# FIN2020 Homework 1

Xue Zhongkai (122090636)

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**Question 1.**

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**(a)** We have the Lagrangian multiplier for

$$L(c_0, c_1, c_2, \lambda) = ln(c_0) + ln(c_1) + ln(c_2) + \lambda[Y - (p_0c_0 + p_1c_1 + p_2c_2)]$$

For the first-order condition of choice variables, we have

$$\frac{\partial L}{\partial c_0} = \frac{1}{c_0} - \lambda p_0, \quad \frac{\partial L}{\partial c_1} = \frac{1}{c_1} - \lambda p_1, \quad \frac{\partial L}{\partial c_2} = \frac{1}{c_2} - \lambda p_2$$

**(b)** By **Kuhn-Tucker Theorem**, we have

$$\frac{\partial L}{\partial c_i} = \frac{1}{c_i} - \lambda p_i = 0, \quad i = 0, 1, 2 \tag{1}$$

and the slackness condition

$$\lambda[Y - (p_0c_0 + p_1c_1 + p_2c_2)] = 0 \tag{2}$$

If $\lambda = 0$, by (i) we will not have a feasible real value for $c_i$, thus $\lambda \neq 0$.
We also have

$$p_ic_i = \frac{1}{\lambda} \quad (\lambda \neq 0)$$

Taking it into (ii), we have

$$\lambda(Y - \frac{3}{\lambda}) = 0 \quad (\lambda \neq 0)$$

So the solution for $\lambda^*$ is

$$\lambda^* = \frac{3}{Y}$$

With a practical meaning $Y > 0$,

$$\lambda^* > 0$$

As a result, the constraint binds as $\lambda^* > 0$.

**(c)** Taking $\lambda^* = \frac{3}{Y}$ in, we have

$$c_i^* = \frac{Y}{3p_i}$$

Thus we have the optimal

$$L^* = ln(c_0c_1c_2) = ln(\frac{Y^3}{27p_0p_1p_2})$$

The larger $p_i$ is, the smaller our result; the larger $Y$ is, the larger our result.

**(d)** Let's define the fraction to be $F_i$. We have

$$F_i = \frac{c_i p_i}{Y} = \frac{1}{3}, \quad i = 0, 1, 2$$

## Question 2.

**(a)** We have the Lagrangian multiplier for

$$L(y, k, l) = py - rk - wl + \lambda(k^a l^b - y)$$

For the first-order condition of choice variables, we have

$$\frac{\partial L}{\partial y} = p - \lambda, \quad \frac{\partial L}{\partial k} = -r + \lambda l^b a k^{a-1}, \quad \frac{\partial L}{\partial l} = -w + \lambda k^a b l^{b-1}$$

**(b)** By **Kuhn-Tucker Theorem**, we have

$$\frac{\partial L}{\partial y} = \frac{\partial L}{\partial k} = \frac{\partial L}{\partial l} = 0 \tag{3}$$

and the slackness condition

$$\lambda(k^a l^b - y) = 0 \tag{4}$$

**(c)** The constraint binds at the optimum when

$$\lambda^* > 0$$

And we have the solution set to be

$$k^* = (w^b a^{b-1} b^{-b} r^{1-b} p^{-1})^{1-a-b}$$

$$l^* = (w^{1-a} a^{b-1} b^{-b} r^a p^{-1})^{1-a-b}$$

$$y^* = [w^b a^{(a+b-1)b-a} b^{-b(a+b)} r^a p^{-a-b}]^{1-a-b}$$

**(d)** Holding others the same:

(i) $p$ rises $\rightarrow$ $y^*$, $k^*$, $l^*$ all falls.

(ii) $r$ rises $\rightarrow$ $y^*$ increases, $k^*$ increases, and $l^*$ increases.

(iii) $w$ rises $\rightarrow$ $y^*$ increases, $k^*$ increases, and $l^*$ falls.

(iv) $p$, $r$, $w$ all double $\rightarrow$ $y^*$, $k^*$, $l^*$ stays the same, as the sum of powers of $p$, $r$ and $w$ is 0.

## Question 3.

**(a)** We have the *pdf* of $X$ to be

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp[-\frac{(x-\mu)^2}{2\sigma^2}]dx$$

Thus

$$E[u(X)] = \int_{-\infty}^{\infty} u(x)f(x)dx = \int_{-\infty}^{\infty}(-e^{vX})\frac{1}{\sigma\sqrt{2\pi}}\exp[-\frac{(x-\mu)^2}{2\sigma^2}]dx$$

Recall: the *mgf* of the normal distribution is

$$M_x(t) = e^{\mu t-\frac{1}{2}t^2\sigma^2} = E[e^{tx}]$$

Similarly, we have

$$E[u(X)] = -e^{-\mu v-\frac{1}{2}v^2\sigma^2}$$

As we have $E(X) = \mu$,

$$u(E[X]) = u(\mu) = -e^{-\mu v}$$

They are not equal. Obviously, we have

$$E[u(X)] > u(E[X])$$

**(b)** Like what has been introduced above, we have

$$E[u(aX)] = -e^{-\mu av-\frac{1}{2}v^2a^2\sigma^2} \quad \text{(parameter } v > 0)$$

The optimization problems towards $E[u(aX)]$ and $\mu a - \frac{1}{2}va^2\sigma^2$ are thus **equivalent**.

*****This is the end of Homework 1.*****