

COMBINATORIAL AUCTIONS WITH RESTRICTED COMPLEMENTS

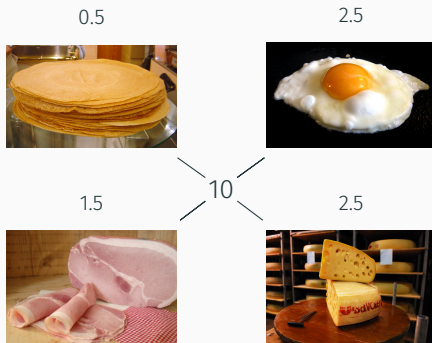
AN ARTICLE BY I. ABRAHAM, M. BABAI OFF, S. DUGHMI AND T. ROUGHGARDEN

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INTRODUCTION

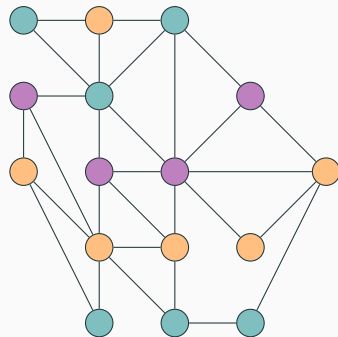
- Auction with n bidders and m items
- Some **item groups** gain added value when purchased together
- Modeled with **hyperedges**



1. Polynomial approximation for **planar graphs**?
2. Polynomial approximation for **hypergraphs**?
3. **Truthful version** of this approximation?

POLYNOMIAL APPROXIMATION FOR PLANAR GRAPHS

BAKER'S TECHNIQUE FOR POLYNOMIAL APPROXIMATION

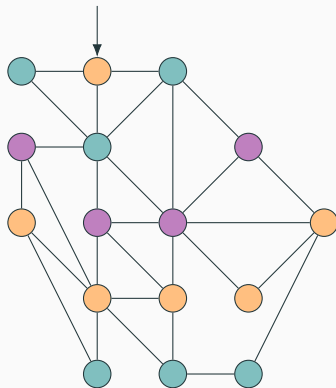


BAKER'S TECHNIQUE FOR POLYNOMIAL APPROXIMATION

Baker's decomposition [Baker, 1994]

A **planar graph** G can be partitioned into P_0, \dots, P_k such that for each i , $G \setminus P_i$ has a treewidth $\leq 3k$.

↪ very useful for **polynomial approximations**



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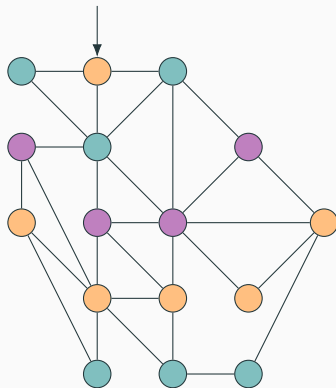
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Generalization with minors [DeVos, 2004]

The same method can be applied for any graph that **excludes a given minor** X .

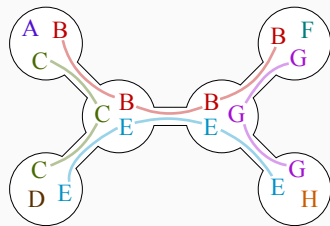
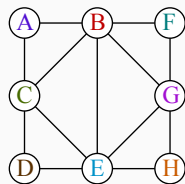


POLYNOMIAL-TIME APPROXIMATION FOR PLANAR GRAPHS

Algorithm

1. **Partition** G into P_0, \dots, P_k such that each $G_i = G \setminus P_i$ has a fixed **treewidth**
2. Compute the **tree decomposition** of each G_i
3. Compute the optimal allocation A_i for each tree with **dynamic programming**
4. Return $A^* = \arg \max_{A_i} \sum_{j \in \{\text{bidders}\}} v_j(A_i^{-1}(j))$

\hookrightarrow polynomial $(1 + \epsilon)$ -approximation



POLYNOMIAL-TIME APPROXIMATION FOR PLANAR GRAPHS

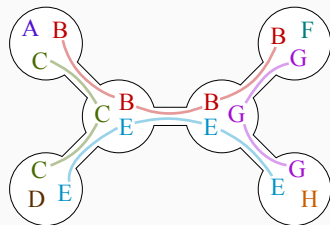
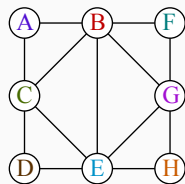
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Truthfulness

Once combined with **VCG payment**, this mechanism is **truthful**.



CAN WE GENERALIZE THIS APPROACH FOR HYPERGRAPHS?

APPROXIMATE WELFARE MAXIMIZATION ALGORITHM

We generalize to r -hypergraph with **nonnegative hyperedge weight**

Polynomial time r -approximation

The welfare maximization problem can be r -approximated in polynomial time.

Known lower bound [Trevisan, 2001]

There is no $r/2^{\mathcal{O}(\sqrt{\log r})}$ -approximation.

2 step algorithm:

- (real) linear programming
- randomized assignment

$$\text{maximize: } \sum_{i \in \{\text{bidders}\}} \left(\sum_{j \in \{\text{items}\}} w_{ij} x_{ij} + \sum_{e \in \{\text{edges}\}_i} w_{ie} z_{ie} \right)$$

with

$$\sum_{i \in \{\text{bidders}\}} x_{ij} = 1 \quad \text{for all item } j$$

$$z_{ie} \leq x_{ij} \quad \text{for all player } i, \text{ edge } e \text{ and item } j$$

$$x_{ij} \geq 0 \quad \text{for all player } i \text{ and item } j$$

$$z_{ie} \geq 0 \quad \text{for all player } i \text{ and edge } e$$

↪ Real-valued valuation in polynomial time.

THE RANDOMIZED ROUNDING ALGORITHM

Input: an optimal (real) solution (x^*, z^*) of the linear program

While there exists an unassigned item

1. choose a player i at random
 2. choose a threshold $t \in [0, 1]$ at random
 3. assign to i every unassigned item with $x_{ij}^* \geq t$
-

Probability that player i gets edge e greater than $\frac{z_{ie}^*}{|e|} \geq \frac{z_{ie}^*}{r}$
 $\hookrightarrow r$ -approximation

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Not Truthful

CAN WE FIND A TRUTHFUL VERSION OF THIS APPROACH?

Algorithm

1. Compute an oracle in polynomial time
 2. Solve a linear program of exponential size in polynomial time
 3. Choose an integral solution from a set of feasible integral solutions
 4. Random allocation similar to the precedent algorithm
- Expected value is optimal for the feasible solutions
 \hookrightarrow truthful with VCG payments
 - $\mathcal{O}(\log^r m)$ approximation

- First step for auctions with complements
- Polynomial time approximations
- Truthful in some cases

Limits:

- Expensive complexity despite polynomial time : $n^{\mathcal{O}(\frac{1}{\varepsilon})}$
- Limitation to nonnegative-weight hyperedges

APPENDIX

PLANAR GRAPHS CHARACTERIZATION WITH MINORS

Kuratowski's theorem

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

