Table of spherical harmonics

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This is a **table of orthonormalized spherical harmonics** that employ the Condon-Shortley phase up to degree $l = 10$. Some of these formulas give the "Cartesian" version. This assumes x, y, z, and r are related to θ and φ through the usual spherical-to-Cartesian coordinate transformation:

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}$$

Contents

- 1 Spherical harmonics
  - 1.1 l = 0[1]
  - 1.2 l = 1[1]
  - 1.3 l = 2[1]
  - 1.4 l = 3[1]
  - 1.5 l = 4[1]
  - 1.6 l = 5[1]
  - 1.7 l = 6
  - 1.8 l = 7
  - 1.9 l = 8
  - 1.10 l = 9
  - 1.11 l = 10
- 2 Real spherical harmonics
  - 2.1 l = 0[2][3]
  - 2.2 l = 1[2][3]
  - 2.3 l = 2[2][3]
  - 2.4 l = 3[2]
  - 2.5 l = 4
- 3 See also
- 4 External links
- 5 References

Spherical harmonics

l = 0[1]

$$Y_0^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

l = 1[1]

$$\begin{aligned} Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta &&= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x-iy)}{r} \\ Y_1^{0}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\cos\theta &&= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\frac{z}{r} \\ Y_1^{1}(\theta,\varphi) &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta &&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{(x+iy)}{r} \end{aligned}$$

l = 2[1]

$$Y_2^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x-iy)^2}{r^2}$$

$$Y_2^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \cdot \cos\theta = \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x-iy)z}{r^2}$$

$$Y_2^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot (3\cos^2\theta - 1) = \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot \frac{(2z^2-x^2-y^2)}{r^2}$$

$$Y_2^{1}(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot \cos\theta = -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x+iy)z}{r^2}$$

$$Y_2^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x+iy)^2}{r^2}$$

l = 3[1]

$$Y_3^{-3}(\theta,\varphi) = \frac{1}{8}\sqrt{\frac{35}{\pi}} \cdot e^{-3i\varphi} \cdot \sin^3\theta = \frac{1}{8}\sqrt{\frac{35}{\pi}} \cdot \frac{(x-iy)^3}{r^3}$$

$$Y_3^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta \cdot \cos\theta = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \cdot \frac{(x-iy)^2 z}{r^3}$$

$$Y_3^{-1}(\theta,\varphi) = \frac{1}{8}\sqrt{\frac{21}{\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \cdot (5\cos^2\theta - 1) = \frac{1}{8}\sqrt{\frac{21}{\pi}} \cdot \frac{(x-iy)(4z^2-x^2-y^2)}{r^3}$$

$$Y_3^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{7}{\pi}} \cdot (5\cos^3\theta - 3\cos\theta) = \frac{1}{4}\sqrt{\frac{7}{\pi}} \cdot \frac{z(2z^2-3x^2-3y^2)}{r^3}$$

$$Y_3^{1}(\theta,\varphi) = -\frac{1}{8}\sqrt{\frac{21}{\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot (5\cos^2\theta - 1) = \frac{-1}{8}\sqrt{\frac{21}{\pi}} \cdot \frac{(x+iy)(4z^2-x^2-y^2)}{r^3}$$

$$Y_3^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta \cdot \cos\theta = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \cdot \frac{(x+iy)^2 z}{r^3}$$

$$Y_3^{3}(\theta,\varphi) = -\frac{1}{8}\sqrt{\frac{35}{\pi}} \cdot e^{3i\varphi} \cdot \sin^3\theta = \frac{-1}{8}\sqrt{\frac{35}{\pi}} \cdot \frac{(x+iy)^3}{r^3}$$

l = 4[1]

$$Y_4^{-4}(\theta,\varphi) = \frac{3}{16}\sqrt{\frac{35}{2\pi}} \cdot e^{-4i\varphi} \cdot \sin^4\theta = \frac{3}{16}\sqrt{\frac{35}{2\pi}} \cdot \frac{(x-iy)^4}{r^4}$$

$$Y_4^{-3}(\theta,\varphi) = \frac{3}{8}\sqrt{\frac{35}{\pi}} \cdot e^{-3i\varphi} \cdot \sin^3\theta \cdot \cos\theta = \frac{3}{8}\sqrt{\frac{35}{\pi}} \cdot \frac{(x-iy)^3 z}{r^4}$$

$$Y_4^{-2}(\theta,\varphi) = \frac{3}{8}\sqrt{\frac{5}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta \cdot (7\cos^2\theta - 1) = \frac{3}{8}\sqrt{\frac{5}{2\pi}} \cdot \frac{(x-iy)^2 \cdot (7z^2-r^2)}{r^4}$$

$$Y_4^{-1}(\theta,\varphi) = \frac{3}{8}\sqrt{\frac{5}{\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \cdot (7\cos^3\theta - 3\cos\theta) = \frac{3}{8}\sqrt{\frac{5}{\pi}} \cdot \frac{(x-iy) \cdot z \cdot (7z^2-3r^2)}{r^4}$$

$$Y_4^{0}(\theta,\varphi) = \frac{3}{16}\sqrt{\frac{1}{\pi}} \cdot (35\cos^4\theta - 30\cos^2\theta + 3) = \frac{3}{16}\sqrt{\frac{1}{\pi}} \cdot \frac{(35z^4-30z^2r^2+3r^4)}{r^4}$$

$$Y_4^{1}(\theta,\varphi) = \frac{-3}{8}\sqrt{\frac{5}{\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot (7\cos^3\theta - 3\cos\theta) = \frac{-3}{8}\sqrt{\frac{5}{\pi}} \cdot \frac{(x+iy) \cdot z \cdot (7z^2-3r^2)}{r^4}$$

$$Y_4^{2}(\theta,\varphi) = \frac{3}{8}\sqrt{\frac{5}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta \cdot (7\cos^2\theta - 1) = \frac{3}{8}\sqrt{\frac{5}{2\pi}} \cdot \frac{(x+iy)^2 \cdot (7z^2-r^2)}{r^4}$$

$$Y_4^{3}(\theta,\varphi) = \frac{-3}{8}\sqrt{\frac{35}{\pi}} \cdot e^{3i\varphi} \cdot \sin^3\theta \cdot \cos\theta = \frac{-3}{8}\sqrt{\frac{35}{\pi}} \cdot \frac{(x+iy)^3 z}{r^4}$$

$$Y_4^{4}(\theta,\varphi) = \frac{3}{16}\sqrt{\frac{35}{2\pi}} \cdot e^{4i\varphi} \cdot \sin^4\theta = \frac{3}{16}\sqrt{\frac{35}{2\pi}} \cdot \frac{(x+iy)^4}{r^4}$$

l = 5[1]

$$Y_5^{-5}(\theta,\varphi) = \frac{3}{32}\sqrt{\frac{77}{\pi}} \cdot e^{-5i\varphi} \cdot \sin^5\theta$$

$$Y_5^{-4}(\theta,\varphi) = \frac{3}{16}\sqrt{\frac{385}{2\pi}} \cdot e^{-4i\varphi} \cdot \sin^4\theta \cdot \cos\theta$$

$$Y_5^{-3}(\theta,\varphi) = \frac{1}{32}\sqrt{\frac{385}{\pi}} \cdot e^{-3i\varphi} \cdot \sin^3\theta \cdot (9\cos^2\theta - 1)$$

$$Y_5^{-2}(\theta,\varphi) = \frac{1}{8}\sqrt{\frac{1155}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta \cdot (3\cos^3\theta - \cos\theta)$$

$$Y_5^{-1}(\theta,\varphi) = \frac{1}{16}\sqrt{\frac{165}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \cdot (21\cos^4\theta - 14\cos^2\theta + 1)$$

$$Y_5^{0}(\theta,\varphi) = \frac{1}{16}\sqrt{\frac{11}{\pi}} \cdot (63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)$$

$$Y_5^{1}(\theta,\varphi) = \frac{-1}{16}\sqrt{\frac{165}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot (21\cos^4\theta - 14\cos^2\theta + 1)$$

$$Y_5^{2}(\theta,\varphi) = \frac{1}{8}\sqrt{\frac{1155}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta \cdot (3\cos^3\theta - \cos\theta)$$

$$Y_5^{3}(\theta,\varphi) = \frac{-1}{32}\sqrt{\frac{385}{\pi}} \cdot e^{3i\varphi} \cdot \sin^3\theta \cdot (9\cos^2\theta - 1)$$

$$Y_5^{4}(\theta,\varphi) = \frac{3}{16}\sqrt{\frac{385}{2\pi}} \cdot e^{4i\varphi} \cdot \sin^4\theta \cdot \cos\theta$$

$$Y_5^{5}(\theta,\varphi) = \frac{-3}{32}\sqrt{\frac{77}{\pi}} \cdot e^{5i\varphi} \cdot \sin^5\theta$$

l = 6

$$Y_6^{-6}(\theta,\varphi)=\frac{1}{64}\sqrt{\frac{3003}{\pi}}\cdot e^{-6i\varphi}\cdot\sin^6\theta$$
$$Y_6^{-5}(\theta,\varphi)=\frac{3}{32}\sqrt{\frac{1001}{\pi}}\cdot e^{-5i\varphi}\cdot\sin^5\theta\cdot\cos\theta$$
$$Y_6^{-4}(\theta,\varphi)=\frac{3}{32}\sqrt{\frac{91}{2\pi}}\cdot e^{-4i\varphi}\cdot\sin^4\theta\cdot(11\cos^2\theta-1)$$
$$Y_6^{-3}(\theta,\varphi)=\frac{1}{32}\sqrt{\frac{1365}{\pi}}\cdot e^{-3i\varphi}\cdot\sin^3\theta\cdot(11\cos^3\theta-3\cos\theta)$$
$$Y_6^{-2}(\theta,\varphi)=\frac{1}{64}\sqrt{\frac{1365}{\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta\cdot(33\cos^4\theta-18\cos^2\theta+1)$$
$$Y_6^{-1}(\theta,\varphi)=\frac{1}{16}\sqrt{\frac{273}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot(33\cos^5\theta-30\cos^3\theta+5\cos\theta)$$
$$Y_6^{0}(\theta,\varphi)=\frac{1}{32}\sqrt{\frac{13}{\pi}}\cdot(231\cos^6\theta-315\cos^4\theta+105\cos^2\theta-5)$$
$$Y_6^{1}(\theta,\varphi)=-\frac{1}{16}\sqrt{\frac{273}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot(33\cos^5\theta-30\cos^3\theta+5\cos\theta)$$
$$Y_6^{2}(\theta,\varphi)=\frac{1}{64}\sqrt{\frac{1365}{\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta\cdot(33\cos^4\theta-18\cos^2\theta+1)$$
$$Y_6^{3}(\theta,\varphi)=-\frac{1}{32}\sqrt{\frac{1365}{\pi}}\cdot e^{3i\varphi}\cdot\sin^3\theta\cdot(11\cos^3\theta-3\cos\theta)$$
$$Y_6^{4}(\theta,\varphi)=\frac{3}{32}\sqrt{\frac{91}{2\pi}}\cdot e^{4i\varphi}\cdot\sin^4\theta\cdot(11\cos^2\theta-1)$$
$$Y_6^{5}(\theta,\varphi)=-\frac{3}{32}\sqrt{\frac{1001}{\pi}}\cdot e^{5i\varphi}\cdot\sin^5\theta\cdot\cos\theta$$
$$Y_6^{6}(\theta,\varphi)=\frac{1}{64}\sqrt{\frac{3003}{\pi}}\cdot e^{6i\varphi}\cdot\sin^6\theta$$

l = 7

$$Y_7^{-7}(\theta,\varphi)=\frac{3}{64}\sqrt{\frac{715}{2\pi}}\cdot e^{-7i\varphi}\cdot\sin^7\theta$$

$$Y_7^{-6}(\theta,\varphi)=\frac{3}{64}\sqrt{\frac{5005}{\pi}}\cdot e^{-6i\varphi}\cdot\sin^6\theta\cdot\cos\theta$$

$$Y_7^{-5}(\theta,\varphi)=\frac{3}{64}\sqrt{\frac{385}{2\pi}}\cdot e^{-5i\varphi}\cdot\sin^5\theta\cdot(13\cos^2\theta-1)$$

$$Y_7^{-4}(\theta,\varphi)=\frac{3}{32}\sqrt{\frac{385}{2\pi}}\cdot e^{-4i\varphi}\cdot\sin^4\theta\cdot(13\cos^3\theta-3\cos\theta)$$

$$Y_7^{-3}(\theta,\varphi)=\frac{3}{64}\sqrt{\frac{35}{2\pi}}\cdot e^{-3i\varphi}\cdot\sin^3\theta\cdot(143\cos^4\theta-66\cos^2\theta+3)$$

$$Y_7^{-2}(\theta,\varphi)=\frac{3}{64}\sqrt{\frac{35}{\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta\cdot(143\cos^5\theta-110\cos^3\theta+15\cos\theta)$$

$$Y_7^{-1}(\theta,\varphi)=\frac{1}{64}\sqrt{\frac{105}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot(429\cos^6\theta-495\cos^4\theta+135\cos^2\theta-5)$$

$$Y_7^{0}(\theta,\varphi)=\frac{1}{32}\sqrt{\frac{15}{\pi}}\cdot(429\cos^7\theta-693\cos^5\theta+315\cos^3\theta-35\cos\theta)$$

$$Y_7^{1}(\theta,\varphi)=-\frac{1}{64}\sqrt{\frac{105}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot(429\cos^6\theta-495\cos^4\theta+135\cos^2\theta-5)$$

$$Y_7^{2}(\theta,\varphi)=\frac{3}{64}\sqrt{\frac{35}{\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta\cdot(143\cos^5\theta-110\cos^3\theta+15\cos\theta)$$

$$Y_7^{3}(\theta,\varphi)=-\frac{3}{64}\sqrt{\frac{35}{2\pi}}\cdot e^{3i\varphi}\cdot\sin^3\theta\cdot(143\cos^4\theta-66\cos^2\theta+3)$$

$$Y_7^{4}(\theta,\varphi)=\frac{3}{32}\sqrt{\frac{385}{2\pi}}\cdot e^{4i\varphi}\cdot\sin^4\theta\cdot(13\cos^3\theta-3\cos\theta)$$

$$Y_7^{5}(\theta,\varphi)=-\frac{3}{64}\sqrt{\frac{385}{2\pi}}\cdot e^{5i\varphi}\cdot\sin^5\theta\cdot(13\cos^2\theta-1)$$

$$Y_7^{6}(\theta,\varphi)=\frac{3}{64}\sqrt{\frac{5005}{\pi}}\cdot e^{6i\varphi}\cdot\sin^6\theta\cdot\cos\theta$$

$$Y_7^{7}(\theta,\varphi)=-\frac{3}{64}\sqrt{\frac{715}{2\pi}}\cdot e^{7i\varphi}\cdot\sin^7\theta$$

l = 8

$$Y_8^{-8}(\theta,\varphi) = \frac{3}{256}\sqrt{\frac{12155}{2\pi}} \cdot e^{-8i\varphi} \cdot \sin^8\theta$$
$$Y_8^{-7}(\theta,\varphi) = \frac{3}{64}\sqrt{\frac{12155}{2\pi}} \cdot e^{-7i\varphi} \cdot \sin^7\theta \cdot \cos\theta$$
$$Y_8^{-6}(\theta,\varphi) = \frac{1}{128}\sqrt{\frac{7293}{\pi}} \cdot e^{-6i\varphi} \cdot \sin^6\theta \cdot (15\cos^2\theta - 1)$$
$$Y_8^{-5}(\theta,\varphi) = \frac{3}{64}\sqrt{\frac{17017}{2\pi}} \cdot e^{-5i\varphi} \cdot \sin^5\theta \cdot (5\cos^3\theta - \cos\theta)$$
$$Y_8^{-4}(\theta,\varphi) = \frac{3}{128}\sqrt{\frac{1309}{2\pi}} \cdot e^{-4i\varphi} \cdot \sin^4\theta \cdot (65\cos^4\theta - 26\cos^2\theta + 1)$$
$$Y_8^{-3}(\theta,\varphi) = \frac{1}{64}\sqrt{\frac{19635}{2\pi}} \cdot e^{-3i\varphi} \cdot \sin^3\theta \cdot (39\cos^5\theta - 26\cos^3\theta + 3\cos\theta)$$
$$Y_8^{-2}(\theta,\varphi) = \frac{3}{128}\sqrt{\frac{595}{\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta \cdot (143\cos^6\theta - 143\cos^4\theta + 33\cos^2\theta - 1)$$
$$Y_8^{-1}(\theta,\varphi) = \frac{3}{64}\sqrt{\frac{17}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \cdot (715\cos^7\theta - 1001\cos^5\theta + 385\cos^3\theta - 35\cos\theta)$$
$$Y_8^{0}(\theta,\varphi) = \frac{1}{256}\sqrt{\frac{17}{\pi}} \cdot (6435\cos^8\theta - 12012\cos^6\theta + 6930\cos^4\theta - 1260\cos^2\theta + 35)$$
$$Y_8^{1}(\theta,\varphi) = \frac{-3}{64}\sqrt{\frac{17}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot (715\cos^7\theta - 1001\cos^5\theta + 385\cos^3\theta - 35\cos\theta)$$
$$Y_8^{2}(\theta,\varphi) = \frac{3}{128}\sqrt{\frac{595}{\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta \cdot (143\cos^6\theta - 143\cos^4\theta + 33\cos^2\theta - 1)$$
$$Y_8^{3}(\theta,\varphi) = \frac{-1}{64}\sqrt{\frac{19635}{2\pi}} \cdot e^{3i\varphi} \cdot \sin^3\theta \cdot (39\cos^5\theta - 26\cos^3\theta + 3\cos\theta)$$
$$Y_8^{4}(\theta,\varphi) = \frac{3}{128}\sqrt{\frac{1309}{2\pi}} \cdot e^{4i\varphi} \cdot \sin^4\theta \cdot (65\cos^4\theta - 26\cos^2\theta + 1)$$
$$Y_8^{5}(\theta,\varphi) = \frac{-3}{64}\sqrt{\frac{17017}{2\pi}} \cdot e^{5i\varphi} \cdot \sin^5\theta \cdot (5\cos^3\theta - \cos\theta)$$
$$Y_8^{6}(\theta,\varphi) = \frac{1}{128}\sqrt{\frac{7293}{\pi}} \cdot e^{6i\varphi} \cdot \sin^6\theta \cdot (15\cos^2\theta - 1)$$
$$Y_8^{7}(\theta,\varphi) = \frac{-3}{64}\sqrt{\frac{12155}{2\pi}} \cdot e^{7i\varphi} \cdot \sin^7\theta \cdot \cos\theta$$
$$Y_8^{8}(\theta,\varphi) = \frac{3}{256}\sqrt{\frac{12155}{2\pi}} \cdot e^{8i\varphi} \cdot \sin^8\theta$$

l = 9

$$Y_9^{-9}(\theta,\varphi)=\frac{1}{512}\sqrt{\frac{230945}{\pi}}\cdot e^{-9i\varphi}\cdot\sin^9\theta$$
$$Y_9^{-8}(\theta,\varphi)=\frac{3}{256}\sqrt{\frac{230945}{2\pi}}\cdot e^{-8i\varphi}\cdot\sin^8\theta\cdot\cos\theta$$
$$Y_9^{-7}(\theta,\varphi)=\frac{3}{512}\sqrt{\frac{13585}{\pi}}\cdot e^{-7i\varphi}\cdot\sin^7\theta\cdot(17\cos^2\theta-1)$$
$$Y_9^{-6}(\theta,\varphi)=\frac{1}{128}\sqrt{\frac{40755}{\pi}}\cdot e^{-6i\varphi}\cdot\sin^6\theta\cdot(17\cos^3\theta-3\cos\theta)$$
$$Y_9^{-5}(\theta,\varphi)=\frac{3}{256}\sqrt{\frac{2717}{\pi}}\cdot e^{-5i\varphi}\cdot\sin^5\theta\cdot(85\cos^4\theta-30\cos^2\theta+1)$$
$$Y_9^{-4}(\theta,\varphi)=\frac{3}{128}\sqrt{\frac{95095}{2\pi}}\cdot e^{-4i\varphi}\cdot\sin^4\theta\cdot(17\cos^5\theta-10\cos^3\theta+\cos\theta)$$
$$Y_9^{-3}(\theta,\varphi)=\frac{1}{256}\sqrt{\frac{21945}{\pi}}\cdot e^{-3i\varphi}\cdot\sin^3\theta\cdot(221\cos^6\theta-195\cos^4\theta+39\cos^2\theta-1)$$
$$Y_9^{-2}(\theta,\varphi)=\frac{3}{128}\sqrt{\frac{1045}{\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta\cdot(221\cos^7\theta-273\cos^5\theta+91\cos^3\theta-7\cos\theta)$$
$$Y_9^{-1}(\theta,\varphi)=\frac{3}{256}\sqrt{\frac{95}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot(2431\cos^8\theta-4004\cos^6\theta+2002\cos^4\theta-308\cos^2\theta+7)$$
$$Y_9^{0}(\theta,\varphi)=\frac{1}{256}\sqrt{\frac{19}{\pi}}\cdot(12155\cos^9\theta-25740\cos^7\theta+18018\cos^5\theta-4620\cos^3\theta+315\cos\theta)$$
$$Y_9^{1}(\theta,\varphi)=\frac{-3}{256}\sqrt{\frac{95}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot(2431\cos^8\theta-4004\cos^6\theta+2002\cos^4\theta-308\cos^2\theta+7)$$
$$Y_9^{2}(\theta,\varphi)=\frac{3}{128}\sqrt{\frac{1045}{\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta\cdot(221\cos^7\theta-273\cos^5\theta+91\cos^3\theta-7\cos\theta)$$
$$Y_9^{3}(\theta,\varphi)=\frac{-1}{256}\sqrt{\frac{21945}{\pi}}\cdot e^{3i\varphi}\cdot\sin^3\theta\cdot(221\cos^6\theta-195\cos^4\theta+39\cos^2\theta-1)$$
$$Y_9^{4}(\theta,\varphi)=\frac{3}{128}\sqrt{\frac{95095}{2\pi}}\cdot e^{4i\varphi}\cdot\sin^4\theta\cdot(17\cos^5\theta-10\cos^3\theta+\cos\theta)$$
$$Y_9^{5}(\theta,\varphi)=\frac{-3}{256}\sqrt{\frac{2717}{\pi}}\cdot e^{5i\varphi}\cdot\sin^5\theta\cdot(85\cos^4\theta-30\cos^2\theta+1)$$
$$Y_9^{6}(\theta,\varphi)=\frac{1}{128}\sqrt{\frac{40755}{\pi}}\cdot e^{6i\varphi}\cdot\sin^6\theta\cdot(17\cos^3\theta-3\cos\theta)$$
$$Y_9^{7}(\theta,\varphi)=\frac{-3}{512}\sqrt{\frac{13585}{\pi}}\cdot e^{7i\varphi}\cdot\sin^7\theta\cdot(17\cos^2\theta-1)$$
$$Y_9^{8}(\theta,\varphi)=\frac{3}{256}\sqrt{\frac{230945}{2\pi}}\cdot e^{8i\varphi}\cdot\sin^8\theta\cdot\cos\theta$$
$$Y_9^{9}(\theta,\varphi)=\frac{-1}{512}\sqrt{\frac{230945}{\pi}}\cdot e^{9i\varphi}\cdot\sin^9\theta$$

l = 10

$$Y_{10}^{-10}(\theta,\varphi)=\frac{1}{1024}\sqrt{\frac{969969}{\pi}}\cdot e^{-10i\varphi}\cdot\sin^{10}\theta$$
$$Y_{10}^{-9}(\theta,\varphi)=\frac{1}{512}\sqrt{\frac{4849845}{\pi}}\cdot e^{-9i\varphi}\cdot\sin^{9}\theta\cdot\cos\theta$$
$$Y_{10}^{-8}(\theta,\varphi)=\frac{1}{512}\sqrt{\frac{255255}{2\pi}}\cdot e^{-8i\varphi}\cdot\sin^{8}\theta\cdot(19\cos^2\theta-1)$$
$$Y_{10}^{-7}(\theta,\varphi)=\frac{3}{512}\sqrt{\frac{85085}{\pi}}\cdot e^{-7i\varphi}\cdot\sin^{7}\theta\cdot(19\cos^3\theta-3\cos\theta)$$
$$Y_{10}^{-6}(\theta,\varphi)=\frac{3}{1024}\sqrt{\frac{5005}{\pi}}\cdot e^{-6i\varphi}\cdot\sin^{6}\theta\cdot(323\cos^4\theta-102\cos^2\theta+3)$$
$$Y_{10}^{-5}(\theta,\varphi)=\frac{3}{256}\sqrt{\frac{1001}{\pi}}\cdot e^{-5i\varphi}\cdot\sin^{5}\theta\cdot(323\cos^5\theta-170\cos^3\theta+15\cos\theta)$$
$$Y_{10}^{-4}(\theta,\varphi)=\frac{3}{256}\sqrt{\frac{5005}{2\pi}}\cdot e^{-4i\varphi}\cdot\sin^{4}\theta\cdot(323\cos^6\theta-255\cos^4\theta+45\cos^2\theta-1)$$
$$Y_{10}^{-3}(\theta,\varphi)=\frac{3}{256}\sqrt{\frac{5005}{\pi}}\cdot e^{-3i\varphi}\cdot\sin^{3}\theta\cdot(323\cos^7\theta-357\cos^5\theta+105\cos^3\theta-7\cos\theta)$$
$$Y_{10}^{-2}(\theta,\varphi)=\frac{3}{512}\sqrt{\frac{385}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^{2}\theta\cdot(4199\cos^8\theta-6188\cos^6\theta+2730\cos^4\theta-364\cos^2\theta+7)$$
$$Y_{10}^{-1}(\theta,\varphi)=\frac{1}{256}\sqrt{\frac{1155}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot(4199\cos^9\theta-7956\cos^7\theta+4914\cos^5\theta-1092\cos^3\theta+63\cos\theta)$$
$$Y_{10}^{0}(\theta,\varphi)=\frac{1}{512}\sqrt{\frac{21}{\pi}}\cdot(46189\cos^{10}\theta-109395\cos^8\theta+90090\cos^6\theta-30030\cos^4\theta+3465\cos^2\theta-63)$$
$$Y_{10}^{1}(\theta,\varphi)=\frac{-1}{256}\sqrt{\frac{1155}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot(4199\cos^9\theta-7956\cos^7\theta+4914\cos^5\theta-1092\cos^3\theta+63\cos\theta)$$
$$Y_{10}^{2}(\theta,\varphi)=\frac{3}{512}\sqrt{\frac{385}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^{2}\theta\cdot(4199\cos^8\theta-6188\cos^6\theta+2730\cos^4\theta-364\cos^2\theta+7)$$
$$Y_{10}^{3}(\theta,\varphi)=\frac{-3}{256}\sqrt{\frac{5005}{\pi}}\cdot e^{3i\varphi}\cdot\sin^{3}\theta\cdot(323\cos^7\theta-357\cos^5\theta+105\cos^3\theta-7\cos\theta)$$
$$Y_{10}^{4}(\theta,\varphi)=\frac{3}{256}\sqrt{\frac{5005}{2\pi}}\cdot e^{4i\varphi}\cdot\sin^{4}\theta\cdot(323\cos^6\theta-255\cos^4\theta+45\cos^2\theta-1)$$
$$Y_{10}^{5}(\theta,\varphi)=\frac{-3}{256}\sqrt{\frac{1001}{\pi}}\cdot e^{5i\varphi}\cdot\sin^{5}\theta\cdot(323\cos^5\theta-170\cos^3\theta+15\cos\theta)$$
$$Y_{10}^{6}(\theta,\varphi)=\frac{3}{1024}\sqrt{\frac{5005}{\pi}}\cdot e^{6i\varphi}\cdot\sin^{6}\theta\cdot(323\cos^4\theta-102\cos^2\theta+3)$$
$$Y_{10}^{7}(\theta,\varphi)=\frac{-3}{512}\sqrt{\frac{85085}{\pi}}\cdot e^{7i\varphi}\cdot\sin^{7}\theta\cdot(19\cos^3\theta-3\cos\theta)$$
$$Y_{10}^{8}(\theta,\varphi)=\frac{1}{512}\sqrt{\frac{255255}{2\pi}}\cdot e^{8i\varphi}\cdot\sin^{8}\theta\cdot(19\cos^2\theta-1)$$
$$Y_{10}^{9}(\theta,\varphi)=\frac{-1}{512}\sqrt{\frac{4849845}{\pi}}\cdot e^{9i\varphi}\cdot\sin^{9}\theta\cdot\cos\theta$$
$$Y_{10}^{10}(\theta,\varphi)=\frac{1}{1024}\sqrt{\frac{969969}{\pi}}\cdot e^{10i\varphi}\cdot\sin^{10}\theta$$

Real spherical harmonics

For each real spherical harmonic, the corresponding atomic orbital symbol (*s*, *p*, *d*, *f*, *g*) is reported as well.

l = 0[2][3]

$$Y_{00}=s=Y_0^0=\frac{1}{2}\sqrt{\frac{1}{\pi}}$$

l = 1[2][3]

$$Y_{1,-1} = p_y = i\sqrt{\frac{1}{2}}\left(Y_1^{-1} + Y_1^1\right) = \sqrt{\frac{3}{4\pi}} \cdot \frac{y}{r}$$

$$Y_{1,0} = p_z = Y_1^0 = \sqrt{\frac{3}{4\pi}} \cdot \frac{z}{r}$$

$$Y_{1,1} = p_x = \sqrt{\frac{1}{2}}\left(Y_1^{-1} - Y_1^1\right) = \sqrt{\frac{3}{4\pi}} \cdot \frac{x}{r}$$

l = 2[2][3]

$$Y_{2,-2} = d_{xy} = i\sqrt{\frac{1}{2}}\left(Y_2^{-2} - Y_2^2\right) = \frac{1}{2}\sqrt{\frac{15}{\pi}} \cdot \frac{xy}{r^2}$$

$$Y_{2,-1} = d_{yz} = i\sqrt{\frac{1}{2}}\left(Y_2^{-1} + Y_2^1\right) = \frac{1}{2}\sqrt{\frac{15}{\pi}} \cdot \frac{yz}{r^2}$$

$$Y_{2,0} = d_{z^2} = Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot \frac{-x^2 - y^2 + 2z^2}{r^2}$$

$$Y_{2,1} = d_{xz} = \sqrt{\frac{1}{2}}\left(Y_2^{-1} - Y_2^1\right) = \frac{1}{2}\sqrt{\frac{15}{\pi}} \cdot \frac{zx}{r^2}$$

$$Y_{2,2} = d_{x^2-y^2} = \sqrt{\frac{1}{2}}\left(Y_2^{-2} + Y_2^2\right) = \frac{1}{4}\sqrt{\frac{15}{\pi}} \cdot \frac{x^2 - y^2}{r^2}$$

l = 3[2]

$$Y_{3,-3} = f_{y(3x^2-y^2)} = i\sqrt{\frac{1}{2}}\left(Y_3^{-3} + Y_3^3\right) = \frac{1}{4}\sqrt{\frac{35}{2\pi}} \cdot \frac{\left(3x^2 - y^2\right)y}{r^3}$$

$$Y_{3,-2} = f_{xyz} = i\sqrt{\frac{1}{2}}\left(Y_3^{-2} - Y_3^2\right) = \frac{1}{2}\sqrt{\frac{105}{\pi}} \cdot \frac{xyz}{r^3}$$

$$Y_{3,-1} = f_{yz^2} = i\sqrt{\frac{1}{2}}\left(Y_3^{-1} + Y_3^1\right) = \frac{1}{4}\sqrt{\frac{21}{2\pi}} \cdot \frac{y(4z^2 - x^2 - y^2)}{r^3}$$

$$Y_{3,0} = f_{z^3} = Y_3^0 = \frac{1}{4}\sqrt{\frac{7}{\pi}} \cdot \frac{z(2z^2 - 3x^2 - 3y^2)}{r^3}$$

$$Y_{3,1} = f_{xz^2} = \sqrt{\frac{1}{2}}\left(Y_3^{-1} - Y_3^1\right) = \frac{1}{4}\sqrt{\frac{21}{2\pi}} \cdot \frac{x(4z^2 - x^2 - y^2)}{r^3}$$

$$Y_{3,2} = f_{z(x^2-y^2)} = \sqrt{\frac{1}{2}}\left(Y_3^{-2} + Y_3^2\right) = \frac{1}{4}\sqrt{\frac{105}{\pi}} \cdot \frac{\left(x^2 - y^2\right)z}{r^3}$$

$$Y_{3,3} = f_{x(x^2-3y^2)} = \sqrt{\frac{1}{2}}\left(Y_3^{-3} - Y_3^3\right) = \frac{1}{4}\sqrt{\frac{35}{2\pi}} \cdot \frac{\left(x^2 - 3y^2\right)x}{r^3}$$

l = 4

$$Y_{4,-4} = g_{xy(x^2-y^2)} = i\sqrt{\frac{1}{2}}\left(Y_4^{-4} - Y_4^4\right) = \frac{3}{4}\sqrt{\frac{35}{\pi}} \cdot \frac{xy\left(x^2-y^2\right)}{r^4}$$

$$Y_{4,-3} = g_{zy^3} = i\sqrt{\frac{1}{2}}\left(Y_4^{-3} + Y_4^3\right) = \frac{3}{4}\sqrt{\frac{35}{2\pi}} \cdot \frac{(3x^2-y^2)yz}{r^4}$$

$$Y_{4,-2} = g_{z^2xy} = i\sqrt{\frac{1}{2}}\left(Y_4^{-2} - Y_4^2\right) = \frac{3}{4}\sqrt{\frac{5}{\pi}} \cdot \frac{xy \cdot (7z^2-r^2)}{r^4}$$

$$Y_{4,-1} = g_{z^3y} = i\sqrt{\frac{1}{2}}\left(Y_4^{-1} + Y_4^1\right) = \frac{3}{4}\sqrt{\frac{5}{2\pi}} \cdot \frac{yz \cdot (7z^2-3r^2)}{r^4}$$

$$Y_{4,0} = g_{z^4} = Y_4^0 = \frac{3}{16}\sqrt{\frac{1}{\pi}} \cdot \frac{(35z^4-30z^2r^2+3r^4)}{r^4}$$

$$Y_{4,1} = g_{z^3x} = \sqrt{\frac{1}{2}}\left(Y_4^{-1} - Y_4^1\right) = \frac{3}{4}\sqrt{\frac{5}{2\pi}} \cdot \frac{xz \cdot (7z^2-3r^2)}{r^4}$$

$$Y_{4,2} = g_{z^2xy} = \sqrt{\frac{1}{2}}\left(Y_4^{-2} + Y_4^2\right) = \frac{3}{8}\sqrt{\frac{5}{\pi}} \cdot \frac{(x^2-y^2) \cdot (7z^2-r^2)}{r^4}$$

$$Y_{4,3} = g_{zx^3} = \sqrt{\frac{1}{2}}\left(Y_4^{-3} - Y_4^3\right) = \frac{3}{4}\sqrt{\frac{35}{2\pi}} \cdot \frac{(x^2-3y^2)xz}{r^4}$$

$$Y_{4,4} = g_{x^4+y^4} = \sqrt{\frac{1}{2}}\left(Y_4^{-4} + Y_4^4\right) = \frac{3}{16}\sqrt{\frac{35}{\pi}} \cdot \frac{x^2\left(x^2-3y^2\right) - y^2\left(3x^2-y^2\right)}{r^4}$$

See also

- Spherical harmonics

External links

- Spherical Harmonic (http://mathworld.wolfram.com/SphericalHarmonic.html) at MathWorld

References

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General references

- See section 3 in Mathar, R. J. (2009). "Zernike basis to cartesian transformations". *Serbian Astronomical Journal*. **179** (179): 107–120. Bibcode:2009SerAj.179..107M (http://adsabs.harvard.edu/abs/2009SerAj.179..107M). arXiv:0809.2368 (https://arxiv.org/abs/0809.2368). doi:10.2298/SAJ0979107M (https://doi.org/10.2298%2FSAJ0979107M). (see section 3.3)
- For complex spherical harmonics, see also SphericalHarmonicY[l,m,theta,phi] at Wolfram Alpha (http://www.wolframalpha.com/input/?i=SphericalHarmonicY%5Bl,m,theta,phi%5D), especially for specific values of l and m.