MAT2040 Homework 1

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Question 1.

1.1) **False.** The system is inconsistent if it has no solutions.

1.2) **True.**

1.3) **False.** One possible non-e.g.: $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

1.4) **True.**

1.5) **False.** $AB \neq BA$.

Question 2.

Written in the form of **augmented matrix**, we have

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to \frac{R_2 - 2R_1}{3}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to \frac{R_2 - 2R_1}{3}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have identical RREFs, which indicates **equivalent** linear systems.

Question 3.

3.1) We have coefficient matrix $\begin{bmatrix} 4 & 5 & 3 & 3 & 4 \\ 2 & 3 & 1 & 0 & 1 \\ 3 & 4 & 2 & 1 & 1 \end{bmatrix}$ and the augmented matrix $\left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 2 & 3 & 1 & 0 & 1 & -3 \\ 3 & 4 & 2 & 1 & 1 & -1 \end{array}\right]$.

3.2) We have

$$\left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 2 & 3 & 1 & 0 & 1 & -3 \\ 3 & 4 & 2 & 1 & 1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 1 & -3 \\ 4 & 5 & 3 & 3 & 4 & -5 \\ 3 & 4 & 2 & 1 & 1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 2 & -1 & 7 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & -3 & -2 & -1 \\ 0 & 0 & 0 & 1 & 3 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 0 & 7 & -19 \\ 0 & 0 & 0 & 1 & 3 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -10 & 27 \\ 0 & 1 & -1 & 0 & 7 & -19 \\ 0 & 0 & 0 & 1 & 3 & -6 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = 27 - 2x_3 + 10x_5 \\ x_2 = -19 + x_3 - 7x_5 \\ x_4 = -6 - 3x_5 \end{cases}$$

Question 4.

We have the matrix to be

$$\left[\begin{array}{ccc|c} 3 & k & 1 & 0 \\ 0 & 4 & 1 & 0 \\ k & -5 & -1 & 0 \end{array}\right]$$

If $k = 0$, we have

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{(rejected)}$$

If $k \neq 0$, it could be reduced to the form of

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{3}k & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & k^2 - 4k + 3 & 0 \end{array}\right]$$

In order for the RREF to have the property that $r < n$, we have

$$k^2 - 4k + 3 = 0$$

such that it will become a 3×2 matrix of RREF, which has infinite solutions.
As a result,

$$k_1 = 1, \quad k_2 = 3$$

Question 5.

We have the matrix to be

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & p & q \end{array}\right]$$

By row operations, it could be reduced to the form of

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & p+4 & q-5 \end{array}\right]$$

5.1) For the RREF to become an inconsistent one, we have

$$p + 4 = 0$$

As a result,

$$p = -4$$

5.2) For the RREF to have a unique solution, we have

$$\begin{cases} p + 4 \neq 0 \\ q - 5 \neq 0 \end{cases}$$

As a result,

$$p \neq -4$$

5.3) For the RREF to have infinitely many solutions, we have

$$\begin{cases} p + 4 = 0 \\ q - 5 = 0 \end{cases}$$

As a result,

$$p = -4, \quad q = 5$$

Question 6.

We have the matrix to be

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 2 & 4 & b \end{bmatrix}$$

By row operations, it can be reduced to the form of

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & b-2 \end{bmatrix}$$

By **Thm**: An underdetermined homogeneous linear system has infinite solutions, we have

$$b - 2 = 0$$

As a result,

$$b = 2$$

Question 7.

We have the matrix to be

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & a & -1 \\ 2 & 3 & 0 & b \end{array}\right]$$

By row operations, it could be reduced to the form of

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & b-4 \\ 0 & 0 & a+1 & b-1 \end{array}\right]$$

7.1) For the RREF to become an inconsistent one, we have

$$\begin{cases} a+1=0 \\ b-1\neq 0 \end{cases}$$

As a result,

$$a=-1, \quad b\neq 1$$

7.2) For the RREF to have a unique solution, we have

$$a+1\neq 0$$

As a result,

$$a\neq -1$$

7.3) For the RREF to have infinitely many solutions, we have

$$\begin{cases} a+1=0 \\ b-1=0 \end{cases}$$

As a result,

$$a=-1, \quad b=1$$

Question 8.

We have the matrix to be

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & a & 0 \\ 1 & 4 & a^2 & 0 \\ 1 & 2 & 1 & a-1 \end{array}\right]$$

By row operations, it could be reduced to the form of

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & a-1 \\ 0 & 0 & a^2-3a+2 & 0 \\ 0 & 0 & -1+a & 1-a \end{array}\right]$$

For the RREF to have at least one solution: in the case $a = 1$, we have more than one solutions. In the case $a \neq 1$, we have

$$a^2 - 3a + 2 = 0 \Rightarrow a = 2$$

As a result,

$$a = 2 \text{ or } 1$$

When $a = 2$,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

When $a = 1$,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = 0 \end{cases}$$

Question 9.

We find A^2 to be

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

For an odd number k,

$$A^k = \underbrace{(AA)(AA)...(AA)}_{\frac{k-1}{2} groups} A = I^{\frac{k-1}{2}} A = IA = A$$

For a even number k,

$$A^k = \underbrace{(AA)(AA)...(AA)}_{\frac{k}{2} groups} = I^{\frac{k}{2}} = I$$

Question 10.

10.1) It is symmetric, as

$$(A^2 - B^2)^T = (A^2)^T - (B^2)^T = (A^T)^2 - (B^T)^2 = A^2 - B^2$$

10.2) It is NOT symmetric, as we have

$$(A + B)(A - B) = A^2 + BA - AB - B^2$$

Given $A^2 - B^2$ is symmetric,

$$(BA - AB)^T = (BA)^T - (AB)^T = A^T B^T - B^T A^T = AB - BA \neq BA - AB$$

10.3) It is symmetric, as

$$(ABA)^T = ((AB)A)^T = A^T(AB)^T = A^TB^TA^T = ABA$$

10.4) It is NOT symmetric, as

$$(ABAB)^T = (AB)^T(AB)^T = B^TA^TB^TA^T = BABA \neq ABAB$$

Question 11.

We have

$$A = ab^T = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 0 \end{bmatrix}$$

Thus we have

$$A^{11} = \underbrace{(ab^T)(ab^T)...(ab^T)}_{10groups} = a(b^Ta)^10b^T = 1024 \times \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1024 & 512 & 0 \\ 2048 & 1024 & 0 \\ 4096 & 2048 & 0 \end{bmatrix}$$

Question 12.

We find A^2 to be a diagonal matrix, as

$$A^2 = AA = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Thus

$$A^8 = (A^2)^4 = \begin{bmatrix} 6561 & 0 & 0 \\ 0 & 6561 & 0 \\ 0 & 0 & 6561 \end{bmatrix}$$

As a result,

$$A^8 - 6400I = \begin{bmatrix} 6561 & 0 & 0 \\ 0 & 6561 & 0 \\ 0 & 0 & 6561 \end{bmatrix} - 6400 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 161 & 0 & 0 \\ 0 & 161 & 0 \\ 0 & 0 & 161 \end{bmatrix}$$

Question 13.

As $A^k = 0$, we have

$$\begin{aligned}
&(I - A)(I + A + A^2 + ... + A^{k-1}) \\
&= (I^2 + IA + IA^2 + ... + IA^{k-1}) + (-AI - A^2 - ...A^k) \\
&= I^2 + A + A^2 + ... + A^{k-1} - A - A^2 - ... - A^k \\
&= I^2 \\
&= I
\end{aligned}$$

Question 14.

Assume a = $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, b= $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, we have

$$ab^T = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} = \begin{bmatrix} 65 & 91 & 78 \\ 70 & 98 & 84 \\ 60 & 84 & 72 \end{bmatrix}$$

Thus we have

$$a^Tb = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3 = 65 + 98 + 72 = 235$$

Question 15.

15.1)

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

15.2)

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Question 16.

We have

$$A = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$$

Question 17.

For $a_{i,j}$ in matrix $A_{m\times n}$, we have

$$A^TA = O \Rightarrow O_{ii} = \sum_{j=1}^{n} a_{ji}^2 = 0 \text{ for } i = 1, 2...n$$

As $a_{ji}^2 \geq 0$, we have

$$a_{ji} = 0 \text{ for } j = 1, 2, 3...n$$

As a result,

$$A_{ij} = 0 \quad \text{ for } i = 1, 2, 3...n;\ j = 1, 2, 3...n$$

It is equivalent to

$$A = O$$

Question 18.

Based on $(A+B)^2 = A^2 + B^2$,

$$(A+B)^2 = A^2 + B^2 + AB + BA = A^2 + B^2$$

which indicates that

$$AB + BA = 0$$

Thus

$$AB(A+I) = ABA + ABI = A(BA) + (AB)I = -A^2B + AB$$

As $A^2 = A$, we have

$$AB(A+I) = -AB + AB = O$$

Question 19.

We have

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Also we have

$$A^3 = A^2 A = IA = A$$

Thus we have

$$A^{2n} = I^n = I, \quad A^{2n+1} = A^{2n} A = IA = A$$

Question 20.

We have

$$(I - xy^T)(I + xy^T) = I - xy^T + xy^T - (xy^T)^2 = I - (xy^T)(xy^T) = I - x(y^T x)y^T = I$$

Thus we have

$$(I - xy^T)^{-1} = (I + xy^T)$$

****This is the end of Homework 1.****