MAT2040 Homework 2

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Question 1.

Define the matrix to be $B = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ $(a, b, c \neq 0)$, then

$$B^2 = \begin{bmatrix} a^2 + b^2 & ab + bc \\ ab + bc & b^2 + c^2 \end{bmatrix}.$$

If $B^2 = O$, we have

$$\begin{cases} a^2 + b^2 \neq 0 \\ ab + bc \neq 0 \\ b^2 + c^2 \neq 0 \end{cases} \Rightarrow \begin{cases} a \neq 0 \\ b \neq 0 \\ c \neq 0 \end{cases},$$

which is a contradiction to the basic assumption $a, b, c \neq 0$.

Thus it is NOT possible for a nonzero symmetric 2×2 matrix to have this property.

Question 2.

(a) We have

$$A^T = (C + C^T)^T = C^T + (C^T)^T = C^T + C = A$$

Thus it must be necessarily symmetric.

(b) We have

$$B^T = (C - C^T)^T = C^T - (C^T)^T = C^T - C \neq A$$

Thus it could possibly be non-symmetric.

(c) We have

$$D^T = (C^T C)^T = C^T (C^T)^T = C^T C = D$$

Thus it must be necessarily symmetric.

(d) We have

$$E^T = (C^T C - CC^T)^T = (C^T C)^T - (CC^T)^T = C^T C - CC^T = E$$

Thus it must be necessarily symmetric.

(e) We have

$$F^T = [(I + C)(I + C^T)]^T = (I + C^T)^T (I + C)^T = (I + C)(I + C^T) = F$$

Thus it must be necessarily symmetric.

(f) We have

$$F^T = [(I+C)(I-C^T)]^T = (I-C^T)^T(I+C)^T = (I-C)(I+C^T) = F$$

Thus it could possibly be non-symmetric.

Question 3.

We have the assumption that

$$M = \frac{1}{d} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

It is equivalent to show

$$A^{-1} = M,$$

which is equivalent to

$$MA = I$$

Thus we have

$$\begin{aligned} MA &= \frac{1}{d} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22}a_{11} - a_{12}a_{21} & 0 \\ 0 & a_{22}a_{11} - a_{12}a_{21} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

As a result, the proposition holds true.

Question 4.

If A is NOT singular, then there exists A^{-1}.

Multiplying A^{-1} on both sides,

$$A^{-1}AB = A^{-1}A$$

thus we have

$$B = I$$

It is a contradiction to $B \neq I$, thus A must be singular.

Question 5.

If A is non-singular, then

$$AB = I$$

It is equivalent to

$$(AB)^T = I^T$$

then

$$B^T A^T = I$$

Thus A^T is non-singular as well. We have

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

As the transpose of a matrix is unique, we have

$$(A^{-1})^T = (A^T)^{-1}$$

Question 6.

We have

$$(A^T A)^T = A^T (A^T)^T = A^T A$$

and

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

Thus $A^T A$ and AA^T are both symmetric.

Question 7.

(a) We have the elementary row operation to be

$$R_3 \to R_1 + R_3$$

Thus we have

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) We have the elementary row operation to be

$$R_2 \to -R_3 + R_2$$

Thus we have

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) For a given matrix A, performing row operations for A is equivalent to pre-multiplying A by the corresponding elementary matrix, thus C must be row equivalent to A.

Question 8.

(a) We have the matrix to be

$$\begin{bmatrix}2&1&1\\6&4&5\\4&1&3\end{bmatrix}\xrightarrow[R_3\to -2R_1+R_3]{R_2\to -3R_1+R_2}\begin{bmatrix}2&1&1\\0&1&2\\0&-1&1\end{bmatrix}\xrightarrow{R_3\to R_2+R_3}\begin{bmatrix}2&1&1\\0&1&2\\0&0&3\end{bmatrix}=U$$

The corresponding elementary matrices are

$$E_1=\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix},\quad E_2=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix},\quad E_3=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}$$

(b) We have inverse of matrices E_1, E_2, E_3 to be

$$\begin{cases}E_1=E_{-3R_1+R_2}\\E_2=E_{-2R_1+R_3}\\E_3=E_{R_2+R_3}\end{cases}\Rightarrow\begin{cases}E_1^{-1}=E_{3R_1+R_2}\\E_2^{-1}=E_{2R_1+R_3}\\E_3^{-1}=E_{-R_2+R_3}\end{cases}$$

The corresponding elementary matrices are

$$E_1^{-1}=\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix},\quad E_2^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix},\quad E_3^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}$$

We have

$$L=E_1^{-1}E_2^{-1}E_3^{-1}=\begin{bmatrix}1&0&0\\3&1&0\\2&-1&1\end{bmatrix}$$

For verification, we have

$$LU=\begin{bmatrix}1&0&0\\3&1&0\\2&-1&1\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&2\\0&0&3\end{bmatrix}=\begin{bmatrix}2&1&1\\6&4&5\\4&1&3\end{bmatrix}=A$$

Question 9.

(a) First we calculate A^{-1}, as

$$[A|I]=\left[\begin{array}{cc|cc}2&1&1&0\\6&4&0&1\end{array}\right]\Rightarrow\left[\begin{array}{cc|cc}1&0&2&-\frac{1}{2}\\0&1&-3&1\end{array}\right]$$

We have row operations as

$$A^{-1}=\begin{bmatrix}2&-\frac{1}{2}\\-3&1\end{bmatrix}\xrightarrow{R_1\to\frac{1}{2}R_1}\begin{bmatrix}1&-\frac{1}{4}\\-3&1\end{bmatrix}\xrightarrow{R_2\to 3R_1+R_2}\begin{bmatrix}1&-\frac{1}{4}\\0&\frac{1}{4}\end{bmatrix}\xrightarrow{R_2\to 4R_2}\begin{bmatrix}1&-\frac{1}{4}\\0&1\end{bmatrix}\xrightarrow{R_1\to\frac{1}{4}R_2+R_1}\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Thus we have

$$A^{-1} = E_{\frac{1}{2}R_1}^{-1} \cdot E_{3R_1+R_2}^{-1} \cdot E_{4R_2}^{-1} \cdot E_{\frac{1}{4}R_2+R_1}^{-1}$$

That is,

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & 1 \end{bmatrix}$$

(b) We have

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{1}{2} \\ 6 & 4 \end{bmatrix} \xrightarrow{R_2 \to -6R_1+R_2} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to -\frac{1}{2}R_2+R_1} \begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{4}R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus,

$$A = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & 1 \end{bmatrix}$$

Question 10.

(a) We have the row operation as

$$\begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix} \xrightarrow{R_2 \to -3R_1+R_2} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

Thus we have

$$\begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

(b) We have the row operation as

$$\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_1+R_2} \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix}$$

Thus we have

$$\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix}$$

(c) We have the row operations as

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{bmatrix} \xrightarrow[R_3 \to 2R_1+R_3]{R_2 \to -3R_1+R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 9 \end{bmatrix} \xrightarrow{R_3 \to -2R_2+R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Thus we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

(d) We have the row operations as

$$\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix} \xrightarrow[R_3 \to -3R_1+R_3]{R_2 \to 2R_1+R_2} \begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & -6 & -2 \end{bmatrix} \xrightarrow{R_3 \to -6R_2+R_3} \begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & -14 \end{bmatrix}$$

Thus we have

$$\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & -14 \end{bmatrix}$$

Question 11.

(a) If A is non-singular, then

$$AB = I$$

It is equivalent to

$$(AB)^{-1} = I^{-1}$$

then

$$B^{-1}A^{-1} = I$$

Thus A^{-1} is non-singular as well. We have

$$AA^{-1}x = I$$

and

$$(A^{-1})^{-1}A^{-1} = (AA^{-1})^{-1} = I^{-1} = I$$

As the inverse of a matrix is unique, we have

$$(A^{-1})^{-1} = A$$

(b) If A is non-singular, then

$$AB = I$$

It is equivalent to

$$(AB)^T = I^T$$

then

$$B^T A^T = I$$

Thus A^T is non-singular as well. We have

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

As the transpose of a matrix is unique, we have

$$(A^{-1})^T = (A^T)^{-1}$$

Question 12.

The system has infinitely many solutions. The solution is equivalent to

$$A\begin{bmatrix} 2m \\ m \\ -4m \end{bmatrix} = 0$$

and it still holds for any other $\mathbf{x}$ satisfying the given condition.

If A is invertible, then the system has a unique solution for $\mathbf{x}$, but we have infinite solutions.

Thus we have infinitely many solutions, and A is singular.

Question 13.

Since A is symmetric and non-singular, we have

$$A^T = A$$

and

$$AA^{-1} = I$$

First, we have

$$(A^{-1})^T A^T = (AA^{-1})^T = I$$

Substitute A^T with A, we have

$$(A^{-1})^T A = I \tag{1}$$

We also have

$$A^{-1}A = I \tag{2}$$

By (1) and (2), we have

$$(A^{-1})^T A = A^{-1}A$$

Multiply A^{-1} on both sides, we have

$$(A^{-1})^T = A^{-1}$$

Thus A^{-1} is also symmetric.

Question 14.

First prove: B is row equivalent to $A \Rightarrow B = MA$ (M is non-singular)

Since A is row equivalent to B, there exists a set of elementary matrices such that

$$B = E_k E_{k-1} ... E_1 A$$

As the product of elementary matrices is invertible, we can rewrite like

$$B = MA \quad (M \text{ is non-singular})$$

Also prove: $B = MA$ (M is non-singular) $\Rightarrow$ B is row equivalent to A

As a non-singular matrix, it could be decomposed as elementary matrices like

$$M = E_k E_{k-1} ... E_1$$

Thus we have

$$B = E_k E_{k-1} ... E_1 A$$

which indicates that A is row equivalent to B.

Thus the proposition is proved.

Question 15.

(a) We have the multiplication to be

$$\begin{aligned} \text{Multiplication} &= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 12 & 1 & 7 \end{bmatrix} + \begin{bmatrix} -1 & -2 & -3 \\ -1 & -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & 1 \\ 11 & -1 & 4 \end{bmatrix} \end{aligned}$$

(b) We have the multiplication to be

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & -2 \\ 8 & 5 & 8 & -5 \\ \hline 3 & 2 & 3 & -2 \\ 5 & 3 & 5 & -3 \end{array}\right]$$

(c) We have

$$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

Thus we have the multiplication to be

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

(d) We have

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 2 & -2 \\ 1 & -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -4 \\ 5 & -5 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 4 & -4 \end{bmatrix}$$

Thus we have the multiplication to be

$$\left[\begin{array}{cc} 3 & -3 \\ 2 & -2 \\ 1 & -1 \\ \hline 5 & -5 \\ 4 & -4 \end{array}\right]$$

Question 16.

Given B is singular, there exists a non-zero vector$\mathbf{x}$ such that

$$B\mathbf{x} = 0$$

If $C = AB$, then we have

$$C\mathbf{x} = AB\mathbf{x} = A0 = 0$$

Thus C is also singular.

Question 17.

(a) Since A_{11} and A_{12} are non-singular, we define

$$C = -A_{11}^{-1}A_{12}A_{22}^{-1}$$

Hence we have

$$\begin{bmatrix} A_{11}^{-1} & C \\ 0 & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & C \\ 0 & A_{22}^{-1} \end{bmatrix} = I_{2n}$$

(b) As stated,

$$C = -A_{11}^{-1}A_{12}A_{22}^{-1}$$

Question 18.

We have

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

and

$$U = D\hat{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$A = LD\hat{U} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

****This is the end of Homework 2.****