in ε_{ijk}, the values 1, 2, 3 occurring in the ▯ cyclic order (1, 2, 3) correspond to $\varepsilon = +1$, while occurring in the ▮ reverse cyclic order correspond to $\varepsilon = -1$, otherwise $\varepsilon = 0$.

Four dimensions

In four dimensions, the Levi-Civita symbol is defined by:

$$\varepsilon_{ijkl} = \begin{cases} +1 & \text{if } (i,j,k,l) \text{ is an even permutation of } (1,2,3,4) \\ -1 & \text{if } (i,j,k,l) \text{ is an odd permutation of } (1,2,3,4) \\ 0 & \text{otherwise} \end{cases}$$

These values can be arranged into a 4 × 4 × 4 × 4 array, although in 4 dimensions and higher this is difficult to draw.

Some examples:

$$\varepsilon_{1432} = -\varepsilon_{1234} = -1$$
$$\varepsilon_{2134} = -\varepsilon_{1234} = -1$$
$$\varepsilon_{4321} = -\varepsilon_{1324} = -(-\varepsilon_{1234}) = 1$$
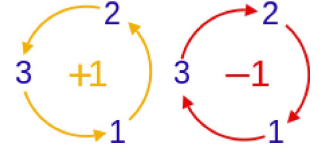$$\varepsilon_{3243} = -\varepsilon_{3243} = 0$$

Generalization to n dimensions

More generally, in n dimensions, the Levi-Civita symbol is defined by:[4]

$$\varepsilon_{a_1 a_2 a_3 \ldots a_n} = \begin{cases} +1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an even permutation of } (1,2,3,\ldots,n) \\ -1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an odd permutation of } (1,2,3,\ldots,n) \\ 0 & \text{otherwise} \end{cases}$$

Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise.

Using the capital pi notation $\prod$ for ordinary multiplication of numbers, an explicit expression for the symbol is:

$$\varepsilon_{a_1 a_2 a_3 \ldots a_n} = \prod_{1 \le i < j \le n} \operatorname{sgn}(a_j - a_i)$$
$$= \operatorname{sgn}(a_2 - a_1) \operatorname{sgn}(a_3 - a_1) \ldots \operatorname{sgn}(a_n - a_1) \operatorname{sgn}(a_3 - a_2) \operatorname{sgn}(a_4 - a_2) \ldots \operatorname{sgn}(a_n - a_2) \ldots \operatorname{sgn}(a_n - a_{n-1})$$

where the signum function (denoted sgn) returns the sign of its argument while discarding the absolute value if nonzero. The formula is valid for all index values, and for any n (when $n = 0$ or $n = 1$, this is the empty product). However, computing the formula above naively has a time complexity of $O(n^2)$, whereas the sign can be computed from the parity of the permutation from its disjoint cycles in only $O(n \log(n))$ cost.

Properties

A tensor whose components in an orthonormal basis are given by the Levi-Civita symbol (a tensor of covariant rank n) is sometimes called a **permutation tensor**.

Under the ordinary transformation rules for tensors the Levi-Civita symbol is unchanged under pure rotations, consistent with that it is (by definition) the same in all coordinate systems related by orthogonal transformations. However, the Levi-Civita symbol is a pseudotensor because under an orthogonal transformation of Jacobian determinant −1, e.g. a reflection in an odd number of dimensions, it *should* acquire a minus sign if it were a tensor. As it does not change at all, the Levi-Civita symbol is, by definition, a pseudotensor.

As the Levi-Civita symbol is a pseudotensor, the result of taking a cross product is a pseudovector, not a vector.[5]

Under a general coordinate change, the components of the permutation tensor are multiplied by the Jacobian of the transformation matrix. This implies that in coordinate frames different from the one in which the tensor was defined, its components can differ from those of the Levi-Civita symbol by an overall factor. If the frame is orthonormal, the factor will be ±1 depending on whether the orientation of the frame is the same or not.[5]

In index-free tensor notation, the Levi-Civita symbol is replaced by the concept of the Hodge dual.

In a context where tensor index notation is used to manipulate tensor components, the Levi-Civita symbol may be written with its indices as either subscripts or superscripts with no change in meaning, as might be convenient. Thus, one could write

$$\varepsilon^{ij\ldots k} = \varepsilon_{ij\ldots k}.$$

In these examples, superscripts should be considered equivalent with subscripts.

Summation symbols can be eliminated by using Einstein notation, where an index repeated between two or more terms indicates summation over that index. For example,

$$\varepsilon_{ijk}\varepsilon^{imn} \equiv \sum_{i=1,2,3} \varepsilon_{ijk}\varepsilon^{imn}.$$

In the following examples, Einstein notation is used.

Two dimensions

In two dimensions, when all i, j, m, n each take the values 1 and 2,[3]

$$\varepsilon_{ij}\varepsilon^{mn} = \delta_i{}^m \delta_j{}^n - \delta_i{}^n \delta_j{}^m \tag{1}$$

$$\varepsilon_{ij}\varepsilon^{in} = \delta_j{}^n \tag{2}$$

$$\varepsilon_{ij}\varepsilon^{ij} = 2. \tag{3}$$

Three dimensions

Index and symbol values

In three dimensions, when all i, j, k, m, n each take values 1, 2, and 3:[3]

$$\varepsilon_{ijk}\varepsilon^{imn} = \delta_j{}^m \delta_k{}^n - \delta_j{}^n \delta_k{}^m \tag{4}$$

$$\varepsilon_{jmn}\varepsilon^{imn} = 2\delta_j{}^i \tag{5}$$

$$\varepsilon_{ijk}\varepsilon^{ijk} = 6. \tag{6}$$

Product

The Levi-Civita symbol is related to the Kronecker delta. In three dimensions, the relationship is given by the following equations (vertical lines denote the determinant):[4]

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$
$$= \delta_{il}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right) + \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right).$$

A special case of this result is (**4**):

$$\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

sometimes called the "contracted epsilon identity".

In Einstein notation, the duplication of the i index implies the sum on i. The previous is then denoted $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}$$

n dimensions

Index and symbol values

In n dimensions, when all $i_1, \ldots, i_n, j_1, \ldots, j_n$ take values 1, 2, ..., n:

$$\varepsilon_{i_1\ldots i_n}\varepsilon^{j_1\cdots j_n} = n!\delta_{[i_1}^{j_1}\ldots\delta_{i_n]}^{j_n} = \delta_{i_1\ldots i_n}^{j_1\cdots j_n} \qquad\qquad (7)$$

$$\varepsilon_{i_1\ldots i_k\, i_{k+1}\ldots i_n}\varepsilon^{i_1\cdots i_k\, j_{k+1}\cdots j_n} = k!(n-k)!\,\delta_{[i_{k+1}}^{j_{k+1}}\ldots\delta_{i_n]}^{j_n} = k!\,\delta_{i_{k+1}\ldots i_n}^{j_{k+1}\ldots j_n} \qquad\qquad (8)$$

$$\varepsilon_{i_1\ldots i_n}\varepsilon^{i_1\ldots i_n} = n! \qquad\qquad (9)$$

where the exclamation mark (!) denotes the factorial, and $\delta_{\beta\ldots}^{\alpha\ldots}$ is the generalized Kronecker delta. For any n, the property

$$\sum_{i,j,k,\cdots=1}^{n} \varepsilon_{ijk\ldots}\varepsilon_{ijk\ldots} = n!$$

follows from the facts that

- every permutation is either even or odd,
- $(+1)^2 = (-1)^2 = 1$, and
- the number of permutations of any n-element set number is exactly $n!$.

Product

In general, for n dimensions, one can write the product of two Levi-Civita symbols as:

$$\varepsilon_{i_1 i_2\ldots i_n}\varepsilon_{j_1 j_2\ldots j_n} = \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \cdots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \cdots & \delta_{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \cdots & \delta_{i_n j_n} \end{vmatrix}.$$

Proofs

For (**1**), both sides are antisymmetric with respect of ij and mn. We therefore only need to consider the case $i \neq j$ and $m \neq n$. By substitution, we see that the equation holds for $\varepsilon_{12}\varepsilon^{12}$, i.e., for $i = m = 1$ and $j = n = 2$. (Both sides are then one). Since the equation is antisymmetric in ij and mn, any set of values for these can be reduced to the above case (which holds). The equation thus holds for all values of ij and mn.

Using (**1**), we have for (**2**)

$$\varepsilon_{ij}\varepsilon^{in} = \delta_i{}^i\delta_j{}^n - \delta_i{}^n\delta_j{}^i = 2\delta_j{}^n - \delta_j{}^n = \delta_j{}^n\,.$$

Here we used the Einstein summation convention with i going from 1 to 2. Next, (**3**) follows similarly from (**2**).

To establish (**5**), notice that both sides vanish when $i \neq j$. Indeed, if $i \neq j$, then one can not choose m and n such that both permutation symbols on the left are nonzero. Then, with $i = j$ fixed, there are only two ways to choose m and n from the remaining two indices. For any such indices, we have

$$\varepsilon_{jmn}\varepsilon^{imn} = \left(\varepsilon^{imn}\right)^2 = 1$$

(no summation), and the result follows.

Then (**6**) follows since 3! = 6 and for any distinct indices i, j, k taking values 1, 2, 3, we have

$$\varepsilon_{ijk}\varepsilon^{ijk} = 1 \qquad \text{(no summation, distinct } i, j, k)$$

Applications and examples

Determinants

In linear algebra, the determinant of a 3 × 3 square matrix $\mathbf{A} = [a_{ij}]$ can be written[6]

$$\det(\mathbf{A}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

Similarly the determinant of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ can be written as[5]

$$\det(\mathbf{A}) = \varepsilon_{i_1 \ldots i_n} a_{1i_1} \cdots a_{ni_n},$$

where each i_r should be summed over $1, \ldots, n$, or equivalently:

$$\det(\mathbf{A}) = \frac{1}{n!} \varepsilon_{i_1 \ldots i_n} \varepsilon_{j_1 \ldots j_n} a_{i_1 j_1} \cdots a_{i_n j_n},$$

where now each i_r and each j_r should be summed over $1, \ldots, n$. More generally, we have the identity[5]

$$\sum_{i_1, i_2, \ldots} \varepsilon_{i_1 \ldots i_n} a_{i_1 j_1} \cdots a_{i_n j_n} = \det(\mathbf{A}) \varepsilon_{j_1 \ldots j_n}$$

Vector cross product

Cross product (two vectors)

If $\mathbf{a} = (a^1, a^2, a^3)$ and $\mathbf{b} = (b^1, b^2, b^3)$ are vectors in $\mathbb{R}^3$ (represented in some right-handed coordinate system using an orthonormal basis), their cross product can be written as a determinant:[5]

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} \mathbf{e}_i a^j b^k$$

hence also using the Levi-Civita symbol, and more simply:

$$(\mathbf{a} \times \mathbf{b})^i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a^j b^k.$$

In Einstein notation, the summation symbols may be omitted, and the ith component of their cross product equals[4]

$$(\mathbf{a} \times \mathbf{b})^i = \varepsilon_{ijk} a^j b^k.$$

The first component is

$$(\mathbf{a} \times \mathbf{b})^1 = a^2 b^3 - a^3 b^2,$$

then by cyclic permutations of 1, 2, 3 the others can be derived immediately, without explicitly calculating them from the above formulae:

$$(\mathbf{a} \times \mathbf{b})^2 = a^3 b^1 - a^1 b^3,$$
$$(\mathbf{a} \times \mathbf{b})^3 = a^1 b^2 - a^2 b^1.$$

Triple scalar product (three vectors)

From the above expression for the cross product, we have:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$

If $\mathbf{c} = (c^1, c^2, c^3)$ is a third vector, then the triple scalar product equals

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \varepsilon_{ijk} a^i b^j c^k.$$

From this expression, it can be seen that the triple scalar product is antisymmetric when exchanging any pair of arguments. For example,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}).$$

Curl (one vector field)

If $\mathbf{F} = (F^1, F^2, F^3)$ is a vector field defined on some open set of $\mathbb{R}^3$ as a function of position $\mathbf{x} = (x^1, x^2, x^3)$ (using Cartesian coordinates). Then the ith component of the curl of $\mathbf{F}$ equals[4]

$$(\nabla \times \mathbf{F})^i(\mathbf{x}) = \varepsilon^{ijk} \frac{\partial}{\partial x^j} F_k(\mathbf{x}),$$

which follows from the cross product expression above, substituting components of the gradient vector operator (nabla).

Tensor density

In any arbitrary [curvilinear coordinate system](#) and even in the absence of a [metric](#) on the [manifold](#), the Levi-Civita symbol as defined above may be considered to be a [tensor density](#) field in two different ways. It may be regarded as a [contravariant](#) tensor density of weight +1 or as a [covariant](#) tensor density of weight −1. In n dimensions using the [generalized Kronecker delta](#),[7][8]

$$\varepsilon^{\mu_1\cdots\mu_n} = \delta^{\mu_1\cdots\mu_n}_{1\ldots n}$$
$$\varepsilon_{\nu_1\ldots\nu_n} = \delta^{1\ldots n}_{\nu_1\ldots\nu_n}.$$

Notice that these are numerically identical. In particular, the sign is the same.

Levi-Civita tensors

On a [pseudo-Riemannian manifold](#), one may define a coordinate-invariant covariant tensor field whose coordinate representation agrees with the Levi-Civita symbol wherever the coordinate system is such that the basis of the tangent space is orthonormal with respect to the metric and matches a selected orientation. This tensor should not be confused with the tensor density field mentioned above. The presentation in this section closely follows [Carroll 2004](#).

The covariant Levi-Civita tensor (also known as the [Riemannian volume form](#)) in any coordinate system that matches the selected orientation is

$$E_{a_1\ldots a_n} = \sqrt{|\det[g_{ab}]|}\,\varepsilon_{a_1\ldots a_n}\,,$$

where g_{ab} is the representation of the metric in that coordinate system. We can similarly consider a contravariant Levi-Civita tensor by raising the indices with the metric as usual,

$$E^{a_1\ldots a_n} = E_{b_1\ldots b_n}\prod_{i=1}^{n} g^{a_i b_i}\,,$$

but notice that if the [metric signature](#) contains an odd number of negatives q, then the sign of the components of this tensor differ from the standard Levi-Civita symbol:

$$E^{a_1\ldots a_n} = \frac{\operatorname{sgn}(\det[g_{ab}])}{\sqrt{|\det[g_{ab}]|}}\,\varepsilon^{a_1\ldots a_n}\,,$$

where $\operatorname{sgn}(\det[g_{ab}]) = (-1)^q$, and $\varepsilon^{a_1\ldots a_n}$ is the usual Levi-Civita symbol discussed in the rest of this article. More explicitly, when the tensor and basis orientation are chosen such that $E_{01\ldots n} = +\sqrt{|\det[g_{ab}]|}$, we have that $E^{01\ldots n} = \frac{\operatorname{sgn}(|\det[g_{ab}]|)}{\sqrt{|\det[g_{ab}]|}}$.

Example: Minkowski Space

In [Minkowski space](#) (the four-dimensional [spacetime](#) of [special relativity](#)), the covariant Levi-Civita tensor is

$$E_{\alpha\beta\gamma\delta} = \pm\sqrt{|\det[g_{\mu\nu}]|}\,\varepsilon_{\alpha\beta\gamma\delta}\,,$$

where the sign depends on the orientation of the basis. The contravariant Levi-Civita tensor is

$$E^{\alpha\beta\gamma\delta} = g^{\alpha\zeta}g^{\beta\eta}g^{\gamma\theta}g^{\delta\iota}E_{\zeta\eta\theta\iota}\,.$$

From this we can infer the identity, valid in Minkowski space (where $n = 4$) for both sign conventions:

$$E^{\mu_1\ldots\mu_p\alpha_1\ldots\alpha_{n-p}}E_{\mu_1\ldots\mu_p\beta_1\ldots\beta_{n-p}} = -p!\,\delta^{\alpha_1\ldots\alpha_{n-p}}_{\beta_1\ldots\beta_{n-p}}\,,$$

where $\delta^{\alpha_1\ldots\alpha_{n-p}}_{\beta_1\ldots\beta_{n-p}}$ is the [generalized Kronecker delta](#). The negative sign arises from the odd number of negatives in the signature of the metric tensor. The following are examples of this identity:

$$E_{\alpha\beta\gamma\delta}E_{\rho\sigma\mu\nu} = g_{\alpha\zeta}g_{\beta\eta}g_{\gamma\theta}g_{\delta\iota}\delta^{\zeta\eta\theta\iota}_{\rho\sigma\mu\nu}$$
$$E^{\alpha\beta\gamma\delta}E^{\rho\sigma\mu\nu} = g^{\alpha\zeta}g^{\beta\eta}g^{\gamma\theta}g^{\delta\iota}\delta^{\rho\sigma\mu\nu}_{\zeta\eta\theta\iota}$$
$$E^{\alpha\beta\gamma\delta}E_{\alpha\beta\gamma\delta} = -24$$
$$E^{\alpha\beta\gamma\delta}E_{\rho\beta\gamma\delta} = -6\delta^{\alpha}_{\rho}$$
$$E^{\alpha\beta\gamma\delta}E_{\rho\sigma\gamma\delta} = -2\delta^{\alpha\beta}_{\rho\sigma}$$
$$E^{\alpha\beta\gamma\delta}E_{\rho\sigma\theta\delta} = -\delta^{\alpha\beta\gamma}_{\rho\sigma\theta}\,.$$

See also

- [Antisymmetric tensor](#)
- [Kronecker delta](#)
- [List of permutation topics](#)
- [Symmetric tensor](#)

Notes

1. Labelle, P. (2010). *Supersymmetry*. Demystified. McGraw-Hill. pp. 57–58. [ISBN](#) [978-0-07-163641-4](#).

2. Hadrovich, F. "Twistor Primer" (http://users.ox.ac.uk/~tweb/00004/index.shtml). Retrieved 2013-09-03.

3. Tyldesley, J. R. (1973). *An introduction to Tensor Analysis: For Engineers and Applied Scientists*. Longman. ISBN 0-582-44355-5.

4. Kay, D. C. (1988). *Tensor Calculus*. Schaum's Outlines. McGraw Hill. ISBN 0-07-033484-6.

5. Riley, K. F.; Hobson, M. P.; Bence, S. J. (2010). *Mathematical Methods for Physics and Engineering*. Cambridge University Press. ISBN 978-0-521-86153-3.

6. Lipcshutz, S.; Lipson, M. (2009). *Linear Algebra*. Schaum's Outlines (4th ed.). McGraw Hill. ISBN 978-0-07-154352-1.

7. Murnaghan, F. D. (1925), "The generalized Kronecker symbol and its application to the theory of determinants", *Amer. Math. Monthly*, **32**: 233–241, doi:10.2307/2299191 (https://doi.org/10.2307%2F2299191)

8. Lovelock, David; Rund, Hanno (1989). *Tensors, Differential Forms, and Variational Principles*. Courier Dover Publications. p. 113. ISBN 0-486-65840-6.

References

- Wheeler, J. A.; Misner, C.; Thorne, K. S. (1973). *Gravitation*. W. H. Freeman & Co. pp. 85–86, §3.5. ISBN 0-7167-0344-0.
- Neuenschwander, D. E. (2015). *Tensor Calculus for Physics*. Johns Hopkins University Press. pp. 11, 29, 95. ISBN 978-1-4214-1565-9.
- Carroll, Sean M. (2004), *Spacetime and Geometry* (http://spacetimeandgeometry.net), Addison-Wesley, ISBN 0-8053-8732-3

External links

This article incorporates material from *Levi-Civita permutation symbol* (http://planetmath.org/sites/default/files/texpdf/34116.pdf) on *PlanetMath*, which is licensed under the *Creative Commons Attribution/Share-Alike License*.

- Weisstein, Eric W. "Permutation Tensor" (http://mathworld.wolfram.com/PermutationTensor.html). *MathWorld*.