Bimatrix with Fixed Flowing Number

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Suppose the number of pure NE is m and the number of mixed NE is n. Let

$$A_{ij} = \begin{cases} n+1, & i \le j \le m \\ k, & i \le j, j = m+2n-2k+1, i+j \text{ is even} \\ k, & i \le j, j = m+2n-2k+2, i+j \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{ij} = \begin{cases} n+1, & j \le i \le m \\ k, & j \le i+1, i = m+2n-2k+1, i+j \text{ is odd} \\ k, & j \le i-1, i = m+2n-2k+2, i+j \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

For example, if $m = 4, n = 2$, the bimatrix should be

$$A = \begin{bmatrix} 3 & 3 & 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 3 & 3 & 3 & 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Now, let

$$A_1 = A[1:m, 1:m], A_2 = A[m+1:m+n, m+1:m+n]$$

$$B_1 = B[1:m, 1:m], B_2 = B[m+1:m+n, m+1:m+n]$$

$$P = A[1:m, m+1:m+n], Q = B[m+1:m+n, 1:m]$$

Hence,

$$A = \begin{bmatrix} A_1 & P \\ O & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1 & O \\ Q & B_2 \end{bmatrix}$$

Theorem 1 *Bimatrix (A_1, B_1) has m pure NE and 0 mixed NE.*

Proof: It's clearly that (A_1, B_1) has m pure NE. Suppose (A_1, B_1) has an NE $(\mathbf{x}, \mathbf{y})$. Let $i^* = \min SI(\mathbf{x})$. For any $i > i^*$, we can derive that

$$(\mathbf{x}^T B_1)_{i^*} > (\mathbf{x}^T B_1)_i$$

Hence,

$$\max BRI(\mathbf{x}) \leq i^*$$

Since $SI(\mathbf{y}) \subseteq BRI(\mathbf{x})$, there must be

$$\max SI(\mathbf{y}) \leq i^*$$

Let $j^* = \min SI(\mathbf{y})$. Similarly, we can derive that

$$\max SI(\mathbf{x}) \leq j^*$$

Thus,

$$\max SI(\mathbf{x}) \leq \min SI(\mathbf{y}) \leq \max SI(\mathbf{y}) \leq \min SI(\mathbf{x})$$

which leads to

$$|SI(\mathbf{x})| = |SI(\mathbf{y})| = 1$$

$(\mathbf{x}, \mathbf{y})$ is always a pure NE. $\square$

Here we define a new notation:

$$SI_1(\mathbf{x}) = SI(\mathbf{x}) \cap \{2k - 1 \mid k \in \mathbb{Z}^+\}$$

$$SI_2(\mathbf{x}) = SI(\mathbf{x}) \cap \{2k \mid k \in \mathbb{Z}^+\}$$

$$SI_1(\mathbf{y}) = SI(\mathbf{y}) \cap \{2k - 1 \mid k \in \mathbb{Z}^+\}$$

$$SI_2(\mathbf{y}) = SI(\mathbf{y}) \cap \{2k \mid k \in \mathbb{Z}^+\}$$

Lemma 2 *For any NE $(\mathbf{x}, \mathbf{y})$ of bimatrix (A_2, B_2). None of $SI_1(\mathbf{x})$, $SI_2(\mathbf{x})$, $SI_1(\mathbf{y})$, and $SI_2(\mathbf{y})$ is empty.*

Proof:

1. If $SI_1(\mathbf{x})$ is empty, for any even number j,

$$(\mathbf{x}^T B_2)_j = 0$$

   Hence, $SI_2(\mathbf{y}) = \emptyset$.

2. If $SI_2(\mathbf{y})$ is empty, for any even number i,

$$(A_2 \mathbf{y})_i = 0$$

   Hence, $SI_2(\mathbf{x}) = \emptyset$.

3. If $SI_2(\mathbf{x})$ is empty, for any odd number j,

$$(\mathbf{x}^T B_2)_j = 0$$

   Hence, $SI_1(\mathbf{y}) = \emptyset$.

4. If $SI_1(\mathbf{y})$ is empty, for any odd number i,

$$(A_2 \mathbf{y})_i = 0$$

   Hence, $SI_1(\mathbf{x}) = \emptyset$.

As a result, if any of these four sets is empty, all of them are empty. However, neither $SI(\mathbf{x})$ nor $SI(\mathbf{y})$ is an empty set. Contradiction. $\square$

Theorem 3 *Bimatrix (A_2, B_2) has 0 pure NE and n mixed NE.*

Proof: The proof of (A_2, B_2) has 0 pure NE is trivial.

According to Lemma 2, all of $SI_1(\mathbf{x})$, $SI_2(\mathbf{x})$, $SI_1(\mathbf{y})$, and $SI_2(\mathbf{y})$ are non-empty sets.

Let $i_1 = \min SI_1(\mathbf{x})$. For any $j > i_1 + 2k + 1 (k \in \mathbb{Z}^+)$,

$$(\mathbf{x}^T B_2)_{i_1+1} > (\mathbf{x}^T B_2)_j$$

Hence,

$$\max SI_2(\mathbf{y}) \leq i_1 + 1$$

Let $j_2 = \min SI_2(\mathbf{y})$. For any $i > j_2 + 2k (k \in \mathbb{Z}^+)$,

$$(A_2\mathbf{y})_{j_2} > (A_2\mathbf{y})_i$$

Hence,

$$\max SI_2(\mathbf{x}) \leq j_2$$

Let $i_2 = \min SI_2(\mathbf{x})$. For any $j > i_2 + 2k - 1 (k \in \mathbb{Z}^+)$,

$$(\mathbf{x}^T B_2)_{i_2-1} > (\mathbf{x}^T B_2)_j$$

Hence,

$$\max SI_1(\mathbf{y}) \leq i_2 - 1$$

Let $j_1 = \min SI_1(\mathbf{y})$. For any $i > j_1 + 2k (k \in \mathbb{Z}^+)$,

$$(A_2\mathbf{y})_{j_1} > (A_2\mathbf{y})_i$$

Hence,

$$\max SI_1(\mathbf{x}) \leq j_1$$

As a result,

$$\begin{aligned} &\max SI_1(\mathbf{x}) \leq \min SI_1(\mathbf{y}) \leq \max SI_1(\mathbf{y}) \leq \min SI_2(\mathbf{x}) - 1 \\ \leq &\max SI_2(\mathbf{x}) - 1 \leq \min SI_2(\mathbf{y}) - 1 \leq \max SI_2(\mathbf{y}) - 1 \leq \min SI_1(\mathbf{x}) \end{aligned}$$

Therefore, all of these four sets have exactly one element satisfy that

$$i_1 = j_1 = i_2 - 1 = j_2 - 1$$

where $i_1 \in SI_1(\mathbf{x}), i_2 \in SI_2(\mathbf{x}), j_1 \in SI_1(\mathbf{y})$, and $j_2 \in SI_2(\mathbf{y})$.

Moreover, it's obviously that for any odd number i_1, the NE is unique. Hence, bimatrix (A_2, B_2) contains n mixed NE. $\square$

Lemma 4 *If* $(\mathbf{x}, \mathbf{y})$ *is the NE of bimatrix* (A, B) *with* $SI(\mathbf{x}) \cap \{2, \cdots, m\} \neq \emptyset$ *or* $SI(\mathbf{y}) \cap \{2, \cdots, m\} \neq \emptyset$ *,* $(\mathbf{x}, \mathbf{y})$ *is a pure NE.*

Proof: Let i be one of the elements of $SI(\mathbf{x}) \cap \{2, \cdots, m\} \neq \emptyset$. Here, $i \geq 2$.

If m is odd,

$$(\mathbf{x}^T B)_1 > (\mathbf{x}^T B)_{m+2} \geq (\mathbf{x}^T B)_{m+4} \geq \cdots$$

$$(\mathbf{x}^T B)_2 > (\mathbf{x}^T B)_{m+1} \geq (\mathbf{x}^T B)_{m+3} \geq \cdots$$

If m is even,

$$(\mathbf{x}^T B)_1 > (\mathbf{x}^T B)_{m+1} \geq (\mathbf{x}^T B)_{m+3} \geq \cdots$$

$$(\mathbf{x}^T B)_2 > (\mathbf{x}^T B)_{m+2} \geq (\mathbf{x}^T B)_{m+4} \geq \cdots$$

Hence,

$$\max SI(\mathbf{y}) \leq \max BRI(\mathbf{x}) \leq m$$

Since the second player only chooses coloms among the first m coloms, the first player has to choose rows among the first m rows. Otherwise, he will get nothing.

Now, they will play the game with bimatrix (A_1, B_1). According to Theorem 1, the NE should be pure.

The proof of $\mathbf{y}$ is quite similar. □

Theorem 5 *Bimatrix* (A, B) *has* m *pure NE and* n *mixed NE.*

Proof: It's obviously that the number of pure NE is m. Here we need to show that (A, B) has n mixed NE.

Case 1: $\min SI(\mathbf{x}) > m$ and $\min SI(\mathbf{y}) > m$. Under this situation, two players play the game with bimatrix (A_2, B_2), which has n mixed NE.

Case 2: $\min SI(\mathbf{x}) \leq m$. According to Lemma 4, $SI(\mathbf{x}) \cap \{2, \cdots, m\}$ has to be $\emptyset$ to generate mixed NE. Hence, 1 is the only element in $SI(\mathbf{x})$ which is not greater than m.

Here we suppose m is odd (the proof of another occasion is quite similar). For any $j = m + 2k, k \in \{1, 2, \cdots, n\}$,

$$(\mathbf{x}^T B)_1 > (\mathbf{x}^T B)_j$$

Hence, for any $j \in SI(\mathbf{y}) \cap \{m+1, m+2, \cdots, m+2n\}$, j must be even.

Also, $SI(\mathbf{y}) \cap \{1, 2, \cdots, m\} \subseteq \{1\}$ according to Lemma 4.

Here we can conclude that if $SI(\mathbf{x}) \cap \{1, 2, \cdots, m\} = \{1\}$, $SI(\mathbf{y})$ is the subset of $\{1, m + 2k - 1 : k \in \{1, 2, \cdots, n\}\}$.

Now, we can observe that for any odd number i in $\{m, m+1, \cdots, m+2n\}$,

$$(A\mathbf{y})_i = 0$$

Hence, for any $i \in SI(\mathbf{x}) \cap \{m + 1, m + 2, \cdots, m + 2n\}$, i must be even, which leads to for any $j \in \{m + 2k - 1 : k \in \{1, 2, \cdots, n\}\}$,

$$(\mathbf{x}^T B)_j = 0$$

As a result, $SI(\mathbf{y}) = \{1\}$, which leads to $SI(\mathbf{x}) = \{1\}$. It is a pure NE.

Case 3: $\min SI(\mathbf{y}) \leq m$. The proof is quite similar with the previous case.

As a result, Bimatrix (A, B) has m pure NE and n mixed NE. $\square$