Combinatorial Auctions with Restricted Complements

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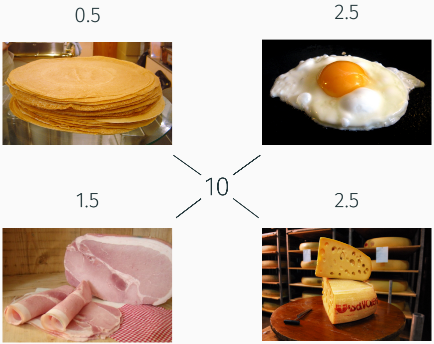
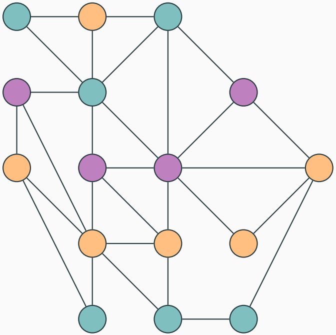
- Auction with n bidders and m items
- Some item groups gain added value when purchased together
- Modeled with hyperedges



1. Polynomial approximation for planar graphs?
2. Polynomial approximation for hypergraphs?
3. Truthful version of this approximation?

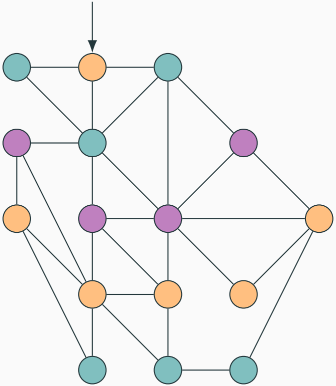
Polynomial approximation for planar graphs



Baker's decomposition [Baker, 1994]

A planar graph G can be partitioned into $P_0, \ldots, P_k$ such that for each i, $G \setminus P_i$ has a treewidth $\leqslant 3k$.

$\hookrightarrow$ very useful for polynomial approximations



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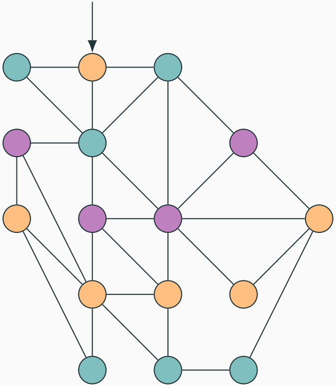
Generalization with minors [DeVos, 2004]

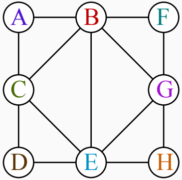
The same method can be applied for any graph that excludes a given minor X.

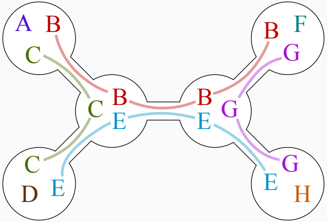


Algorithm

1. Partition G into $P_0, \ldots, P_k$ such that each $G_i = G \setminus P_i$ has a fixed treewidth
2. Compute the tree decomposition of each G_i
3. Compute the optimal allocation A_i for each tree with dynamic programming
4. Return $A^* = \arg\max_{A_i} \sum_{j \in \{\text{bidders}\}} v_j(A_i^{-1}(j))$

$\hookrightarrow$ polynomial $(1+\varepsilon)$-approximation





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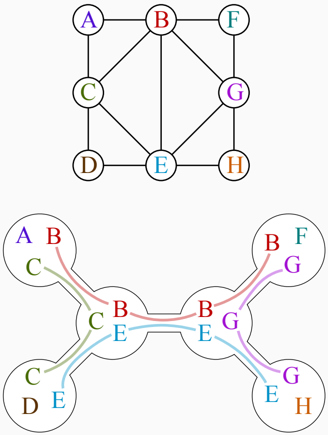
Truthfulness

Once combined with VCG payment, this mechanism is truthful.



Can we generalize this approach for hypergraphs?

We generalize to r-hypergraph with nonnegative hyperedge weight

Polynomial time r-approximation

The welfare maximization problem can be r-approximated in polynomial time.

Known lower bound [Trevisan, 2001]

There is no $r/2^{\mathcal{O}(\sqrt{\log r})}$-approximation.

2 step algorithm:

- (real) linear programming
- randomized assignment

The linear program

$$\text{maximize: } \sum_{i\in\{\text{bidders}\}} \left(\sum_{j\in\{\text{items}\}} w_{ij}\, x_{ij} + \sum_{e\in\{\text{edges}\}_i} w_{ie}\, z_{ie} \right)$$

with

$$\begin{aligned} \sum_{i\in\{\text{bidders}\}} x_{ij} &= 1 && \text{for all item } j \\ z_{ie} &\leqslant x_{ij} && \text{for all player } i\text{, edge } e \text{ and item } j \\ x_{ij} &\geqslant 0 && \text{for all player } i \text{ and item } j \\ z_{ie} &\geqslant 0 && \text{for all player } i \text{ and edge } e \end{aligned}$$

$\hookrightarrow$ Real-valued valuation in polynomial time.

Input: an optimal (real) solution (x^*, z^*) of the linear program

While there exists an unassigned item

1. choose a player i at random
2. choose a threshold $t \in [0, 1]$ at random
3. assign to i every unassigned item with $x^*_{ij} \geqslant t$

Probability that player i gets edge e greater than $\frac{z^*_{ie}}{|e|} \geqslant \frac{z^*_{ie}}{r}$
$\hookrightarrow$ r-approximation

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Not Truthful

Can we find a truthful version of this approach?

Algorithm

1. Compute an oracle in polynomial time
2. Solve a linear program of exponential size in polynomial time
3. Choose an integral solution from a set of feasible integral solutions
4. Random allocation similar to the precedent algorithm

- Expected value is optimal for the feasible solutions
  $\hookrightarrow$ truthful with VCG payments
- $\mathcal{O}(\log^r m)$ approximation

Conclusion

- First step for auctions with complements
- Polynomial time approximations
- Truthful in some cases

Limits:

- Expensive complexity despite polynomial time : $n^{\mathcal{O}\left(\frac{1}{\varepsilon}\right)}$
- Limitation to nonnegative-weight hyperedges

Appendix

Kuratowski's theorem

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.