MAT3007 Homework 1

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Problem 1.

We introduce a new variable D to represent the maximum 1-distance between the post office and the towns. Our objective is to minimize D.

Objective function:

$$\text{Minimize } D$$

Constraints:

$$|x-0|+|y-0| \leq D \tag{1}$$
$$|x-0|+|y-5| \leq D \tag{2}$$
$$|x-2|+|y-2| \leq D \tag{3}$$

To convert these absolute value constraints into linear constraints, we introduce additional constraints for each absolute value term.

For the first town $(0,0)$:

$$x+y \leq D \tag{4}$$
$$-x+y \leq D \tag{5}$$
$$x-y \leq D \tag{6}$$
$$-x-y \leq D \tag{7}$$

For the second town $(0,5)$:

$$x+(y-5) \leq D \tag{8}$$
$$-x+(y-5) \leq D \tag{9}$$
$$x-(y-5) \leq D \tag{10}$$
$$-x-(y-5) \leq D \tag{11}$$

For the third town $(2,2)$:

$$(x-2)+(y-2) \leq D \tag{12}$$
$$-(x-2)+(y-2) \leq D \tag{13}$$
$$(x-2)-(y-2) \leq D \tag{14}$$
$$-(x-2)-(y-2) \leq D \tag{15}$$

Combining all these constraints, we get the following linear programming problem:

Objective function:

$$\text{Minimize } D$$

Constraints:

$$x + y \leq D \tag{16}$$
$$-x + y \leq D \tag{17}$$
$$x - y \leq D \tag{18}$$
$$-x - y \leq D \tag{19}$$
$$x + y - 5 \leq D \tag{20}$$
$$-x + y - 5 \leq D \tag{21}$$
$$x - y + 5 \leq D \tag{22}$$
$$-x - y + 5 \leq D \tag{23}$$
$$x + y - 4 \leq D \tag{24}$$
$$-x + y - 2 \leq D \tag{25}$$
$$x - y - 2 \leq D \tag{26}$$
$$-x - y \leq D \tag{27}$$

Problem 2.

(a)

Let:

- x_1 be the number of products of the first type produced.
- x_2 be the number of products of the second type produced.

Objective function (maximize daily profit):

$$\text{Maximize } Z = 8x_1 + 6x_2 - 1.2x_1 - 0.9x_2 = 6.8x_1 + 5.1x_2$$

Subject to:

$$\begin{aligned}
&\frac{1}{8}x_1 + \frac{1}{2}x_2 \leq 80 \quad \text{(assembly labor constraint)} \\
&\frac{1}{4}x_1 + \frac{1}{6}x_2 \leq 60 \quad \text{(testing constraint)} \\
&x_1 \geq 0 \\
&x_2 \geq 0
\end{aligned}$$

(b)

Standard form of the linear programming problem:

Objective function:

$$\text{Maximize } Z = 6.8x_1 + 5.1x_2$$

Subject to:

$$\begin{aligned}
&\frac{1}{8}x_1 + \frac{1}{2}x_2 \le 80 \\
&\frac{1}{4}x_1 + \frac{1}{6}x_2 \le 60 \\
&-x_1 \le 0 \\
&-x_2 \le 0
\end{aligned}$$

(c)

Suppose up to 30 hours of overtime assembly labor can be scheduled, at a cost of \$7 per hour. Let:

- y be the number of overtime assembly labor hours.

New objective function:

$$\text{Maximize } Z = 6.8x_1 + 5.1x_2 - 7y$$

New constraints:

$$\begin{aligned}
&\frac{1}{8}x_1 + \frac{1}{2}x_2 \le 80 + y \\
&y \le 30 \\
&y \ge 0 \\
&\frac{1}{4}x_1 + \frac{1}{6}x_2 \le 60 \\
&x_1 \ge 0 \\
&x_2 \ge 0
\end{aligned}$$

(d)

Using the ‘cvxpy’ library in Python, we solved the linear programming problem for the original constraints. The optimal value and the number of products to produce are as follows:

Optimal value: \$1700.00
Number of first type products: 160
Number of second type products: 120

The following Python script (refer to the attached `products.py`) was used to obtain the results:

```
import cvxpy as cp

x1 = cp.Variable()
x2 = cp.Variable()

objective = cp.Maximize(6.8 * x1 + 5.1 * x2)
```

```
constraints = [
    (1/8) * x1 + (1/2) * x2 <= 80,
    (1/4) * x1 + (1/6) * x2 <= 60,
    x1 >= 0,
    x2 >= 0
]

problem = cp.Problem(objective, constraints)

result = problem.solve()

print(f"Optimal value: {result:.2f}")
print(f"Number of first type products: {x1.value:.2f}")
print(f"Number of second type products: {x2.value:.2f}")
```

Problem 3.

Using the 'cvxpy' library in Python, we solved the linear programming problem for the original constraints. The optimal value and the number of products to produce are as follows:

The optimal path is S $\rightarrow$ 3 $\rightarrow$ 5 $\rightarrow$ 7 $\rightarrow$ T, with the shortest length of the path to be 8.

The following Python script (refer to the attached `shortest.py`) was used to obtain the results:

```
import cvxpy as cp
import numpy as np

inf = 1000

W = np.array([
    [  inf,    5,    4, inf, inf, inf, inf, inf],
    [  5,   inf, inf,    3, inf,    7, inf, inf],
    [  4, inf,    inf, inf,    1,    2, inf, inf],
    [inf,    3, inf,    inf,    2, inf, inf, inf],
    [inf, inf,    1,    2,    inf, inf,    2,    5],
    [inf,    7,    2, inf, inf,    inf, inf,    3],
    [inf, inf, inf, inf,    2, inf,    inf,    1],
    [inf, inf, inf, inf,    5,    3,    1,    inf]
])

num = 8
x = cp.Variable((num, num), boolean=True)

objective = cp.Minimize(cp.sum(cp.multiply(W, x)))

constraints = []
constraints.append(cp.sum(x[0, :]) == 1)
```

```
constraints.append(cp.sum(x[:, num - 1]) == 1)
for i in range(1, num - 1):
    constraints.append(cp.sum(x[i, :]) - cp.sum(x[:, i]) == 0)

problem = cp.Problem(objective, constraints)
problem.solve()

path = []
current = 0
while current != num - 1:
    next_num = np.argmax(x.value[current])
    path.append((current + 1, next_num + 1))
    current = next_num

print("Optimal path:", path)
print("Minimum path length:", problem.value)
```

Problem 4.

(a)

Original problem:

$$\begin{aligned} &\min 3x + 4|y - x| \\ &\text{s.t.} \quad |x + 1| + |y| \le 6 \end{aligned}$$

where $x, y \in \mathbb{R}$.

We can introduce auxiliary variables z, t, and v to replace the absolute values:

$$\begin{aligned} &\min 3x + 4z \\ &\text{s.t.} \\ &-z \le y - x \le z \\ &-t \le x + 1 \le t \\ &-v \le y \le v \\ &t + v \le 6 \\ &z \ge 0,\ t \ge 0,\ v \ge 0 \end{aligned}$$

(b)

Original problem:

$$\begin{aligned} &\min c^\top x + f(d^\top x) \\ &\text{s.t.} \quad Ax \ge b \end{aligned}$$

where $x, c, d \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $f(\alpha) = \max\{\alpha, 2, 3\alpha - 5\}$ for $\alpha \in \mathbb{R}$.

We can introduce an auxiliary variable z to replace $f(d^\top x)$, and express it with linear constraints:

$$\begin{aligned}
&\min c^\top x + z \\
&\text{s.t.} \\
&Ax \geq b \\
&z \geq d^\top x \\
&z \geq 2 \\
&z \geq 3(d^\top x) - 5
\end{aligned}$$

(c)

Original problem:

$$\begin{aligned}
&\min c^\top x \\
&\text{s.t.} \quad \|Ax - b\|_\infty \leq \delta \\
&x \geq 0
\end{aligned}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\|y\|_\infty = \max_i |y_i|$ for $y \in \mathbb{R}^n$.

We can introduce an auxiliary variable t to replace $\|Ax - b\|_\infty$:

$$\begin{aligned}
&\min c^\top x \\
&\text{s.t.} \\
&-t \leq Ax - b \leq t \\
&t \leq \delta \\
&x \geq 0 \\
&t \geq 0
\end{aligned}$$

Problem 5.

Define the variables:

x_p is the number of times pattern p is used.

Objective function:

$$\min \sum_{p=1}^{P} x_p$$

Subject to:

$$\sum_{p=1}^{P} a_{pi} x_p \geq d_i \quad \forall i = 1, \ldots, I$$

$$x_p \geq 0 \quad \forall p = 1, \ldots, P$$