Homework 2 - Thibaud Bruyelle

Problem 1

Let us write:

$$\forall n \,,\; R_n(\alpha) = \sum_{i=0}^{n-1} e^{-\alpha i} Z_i \,.$$

Besides, $\forall i \geq 0$,

$$\begin{aligned} Z_i &= a Z_{i-1} + \beta_i \\ &= a(a Z_{i-2} + \beta_{i-1}) + \beta_i \\ &= \dots \\ &= a^i Z_0 + \sum_{k=0}^{i-1} a^k \beta_{i-k} \,. \end{aligned}$$

Then we can write $\forall i \geq 0$:

$$X_i := e^{-\alpha i} Z_i$$

$$= e^{-\alpha i} \left(a^i Z_0 + \sum_{k=0}^{i-1} a^k \beta_{i-k} \right)$$

$$\mathbb{E}(X_i) = e^{-\alpha i} a^i \, \mathbb{E}(Z_0)$$

$$+ e^{-\alpha i} \frac{1-a^i}{1-a} \mathbb{E}(\beta_1) .$$

$$\mathbb{E}(R_n(\alpha)) = \left(\sum_{i=0}^{n-1} (e^{-\alpha} a)^i \right) \mathbb{E}(Z_0)$$

$$+ \left[\frac{1}{1-a} \sum_{i=0}^{n-1} e^{-\alpha i} + \frac{1}{(1-a)a} \sum_{i=0}^{n-1} (e^{-\alpha} a)^i \right] \mathbb{E}(\beta_1)$$

$$\mathbb{E}(R_n(\alpha)) \xrightarrow[n \to +\infty]{} \frac{\mathbb{E}(\varepsilon_0)}{1-e^{-\alpha}a}$$

$$+ \left(\frac{1}{1-a} \times \frac{1}{1-e^{-\alpha}} + \frac{1}{(1-a)a} \times \frac{1}{1-e^{-\alpha}a} \right) \mathbb{E}(P_1)$$

Let us write this limit $\mathbb{E}(R(\alpha))$.

$$\left(\begin{cases} |e^{-\alpha}| < 1 \\ |a| < 1 \end{cases} \Rightarrow |e^{-\alpha} a| < 1 \right)$$

In the same way:

$$\mathbb{V}(R_n(\alpha)) = \sum_{i,j} \mathrm{cov}(X_i, X_j)$$

$$= \sum_{i=0}^{n-1} \mathbb{V}(X_i) + \sum_{i \neq j}^{n-1} \mathrm{cov}(X_i, X_j).$$

first:

$$\mathbb{V}(X_i) = e^{-2\alpha i} a^{2i} \mathbb{V}(Z_0)$$

$$+ e^{-\alpha i} \mathrm{cov}\left(a^i Z_0, \sum_{h=0}^{i-1} a^h \beta_{i-h}\right)$$

$$+ e^{-2\alpha i} \mathbb{V}\left(\sum_{h=0}^{i-1} a^h \beta_{i-h}\right)$$

$$= \left(e^{-2\alpha} a^2\right)^i \mathbb{V}(Z_0)$$

$$+ e^{-2\alpha i} \sum_{k,l}^{i-1} a^{k+l} \mathrm{cov}(\beta_{i-k}, \beta_{i-l})$$

$$\left(\mathrm{cov}(Z_0, \beta_{i-h}) = 0 \quad \forall h \geq 1\right)$$

$$\sum_{k,l}^{i-1} a^{k+l} \mathrm{cov}(\beta_{i-k}, \beta_{i-l}) =$$

$$\sum_{\substack{1 \leq k \leq i \\ 1 \leq l \leq i}} a^{i-k+i-l} \operatorname{cov}(\beta_k, \beta_l)$$

$$= a^{2i} \sum a^{-2k} V(\beta_1)$$

$$= a^{2i} V(\beta_1) \times \frac{1 - a^{-2i}}{1 - a^{-2}}$$

$$\boxed{\begin{aligned} V(X_i) &= \left(e^{-2\alpha} a^2\right)^i V(Z_0) \\ &\quad + e^{-2\alpha i} \times a^{2i} \left(\frac{1 - a^{-2i}}{1 - a^{-2}}\right) V(\beta_1) \end{aligned}}$$

and

$$\operatorname{cov}(X_i, X_j) =$$

$$e^{-\alpha(i+j)} \operatorname{cov}\left(a^i Z_0 + \sum a^{i-k} \beta \dots\right.$$

$$e \cdot \operatorname{cov}\left(a^{i} Z_0 + \sum_{k=1}^{i} a^{i-k} \beta_k \, , \right.$$

$$\left. a^{j} Z_0 + \sum_{\ell=1}^{j} a^{j-\ell} \beta_\ell \right)$$

$$= e^{-\alpha(i+j)} \times a^{i+j} \, \mathbb{V}(Z_0)$$

$$+ e^{-\alpha(i+j)} \times \sum_{\substack{1 \le k \le i \\ 1 \le \ell \le j}} a^{i-k+j-\ell} \operatorname{cov}(\beta_k, \beta_\ell)$$

$$\boxed{\begin{aligned} &= e^{-\alpha(i+j)} \cdot a^{i+j} \, \mathbb{V}(Z_0) \\ &+ e^{-\alpha(i+j)} \times \sum_{1 \le k \le \min(i,j)} a^{i+j-2k} \, \mathbb{V}(\beta_1) \end{aligned}}$$

So $\displaystyle\sum_{i \neq j} \operatorname{cov}(X_i, X_j) = 2 \sum_{i > j} \operatorname{cov}(X_i, X_j)$

$$2 \times \mathbb{V}(Z_0) \sum_{i > j}^{n-1} e^{-\alpha(i+j)} a^{i+j}$$

$$+2 \times V(\beta_1) \times \sum_{i>j}^{n-1} e^{-\alpha(i+j)} a^{i+j} \times \frac{1-a^{+\min(i,j)}}{1-a}$$

$$\bullet \sum_{i>j}^{n-1} (e^{-\alpha} a)^{i+j} = \sum_{j=0}^{n-1} \sum_{i=j+1}^{n-1} (e^{-\alpha} a)^{i+j}$$

$$= \sum_{j=0}^{n-1} (e^{-\alpha} a)^{j} \times \frac{1-(e^{-\alpha} a)^{n-j-1}}{1-e^{-\alpha} a}$$

$$:= A_n$$

$$\boxed{\frac{1}{1-ae^{-\alpha}} \times \left[\frac{1-(ae^{-\alpha})^{n}}{1-ae^{-\alpha}} - n(e^{-\alpha} a)^{n-1} \right]}$$

$$\bullet \sum_{j=0}^{n-1} (ae^{-\alpha})^{j} \times \frac{1-a^{j}}{1-a} \times \sum_{i=j+1}^{n-1} (ae^{-\alpha})^{j}$$

$$= \frac{1}{1-a} \sum_{j=0}^{n-1} (ae^{-\alpha})^{j} (1-a^{j}) \times \frac{1-(ae^{-\alpha})^{n-j+1}}{1-ae^{-\alpha}}$$

$j=0$

$$= \frac{1}{(1-a)(1-ae^{-\alpha})} \times B_n$$

such that $\lim_{n \to +\infty} B_n$ is finite.

Then

$$V(R_n(\alpha)) = 2V(Z_0)A_n + 2V(P_1)B_n + \sum_{i=0}^{n-1} V(X_i)$$

We can show using the values of $V(X_i)$, A_n and B_n that

$$V(R_n(\alpha)) \longrightarrow V(R(\alpha))$$

$n \to +\infty$

where $\mathbb{V}(R(\alpha)) < +\infty$.

Using the CLT, we could estimate that for n large enough:

$$\boxed{R(\alpha) \overset{\mathcal{D}}{\approx} \mathcal{N}\big(\mathbb{E}(R(\alpha)), \mathbb{V}(R(\alpha))\big)}$$

In order to hold, we need α very small so there is X_i that dominates the other value.

Problem 2

Let us define : $K_n := \sum_{i=1}^{n} \left(Z_i - E(Z_i | X_i) \right)$

$$L_n := \sum_{i=1}^{n} E(Z_{i+1} | X_{i+1}) - E(Z_1)$$

We then have :

$$M_n = K_n + L_n = \sum_{i=1}^{n} \left[Z_i - E(Z_1) \right] + E(Z_{n+1} | X_{n+1}) - E(Z_1 | X_1) .$$

- $E(K_{n+1} | X_1, \dots, X_{n+1}) = E(Z_{n+1} - E(Z_{n+1} | X_{n+1}) | X_1, \dots, X_n) + \sum_{i=1}^{n} E(Z_i - E(Z_i | X_i) | X_1, \dots, X_{n+1})$

$= \sum_{i=1}^{n} (Z_i - E(Z_i | X_i))$ because

$(Z_1, \dots, Z_n)$ are $(X_1, \dots, X_{n+1})$-measurable

Finally $\boxed{K_n \text{ is a } (X_1, \dots, X_{n+1}) \text{ martingale}}$

- $E(L_{n+1} \mid X_1, \dots, X_{n+1}) = E(E(Z_{n+2} \mid X_{n+2}) \mid X_1, \dots, X_{n+1})$

$- E(Z_1) + \sum_{i=1}^{n} E(E(Z_{i+1} \mid X_{i+1}) \mid X_1, \dots, X_{n+1})$

$- nE(Z_1)$

$= 0 + \sum_{i=1}^{n} E(Z_{i+1} \mid X_{i+1})$

($E(Z_{i+1} \mid X_{i+1})$ is $(X_1, \dots, X_{i+1})$-mesurable.)

Finally $\boxed{L_n \text{ is a } (X_1, \dots, X_{n+1}) \text{ martingale}}$

M_n is a $(X_1, \dots, X_{n+1})$ martingale and:

- $D_n = M_{n+1} - M_n$

$= Z_{n+1} - E(Z_1) + E(Z_{n+2} \mid X_{n+2}) - E(Z_{n+1} \mid X_{n+1})$

- $E(D_n^2) = E[Z_{n+1}^2] + E(Z_1^2) +$

$E(E(Z_{n+2} \mid X_{n+2})^2) + E(E(Z_{n+1} \mid X_{n+1})^2)$

$$- 2\,\mathbb{E}(Z_1)\,\mathbb{E}(Z_{n+1}) + 2\,\mathbb{E}\big(Z_{n+1}\, E(Z_{n+2}|X_{n+2})\big)\Big]$$

$$- 2\,\mathbb{E}\big(Z_{n+1}\, E(Z_{n+1}|X_{n+1})\big) - 2\,\mathbb{E}(Z_1)\,\mathbb{E}\big(E(Z_{n+2}|X_{n+2})\big)$$

$$+ 2\,\mathbb{E}(Z_1)\,\mathbb{E}\big(E(Z_{n+1}|X_{n+1})\big) +$$

$$2\,\mathbb{E}\big(E(Z_{n+2}|X_{n+2})\, E(Z_{n+1}|X_{n+1})\big)\Big]$$

$$= \big[1+1+1+1-2+2-4+4\big]\cdot \mathbb{E}(Z_1^2)$$

$$= 4\cdot \mathbb{E}(Z_1^2) < +\infty \quad (f \text{ bounded})$$

Consequently, <u>$(D_k^2)_{k\geq 1}$ is a one-dependent</u> sequence and since $(X_k)_{k\geq 1}$ are iid, <u>$(D_k^2)_{k\geq 1}$ is stationary.</u> Thus, we can use the LLN in the stationary and ergodic case:

$$\boxed{\sum_{i=1}^{n} D_i^2 \xrightarrow[n\to+\infty]{\mathbb{P}} 4\,\mathbb{E}(Z_1^2)}$$

∘ $|D_i| < 4 \max_{(x,y)\in\mathbb{R}^2} f(x,y)$ as

$(\forall i \geq 1)$ so $\max_i |D_i| < 4K$ as

so we can use the DCT to show:

$$\lim_{n\to+\infty} \frac{1}{\sqrt{n}} \mathbb{E}\left(\max_i |D_i|\right) = \mathbb{E}\left(\lim_{n\to+\infty} \frac{\max |D_i|}{\sqrt{n}}\right) = 0.$$

Then $\boxed{\frac{M_n}{\sqrt{n}} \xrightarrow[n\to+\infty]{\mathcal{D}} \sqrt{\mathbb{E}(D_1^2)}\, \mathcal{N}(0,1).}$

Furthermore, as f is bounded,

$$E(Z_{m+1}|X_{m+1}) - E(Z_1|X_1) \xrightarrow[m\to+\infty]{\mathbb{P}} 0$$

so by using Slutsky's theorem:

$$\underline{\left(M_m, E(Z_{m+1}|X_{m+1}) - E(Z_1|X_1)\right) \xrightarrow{\mathcal{D}} \left(\sqrt{E(D_1^2)}\,\mathcal{N}(0,1), \delta_0\right)}$$

By using $h : (x, y) \to y - x$ (measurable function)
we get:

$$\boxed{\frac{\sum_{i=1}^{n} (Z_i - n\,\mathbb{E}(Z_1))}{\sqrt{n}} \xrightarrow[n \to +\infty]{\mathcal{D}} \eta \cdot \mathcal{N}(0,1)}$$

where

$$\eta = \sqrt{\mathbb{E}(D^2)}$$

Problem 5

a) X has a unique mode so:

$\exists\, \alpha \in \mathbb{R},\ f'(\alpha) = 0$

where f is the density of X.

Let $b \in \mathbb{R}$ a prediction of X.

$$\ell(X, b) = \mathbb{E}\left(\mathbb{1}_{\{|X-b| \geq \varepsilon\}}\right)$$

$$= \int_{-\infty}^{+\infty} \mathbb{1}_{\{x-b \geq \varepsilon\} \oplus \{x-b \leq -\varepsilon\}} f(x)\,dx$$

$$= \int_{b+\varepsilon}^{b-\varepsilon} f(x)\,dx.$$

$$\Rightarrow \ell'(X, b) = f(b-\varepsilon) - f(b+\varepsilon)$$

So the optimal predictor will satisfy:

$$f(b^* - \varepsilon) \quad f(b^* + \varepsilon)$$

$$f(\alpha - \varepsilon) = f(\alpha + \varepsilon).$$

Since the mode of X is such that

$$f'(\alpha) = 0$$

$$\Rightarrow \frac{f(\alpha+\varepsilon) - f(\alpha - \varepsilon)}{2\varepsilon} = 0$$

$$\Rightarrow f(\alpha + \varepsilon) = f(\alpha - \varepsilon).$$

We can conclude that

$$\boxed{b^* = \alpha.}$$

b) Here:

$$l(X,b) = \mathbb{E}\left[c(b-X)\mathbb{1}_{\{X \le b\}}\right.$$

$$+ d(X-b)\,\mathbb{1}_{\{X\geq b\}} \quad]$$

$$= \int_{-\infty}^{b} c(b-x)\,f(x)\,dx$$

$$+ \int_{b}^{+\infty} d(x-b)\,f(x)\,dx.$$

So we have:

$$\ell'(X,b) = c\int_{-\infty}^{b} f(x)\,dx + \cancel{cb f(b)}$$

$$- \cancel{c b f(b)}$$

$$- \cancel{d b f(b)} - d\int_{b}^{+\infty} f(x)\,dx$$

$$+ \cancel{d b f(b)}$$

So b^* satisfies:

$$\ell'(X, b^*) = 0$$

$$\Rightarrow \quad d\left(1 - \int_{-\infty}^{b} f(x)\,dx\right) = c \int_{-\infty}^{b} f(x)\,dx$$

$$\Rightarrow \quad \boxed{\mathbb{P}(X \leq b^*) = \frac{d}{c+d}}$$

In other words b^* should be the $\frac{d}{c+d}$ th quantile of X's distribution.

c) We now have $f_{X,Z}(\cdot)$ and

can observe Z.

Since

$$f_{(X,Z)}(x,z) = f_Z(z) \times f_{X|Z=z}(x)$$

$$\Rightarrow f_{X|Z=z} = \frac{f_{(X,Z)}(x,z)}{f_Z(z)}$$

And I would use this function instead of $f(\cdot)$.

Problem 6

a) $L(\theta, X) = \frac{1}{\theta} \times \dots \times \frac{1}{\theta}$

$$= \frac{1}{\theta^n}$$

$$\Rightarrow \frac{\partial \log}{\partial \theta}(L(\theta, X)) = -\frac{n}{\theta}.$$

This is a convex function so it reaches its minimum with $\boxed{\hat{\theta}_{MLE} = \max_{1 \leq i \leq n} x_i}$

b) Let $t \in]0, \theta[$.

$$\mathbb{P}(\hat{\theta}_{MLE} \leq t) = \mathbb{P}(X_1 \leq t)^n$$

$$= \left(\frac{t}{\theta}\right)^n$$

So

$$f(t) = n \frac{t^{n-1}}{\theta^n}$$

$$= \frac{n}{\theta} \times \left(\frac{t}{\theta}\right)^{n-1}$$

and finally

$$\mathbb{E}(\hat{\theta}_{MLE}) = \int_0^\theta n \frac{t^n}{\theta^n} dt$$

$$= \frac{n}{\theta^n} \left[\frac{1}{n+1} t^{n+1}\right]_0^\theta$$

$$= \frac{n}{n+1} \theta.$$

So yes this estimator is biased but as n grows, the bias tends toward 0.

c) $\hat{\theta}^{\,unbiased}_{MLE} = \frac{n}{n+1} \hat{\theta}_{MLE}$

and this estimator will be unbiased.

Problem 7

a) $\operatorname{cov}(U, V \mid W) =$

$$\mathbb{E}(\overset{(1)}{UV} \mid W) - \mathbb{E}(\mathbb{E}(U \mid W)\overset{(2)}{V} \mid W)$$

$$\mathbb{E}(U\,\mathbb{E}(V \mid W) \mid W)$$

$$- \mathbb{E}\left(U\, \mathbb{E}(V|W) \,|\, W \right)$$

$$+ \mathbb{E}\left(E(U|W) E(V|W) \,|\, W \right)$$

and

$$\operatorname{cov}\left(E(U|W), E(V|W) \right)$$

$$= \mathbb{E}\Big[\left(E(\overset{(3)}{U}|W) - \mathbb{E}\left(E(U|W)\right) \right) \times$$

$$\left(E(V|W) - \overset{(4)}{\mathbb{E}}\left(E(V|W) \right) \right) \Big]$$

Let us recall that:

$\ast^{(1)}$ $$\mathbb{E}\left[E(UV|W) \right] = \mathbb{E}\left[UV \right]$$

$\ast^{(2)}$ $$\mathbb{E}\left[E\left(E(U|W) V | W \right) \right]$$

$$= \mathbb{E}\left[E(U|W) V \right]$$

$$(3)\times(4) - \mathbb{E}\left(E(U|W)\right)\times\mathbb{E}\left(E(V|W)\right)$$

$$= \mathbb{E}\left(E(V|W)\right)\times\mathbb{E}\left[E(U|W)\right]$$

$$= -\mathbb{E}(V)\times\mathbb{E}(U).$$

In a similar way for the other parts of the equation, we have the results.

b) X is such that $\begin{cases}\mathbb{E}(X)=\mu\\ \mathbb{V}(X)=\sigma^2\end{cases}$

and $Y \sim \mathcal{E}\left(\frac{1}{X}\right)$.

$\mathrm{cov}(Y_1, Y_5)$?

Let us use the formula from a):

$$\operatorname{cov}(Y_1, Y_5) = E\big(\operatorname{cov}(Y_1, Y_5 \mid X)\big) + \operatorname{cov}\big(E(Y_1 \mid X), E(Y_5 \mid X)\big).$$

- $$E(Y_1 \mid X = x) = \int_0^{+\infty} \frac{t}{x} e^{-\frac{t}{x}} \, dt$$

$$= x.$$

So then:

$$E(Y_1 \mid X) = X.$$

In a similar way, we can write:

$$Y_5 = Y_1 + \underbrace{\tilde{Y}_2 + \tilde{Y}_3 + \tilde{Y}_4 + \tilde{Y}_5}_{= \text{time between two consecutive clashes}}$$

$$E(Y_5 \mid X) = 5X$$

so :

$$\boxed{\begin{aligned}\operatorname{cov}(E(Y_1|X), E(Y_5|X)) &= 5\,V(X)\\ &= 5\sigma^2\end{aligned}}$$

Then :

$$\operatorname{cov}(Y_1, Y_2 | X) = E\Big[(Y_1 - X)\Big(Y_1 + \sum_{i=2}^{5} \tilde{Y}_i - 5X\Big) \,\Big|\, X\Big]$$

$$= E(Y_1^2 | X) - 5X E(Y_1 | X)$$

$$+ E\Big(Y_1 \cdot \Big(\sum_{i=2}^{5} \tilde{Y}_i\Big) \Big| X\Big) - X E(Y_1 | X)$$

$$- X E\Big(\sum_{i=2}^{5} \tilde{Y}_i | X\Big) + 5X^2$$

$$= 2X^2 - 5X^2 + 5X^2 - X^2 - 4X^2 + 4X^2$$

$$= X^2$$

<u>And</u>

$$\mathbb{E}(X^2) = \mu^2 + \sigma^2$$

S. Brindley :

So finally:

$$\boxed{\mathrm{cov}(Y_1, Y_3) = 6\sigma^2 + \mu^2.}$$

c) The number of throws until we get an odd number is a r.v. X such that: $X \sim \mathcal{G}(\frac{1}{2})$.

So $\mathbb{E}(X) = 2$.

die value

$$\mathbb{E}(S) = \mathbb{E}(E(S|X))$$

$$= \mathbb{E}\left(E\left(\sum_{i=1}^{X} Y_i \,\middle|\, X\right)\right)$$

$$= \mathbb{E}(X \times \underbrace{\mathbb{E}(Y_i)}_{=4}) \quad (Y_i \perp\!\!\!\perp X)$$

$$= 4 \times \mathbb{E}(X)$$

$$= 8.$$

Finally $\boxed{\mathbb{E}(S) = 8}$

Finally $\boxed{\mathbb{E}(S) = 6}$

$$
\begin{aligned}
\operatorname{var}(S) &= \operatorname{cov}(S, S) \\
&= \mathbb{E}(\operatorname{var}(S \mid X)) + \operatorname{cov}(E(S \mid X), E(S \mid X)) \\
&= \mathbb{E}(\cancel{X} \operatorname{Var}(Y_1)) + 16 \operatorname{var}(X) \\
&= \frac{8}{3} \times 2 + 16 \times 4 \quad \left(\operatorname{var}(Y_1) = \frac{56}{3} - 16\right) \\
&= 2\left(\frac{8}{3} + 32\right) \\
&= 2\left(\frac{104}{3}\right)
\end{aligned}
$$

$$\Rightarrow \boxed{V(S) = 2 \cdot \frac{104}{3}}$$

hw2

May 1, 2020

1 Homework 2 - T. Bruyelle

[1]:
```python
import matplotlib.pyplot as plt
import numpy as np
from numpy.random import exponential
import statsmodels
import scipy.stats
```

1.1 Problem 3

longest_path_length is a function such that :

$$(z_1, \ldots, z_6) \mapsto max(z_1 + z_2 + z_4, z_1 + z_3, z_5 + z_6)$$

This function is used to sample the rv L since we want to estimate $\mathbb{E}(L)$.

1.1.1 i - Monte Carlo Standard

Here our estimator is :

$$\alpha_{MC}^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} L_i$$

where $(L_0, \ldots, L_{n-1})$ are *iid* observations of L.

[2]:
```python
n = 1000
lambda_1 = 1
lambda_2 = 1 / 2
lambda_3 = 1 / 3
lambda_4 = 1 / 4
lambda_5 = 1 / 15
lambda_6 = 1 / 6

def longest_path_length(z1, z2, z3, z4, z5, z6):
    return max(z1 + z2 + z4, z1 + z3, z5 + z6)

def inverse_cdf_exponential_law(x,mean) :
    return mean * np.log(1 / (1-x))
```

```
## Simulations
Z1 = exponential(1/lambda_1,n)
Z2 = exponential(1/lambda_2,n)
Z3 = exponential(1/lambda_3,n)
Z4 = exponential(1/lambda_4,n)
Z5 = exponential(1/lambda_5,n)
Z6 = exponential(1/lambda_6,n)

L = []
for i in range(n) :
    L.append(longest_path_length(Z1[i],Z2[i],Z3[i],Z4[i],Z5[i],Z6[i]))
L = np.array(L)

# 1 - MC Standard
print("MC (standard) estimator : ", np.mean(L))
print("MC (standard) estimated variance : ", (1/n) * np.var(L))
```

```
MC (standard) estimator :  22.02151836703495
MC (standard) estimated variance :  0.2642557054400294
```

1.1.2 ii - Monte-Carlo with Control Variate

Let us consider the control variate :

$$C = (Z_1, \ldots, Z_6)^T$$

With this setting we would like to estimate $\mathbb{E}(X)$ where :

$$X(\lambda) := L - \lambda^T (C - \mathbb{E}(C))$$

In order to have the smallest variance, I took :

$$\lambda^* = \Sigma^{-1} \begin{bmatrix} cov(L, Z_1 - \mathbb{E}(Z_1)) \\ \vdots \\ cov(L, Z_2 - \mathbb{E}(Z_2)) \end{bmatrix}$$

[3]:
```
C = np.array([Z1 - np.mean(Z1),Z2 - np.mean(Z2), Z3 - np.mean(Z3),
              Z4 - np.mean(Z4),Z5 - np.mean(Z5), Z6 - np.mean(Z6)])

cov_vec = np.array([np.cov(L,C[0,])[0,1],np.cov(L,C[1,])[0,1],np.
 ↪cov(L,C[2,])[0,1],
                    np.cov(L,C[3,])[0,1],np.cov(L,C[4,])[0,1],np.
 ↪cov(L,C[5,])[0,1]])
Sigma = np.cov([C[0,],C[1,],C[2,],C[3,],C[4,],C[5,]])
lambda_star = np.dot(np.linalg.inv(Sigma), cov_vec)

L_var_red = L - np.dot(lambda_star, C)
```

```
print("MC (control variate) estimator : ", np.mean(L_var_red))
print("MC (control variate) estimated variance : ", (1/n) * np.var(L_var_red))

# (1-alpha) % confidence interval.
alpha = 0.1
z = scipy.stats.norm.ppf(1-alpha/2)
inf = np.mean(L_var_red) - (z/np.sqrt(n)) * \
      np.sqrt(np.var(L_var_red))
sup = np.mean(L_var_red) + (z/np.sqrt(n)) * \
      np.sqrt(np.var(L_var_red))
print("({})% confidence interval : ".format((1-alpha)*100),"[",inf,",", sup,␣
 ↪"]")
```

```
MC (control variate) estimator :  22.021518367034954
MC (control variate) estimated variance :  0.00392018372895707
(90.0)% confidence interval :  [ 21.91853182700796 , 22.124504907061947 ]
```

1.1.3 iii - Antithetic Variate

As $F_X^{-1}(U) \sim X$ when $U \sim \mathcal{U}([0,1])$, we can simulate an exponential law since its cumulative distribution function is easily invertible.

Since $U \sim 1-U$, we can compute the Antithetic Variate Estimator :

$$\alpha_{AV}^{(n)} = \frac{1}{2n} \sum_{i=0}^{n-1} \phi(U_i) + \phi(1-U_i)$$

where $\phi : u \mapsto \max(F_{\mathcal{E}(1)}^{-1}(u) + F_{\mathcal{E}(2)}^{-1}(u) + F_{\mathcal{E}(4)}^{-1}(u), F_{\mathcal{E}(1)}^{-1}(u) + F_{\mathcal{E}(3)}^{-1}(u), F_{\mathcal{E}(5)}^{-1}(u) + F_{\mathcal{E}(6)}^{-1}(u))$

[4]:
```
U = np.random.uniform(0,1,n)

Z1_bis = [] ; Z1_bis_trans = []
Z2_bis = [] ; Z2_bis_trans = []
Z3_bis = [] ; Z3_bis_trans = []
Z4_bis = [] ; Z4_bis_trans = []
Z5_bis = [] ; Z5_bis_trans = []
Z6_bis = [] ; Z6_bis_trans = []

for i in range(n) :
    Z1_bis.append(inverse_cdf_exponential_law(U[i], 1 / lambda_1))
    Z2_bis.append(inverse_cdf_exponential_law(U[i], 1 / lambda_2))
    Z3_bis.append(inverse_cdf_exponential_law(U[i], 1 / lambda_3))
    Z4_bis.append(inverse_cdf_exponential_law(U[i], 1 / lambda_4))
    Z5_bis.append(inverse_cdf_exponential_law(U[i], 1 / lambda_5))
    Z6_bis.append(inverse_cdf_exponential_law(U[i], 1 / lambda_6))
    # antithetic tranformation of the sample U
    Z1_bis_trans.append(inverse_cdf_exponential_law(1-U[i], 1 / lambda_1))
    Z2_bis_trans.append(inverse_cdf_exponential_law(1-U[i], 1 / lambda_2))
```

```
    Z3_bis_trans.append(inverse_cdf_exponential_law(1-U[i], 1 / lambda_3))
    Z4_bis_trans.append(inverse_cdf_exponential_law(1-U[i], 1 / lambda_4))
    Z5_bis_trans.append(inverse_cdf_exponential_law(1-U[i], 1 / lambda_5))
    Z6_bis_trans.append(inverse_cdf_exponential_law(1-U[i], 1 / lambda_6))

L_anti = []
L_anti_trans = []
for i in range(n) :
    L_anti.append(longest_path_length(Z1_bis[i],Z2_bis[i],Z3_bis[i],
                                      Z4_bis[i],Z5_bis[i],Z6_bis[i]))
    L_anti_trans.append(longest_path_length(Z1_bis_trans[i], Z2_bis_trans[i],
                                            Z3_bis_trans[i], Z4_bis_trans[i],
                                            Z5_bis_trans[i], Z6_bis_trans[i]))
L_anti = np.array(L_anti)
L_anti_trans = np.array(L_anti_trans)

print("MC (antithetic variate) estimator : ",
      0.5 * np.mean(L_anti + L_anti_trans))
print("MC (antithetic variate) estimated variance : ",
      (0.25 / n) * np.var(L_anti + L_anti_trans))

# (1-alpha) % confidence interval.
alpha = 0.1
z = scipy.stats.norm.ppf(1-alpha/2)
inf = 0.5 * np.mean(L_anti + L_anti_trans) - \
      (z/np.sqrt(n)) * np.sqrt(np.var(L_var_red))
sup = 0.5 * np.mean(L_anti + L_anti_trans) + \
      (z/np.sqrt(n)) * np.sqrt(np.var(L_var_red))
print("({})% confidence interval : ".format((1-alpha)*100),"[",inf,",", sup,␣
 ↪"]")
```

```
MC (antithetic variate) estimator :  21.154668733328624
MC (antithetic variate) estimated variance :  0.0820312271911094
(90.0)% confidence interval :  [ 21.05168219330163 , 21.257655273355617 ]
```

1.2 Problem 4

Let us write :

$$A := \{(x, y) \in \mathbb{R}^2 | x \geq 3 \quad and \quad y \geq 3\}$$

.

We want to compute :

$$p := \mathbb{P}((X, Y) \in A) = \mathbb{E}(\mathbb{1}_{\{(X,Y)\in A\}})$$

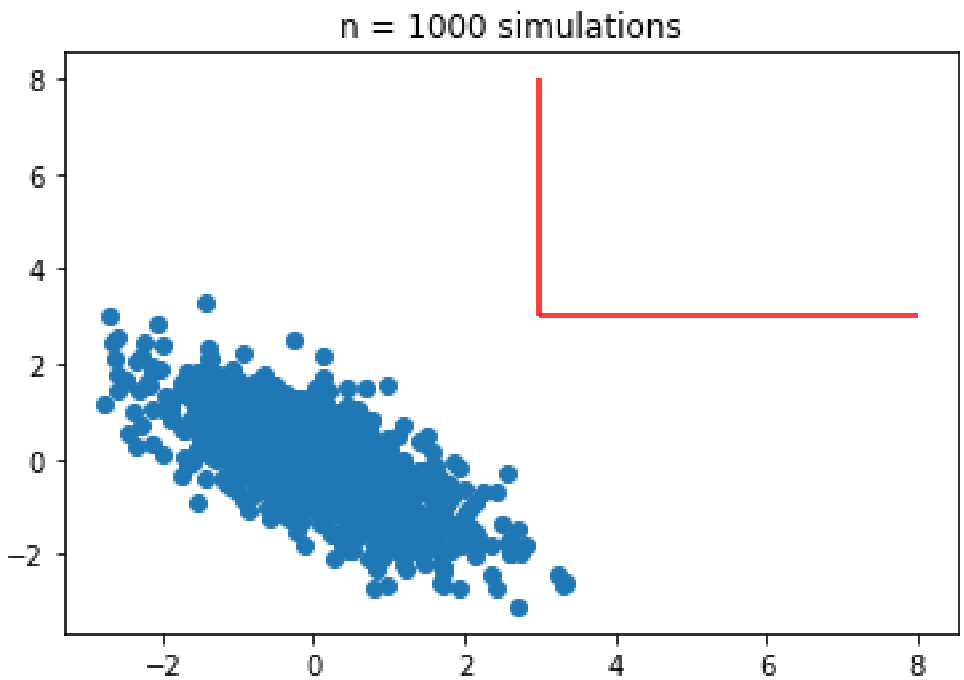
.

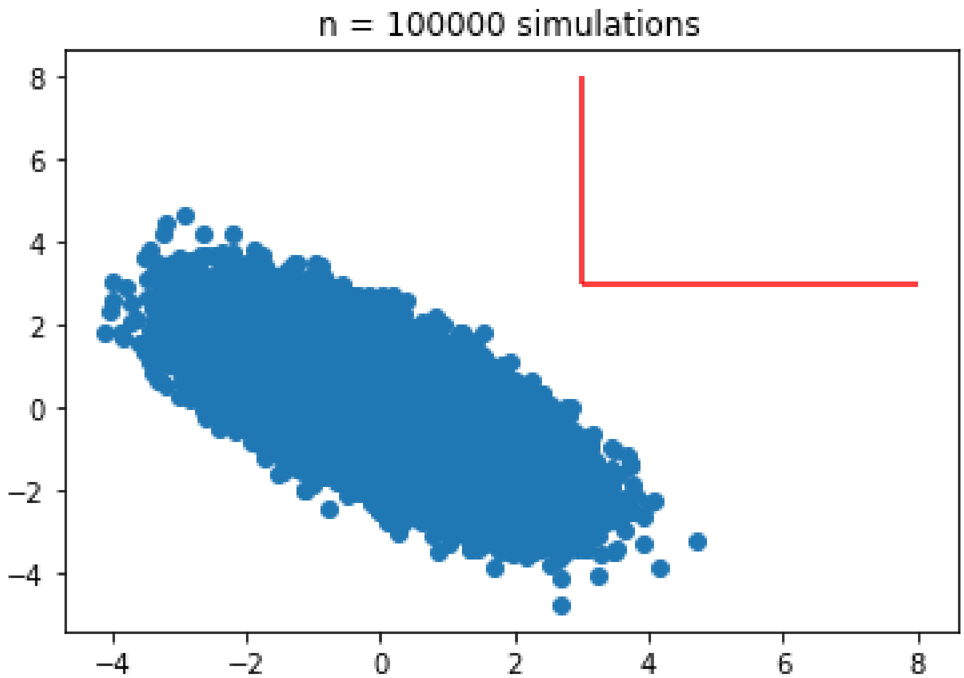
Consequently, we can define the Monte Carlo standard estimator :

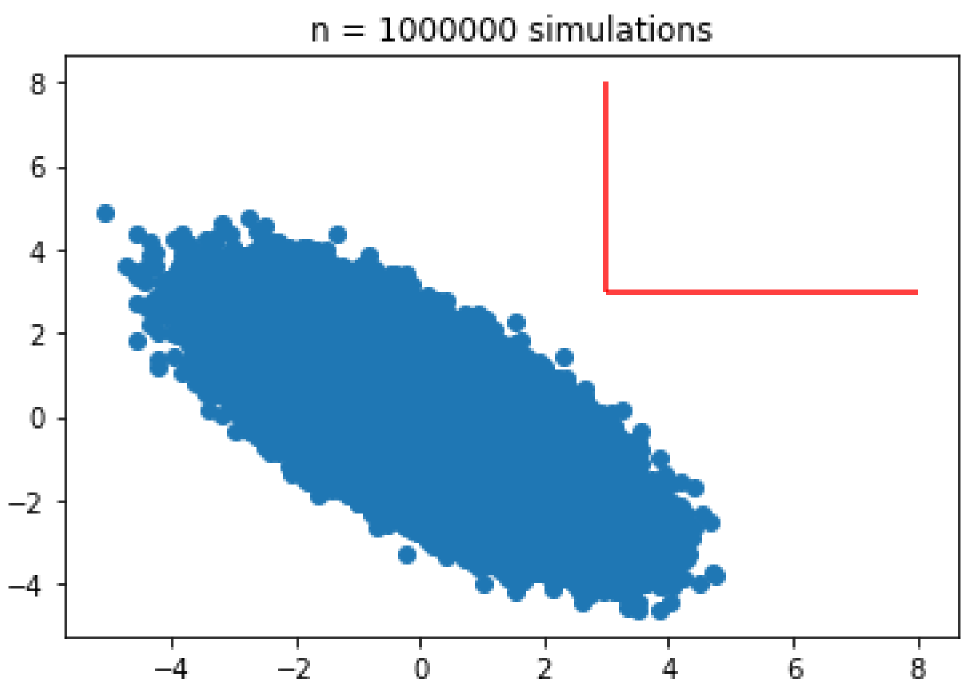
$$p_n := \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_{\{(x_i,y_i)\in A\}}.$$

[5]:

```
mean_1 = np.array([0,0])
cov = np.array([[1,-0.7],[-0.7,1]])
# Simulations
for n in [1000, 100000, 1000000] :
    X_Y = np.random.multivariate_normal(mean_1, cov, size=n)
    # np.cov(X_Y[:,0], X_Y[:,1])
    # Plots
    plt.scatter(x=X_Y[:,0], y=X_Y[:,1])
    plt.vlines(3, ymin=3, ymax = 8, color = "red", linestyles='solid')
    plt.hlines(3, xmin=3, xmax = 8, color = "red")
    plt.title("n = {} simulations".format(n))
    plt.show()
```







In each case we have \$p_{n} = 0 \$ so we cannot compute any confidence interval. In order to use importance sampling, let us consider g the pdf of $\mathcal{N}(\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 & -0.7 \\ -0.7 & 1 \end{bmatrix})$ so we compute :

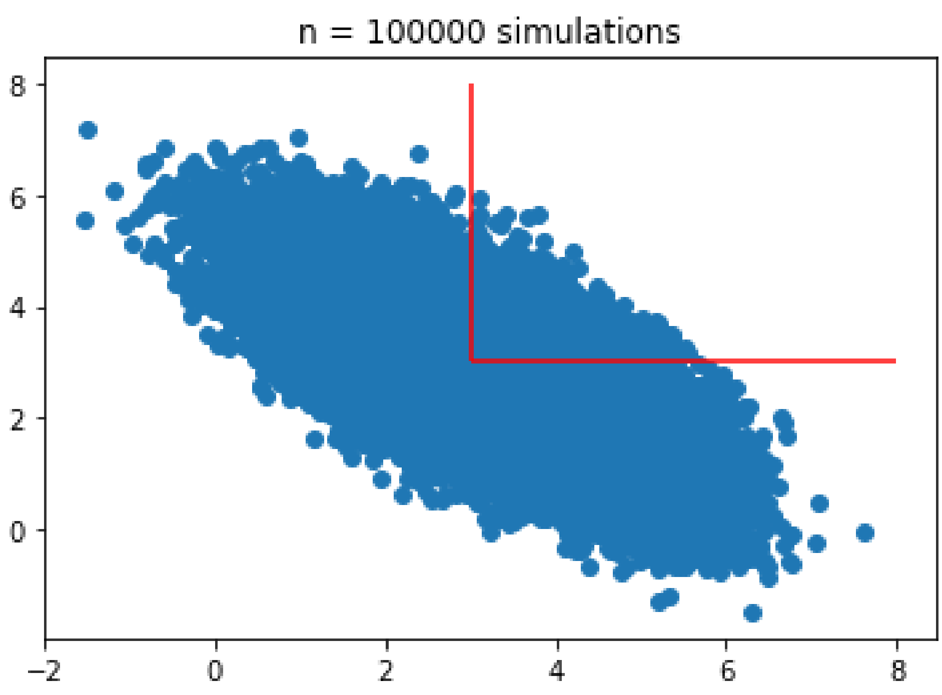
$$\mathbb{E}_g(\mathbb{1}(Z \in A)\frac{f(Z)}{g(Z)})$$

where $Z = (X, Y)$ and f is the pdf of $\mathcal{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -0.7 \\ -0.7 & 1 \end{bmatrix})$.

[6]:
```
# Importance Sampling
n = 100000
mean_2 = np.array([3,3])
Z = np.random.multivariate_normal(mean_2, cov, size=n)
f = scipy.stats.multivariate_normal(mean=[0,0], cov=[[1,-0.7],[-0.7,1]])
g = scipy.stats.multivariate_normal(mean=[3,3], cov=[[1,-0.7],[-0.7,1]])

# Plots
plt.scatter(x=Z[:,0], y=Z[:,1])
plt.vlines(3, ymin=3, ymax = 8, color = "red", linestyles='solid')
plt.hlines(3, xmin=3, xmax = 8, color = "red")
plt.title("n = {} simulations".format(n))
plt.show()

ratio = [] # only if Z belongs to A
for i in range(n) :
    if Z[i][0] >= 3 and Z[i][1] > 3 :
        ratio.append(f.pdf(Z[i]) / g.pdf(Z[i]))
ratio = np.array(ratio)
```



[7]:
```
print("{} simulations".format(n))
print("p_n = ", np.mean(ratio))
print("Estimated variance of the estimator : ", (1/n) * np.var(ratio))

# (1-alpha) % confidence interval.
alpha = 0.1
z = scipy.stats.norm.ppf(1-alpha/2)
inf = np.mean(ratio) - (z/np.sqrt(n)) * np.var(ratio)
sup = np.mean(ratio) + (z/np.sqrt(n)) * np.var(ratio)
print("({})% confidence interval : ".format((1-alpha)*100),"[",inf,",", sup,␣
 ↪"]")
```

```
100000 simulations
p_n =  1.4851135794924825e-15
Estimated variance of the estimator :  3.236071401514026e-34
(90.0)% confidence interval :  [ 1.4851135794924823e-15 , 1.4851135794924827e-15
]
```