Calculus II: Improper Integrals

Definition (Improper Integral of Type I)

Integrals with infinite limits of integration are **Improper Integrals of Type I**.

1) If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

2) If $f(x)$ is continuous on $(-\infty,b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

3) If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

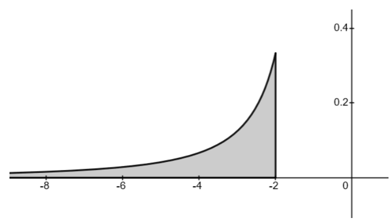
where c is any real number.

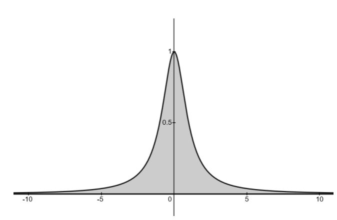
In each case, if the limit exists and is finite, we say that the improper integral **converges** and that limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Note If $f(x) \geqslant 0$ on the interval of integration, then we can interpret an improper integral as an area under the curve $y = f(x)$.

e.g. Below are some examples of convergent improper integrals:

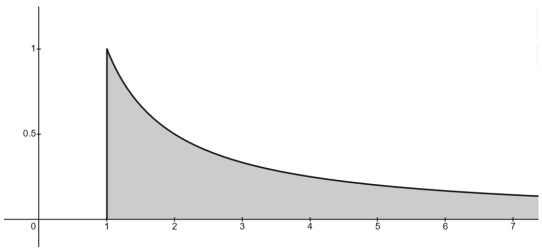
$$\int_1^\infty \frac{\ln x}{x^2}\,dx \qquad \int_{-\infty}^{-2} \frac{1}{x^2-1}\,dx \qquad \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$





Question What happens when an improper integral diverges? What does the region under the curve look like?

e.g. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges!



Let's compare $\int_1^\infty \frac{1}{x}\,dx$ and $\int_1^\infty \frac{1}{x^2}\,dx$.

Example Show that the improper integral diverges.

$$\int_1^\infty \frac{1}{x}\,dx$$