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On the rank of Hankel matrices over finite fields

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ABSTRACT

Given three nonnegative integers p, q, r and a finite field F, how many Hankel matrices $(x_{i+j})_{0 \leq i \leq p,\ 0 \leq j \leq q}$ over F have rank $\leq r$? This question is classical, and the answer (q^{2r} when $r \leq \min\{p, q\}$) has been obtained independently by various authors using different tools ([3, Theorem 1 for $m = n$], [4, (26)], [5, Theorem 5.1]). In this note, we will study a refinement of this result: We will show that if we fix the first k of the entries $x_0, x_1, \ldots, x_{k-1}$ for some $k \leq r \leq \min\{p, q\}$, then the number of ways to choose the remaining $p+q-k+1$ entries $x_k, x_{k+1}, \ldots, x_{p+q}$ such that the resulting Hankel matrix $(x_{i+j})_{0 \leq i \leq p,\ 0 \leq j \leq q}$ has rank $\leq r$ is q^{2r-k}. This is exactly the answer that one would expect if the first k entries had no effect on the rank, but of course the situation is not this simple (and we had to combine some ideas from [4, (26)] and from [5, Theorem 5.1 for $r = n$] to obtain our proof). The refined result generalizes (and provides an alternative proof of) [1, Corollary 6.4].

1. Results

We let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$.

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Fix a field F. For any $n \in \mathbb{N}$, any $(n+1)$-tuple $x = (x_0, x_1, \ldots, x_n) \in F^{n+1}$, and any two integers $p, q \in \{-1, 0, 1, \ldots\}$ satisfying $p+q \leq n$, we define a $(p+1) \times (q+1)$-matrix $H_{p,q}(x)$ by

$$H_{p,q}(x) := (x_{i+j})_{0 \leq i \leq p,\ 0 \leq j \leq q} = \begin{pmatrix} x_0 & x_1 & \cdots & x_q \\ x_1 & x_2 & \cdots & x_{q+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_p & x_{p+1} & \cdots & x_{p+q} \end{pmatrix} \in F^{(p+1)\times(q+1)}.$$

Such a matrix $H_{p,q}(x)$ is called a *Hankel matrix*. The study of Hankel matrices has a long history in linear algebra (see, e.g., [6]) and relates to linearly recurrent sequences ([4], [10, §8.6]), coprime polynomials ([5]), determinants ([11, Section XII.II]), orthogonal polynomials and continued fractions ([9, §2.7]), total positivity ([8]), and various applications such as x-ray imaging ([12, §V.5]), as well as the recent resolution of the t-adic Littlewood conjecture ([2]).[1] Numerous results have been obtained about their ranks in particular ([6, §11]). When the field F is finite, a strikingly simple formula can be given for the number of Hankel matrices of a given rank (more precisely, of rank $\leq$ to a given number):

Theorem 1.1. *Assume that F is finite. Let $q = |F|$. Let $r, m, n \in \mathbb{N}$ satisfy $r \leq m$ and $r \leq n$. The number of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying* $\operatorname{rank}(H_{m,n}(x)) \leq r$ *is q^{2r}.*

Example 1.2. For a simple example, let $r = 1$ and $m = 2$ and $n = 3$. Thus, for every $x = (x_0, x_1, x_2, x_3, x_4, x_5) \in F^6$, we have

$$H_{m,n}(x) = H_{2,3}(x) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix}.$$

Theorem 1.1 yields that the number of 6-tuples $x \in F^6$ satisfying $\operatorname{rank}(H_{2,3}(x)) \leq 1$ is $q^{2 \cdot 1} = q^2$. These 6-tuples can indeed be described explicitly:

- Any 6-tuple of the form $\left(u, uv, uv^2, uv^3, uv^4, uv^5\right)$ with $u \in F \setminus \{0\}$ and $v \in F$ is such a 6-tuple x. This gives a total of $|F \setminus \{0\}| \cdot |F| = (q-1)\,q$ many such 6-tuples.
- Any 6-tuple of the form $(0, 0, 0, 0, 0, w)$ with $w \in F$ is such a 6-tuple x. This gives a total of $|F| = q$ many such 6-tuples.

For higher values of r, it is harder to describe all the q^{2r} pertinent tuples.

[1] Some of these references are studying *Toeplitz matrices* instead of Hankel matrices. However, this is equivalent, since a Toeplitz matrix is just a Hankel matrix turned upside down (i.e., the result of reversing the order of the rows in a Hankel matrix).

To our knowledge, Theorem 1.1 has not appeared in this exact form in the literature; however, it is easily seen to be equivalent to the following variant, which has appeared in [3, Theorem 1]:

Corollary 1.3. *Assume that F is finite. Let $q = |F|$. Let $r, m, n \in \mathbb{N}$ satisfy $m \leq n$. The number of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $\operatorname{rank}(H_{m,n}(x)) = r$ is*

$$\begin{cases} 1, & \text{if } r = 0; \\ q^{2r-2}\left(q^2 - 1\right), & \text{if } 0 < r \leq m; \\ q^{2r-2}\left(q^{n-m+1} - 1\right), & \text{if } r = m+1; \\ 0, & \text{if } r > m+1. \end{cases}$$

The particular case of Corollary 1.3 for $m = n$ also appears in [5, Theorem 5.1][2] and [4, (26)]. The particular case when $r = m = n$ appears in [7, Corollary 3] as well.

Another setting in which Hankel matrices appear is the theory of symmetric functions, specifically Schur functions (see, e.g., [13, Chapter 7]). While we will not use this setting to prove our main results, it has provided the main inspiration for this note, so we shall briefly recall it now. The Jacobi–Trudi formula [13, Theorem 7.16.1] expresses a Schur function s_λ as the determinant of a matrix, which is a Hankel matrix when the partition λ is rectangle-shaped. The recent result [1, Corollary 6.4] by Anzis, Chen, Gao, Kim, Li and Patrias can thus be framed as a formula for the probability of a certain $(n+1) \times (n+1)$ Hankel matrix over a finite field to have determinant 0 (that is, rank $\leq n$). This would be a particular case of Theorem 1.1 if not for the fact that the entries of the relevant Hankel matrix are not chosen uniformly at random; instead, the first few of them are fixed, while the rest are chosen uniformly at random.[3] This suggests a generalization of Theorem 1.1 in which the first few entries[4] of the $(m+n+1)$-tuples $x \in F^{m+n+1}$ are fixed. The existence of such a generalization was suggested to us by Peter Scholze.

This generalization indeed exists, and will be the main result of this note. In stating it, we will use the following notation:

Definition 1.4. Let $n \in \mathbb{N}$. Let $x = (x_0, x_1, \ldots, x_n)$ be any $(n+1)$-tuple of any kinds of objects. Let $i \in \{0, 1, \ldots, n+1\}$. Then, $x_{[0,i)}$ denotes the i-tuple $(x_0, x_1, \ldots, x_{i-1})$.

For instance, $(a, b, c, d, e)_{[0,3)} = (a, b, c)$.

We can now state our generalization of Theorem 1.1:

[2] Note that [5, Theorem 5.1] works with Toeplitz matrices instead of Hankel matrices, but this makes no real difference, since a Toeplitz matrix is just a Hankel matrix turned upside down (and this operation clearly does not change the rank of the matrix).

[3] See Section 5 for concrete examples of such matrices.

[4] Specifically, "first few" means "at most m".

Theorem 1.5. *Assume that F is finite. Let $q = |F|$. Let $k, r, m, n \in \mathbb{N}$ satisfy $k \leq r \leq m$ and $r \leq n$. Fix any k-tuple $a = (a_0, a_1, \ldots, a_{k-1}) \in F^k$. The number of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ and $\operatorname{rank}(H_{m,n}(x)) \leq r$ is q^{2r-k}.*

Example 1.6. For an example, let $k = 2$, $r = 2$, $m = 3$ and $n = 3$. Let $a = (a_0, a_1) \in F^2$. Then, Theorem 1.5 yields that the number of 7-tuples $x \in F^7$ satisfying $x_{[0,2)} = a$ and $\operatorname{rank}(H_{3,3}(x)) \leq 2$ is $q^{2 \cdot 2 - 2} = q^2$. Note that a 7-tuple $x \in F^7$ satisfying $x_{[0,2)} = a$ is nothing but a 7-tuple $x \in F^7$ that begins with the entries a_0 and a_1; thus, we could just as well be counting the 5-tuples $(x_2, x_3, x_4, x_5, x_6) \in F^5$ satisfying $\operatorname{rank}(H_{3,3}(a_0, a_1, x_2, x_3, x_4, x_5, x_6)) \leq 2$.

Clearly, Theorem 1.1 is the particular case of Theorem 1.5 for $k = 0$, since the 0-tuple $a = () \in F^0$ automatically satisfies $x_{[0,0)} = a$ for every $x \in F^{m+n+1}$.

By specializing Theorem 1.1 to the case $r = m = n$ (and recalling that a square matrix has determinant 0 if and only if it has less-than-full rank), we can easily obtain the following:

Corollary 1.7. *Assume that F is finite. Let $q = |F|$. Let $k, n \in \mathbb{N}$ satisfy $k \leq n$. Fix any k-tuple $a = (a_0, a_1, \ldots, a_{k-1}) \in F^k$. The number of $(2n+1)$-tuples $x \in F^{2n+1}$ satisfying $x_{[0,k)} = a$ and $\det(H_{n,n}(x)) = 0$ is q^{2n-k}.*

We shall prove Theorem 1.5 in Section 4; we will then derive Theorem 1.1, Corollary 1.3 and Corollary 1.7 from it. Finally, in Section 5, we will explain how Corollary 1.7 generalizes [1, Corollary 6.4].

Remark 1.8. Theorem 1.5 also holds if we replace the assumptions "$k \leq r \leq m$ and $r \leq n$" by "$k \leq r \leq m \leq n+1$". In fact, the only case covered by the latter assumptions but not by the former is the case when $k \leq r = m = n+1$; however, Theorem 1.5 is easy to prove directly in this case. (To wit, if $k \leq r = m = n+1$, then **every** $(m+n+1)$-tuple $x \in F^{m+n+1}$ satisfies $\operatorname{rank}(H_{m,n}(x)) \leq r$, since the matrix $H_{m,n}(x)$ has $n+1$ columns and therefore has rank $\leq n+1 = r$. Hence, the number of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ and $\operatorname{rank}(H_{m,n}(x)) \leq r$ equals the number of **all** $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ in this case. But this number is easily seen to be $q^{m+n+1-k} = q^{2r-k}$ (since $\underbrace{m}_{=r} + \underbrace{n+1}_{=r} = r + r = 2r$). Thus, Theorem 1.5 is proved in the case when $k \leq r = m = n+1$.)

As a consequence, Theorem 1.1 also holds if we replace the assumptions "$r \leq m$ and $r \leq n$" by "$r \leq m \leq n+1$". Hence, Corollary 1.3 still holds if we replace the assumption "$m \leq n$" by "$m \leq n+1$". However, we gain nothing significantly new in this way, since the newly covered cases can also be easily obtained from the old ones.

2. Rank lemmas

Before we come to the proof of Theorem 1.5, we are going to build a toolbox of general lemmas about ranks of the Hankel matrices $H_{p,q}(x)$. We note that none of these lemmas requires F to be finite; they can equally well be applied to fields like $\mathbb{R}$ and $\mathbb{C}$.

Lemma 2.1. *Let $n \in \mathbb{N}$. Let $p, q \in \mathbb{N}$ be such that $p + q \leq n + 1$. If $x \in F^{n+1}$ satisfies $\operatorname{rank}(H_{p,q-1}(x)) \leq p$, then*

$$\operatorname{rank}(H_{p,q-1}(x)) \leq \operatorname{rank}(H_{p-1,q}(x)).$$

Proof of Lemma 2.1. We proceed by induction on p (without fixing x):

Induction base: Proving Lemma 2.1 in the case when $p = 0$ is easy: In this case, the assumption $\operatorname{rank}(H_{p,q-1}(x)) \leq p$ rewrites as $\operatorname{rank}(H_{p,q-1}(x)) \leq 0$, which immediately yields the claim.

Induction step: Let p be a positive integer. Assume (as the induction hypothesis) that Lemma 2.1 holds for $p - 1$ instead of p. Our goal is now to prove Lemma 2.1 for p.

Let $q \in \mathbb{N}$ be such that $p + q \leq n + 1$. Let $x \in F^{n+1}$ satisfy $\operatorname{rank}(H_{p,q-1}(x)) \leq p$. We must thus prove that

$$\operatorname{rank}(H_{p,q-1}(x)) \leq \operatorname{rank}(H_{p-1,q}(x)).$$

Write the $(n + 1)$-tuple $x \in F^{n+1}$ as $x = (x_0, x_1, \ldots, x_n)$. Then,

$$H_{p,q-1}(x) = \begin{pmatrix} x_0 & x_1 & \cdots & x_{q-1} \\ x_1 & x_2 & \cdots & x_q \\ \vdots & \vdots & \ddots & \vdots \\ x_p & x_{p+1} & \cdots & x_{p+q-1} \end{pmatrix} \qquad \text{and}$$

$$H_{p-1,q}(x) = \begin{pmatrix} x_0 & x_1 & \cdots & x_q \\ x_1 & x_2 & \cdots & x_{q+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p-1} & x_p & \cdots & x_{p+q-1} \end{pmatrix} \qquad \text{and}$$

$$H_{p-1,q-1}(x) = \begin{pmatrix} x_0 & x_1 & \cdots & x_{q-1} \\ x_1 & x_2 & \cdots & x_q \\ \vdots & \vdots & \ddots & \vdots \\ x_{p-1} & x_p & \cdots & x_{p+q-2} \end{pmatrix}.$$

Hence, the matrix $H_{p,q-1}(x)$ is $H_{p-1,q-1}(x)$ with one extra row inserted at the bottom, whereas the matrix $H_{p-1,q}(x)$ is $H_{p-1,q-1}(x)$ with one extra column inserted at the right end.

For any matrix A that has at least one row, we let $\overline{A}$ denote the matrix A with its first row removed. The following properties of $\overline{A}$ are well-known:

- If the first row of A is a linear combination of the remaining rows, then

$$\operatorname{rank} A = \operatorname{rank} \overline{A}. \tag{1}$$

- If the first row of A is **not** a linear combination of the remaining rows, then

$$\operatorname{rank} A = \operatorname{rank} \overline{A} + 1. \tag{2}$$

It is furthermore well-known that if A is any matrix, and if B is any submatrix of A, then $\operatorname{rank} B \leq \operatorname{rank} A$. However, the matrix $\overline{H_{p,q-1}(x)}$ is a submatrix of $H_{p-1,q}(x)$ (indeed, it can be obtained from $H_{p-1,q}(x)$ by removing the first column). Hence,

$$\operatorname{rank}\left(\overline{H_{p,q-1}(x)}\right) \leq \operatorname{rank}\left(H_{p-1,q}(x)\right).$$

Let $\overline{x}$ denote the n-tuple $(x_1, x_2, \ldots, x_n) \in F^n$. It is easy to see that

$$\overline{H_{u,v}(x)} = H_{u-1,v}(\overline{x}) \tag{3}$$

for all $u \in \mathbb{N}$ and $v \in \{-1, 0, 1, \ldots\}$ satisfying $u + v \leq n$. Thus, in particular,

$$\overline{H_{p,q-1}(x)} = H_{p-1,q-1}(\overline{x}) \tag{4}$$

and

$$\overline{H_{p-1,q}(x)} = H_{p-2,q}(\overline{x}). \tag{5}$$

If the first row of the matrix $H_{p,q-1}(x)$ is a linear combination of the remaining rows, then (1) yields

$$\operatorname{rank}\left(H_{p,q-1}(x)\right) = \operatorname{rank}\left(\overline{H_{p,q-1}(x)}\right) \leq \operatorname{rank}\left(H_{p-1,q}(x)\right),$$

which is precisely what we wanted to show. Hence, for the rest of this proof, we assume without loss of generality that the first row of the matrix $H_{p,q-1}(x)$ is **not** a linear combination of the remaining rows. Thus, (2) yields

$$\operatorname{rank}\left(H_{p,q-1}(x)\right) = \operatorname{rank}\left(\overline{H_{p,q-1}(x)}\right) + 1.$$

In view of (4), this rewrites as

$$\operatorname{rank}\left(H_{p,q-1}(x)\right) = \operatorname{rank}\left(H_{p-1,q-1}(\overline{x})\right) + 1. \tag{6}$$

Hence,

$$\operatorname{rank}\left(H_{p-1,q-1}\left(\overline{x}\right)\right) = \underbrace{\operatorname{rank}\left(H_{p,q-1}\left(x\right)\right)}_{\leq p} - 1 \leq p-1.$$

Recall that the first row of the matrix $H_{p,q-1}(x)$ is not a linear combination of the remaining rows. This entails that the first row of the matrix $H_{p-1,q-1}(x)$ is not a linear combination of the remaining rows (since the matrix $H_{p-1,q-1}(x)$ is the same as $H_{p,q-1}(x)$ without the last row). Therefore, the first row of the matrix $H_{p-1,q}(x)$ is not a linear combination of the remaining rows (since the matrix $H_{p-1,q}(x)$ is just $H_{p-1,q-1}(x)$ with an extra column). Thus, (2) yields

$$\operatorname{rank}\left(H_{p-1,q}\left(x\right)\right) = \operatorname{rank}\left(\overline{H_{p-1,q}\left(x\right)}\right) + 1.$$

In view of (5), this rewrites as

$$\operatorname{rank}\left(H_{p-1,q}\left(x\right)\right) = \operatorname{rank}\left(H_{p-2,q}\left(\overline{x}\right)\right) + 1. \tag{7}$$

However, our induction hypothesis shows that we can apply Lemma 2.1 to $n-1$, $p-1$ and $\overline{x}$ instead of n, p and x (since $\operatorname{rank}\left(H_{p-1,q-1}\left(\overline{x}\right)\right) \leq p-1$). We thus obtain

$$\operatorname{rank}\left(H_{p-1,q-1}\left(\overline{x}\right)\right) \leq \operatorname{rank}\left(H_{p-2,q}\left(\overline{x}\right)\right).$$

Adding 1 to both sides of this inequality, we find

$$\operatorname{rank}\left(H_{p-1,q-1}\left(\overline{x}\right)\right) + 1 \leq \operatorname{rank}\left(H_{p-2,q}\left(\overline{x}\right)\right) + 1.$$

In view of (6) and (7), this rewrites as $\operatorname{rank}\left(H_{p,q-1}(x)\right) \leq \operatorname{rank}\left(H_{p-1,q}(x)\right)$. This completes the induction step. Thus, Lemma 2.1 is proved. $\square$

Lemma 2.2. *Let $n \in \mathbb{N}$. Let $p, q \in \mathbb{N}$ be such that $p + q \leq n + 1$. If $x \in F^{n+1}$ satisfies $\operatorname{rank}\left(H_{p-1,q}(x)\right) \leq q$, then*

$$\operatorname{rank}\left(H_{p-1,q}\left(x\right)\right) \leq \operatorname{rank}\left(H_{p,q-1}\left(x\right)\right).$$

Proof of Lemma 2.2. This is just a restatement of Lemma 2.1 (applied to p and q instead of q and p), since the matrices $H_{p-1,q}(x)$ and $H_{p,q-1}(x)$ are the transposes of the matrices $H_{q,p-1}(x)$ and $H_{q-1,p}(x)$. (Alternatively, you can prove it by the same argument as we used to prove Lemma 2.1, except that rows and columns switch roles.) $\square$

Lemma 2.3. *Let $n \in \mathbb{N}$. Let $p, q \in \mathbb{N}$ be such that $p + q \leq n + 1$. If $x \in F^{n+1}$ satisfies $\operatorname{rank}\left(H_{p,q-1}(x)\right) \leq p$ and $\operatorname{rank}\left(H_{p-1,q}(x)\right) \leq q$, then*

$$\operatorname{rank}\left(H_{p,q-1}\left(x\right)\right) = \operatorname{rank}\left(H_{p-1,q}\left(x\right)\right).$$

Proof of Lemma 2.3. This follows by combining Lemma 2.1 with Lemma 2.2. $\square$

Our next lemma is a simple corollary of Lemma 2.3:

Lemma 2.4. *Let $n \in \mathbb{N}$. Let $p, q \in \mathbb{N}$ be such that $p + q \leq n + 1$. Let $r \in \mathbb{N}$ satisfy $r + 1 \leq p$ and $r + 1 \leq q$. Let $x \in F^{n+1}$. Then, we have the logical equivalence*

$$(\operatorname{rank}(H_{p,q-1}(x)) \leq r) \iff (\operatorname{rank}(H_{p-1,q}(x)) \leq r).$$

Proof of Lemma 2.4. We assume that $\operatorname{rank}(H_{p,q-1}(x)) \leq r$; we will show that $\operatorname{rank}(H_{p-1,q}(x)) \leq r$. (This will prove only the "$\Longrightarrow$" direction of Lemma 2.4, but the other direction is completely analogous.)

The matrix $H_{p-1,q-1}(x)$ is a submatrix of $H_{p,q-1}(x)$, and thus its rank cannot surpass the rank of $H_{p,q-1}(x)$. In other words, we have $\operatorname{rank}(H_{p-1,q-1}(x)) \leq \operatorname{rank}(H_{p,q-1}(x))$.

However, the matrix $H_{p-1,q}(x)$ can be viewed as being the matrix $H_{p-1,q-1}(x)$ with one extra column attached to it (at its right end). Thus,

$$\operatorname{rank}(H_{p-1,q}(x)) \leq \operatorname{rank}(H_{p-1,q-1}(x)) + 1$$

(since attaching one column cannot increase the rank of a matrix by more than 1). Hence,

$$\operatorname{rank}(H_{p-1,q}(x)) \leq \underbrace{\operatorname{rank}(H_{p-1,q-1}(x))}_{\leq \operatorname{rank}(H_{p,q-1}(x)) \leq r} + 1 \leq r + 1 \leq q.$$

Moreover, $\operatorname{rank}(H_{p,q-1}(x)) \leq r \leq r + 1 \leq p$. Hence, we can apply Lemma 2.3, and conclude that $\operatorname{rank}(H_{p,q-1}(x)) = \operatorname{rank}(H_{p-1,q}(x))$. Thus, of course, $\operatorname{rank}(H_{p-1,q}(x)) \leq r$ follows immediately from our assumption $\operatorname{rank}(H_{p,q-1}(x)) \leq r$. Hence, the "$\Longrightarrow$" direction of Lemma 2.4 is proved. The "$\Longleftarrow$" direction is analogous, so the proof is complete. $\square$

The following lemma is a (much simpler) counterpart to Lemma 2.1 that replaces the assumption $\operatorname{rank}(H_{p,q-1}(x)) \leq p$ by the reverse inequality:

Lemma 2.5. *Let $n \in \mathbb{N}$. Let $p, q \in \mathbb{N}$ be such that $p + q \leq n + 1$. If $x \in F^{n+1}$ satisfies $\operatorname{rank}(H_{p,q-1}(x)) > p$, then*

$$\operatorname{rank}(H_{p-1,q}(x)) = p.$$

Proof of Lemma 2.5. Let $x \in F^{n+1}$ satisfy $\operatorname{rank}(H_{p,q-1}(x)) > p$. The assumption $\operatorname{rank}(H_{p,q-1}(x)) > p$ shows that the $p + 1$ rows of the matrix $H_{p,q-1}(x)$ are linearly independent. Hence, in particular, the p rows of the matrix $H_{p-1,q-1}(x)$ are linearly independent (since these p rows are simply the first p rows of the matrix $H_{p,q-1}(x)$). Therefore, the p rows of the matrix $H_{p-1,q}(x)$ are linearly independent as well (since the matrix $H_{p-1,q}(x)$ is just $H_{p-1,q-1}(x)$ with an extra column, and therefore the rows of the former contain the rows of the latter as subsequences). In other words, $\operatorname{rank}(H_{p-1,q}(x)) = p$. This proves Lemma 2.5. $\square$

Our above lemmas have related ranks of the "adjacent" Hankel matrices $\operatorname{rank}(H_{p,q-1}(x))$ and $\operatorname{rank}(H_{p-1,q}(x))$. By induction, we shall now extend these to further-apart Hankel matrices:

Lemma 2.6. *Let $u \in \mathbb{N}$. Let $m, n, r \in \mathbb{N}$ be such that $m + n \leq u$ and $r \leq m$ and $r \leq n$. Let $s = m + n - r$. Let $x \in F^{u+1}$ be arbitrary. Then, we have the logical equivalence*

$$(\operatorname{rank}(H_{m,n}(x)) \leq r) \iff (\operatorname{rank}(H_{r,s}(x)) \leq r).$$

Before we prove this lemma, let us comment on its significance (even though we will use it rather directly): If one wants to determine the rank of a $(m+1) \times (n+1)$-matrix A, it suffices to probe for each $r \in \{0, 1, \ldots, \min\{m, n\}\}$ whether $\operatorname{rank} A \leq r$ is true (since $0 \leq \operatorname{rank} A \leq \min\{m, n\} + 1$). Thus, Lemma 2.6 allows us to determine the ranks of the various matrices $H_{m,n}(x)$ for a given $x \in F^{u+1}$ if we know which pairs (r, s) satisfy $\operatorname{rank}(H_{r,s}(x)) \leq r$.

Proof of Lemma 2.6. From $s = m + n - r = (m - r) + n$, we obtain $s - (m - r) = n$. Furthermore, $s = m + n - \underbrace{r}_{\leq m} \geq m + n - m = n$ and similarly $s \geq m$. Hence, $m \leq s$.

We now claim that the equivalence

$$(\operatorname{rank}(H_{r+i,s-i}(x)) \leq r) \iff (\operatorname{rank}(H_{r,s}(x)) \leq r) \tag{8}$$

holds for each $i \in \{0, 1, \ldots, s - r\}$.

[**Proof of (8):** We proceed by induction on i:

Induction base: Clearly, (8) holds for $i = 0$, since we have $H_{r+i,s-i}(x) = H_{r+0,s-0}(x) = H_{r,s}(x)$ in this case.

Induction step: Let $j \in \{1, 2, \ldots, s - r\}$. Assume (as the induction hypothesis) that (8) holds for $i = j - 1$. We must prove that (8) holds for $i = j$. In other words, we must prove the equivalence

$$(\operatorname{rank}(H_{r+j,s-j}(x)) \leq r) \iff (\operatorname{rank}(H_{r,s}(x)) \leq r). \tag{9}$$

However, our induction hypothesis tells us that the equivalence

$$\left(\operatorname{rank}\left(H_{r+(j-1),s-(j-1)}(x)\right) \leq r\right) \iff (\operatorname{rank}(H_{r,s}(x)) \leq r) \tag{10}$$

holds.

We have

$$(r+j)+(s-j+1)=r+\underbrace{s}_{=m+n-r}+1=r+(m+n-r)+1=\underbrace{m+n}_{\leq u}+1\leq u+1.$$

Furthermore, we have $j\in\{1,2,\ldots,s-r\}$, so that $1\leq j\leq s-r$. From $j\leq s-r$, we obtain $r\leq s-j$, so that $r+1\leq s-j+1$. This entails $s-j+1\in\mathbb{N}$ (since $r+1\in\mathbb{N}$). Also, $r+1\leq r+j$ (since $j\geq 1$). Hence, Lemma 2.4 (applied to $p=r+j$ and $n=s-j+1$) yields that we have the logical equivalence

$$(\operatorname{rank}(H_{r+j,s-j+1-1}(x))\leq r)\iff(\operatorname{rank}(H_{r+j-1,s-j+1}(x))\leq r).$$

In other words, we have the equivalence

$$(\operatorname{rank}(H_{r+j,s-j}(x))\leq r)\iff\left(\operatorname{rank}\left(H_{r+(j-1),s-(j-1)}(x)\right)\leq r\right)$$

(since $s-j+1-1=s-j$ and $r+j-1=r+(j-1)$ and $s-j+1=s-(j-1)$). Combining this equivalence with (10), we obtain precisely the equivalence (9) that we were meaning to prove.

Thus, we have shown that (8) holds for $i=j$. This completes the induction step, so that (8) is proven.]

Now, $m\in\{r,r+1,\ldots,s\}$ (since $r\leq m\leq s$), so that $m-r\in\{0,1,\ldots,s-r\}$. Hence, we can apply (8) to $i=m-r$. As a result, we obtain that the equivalence

$$\left(\operatorname{rank}\left(H_{r+m-r,s-(m-r)}(x)\right)\leq r\right)\iff(\operatorname{rank}(H_{r,s}(x))\leq r)$$

holds. In other words, the equivalence

$$(\operatorname{rank}(H_{m,n}(x))\leq r)\iff(\operatorname{rank}(H_{r,s}(x))\leq r)$$

holds (since $r+m-r=m$ and $s-(m-r)=n$). This proves Lemma 2.6. $\square$

3. Auxiliary enumerative results

3.1. Assumptions and notations

From now on, we assume that the field F is finite. We set $q=|F|$.

We shall use the so-called *Iverson bracket notation*:

Definition 3.1. If $\mathcal{A}$ is any logical statement, then we define an integer $[\mathcal{A}]\in\{0,1\}$ by

$$[\mathcal{A}]=\begin{cases}1, & \text{if } \mathcal{A} \text{ is true;}\\ 0, & \text{if } \mathcal{A} \text{ is false.}\end{cases}$$

For example, $[2+2=4]=1$ but $[2+2=5]=0$.

If $\mathcal{A}$ is any logical statement, then $[\mathcal{A}]$ is known as the *truth value* of $\mathcal{A}$.

The following fact ("counting by roll-call") makes truth values useful to us:

Proposition 3.2. *Let S be a finite set. Let $\mathcal{A}(s)$ be a logical statement for each $s \in S$. Then, $\sum_{s\in S} [\mathcal{A}(s)]$ equals the number of elements $s \in S$ satisfying $\mathcal{A}(s)$.*

3.2. Sums over v for fixed x

The following proposition is a restatement of [4, Proposition 2] (but we shall prove it nevertheless to keep this note self-contained):

Proposition 3.3. *Let $m, n \in \mathbb{N}$ satisfy $m \leq n+1$. Let $x \in F^{m+n+1}$ be a $(m+n+1)$-tuple. Then,*

$$\begin{aligned}
&(q-1) \cdot [\operatorname{rank}(H_{m,n}(x)) \leq m] \\
&= \sum_{\substack{v \in F^{1\times(m+1)}; \\ v \neq 0}} [v\ H_{m,n}(x) = 0] - q \sum_{\substack{v \in F^{1\times m}; \\ v \neq 0}} [v\ H_{m-1,n+1}(x) = 0].
\end{aligned}$$

Before we prove this proposition, a few words about its significance are worth saying. Assume that, as a first step towards proving Theorem 1.5, we want to count the $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $\operatorname{rank}(H_{m,n}(x)) \leq m$. (This is just an interim goal; we will later generalize this inequality to $\operatorname{rank}(H_{m,n}(x)) \leq r$ and impose the additional condition $x_{[0,k)} = a$.) In view of Proposition 3.2, this boils down to computing $\sum_{x\in F^{m+n+1}} [\operatorname{rank}(H_{m,n}(x)) \leq m]$. Using Proposition 3.3, we can rewrite the addends $[\operatorname{rank}(H_{m,n}(x)) \leq m]$ in this sum in terms of other truth values, which are more "local" (one can think of "$\operatorname{rank}(H_{m,n}(x)) \leq m$" as a "global" statement about the matrix $H_{m,n}(x)$, whereas the statements "$v\ H_{m,n}(x) = 0$" and "$v\ H_{m-1,n+1}(x) = 0$" are local in the sense that they only "sample" the matrix at a single vector each) and thus (as we will soon see) are easier to sum.

Proof of Proposition 3.3. We are in one of the following two cases:

Case 1: We have $\operatorname{rank}(H_{m,n}(x)) > m$.

Case 2: We have $\operatorname{rank}(H_{m,n}(x)) \leq m$.

Let us first consider Case 1. In this case, we have $\operatorname{rank}(H_{m,n}(x)) > m$. Thus, $\operatorname{rank}(H_{m,n}(x)) = m+1$ (since $H_{m,n}(x)$ is an $(m+1)\times(n+1)$-matrix). Therefore, the rows of the matrix $H_{m,n}(x)$ are linearly independent. Hence, there exists no nonzero $v \in F^{1\times(m+1)}$ satisfying $v\ H_{m,n}(x) = 0$. Therefore,

$$\sum_{\substack{v\in F^{1\times(m+1)};\\ v\neq 0}} [v\ H_{m,n}(x)=0]=0. \tag{11}$$

Moreover, Lemma 2.5 (applied to $m+n$, m and $n+1$ instead of n, p and q) yields that $\operatorname{rank}(H_{m-1,n+1}(x))=m$ (since $\operatorname{rank}(H_{m,n}(x))>m$). In other words, the $m\times(n+2)$-matrix $H_{m-1,n+1}(x)$ has rank m. Hence, there exists no nonzero $v\in F^{1\times m}$ satisfying $v\ H_{m-1,n+1}(x)=0$. Therefore,

$$\sum_{\substack{v\in F^{1\times m};\\ v\neq 0}} [v\ H_{m-1,n+1}(x)=0]=0. \tag{12}$$

Finally, $[\operatorname{rank}(H_{m,n}(x))\leq m]=0$ (since $\operatorname{rank}(H_{m,n}(x))>m$). In view of this equality, as well as (11) and (12), the equality that we are trying to prove rewrites as $(q-1)\cdot 0=0-q\cdot 0$, which is clearly true. Thus, Proposition 3.3 is proved in Case 1.

Let us now consider Case 2. In this case, we have $\operatorname{rank}(H_{m,n}(x))\leq m$. Also, the matrix $H_{m-1,n+1}(x)$ has m rows; thus, $\operatorname{rank}(H_{m-1,n+1}(x))\leq m\leq n+1$. Therefore, Lemma 2.3 (applied to $m+n$, m and $n+1$ instead of n, p and q) yields

$$\operatorname{rank}(H_{m,n}(x))=\operatorname{rank}(H_{m-1,n+1}(x)). \tag{13}$$

Now, Proposition 3.2 shows that $\sum_{v\in F^{1\times(m+1)}} [v\ H_{m,n}(x)=0]$ is the number of all $v\in F^{1\times(m+1)}$ satisfying $v\ H_{m,n}(x)=0$. In other words, $\sum_{v\in F^{1\times(m+1)}} [v\ H_{m,n}(x)=0]$ is the size of the left kernel[5] of the matrix $H_{m,n}(x)$. But the dimension of this left kernel is $(m+1)-\operatorname{rank}(H_{m,n}(x))$ (by the rank-nullity theorem[6]); hence, the size of this left kernel is $q^{(m+1)-\operatorname{rank}(H_{m,n}(x))}$. Thus,

$$\sum_{v\in F^{1\times(m+1)}} [v\ H_{m,n}(x)=0]=q^{(m+1)-\operatorname{rank}(H_{m,n}(x))}. \tag{14}$$

The same reasoning shows that

$$\sum_{v\in F^{1\times m}} [v\ H_{m-1,n+1}(x)=0]=q^{m-\operatorname{rank}(H_{m-1,n+1}(x))}. \tag{15}$$

Now, (13) yields

$$q^{(m+1)-\operatorname{rank}(H_{m,n}(x))}=q^{(m+1)-\operatorname{rank}(H_{m-1,n+1}(x))}=q\cdot q^{m-\operatorname{rank}(H_{m-1,n+1}(x))}.$$

[5] The *left kernel* of an $s\times t$-matrix $A\in F^{s\times t}$ is defined to be the set of all row vectors $v\in F^{1\times s}$ satisfying $vA=0$. This is a vector subspace of $F^{1\times s}$.

[6] The *rank-nullity theorem* (in the form we are using it here) says that the dimension of the left kernel of a matrix $A\in F^{s\times t}$ equals $s-\operatorname{rank} A$.

In view of

$$\begin{aligned}
& q^{(m+1)-\operatorname{rank}(H_{m,n}(x))} \\
& = \sum_{v \in F^{1\times(m+1)}} [v\ H_{m,n}(x) = 0] \qquad \text{(by (14))} \\
& = 1 + \sum_{\substack{v \in F^{1\times(m+1)}; \\ v \neq 0}} [v\ H_{m,n}(x) = 0] \qquad (\text{since } 0\ H_{m,n}(x) = 0)
\end{aligned}$$

and

$$\begin{aligned}
& q^{m-\operatorname{rank}(H_{m-1,n+1}(x))} \\
& = \sum_{v \in F^{1\times m}} [v\ H_{m-1,n+1}(x) = 0] \qquad \text{(by (15))} \\
& = 1 + \sum_{\substack{v \in F^{1\times m}; \\ v \neq 0}} [v\ H_{m-1,n+1}(x) = 0] \qquad (\text{since } 0\ H_{m-1,n+1}(x) = 0),
\end{aligned}$$

we can rewrite this as

$$1 + \sum_{\substack{v \in F^{1\times(m+1)}; \\ v \neq 0}} [v\ H_{m,n}(x) = 0] = q \cdot \left(1 + \sum_{\substack{v \in F^{1\times m}; \\ v \neq 0}} [v\ H_{m-1,n+1}(x) = 0] \right).$$

In other words,

$$\sum_{\substack{v \in F^{1\times(m+1)}; \\ v \neq 0}} [v\ H_{m,n}(x) = 0] - q \sum_{\substack{v \in F^{1\times m}; \\ v \neq 0}} [v\ H_{m-1,n+1}(x) = 0] = q - 1.$$

Comparing this with

$$(q-1) \cdot \underbrace{[\operatorname{rank}(H_{m,n}(x)) \leq m]}_{\substack{=1 \\ (\text{since } \operatorname{rank}(H_{m,n}(x)) \leq m)}} = q - 1,$$

we obtain precisely the claim of Proposition 3.3. Thus, Proposition 3.3 is proved in Case 2.

We have now proved Proposition 3.3 in both Cases 1 and 2. Thus, Proposition 3.3 always holds. $\square$

3.3. Sums over x for fixed v

We need another definition. Namely, if $m \in \mathbb{N}$, and if $v = (v_0, v_1, \ldots, v_m) \in F^{1\times(m+1)}$ is a row vector of size $m+1$, then $\operatorname{last} v$ will denote v_m (that is, the last entry of v). There is a bijection

$$R : \left\{ v \in F^{1\times(m+1)} \mid \operatorname{last} v = 0 \right\} \to F^{1\times m},$$
$$(v_0, v_1, \ldots, v_m) \mapsto (v_0, v_1, \ldots, v_{m-1}).$$

Its inverse map R^{-1} sends each row vector $(v_0, v_1, \ldots, v_{m-1}) \in F^{1\times m}$ to the row vector $(v_0, v_1, \ldots, v_{m-1}, 0)$.

Lemma 3.4. *Let $k, m, n \in \mathbb{N}$ satisfy $k \leq m$. Let $v \in F^{1\times(m+1)}$ be a row vector of size $m+1$ such that* $\operatorname{last} v \neq 0$. *Fix any k-tuple $a \in F^k$. Then,*

$$\sum_{\substack{x \in F^{m+n+1}; \\ x_{[0,k)} = a}} [v\ H_{m,n}(x) = 0] = q^{m-k}.$$

Proof of Lemma 3.4. Proposition 3.2 shows that $\sum_{\substack{x \in F^{m+n+1}; \\ x_{[0,k)} = a}} [v\ H_{m,n}(x) = 0]$ is the number of all $x \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ and $v\ H_{m,n}(x) = 0$. Thus, we must prove that this number is q^{m-k}.

Write v and a as $v = (v_0, v_1, \ldots, v_m)$ and $a = (a_0, a_1, \ldots, a_{k-1})$, respectively. Thus, $\operatorname{last} v = v_m$, so that $v_m = \operatorname{last} v \neq 0$.

Now, we are looking for an $x = (x_0, x_1, \ldots, x_{m+n}) \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ and $v\ H_{m,n}(x) = 0$. The condition $x_{[0,k)} = a$ says that the first k entries of x equal the respective entries of a; that is, $x_i = a_i$ for each $i \in \{0, 1, \ldots, k-1\}$. Thus, $x_0, x_1, \ldots, x_{k-1}$ are uniquely determined. The condition $v\ H_{m,n}(x) = 0$ is equivalent to x satisfying the following system of linear equations:

$$\begin{cases} v_0 x_0 + v_1 x_1 + \cdots + v_m x_m = 0; \\ v_0 x_1 + v_1 x_2 + \cdots + v_m x_{m+1} = 0; \\ v_0 x_2 + v_1 x_3 + \cdots + v_m x_{m+2} = 0; \\ \cdots; \\ v_0 x_n + v_1 x_{n+1} + \cdots + v_m x_{m+n} = 0. \end{cases} \tag{16}$$

Since $v_m \neq 0$, this latter system of equations can be uniquely solved for the unknowns $x_m, x_{m+1}, \ldots, x_{m+n}$ (by recursive substitution) when the m entries $x_0, x_1, \ldots, x_{m-1}$ are given. Hence, each $(m+n+1)$-tuple $x = (x_0, x_1, \ldots, x_{m+n}) \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ and $v\ H_{m,n}(x) = 0$ can be constructed as follows:

- First, we set $x_i = a_i$ for each $i \in \{0, 1, \ldots, k-1\}$. This determines the first k entries $x_0, x_1, \ldots, x_{k-1}$ of x.
- Then, we choose arbitrary values for the next $m-k$ entries $x_k, x_{k+1}, \ldots, x_{m-1}$.
- Finally, we uniquely determine the remaining entries $x_m, x_{m+1}, \ldots, x_{m+n}$ by solving the system (16).

Clearly, the number of ways to perform this construction is q^{m-k} (since there are $|F| = q$ many options for each of the $m-k$ entries $x_k, x_{k+1}, \ldots, x_{m-1}$). Thus, the number of all $x \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ and $v\ H_{m,n}(x) = 0$ is q^{m-k}. This proves Lemma 3.4. $\square$

Lemma 3.5. *Let $k, m, n \in \mathbb{N}$ satisfy $k \leq n+1$. Let $v \in F^{1\times(m+1)}$ be a nonzero row vector of size $m+1$ such that* $\operatorname{last} v = 0$*. Fix any k-tuple $a \in F^k$. Then,*

$$\sum_{\substack{x \in F^{m+n+1};\\ x_{[0,k)}=a}} [v\ H_{m,n}(x) = 0] = q \sum_{\substack{x \in F^{m+n+1};\\ x_{[0,k)}=a}} [R(v)\ H_{m-1,n+1}(x) = 0]. \qquad (17)$$

Proof of Lemma 3.5. The sum on the left hand side of (17) is the number of all $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ and $v\ H_{m,n}(x) = 0$ (because of Proposition 3.2). Let us refer to such $(m+n+1)$-tuples x as *weakly nice tuples.*

The sum on the right hand side of (17) is the number of all $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ and $R(v)\ H_{m-1,n+1}(x) = 0$ (because of Proposition 3.2). Let us refer to such $(m+n+1)$-tuples x as *strongly nice tuples.*

We thus need to prove that the number of weakly nice tuples equals q times the number of strongly nice tuples.

We shall achieve this by constructing a bijection

$$\{\text{weakly nice tuples}\} \to F \times \{\text{strongly nice tuples}\}.$$

Indeed, let us unravel the definitions of weakly and strongly nice tuples.

Write v and a as $v = (v_0, v_1, \ldots, v_m)$ and $a = (a_0, a_1, \ldots, a_{k-1})$, respectively. Thus, $\operatorname{last} v = v_m$, so that $v_m = \operatorname{last} v = 0$. Consider the **largest** $j \in \{0, 1, \ldots, m\}$ satisfying $v_j \neq 0$. (This exists, since v is nonzero.) Thus, $v_j \neq 0$ but $v_{j+1} = v_{j+2} = \cdots = v_m = 0$. Also, the definition of R yields $R(v) = (v_0, v_1, \ldots, v_{m-1})$.

We have $j \neq m$ (since $v_j \neq 0$ but $v_m = 0$). Thus, $j \leq m-1$. Furthermore,

$$\underbrace{j}_{\geq 0} + n + 1 \geq n + 1 > n \geq k - 1 \qquad (\text{since } k \leq n+1).$$

The weakly nice tuples are the $(m+n+1)$-tuples $x = (x_0, x_1, \ldots, x_{m+n}) \in F^{m+n+1}$ satisfying

$$x_i = a_i \qquad \text{for each } i \in \{0, 1, \ldots, k-1\} \qquad (18)$$

as well as

$$\begin{cases} v_0x_0 + v_1x_1 + \cdots + v_mx_m = 0; \\ v_0x_1 + v_1x_2 + \cdots + v_mx_{m+1} = 0; \\ v_0x_2 + v_1x_3 + \cdots + v_mx_{m+2} = 0; \\ \cdots; \\ v_0x_n + v_1x_{n+1} + \cdots + v_mx_{m+n} = 0 \end{cases} \tag{19}$$

(because the condition "$x_{[0,k)} = a$" is equivalent to (18), whereas the condition "$v \ H_{m,n}(x) = 0$" is equivalent to (19)). In view of $v_{j+1} = v_{j+2} = \cdots = v_m = 0$, we can rewrite this as follows: The weakly nice tuples are the $(m+n+1)$-tuples $x = (x_0, x_1, \ldots, x_{m+n}) \in F^{m+n+1}$ satisfying

$$x_i = a_i \qquad \text{for each } i \in \{0, 1, \ldots, k-1\}$$

as well as

$$\begin{cases} v_0x_0 + v_1x_1 + \cdots + v_jx_j = 0; \\ v_0x_1 + v_1x_2 + \cdots + v_jx_{j+1} = 0; \\ v_0x_2 + v_1x_3 + \cdots + v_jx_{j+2} = 0; \\ \cdots; \\ v_0x_n + v_1x_{n+1} + \cdots + v_jx_{j+n} = 0. \end{cases} \tag{20}$$

A similar argument (using $R(v) = (v_0, v_1, \ldots, v_{m-1})$) shows that the strongly nice tuples are the $(m+n+1)$-tuples $x = (x_0, x_1, \ldots, x_{m+n}) \in F^{m+n+1}$ satisfying

$$x_i = a_i \qquad \text{for each } i \in \{0, 1, \ldots, k-1\}$$

as well as

$$\begin{cases} v_0x_0 + v_1x_1 + \cdots + v_jx_j = 0; \\ v_0x_1 + v_1x_2 + \cdots + v_jx_{j+1} = 0; \\ v_0x_2 + v_1x_3 + \cdots + v_jx_{j+2} = 0; \\ \cdots; \\ v_0x_n + v_1x_{n+1} + \cdots + v_jx_{j+n} = 0; \\ v_0x_{n+1} + v_1x_{n+2} + \cdots + v_jx_{j+n+1} = 0. \end{cases} \tag{21}$$

These characterizations of weakly and strongly nice tuples are very similar: The system (21) consists of all the equations of (20) as well as one extra equation

$$v_0x_{n+1} + v_1x_{n+2} + \cdots + v_jx_{j+n+1} = 0. \tag{22}$$

This latter equation (22) uniquely determines the entry x_{j+n+1} in terms of the other entries of x (since $v_j \neq 0$), whereas x_{j+n+1} is entirely unconstrained by the system

(20). Thus, the entry x_{j+n+1} is uniquely determined (in terms of the other entries of x) in a strongly nice tuple x, while being entirely unconstrained in a weakly nice tuple.[7] Informally speaking, this shows that a weakly nice tuple has "one more degree of freedom" than a strongly nice tuple (and this degree of freedom is the entry x_{j+n+1}, which can take q possible values in a weakly nice tuple). This easily entails that the number of weakly nice tuples equals q times the number of strongly nice tuples.[8] This proves Lemma 3.5. □

Lemma 3.5 and Lemma 3.4 combined lead to the following:

Lemma 3.6. *Let $k, m, n \in \mathbb{N}$ satisfy $k \leq m$ and $k \leq n+1$. Fix any k-tuple $a \in F^k$. Then,*

$$\sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v\in F^{1\times(m+1)};\\ v\neq 0}} [v\ H_{m,n}(x)=0] - q \sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v\in F^{1\times m};\\ v\neq 0}} [v\ H_{m-1,n+1}(x)=0]$$
$$= (q-1)\, q^{2m-k}.$$

Proof of Lemma 3.6. We first observe that

$$\begin{aligned}&\left(\text{the number of all vectors } v\in F^{1\times(m+1)} \text{ satisfying } \operatorname{last} v\neq 0\right)\\ &=(q-1)\,q^m \end{aligned} \tag{23}$$

[7] Here we are using the fact that the "$x_i = a_i$ for each $i \in \{0, 1, \ldots, k-1\}$" conditions don't constrain x_{j+n+1} either (since $j+n+1 > k-1$).

[8] Here is a rigorous way to show this: Consider the map

$$\alpha : F \times \{\text{strongly nice tuples}\} \to \{\text{weakly nice tuples}\},$$
$$(y, (x_0, x_1, \ldots, x_{m+n})) \mapsto (x_0, x_1, \ldots, x_{j+n}, y, x_{j+n+2}, x_{j+n+3}, \ldots, x_{m+n}),$$

which simply replaces the entry x_{j+n+1} of the strongly nice tuple $(x_0, x_1, \ldots, x_{m+n})$ by the element y. Consider the map

$$\beta : \{\text{weakly nice tuples}\} \to F \times \{\text{strongly nice tuples}\},$$
$$(x_0, x_1, \ldots, x_{m+n}) \mapsto (x_{j+n+1}, (x_0, x_1, \ldots, x_{j+n}, z, x_{j+n+2}, x_{j+n+3}, \ldots, x_{m+n})),$$

where z is the unique element of F that would make the equation (22) valid when it is substituted for x_{j+n+1} (that is, explicitly, z is given by the formula $z = -(v_0 x_{n+1} + v_1 x_{n+2} + \cdots + v_{j-1} x_{j+n}) / v_j$). Our above characterizations of weakly nice and strongly nice tuples show that these two maps α and β are mutually inverse. Hence, α and β are bijections. Thus,

$$\begin{aligned}|\{\text{weakly nice tuples}\}| &= |F \times \{\text{strongly nice tuples}\}|\\ &= q \cdot |\{\text{strongly nice tuples}\}|.\end{aligned}$$

In other words, the number of weakly nice tuples equals q times the number of strongly nice tuples.

(since a vector $v \in F^{1\times(m+1)}$ satisfying $\operatorname{last} v \neq 0$ can be constructed by choosing its last entry from the $(q-1)$-element set $F \setminus \{0\}$ and then choosing its remaining m entries from the q-element set F).

For any row vector $v \in F^{1\times(m+1)}$, we define a number

$$\chi_v := \sum_{\substack{x \in F^{m+n+1};\\ x_{[0,k)}=a}} [v\ H_{m,n}(x) = 0]. \tag{24}$$

Thus, if $v \in F^{1\times(m+1)}$ is a row vector satisfying $\operatorname{last} v \neq 0$, then

$$\chi_v = \sum_{\substack{x \in F^{m+n+1};\\ x_{[0,k)}=a}} [v\ H_{m,n}(x) = 0] = q^{m-k} \tag{25}$$

(by Lemma 3.4).

Recall that $R : \left\{v \in F^{1\times(m+1)} \mid \operatorname{last} v = 0\right\} \to F^{1\times m}$ is a bijection. This bijection sends 0 to 0, and therefore restricts to a bijection

$$\left\{v \in F^{1\times(m+1)} \mid \operatorname{last} v = 0 \text{ and } v \neq 0\right\} \to \left\{v \in F^{1\times m} \mid v \neq 0\right\},$$
$$v \mapsto R(v).$$

Hence, given any $x \in F^{m+n+1}$, we can substitute $R(v)$ for v in the sum $\sum_{\substack{v \in F^{1\times m};\\ v\neq 0}} [v\ H_{m-1,n+1}(x) = 0]$, and thus obtain

$$\sum_{\substack{v \in F^{1\times m};\\ v\neq 0}} [v\ H_{m-1,n+1}(x) = 0] = \sum_{\substack{v \in F^{1\times(m+1)};\\ \operatorname{last} v=0;\\ v\neq 0}} [R(v)\ H_{m-1,n+1}(x) = 0].$$

Thus,

$$\begin{aligned}
&q \sum_{\substack{x \in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v \in F^{1\times m};\\ v\neq 0}} [v\ H_{m-1,n+1}(x) = 0] \\
&= q \sum_{\substack{x \in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v \in F^{1\times(m+1)};\\ \operatorname{last} v=0;\\ v\neq 0}} [R(v)\ H_{m-1,n+1}(x) = 0] \\
&= \sum_{\substack{v \in F^{1\times(m+1)};\\ \operatorname{last} v=0;\\ v\neq 0}} \underbrace{q \sum_{\substack{x \in F^{m+n+1};\\ x_{[0,k)}=a}} [R(v)\ H_{m-1,n+1}(x) = 0]}_{\substack{= \sum_{\substack{x \in F^{m+n+1};\\ x_{[0,k)}=a}} [v\ H_{m,n}(x)=0] \\ \text{(by Lemma 3.5)}}}
\end{aligned}$$

$$= \sum_{\substack{v \in F^{1\times(m+1)};\\ \text{last } v=0;\\ v\neq 0}} \underbrace{\sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} [v\ H_{m,n}(x)=0]}_{\substack{=\chi_v\\ \text{(by (24))}}}$$

$$= \sum_{\substack{v \in F^{1\times(m+1)};\\ \text{last } v=0;\\ v\neq 0}} \chi_v = \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0;\\ \text{last } v=0}} \chi_v. \tag{26}$$

On the other hand,

$$\sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0}} [v\ H_{m,n}(x)=0]$$

$$= \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0}} \underbrace{\sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} [v\ H_{m,n}(x)=0]}_{\substack{=\chi_v\\ \text{(by (24))}}}$$

$$= \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0}} \chi_v. \tag{27}$$

Subtracting the equality (26) from the equality (27), we obtain

$$\sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0}} [v\ H_{m,n}(x)=0] - q \sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v \in F^{1\times m};\\ v\neq 0}} [v\ H_{m-1,n+1}(x)=0]$$

$$= \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0}} \chi_v - \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0;\\ \text{last } v=0}} \chi_v = \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0;\\ \text{last } v\neq 0}} \chi_v$$

(since $\sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0}} \rho_v - \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0;\\ \text{last } v=0}} \rho_v = \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0;\\ \text{last } v\neq 0}} \rho_v$ for any numbers ρ_v). This further becomes

$$\sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0}} [v\ H_{m,n}(x)=0] - q \sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v \in F^{1\times m};\\ v\neq 0}} [v\ H_{m-1,n+1}(x)=0]$$

$$= \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0;\\ \text{last } v\neq 0}} \underbrace{\chi_v}_{\substack{=q^{m-k}\\ \text{(by (25))}}} = \sum_{\substack{v \in F^{1\times(m+1)};\\ v\neq 0;\\ \text{last } v\neq 0}} q^{m-k} = \sum_{\substack{v \in F^{1\times(m+1)};\\ \text{last } v\neq 0}} q^{m-k}$$

(here, we have removed the condition "$v \neq 0$" from under the summation sign, since any vector $v \in F^{1\times(m+1)}$ satisfying $\operatorname{last} v \neq 0$ automatically satisfies $v \neq 0$). This further simplifies to

$$\begin{aligned}
&\sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v\in F^{1\times(m+1)};\\ v\neq 0}} [v\ H_{m,n}(x)=0] - q \sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \sum_{\substack{v\in F^{1\times m};\\ v\neq 0}} [v\ H_{m-1,n+1}(x)=0] \\
&= \sum_{\substack{v\in F^{1\times(m+1)};\\ \operatorname{last} v\neq 0}} q^{m-k} \\
&= \underbrace{\left(\text{the number of all vectors } v\in F^{1\times(m+1)} \text{ satisfying } \operatorname{last} v\neq 0\right)}_{\substack{=(q-1)q^m\\ (\text{by (23)})}} \cdot q^{m-k} \\
&= (q-1)\underbrace{q^m\cdot q^{m-k}}_{=q^{2m-k}} = (q-1)\,q^{2m-k}.
\end{aligned}$$

This proves Lemma 3.6. $\square$

3.4. *Theorem 1.5 for $r = m$*

Before we prove Theorem 1.5 in full generality, let us first show it in the particular case when $r = m$:

Lemma 3.7. *Let $k, m, n \in \mathbb{N}$ satisfy $k \leq m \leq n+1$. Fix any k-tuple $a \in F^k$. The number of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ and $\operatorname{rank}(H_{m,n}(x)) \leq m$ is q^{2m-k}.*

Proof of Lemma 3.7. Write the k-tuple a in the form $a = (a_0, a_1, \ldots, a_{k-1})$.

We have

$$\begin{aligned}
&(q-1)\cdot \sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} [\operatorname{rank}(H_{m,n}(x))\leq m] \\
&= \sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \underbrace{(q-1)\cdot[\operatorname{rank}(H_{m,n}(x))\leq m]}_{\substack{=\sum_{\substack{v\in F^{1\times(m+1)};\\ v\neq 0}} [v\ H_{m,n}(x)=0] - q\sum_{\substack{v\in F^{1\times m};\\ v\neq 0}} [v\ H_{m-1,n+1}(x)=0]\\ (\text{by Proposition 3.3})}} \\
&= \sum_{\substack{x\in F^{m+n+1};\\ x_{[0,k)}=a}} \left(\sum_{\substack{v\in F^{1\times(m+1)};\\ v\neq 0}} [v\ H_{m,n}(x)=0] - q\sum_{\substack{v\in F^{1\times m};\\ v\neq 0}} [v\ H_{m-1,n+1}(x)=0]\right)
\end{aligned}$$

$$= \sum_{\substack{x \in F^{m+n+1}; \\ x_{[0,k)}=a}} \sum_{\substack{v \in F^{1\times(m+1)}; \\ v \neq 0}} [v\ H_{m,n}(x) = 0] - q \sum_{\substack{x \in F^{m+n+1}; \\ x_{[0,k)}=a}} \sum_{\substack{v \in F^{1\times m}; \\ v \neq 0}} [v\ H_{m-1,n+1}(x) = 0]$$

$$= (q-1)\, q^{2m-k} \qquad \text{(by Lemma 3.6)}.$$

Canceling $q-1$ from this equality (since $q-1 \neq 0$), we obtain

$$\sum_{\substack{x \in F^{m+n+1}; \\ x_{[0,k)}=a}} [\operatorname{rank}(H_{m,n}(x)) \leq m] = q^{2m-k}.$$

But the left hand side of this equality is the number of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $x_{[0,k)} = a$ and $\operatorname{rank}(H_{m,n}(x)) \leq m$ (because of Proposition 3.2). Thus, this number is q^{2m-k}. This proves Lemma 3.7. □

4. Proofs of the main results

We can now prove the results from Section 1 in their full generality.

Proof of Theorem 1.5. Let $s = m+n-r$. Then, $r+s = m+n$. Also, $r \leq s$ (since $\underbrace{s}_{=m+n-r} - r = m+n - \underbrace{r}_{\leq m} - \underbrace{r}_{\leq n} \geq m+n-m-n = 0$), so that $r \leq s \leq s+1$ and thus $k \leq r \leq s+1$.

Lemma 2.6 (applied to $u = m+n$) yields the logical equivalence

$$(\operatorname{rank}(H_{m,n}(x)) \leq r) \iff (\operatorname{rank}(H_{r,s}(x)) \leq r)$$

for any $(m+n+1)$-tuple $x \in F^{m+n+1}$. Thus,[9]

$$\begin{aligned}
&\big(\#\text{ of all } (m+n+1)\text{-tuples } x \in F^{m+n+1} \text{ satisfying } x_{[0,k)} = a \\
&\qquad \text{and } \operatorname{rank}(H_{m,n}(x)) \leq r\big) \\
&= \big(\#\text{ of all } (m+n+1)\text{-tuples } x \in F^{m+n+1} \text{ satisfying } x_{[0,k)} = a \\
&\qquad \text{and } \operatorname{rank}(H_{r,s}(x)) \leq r\big) \\
&= \big(\#\text{ of all } (r+s+1)\text{-tuples } x \in F^{r+s+1} \text{ satisfying } x_{[0,k)} = a \\
&\qquad \text{and } \operatorname{rank}(H_{r,s}(x)) \leq r\big) \qquad \text{(since } m+n = r+s\text{)} \\
&= q^{2r-k} \qquad \text{(by Lemma 3.7, applied to } r \text{ and } s \text{ instead of } m \text{ and } n\text{)}.
\end{aligned}$$

This proves Theorem 1.5. □

[9] The symbol "#" means "number".

Proof of Theorem 1.1. Let a be the 0-tuple $() \in F^0$. Thus, Theorem 1.5 (applied to $k = 0$) yields that the number of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $x_{[0,0)} = a$ and $\operatorname{rank}(H_{m,n}(x)) \leq r$ is q^{2r-0}. We can remove the "$x_{[0,0)} = a$" condition from the previous sentence (since **every** $(m+n+1)$-tuple $x \in F^{m+n+1}$ satisfies $x_{[0,0)} = () = a$), and thus obtain the following: The number of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $\operatorname{rank}(H_{m,n}(x)) \leq r$ is q^{2r-0}. But this is precisely the claim of Theorem 1.1 (since $2r - 0 = 2r$). Thus, Theorem 1.1 is proved. $\square$

Proof of Corollary 1.3. We need to prove the following four claims[10]:

Claim 1. *If $r = 0$, then the # of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $\operatorname{rank}(H_{m,n}(x)) = r$ is 1.*

Claim 2. *If $0 < r \leq m$, then the # of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $\operatorname{rank}(H_{m,n}(x)) = r$ is $q^{2r-2}(q^2 - 1)$.*

Claim 3. *If $r = m+1$, then the # of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $\operatorname{rank}(H_{m,n}(x)) = r$ is $q^{2r-2}(q^{n-m+1} - 1)$.*

Claim 4. *If $r > m+1$, then the # of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $\operatorname{rank}(H_{m,n}(x)) = r$ is 0.*

[**Proof of Claim 1:** We need to show that the # of $(m+n+1)$-tuples $x \in F^{m+n+1}$ satisfying $\operatorname{rank}(H_{m,n}(x)) = 0$ is 1. In other words, we need to show that there is exactly one $(m+n+1)$-tuple $x \in F^{m+n+1}$ satisfying $\operatorname{rank}(H_{m,n}(x)) = 0$. But this is rather simple: The $(m+n+1)$-tuple $(0, 0, \ldots, 0) \in F^{m+n+1}$ does satisfy $\operatorname{rank}(H_{m,n}(x)) = 0$ (since $H_{m,n}(x)$ is the zero matrix when x is this $(m+n+1)$-tuple), and no other $(m+n+1)$-tuple does this (because if $x \in F^{m+n+1}$ is not $(0, 0, \ldots, 0)$, then the matrix $H_{m,n}(x)$ has at least one nonzero entry, and therefore its rank cannot be 0). Thus, Claim 1 is proved.]

[**Proof of Claim 2:** Assume that $0 < r \leq m$. Thus, r and $r-1$ are elements of $\mathbb{N}$ and satisfy $r \leq m \leq n$ and $r - 1 \leq r \leq m \leq n$. Hence:

- Theorem 1.1 yields that

$$\begin{aligned}&\left(\#\text{ of }(m+n+1)\text{-tuples } x \in F^{m+n+1} \text{ satisfying } \operatorname{rank}(H_{m,n}(x)) \leq r\right)\\&= q^{2r}.\end{aligned}$$

- Theorem 1.1 (applied to $r-1$ instead of r) yields that

$$\left(\#\text{ of }(m+n+1)\text{-tuples } x \in F^{m+n+1} \text{ satisfying } \operatorname{rank}(H_{m,n}(x)) \leq r-1\right)$$

[10] The symbol "#" means "number".

$$= q^{2(r-1)}.$$

However, a matrix A satisfies $\operatorname{rank} A = r$ if and only if it satisfies $\operatorname{rank} A \leq r$ but not $\operatorname{rank} A \leq r-1$. Hence,

$$\begin{aligned}
&\left(\# \text{ of } (m+n+1)\text{-tuples } x \in F^{m+n+1} \text{ satisfying } \operatorname{rank}(H_{m,n}(x)) = r\right) \\
&= \underbrace{\left(\# \text{ of } (m+n+1)\text{-tuples } x \in F^{m+n+1} \text{ satisfying } \operatorname{rank}(H_{m,n}(x)) \leq r\right)}_{=q^{2r}} \\
&\qquad - \underbrace{\left(\# \text{ of } (m+n+1)\text{-tuples } x \in F^{m+n+1} \text{ satisfying } \operatorname{rank}(H_{m,n}(x)) \leq r-1\right)}_{=q^{2(r-1)}} \\
&= q^{2r} - q^{2(r-1)} = q^{2r-2}\left(q^2 - 1\right).
\end{aligned}$$

This proves Claim 2.]

[**Proof of Claim 3:** Assume that $r = m+1$. Thus, $2r = 2(m+1) = 2m+2$, so that $2m = 2r-2$. The matrix $H_{m,n}(x)$ (for any given x) is an $(m+1) \times (n+1)$-matrix; thus, its rank is always $\leq m+1$. Hence, it has rank $m+1$ if and only if it does not have rank $\leq m$. Thus,

$$\begin{aligned}
&\left(\# \text{ of } (m+n+1)\text{-tuples } x \in F^{m+n+1} \text{ satisfying } \operatorname{rank}(H_{m,n}(x)) = m+1\right) \\
&= \underbrace{\left(\# \text{ of all } (m+n+1)\text{-tuples } x \in F^{m+n+1}\right)}_{\substack{=q^{m+n+1} \\ (\text{since } |F|=q)}} \\
&\qquad - \underbrace{\left(\# \text{ of } (m+n+1)\text{-tuples } x \in F^{m+n+1} \text{ satisfying } \operatorname{rank}(H_{m,n}(x)) \leq m\right)}_{\substack{=q^{2m} \\ (\text{by Theorem 1.1, applied to } m \text{ instead of } r)}} \\
&= q^{m+n+1} - q^{2m} = q^{2m}\left(q^{n-m+1} - 1\right) = q^{2r-2}\left(q^{n-m+1} - 1\right)
\end{aligned}$$

(since $2m = 2r-2$). But this is precisely the claim of Claim 3 (since $r = m+1$). Thus, Claim 3 is proven.]

[**Proof of Claim 4:** Assume that $r > m+1$. The matrix $H_{m,n}(x)$ (for any given x) is an $(m+1) \times (n+1)$-matrix; thus, its rank is always $\leq m+1$. Hence, its rank is never r (because $r > m+1$). Thus,

$$\left(\# \text{ of } (m+n+1)\text{-tuples } x \in F^{m+n+1} \text{ satisfying } \operatorname{rank}(H_{m,n}(x)) = r\right) = 0.$$

This proves Claim 4.]

Having proved all four claims, we thus have completed the proof of Corollary 1.3. $\square$

Proof of Corollary 1.7. If $x \in F^{2n+1}$ is any $(2n+1)$-tuple, then the condition "$\det(H_{n,n}(x)) = 0$" is equivalent to "$\operatorname{rank}(H_{n,n}(x)) \leq n$" (since $H_{n,n}(x)$ is an

$(n+1) \times (n+1)$-matrix, and thus its determinant vanishes if and only if its rank is $\leq n$). Hence, the number of $(2n+1)$-tuples $x \in F^{2n+1}$ satisfying $x_{[0,k)} = a$ and $\det(H_{n,n}(x)) = 0$ is precisely the number of $(2n+1)$-tuples $x \in F^{2n+1}$ satisfying $x_{[0,k)} = a$ and $\operatorname{rank}(H_{n,n}(x)) \leq n$. But Theorem 1.5 (applied to $m = n$ and $r = n$) shows that the latter number is q^{2n-k}. This proves Corollary 1.7. □

5. Application to Jacobi–Trudi matrices

Let us now discuss how [1, Corollary 6.4] follows from Corollary 1.7. For the sake of simplicity, we shall first restate [1, Corollary 6.4] in a self-contained form that does not rely on the concepts of symmetric functions:

Corollary 5.1. *Assume that F is finite. Let $q = |F|$. Let $u, v \in \mathbb{N}$. For each $(u+v-1)$-tuple $y = (y_1, y_2, \ldots, y_{u+v-1}) \in F^{u+v-1}$, we define the matrix*

$$J_{u,v}(y) := (y_{u-i+j})_{1 \leq i \leq v,\ 1 \leq j \leq v} \in F^{v \times v},$$

where we set $y_0 := 1$ and $y_k := 0$ for all $k < 0$.

Then, the number of all $(u+v-1)$-tuples $y \in F^{u+v-1}$ satisfying $\det(J_{u,v}(y)) = 0$ is q^{u+v-2}.

Example 5.2. (a) If $u = 1$ and $v = 3$, then each 3-tuple $y = (y_1, y_2, y_3) \in F^3$ satisfies

$$J_{u,v}(y) = J_{1,3}(y) = (y_{1-i+j})_{1 \leq i \leq 3,\ 1 \leq j \leq 3} = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_0 & y_1 & y_2 \\ y_{-1} & y_0 & y_1 \end{pmatrix}$$

$$= \begin{pmatrix} y_1 & y_2 & y_3 \\ 1 & y_1 & y_2 \\ 0 & 1 & y_1 \end{pmatrix} \qquad (\text{since } y_0 = 1 \text{ and } y_{-1} = 0)$$

and thus $\det(J_{u,v}(y)) = y_3 + y_1^3 - 2y_1 y_2$.

(b) If $u = 4$ and $v = 3$, then each 6-tuple $y = (y_1, y_2, \ldots, y_6) \in F^6$ satisfies

$$J_{u,v}(y) = J_{4,3}(y) = (y_{4-i+j})_{1 \leq i \leq 3,\ 1 \leq j \leq 3} = \begin{pmatrix} y_4 & y_5 & y_6 \\ y_3 & y_4 & y_5 \\ y_2 & y_3 & y_4 \end{pmatrix}$$

and thus $\det(J_{u,v}(y)) = y_6 y_3^2 - 2y_3 y_4 y_5 + y_4^3 - y_2 y_6 y_4 + y_2 y_5^2$.

Why is Corollary 5.1 equivalent to [1, Corollary 6.4]? In fact, Corollary 5.1 can be restated in probabilistic terms; then it says that a uniformly random $(u+v-1)$-tuple $y \in F^{u+v-1}$ satisfies $\det(J_{u,v}(y)) = 0$ with a probability of $\dfrac{q^{u+v-2}}{q^{u+v-1}} = \dfrac{1}{q}$. However, the matrix $J_{u,v}(y)$ in Corollary 5.1 is precisely the Jacobi–Trudi matrix[11] corresponding to the rectangle-shaped partition (u^v), except that the entries of y have been substituted for the complete homogeneous symmetric functions $h_1, h_2, \ldots, h_{u+v-1}$. The determinant $\det(J_{u,v}(y))$ therefore is the image of the Schur function $s_{(u^v)}$ under this substitution. Thus, Corollary 5.1 says that when a uniformly random $(u+v-1)$-tuple of elements of F is substituted for $(h_1, h_2, \ldots, h_{u+v-1})$, the Schur function $s_{(u^v)}$ becomes 0 with a probability of $\frac{1}{q}$. This is precisely the claim of [1, Corollary 6.4].

We shall now sketch (on an example) how Corollary 5.1 can be derived from our Corollary 1.7:

Proof of Corollary 5.1 (sketched). For a sufficiently representative example, we pick the case when $u = 2$ and $v = 5$; the reader will not find any difficulty in generalizing our reasoning to the general case.

Thus, we must show that the number of all 6-tuples $y \in F^6$ satisfying $\det(J_{2,5}(y)) = 0$ is q^5. Let $y = (y_1, y_2, \ldots, y_6) \in F^6$ be any 6-tuple. Then,

$$J_{2,5}(y) = \begin{pmatrix} y_2 & y_3 & y_4 & y_5 & y_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ y_0 & y_1 & y_2 & y_3 & y_4 \\ y_{-1} & y_0 & y_1 & y_2 & y_3 \\ y_{-2} & y_{-1} & y_0 & y_1 & y_2 \end{pmatrix} = \begin{pmatrix} y_2 & y_3 & y_4 & y_5 & y_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ 1 & y_1 & y_2 & y_3 & y_4 \\ 0 & 1 & y_1 & y_2 & y_3 \\ 0 & 0 & 1 & y_1 & y_2 \end{pmatrix}$$

(since $y_0 = 1$ and $y_{-1} = 0$ and $y_{-2} = 0$). If we turn the matrix $J_{2,5}(y)$ upside down (i.e., we reverse the order of its rows), then we obtain the matrix

$$\begin{pmatrix} 0 & 0 & 1 & y_1 & y_2 \\ 0 & 1 & y_1 & y_2 & y_3 \\ 1 & y_1 & y_2 & y_3 & y_4 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ y_2 & y_3 & y_4 & y_5 & y_6 \end{pmatrix},$$

which is precisely the Hankel matrix $H_{4,4}(x)$ for the 9-tuple

$$x = (0, 0, 1, y_1, y_2, y_3, y_4, y_5, y_6).$$

Hence, this 9-tuple x satisfies $\det(H_{4,4}(x)) = \pm\det(J_{2,5}(y))$ (since the determinant of a matrix is multiplied by ± 1 when the rows of the matrix are permuted). Therefore, the condition "$\det(J_{2,5}(y)) = 0$" is equivalent to the condition "$\det(H_{4,4}(x)) = 0$" for this 9-tuple x. Hence, the number of all 6-tuples $y \in F^6$ satisfying $\det(J_{2,5}(y)) = 0$ is precisely the number of all 9-tuples $x \in F^9$ that start with the entries $0, 0, 1$ and satisfy

[11] We are using the terminology of [1] here.

$\det(H_{4,4}(x)) = 0$. In other words, it is precisely the number of all 9-tuples $x \in F^9$ satisfying $x_{[0,3)} = (0, 0, 1)$ and $\det(H_{4,4}(x)) = 0$. However, Corollary 1.7 (applied to $k = 3$ and $n = 4$ and $a = (0, 0, 1)$) shows that the latter number is $q^{2\cdot 4-3} = q^5$. This is precisely what we wanted to show. Thus, Corollary 5.1 is proved. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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