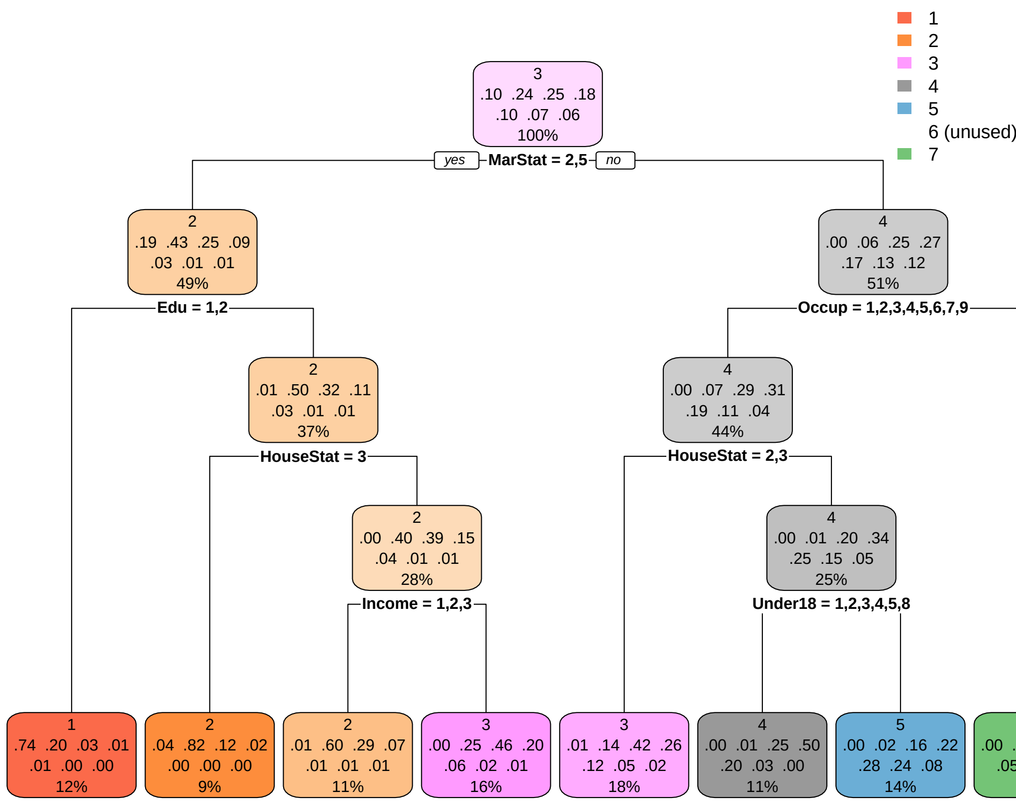
STATS305B - HW1

Thibaud Bruyelle

5/1/2020

Question 1

```r
library(rpart)
library(rpart.plot)
# Question 1
age <- read.csv("handout/Data/age_stats315B.csv", header= T)
for (i in 1:ncol(age)){
  age[,i] <- as.factor(age[,i])
}
tree <- rpart(age ~., data = age, method = "class")
rpart.plot(tree)
```




```r
summary(tree)
```

```
## Call:
## rpart(formula = age ~ ., data = age, method = "class")
##   n= 8710
##
##           CP nsplit rel error    xerror        xstd
## 1 0.12821297      0 1.0000000 1.0000000 0.006179661
## 2 0.08629131      1 0.8717870 0.8717870 0.006791529
## 3 0.06058752      2 0.7854957 0.7854957 0.007024341
```

```
## 4 0.03855569      3 0.7249082 0.7249082 0.007111824
## 5 0.02210832      4 0.6863525 0.6788556 0.007138222
## 6 0.01269890      6 0.6421359 0.6456548 0.007136056
## 7 0.01000000      7 0.6294370 0.6367809 0.007132473
##
## Variable importance
##       Edu     Occup   MarStat HouseStat   DualInc    Income   Under18  TypeHome
##        20        18        17        15        11         9         6         2
##   Persons
##         2
##
## Node number 1: 8710 observations,    complexity param=0.128213
##   predicted class=3  expected loss=0.7504018  P(node) =1
##     class counts:   828  2083  2174  1556   870   636   563
##    probabilities: 0.095 0.239 0.250 0.179 0.100 0.073 0.065
##   left son=2 (4268 obs) right son=3 (4442 obs)
##   Primary splits:
##       MarStat   splits as  RLRRL,     improve=555.6238, (0 missing)
##       HouseStat splits as  RRL,       improve=511.5696, (180 missing)
##       Edu       splits as  LLRRRR,    improve=499.3220, (76 missing)
##       Occup     splits as  RLRRRLLRL, improve=403.8474, (0 missing)
##       Income    splits as  LRRRRRRRR, improve=354.8621, (338 missing)
##   Surrogate splits:
##       DualInc   splits as  LRR,       agree=0.832, adj=0.658, (0 split)
##       HouseStat splits as  RLL,       agree=0.724, adj=0.437, (0 split)
##       Occup     splits as  RLLRRLRRL, agree=0.699, adj=0.386, (0 split)
##       Income    splits as  LLLLRRRRR, agree=0.663, adj=0.312, (0 split)
##       Edu       splits as  LLRRRR,    agree=0.600, adj=0.184, (0 split)
##
## Node number 2: 4268 observations,    complexity param=0.08629131
##   predicted class=2  expected loss=0.5707591  P(node) =0.4900115
##     class counts:   816  1832  1063   376   111    44    26
##    probabilities: 0.191 0.429 0.249 0.088 0.026 0.010 0.006
##   left son=4 (1050 obs) right son=5 (3218 obs)
##   Primary splits:
##       Edu       splits as  LLRRRR,     improve=562.3539, (38 missing)
##       HouseStat splits as  RRL,        improve=310.8559, (92 missing)
##       Under18   splits as  RLLLLLLLLR, improve=295.2636, (0 missing)
##       Income    splits as  LRRRRRRRR,  improve=261.9943, (178 missing)
##       Occup     splits as  RLRRRLRRL,  improve=229.7293, (0 missing)
##   Surrogate splits:
##       Under18 splits as  RRLLLLRLLR, agree=0.806, adj=0.210, (38 split)
##       Occup   splits as  RRRRRLRRR,  agree=0.758, adj=0.016, (0 split)
##
## Node number 3: 4442 observations,    complexity param=0.06058752
##   predicted class=4  expected loss=0.7343539  P(node) =0.5099885
##     class counts:    12   251  1111  1180   759   592   537
##    probabilities: 0.003 0.057 0.250 0.266 0.171 0.133 0.121
##   left son=6 (3806 obs) right son=7 (636 obs)
##   Primary splits:
##       Occup     splits as  LLLLLLLRL,  improve=322.67010, (0 missing)
##       Under18   splits as  RLLLLLLLLR, improve=145.86890, (0 missing)
##       HouseStat splits as  RLL,        improve=101.76260, (88 missing)
##       DualInc   splits as  RLR,        improve= 94.68224, (0 missing)
```

```
##       Persons   splits as  RRLLLLLLL,  improve= 93.83632, (122 missing)
##   Surrogate splits:
##       MarStat splits as  L-LR-,      agree=0.862, adj=0.033, (0 split)
##       Under18 splits as  LLLLLLLLLR, agree=0.857, adj=0.002, (0 split)
##
## Node number 4: 1050 observations
##   predicted class=1  expected loss=0.2590476  P(node) =0.1205511
##     class counts:   778   214    33    11     8     5     1
##    probabilities: 0.741 0.204 0.031 0.010 0.008 0.005 0.001
##
## Node number 5: 3218 observations,    complexity param=0.02210832
##   predicted class=2  expected loss=0.4972032  P(node) =0.3694604
##     class counts:    38  1618  1030   365   103    39    25
##    probabilities: 0.012 0.503 0.320 0.113 0.032 0.012 0.008
##   left son=10 (817 obs) right son=11 (2401 obs)
##   Primary splits:
##       HouseStat splits as  RRL,       improve=174.1940, (83 missing)
##       Edu       splits as  --LLRR,    improve=165.2050, (27 missing)
##       Occup     splits as  RLRRRLLRL, improve=164.6654, (0 missing)
##       Persons   splits as  RRLLLLLLL, improve=159.3146, (153 missing)
##       Income    splits as  LLRRRRRRR, improve= 97.9920, (114 missing)
##   Surrogate splits:
##       Under18 splits as  RLLLLLLLR-R, agree=0.769, adj=0.100, (83 split)
##       Persons splits as  RRRLLLRLL,  agree=0.746, adj=0.011, (0 split)
##
## Node number 6: 3806 observations,    complexity param=0.03855569
##   predicted class=4  expected loss=0.6918024  P(node) =0.436969
##     class counts:    11   249  1111  1173   726   402   134
##    probabilities: 0.003 0.065 0.292 0.308 0.191 0.106 0.035
##   left son=12 (1596 obs) right son=13 (2210 obs)
##   Primary splits:
##       HouseStat splits as  RLL,        improve=90.42640, (70 missing)
##       Under18   splits as  RLLLLLLLL-, improve=73.85239, (0 missing)
##       TypeHome  splits as  RRLRL,      improve=48.33088, (0 missing)
##       LiveBA    splits as  LLLLR,      improve=36.00277, (381 missing)
##       Persons   splits as  RRLLLLLLR,  improve=35.43460, (87 missing)
##   Surrogate splits:
##       TypeHome splits as  RRLRL,     agree=0.801, adj=0.526, (70 split)
##       Income   splits as  LLLLLRRRR, agree=0.691, adj=0.262, (0 split)
##       MarStat  splits as  R-LR-,     agree=0.645, adj=0.153, (0 split)
##       DualInc  splits as  LRR,       agree=0.641, adj=0.142, (0 split)
##       Occup    splits as  RLLLRLL-L, agree=0.632, adj=0.121, (0 split)
##
## Node number 7: 636 observations
##   predicted class=7  expected loss=0.3663522  P(node) =0.07301952
##     class counts:     1     2     0     7    33   190   403
##    probabilities: 0.002 0.003 0.000 0.011 0.052 0.299 0.634
##
## Node number 10: 817 observations
##   predicted class=2  expected loss=0.1811506  P(node) =0.09380023
##     class counts:    31   669    95    15     2     4     1
##    probabilities: 0.038 0.819 0.116 0.018 0.002 0.005 0.001
##
## Node number 11: 2401 observations,    complexity param=0.02210832
```

```
##   predicted class=2  expected loss=0.604748  P(node) =0.2756602
##     class counts:     7   949   935   350   101    35    24
##    probabilities: 0.003 0.395 0.389 0.146 0.042 0.015 0.010
##   left son=22 (976 obs) right son=23 (1425 obs)
##   Primary splits:
##       Income  splits as  LLLRRRRRR, improve=96.70470, (68 missing)
##       Occup   splits as  RLRRRLLRR, improve=85.93692, (0 missing)
##       Edu     splits as  --LLRR,    improve=83.32219, (19 missing)
##       Persons splits as  RRRLLLLLL, improve=38.69356, (117 missing)
##       LiveBA  splits as  LLLRR,     improve=19.48925, (221 missing)
##   Surrogate splits:
##       Occup    splits as  RRRRLLLRL,  agree=0.700, adj=0.263, (68 split)
##       Edu      splits as  --LRRR,     agree=0.625, adj=0.077, (0 split)
##       TypeHome splits as  RRRLL,      agree=0.611, adj=0.043, (0 split)
##       Under18  splits as  RRRRLRLR-R, agree=0.595, adj=0.004, (0 split)
##
## Node number 12: 1596 observations
##   predicted class=3  expected loss=0.5814536  P(node) =0.1832377
##     class counts:     9   216   668   416   184    73    30
##    probabilities: 0.006 0.135 0.419 0.261 0.115 0.046 0.019
##
## Node number 13: 2210 observations,    complexity param=0.0126989
##   predicted class=4  expected loss=0.6574661  P(node) =0.2537313
##     class counts:     2    33   443   757   542   329   104
##    probabilities: 0.001 0.015 0.200 0.343 0.245 0.149 0.047
##   left son=26 (975 obs) right son=27 (1235 obs)
##   Primary splits:
##       Under18 splits as  RLLLLLR-L-, improve=78.61524, (0 missing)
##       Persons splits as  RRLLLLRLR,  improve=40.62282, (46 missing)
##       DualInc splits as  RLR,        improve=20.79117, (0 missing)
##       LiveBA  splits as  RLLLR,      improve=15.60334, (219 missing)
##       MarStat splits as  L-LR-,      improve=12.78300, (0 missing)
##   Surrogate splits:
##       Persons  splits as  RRLLLLLRL, agree=0.833, adj=0.622, (0 split)
##       Ethnic   splits as  RLLRLRRR,  agree=0.573, adj=0.032, (0 split)
##       Occup    splits as  RRLRLLR-R, agree=0.566, adj=0.015, (0 split)
##       TypeHome splits as  RRLRR,     agree=0.561, adj=0.004, (0 split)
##
## Node number 22: 976 observations
##   predicted class=2  expected loss=0.397541  P(node) =0.1120551
##     class counts:     5   588   285    65    12     9    12
##    probabilities: 0.005 0.602 0.292 0.067 0.012 0.009 0.012
##
## Node number 23: 1425 observations
##   predicted class=3  expected loss=0.5438596  P(node) =0.1636051
##     class counts:     2   361   650   285    89    26    12
##    probabilities: 0.001 0.253 0.456 0.200 0.062 0.018 0.008
##
## Node number 26: 975 observations
##   predicted class=4  expected loss=0.4974359  P(node) =0.1119403
##     class counts:     1    14   241   490   192    34     3
##    probabilities: 0.001 0.014 0.247 0.503 0.197 0.035 0.003
##
## Node number 27: 1235 observations
```

```
##   predicted class=5  expected loss=0.7165992  P(node) =0.141791
##     class counts:     1    19   202   267   350   295   101
##    probabilities: 0.001 0.015 0.164 0.216 0.283 0.239 0.082
```

```
# pruned <- prune(tree, cp = 0.012699)
# rpart.plot(pruned)
```

The plot of the tree shows 7 splits and 15 nodes (leaves included). We also notice that there is no prediction for people who are between 55 and 64 years old. It seems relevant that the marital status is a great split variable to classify since the overall population gets married/dies at approximately the same age. Furthermore, education and occupation are also good split variables because people are gathered according to their age. For instance, most people at highschool have the same age.

(a)

Yes, some surrogates variables were used during the construction. Let us give an explanation of the output from `summary(tree)` : for the root node that uses the marital status to do the split, there were no missing values so there was no need to use any surrogate variables. Conversely, for node 2 (`Education`) there were 38 missing values and the surrogate variable `Under18` was used to handle these missing values.

(b)

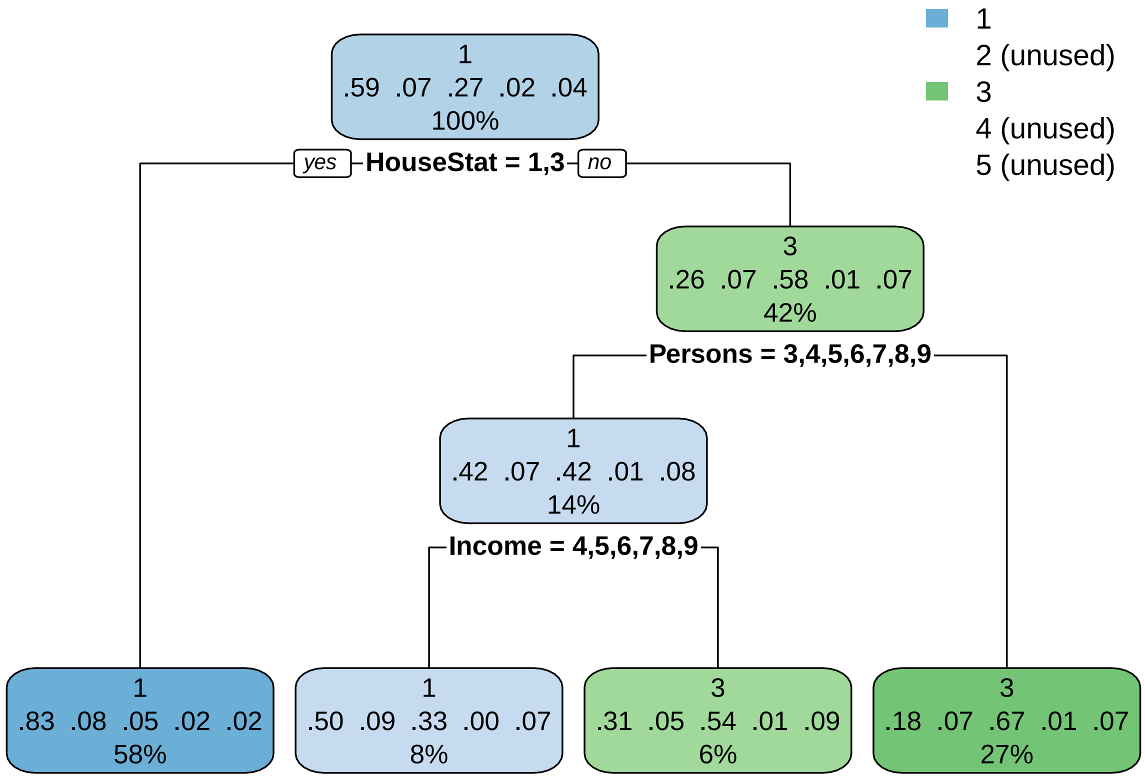
```
new = matrix(ncol = 13, nrow =1)
new = data.frame(new)
new[1,] = as.factor(c(6,3,1,5,6,1,1,1,4,0,2,7,3))
colnames(new) <- colnames(age)[-1]
print(predict(tree, new))
```

```
##             1         2         3          4          5           6          7
## 1 0.005122951 0.602459 0.2920082 0.06659836 0.01229508 0.009221311 0.01229508
```

Therefore my predicted age is 18 - 24 years olds which is great because I am 23.

Question 2

```
housetype <- read.csv("handout/Data/housetype_stats315B.csv", header =T)
for (i in 1:ncol(housetype)){
  housetype[,i] <- as.factor(housetype[,i])
}
housetype_tree <- rpart(TypeHome ~. , data = housetype, method = "class")
rpart.plot(housetype_tree)
```




```
fit.val <- predict(housetype_tree, housetype[,-1], type = "class")
table(housetype$TypeHome, fit.val)
```

```
##    fit.val
##        1    2    3    4    5
##   1 4707    0  612    0    0
##   2  475    0  199    0    0
##   3  510    0 1944    0    0
##   4  132    0   30    0    0
##   5  181    0  223    0    0
```

```
1 - (sum(diag(table(housetype$TypeHome, fit.val))))/length(fit.val)
```

```
## [1] 0.2620659
```

```
# pruned <- prune(housetype_tree, cp = 0.1)
# rpart.plot(pruned)
# fit.val_pruned <- predict(pruned, housetype[,-1], type="class")
# c = table(actual = housetype$TypeHome, fitted = fit.val_pruned)
# 1 - (sum(diag(c)))/length(fit.val_pruned)
```

The model returns an optimal tree with 3 splits and 7 nodes. It is surprising to notice that only class 1 and 3 are predicted by the model. It is probably due to the fact that 86 % of the data represents these two classes but this is still a weakness of the model. We can also see that the prediction of class 1 is straightfoward in most cases. Indeed, when people fall in the 'Own' category, they are directly predicted as "House", which makes sense. The misclassification error with the optimal tree is : 0.2620659. As an alternative example, with a pruned tree that provides 2 splits, the misclassification error becomes 0.2763786.

Question 3

- Reason 1 : If our model is not trained enough, we will underfit the data and consequently, when trying to do predicitions on another set of data, we will get large errors. In other words, there is an important bias in our model because of too restrictive assumptions.
- Reason 2 : Conversely, if our model is too much trained, it will overfit the data used to build it. Therefore, when testing it, we will also get large errors because our model will do predictions by only using the training dataset structure and relationships. This error is due to a high variance in our model that is responsible for high sensitivity to small fluctuations.

Question 4

We cannot chose a predicition function among all possible functions for complexity purposes. We need to put constraints and restrictions when we search for the best predicitor because otherwise it would be beyond our computational abilities.

Question 5

The target function f^* can be defiend as : $arg\min_f \mathbb{E}(l(f(X), Y))$ where l is the loss function. In other words, the target function is the function that, for a given measure of the loss, will minimize the error in our predictions. The accuracy of a target function depends on the constraints of the class of functions we are working on and also to the nature of the problem.

Question 6

No it cannot always be a good surrogate for prediction risk. Indeed, prediction error on the training data can be very low but if our model is overfitted, then the error on the actual population will be much higher. As an example, classification trees are prone to high variance so they easily overfit.If there is no overfitting and underfitting, it might be an option to use the empirical risk classification.

Question 7

Let us assume that the misclassification loss $l_{l,k} = l(c_l, c_k)$ is such that : $\boxed{l_{l,k} = I(k \neq l}$). Then the misclassification risk when predicting $c(\underline{X}) = c_k$ is given by $r_k = \mathbb{E}_{Y,\underline{X}}(l(Y, c_k)) = \sum_{l=1}^{K} l_{l,k}\mathbb{P}(\{Y = c_l|\underline{X}\}) = 1 - \mathbb{P}(Y = c_k|\underline{X}) = \mathbb{P}(Y \neq c_k|\underline{X})$. The latter is equal to the error rate. Then Bayes optimal prediction rule satisfies : $\boxed{k^* = arg \min_{1 \leq k \leq K} \mathbb{P}(Y \neq c_k|\underline{X})}$ with the optimal classifier $c^*(\underline{X}) = k^*$.

Question 8

It is not always true because wrong estimates of $(\mathbb{P}(Y \neq c_k|\underline{X}))_{1 \leq k \leq K}$ can lead to a low error rate (by choosing the wrong optimal rule and then computing the wrong error rate). In this case, we would be mistaken if we thought that our estimations og these probabilities are good.

Question 9

We are always looking for models with small bias (when we do too restrictive assumptions) and small variance (high sensitivity to small fluctuations usually caused by overfitting). However models with small variance usually have a high bias and on the contrary, models with small bias have a high variance. As a consequence, there is tradeoff between these two effects that we want to minimize.

Question 10

Surrogate variables are meant to mimic the split of a primary variable so it makes no sense to use them as primary split variables because the split is not computed with respect to the same criteria. A good surrogate variable may not behave as a good primary variable. Sometimes a variable can be both a primary split variable and a surrogate split variable. We will notice that the way this variable is splitted in both cases is different because it is not meant to have the same functionalities.

Question 11

Let us define $\alpha_N := \displaystyle\sum_{i=1}^{N}[y_i^2 - 2y_i \sum_{m=1}^{M} c_m I(\boldsymbol{x}_i \in \mathcal{R}_m) + \sum_{1 \leq l,m \leq M} c_m c_l I(\boldsymbol{x}_i \in \mathcal{R}_m) I(\boldsymbol{x}_i \in \mathcal{R}_l\})]$. Then we have for $m \in \{1, \ldots, M\}$: $\frac{\partial \alpha_N}{\partial c_m} = 0 - 2 \displaystyle\sum_{i=1}^{N}[I(\boldsymbol{x}_i \in \mathcal{R}_{\Updownarrow}) + c_m I(\boldsymbol{x}_i \in \mathcal{R}_m)]$. Finally since $\hat{c}_m$ satisfies $\frac{\partial \alpha_N}{\partial c_m} = 0$, we have the result.

Question 12

After such a split, $F(\boldsymbol{x})$ becomes $G(\boldsymbol{x}) = \displaystyle\sum_{l=1}^{M+1} c_m I(\boldsymbol{x} \in \mathcal{R}_m)$. The difference of estimated risk is :

$$\hat{r}_F - \hat{r}_G = \sum_i (y_i - \hat{F}(\boldsymbol{x}_i))^2 - (y_i - \hat{G}(\boldsymbol{x}_i))^2 \quad (1)$$

We can notice that : $\hat{c}_{l,r} = \overline{y}_{l,r}$ and that $\hat{c}_m = \frac{1}{n}(n_l \overline{y_l} + n_r \overline{y_r})$. So finally we can rewrite (1) as :

$$\sum_{i=1}^{N}[2y_i(c_l I(\boldsymbol{x}_i \in \mathcal{R}_l) + c_r I(\boldsymbol{x}_i \in \mathcal{R}_r) - c_m I(\boldsymbol{x}_i \in \mathcal{R}_m)) + c_m^2 I(\boldsymbol{x}_i \in \mathcal{R}_m) - c_l^2 I(\boldsymbol{x}_i \in \mathcal{R}_l) - c_r^2 I(\boldsymbol{x}_i \in \mathcal{R}_r)]$$

By replacing c by $\hat{c}$, we get :

$$2[n_l \overline{y}_l^2 + n_r \overline{y}_r^2 - \frac{1}{n}(n_l \overline{y}_l + n_r \overline{y}_r)^2] - n \times \frac{1}{n^2}(n_l \overline{y}_l + n_r \overline{y}_r)^2 - n_l \overline{y}_l^2 - n_r \overline{y}_r^2$$

It yields to :

$$-\frac{2}{n} n_l n_r \overline{y}_l \overline{y}_r + (n_l - \frac{n_l^2}{n})\overline{y}_l^2 - (n_r - \frac{n_r^2}{n})\overline{y}_r^2 = \frac{n_l n_r}{n}(\overline{y}_l - \overline{y}_r)^2$$

since $n = n_l + n_r$.

Question 13

Let us assume that y_o changes from $\mathcal{R}_l$ to $\mathcal{R}_r$. Then $\overline{y}_{l,new} \leftarrow \frac{n_l}{n_l - 1}\overline{y}_{l,old} - \frac{y_o}{n_l - 1}$ and $\overline{y}_{r,new} \leftarrow \frac{n_r}{n_r + 1}\overline{y}_{r,old} + \frac{y_o}{n_r + 1}$. As a consequence, the new improvement can be written as :

$$\boxed{\frac{(n_l - 1)(n_r + 1)}{n}(\frac{n_l}{n_l - 1}\overline{y}_{l,old} - \frac{y_o}{n_l - 1} - \frac{n_r}{n_r + 1}\overline{y}_{r,old} - \frac{y_o}{n_r + 1})^2}$$

Question 14

Enlarging the class of functions to get a better MSE is good idea as long as it requires affordable computational cost. Usually it will reduce the MSE on future data but if the *true* function holds in a smaller class (*e.g.* linear function when we look for more complex polynomial functions), it will overfit the data and MSE will not be better on these data. Conversely, reducing the class of functions can be great for complexity purposes. Nonetheless it implies that our model will be more biased and probably the MSE will be high on future data.

Question 15

One advantadge would be the ability to predict more than two subgroups at each node of the tree. It will be a means to represent more complex patterns in the data. Nonetheless the splits at such a node could become meaningless and less effective. Knowing whether we should do such a split appears to be another issue.

Question 16

With such relationships, the split could approximate linear patterns that exist within the training data which can be great in some cases. If such relationships between the inputs do not exist (because it might simpler or even more complex, ie, quadratic), the model will be error-proned or with a really high variance.