STA2002 - Homework 2

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PROBLEM 1.

(a) For point estimator $\hat{\mu_1}$, we have

$$\mathbf{Bias} = \mathbb{E}(\frac{X_1}{3} + \frac{X_2}{3} + \frac{X_3}{3}) - \mu_1 = \frac{\mu_1}{3} + \frac{\mu_1}{3} + \frac{\mu_1}{3} - \mu_1 = 0$$

For point estimator $\hat{\mu_2}$, we have

$$\mathbf{Bias} = \mathbb{E}(\frac{X_1}{4} + \frac{X_2}{3} + \frac{X_3}{5}) - \mu_2 = \frac{\mu_2}{4} + \frac{\mu_2}{3} + \frac{\mu_2}{5} - \mu_2 = -\frac{13}{60}\mu_2$$

As a result, $\hat{\mu_1}$ is unbiased.

(b) For point estimator $\hat{\mu_1}$, we have

$$\mathbf{Var}(\hat{\mu_1}) = \mathbf{Var}(\frac{X_1}{3}) + \mathbf{Var}(\frac{X_2}{3}) + \mathbf{Var}(\frac{X_3}{3}) = \frac{\mathbf{Var}(X_1)}{9} + \frac{\mathbf{Var}(X_2)}{9} + \frac{\mathbf{Var}(X_3)}{9} = \frac{40}{9} \approx 4.444$$

For point estimator $\hat{\mu_2}$, we have

$$\mathbf{Var}(\hat{\mu_1}) = \mathbf{Var}(\frac{X_1}{4}) + \mathbf{Var}(\frac{X_2}{3}) + \mathbf{Var}(\frac{X_3}{5}) = \frac{\mathbf{Var}(X_1)}{16} + \frac{\mathbf{Var}(X_2)}{9} + \frac{\mathbf{Var}(X_3)}{25} = \frac{1931}{720} \approx 2.682$$

As a result, $\hat{\mu_2}$ has the smaller variance.

(c)By equation $\mathbf{MSE} = \mathbf{Bias}^2 + \mathbf{Var}$, we have

$$\mathbf{MSE_1} = \mathbf{Bias_1}^2 + \mathbf{Var_1} = 0 + \frac{40}{9} = \frac{40}{9} \approx 4.444$$

$$\mathbf{MSE_2} = (\mathbf{Bias_2})^2 + \mathbf{Var_2} = -\frac{169}{3600}\mu_2 + \frac{1931}{720}$$

For $\mu = 3$, we further have

$$\mathbf{MSE_2} = \frac{2287}{900} \approx 2.541 < \mathbf{MSE_1}$$

As a result, $\hat{\mu_2}$ has the smaller MSE.

Problem 2.

Assume all $X_i \geq \theta$, or the likelihood function is equal to 0 and it becomes trivial.

$$\mathbf{L}(\theta) = f(x_1;\theta)f(x_2;\theta)\ldots f(x_n;\theta) = \exp(n\theta - \sum_{i=1}^{n} x_i)$$

Taking the logarithm, we have

$$\mathbf{l}(\theta) = n\theta - \sum_{i=1}^{n} x_i$$

Taking the derivative, we have

$$\frac{\partial}{\partial\theta}\mathbf{l}(\theta) = n > 0 \ ,$$

which indicates $\mathbf{L}(\theta)$ is increasing mononiacally.

To maximize $\mathbf{l}(\theta)$, θ should be as large as possible.

As a result,

$$\hat{\theta}_{mle} = \mathbf{min}(X_i) \ .$$

Problem 3.

(a) With the law of total probability, we have the joint probability

$$f(X, K) = f(X|K = 0)P(k = 0) + f(X|K = 1)P(k = 1)$$

Since $X|K = 0 \sim N(\mu_0, {\sigma_0}^2)$ and $X|K = 1 \sim N(\mu_1, {\sigma_1}^2)$,

$$f(X, K) = \begin{cases} \frac{1}{\sigma_0\sqrt{2\pi}} \exp[-\frac{(x-\mu_0)^2}{2\sigma_0^2}]\pi_0, & K_i = 0 \\ \frac{1}{\sigma_1\sqrt{2\pi}} \exp[-\frac{(x-\mu_1)^2}{2\sigma_1^2}]\pi_1, & K_i = 1. \end{cases}$$

(b) Denote the distribution of (X, K) as $\mathbf{GMM}(\pi_0, \mu_0, \sigma_0^2, \mu_1, \sigma_1^2)$.

By **MLE**, we can write the likelihood function as

$$\mathbf{L}(\pi_0, \mu_0, {\sigma_0}^2, \mu_1, {\sigma_1}^2) = \prod_{i=1}^{n}\{\frac{1}{\sigma_0\sqrt{2\pi}} \exp[-\frac{(x-\mu_0)^2}{2\sigma_0^2}]\pi_0\}^{1-K_i}\{\frac{1}{\sigma_0\sqrt{2\pi}} \exp[-\frac{(x-\mu_1)^2}{2\sigma_1^2}]\pi_1\}^{K_i}$$

Taking the logarithm,

$$\mathbf{l} = \ln \mathbf{L} = \sum_{i=1}^{n}(1 - K_i)[\ln(\frac{\pi_0}{\sigma_0\sqrt{2\pi}}) - \frac{(x-\mu_0)^2}{2\sigma_0^2}] + \sum_{i=1}^{n} K_i[\ln(\frac{\pi_1}{\sigma_1\sqrt{2\pi}}) - \frac{(x-\mu_1)^2}{2\sigma_1^2}]$$

Let n_0 and n_1 be the number of X_i that belongs to group 0 and group 1 respectively. Taking the derivative $w.r.t$ each variable,

$$\begin{cases} \frac{\partial}{\partial \pi_0}\mathbf{l} = \sum_{i=1}^n \frac{1-K_i}{\pi_0} + \sum_{i=1}^n \frac{K_i}{1-\pi_0} = \frac{n_0}{\pi_0} + \frac{n_1}{1-\pi_0} = 0 \\ \frac{\partial}{\partial \mu_0}\mathbf{l} = \sum_{i=1}^n (1-K_i)n_i - \mu_0 \sum_{i=1}^n (1-K_i) = 0 \\ \frac{\partial}{\partial \mu_1}\mathbf{l} = \sum_{i=1}^n K_i n_i - \mu_1 \sum_{i=1}^n K_i = 0 \\ \frac{\partial}{\partial \sigma_0^2}\mathbf{l} = -\sum_{i=1}^n \frac{1-K_i}{2\sigma_0^2} + \sum_{i=1}^n (1-K_i)\frac{(x_i-\mu_0)^2}{2\sigma_0^2} = 0 \\ \frac{\partial}{\partial \sigma_1^2}\mathbf{l} = -\sum_{i=1}^n \frac{K_i}{2\sigma_1^2} + \sum_{i=1}^n (K_i)\frac{(x_i-\mu_1)^2}{2\sigma_1^2} = 0 \end{cases}$$

Solving the equation, we have

$$\hat{\pi_0} = \frac{n_0}{n_0 - n_1}, \quad \hat{\mu_0} = \frac{1}{n_0}\sum_{i=1}^n (1-K_i)x_i, \quad \hat{\mu_1} = \frac{1}{n_1}\sum_{i=1}^n K_i x_i$$

$$\hat{\sigma_0^2} = \frac{1}{n_0}\sum_{i=1}^n (1-K_i)(x_i - \hat{\mu_0})^2, \quad \hat{\sigma_1^2} = \frac{1}{n_1}\sum_{i=1}^n K_i(x_i - \hat{\mu_1})^2$$

PROBLEM 4.

(a) Given $\sigma = 0.5$, $n = 20$, $1 - \alpha = 0.99$, two-sided estimation,

$$\mathbf{z}_{\alpha/2} = \mathbf{z}_{0.005} = 2.5758, \quad \bar{x} = \frac{1}{11}\sum_{i=1}^{11} = 13.755$$

For lower bound of CI,

$$\mu \geq \bar{x} - \mathbf{z}_\alpha \times \frac{\sigma}{\sqrt{n}} = 13.755 - 2.5758 \times \frac{0.5}{\sqrt{20}} = 13.467$$

For upper bound of CI,

$$\mu \leq \bar{x} + \mathbf{z}_\alpha \times \frac{\sigma}{\sqrt{n}} = 13.755 + 2.5758 \times \frac{0.5}{\sqrt{20}} = 14.043$$

As a result, the confidence interval is $[13.467, 14.043]$.

(b) Since $1 - \alpha = 0.95$, one-sided estimation,

$$\mathbf{z}_\alpha = \mathbf{z}_{0.05} = 1.6449$$

For lower bound of CI,

$$\mu \geq \bar{x} - \mathbf{z}_\alpha \times \frac{\sigma}{\sqrt{n}} = 13.755 - 1.6449 \times \frac{0.5}{\sqrt{20}} = 13.571$$

As a result, the confidence interval is $[13.571, +\infty)$.

(c) If $\bar{x}$ is used as an estimate of μ, we can be 95% (which implies, $\alpha = 0.05$) confident that the error $|\bar{x} - \mu|$ will not exceed the specific amount E when the sample size is

$$n = (\frac{\mathbf{z}_{\alpha/2}\sigma}{E})^2 = (\frac{1.96 \times 0.5}{2})^2 = 0.2401 \approx 1$$

Therefore, a sample size of at least 1 would be required to achieve a 95% CI.

PROBLEM 5.

Since $n = 12$ and is sufficiently large, **CLT** could be applied.

Using chi-square, we can estimate a CI for the variance σ^2

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

That is,

$$P[\chi^2_{1-\alpha/2}(n-1) \leq \frac{(n-1)s^2}{\sigma^2} \leq \chi^2_{\alpha/2}(n-1)] = 1 - \alpha$$

Construct a two-sided CI with $1 - \alpha = 95\%$. That is, $\alpha = 5\%$.

For lower bound of CI,

$$\sigma^2 \geq \frac{(n-1)s^2}{\chi^2_{\alpha/2}(n-1)} = \frac{(12-1)s^2}{\chi^2_{0.025}(12-1)} = \frac{11 \times 0.02445}{21.920} = 0.0123$$

For upper bound of CI,

$$\sigma^2 \geq \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}(n-1)} = \frac{(12-1)s^2}{\chi^2_{0.975}(12-1)} = \frac{11 \times 0.02445}{3.816} = 0.0705$$

As a result, the confidence interval is $[0.0123, 0.0705]$.

PROBLEM 6. According to the poll, the proportion of opposing is 32%, and others is 68%. For each of the individual as p = 32%,

$$X_i \sim Bernoulli(p)$$

Since $n = 1346$ and is sufficiently large, **CLT** could be applied as

$$\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n} \sim N(p, \frac{p(1-p)}{n})$$

Build a two-sided CI with $1-\alpha = 95\%$, $\mathbf{z}_{\alpha/2} = \mathbf{z}_{0.025} = 1.96$,

$$p \geq \hat{p} - \mathbf{z}_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.32 - 1.96\sqrt{\frac{0.32(1-0.32)}{1346}} \approx 0.295$$

$$p \geq \hat{p} + \mathbf{z}_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.32 + 1.96\sqrt{\frac{0.32(1-0.32)}{1346}} \approx 0.345$$

As a result, the confidence interval is $[0.295, 0.345]$.

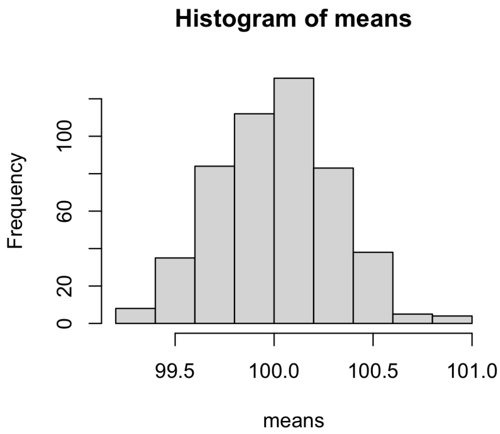
That is, we can be 95% confident to confirm that the proportion of all American adults who oppose the legalization falls in the section of $[0.295, 0.345]$.

PROBLEM 7.

(a)(b)(c) The graph is as follows:

```
> # (a)
> x = rnorm(100, mean=100, sd=3)
> mean(x)
[1] 99.701
> var(x)
[1] 6.137363
> # (b)
> n = 100 # number of observations in one sample
> S = 500 # number of simulations
> X = matrix(0, nrow=S, ncol=n)
> for (i in 1:S){
+     X[i,] = rnorm(n, mean=100, sd=3)
+ }
> means = apply(X, 1, mean)
> # (c)
> hist(means)
>
```

(a) Code



(b) Plot

(d)For the theoretical sampling distribution with $s^2 = \frac{\sigma^2}{500} = 0.018$,

$$\hat{\mu} \sim N(100, 0.018)$$

The hisgram seems quite normal, with a standard symmetry bell-characteristic.

As a result, the hisgram could approximate the sampling well.