Summary

Dynamic model of manipulators

Robot Dynamics $\Longrightarrow$ Study of the relation between the applied forces/torques and the resulting motion of an industrial manipulator.

Similarly to kinematics, also for the dynamics it is possible to define two "models":

- **Direct model**: once the forces/torques applied to the joints, as well as the joint positions and velocities are known, compute the joint accelerations

$$\ddot{\mathbf{q}} = f(\mathbf{q}, \dot{\mathbf{q}}, \tau)$$

  and then

$$\dot{\mathbf{q}} = \int \ddot{\mathbf{q}}\, dt, \qquad \mathbf{q} = \int \dot{\mathbf{q}}\, dt$$

- **Inverse model**: once the joint accelerations, velocities and positions are known, compute the corresponding forces/torques

$$\tau = f^{-1}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) = g(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q})$$

Dynamic model of manipulators

Normally, a manipulator is composed by an open kinematic chain, and its dynamic model is affected by several “drawbacks（缺陷）”:

- low rigidity (elasticity in the structure and in the joints)
- potentially unknown parameters (dimensions, inertia, mass, . . .)
- dynamic coupling among links.

Other non linear effects are usually introduced by the actuation system:

- friction
- dead zones
- . . .

In any case, in the derivation of the dynamic model, an ideal case of a **series of connected rigid bodies** is made.

Dynamic model of manipulators

Two problems may be defined for the study of the dynamic model:

- **Direct dynamic model**: computation of the time evolution of $\ddot{\mathbf{q}}(t)$ (and then of $\mathbf{q}(t), \dot{\mathbf{q}}(t)$), given the vector of generalized forces (torques and/or forces) $\boldsymbol{\tau}(t)$ applied to the joints and, in case, the external forces applied to the end-effector, and the initial conditions $\mathbf{q}(t = t_0), \dot{\mathbf{q}}(t = t_0)$.

$$\boldsymbol{\tau}(t) \quad \Longrightarrow \quad \ddot{\mathbf{q}}(t) \quad (\ \dot{\mathbf{q}}(t), \mathbf{q}(t)\ )$$

- **Inverse dynamic problem**: computation of the vector $\boldsymbol{\tau}(t)$ necessary to obtain a desired trajectory $\ddot{\mathbf{q}}(t), \dot{\mathbf{q}}(t), \mathbf{q}(t)$, once the forces applied of the end-effector are known.

$$\ddot{\mathbf{q}}(t), \quad \dot{\mathbf{q}}(t), \quad \mathbf{q}(t) \quad \Longrightarrow \quad \boldsymbol{\tau}(t)$$

Dynamic model of manipulators

There are several reasons for studying the dynamics of a manipulator:

- *simulation:* test desired motions without resorting to real experimentation
- *analysis and synthesis* of suitable *control algorithms*
- *analysis of the structural properties* of the manipulator since the design phase.

Two approaches for the definition of the dynamic model:

- **EULER-LAGRANGE** approach.
  First approach to be developed. The dynamic model obtained in this manner is simpler and more intuitive, and also more suitable to understand the effects of changes in the mechanical parameters. The links are considered altogether, and the model is obtained analytically. Drawbacks: the model is obtained starting from the kinetic and potential energies (non intuitive); the model is not computationally efficient.
- **NEWTON-EULER** approach.
  Based on a computationally efficient recursive technique that exploits the serial structure of an industrial manipulator. On the other hand, the mathematical model is not expressed in closed form.

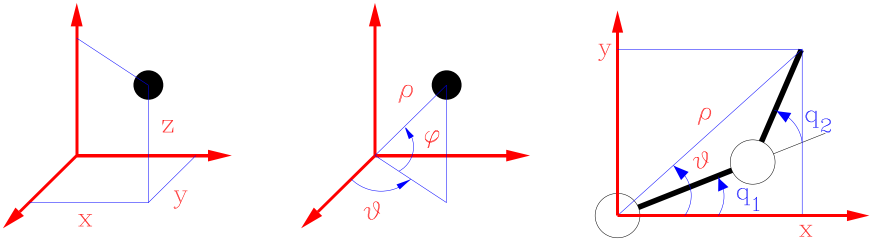
Obviously, the two techniques are equivalent (provide the same results).

Euler-Lagrange model

Generalized variables, or *Lagrange coordinates*:

Independent variables used to describe the position of rigid bodies in the space.

For the same physical system, more choices for the Lagrangian coordinates are usually possible.



In robotics $\Longrightarrow$ *joint variables* $q_1, q_2, \ldots q_n$.

Euler-Lagrange model

From physics, we know that it is possible to define:

1. The *Kinetic Energy* function $K(q, \dot{q})$
2. The *Potential Energy* function $P(q)$

and therefore the *Lagrangian function* $\mathcal{L}(q, \dot{q}) = K(q, \dot{q}) - P(q)$

The Euler-Lagrange equations are

$$\psi_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} \qquad i = 1, \ldots, n$$

being ψ_i the non-conservative (external or dissipative) generalized forces performing work on q_i. In robotics, we consider:

τ_i	joint actuator torque
$\left[\mathbf{J}^T \mathbf{F}_c\right]_i$	term due to external (contact) forces
$d_{ii}\ \dot{q}_i$	joint friction torque

Therefore: $\psi_i = \tau_i + \left[\mathbf{J}^T \mathbf{F}_c\right]_i - d_{ii}\dot{q}_i.$

Euler-Lagrange model

Since the potential energy does not depend on the velocity, the Euler-Lagrange equations can be rewritten as

$$\psi_i = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i} \qquad i = 1, \ldots, n \tag{1}$$

This formulation is more convenient since in robotics it is possible to compute quite easily the terms K and P from the geometric properties of the manipulator. Then, by applying (1), the dynamic model is obtained.

Note that

$$K = \sum_{i=1}^{n} K_i \qquad P = \sum_{i=1}^{n} P_i$$

Euler-Lagrange model

Computation of the Kinetic Energy. For a rigid body B:

- The mass can be computed by

$$m = \int_B \rho(x, y, z)\, dx\, dy\, dz$$

where $\rho(x, y, z)$ is the mass density (assumed constant): $\rho(x, y, z) = \rho$.

- The center of mass (CoM) is

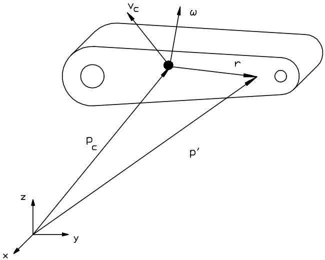
$$\mathbf{p}_C = \frac{1}{m} \int_B \mathbf{p}(x, y, z) \rho\, dx\ dy\ dz = \frac{1}{m} \int_B \mathbf{p}(x, y, z)\, dm$$

- The kinetic energy results as

$$\begin{aligned} K &= \frac{1}{2} \int_B \mathbf{v}^T(x, y, z) \mathbf{v}(x, y, z) \rho\, dx\, dy\, dz \\ &= \frac{1}{2} \int_B \mathbf{v}^T(x, y, z) \mathbf{v}(x, y, z)\, dm \end{aligned}$$

Euler-Lagrange model

Let assume that $\mathbf{v}_C$ and $\boldsymbol{\omega}$, i.e. the translational and rotational velocities of the center of mass, are known with respect to an inertial frame $\mathcal{F}_0$.



The velocity of a generic point $\mathbf{p}'$ of the body is

$$\mathbf{v} = \mathbf{v}_C + \omega \times (\mathbf{p}' - \mathbf{p}_C) = \mathbf{v}_C + \omega \times \mathbf{r} \tag{2}$$

The velocity expressed in a frame $\mathcal{F}$' fixed to the rigid body may be computed by introducing the rotation matrix $\mathbf{R}$ betweeen $\mathcal{F}$' and $\mathcal{F}_0$

$$\mathbf{R}^T\mathbf{v} = \mathbf{R}^T(\mathbf{v}_C + \omega \times \mathbf{r}) = \mathbf{R}^T\mathbf{v}_C + (\mathbf{R}^T\omega) \times (\mathbf{R}^T\mathbf{r})$$

Therefore

$$\mathbf{v}' = \mathbf{v}'_C + \omega' \times \mathbf{r}'$$

Euler-Lagrange model

Since the product $\omega \times \mathbf{r}$ in (2) can be expressed as $\mathbf{S}(\boldsymbol{\omega})\,\mathbf{r}$, we have:

$$
\begin{aligned}
K &= \frac{1}{2}\int_B \mathbf{v}^T(x,y,z)\mathbf{v}(x,y,z)\,dm \\
&= \frac{1}{2}\int_B (\mathbf{v}_C + \mathbf{S}\mathbf{r})^T(\mathbf{v}_C + \mathbf{S}\mathbf{r})\;\,dm \\
&= \frac{1}{2}\int_B \mathbf{v}_C^T\mathbf{v}_C\;\,dm + \frac{1}{2}\int_B \mathbf{r}^T\mathbf{S}^T\mathbf{S}\mathbf{r}\;\,dm + \int_B \mathbf{v}_C^T\mathbf{S}\mathbf{r}\;\,dm \\
&= \frac{1}{2}\int_B \mathbf{v}_C^T\mathbf{v}_C\;\,dm + \frac{1}{2}\int_B \mathbf{r}^T\mathbf{S}^T\mathbf{S}\mathbf{r}\;\,dm
\end{aligned}
$$

As a matter of fact, from the definition of CoM ($\int_B \mathbf{r}\,dm = \int_B(\mathbf{p}_C - \mathbf{p})dm = 0$):

$$
\int_B \mathbf{v}_C^T\mathbf{S}\mathbf{r}\;\,dm = \mathbf{v}_C^T\mathbf{S}\int_B \mathbf{r}\;\,dm = 0
$$

Euler-Lagrange model

In conclusion

$$K = \frac{1}{2}\int_B \mathbf{v}_C^T \mathbf{v}_C \; dm + \frac{1}{2}\int_B \mathbf{r}^T \mathbf{S}^T \mathbf{S} \mathbf{r} \; dm$$

The first term depends on the linear velocity $\mathbf{v}_C$ of the center of mass

$$\frac{1}{2}\int_B \mathbf{v}_C^T \mathbf{v}_C \; dm = \frac{1}{2} m \, \mathbf{v}_C^T \mathbf{v}_C$$

For the second term, considering that $\mathbf{a}^T \mathbf{b} = Tr(\mathbf{a}\;\mathbf{b}^T)$ and the particular structure of matrix $\mathbf{S}$, we have

$$\begin{aligned}
\frac{1}{2}\int_B \mathbf{r}^T \mathbf{S}^T \mathbf{S} \mathbf{r} \; dm &= \frac{1}{2}\int_B Tr(\mathbf{S}\mathbf{r}\;\mathbf{r}^T \mathbf{S}^T) \; dm = \frac{1}{2} Tr(\mathbf{S} \int_B \mathbf{r}\mathbf{r}^T \; dm \; \mathbf{S}^T) \\
&= \frac{1}{2} Tr(\mathbf{S}\;\mathbf{J}\;\mathbf{S}^T) = \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I}\; \boldsymbol{\omega} \qquad \mathbf{I} : \textit{body inertia matrix}
\end{aligned}$$

Also this term depends on the velocity of the center of mass (in this case $\boldsymbol{\omega}$).

Euler-Lagrange model

Matrix $\mathbf{J}$, *Euler matrix*, and matrix $\mathbf{I}$, *body inertia matrix*, are symmetric, and have the following general expressions:

$$\mathbf{J} = \begin{bmatrix} \int r_x^2\, dm & \int r_x r_y\, dm & \int r_x r_z\, dm \\ \int r_x r_y\, dm & \int r_y^2\, dm & \int r_y r_z\, dm \\ \int r_x r_z\, dm & \int r_y r_z\, dm & \int r_z^2\, dm \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} \int (r_y^2 + r_z^2)\, dm & -\int r_x r_y\, dm & -\int r_x r_z\, dm \\ -\int r_x r_y\, dm & \int (r_x^2 + r_z^2)\, dm & -\int r_y r_z\, dm \\ -\int r_x r_z\, dm & -\int r_y r_z\, dm & \int (r_x^2 + r_y^2)\, dm \end{bmatrix}$$

The elements of both matrices $\mathbf{J}$ ed $\mathbf{I}$ depend on vector $\mathbf{r}$, i.e. on the position of the generic point of the i-th link with respect to its center of mass, defined in the base frame.

Since the position of the i-th link depends on the configuration of the manipulator, matrices $\mathbf{J}$ and $\mathbf{I}$ are in general functions of the joint variables $\mathbf{q}$!

Examples of body inertia matrices

Homogeneous bodies of mass m, with axes of symmetry.

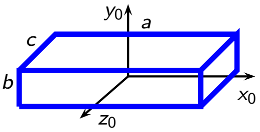
- Parallelepiped with sides a (length/height), b, c (base)

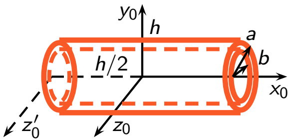
$$I = \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix} = \begin{bmatrix} \frac{1}{12}m(b^2+c^2) & & \\ & \frac{1}{12}m(a^2+c^2) & \\ & & \frac{1}{12}m(a^2+b^2) \end{bmatrix}$$

- Empty cylinder with length h, and external/internal radii a, b

$$I = \begin{bmatrix} \frac{1}{2}m(a^2+b^2) & & \\ & \frac{1}{12}m\left(3(a^2+b^2)+h^2\right) & \\ & & I_{zz} \end{bmatrix}, \qquad \begin{aligned} I_{zz} &= I_{yy} \\ I'_{zz} &= I_{zz} + m\left(\frac{h}{2}\right)^2 \end{aligned}$$

single axis translation theorem





Steiner theorem: changes of body inertia due to translation $\mathbf{p}$ of the frame of computation:

$$I = I_c + m(\mathbf{p}^T\mathbf{p}\,\mathbf{E}_{3\times 3} - \mathbf{p}\mathbf{p}^T)$$

I_c body inertia matrix wrt the center of mass

$\mathbf{E}_{3\times 3}$ (3×3) identity matrix

Euler-Lagrange model

In conclusion, the kinetic energy of a rigid body is defined as (Konig Theorem)

$$K = \frac{1}{2} m \, \mathbf{v}_C^T \mathbf{v}_C + \frac{1}{2} \omega^T \mathbf{I} \omega \tag{3}$$

Both terms depend only on the velocity of the rigid body.

The first term, being related to the magnitude of a vector ($\|\mathbf{v}_C\| = \mathbf{v}_C^T \mathbf{v}_C$), is invariant with respect to the reference frame used to express the velocity:

$$\mathbf{v}_C^T \, \mathbf{v}_C = (\mathbf{R}\mathbf{v}_C)^T (\mathbf{R}\mathbf{v}_C) = \mathbf{v}_C^T (\mathbf{R}^T \mathbf{R}) \mathbf{v}_C, \qquad \forall \, \mathbf{R}$$

This property holds also for the second term: the product $\boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega}$ is invariant with respect to the reference frame. As a matter of fact, the body inertia matrix is transformed according to the following relation:

$$\mathbf{I}' = \mathbf{R} \; \mathbf{I} \; \mathbf{R}^T$$

Then: $\quad \boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega} = \boldsymbol{\omega}'^T \mathbf{I}' \boldsymbol{\omega}' = (\mathbf{R} \; \boldsymbol{\omega})^T (\mathbf{R} \mathbf{I} \mathbf{R}^T)(\mathbf{R} \boldsymbol{\omega}) = \boldsymbol{\omega}^T (\mathbf{R}^T \mathbf{R}) \mathbf{I} (\mathbf{R}^T \mathbf{R}) \boldsymbol{\omega}.$

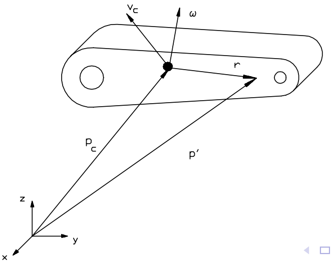
Therefore, being (3) invariant with respect to the reference frame, $\mathcal{F}$ can be chosen in order to simplify the computations required for the evaluation of K.

Euler-Lagrange model

Since the kinetic energy K_i of the generic i-th link is invariant with respect to the reference frame (used to express $\mathbf{v}_C, \boldsymbol{\omega}, \mathbf{I}$), it is convenient to chose a frame $\mathcal{F}_i$ fixed to the link itself, with origin in the center of mass.

In this manner matrix $\mathbf{I}$ is constant and simple to be computed on the basis of the geometric properties of the link.

On the other hand, normally the rotational velocity $\boldsymbol{\omega}$ is defined in the base frame $\mathcal{F}_0$, and therefore it is needed to transform it according to $\mathbf{R}^T\boldsymbol{\omega}$, being $\mathbf{R}$ the rotation matrix between $\mathcal{F}_i$ and $\mathcal{F}_0$ (known from the kinematic model of the manipulator).



Euler-Lagrange model

In conclusion, the kinetic energy of a manipulator can be determined when, for each link, the following quantities are known:

- the link mass m_i;
- the inertia matrix $\mathbf{I}_i$, computed wrt a frame $\mathcal{F}_i$ fixed to the center of mass in which it has a constant expression $\tilde{\mathbf{I}}_i$;
- the linear velocity $\mathbf{v}_{Ci}$ of the center of mass, and the rotational velocity $\boldsymbol{\omega}_i$ of the link (both expressed in $\mathcal{F}_0$);
- the rotation matrix $\mathbf{R}_i$ between the frame fixed to the link and $\mathcal{F}_0$.

The kinetic energy K_i of the i-th link has the form:

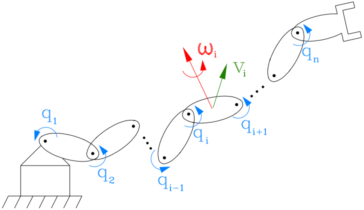
$$K_i = \frac{1}{2} m_i \mathbf{v}_{Ci}^T \mathbf{v}_{Ci} + \frac{1}{2} \omega_i^T \mathbf{R}_i \tilde{\mathbf{I}}_i \mathbf{R}_i^T \omega_i$$

It is now necessary to compute the linear and rotational velocities ($\mathbf{v}_{Ci}$ and $\boldsymbol{\omega}_i$) as functions of the Lagrangian coordinates (i.e. the joint variables $\mathbf{q}$).

Euler-Lagrange model

The end-effector velocity may be computed as a function of the joint velocities $\dot{q}_1, \ldots, \dot{q}_n$ through the Jacobian matrix $\mathbf{J}$. The same methodology can be used to compute the velocity of a generic point of the manipulator, and in particular the velocity $\mathbf{v}_{Ci} = \dot{\mathbf{p}}_{Ci}$ of the center of mass $\mathbf{p}_{Ci}$, that results function of the joint velocities $\dot{q}_1, \ldots, \dot{q}_i$ only:

$$\begin{aligned} \dot{\mathbf{p}}_{Ci} &= \mathbf{J}^i_{v1}\dot{q}_1 + \mathbf{J}^i_{v2}\dot{q}_2 + \ldots + \mathbf{J}^i_{vi}\dot{q}_i = \mathbf{J}^i_v \dot{\mathbf{q}} \\ \boldsymbol{\omega}_i &= \mathbf{J}^i_{\omega 1}\dot{q}_1 + \mathbf{J}^i_{\omega 2}\dot{q}_2 + \ldots + \mathbf{J}^i_{\omega i}\dot{q}_i = \mathbf{J}^i_\omega \dot{\mathbf{q}} \end{aligned}$$



where

$$\begin{aligned} \mathbf{J}^i_v &= \begin{bmatrix} \mathbf{J}^i_{v1} & \ldots & \mathbf{J}^i_{vi} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix} \\ \mathbf{J}^i_\omega &= \begin{bmatrix} \mathbf{J}^i_{\omega 1} & \ldots & \mathbf{J}^i_{\omega i} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix} \end{aligned}$$

with

$$\begin{bmatrix} \mathbf{J}^i_{vj} \\ \mathbf{J}^i_{\omega j} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_{j-1} \times (\mathbf{p}_{Ci} - \mathbf{p}_{j-1}) \\ \mathbf{z}_{j-1} \end{bmatrix} \quad \text{rotational joint}$$

$$\begin{bmatrix} \mathbf{J}^i_{vj} \\ \mathbf{J}^i_{\omega j} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_{j-1} \\ \mathbf{0} \end{bmatrix} \quad \text{linear joint}$$

being $\mathbf{p}_{j-1}$ the position of the origin of the frame associated to the j-th link.

Euler-Lagrange model

In conclusion, for a n dof manipulator we have:

$$
\begin{aligned}
K &= \frac{1}{2}\sum_{i=1}^{n} m_i \mathbf{v}_{Ci}^T \mathbf{v}_{Ci} + \frac{1}{2}\sum_{i=1}^{n} \omega_i^T \mathbf{R}_i \tilde{\mathbf{I}}_i \mathbf{R}_i^T \omega \\
&= \frac{1}{2}\dot{\mathbf{q}}^T \sum_{i=1}^{n} \left[m_i \mathbf{J}_v^{i\;T}(\mathbf{q}) \mathbf{J}_v^i(\mathbf{q}) + \mathbf{J}_\omega^{i\;T}(\mathbf{q}) \mathbf{R}_i \tilde{\mathbf{I}}_i \mathbf{R}_i^T \mathbf{J}_\omega^i(\mathbf{q}) \right] \dot{\mathbf{q}} \\
&= \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j
\end{aligned}
$$

where $\mathbf{M}(\mathbf{q})$ is a $n \times n$, symmetric and positive definite matrix, function of the manipulator configuration $\mathbf{q}$.

Matrix $\mathbf{M}(\mathbf{q})$ is called the **Inertia Matrix** of the manipulator.

Euler-Lagrange model

Computation of the Potential Energy. For rigid bodies, the only potential energy taken into account in the dynamics is due to the gravitational field $\mathbf{g}$. For the generic i-th link

$$P_i = \int_{L_i} \mathbf{g}^T \mathbf{p} \, dm = \mathbf{g}^T \int_{L_i} \mathbf{p} \, dm = \mathbf{g}^T \mathbf{p}_{Ci} m_i$$

The potential energy does not depend on the joint velocities $\dot{\mathbf{q}}$, and may be expressed as a function of the position of the centers of mass. For the whole manipulator:

$$P = \sum_{i=1}^{n} \mathbf{g}^T \mathbf{p}_{Ci} m_i$$

In case of flexible link, one should consider also terms due to elastic forces.

K e P are computed (once m_i and $\tilde{\mathbf{I}}$ are known) with a procedure similar to the one adopted for the forward kinematic model:

- $K \rightarrow$ matrices $\mathbf{J}^i$ e $\mathbf{R}_i$,
- $P \rightarrow \mathbf{p}_{Ci}$ position of the centers of mass

Euler-Lagrange model

Once K and P are known, it is possible to compute the dynamic model of the manipulator. The dynamics is expressed by

$$\psi_k = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} \qquad k = 1, \dots, n$$

The Lagrangian function is

$$\mathcal{L} = K - P = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\dot{q}_i\dot{q}_j - \sum_{i=1}^{n} \mathbf{g}^T \mathbf{p}_{Ci} m_i$$

Then

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{\partial K}{\partial \dot{q}_k} = \sum_{j=1}^{n} M_{kj}\dot{q}_j$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_{j=1}^{n} M_{kj}\ddot{q}_j + \sum_{j=1}^{n} \frac{d\ M_{kj}}{dt}\dot{q}_j = \sum_{j=1}^{n} M_{kj}\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial M_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

Moreover

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial M_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}$$

Euler-Lagrange model

The Lagrangian equations have the following formulation

$$\sum_{j=1}^{n} M_{kj}\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial M_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial M_{ij}}{\partial q_k}\right]\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \psi_k \qquad k = 1,\dots,n$$

By defining the term $h_{kji}(\mathbf{q})$ as

$$h_{kji}(\mathbf{q}) = \frac{\partial M_{kj}(\mathbf{q})}{\partial q_i} - \frac{1}{2}\frac{\partial M_{ij}(\mathbf{q})}{\partial q_k}$$

and $g_k(\mathbf{q})$ as

$$g_k(\mathbf{q}) = \frac{\partial P(\mathbf{q})}{\partial q_k}$$

the following equations are finally obtained

$$\sum_{j=1}^{n} M_{kj}(\mathbf{q})\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} h_{kji}(\mathbf{q})\dot{q}_i\dot{q}_j + g_k(\mathbf{q}) = \psi_k \qquad k = 1,\dots,n$$

Euler-Lagrange model

The elements $M_{kj}(\mathbf{q})$, $h_{ijk}(\mathbf{q})$, $g_k(\mathbf{q})$ are function of the joint position only, and therefore their computation is relatively simple once the manipulator's configuration is known. They have the following physical meaning:

For the acceleration terms:

- M_{kk} is the *moment of inertia* about the k-th joint axis, in a given configuration and considering blocked all the other joints
- M_{kj} is the *inertia coupling*, accounting the effect of acceleration of joint j on joint k

For the quadratic velocity terms:

- $h_{kjj}\dot{q}_j^2$ represents the *centrifugal effect* induced on joint k by the velocity of joint j (notice that $h_{kkk} = \frac{\partial M_{kk}}{\partial q_k} = 0$)
- $h_{kji}\dot{q}_i\dot{q}_j$ represents the *Coriolis effect* induced on joint k by the velocities of joints i and j

For the configuration-dependent terms:

- g_k represents the *torque* generated on joint k by the gravity force acting on the manipulator in the current configuration.

Euler-Lagrange model

Recalling that the non-conservative forces ψ_k are in general composed by

τ_k	joint actuator torque
$\left[\mathbf{J}^T \mathbf{F}_c\right]_k$	external (contact) forces
$d_{kk}\ \dot{q}_k$	joint friction torque

the Lagrangian equations

$$\sum_{j=1}^{n} M_{kj}(\mathbf{q})\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} h_{kji}(\mathbf{q})\dot{q}_i\dot{q}_j + g_k(\mathbf{q}) = \psi_k \qquad k = 1, \ldots, n$$

can be written in matrix form as

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} + \mathbf{J}^T(\mathbf{q})\mathbf{F}_c$$

This matrix equation is known as the **dynamic model of the manipulator**.

Euler-Lagrange model - Some considerations

The product $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \sum_{i=1}^{n} \sum_{j=1}^{n} h_{kji}(\mathbf{q})\dot{q}_i \dot{q}_j$ is a $(n \times 1)$ vector $\mathbf{v}$ whose elements are quadratic functions of the joint velocities $\dot{q}_j$.

The k-th element v_k of this vector is:

$$v_k = \sum_{j=1}^{n} C_{kj} \dot{q}_j$$

where the elements C_{kj} are computed as

$$C_{kj} = \sum_{i=1}^{n} c_{ijk} \dot{q}_i$$

with

$$c_{ijk} = \frac{1}{2} \left[\frac{\partial M_{kj}}{\partial q_i} + \frac{\partial M_{ki}}{\partial q_j} - \frac{\partial M_{ij}}{\partial q_k} \right] \qquad \textit{Christoffel Symbols} \qquad (4)$$

Euler-Lagrange model - Some considerations

The elements of matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ are computed as follows. From

$$\sum_{i=1}^{n} \sum_{j=1}^{n} h_{kji} \dot{q}_i \dot{q}_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial M_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial M_{ij}}{\partial q_k} \right] \dot{q}_i \dot{q}_j$$

by exchanging the sum (i, j) and exploiting the symmetry one obtains

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial M_{kj}}{\partial q_i} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial M_{kj}}{\partial q_i} + \frac{\partial M_{ki}}{\partial q_j} \right]$$

and then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial M_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial M_{ij}}{\partial q_k} \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} \left[\frac{\partial M_{kj}}{\partial q_i} + \frac{\partial M_{ki}}{\partial q_j} - \frac{\partial M_{ij}}{\partial q_k} \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ijk}$$

where $c_{ijk} = \frac{1}{2} \left[\frac{\partial M_{kj}}{\partial q_i} + \frac{\partial M_{ki}}{\partial q_j} - \frac{\partial M_{ij}}{\partial q_k} \right]$ are the so-called *Christoffel Symbols*.

Since matrix $\mathbf{M}(\mathbf{q})$ is symmetric, for a given k then $c_{ijk} = c_{jik}$.

The elements of matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$are then computed as

$$[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})]_{k,j} = \sum_{i=1}^{n} c_{ijk} \dot{q}_i \qquad (5)$$

Euler-Lagrange model - Some considerations

This is not the only possible expression for matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$. In general, any matrix such that

$$\sum_{j=1}^{n} c_{ij} \dot{q}_j = \sum_{j=1}^{n} \sum_{k=1}^{n} h_{ijk} \dot{q}_k \dot{q}_j$$

can be considered. The choice (4) is preferred since in this case the following property is verified.

Property. Matrix $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})$, defined as

$$\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{M}}(\mathbf{q}, \dot{\mathbf{q}}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \tag{6}$$

in which the elements of $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ are defined as

$$c_{ijk} = \frac{1}{2} \left[\frac{\partial M_{kj}}{\partial q_i} + \frac{\partial M_{ki}}{\partial q_j} - \frac{\partial M_{ij}}{\partial q_k} \right] \qquad [\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})]_{k,j} = \sum_{i=1}^{n} c_{ijk} \dot{q}_i$$

results **skew-symmetric**, i.e. $n_{kj} = -n_{jk}, \; n_{kk} = 0$.

Euler-Lagrange model - Some considerations

In fact, by considering the generic element n_{kj}, one obtains

$$
\begin{aligned}
n_{kj} &= \frac{d\ M_{kj}}{dt} - 2[\mathbf{C}]_{kj} \\
&= \sum_{i=1}^{n} \left[\frac{\partial M_{kj}}{\partial q_i} - \left(\frac{\partial M_{kj}}{\partial q_i} + \frac{\partial M_{ki}}{\partial q_j} - \frac{\partial M_{ij}}{\partial q_k} \right) \right] \dot{q}_i \\
&= \sum_{i=1}^{n} \left[\frac{\partial M_{ij}}{\partial q_k} - \frac{\partial M_{ki}}{\partial q_j} \right] \dot{q}_i
\end{aligned}
$$

from which it follows (if indices k and j are exchanged, because of the symmetry of $\mathbf{M}(\mathbf{q})$) that $n_{kj} = -n_{jk}$.

Since matrix $\mathbf{N}$ is skew-symmetrix, then

$$
\mathbf{x}^T\ \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})\ \mathbf{x} = 0, \qquad \forall \mathbf{x}
$$

Euler-Lagrange model - Some considerations

The condition

$$\mathbf{x}^T\ \mathbf{N}(\mathbf{q},\dot{\mathbf{q}})\ \mathbf{x} = 0, \qquad\qquad \forall \mathbf{x}$$

holds since $\mathbf{N}(\mathbf{q},\dot{\mathbf{q}})$ is skew-symmetric, due to the *particular* choice of the elements of matrix $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$. On the other hand, the condition

$$\dot{\mathbf{q}}^T\ \mathbf{N}(\mathbf{q},\dot{\mathbf{q}})\ \dot{\mathbf{q}} = 0$$

holds for *any choice of matrix* $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ (from the energy conservation principle).

The evolution over time of the kinetic energy K must be equal to the work generated by the forces acting at joints:

$$\frac{d\ K}{dt} = \frac{1}{2}\frac{d}{dt}\left(\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}\right) = \dot{\mathbf{q}}^T(\tau - \mathbf{D}\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q}) - \mathbf{J}^T\mathbf{F})$$

The first element is (from the dynamic model $\quad \mathbf{M}\ddot{\mathbf{q}} = -\mathbf{C}\dot{\mathbf{q}} - \mathbf{D}\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q}) + \tau + \mathbf{J}^T\mathbf{F}$):

$$\frac{1}{2}\frac{d}{dt}\left(\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}\right) = \frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{M}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T\mathbf{M}\ddot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{M}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T(-\mathbf{C}\dot{\mathbf{q}} - \mathbf{D}\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q}) + \tau + \mathbf{J}^T\mathbf{F})$$

Euler-Lagrange model - Some considerations

Then

$$\frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{M}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T(-\mathbf{C}\dot{\mathbf{q}} - \mathbf{D}\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q}) + \tau - \mathbf{J}^T\mathbf{F}) = \dot{\mathbf{q}}^T(\tau - \mathbf{D}\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q}) - \mathbf{J}^T\mathbf{F})$$

from which

$$\frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{M}}\dot{\mathbf{q}} = \dot{\mathbf{q}}^T\mathbf{C}\dot{\mathbf{q}} \quad \Longrightarrow \quad \dot{\mathbf{q}}^T(\dot{\mathbf{M}} - 2\mathbf{C})\dot{\mathbf{q}} = 0$$

This equation holds $\forall\dot{\mathbf{q}}$ and without any assumption on matrix $\mathbf{C}(\dot{\mathbf{q}}, \mathbf{q})$ (it holds also if $\mathbf{C}$ is not based on the Cristoffel symbols).

Euler-Lagrange model - Some considerations

In deriving the dynamic model, the actuation system has not been taken into account. This is normally composed by:

- motors
- reduction gears
- trasmission system.

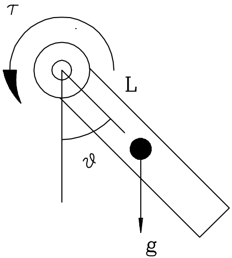
The actuation system has several effects on the dynamics:

- if motors are installed on the links, then masses and inertia are changed
- it introduces its own dynamics (electromechanical, pneumatic, hydraulic, ...) that may be non negligible (e.g. in case of lightweight manipulators)
- it introduces additional nonlinear effects such as backslash, friction, elasticity, ...

Notice that these effects could be considered by introducing suitable terms in the dynamic model derived on the basis of the Euler-Lagrangian formulation.

Example - 1

Dynamic model of a pendulum (one dof manipulator).



Consider

- θ joint variable,
- τ joint torque,
- m mass,
- L distance between center of mass and joint,
- d viscous friction coefficient,
- I inertia seen at the rotation axis.

Kinetic energy: $K = \frac{1}{2} I \dot{\theta}^2$

Potenzial energy: $P = mgL(1 - \cos\theta)$

Lagrangian function $\mathcal{L}$: $\mathcal{L} = \frac{1}{2} I \dot{\theta}^2 - mgL(1 - \cos\theta)$

Example - 1

Lagrangian function: $\mathcal{L} = \frac{1}{2} I \dot{\theta}^2 - mgL(1 - \cos\theta)$

from which

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\dot{\theta}, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\ddot{\theta}, \qquad \frac{\partial \mathcal{L}}{\partial \theta} = -mgL\sin\theta$$

The generalized Lagrangian force in this case must account for the torque applied to the joint and for the friction effect:

$$\psi = \tau - d\dot{\theta}$$

From the general expression

$$\psi = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta}$$

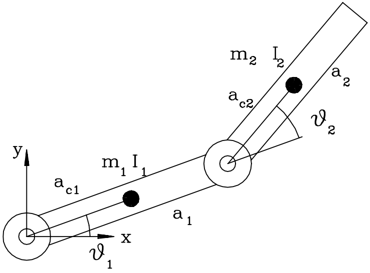
we have the following second order differential equation

$$I\ddot{\theta} + d\dot{\theta} + mgL\sin\theta = \tau$$

Example - 2

Dynamic model of a 2 dof manipulator. Consider:

- θ_i i-th joint variable;
- m_i i-th link mass ;
- $\tilde{I}_i$ i-th link inertia, about an axis through the CoM and parallel to $\mathbf{z}$;
- a_i i-th link length;
- a_{Ci} distance between joint i and the CoM of the i-th link;
- τ_i torque on joint i;
- g gravity force along $\mathbf{y}$;
- P_i, K_i potential and kinetic energy of the i-th link.



The dynamic equations will be obtained in two manners:

a) with the "classic" approach, deriving the Lagrangian function (based on the kinetic and potential energy K, P)

b) exploiting the particular structure of a manipulator (Jacobian, ...).

Example - 2 (classic approach)

We chose as generalized coordinates the joint variables $q_1 = \theta_1;\ q_2 = \theta_2$. The kinetic and potential energies K_i and P_i are:

- link 1:

$$K_1 = \frac{1}{2} m_1 a_{C1}^2 \dot{\theta_1}^2 + \frac{1}{2} \tilde{I}_1 \dot{\theta_1}^2, \qquad P_1 = m_1 g a_{C1} S_1$$

- link 2: in this case, the position and velocity of the CoM are

$$\begin{cases} p_{C2x} = a_1 C_1 + a_{C2} C_{12} \\ p_{C2y} = a_1 S_1 + a_{C2} S_{12} \end{cases} \qquad \begin{cases} \dot{p}_{C2x} = -a_1 S_1 \dot{\theta}_1 - a_{C2} S_{12} (\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{p}_{C2y} = a_1 C_1 \dot{\theta}_1 + a_{C2} C_{12} (\dot{\theta}_1 + \dot{\theta}_2) \end{cases}$$

then

$$K_2 = \frac{1}{2} m_2 \dot{\mathbf{p}}_{C2}^T \dot{\mathbf{p}}_{C2} + \frac{1}{2} \tilde{I}_2 (\dot{\theta}_1 + \dot{\theta}_2)^2, \qquad P_2 = m_2 g (a_1 S_1 + a_{C2} S_{12})$$

where

$$\dot{\mathbf{p}}_{C2}^T \dot{\mathbf{p}}_{C2} = a_1^2 \dot{\theta_1}^2 + a_{C2}^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 a_1 a_{C2} C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2)$$

Example - 2 (classic approach)

Therefore, $\mathcal{L} = K_1 + K_2 - P_1 - P_2$ and

$$
\begin{aligned}
\tau_1 \quad = \quad & [m_1 a_{C1}^2 + \tilde{I}_1 + m_2(a_1^2 + a_{C2}^2 + 2a_1 a_{C2} C_2) + \tilde{I}_2]\ddot{\theta}_1 + \\
& +[m_2(a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{I}_2]\ddot{\theta}_2 - m_2 a_1 a_{C2} S_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) + \\
& +m_1 g a_{C1} C_1 + m_2 g(a_1 C_1 + a_{C2} C_{12})
\end{aligned}
$$

$$
\begin{aligned}
\tau_2 \quad = \quad & [m_2(a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{I}_2]\ddot{\theta}_1 + (m_2 a_{C2}^2 + \tilde{I}_2)\ddot{\theta}_2 + \\
& m_2 a_1 a_{C2} S_2 \dot{\theta}_1^2 + m_2 g a_{C2} C_{12}
\end{aligned}
$$

Example - 2 ('robotics' approach)

The structural properties of the manipulator are exploited for the computation of the kinematic and potential energies. For the computation of the velocities of the CoM, one obtains:

$$\mathbf{J}_v^1 = \begin{bmatrix} -a_{C1}S_1 & 0 \\ a_{C1}C_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{J}_v^2 = \begin{bmatrix} -a_1S_1 - a_{C2}S_{12} & -a_{C2}S_{12} \\ a_1C_1 + a_{C2}C_{12} & a_{C2}C_{12} \\ 0 & 0 \end{bmatrix}$$

and

$$\mathbf{J}_\omega^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{J}_\omega^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

In this particular case, the frames associated to link 1 and 2 have $\mathbf{z}$ axes parallel to the same axis of $\mathcal{F}_0$, and therefore it is not necessary to consider the rotation matrices $\mathbf{R}_1, \mathbf{R}_2$ $(\omega = \omega_z)$.

Example - 2 ('robotics' approach)

The kinetic energy is computed as

$$K = \frac{1}{2}\dot{\mathbf{q}}^T \left[m_1 \mathbf{J}_v^1{}^T \mathbf{J}_v^1 + m_2 \mathbf{J}_v^2{}^T \mathbf{J}_v^2 + \mathbf{J}_\omega^1{}^T \tilde{I}_1 \mathbf{J}_\omega^1 + \mathbf{J}_\omega^2{}^T \tilde{I}_2 \mathbf{J}_\omega^2 \right] \dot{\mathbf{q}}$$

being

$$\mathbf{J}_\omega^1{}^T \tilde{I}_1 \mathbf{J}_\omega^1 + \mathbf{J}_\omega^1{}^T \tilde{I}_2 \mathbf{J}_\omega^2 = \begin{bmatrix} \tilde{I}_1 + \tilde{I}_2 & \tilde{I}_2 \\ \tilde{I}_2 & \tilde{I}_2 \end{bmatrix}$$

The elements of the inertia matrix $\mathbf{M}(\mathbf{q})$ are

$$\begin{aligned} M_{11} &= m_1 a_{C1}^2 + m_2(a_1^2 + a_{C2}^2 + 2a_1 a_{C2} C_2) + \tilde{I}_1 + \tilde{I}_2 \\ M_{12} &= m_2(a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{I}_2 \\ M_{22} &= m_2 a_{C2}^2 + \tilde{I}_2 \end{aligned}$$

Example - 2 ('robotics' approach)

From
$$\begin{aligned} M_{11} &= m_1 a_{C1}^2 + m_2(a_1^2 + a_{C2}^2 + 2a_1 a_{C2} C_2) + \tilde{I}_1 + \tilde{I}_2 \\ M_{12} &= m_2(a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{I}_2 \\ M_{22} &= m_2 a_{C2}^2 + \tilde{I}_2 \end{aligned}$$

The Christoffel symbols $c_{ijk} = \frac{1}{2}\left[\frac{\partial M_{kj}}{\partial q_i} + \frac{\partial M_{ki}}{\partial q_j} - \frac{\partial M_{ij}}{\partial q_k}\right]$ are

$$\begin{aligned} c_{111} &= \frac{1}{2}\frac{\partial M_{11}}{\partial q_1} = 0 \\ c_{121} &= c_{211} = \frac{1}{2}\frac{\partial M_{11}}{\partial q_2} = -m_2 a_1 a_{C2} S_2 = h \\ c_{221} &= \frac{\partial M_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial M_{22}}{\partial q_1} = h \\ c_{112} &= \frac{\partial M_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial M_{11}}{\partial q_2} = -h \\ c_{122} &= c_{212} = \frac{\partial M_{22}}{\partial q_1} = 0 \\ c_{222} &= \frac{\partial M_{22}}{\partial q_2} = 0 \end{aligned}$$

$\Longrightarrow$ Matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} h\dot{\theta}_2 & h(\dot{\theta}_1 + \dot{\theta}_2) \\ -h\dot{\theta}_1 & 0 \end{bmatrix}$$

Example - 2 (‘robotics’ approach)

Matrix $\mathbf{N}(\mathbf{q})$ is

$$\begin{aligned}\mathbf{N}(\mathbf{q}) &= \dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \\ &= \begin{bmatrix} 2h\dot{\theta}_2 & h\dot{\theta}_2 \\ h\dot{\theta}_2 & 0 \end{bmatrix} - 2\begin{bmatrix} h\dot{\theta}_2 & h(\dot{\theta}_1 + \dot{\theta}_2) \\ -h\dot{\theta}_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2h\dot{\theta}_1 - h\dot{\theta}_2 \\ 2h\dot{\theta}_1 + h\dot{\theta}_2 & 0 \end{bmatrix}\end{aligned}$$

As expected, it results skew-symmetric.

Example - 2 (‘robotics’ approach)

For the potential energy, we have:

$$\begin{aligned} P_1 &= m_1 g a_{C1} S_1 \\ P_2 &= m_2 g (a_1 S_1 + a_{C2} S_{12}) \end{aligned}$$

Then

$$\begin{aligned} P &= P_1 + P_2 = (m_1 a_{C1} + m_2 a_1) g S_1 + m_2 g a_{C2} S_{12} \\ g_1 &= \frac{\partial P}{\partial \theta_1} = (m_1 a_{C1} + m_2 a_1) g C_1 + m_2 g a_{C2} C_{12} \\ g_2 &= \frac{\partial P}{\partial \theta_2} = m_2 g a_{C2} C_{12} \end{aligned}$$

Example - 2 ('robotics' approach)

Summarizing, from $\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$ we have

$$\begin{aligned} M_{11}\ddot{\theta}_1 + M_{12}\ddot{\theta}_2 + c_{121}\dot{\theta}_1\dot{\theta}_2 + c_{211}\dot{\theta}_2\dot{\theta}_1 + c_{221}\dot{\theta}_2^2 + g_1 &= \tau_1 \\ M_{21}\ddot{\theta}_1 + M_{22}\ddot{\theta}_2 + c_{112}\dot{\theta}_1^2 + g_2 &= \tau_2 \end{aligned}$$

or

$$\begin{aligned} [m_1 a_{C1}^2 + m_2(a_1^2 + a_{C2}^2 + 2a_1 a_{C2} C_2) + \tilde{I}_1 + \tilde{I}_2]\ddot{\theta}_1 + [m_2(a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{I}_2]\ddot{\theta}_2 & \\ -m_2 a_1 a_{C2} S_2 \dot{\theta}_2^2 - 2m_2 a_1 a_{C2} S_2 \dot{\theta}_1 \dot{\theta}_2 & \\ +(m_1 a_{C1} + m_2 a_1) g C_1 + m_2 g a_{C2} C_{12} &= \tau_1 \\ [m_2(a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{I}_2]\ddot{\theta}_1 + [m_2 a_{C2}^2 + \tilde{I}_2]\ddot{\theta}_2 & \\ +m_2 a_1 a_{C2} S_2 \dot{\theta}_1^2 & \\ +m_2 g a_{C2} C_{12} &= \tau_2 \end{aligned}$$

$\Longrightarrow$ **Same result!**

Properties of the Euler-Lagrangian dynamic model

The Euler-Lagrange dynamic model is characterized by some structural properties, concerning in particular:

1. The inertia matrix $\mathbf{M}(\mathbf{q})$;
2. The vector $\mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q}$;
3. The vectors $\mathbf{g}(\mathbf{q})$ and $\mathbf{D}\ \dot{\mathbf{q}}$;
4. Linearity with respect to the geometric/mechanical parameters.

Properties of the Euler-Lagrangian dynamic model

1. Properties of matrix $\mathbf{M}(\mathbf{q})$

1. $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n\times n}$ is symmetric, positive definite and depends on the manipulator configuration $\mathbf{q}$
2. $\mathbf{M}(\mathbf{q})$ is upper and lower bounded:

$$\mu_1 \mathbf{I} \le \mathbf{M}(\mathbf{q}) \le \mu_2 \mathbf{I}$$

   that is

$$\mathbf{x}^T(\mathbf{M}(\mathbf{q}) - \mu_1 \mathbf{I})\mathbf{x} \ge \mathbf{0} \qquad\qquad \mathbf{x}^T(\mu_2 \mathbf{I} - \mathbf{M}(\mathbf{q}))\mathbf{x} \ge \mathbf{0}$$

3. also $\mathbf{M}^{-1}(\mathbf{q})$ is bounded

$$\frac{1}{\mu_2}\mathbf{I} \le \mathbf{M}^{-1}(\mathbf{q}) \le \frac{1}{\mu_1}\mathbf{I}$$

4. in case of revolute joints, then μ_1, μ_2 are constant (not function of $\mathbf{q}$) since the elements of $\mathbf{M}(\mathbf{q})$ are functions of $sin(q_i)$ or $cos(q_i)$
5. in case of prismatic joints, μ_1, μ_2 may result scalar functions of $\mathbf{q}$
6. since $\mathbf{M}(\mathbf{q})$ is bounded, then

$$M_1 \le ||\mathbf{M}(\mathbf{q})|| \le M_2$$

   for some properly defined norm (1, 2, p, ∞)

Properties of the Euler-Lagrangian dynamic model

2. Properties of vector $\mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$

1. $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ is a quadratic function of $\dot{\mathbf{q}}$
2. the generic k-th element of vector $\mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ can also be witten as

$$
\begin{aligned}
c_k(\mathbf{q}, \dot{\mathbf{q}}) &= \dot{\mathbf{q}}^T \mathbf{S}_k(\mathbf{q})\dot{\mathbf{q}} \\
\mathbf{S}_k(\mathbf{q}) &= \frac{1}{2}\left(\frac{\partial \mathbf{m}_k}{\partial \mathbf{q}} + \left(\frac{\partial \mathbf{m}_k}{\partial \mathbf{q}}\right)^T - \frac{\partial \mathbf{M}}{\partial q_k}\right) \qquad \mathbf{m}_k = \text{k-th col. of } \mathbf{M}
\end{aligned}
$$

3. it results that

$$||\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}|| \leq v_b ||\dot{\mathbf{q}}||^2$$

4. in case of rotative joints, then v_b is constant (not function of $\mathbf{q}$) since we have only transcendental functions ($sin(q_i)$ or $cos(q_i)$)
5. in case of prismatic joints, then v_b may result a scalar function of $\mathbf{q}$
6. for any choice of $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$, then matrix $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{M}}(\mathbf{q}, \dot{\mathbf{q}}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ verifies:

$$\dot{\mathbf{q}}^T \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{0}$$

7. with a proper choice of the elements of $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ (*Christoffel symbols*), matrix $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{M}}(\mathbf{q}, \dot{\mathbf{q}}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric, i.e.

$$\mathbf{x}^T \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{x} = \mathbf{0} \qquad \forall \mathbf{x}$$

Properties of the Euler-Lagrangian dynamic model

3. Properties of vectors $\mathbf{g}(\mathbf{q})$ and $\mathbf{D}\dot{\mathbf{q}}$

1. the friction term $\mathbf{D}\dot{\mathbf{q}}$ is bounded:

$$||\mathbf{D}\dot{\mathbf{q}}|| \leq d_{max}||\dot{\mathbf{q}}||$$

2. the gravity term $\mathbf{g}(\mathbf{q})$ is bounded

$$||\mathbf{g}(\mathbf{q})|| \leq g_b(\mathbf{q})$$

3. in case of revolute joints, g_b is constant (does not depend on $\mathbf{q}$) since q_i depends on sinusoidal functions ($sin(q_i)$ or $cos(q_i)$)
4. in case of prismatic joints, then g_b may result function of $\mathbf{q}$.

Properties of the Euler-Lagrangian dynamic model

4. Linearity properties (in the geometrical/mechanical parameters)

The dynamic model of a manipulator:

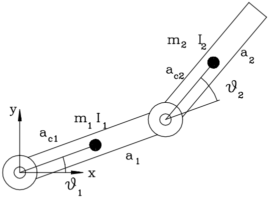
- in general is a non linear function of $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$ and with dynamic coupling effects among the joints,
- is a linear function of the geometrical/mechanical parameters of the links (i.e. masses, inertia, friction, ...)

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau$$

$$\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\alpha = \tau$$

Example

Properties of the dynamic model of a 2 dof manipulator.



Neglecting friction effects we have:

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_1 a_{C1}^2 + m_2(a_1^2 + a_{C2}^2 + 2a_1 a_{C2} C_2) + \tilde{I}_1 + \tilde{I}_2 & m_2(a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{I}_2 \\ m_2(a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{I}_2 & m_2 a_{C2}^2 + \tilde{I}_2 \end{bmatrix}$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = -m_2 a_1 a_{C2} S_2 \begin{bmatrix} 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2 \\ -\dot{\theta}_1^2 \end{bmatrix}, \qquad \mathbf{g}(\mathbf{q}) = \begin{bmatrix} (m_1 a_{C1} + m_2 a_1) g C_1 + m_2 g a_{C2} C_{12} \\ m_2 g a_{C2} C_{12} \end{bmatrix}$$

Consider (for the sake of simplicity) the 1-norm $||\cdot||_1$, and $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$.

Example

1. Bounds on the inertia matrix.

The scalar quantities μ_1, μ_2 can be defined as the minimum and maximum eigenvalues $(\lambda_{min}, \lambda_{max})$ of $\mathbf{M}(\mathbf{q})$, $\forall \mathbf{q}$. Computationally, it is easier to define the scalars M_1, M_2.

The 1–norm of $\mathbf{M}(\mathbf{q})$ is alway defined on the basis of the first column:

$$||\mathbf{M}(\mathbf{q})||_1 = |m_1 a_{C1}^2 + m_2(a_1^2 + a_{C2}^2 + 2a_1 a_{C2} C_2) + \tilde{I}_1 + \tilde{I}_2| + |m_2(a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{I}_2|$$

which is bounded by

$$\begin{aligned} M_1 &= m_1 a_{C1}^2 + m_2(a_1^2 + 2a_{C2}^2) + \tilde{I}_1 + 2\tilde{I}_2 \\ M_2 &= m_1 a_{C1}^2 + m_2(a_1^2 + 2a_{C2}^2 + 3a_1 a_{C2}) + \tilde{I}_1 + 2\tilde{I}_2 \end{aligned}$$

Example

2. Bounds on vector $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}$

It results that

$$\begin{aligned}
||\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}||_1 &= |m_2 a_1 a_{C2} S_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)| + |m_2 a_1 a_{C2} S_2 \dot{\theta}_1^2| \\
&\leq m_2 a_1 a_{C2} |2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2 + \dot{\theta}_1^2| \\
&\leq m_2 a_1 a_{C2} (|\dot{\theta}_1| + |\dot{\theta}_2|)^2 \\
&= v_b ||\dot{\mathbf{q}}||^2
\end{aligned}$$

Moreover $\mathbf{N}(\mathbf{q},\dot{\mathbf{q}}) = \dot{\mathbf{M}}(\mathbf{q},\dot{\mathbf{q}}) - 2\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$, $\quad (h = -m_2 a_1 a_{C2} S_2)$:

$$\begin{aligned}
\mathbf{N}(\mathbf{q}) &= \dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) \\
&= \begin{bmatrix} 2h\dot{\theta}_2 & h\dot{\theta}_2 \\ h\dot{\theta}_2 & 0 \end{bmatrix} - 2 \begin{bmatrix} h\dot{\theta}_2 & h(\dot{\theta}_1 + \dot{\theta}_2) \\ -h\dot{\theta}_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -2h\dot{\theta}_1 - h\dot{\theta}_2 \\ 2h\dot{\theta}_1 + h\dot{\theta}_2 & 0 \end{bmatrix}
\end{aligned}$$

Example

3. Bounds on the gravity vector $\mathbf{g}(\mathbf{q})$

$$
\begin{aligned}
||\mathbf{g}(\mathbf{q})||_1 &= |(m_1 a_{C1} + m_2 a_1) g C_1 + m_2 g a_{C2} C_{12}| + |m_2 g a_{C2} C_{12}| \\
&\leq (m_1 a_{C1} + m_2 a_1) g + 2 m_2 g a_{C2} \\
&= g_b
\end{aligned}
$$

Notice that if one of the joints is prismatic (and therefore a_i, a_{Ci} may vary in time), then v_b, g_b are functions of $\mathbf{q}$.