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Evolving reaction systems

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ABSTRACT

Reaction systems were introduced as a formal model of interactions between biochemical reactions. These interactions, which are based on two mechanisms: facilitation and inhibition, determine the functioning of the living cell. Processes taking place in a reaction system $\mathcal{A}$ are driven by the fixed set A of available reactions provided by $\mathcal{A}$. In this paper we generalize this setup: as a process progresses from a state W to its successor W', the set of available reactions may change from A in W to A' in W'. This new framework of *evolving reaction systems* is introduced and studied in this paper. Also, the notion of enabling equivalence between sets of reactions and the notion of a transformation of a set of reactions are introduced and thoroughly studied.

1. Introduction

Reaction systems were introduced (see [11]) as a formal model of the functioning of the living cell. The underlying idea is that this functioning is determined by the interaction of biochemical reactions in the living cell, where these interactions are driven by two mechanism, facilitation and inhibition – the reactions (through their products) may facilitate or inhibit each other. This model takes into account the basic bioenergetics (flow of energy) of the living cell and the basic fact that the living cell is an open system. Also (because of the level of abstraction it adopts) it is a qualitative rather than a quantitative model.

A biochemical reaction is formalized as a 3-tuple of nonempty sets $b = (R, I, P)$, called a *reaction*, with R and I disjoint, where R is the set of reactants that b needs in order to take place, I is the set of inhibitors – if any of these is present in the current state of the system/cell, then b will not take place, and P is the product set – the set of entities contributed by b to the successor of the current state. Then a *reaction system* is basically a finite set of reactions, which reflects the point of view that the living cell is basically a reactor with a finite set of reactions taking place within it (where the reactor interacts with the environment). Formally, a reaction system is specified as an ordered pair $\mathcal{A} = (S, A)$, where A is a finite set of reactions and S is a finite (background) set containing all entities needed to define reactions in $\mathcal{A}$ and also interactions with the environment.

The notion of reaction system is central for a broad framework of reaction systems, where one considers also various extensions of reaction systems motivated either by biological considerations or by considerations concerned with the need

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to understand the underlying computational nature of the models from this framework, see e.g., the tutorial and survey papers [7–9]. In fact, although the original motivation behind reaction systems came from biology, they became an interesting and novel model of computation, see, e.g., [5,6,10,12,14,16–18].

A central feature of models investigated in the framework of reaction systems is the invariance of the available set of reactions (the set of reactions of the considered reaction system). All processes are supported by this set of reactions, say A: in each state of each process the set of reactions enabled by this state is a subset of A.

In this paper we abandon this "invariance point of view" and consider processes where a transition from a state to state may be accompanied by a change of the available set of reactions. We call this framework "Evolving Reaction Systems". It is motivated by both biological considerations, in particular the evolution of biological systems (Section 7 of this paper deals with this theme), and computational considerations (considering systems where the available transformations change with time is quite traditional in the theory of computation, see, e.g., [4]).

The paper is organized as follows.

In Section 2 we introduce basic notions and notation concerning reactions and reaction systems.

The notion of enabling equivalence of sets of reactions, which is central for this paper, is introduced and analyzed in Section 3.

As discussed above, in this paper we consider the framework of evolving reaction systems where the set of available reactions may change as the given state (say W) is transformed to its successor (say W'). If the set of available reactions at W is A and at W' it is A', then the change from A to A' is governed by a transformation rule as introduced and analyzed in Section 4.

In Section 5 we introduce evolving interactive processes which differ from standard interactive processes considered in reaction systems by the fact that the set of available reactions may change as a process progresses from state to state. These processes are considered in evolving reaction systems.

The main result of this paper, the Invisibility Theorem, is proved in Section 6. It provides conditions under which the changes of the sets of available reactions taking place in an evolving interactive process are not observable (hence invisible), meaning that they are not reflected in the state sequence of the process.

In Section 7 we give an example illustrating the Invisibility Theorem which leads to an interpretation within the framework of evolving reaction systems of the notion of punctuated evolution, see, e.g., [11] and [18].

Finally in Section 8 we provide a brief discussion of the results of this paper.

2. Reactions and reaction systems

Within this paper we will use standard mathematical terminology and notation. More specifically:

The empty set is denoted by $\varnothing$. For sets X and Y, $X \setminus Y$, $X \cup Y$, and $X \cap Y$ denote set difference, set union and set intersection, respectively. Also $X \subseteq Y$ denotes set inclusion and $X \nsubseteq Y$ denotes the negation of set inclusion. For a family $\mathcal{Z}$ of sets, $\bigcup \mathcal{Z}$ denotes the union of the sets from $\mathcal{Z}$.

The formal notion of a reaction captures the basic intuition behind a biochemical reaction: it can take place if all of its reactants and none of its inhibitors are present, and when it takes place it creates its products.

Definition 2.1. A *reaction* is a triplet $b = (R, I, P)$, where R, I, P are finite nonempty sets with $R \cap I = \varnothing$. If S is a set such that $R, I, P \subseteq S$, then b is a *reaction* over S. $\diamond$

The sets R, I, P are also written R_b, I_b, P_b, and called the *reactant set of* b, the *inhibitor set of* b, and the *product set of* b, respectively. Note that if b is a reaction over S, then $|S| \geq 2$. Such finite sets (of cardinality at least 2) are called *background sets*. The set of all reactions over a background set S is denoted by $rac(S)$.

The dynamics of a single reaction and of a set of reactions is given by the following definition.

Definition 2.2. Let S be a background set and let $T \subseteq S$.

1. Let $b \in rac(S)$. Then b is *enabled by* T, denoted by $en_b(T)$, if $R_b \subseteq T$ and $I_b \cap T = \varnothing$. The *result of* b *on* T, denoted by $res_b(T)$, is defined by: $res_b(T) = P_b$ if $en_b(T)$, and $res_b(T) = \varnothing$ otherwise.
2. Let $B \subseteq rac(S)$ be a finite set of reactions. The *result of* B *on* T, denoted by $res_B(T)$, is defined by: $res_B(T) = \bigcup_{b \in B} res_b(T)$. $\diamond$

The above definition says how a reaction or a set of reactions behaves in a state of a biochemical system, where a state is formalized as a set T of biochemical entities (present in this state). Thus a reaction may happen (is enabled) if all of its reactants are present ($R_b \subseteq T$) and none of its inhibitors are present ($I_b \cap T = \varnothing$). If a reaction takes place in T, then it produces its product. Here $res_b(T) = P_b$ means that b contributes P_b to the successor state of T and $res_b(T) = \varnothing$ means that b does not contribute to the successor of T. The result of a set of reactions B in T is cumulative, i.e., it is the union of the results of all reactions from B.

Since $res_B(T)$ is the union of $res_b(T)$ for all reactions b from B which are enabled by T, an entity $x \in S$ is sustained by B in T (i.e., $x \in T$ and $x \in res_B(T)$) if and only if x is produced by (at least) one reaction b from B. This is different from standard models of computation, where if an element from a current state is not "involved" in a transformation of this state, then it will be sustained (present in the successor state).

This *non-permanency* property reflects the basic bioenergetics of the living cell (see e.g., [8] and [15]).

The following notion of equivalence of single reactions and sets of reactions was introduced in [11].

Definition 2.3. Let S be a background set.

1. Reactions $b_1, b_2 \in rac(S)$ are *equivalent (over S)*, denoted by $b_1 eq_S b_2$, if and only if, for all $T \subseteq S$, $res_{b_1}(T) = res_{b_2}(T)$.
2. Sets of reactions $B_1, B_2 \subseteq rac(S)$ are *equivalent (over S)*, denoted by $B_1 eq_S B_2$, if and only if, for all $T \subseteq S$, $res_{B_1}(T) = res_{B_2}(T)$. $\diamond$

Whenever S is clear from the context of considerations we will simplify our terminology and notation and use the term "equivalence" and the notation *eq*.

It turns out that the equivalence of single reactions can be characterized as follows.

Theorem 2.4. [11] *Let S be a background set and $b_1, b_2 \in rac(S)$. Then $b_1 eq_S b_2$ if and only if $R_{b_1} = R_{b_2}$, $I_{b_1} = I_{b_2}$, and $P_{b_1} = P_{b_2}$.*

Thus single reactions are semantically equivalent if and only if they are syntactically equivalent (identical).

The notion of covering of one reaction by another is useful when comparing results of reactions in a given state.

Definition 2.5. Let S be a background set and $b_1, b_2 \in rac(S)$. We say that b_1 covers b_2, denoted by $b_1 \geq_S b_2$, if and only if, for all $T \subseteq S$, $res_{\{b_1, b_2\}}(T) = res_{b_1}(T)$. $\diamond$

Intuitively $b_1 \geq_S b_2$ means that on its own b_1 will accomplish as much as it will together with b_2.

Also the notion of covering gets a syntactic characterization.

Theorem 2.6. [11] *Let S be a background set and $b_1, b_2 \in rac(S)$. Then $b_1 \geq_S b_2$ if and only if $R_{b_1} \subseteq R_{b_2}$, $I_{b_1} \subseteq I_{b_2}$, and $P_{b_2} \subseteq P_{b_1}$.*

We are ready now to recall (see [11]) the formal notion of a reaction system.

Definition 2.7. A *reaction system* is an ordered pair $\mathcal{A} = (S, A)$, where S is a background set and $A \subseteq rac(S)$. $\diamond$

Thus a reaction system is essentially a finite set of reactions A. We also specify the background set S which includes all the entities needed to specify the reactions in A, but it also may include more entities which may be needed to reason about the behavior of $\mathcal{A}$.

Each subset T of S is called a *state* of $\mathcal{A}$, and for each state T, the *result* of applying $\mathcal{A}$ to T, denoted by $res_{\mathcal{A}}(T)$, is defined by $res_{\mathcal{A}}(T) = res_A(T)$.

Note that there is no counting in a reaction system — we deal with sets rather than multisets. Thus a reaction system is a qualitative (rather than quantitative) model which reflects the level of abstraction for modeling interactions between biochemical reactions.

While $\mathcal{A} = (S, A)$ formalizes the static structure of a reaction system, its dynamic behavior is formalized through interactive processes which are defined as follows.

Definition 2.8. Let $\mathcal{A} = (S, A)$ be a reaction system and let n be a positive integer. An *(n-step) interactive process* in $\mathcal{A}$ is an ordered pair $\pi = (\gamma, \delta)$ of finite sequences of finite sets such that $\gamma = C_0, C_1, \ldots, C_n$ and $\delta = D_0, D_1, \ldots, D_n$, where $C_0, C_1, \ldots, C_n, D_0, D_1, \ldots, D_n \subseteq S$, $D_0 = \varnothing$, and $D_i = res_{\mathcal{A}}(D_{i-1} \cup C_{i-1})$ for all $i \in \{1, \ldots, n\}$. $\diamond$

The sequence γ is the *context sequence* of π and the sequence δ is the *result sequence* of π. Then the sequence $\tau = W_0, W_1, \ldots, W_n$ such that $W_i = D_i \cup C_i$ for all $i \in \{0, \ldots, n\}$ is the *state sequence* of π and W_0 is the *initial state* of π. Note that since $D_0 = \varnothing$, $W_0 = C_0$.

Hence the interactive process π runs as follows. It begins in the initial state $W_0 = C_0$. The next state, W_1, consists of D_1, which is the result of applying to W_0 the reactions from A enabled by W_0, and of context C_1, which formalizes the influence/effect of the environment. Then the consecutive states of τ are formed by iterating this procedure: for each $i \in \{1, \ldots, n-1\}$, W_{i+1} is formed by the union of $D_{i+1} = res_{\mathcal{A}}(W_i)$ and C_{i+1}.

3. Enabling equivalence

When we consider the application of a *single* reaction b to a given state T, then we deal with a binary situation: either b is enabled by T or it is not. However, when we consider the application of a *set of* reactions B to T, then the situation is more involved: either *none* of the reactions in B is enabled by T or only *some* of the reactions in B are enabled by T or *all* reactions in B are enabled by T. In this paper we will consider situations when a set of reactions B is acting as if it was a single reaction, which corresponds to considering only states T such that *all* reactions from B are enabled by T. This leads to the following definitions.

Definition 3.1. Let S be a background set and let $B \subseteq rac(S)$.

1. The *reactant set of* B, denoted by R_B, is the set $\bigcup_{b \in B} R_b$; the *inhibitor set of* B, denoted by I_B, is the set $\bigcup_{b \in B} I_b$; and the *product set of* B, denoted by P_B, is the set $\bigcup_{b \in B} P_b$.
2. For $T \subseteq S$, B is *enabled by* T, denoted by $en_B(T)$, if $R_B \subseteq T$ and $I_B \cap T = \varnothing$.
3. B is *consistent* if $R_B \cap I_B = \varnothing$. ◇

Note that if B is consistent by a state T, then *all* reactions from B are enabled by T. On the other hand if B is not consistent, then, for each $T \subseteq S$, B is not enabled by T (recall that if b is a reaction then $R_b \cap I_b = \varnothing$, hence the notion of consistency is incorporated in the definition of a reaction).

When we consider a set of reactions B as one "block" acting as a single reaction we get a different notion of equivalence for sets of reactions.

Definition 3.2. Let S be a background set and let $B_1, B_2 \subseteq rac(S)$. We say that B_1 is *enabling equivalent to* B_2 *(over* S*)*, denoted by $B_1 eeq_S B_2$, if and only if, for each $T \subseteq S$,

(i) $en_{B_1}(T)$ if and only if $en_{B_2}(T)$, and
(ii) if $en_{B_1}(T)$, then $res_{B_1}(T) = res_{B_2}(T)$. ◇

It is easily seen that eeq_S is an equivalence relation.

Whenever S is clear from the context of consideration, we will simplify our terminology and notations, using the term "enabling equivalence" and the notation *eeq*.

Recall that B_1, B_2 are *equivalent* if and only if for each T the results of applying B_1, B_2 to T are equal independently of whether or not the whole sets B_1, B_2 are enabled by T or only parts of them.

On the other hand, B_1, B_2 are *enabling equivalent* if and only if B_1 and B_2 are enabled by the same subsets T of S and on these subsets they give the same result.

First of all we notice that these two notions of equivalence are incomparable as demonstrated by the following examples.

Example 3.3. Let $S = \{x, y, z, w, u\}$ and let $B_1, B_2 \subseteq rac(S)$ be defined as follows:

$$B_1 = \{(\{x\}, \{z\}, \{w\}), (\{y\}, \{z\}, \{u\})\}$$
$$B_2 = \{(\{x, y\}, \{z\}, \{w, u\})\}$$

Since for $T = \{x\}$, $res_{B_1}(T) = \{w\}$ and $res_{B_2}(T) = \varnothing$, B_1 is *not equivalent* to B_2.

Clearly, for each $T \subseteq S$, $en_{B_1}(T)$ if and only if $en_{B_2}(T)$. However, if $en_{B_1}(T)$ (and hence also $en_{B_2}(T)$) holds, then $\{x, y\} \subseteq T$ and $z \notin T$; but then $res_{B_1}(T) = res_{B_2}(T) = \{w, u\}$. Hence B_1 is *enabling equivalent* to B_2. ◇

Example 3.4. Let $S = \{x, y, z\}$ and let $b_1, b_2 \in rac(S)$ be defined as follows:

$$b_1 = (\{x\}, \{z\}, \{y\}) \text{ and } b_2 = (\{x, y\}, \{z\}, \{y\}).$$

Let then $B_1 = \{b_1\}$ and $B_2 = \{b_1, b_2\}$. Since, by Theorem 2.6, $b_1 \geq_S b_2$, for each $T \subseteq S$, $res_{B_1}(T) = res_{B_2}(T)$ and so B_1 is *equivalent* to B_2. However, B_1 is enabled by $T = \{x\}$ while B_2 is not enabled by $T = \{x\}$. Thus B_1 is *not enabling equivalent* to B_2. ◇

The relationship between the notion of consistency for a set of reactions and the notion of enabling equivalence is given by the following result.

Lemma 3.5. *Let S be a background set and let $B_1, B_2 \subseteq rac(S)$.*

1. *If B_1, B_2 are not consistent, then $B_1 eeq_S B_2$.*
2. *If $B_1 eeq_S B_2$, then B_1 is consistent if and only if B_2 is consistent.*

Proof. Follows directly from the definitions (of eeq_S and consistency). □

The notion of enabling equivalence is a *semantic* (behavioral) notion which is global with respect to the space of all states (all subsets of S): to check whether or not two sets of reactions B_1 and B_2 are enabling equivalent, in general one has to test them in *all* states. We will provide now a *syntactic* characterization of enabling equivalence which allows one to test whether or not $B_1 eeq B_2$ by just inspecting the sets B_1 and B_2.

Theorem 3.6. *Let S be a background set and let $B_1, B_2 \subseteq rac(S)$ be consistent. Then $B_1 eeq_S B_2$ if and only if $R_{B_1} = R_{B_2}$, $I_{B_1} = I_{B_2}$, and $P_{B_1} = P_{B_2}$.*

Proof. (1) Assume that $R_{B_1} = R_{B_2}$, $I_{B_1} = I_{B_2}$, and $P_{B_1} = P_{B_2}$.

(i) Since $R_{B_1} = R_{B_2}$ and $I_{B_1} = I_{B_2}$, $en_{B_1}(T)$ if and only if $en_{B_2}(T)$, for each $T \subseteq S$.
(ii) Since $P_{B_1} = P_{B_2}$, $res_{B_1}(T) = res_{B_2}(T)$ whenever $en_{B_1}(T)$ and $en_{B_2}(T)$.

It follows from (i) and (ii) that $B_1 eeq_S B_2$.

(2) Assume that it is not true that ($R_{B_1} = R_{B_2}$, $I_{B_1} = I_{B_2}$, and $P_{B_1} = P_{B_2}$). It follows than that one of the following three cases must hold:

(i) $R_{B_1} \neq R_{B_2}$,
(ii) $I_{B_1} \neq I_{B_2}$, and
(iii) $R_{B_1} = R_{B_2}$, $I_{B_1} = I_{B_2}$, and $P_{B_1} \neq P_{B_2}$.

We will consider each of the three cases separately.

Assume that (i) holds. Without loss of generality we may assume that $R_{B_1} \setminus R_{B_2} \neq \varnothing$. Hence $en_{B_1}(R_{B_2})$ does not hold. However, since B_2 is consistent, $en_{B_2}(R_{B_2})$. Consequently, it is not true that, for each $T \subseteq S$, $en_{B_1}(T)$ if and only if $en_{B_2}(T)$, and therefore $B_1 eeq_S B_2$ does not hold.

Assume that (ii) holds. Without loss of generality we may assume that $I_{B_1} \setminus I_{B_2} \neq \varnothing$. Let then $y \in S$ be such that $y \in I_{B_1} \setminus I_{B_2}$. Hence it is not true that $en_{B_1}(R_{B_2} \cup \{y\})$. However, since B_2 is consistent, $en_{B_2}(R_{B_2} \cup \{y\})$. Consequently, $B_1 eeq_S B_2$ does not hold.

Assume that (iii) holds. Since $R_{B_1} = R_{B_2}$ and $I_{B_1} = I_{B_2}$ (and B_1, B_2 are consistent), $en_{B_1}(R_{B_1})$ and $en_{B_2}(R_{B_1})$. However, since $P_{B_1} \neq P_{B_2}$, $res_{B_1}(R_{B_1}) \neq res_{B_2}(R_{B_1})$. Consequently, $B_1 eeq_S B_2$ does not hold.

Since the cases (i), (ii) and (iii) are exhaustive, it follows that if it is not true that $R_{B_1} = R_{B_2}$, $I_{B_1} = I_{B_2}$, and $P_{B_1} = P_{B_2}$, then $B_1 eeq_S B_2$ does not hold.

The theorem follows now from (1) and (2). □

Note that if B_1 and B_2 are singletons, $B_1 = \{b_1\}$ and $B_2 = \{b_2\}$, then $B_1 eq B_2$ if and only if $B_1 eeq B_2$. Hence, Theorem 3.6 generalizes the characterization of the equivalence of single reactions given in Theorem 2.4: reactions b_1 and b_2 are equivalent if and only if $R_{b_1} = R_{b_2}$, $I_{b_1} = I_{b_2}$, and $P_{b_1} = P_{b_2}$. For single reactions b_1, b_2, this means that b_1 and b_2 are enabling equivalent if and only if they are identical, while two sets of reactions B_1, B_2 may be enabling equivalent even if they are different provided that $R_{B_1} = R_{B_2}$, $I_{B_1} = I_{B_2}$, and $P_{B_1} = P_{B_2}$.

The following technical corollary of Theorem 3.6 will be useful in the remainder of this paper.

Corollary 3.7. *Let S be a background set and let $B_1, B_2, B_3 \subseteq rac(S)$, where B_2 is consistent.*

1. *If $B_1 eeq_S B_2$ and $B_3 eeq_S B_2$, then $(B_1 \cup B_3) eeq_S B_2$.*
2. *If $B_1 \subseteq B_3 \subseteq B_2$ and $B_1 eeq_S B_2$, then $B_1 eeq_S B_3$ and $B_3 eeq_S B_2$.*

Proof. First we note that since B_2 is consistent, the preconditions of (1) and (2) imply that also B_1 and B_3 are consistent (in (1) by Lemma 3.5(2) and in (2) because $B_1 \subseteq B_2$ and $B_3 \subseteq B_2$).

(1) Follows directly from Theorem 3.6.

(2) Since $B_1 eeq_S B_2$, it follows from Theorem 3.6 that

$$R_{B_1} = R_{B_2},\ I_{B_1} = I_{B_2} \text{ and } P_{B_1} = P_{B_2}.$$

Since $B_1 \subseteq B_3 \subseteq B_2$,

$$R_{B_1} \subseteq R_{B_3} \subseteq R_{B_2},\ I_{B_1} \subseteq I_{B_3} \subseteq I_{B_2}, \text{ and } P_{B_1} \subseteq P_{B_3} \subseteq P_{B_2}.$$

Therefore, $R_{B_3} = R_{B_2}$, $I_{B_3} = I_{B_2}$ and $P_{B_3} = P_{B_2}$, and so, by Theorem 3.6, $B_3 eeq_S B_2$. □

4. Transformation rules

In the standard setup an interactive process takes place within a given reaction system $\mathcal{A} = (S, A)$, where the set A of available reactions is invariant: at each stage of the process the set of available reactions is the same, viz., A, and if T is the state of $\mathcal{A}$ at this stage, then reactions from A that are enabled by T will transform T into $res_A(T)$, which together with the context set available at this stage will form the successor state of T.

In this paper we will consider "evolving situations" where the sets of available reactions may change as an interactive process progresses. Hence in an initial state W_0 the set of available reactions is A_0, then at the successor state W_1 it is A_1, at the following state W_2 it is A_2, and so on.

In order to investigate such evolving interactive processes we also need to define a mechanism which transforms A_0 into A_1, A_1 into A_2, and so on.

The transformations that we consider are not arbitrary – they have to satisfy the "safety condition". They have to preserve the enabling equivalence meaning that if a set of reactions A is transformed into a set of reactions B, then A and B must be enabling equivalent. A natural way to transform a set of reactions A into a set of reactions B is to remove some reactions (set of reactions A') and add some new reactions (set of reactions A''), where the order of removing A' and adding A'' is not important. Because of the transformation safety condition, the transformations we will consider preserve enabling equivalence not only between A and B but also between A and intermediate results ($A \setminus A'$ and $A \cup A''$). At any moment of an implementation of transformation (or a sequence of transformations) beginning with a set of reactions A, the current set of reactions must be enabling equivalent to A. For example, for some reasons it may be important that, for a state T and an entity x, $x \notin res_A(T)$. However, it may be the case that $x \notin res_B(T)$ but $x \in res_{A''}(T)$, and so x will pop up in the intermediate state, while it may be important that x is not produced at all!

Formally such transformations are defined as follows.

Definition 4.1. A *transformation rule* is a 4-tuple $q = (S, K, D, E)$, where S is a background set and $K, D, E \subseteq rac(S)$ are such that:

(i) K is consistent,
(ii) $D \subseteq K$ and $E \cap K = \varnothing$, and
(iii) $K eeq_S (K \setminus D)$ and $K eeq_S (K \cup E)$.

The *outcome of* q, denoted by $out(q)$, is defined by $out(q) = (K \setminus D) \cup E$. ◇

For a transformation rule q as above we say that S is the *background set* of q, and that q is a transformation rule *over* S — we use $trr(S)$ to denote the set of transformation rules over S. Also, K is the *kernel* of q, D is the *decrement* of q, and E is the *expansion* of q, and we will use the notations K_q, D_q, and E_q, to denote K, D, and E, respectively. To simplify the notation we may write simply $q = (K, D, E)$ whenever S is understood from the context of considerations.

Note that condition (iii) guarantees the transformation safety condition discussed above.

A transformation rule q is *trivial* if $D_q = E_q = \varnothing$. Obviously, for a trivial transformation rule q, $out(q) = K_q$.

For a transformation rule q over S and $T \subseteq S$, we say that q is *enabled by* T, denoted by $en_q(T)$, if $en_{K_q}(T)$, i.e., if the kernel K_q of q is enabled by T.

The following lemma states a basic property of transformation rules.

Lemma 4.2. *Let $q = (S, K, D, E)$ be a transformation rule. Then*

1. *$K \setminus D$ is consistent,*
2. *$K \cup E$ is consistent,*
3. *$K eeq_S out(q)$, and*
4. *$out(q)$ is consistent.*

Proof. (1) This follows from Lemma 3.5(2), because K is consistent and $K eeq_S (K \setminus D)$.

(2) This follows from Lemma 3.5(2), because K is consistent and $K eeq_S (K \cup E)$.

(3) Note that

$$K \setminus D \subseteq (K \setminus D) \cup E \subseteq K \cup E.$$

Moreover, since $K eeq_S (K \setminus D)$ and $K eeq_S (K \cup E)$, we have $(K \setminus D) eeq_S (K \cup E)$. Therefore, by Corollary 3.7(1), $(K \setminus D) \cup E\, eeq_S (K \setminus D)$ and (since $(K \setminus D) eeq_S K$) $(K \setminus D) \cup E = out(q) eeq_S K$.

(4) This follows, by Lemma 3.5(2), from (3) and the consistency of K. □

Theorem 4.3. *Let $q = (S, K, D, E)$ be a transformation rule. If a 4-tuple $q' = (S, K, D', E')$ is such that $D' \subseteq D$ and $E' \subseteq E$, then q' is also a transformation rule.*

Proof. We will verify that all conditions for q' to be a transformation rule are satisfied.

(i) Since q is a transformation rule, K is consistent.
(ii) Since q is a transformation rule, $D \subseteq K$ and $E \cap K = \varnothing$. Hence, by $D' \subseteq D$ and $E' \subseteq E$, we get $D' \subseteq K$ and $E' \cap K = \varnothing$.
(iii) Since $D' \subseteq D$, $K \setminus D \subseteq K \setminus D'$. Since $(K \setminus D) eeq_S K$, this implies (by Corollary 3.7(2), because $K \setminus D \subseteq K \setminus D' \subseteq K$) that $(K \setminus D') eeq_S K$.
Since $E' \subseteq E$, $K \subseteq K \cup E' \subseteq K \cup E$. Since $K \cup E eeq_S K$, this implies (by Lemma 4.2(2) and Corollary 3.7(2)) that $(K \cup E') eeq_S K$.

Consequently, q' is a transformation rule. □

We will define now how a transformation rule q transforms a set of reactions A — this depends on the state T in which the transformation of A takes place.

Definition 4.4. Let $q = (S, K, D, E)$ be a transformation rule and let $T \subseteq S$.

1. Let $A \subseteq rac(S)$. The *transformation of A by q in T*, denoted by $tr_{q,T}(A)$, is defined by:

$$tr_{q,T}(A) = \begin{cases} (A \setminus K_q) \cup out(q) & \text{if } K_q \subseteq A \text{ and } en_q(T) \\ A & \text{otherwise.} \end{cases}$$

2. Let $\mathcal{A} = (S, A)$ be a reaction system. The *transformation of $\mathcal{A}$ by q in T*, denoted by $tr_{q,T}(\mathcal{A})$, is defined by $tr_{q,T}(\mathcal{A}) = \mathcal{A}'$, where $\mathcal{A}' = (S, A')$ with $A' = tr_{q,T}(A)$. ◇

Note that Theorem 4.3 says that a transformation rule may be implemented piecewise and the order of "pieces" does not matter. For example, one can partition D into nonempty D_1, D_2, D_3 and E into nonempty E_1, E_2. Then, whether one applies the sequence of transformation rules

$$(S, K, D_1, \varnothing),\ (S, K \setminus D_1, D_2, \varnothing),\ (S, K \setminus D_1 \setminus D_2, \varnothing, E_1),\ (S, K \setminus D_1 \setminus D_2 \cup E_1, D_3, E_2),$$

or the sequence of transformation rules

$$(S, K, D_2, E_1),\ (S, K \setminus D_2 \cup E_1, D_1, E_2),\ (S, K \setminus D_2 \setminus D_1 \cup E_1 \cup E_2), D_3, \varnothing)$$

the final outcome of both sequences will be identical — it will be exactly the outcome of the original transformation rule $q = (S, K, D, E)$. Moreover, by Lemma 4.2(3), after each of the sequential steps the outcome of the last transformation is enabling equivalent to the outcome of q!

Since

$$(A \setminus K_q) \cup out(q) = (A \setminus K_q) \cup (K_q \setminus D_q) \cup E_q = (A \setminus D_q) \cup E_q\ ,$$

the definition of $tr_{q,T}(A)$ may be rewritten as

$$tr_{q,T}(A) = \begin{cases} (A \setminus D_q) \cup E_q & \text{if } K_q \subseteq A \text{ and } en_q(T) \\ A & \text{otherwise.} \end{cases}$$

First we consider transformations by trivial transformation rules.

Lemma 4.5. *Let q be a trivial transformation rule over S. Then, for all $T \subseteq S$ and $A \subseteq rac(S)$, $tr_{q,T}(A) = A$.*

Proof. We consider separately three cases.

(i) $K_q \nsubseteq A$. Then $tr_{q,T}(A) = A$.
(ii) $K_q \subseteq A$ but $en_{K_q}(T)$ does not hold. Then $tr_{q,T}(A) = A$.
(iii) $K_q \subseteq A$ and $en_{K_q}(T)$. Since q is trivial, $out(q) = K_q$, and consequently $tr_{q,T}(A) = (A \setminus K_q) \cup K_q = A$. Then $tr_{q,T}(A) = A$.

It follows from (i), (ii), and (iii) that $tr_{q,T}(A) = A$. □

The following result states the fundamental property of transformations of sets of reactions by transformation rules.

Theorem 4.6. *Let q, q' be transformation rules over S, for some background set S. If $K_q = K_{q'}$, then for all $T \subseteq S$ and all $A \subseteq rac(S)$,*

$$res_{tr_{q,T}(A)}(T) = res_{tr_{q',T}(A)}(T) .$$

Proof. Let $K = K_q = K_{q'}$. We consider separately three cases.

Case 1: $K \nsubseteq A$. Then $tr_{q,T}(A) = tr_{q',T}(A) = A$, and consequently

$$res_{tr_{q,T}(A)}(T) = res_{tr_{q',T}(A)}(T) = res_A(T).$$

Case 2: $K \subseteq A$ but $en_q(T)$ does not hold (and hence also $en_{q'}(T)$ does not hold). Then $tr_{q,T}(A) = tr_{q',T}(A) = A$, and consequently

$$res_{tr_{q,T}(A)}(T) = res_{tr_{q',T}(A)}(T) = res_A(T).$$

Case 3: $K \subseteq A$ and $en_q(T)$ (and hence $en_{q'}(T)$). Then $tr_{q,T}(A) = (A \setminus K) \cup out(q)$ and $tr_{q',T}(A) = (A \setminus K) \cup out(q')$. Consequently,

$$\begin{aligned} res_{tr_{q,T}(A)}(T) &= res_{(A\setminus K)\cup out(q)}(T) \\ res_{tr_{q',T}(A)}(T) &= res_{(A\setminus K)\cup out(q')}(T) \end{aligned} \tag{1}$$

Since $K_q eeq_S out(q)$ (by Lemma 4.2(3)), by Theorem 3.6, $P_{K_q} = P_{out(q)}$. Similarly, since $K_{q'} eeq_S out(q')$, $P_{K_{q'}} = P_{out(q')}$. Consequently, since $K_q = K_{q'}$, we get

$$P_{out(q)} = P_{out(q')} \tag{2}$$

By (1),

$$\begin{aligned} res_{tr_{q,T}(A)}(T) &= res_{A\setminus K}(T) \cup res_{out(q)}(T) \\ res_{tr_{q',T}(A)}(T) &= res_{A\setminus K}(T) \cup res_{out(q')}(T) \end{aligned} \tag{3}$$

Since now (in Case 3) $en_q(T)$ and $en_{q'}(T)$, by Lemma 4.2(3) we get $en_{out(q)}(T)$ and $en_{out(q')}(T)$. Consequently, it follows from (3) that

$$\begin{aligned} res_{tr_{q,T}(A)}(T) &= res_{A\setminus K}(T) \cup P_{out(q)} \\ res_{tr_{q',T}(A)}(T) &= res_{A\setminus K}(T) \cup P_{out(q')}. \end{aligned}$$

Therefore, by (2), $res_{tr_{q,T}(A)}(T) = res_{tr_{q',T}(A)}(T)$, and the theorem holds. □

The following corollary relates (for each state T) the effect of a set of reactions A and the effect of the set of reactions resulting from transforming A at T.

Corollary 4.7. *Let q be a transformation rule over S, for some background set S. For all $T \subseteq S$ and $A \subseteq rac(S)$, $res_{tr_{q,T}(A)}(T) = res_A(T)$.*

Proof. Consider the trivial transformation rule $q' = (K_q, \varnothing, \varnothing)$. By Theorem 4.6, for each $T \subseteq S$ and each $A \subseteq rac(S)$,

$$res_{tr_{q,T}(A)}(T) = res_{tr_{q',T}(A)}(T).$$

Since, by Lemma 4.5, $tr_{q',T}(A) = A$, we get $res_{tr_{q,T}(A)}(T) = res_A(T)$. Thus the corollary holds. □

Corollary 4.7 is quite remarkable and perhaps not very intuitive. It says that if a state T is converted by a set of reactions A into $res_A(T)$ and q is a transformation rule, then also $tr_{q,T}(A)$ converts T into $res_A(T)$. This result is an important technical tool for reasoning about chains of transformations and will be essential in the proof of the main result of this paper (Theorem 6.2 in Section 6).

5. Evolving interactive processes

In this section we introduce evolving interactive processes where the underlying set of available reactions (hence the underlying reaction system) may change as the current state of an interactive process changes to the successor state.

Definition 5.1. Let S be a background set. An *evolving interactive process over* S is a 5-tuple $\phi = (\gamma, \delta, \sigma, \rho, \lambda)$ such that, for some $n \geq 1$:

- $\gamma = C_0, C_1, \ldots, C_n$, where, for each $i \in \{0, \ldots, n\}$, $C_i \subseteq S$,

	0	1	2	...	$n-1$	n
γ	C_0	C_1	C_2	...	C_{n-1}	C_n
δ	D_0	D_1	D_2	...	D_{n-1}	D_n
σ	W_0	W_1	W_2	...	W_{n-1}	W_n
ρ	A_0	A_1	A_2	...	A_{n-1}	A_n
λ	q_0	q_1	q_2	...	q_{n-1}	q_n

Fig. 1. An n-step evolving interactive process.

$$\begin{bmatrix} C_0 \\ D_0 \\ W_0 \\ A_0 \\ q_0 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ D_1 \\ W_1 \\ A_1 \\ q_1 \end{bmatrix} \Rightarrow \dots \Rightarrow \begin{bmatrix} C_{n-1} \\ D_{n-1} \\ W_{n-1} \\ A_{n-1} \\ q_{n-1} \end{bmatrix} \Rightarrow \begin{bmatrix} C_n \\ D_n \\ W_n \\ A_n \\ q_n \end{bmatrix}$$

$$f_0 \qquad f_1 \qquad\qquad f_{n-1} \qquad f_n$$

Fig. 2. A sequence of instantaneous descriptions.

- $\delta = D_0, D_1, \dots, D_n$, where, for each $i \in \{0, \dots, n\}$, $D_i \subseteq S$,
- $\sigma = W_0, W_1, \dots, W_n$, where, for each $i \in \{0, \dots, n\}$, $W_i \subseteq S$,
- $\rho = A_0, A_1, \dots, A_n$, where, for each $i \in \{0, \dots, n\}$, $A_i \subseteq rac(S)$, $A_i \neq \varnothing$,
- $\lambda = q_0, q_1, \dots, q_n$, where, for each $i \in \{0, \dots, n\}$, $q_i \in trr(S)$,

and the following relationships hold:

1. $W_i = C_i \cup D_i$, for each $i \in \{0, \dots, n\}$,
2. $D_i = res_{A_{i-1}}(W_{i-1})$, for each $i \in \{1, \dots, n\}$,
3. $K_{q_i} \subseteq A_i$, for each $i \in \{0, \dots, n\}$,
4. $A_i = tr_{q_{i-1}, W_{i-1}}(A_{i-1})$, for each $i \in \{1, \dots, n\}$.

The sequence γ is the *context sequence of* ϕ, denoted by $con(\phi)$; the sequence δ is the *result sequence of* ϕ, denoted by $res(\phi)$; the sequence σ is the *state sequence of* ϕ, denoted by $st(\phi)$; the sequence ρ is the *sequence of sets of reactions of* ϕ, denoted by $sre(\phi)$; and the sequence λ is the *rule sequence of* ϕ, denoted by $rul(\phi)$. The state W_0 is the *initial state* of ϕ (and the initial state of $st(\phi)$), denoted by $in(\phi)$ (and by $in(st(\phi))$). We also say that ϕ is an *n-step evolving interactive process over S*. $\diamond$

Note that it follows from Definition 4.4 (and the comment following it) that if $i \in \{0, \dots, n-1\}$ is such that $en_{q_i}(W_i)$ then, by Definition 5.1 (points 3, 4),

$$A_{i+1} = tr_{q_i, W_i}(A_i) = (A_i \setminus K_{q_i}) \cup out(q_i) = (A_i \setminus D_q) \cup E_q.$$

It is very convenient to represent an evolving n-step interactive process as a $5 \times (n+1)$ matrix, as shown in Fig. 1.

This representation shows that ϕ can be seen as a sequence of columns:

column 0, column 1, ... , column $(n-1)$, *column n*

which portrays very well the intuition of an evolving interactive process. Each column represents the snapshot of a current situation — often called an *instantaneous description* in the theory of computation. An evolving interactive process is then a sequence $f_0, f_1, \dots, f_n$ of such instantaneous descriptions, as illustrated in Fig. 2. We denote the context of f_i by C_i, the transformation rule of f_i by q_i, etc. Here each successor instantaneous description f_{i+1} is obtained from its predecessor f_i by the set of reactions A_i of f_i, the transformation rule q_i of f_i, and the context C_{i+1} of f_{i+1}.

The intuition behind an evolving interactive process ϕ is that the initial snapshot of the situation is the instantaneous description f_0. Here the set of available reactions is A_0, the contribution (influence) of the environment (we deal with open systems) is C_0, which is also the initial state of the system, and q_0 is the transformation rule which determines the set of available reactions in the following (successor) situation f_1. Then, inductively, for each instantaneous description f_i, its set of reactions A_i applied to its state W_i determines the result D_{i+1} of the successor instantaneous description f_{i+1} which together with the context C_{i+1} determines (by union) the state W_{i+1}. The set A_{i+1} of reactions available in f_{i+1} is determined by the rule q_i of f_i (applied to A_i in the state W_i). Thus the context sequence $con(\phi)$ together with the rule sequence $rul(\phi)$ determine ϕ from the initial instantaneous description f_0.

The sequence $A_0, A_1, \dots, A_n$ of sets of reactions of ϕ induces the sequence

$$\mathcal{A}_0 = (S, A_0) \quad \mathcal{A}_1 = (S, A_1) \quad \dots \quad \mathcal{A}_n = (S, A_n)$$

of reaction systems over the same background set S. Thus one can see the pair $(con(\phi), res(\phi))$ as generalizing the notion of an interactive process of a reaction system by allowing the sequence of transformations

$$(C_0, D_0) \to D_1 \quad (C_1, D_1) \to D_2 \quad \ldots \quad (C_{n-1}, D_{n-1}) \to D_n$$

to be carried on by the sequence $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}$ of reaction systems, where each reaction system $\mathcal{A}_{i+1}$ is obtained from the reaction system $\mathcal{A}_i$ (and the state W_i) by the transformation rule q_i.

Definition 5.2. An evolving interactive process $\phi = f_0, f_1, \ldots, f_n$ is *stationary*, if, for each $i \in \{0, \ldots, n\}$, $rul(f_i)$ is trivial. $\diamond$

To simplify our terminology we may use the term "stationary interactive process" rather than "stationary evolving interactive process".

Note that since, for each $i \in \{0, \ldots, n\}$ $rul(f_i)$ is trivial, by Lemma 4.5, the sequence $sre(\phi) = A_0, \ldots, A_n$ is such that $A_0 = A_1 = \ldots = A_n$. Hence (referring to the above intuition of an evolving sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ of reaction systems) ϕ is a process taking place within one reaction system $\mathcal{A} = \mathcal{A}_0$, i.e., $\pi = (con(\phi), res(\phi))$ is an interactive process in $\mathcal{A}_0$. In this way the notion of an evolving interactive process generalizes the notion of an interactive process in reaction systems.

6. The invisibility theorem

In this section we prove the main result of this paper. First we need the following definition.

Definition 6.1. Let S be a background set and let $B \subseteq rac(S)$.

1. A *signature over* S is a triplet (X, Y, Z) of subsets of S. It is called *consistent* if $X, Y, Z \neq \varnothing$ and $X \cap Y = \varnothing$.
2. Let $B \subseteq rac(S)$. A signature $\alpha = (X, Y, Z)$ over S is the *signature of* B, denoted by $sig(B)$, if $X = R_B$, $Y = I_B$ and $Z = P_B$.
3. Let $\alpha = (X, Y, Z)$ be a consistent signature over S. A subset T of S is α-*compatible* if $X \subseteq T$ and $Y \cap T = \varnothing$.
4. Let $q = (S, K, D, E)$ be a transformation rule. A signature $\alpha = (X, Y, Z)$ over S is the *signature of* q, denoted by $sig(q)$, if $\alpha = sig(K)$. $\diamond$

Note that if B is nonempty and consistent (Definition 3.1), then $sig(B)$ is a reaction.

Theorem 6.2. [Invisibility Theorem]. *Let S be a background set and let $\alpha = (X, Y, Z)$ be a consistent signature over S. Let $\phi = f_0, f_1, \ldots, f_n$ be a stationary evolving interactive process, and let $\psi = \overline{f}_0, \overline{f}_1, \ldots, \overline{f}_n$ be an evolving interactive process, where*

$$f_i = (C_i, D_i, W_i, A_i, q_i) \text{ and } \overline{f}_i = (\overline{C}_i, \overline{D}_i, \overline{W}_i, \overline{A}_i, \overline{q}_i),$$

for each $i \in \{0, \ldots, n\}$. Then $res(\phi) = res(\psi)$ and $st(\phi) = st(\psi)$ provided that:

1. *W_i is α-compatible, for each $i \in \{0, \ldots, n\}$,*
2. *$sig(\overline{q}_i) = \alpha$ and $K_{\overline{q}_i} \subseteq \overline{A}_i$, for each $i \in \{0, \ldots, n\}$,*
3. *$con(\phi) = con(\psi)$, and*
4. *$\overline{D}_0 = D_0$ and $\overline{A}_0 = A_0$.* $\diamond$

Note that in Condition 4 we do not require $\overline{C}_0 = C_0$ as this is guaranteed by Condition 3. Also, $\overline{W}_0 = W_0$ follows from $\overline{D}_0 = D_0$ and Condition 3.

Theorem 6.2 states that an evolving interactive process (ψ) can be such that the available sets of reactions ($\overline{A}_0, \overline{A}_1, \ldots, \overline{A}_n$) change as the interactive process progresses from state to state ($\overline{W}_0, \overline{W}_1, \ldots, \overline{W}_n$) but these changes are not observable (hence *invisible*) in the consecutive states ($\overline{W}_0, \overline{W}_1, \ldots, \overline{W}_n$) of the process: a stationary evolving interactive process (ϕ) with the same context sequence ($con(\phi)$) and the same initial situation ($D_0 = \overline{D}_0$, $W_0 = \overline{W}_0$, and $A_0 = \overline{A}_0$) will produce the same state sequence (and the same result sequence). Since (as usual in models of computation) processes are observable through their states, Theorem 6.2 is called the Invisibility Theorem.

We precede the proof of the theorem by a technical lemma considering only one change of the available set of reactions. But first we introduce an auxiliary notion.

Definition 6.3. Let S be a background set. Let $\phi = f_0, \ldots, f_n$ be a stationary evolving interactive process over S, where $f_i = (C_i, D_i, W_i, A_i, q_i)$, for each $i \in \{0, \ldots, n\}$. Let q be a transformation rule over S such that $en_q(W_0)$ and $K_q \subseteq A_0$. Then an evolving interactive process $\psi = f'_0, f'_1, \ldots, f'_n$, where $f'_i = (C'_i, D'_i, W'_i, A'_i, q'_i)$, for each $i \in \{0, \ldots, n\}$, is a *q-change of ϕ* if

1. $f'_0 = (C_0, D_0, W_0, A_0, q)$,
2. $con(\phi) = con(\psi)$, and
3. q'_i is trivial, for each $i \in \{1, \ldots, n\}$. $\diamond$

$$
\begin{array}{r|ccccc|l}
con(\phi) & C_0 & C_1 & C_2 & \dots & C_n & \\
res(\phi) & D_0 & D_1 & D_2 & \dots & D_n & \\
st(\phi) & W_0 & W_1 & W_2 & \dots & W_n & \\
sre(\phi) & A_0 & A_1 & A_2 & \dots & A_n & \textit{all equal to } A_0 \\
rul(\phi) & q_0 & q_1 & q_2 & \dots & q_n & \textit{all trivial} \\
\hline
\phi & f_0 & f_1 & f_2 & \dots & f_n &
\end{array}
$$

$$
\begin{array}{r|ccccc|l}
 & & & & & \textit{as in } \phi & \\
 & C_0 = C'_0 & C'_1 & C'_2 & \dots & C'_n & con(\psi) = con(\phi) \\
 & D_0 = D'_0 & D'_1 & D'_2 & \dots & D'_n & \\
 & W_0 = W'_0 & W'_1 & W'_2 & \dots & W'_n & \\
 & A_0 = A'_0 & A'_1 & A'_2 & \dots & A'_n & \textit{all equal to } A'_1 \\
 & q = q'_0 & q'_1 & q'_2 & \dots & q'_n & \textit{all trivial} \\
\hline
\psi & f'_0 & f'_1 & f'_2 & \dots & f'_n &
\end{array}
$$

Fig. 3. A *stationary* interactive process ϕ and a q-change ψ of ϕ as in Definition 6.3. Note that $K_q \subseteq A_0$,, $A'_1 = tr_{q,W_0}(A_0) = (A_0 \setminus K_q) \cup out(q)$, $D'_1 = D_1$, and $W'_1 = W_1$.

The two processes, ϕ and ψ, appearing in the above definition are depicted in Fig. 3.

Thus ψ results from ϕ by replacing q_0 by q in forming f'_0 and requiring that $con(\psi) = con(\phi)$ and all $q'_1, \dots, q'_n$ are trivial. Since each q'_i is trivial, for $i \in \{1, \dots, n-1\}$, $A'_{i+1} = A'_i$ and, consequently, $sre(\psi) = A_0, A'_1, A'_1, \dots, A'_1$. Hence there is only one initial change (determined by q) here from A_0 to

$$A'_1 = (A_0 \setminus K_q) \cup out(q),$$

after which the available set of reactions does not change anymore (it equals A'_1).

Note that ψ is *a* q-change of ϕ and not *the* q-change of ϕ because the choice of $q'_1, \dots, q'_n$ is "free" provided that all of them are trivial. We cannot require that $q'_1 = q_1, \dots, q'_n = q_n$ because the definition of an evolving interactive process requires that

$$K_{q'_i} \subseteq A'_i = A'_1 \text{ and } K_{q_i} \subseteq A_i = A_0,$$

for each $i \in \{1, \dots, n\}$, and, in general, these conditions are not compatible as A_0 may be different from A'_1!

Lemma 6.4. [One-change Lemma]. *Let S be a background set and let $\alpha = (X, Y, Z)$ be a consistent signature over S. Let $\phi = f_0, f_1, \dots, f_n$ be a stationary evolving interactive process, where $f_i = (C_i, D_i, W_i, A_i, q_i)$, for each $i \in \{0, \dots, n\}$. Let q be a transformation rule such that $K_q \subseteq A_0$ and $sig(q) = \alpha$. Let $\psi = f'_0, f'_1, \dots, f'_n$ be an evolving interactive process such that $f'_i = (C'_i, D'_i, W'_i, A'_i, q'_i)$, for each $i \in \{0, \dots, n\}$, where*

1. $f'_0 = (C_0, D_0, W_0, A_0, q)$,
2. $con(\phi) = con(\psi)$,
3. *q'_i is trivial, for each $i \in \{1, \dots, n\}$, and*
4. *W_i is α-compatible, for each $i \in \{0, \dots, n\}$.*

Then ψ is a q-change of ϕ such that $res(\phi) = res(\psi)$ and $st(\phi) = st(\psi)$. $\diamond$

Note that $sre(\phi) = A_0, A_0, \dots, A_0$ while $sre(\psi) = A_0, A'_1, A'_1, \dots, A'_1$. Thus a possible change of the available set of reactions takes place only once in the transition from the first to the second instantaneous description. Therefore we refer to this lemma as the "one-change lemma".

Proof of Lemma 6.4. Let, for each $0 \le j \le n$,

$$f_j = (C_j, D_j, W_j, A_j, q_j) \text{ and } f'_j = (C'_j, D'_j, W'_j, A'_j, q'_j).$$

Since ϕ is stationary, each A_j equals A_0, and each q_j is a trivial transformation rule. Also, for each $j \ge 1$, each q'_j is trivial and so each $A'_j = A'_1$, while $A'_0 = A_0$ and $q'_0 = q$.

Since W_0 is α-compatible, $sig(q) = \alpha$ implies that $en_q(W_0)$. Since $K_q \subseteq A_0$ and $con(\phi) = con(\psi)$, this implies that ψ is a q-change of ϕ. Therefore

$$A'_1 = tr_{q,W_0}(A_0) = (A_0 \setminus K_q) \cup out(q),$$

and for each $i \in \{1, \dots, n\}$, $A'_i = A'_1$.

We construct now a sequence $\phi^{(0)}, \phi^{(1)}, \dots, \phi^{(n)}$ of evolving interactive processes in $\mathcal{A}$ as follows:

$$\begin{aligned}
\phi^{(0)} &= f_0^{(0)},\ f_1^{(0)},\ \dots,\ f_n^{(0)} \\
\phi^{(1)} &= f_0^{(1)},\ f_1^{(1)},\ \dots,\ f_n^{(1)} \\
&\vdots \\
\phi^{(n)} &= f_0^{(n)},\ f_1^{(n)},\ \dots,\ f_n^{(n)}
\end{aligned}$$

where

(a) $\phi^{(0)}$ is defined as follows (see Fig. 4):
- $(f_0^{(0)}, f_1^{(0)}, \dots, f_{n-1}^{(0)}) = (f_0, f_1, \dots, f_{n-1})$ and
- $f_n^{(0)} = (C_n, D_n, W_n, A_0, q)$,

(b) $\phi^{(1)}$ is defined as follows (see Fig. 4 and Fig. 5):
- $(f_0^{(1)}, f_1^{(1)}, \dots, f_{n-2}^{(1)}) = (f_0, f_1, \dots, f_{n-2})$ and
- $f_{n-1}^{(1)} = (C_{n-1}, D_{n-1}, W_{n-1}, A_0, q)$, $f_n^{(1)} = (C_n, D_n, W_n, A'_1, q'_n)$,

(c) for each $2 \le i \le n$, $\phi^{(i)}$ is defined as follows (see Fig. 6 and Fig. 7):
- $(f_0^{(i)}, f_1^{(i)}, \dots, f_{n-i-1}^{(i)}) = (f_0, f_1, \dots, f_{n-i-1})$ and
- $f_{n-i}^{(i)} = (C_{n-i}, D_{n-i}, W_{n-i}, A_0, q)$,
- $f_{n-i+1}^{(i)} = (C_{n-i+1}, D_{n-i+1}, W_{n-i+1}, A'_1, q'_{n-i+1})$,
- $f_{n-i+j}^{(i)} = (C_{n-i+j}, res_{A'_1}(W_{n-i+j-1}),\ C_{n-j+1} \cup res_{A'_1}(W_{n-i+j-1}),\ A'_1, q'_{n-i+j})$, for each $j \in \{2, \dots, i\}$.

As usual, for each $i, j \in \{0, \dots, n\}$, we use the notation

$$f_j^{(i)} = (C_j^{(i)}, D_j^{(i)}, W_j^{(i)}, A_j^{(i)}, q_j^{(i)}).$$

Since $K_q \subseteq A_0$, it follows indeed from the definition of $\phi^{(0)}, \phi^{(1)}, \dots, \phi^{(n)}$ that, for each $i \in \{1, \dots n\}$, $\phi^{(i)}$ is an evolving interactive process. We also note (see Fig. 6) that $f_0^{(n)} = f'_0, f_1^{(n)} = f'_1, \dots, f_n^{(n)} = f'_n$ and so $\phi^{(n)} = \psi$.

Claim 1. *For all $i \in \{1, \dots, n\}$, $D_{n-i+2}^{(i)} = D_{n-i+2}^{(i-1)}$ and $W_{n-i+2}^{(i)} = W_{n-i+2}^{(i-1)}$.*

Proof of Claim 1. By the definition of the evolving interactive processes $\phi^{(i-1)}$ and $\phi^{(i)}$,

$$D_{n-i+1}^{(i)} = D_{n-i+1}^{(i-1)} \text{ and } W_{n-i+1}^{(i)} = W_{n-i+1}^{(i-1)}.$$

Thus

$$D_{n-i+2}^{(i)} = res_{A'_1}(W_{n-i+1}^{(i)}) = res_{A'_1}(W_{n-i+1}^{(i-1)}).$$

Since $A'_1 = tr_{q,W_{n-i+1}^{i-1}}(A_0)$, by Corollary 4.7,

$$res_{A'_1}(W_{n-i+1}^{(i-1)}) = res_{A_0}(W_{n-i+1}^{(i-1)}) = D_{n-i+2}^{(i-1)}.$$

Consequently, $D_{n-i+2}^{(i)} = D_{n-i+2}^{(i-1)}$. Since the context sequence is the same for $\phi^{(i)}$ and $\phi^{(i-1)}$, i.e., $con(\phi^{(i)}) = con(\phi^{(i-1)})$, we get also $W_{n-i+2}^{(i)} = W_{n-i+2}^{(i-1)}$.

Hence the claim holds. $\square$

Claim 2. *For each $i \in \{1, \dots, n\}$,*

$$res(\phi^{(i)}) = res(\phi^{(i-1)}) \text{ and } st(\phi^{(i)}) = st(\phi^{(i-1)}).$$

$$
\begin{array}{lcccccccl}
\phi & C_0 & C_1 & \dots & C_{n-3} & C_{n-2} & C_{n-1} & C_n & \\
 & D_0 & D_1 & \dots & D_{n-3} & D_{n-2} & D_{n-1} & D_n & \\
 & W_0 & W_1 & \dots & W_{n-3} & W_{n-2} & W_{n-1} & W_n & \\
 & A_0 & A_1 & \dots & A_{n-3} & A_{n-2} & A_{n-1} & A_n & \\
 & q_0 & q_1 & \dots & q_{n-3} & q_{n-2} & q_{n-1} & q_n & \textit{all trivial} \\
\hline
 & f_0 & f_1 & \dots & f_{n-3} & f_{n-2} & f_{n-1} & f_n &
\end{array}
$$

$$
\begin{array}{lcccccccl}
 & & & & & & & \textit{as in } \phi & \\
\phi^{(0)} & C_0^{(0)} & C_1^{(0)} & \dots & C_{n-3}^{(0)} & C_{n-2}^{(0)} & C_{n-1}^{(0)} & C_n^{(0)}{=}C_n & \\
 & D_0^{(0)} & D_1^{(0)} & \dots & D_{n-3}^{(0)} & D_{n-2}^{(0)} & D_{n-1}^{(0)} & D_n^{(0)}{=}D_n & \\
 & W_0^{(0)} & W_1^{(0)} & \dots & W_{n-1}^{(0)} & W_{n-2}^{(0)} & W_{n-1}^{(0)} & W_n^{(0)}{=}W_n & \\
 & A_0^{(0)} & A_1^{(0)} & \dots & A_{n-3}^{(0)} & A_{n-2}^{(0)} & A_{n-1}^{(0)} & A_n^{(0)}{=}A_n & \\
 & q_0^{(0)} & q_1^{(0)} & \dots & q_{n-3}^{(0)} & q_{n-2}^{(0)} & q_{n-1}^{(0)} & q_n^{(0)}{=}\,q & \\
\hline
 & f_0^{(0)} & f_1^{(0)} & \dots & f_{n-3}^{(0)} & f_{n-2}^{(0)} & f_{n-1}^{(0)} & f_n^{(0)} &
\end{array}
$$

$$
\begin{array}{lcccccccl}
 & & & & & & & \textit{as in } \phi & \\
\phi^{(1)} & C_0^{(1)} & C_1^{(1)} & \dots & C_{n-3}^{(1)} & C_{n-2}^{(1)} & C_{n-1}^{(1)} & C_n^{(1)} & \\
 & D_0^{(1)} & D_1^{(1)} & \dots & D_{n-3}^{(1)} & D_{n-2}^{(1)} & D_{n-1}^{(1)} & D_n^{(1)} & \\
 & W_0^{(1)} & W_1^{(1)} & \dots & W_{n-3}^{(1)} & W_{n-2}^{(1)} & W_{n-1}^{(1)} & W_n^{(1)} & \\
 & A_0^{(1)} & A_1^{(1)} & \dots & A_{n-3}^{(1)} & A_{n-2}^{(1)} & A_{n-1}^{(1)}{=}A_0 & A_1' & \\
 & q_0^{(1)} & q_1^{(1)} & \dots & q_{n-3}^{(1)} & q_{n-2}^{(1)} & q & q_n' & \\
\hline
 & f_0^{(1)} & f_1^{(1)} & \dots & f_{n-3}^{(1)} & f_{n-2}^{(1)} & f_{n-1}^{(1)} & f_n^{(1)} &
\end{array}
$$

$$
\begin{array}{lcccccccl}
 & & & & & & & \textit{as in } \phi & \\
\phi^{(2)} & C_0^{(2)} & C_1^{(2)} & \dots & C_{n-3}^{(2)} & C_{n-2}^{(2)} & C_{n-1}^{(2)} & C_n^{(2)} & \\
 & D_0^{(2)} & D_1^{(2)} & \dots & D_{n-3}^{(2)} & D_{n-2}^{(2)} & D_{n-1}^{(2)} & D_n^{(2)} & \\
 & W_0^{(2)} & W_1^{(2)} & \dots & W_{n-3}^{(2)} & W_{n-2}^{(2)} & W_{n-1}^{(2)} & W_n^{(2)} & \\
 & A_0^{(2)} & A_1^{(2)} & \dots & A_{n-3}^{(2)} & A_{n-2}^{(2)}{=}A_0 & A_1' & A_1' & \\
 & q_0^{(2)} & q_1^{(2)} & \dots & q_{n-3}^{(2)} & q & q_{n-1}' & q_n' & \\
\hline
 & f_0^{(2)} & f_1^{(2)} & \dots & f_{n-3}^{(2)} & f_{n-2}^{(2)} & f_{n-1}^{(2)} & f_n^{(2)} &
\end{array}
$$

Fig. 4. The initial stationary interactive process ϕ; a $\phi^{(0)}$ change (the transformation rule q appears in column $f_{n-0}^{(0)} = f_n^{(0)}$ and $A_1' = tr_{q,W_{n-1}}(A_0) = (A_0 \setminus K_q) \cup out(q)$); a $\phi^{(1)}$ change (the transformation rule q appears in column $f_{n-1}^{(1)}$ and $A_1' = tr_{q,W_{n-1}}(A_0) = (A_0 \setminus K_q) \cup out(q)$); and a $\phi^{(2)}$ change (the transformation rule q appears in column $f_{n-2}^{(2)}$).

Proof of Claim 2. Let $i \in \{1, \dots, n\}$.

(1) By the definition of the evolving interactive processes $\phi^{(i)}$ and $\phi^{(i-1)}$, $D_{n-i}^{(i)} = D_{n-i}^{(i-1)}$ and $D_{n-i+1}^{(i)} = D_{n-i+1}^{(i-1)}$. Consequently, because $con(\phi^{(i)}) = con(\phi^{(i-1)})$, $W_{n-i}^{(i)} = W_{n-i}^{(i-1)}$ and $W_{n-i+1}^{(i)} = W_{n-i+1}^{(i-1)}$.

(2) Also, for each $j \in \{2, \dots, i\}$,

$$sre(f_{n-i+j}^{(i)}) = sre(f_{n-i+j}^{(i-1)}) = A_1'.$$

Since by Claim 1,

$$D_{n-i+2}^{(i)} = D_{n-i+2}^{(i-1)} \text{ and } W_{n-i+2}^{(i)} = W_{n-i+2}^{(i-1)},$$

$$
\begin{array}{l|cccccccccc}
 & & & & & & & & & & \textit{as in } \phi \\
\phi^{(i-1)} & C_0 & C_1 & \dots & C_{n-i-1} & C_{n-i} & C^{(i-1)}_{n-i+1} & C^{(i-1)}_{n-i+2} & \dots & C^{(i-1)}_{n-1} & C^{(i-1)}_{n} \\
 & D_0 & D_1 & \dots & D_{n-i-1} & D_{n-i} & D^{(i-1)}_{n-i+1} & D^{(i-1)}_{n-i+2} & \dots & D^{(i-1)}_{n-1} & D^{(i-1)}_{n} \\
 & W_0 & W_1 & \dots & W_{n-i-1} & W_{n-i} & W^{(i-1)}_{n-i+1} & W^{(i-1)}_{n-i+2} & \dots & W^{(i-1)}_{n-1} & W^{(i-1)}_{n} \\
 & A_0 & A_0 & \dots & A_0 & A_0 & A_0 & A'_1 & \dots & A'_1 & A'_1 \\
 & q_0 & q_1 & \dots & q_{n-i-1} & q_{n-i} & q & q^{(i-1)}_{n-i+2} & \dots & q^{(i-1)}_{n-1} & q^{(i-1)}_{n}
\end{array}
$$

$$
\begin{array}{l|cccccccccc}
 & & & & & & & & & & \textit{as in } \phi \\
\phi^{(i)} & C_0 & C_1 & \dots & C_{n-i-1} & C_{n-i} & C^{(i)}_{n-i+1} & C^{(i)}_{n-i+2} & \dots & C^{(i)}_{n-1} & C^{(i)}_{n} \\
 & D_0 & D_1 & \dots & D_{n-i-1} & D_{n-i} & D^{(i)}_{n-i+1} & D^{(i)}_{n-i+2} & \dots & D^{(i)}_{n-1} & D^{(i)}_{n} \\
 & W_0 & W_1 & \dots & W_{n-i-1} & W_{n-i} & W^{(i)}_{n-i+1} & W^{(i)}_{n-i+2} & \dots & W^{(i)}_{n-1} & W^{(i)}_{n} \\
 & A_0 & A_0 & \dots & A_0 & A_0 & A'_1 & A'_1 & \dots & A'_1 & A'_1 \\
 & q_0 & q_1 & \dots & q_{n-i-1} & q & q^{(i)}_{n-i+1} & q^{(i)}_{n-i+2} & \dots & q^{(i)}_{n-1} & q^{(i)}_{n}
\end{array}
$$

Fig. 5. Moving from $\phi^{(i-1)}$ to $\phi^{(i)}$.

$$
\begin{array}{l|ccccc}
 & & & & & \textit{as in } \phi \\
\phi^{(n)} & C^{(n)}_0 & C^{(n)}_1 & C^{(n)}_2 & \dots & C^{(n)}_n \\
 & D^{(n)}_0 & D^{(n)}_1 & D^{(n)}_2 & \dots & D^{(n)}_n \\
 & W^{(n)}_0 & W^{(n)}_1 & W^{(n)}_2 & \dots & W^{(n)}_n \\
 & A^{(n)}_0 & A'_1 & A'_1 & \dots & A'_1 \\
 & q & q'_1 & q'_2 & \dots & q'_n \\
\hline
 & f^{(n)}_0 & f^{(n)}_1 & f^{(n)}_2 & \dots & f^{(n)}_n
\end{array}
$$

Fig. 6. A $\phi^{(n)}$ change (the transformation rule q appears in column $f^{(n)}_{n-n} = f^{(n)}_0$).

$$
\begin{array}{l|ccccccccc}
 & & & & & & & & & \textit{as in } \phi \\
\phi^{(i)} & C^{(i)}_0 & C^{(i)}_1 & \dots & C^{(i)}_{n-i-1} & C^{(i)}_{n-i} & C^{(i)}_{n-i+1} & C^{(i)}_{n-i+2} & \dots & C^{(i)}_{n} \\
 & D^{(i)}_0 & D^{(i)}_1 & \dots & D^{(i)}_{n-i-1} & D^{(i)}_{n-i} & D^{(i)}_{n-i+1} & D^{(i)}_{n-i+2} & \dots & D^{(i)}_{n} \\
 & W^{(i)}_0 & W^{(i)}_1 & \dots & W^{(i)}_{n-i-1} & W^{(i)}_{n-i} & W^{(i)}_{n-i+1} & W^{(i)}_{n-i+2} & \dots & W^{(i)}_{n} \\
 & A^{(i)}_0 & A^{(i)}_1 & \dots & A^{(i)}_{n-i-1} & A^{(i)}_{n-i}{=}A_0 & A'_1 & A'_1 & \dots & A'_1 \\
 & q^{(i)}_0 & q^{(i)}_1 & \dots & q^{(i)}_{n-i-1} & q & q'_{n-i+1} & q'_{n-i+2} & \dots & q'_{n} \\
\hline
 & f^{(i)}_0 & f^{(i)}_1 & \dots & f^{(i)}_{n-i-1} & f^{(i)}_{n-i} & f^{(i)}_{n-i+1} & f^{(i)}_{n-i+2} & \dots & f^{(i)}_{n}
\end{array}
$$

$$
f^{(i)}_{n-i+j} = \left[\begin{array}{r} C_{n-i+j} \\ res_{A'_1}(W_{n-i+j-1}) \\ C_{n-i+j} \cup res_{A'_1}(W_{n-i+j-1}) \\ A'_1 \\ q'_{n-i+j} \end{array}\right] \begin{array}{l} C_{n-i+j} \\ D_{n-i+j} \\ W_{n-i+j} \\ A_{n-i+j} \\ q_{n-i+j} \end{array}
$$

Fig. 7. The definition of $\phi^{(i)}$ change, for $2 \le i \le n$ (the transformation rule q appears in column $f^{(i)}_{n-i}$) and $j \ge 2$.

$$
\begin{array}{llllll}
\mu_0 & : \phi_0 = f_{0,0}\,,\, f_{0,1}\,,\, f_{0,2}\,,\,\dots,\, f_{0,n} & \text{and} & \psi_0 = f'_{0,0}\,,\, f'_{0,1}\,,\, f'_{0,2}\,,\,\dots,\, f'_{0,n}\,, \\
\mu_1 & : \phi_1 = f_{1,1}\,,\, f_{1,2}\,,\,\dots,\, f_{1,n} & \text{and} & \psi_1 = f'_{1,1}\,,\, f'_{1,2}\,,\,\dots,\, f'_{1,n}\,, \\
 & \quad = f'_{0,1}\,,\, f'_{0,2}\,,\,\dots,\, f'_{0,n} & & \\
\mu_2 & : \phi_2 = f_{2,2}\,,\,\dots,\, f_{2,n} & \text{and} & \psi_2 = f'_{2,2}\,,\,\dots,\, f'_{2,n}\,, \\
 & \quad = f'_{1,2}\,,\,\dots,\, f'_{1,n} & & \\
\vdots & \quad \vdots & & \quad \vdots \\
\mu_{n-1} & : \phi_{n-1} = f_{n-1,n-1}\,,\, f_{n-1,n} & \text{and} & \psi_{n-1} = f'_{n-1,n-1}\,,\, f'_{n-1,n}\,. \\
 & \quad = f'_{n-2,n-1}\,,\, f'_{n-2,n} & &
\end{array}
$$

Fig. 8. The construction of μ.

this implies that, for each $j \in \{0, \dots, i\}$, $D^{(i)}_{n-i+j} = D^{(i-1)}_{n-i+j}$ and consequently, because $con(\phi^{(i)}) = con(\phi^{(i-1)})$, $W^{(i)}_{n-i+j} = W^{(i-1)}_{n-i+j}$.

It follows from (1) and (2), that, for each $j \in \{0, \dots, n\}$,

$$res(\phi^{(i)}) = res(\phi^{(i-1)}) \text{ and } st(\phi^{(i)}) = st(\phi^{(i-1)}).$$

Hence the claim holds. □

The lemma follows now from Claim 2 by the fact (mentioned already) that $\phi^{(n)} = \psi$, where, as we proved already, ψ is a q-change of ϕ. (Lemma 6.4) □

We need one more definition before we proceed to the proof of the invisibility theorem, Theorem 6.2.

Definition 6.5. Let $\phi = f_0, f_1, \dots, f_n$ be an evolving interactive process such that $n \geq 2$. The *left cut of* ϕ, denoted by $lcut(\phi)$, is the evolving interactive process $f_1, \dots, f_n$. ◇

Proof of Theorem 6.2. Let then α, ϕ, and ψ be as in the statement of Theorem 6.2. The proof begins by constructing a sequence μ of pairs of evolving interactive processes, $\mu = \mu_0, \mu_1, \dots, \mu_{n-1}$ with $\mu_i = (\phi_i, \psi_i)$ for each $i \in \{0, \dots, n-1\}$, where

- $\phi_0 = \phi$,
- ψ_i is a $\overline{q}_i$-change of ϕ_i, for each $i \in \{0, \dots, n-1\}$, and
- $\phi_i = lcut(\psi_{i-1})$, for each $i \in \{0, \dots, n-1\}$.

We will use the following notation. For each $i \in \{0, \dots, n-1\}$,

$$\phi_i = f_{i,i}, f_{i,i+1}, \dots, f_{i,n} \text{ and } \psi_i = f'_{i,i}, f'_{i,i+1}, \dots, f'_{i,n},$$

where, for each $k \in \{i, i+1, \dots, n\}$,

$$f_{i,k} = (C_{i,k}, D_{i,k}, W_{i,k}, A_{i,k}, q_{i,k}) \text{ and } f'_{i,k} = (C'_{i,k}, D'_{i,k}, W'_{i,k}, A'_{i,k}, q'_{i,k}).$$

The above construction is illustrated in Fig. 8.

Consider now the sequence δ of pairs of result sequences corresponding to the sequence μ, i.e., $\delta = \delta_0, \delta_1, \dots, \delta_{n-1}$ with $\delta_i = (res(\phi_i), res(\psi_i))$ for each $i \in \{0, \dots, n-1\}$. Hence

$$
\begin{array}{rcllrcl}
res(\phi_0) & = & D_{0,0}, D_{0,1}, \dots, D_{0,n} & \text{and} & res(\psi_0) & = & D'_{0,0}, D'_{0,1}, \dots, D'_{0,n}, \\
res(\phi_1) & = & D_{1,1}, \dots, D_{1,n} & \text{and} & res(\psi_1) & = & D'_{1,1}, \dots, D'_{1,n}, \\
 & & & \vdots & & & \\
res(\phi_{n-1}) & = & D_{n-1,n-1}, D_{n-1,n} & \text{and} & res(\psi_{n-1}) & = & D'_{n-1,n-1}, D'_{n-1,n}.
\end{array}
$$

From the construction of μ it follows that:

(1) By the one-change lemma (Lemma 6.4), for $i \in \{0, \dots, n-1\}$ ψ_i is a q_i-change of ϕ_i and $res(\phi_i) = res(\psi_i)$, and
(2) since, for $i \in \{0, \dots, n-2\}$, $\phi_i = lcut(\psi_i)$, we get

$$res(\phi_{i+1}) = D_{i+1,i+1}, D_{i+1,i+2}, \dots, D_{i+1,n} = D'_{i,i+1}, D'_{i,i+2}, \dots, D'_{i,n}.$$

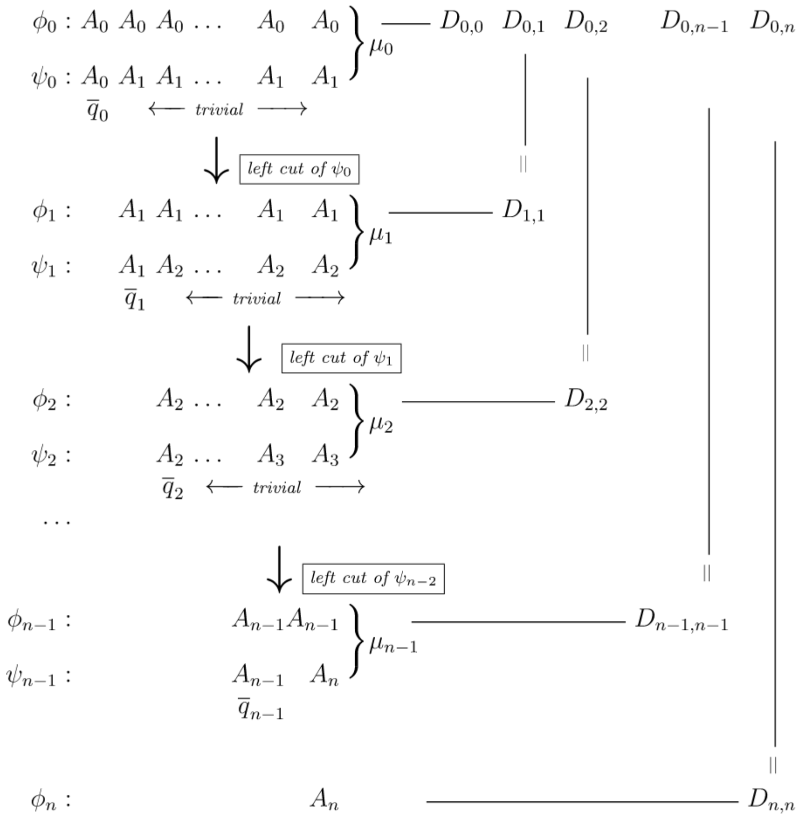

Fig. 9. An illustration of our reasoning that $res(\phi) = res(\psi)$. Here $\phi_n = lcut(\psi_{n-1})$.

(3) The evolving interactive processes ϕ and ψ from the statement of Theorem 6.2 are related to the sequence $\psi_0, \psi_1, \ldots, \psi_{n-1}$ and to ϕ_0 by the following equalities:

$$\overline{f}_0 = f'_{0,0}, \quad \overline{f}_1 = f'_{1,1}, \quad \overline{f}_2 = f'_{2,2}, \ \ldots\ , \overline{f}_{n-1} = f'_{n-1,n-1}, \quad \overline{f}_n = f'_{n,n},$$
$$f_0 = f_{0,0}, \quad f_1 = f_{0,1}, \quad f_2 = f_{0,2}, \ \ldots\ , f_{n-1} = f_{0,n-1}, \quad f_n = f_{0,n},$$

and $\psi = f'_{0,0}, f'_{1,1}, f'_{2,2}, \ldots, f'_{n-1,n-1}, f'_{n,n}$, where $f'_{n,n}$ results from $lcut(\psi_{n-1}) = f'_{n-1,n}$ by replacing the rule component $q'_{n-1,n}$ of $f'_{n-1,n}$ by $\overline{q}_n$.
This together with (1) and (2) implies that:
- $\overline{D}_0 = D'_{0,0} = D_{0,0} = D_0$,
- $\overline{D}_1 = D'_{1,1} = D_{1,1} = D'_{0,1} = D_{0,1} = D_1$,
- $\overline{D}_2 = D'_{2,2} = D_{2,2} = D'_{1,2} = D_{1,2} = D'_{0,2} = D_{0,2} = D_2$,
- $\vdots$
- $\overline{D}_{n-1} = D'_{n-1,n-1} = D_{n-1,n-1} = D'_{n-2,n-1} = \ldots = D'_{0,n-1} = D_{0,n-1} = D_{n-1}$, and
- $\overline{D}_n = D'_{n,n} = D_{n,n} = D'_{n-1,n} = D_{n-1,n} = \ldots = D'_{0,n} = D_{0,n} = D_n$.

Therefore $\overline{D}_0 = D_0, \overline{D}_1 = D_1, \ldots, \overline{D}_n = D_n$. Consequently, $res(\phi) = res(\psi)$ and since $con(\phi) = con(\psi)$, this implies that also $st(\phi) = st(\psi)$. This reasoning is illustrated in Fig. 9 where for each ϕ_i we show just the sequence of sets of reactions $sre(\phi_i)$ and for each ψ_i we show $sre(\psi_i)$ and the rule sequence $rul(\psi_i)$.

Hence the theorem holds. □

7. Example

In this section we will consider an example which will facilitate an interpretation of the Invisibility Theorem related to evolution theory.

Throughout this section we will use the following notation.

Let $l_1, l_2, l_3 \geq 1$.

- $Z^{(i)} = \{z_1^{(i)}, \ldots, z_{l_i}^{(i)}\}$, for $i \in \{1, 2, 3\}$, are pairwise disjoint sets.
- $S = \{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\} \cup Z^{(1)} \cup Z^{(2)} \cup Z^{(3)}$ is a background set such that the set $(Z^{(1)} \cup Z^{(2)} \cup Z^{(3)})$ is disjoint with $\{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\}$.
- A, B, H are sets of reactions over S such that $A = B \cup H$,

$$\begin{aligned} B &= \{(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}, \{z_1, z_2, z_3\})\}, \text{ and} \\ H &= H^{(1)} \cup H^{(2)} \cup H^{(3)}, \end{aligned}$$

  where:

$$\begin{aligned} H^{(1)} &= \{(\{x_1, z_1\}, \{x_2, x_3\}, \{z_j^{(1)}\}) : j \in \{1, \ldots, l_1\}\}, \\ H^{(2)} &= \{(\{x_2, z_2\}, \{x_1, x_3\}, \{z_j^{(2)}\}) : j \in \{1, \ldots, l_2\}\}, \text{ and} \\ H^{(3)} &= \{(\{x_3, z_3\}, \{x_1, x_2\}, \{z_j^{(3)}\}) : j \in \{1, \ldots, l_3\}\}. \end{aligned}$$

- $B' = \{(\{x_1\}, \{y_1\}, \{z_1\}), (\{x_2\}, \{y_2\}, \{z_2\}), (\{x_3\}, \{y_3\}, \{z_3\})\}$.

We want to transform the set of reactions B into B'. We cannot do this in one step, by one transformation rule, as for such a hypothetical transformation rule $q = (S, B, D, E)$ we would have $D = B$ (because B contains only one reaction) and so it would not be true that $Beeq(B \setminus D)$ (because $B \setminus D = \varnothing$) and consequently q could not be a transformation rule.

The desired transformation can be accomplished by a sequence of two transformation rules: $\overline{q}_1 = (S, B, \varnothing, B')$ followed by $\overline{q}_2 = (S, B \cup B', B, \varnothing)$. Note that $out(\overline{q}_1) = B \cup B'$ and $out(\overline{q}_2) = B'$, so indeed the sequence $\overline{q}_1, \overline{q}_2$ accomplishes a transformation of B into B'.

Consider now a stationary interactive process ω:

$$\begin{array}{ccccc} C_0 & C_1 & C_2 & \ldots & C_{j-1} \\ D_0 & D_1 & D_2 & \ldots & D_{j-1} \\ W_0 & W_1 & W_2 & \ldots & W_{j-1} \\ A_0 & A_1 & A_2 & \ldots & A_{j-1} \\ q_0 & q_1 & q_2 & \ldots & q_{j-1} \end{array}$$

where $j \geq 2$,

- $C_0 = C_1 = \ldots = C_{j-1} = \{x_1, x_2, x_3\}$, and
- $A_0 = A$ (and so $A_0 = A_1 = \ldots = A_{j-1} = A$).

Consequently,

- $D_1 = D_2 = \ldots = D_{j-1} = \{z_1, z_2, z_3\}$,
- $W_0 = \{x_1, x_2, x_3\}$, and $W_1 = W_2 = \ldots = W_{j-1} = \{x_1, x_2, x_3, z_1, z_2, z_3\}$.

We extend now ω to an evolving interactive process π:

$$\begin{array}{ccccccccc} C_0 & C_1 & \ldots & C_{j-1} & C_j & C_{j+1} & C_{j+2} & \ldots & C_n \\ D_0 & D_1 & \ldots & D_{j-1} & D_j & D_{j+1} & D_{j+2} & \ldots & D_n \\ W_0 & W_1 & \ldots & W_{j-1} & W_j & W_{j+1} & W_{j+2} & \ldots & W_n \\ A_0 & A_1 & \ldots & A_{j-1} & A_j & A_{j+1} & A_{j+2} & \ldots & A_n \\ q_0 & q_1 & \ldots & q_{j-1} & q_j & q_{j+1} & q_{j+2} & \ldots & q_n \end{array}$$

for some $n \geq j+2$, where

- $C_j = C_{j+1} = C_{j+2} = \ldots = C_n = \{x_1, x_2, x_3\}$,
- $q_j = \overline{q}_1$, $q_{j+1} = \overline{q}_2$, and $q_{j+2}, \ldots, q_n$ are trivial transformation rules.

Let $A' = A_{j+1}$ and $A'' = A_{j+2}$ (thus $A' = B \cup B' \cup H$ and $A'' = B' \cup H$). Hence

- $A_{j+2} = A_{j+3} = \ldots = A_{j+n} = A''$.

Since $C_0 = C_1 = \ldots = C_n$ and $AeeqA'$ and $A'eeqA''$, we note that

- $D_{j-1} = D_j = \ldots = D_n = \{z_1, z_2, z_3\}$ and
- $W_{j-1} = W_j = \ldots = W_n = \{x_1, x_2, x_3, z_1, z_2, z_3\}$ (as predicted by Theorem 6.2).

Now we extend π to three different evolving interaction processes π_1, π_2, π_3 as the context sequence $con(\pi)$ will be extended in such a way that it will split into three different context sequences $con(\pi_1), con(\pi_2), con(\pi_3)$ because of three different continuations of $con(\pi)$:
for some $k \geq 3$,

$$C^{(1)}_{n+1} = C^{(1)}_{n+2} = \ldots = C^{(1)}_{n+k} = \{x_1\},$$
$$C^{(2)}_{n+1} = C^{(2)}_{n+2} = \ldots = C^{(2)}_{n+k} = \{x_2\}, \text{ and}$$
$$C^{(3)}_{n+1} = C^{(3)}_{n+2} = \ldots = C^{(3)}_{n+k} = \{x_3\}.$$

More specifically, for some $k \geq 3$,
(1) π_1 is the evolving interactive process

$$\begin{array}{cccccccccccc} C_0 & C_1 & \ldots & C_j & C_{j+1} & C_{j+2} & \ldots & C_n & C^{(1)}_{n+1} & C^{(1)}_{n+2} & \ldots & C^{(1)}_{n+k} \\ D_0 & D_1 & \ldots & D_j & D_{j+1} & D_{j+2} & \ldots & D_n & D^{(1)}_{n+1} & D^{(1)}_{n+2} & \ldots & D^{(1)}_{n+k} \\ W_0 & W_1 & \ldots & W_j & W_{j+1} & W_{j+2} & \ldots & W_n & W^{(1)}_{n+1} & W^{(1)}_{n+2} & \ldots & W^{(1)}_{n+k} \\ A_0 & A_1 & \ldots & A_j & A_{j+1} & A_{j+2} & \ldots & A_n & A^{(1)}_{n+1} & A^{(1)}_{n+2} & \ldots & A^{(1)}_{n+k} \\ q_0 & q_1 & \ldots & q_j & q_{j+1} & q_{j+2} & \ldots & q_n & q^{(1)}_{n+1} & q^{(1)}_{n+2} & \ldots & q^{(1)}_{n+k} \end{array}$$

where

- $C^{(1)}_{n+1} = C^{(1)}_{n+2} = \ldots = C^{(1)}_{n+k} = \{x_1\}$,
- $A_n = A^{(1)}_{n+1} = A^{(1)}_{n+2} = \ldots = A^{(1)}_{n+k} = A''$, and
- $q^{(1)}_{n+1}, q^{(1)}_{n+2}, \ldots, q^{(1)}_{n+k}$ are trivial rules.

Consequently, because x_1 enables reaction $(\{x_1\}, \{y_1\}, \{z_1\})$ from B' and x_1 inhibits the reactions from $H^{(2)}$ and $H^{(3)}$,

- $D^{(1)}_{n+2} = D^{(1)}_{n+3} = \ldots = D^{(1)}_{n+k} = \{z_1\} \cup Z^{(1)}$, and
- $W^{(1)}_{n+2} = W^{(1)}_{n+3} = \ldots = W^{(1)}_{n+k} = \{x_1\} \cup D^{(1)}_{n+k} = \{x_1, z_1\} \cup Z^{(1)}$.

(2) π_2 is the evolving interactive process

$$\begin{array}{cccccccccccc} C_0 & C_1 & \ldots & C_j & C_{j+1} & C_{j+2} & \ldots & C_n & C^{(2)}_{n+1} & C^{(2)}_{n+2} & \ldots & C^{(2)}_{n+k} \\ D_0 & D_1 & \ldots & D_j & D_{j+1} & D_{j+2} & \ldots & D_n & D^{(2)}_{n+1} & D^{(2)}_{n+2} & \ldots & D^{(2)}_{n+k} \\ W_0 & W_1 & \ldots & W_j & W_{j+1} & W_{j+2} & \ldots & W_n & W^{(2)}_{n+1} & W^{(2)}_{n+2} & \ldots & W^{(2)}_{n+k} \\ A_0 & A_1 & \ldots & A_j & A_{j+1} & A_{j+2} & \ldots & A_n & A^{(2)}_{n+1} & A^{(2)}_{n+2} & \ldots & A^{(2)}_{n+k} \\ q_0 & q_1 & \ldots & q_j & q_{j+1} & q_{j+2} & \ldots & q_n & q^{(2)}_{n+1} & q^{(2)}_{n+2} & \ldots & q^{(2)}_{n+k} \end{array}$$

where

- $C^{(2)}_{n+1} = C^{(2)}_{n+2} = \ldots = C^{(2)}_{n+k} = \{x_2\}$,
- $A^{(2)}_{n+1} = A^{(2)}_{n+2} = \ldots = A^{(2)}_{n+k} = A''$, and
- $q^{(2)}_{n+1}, q^{(2)}_{n+2}, \ldots, q^{(2)}_{n+k}$ are trivial rules.

Consequently, because x_2 enables reaction $(\{x_2\}, \{y_2\}, \{z_2\})$ from B' and x_2 inhibits the reactions from $H^{(1)}$ and $H^{(3)}$,

- $D^{(2)}_{n+2} = D^{(2)}_{n+3} = \ldots = D^{(2)}_{n+k} = \{z_2\} \cup Z^{(2)}$, and
- $W^{(2)}_{n+2} = W^{(2)}_{n+3} = \ldots = W^{(2)}_{n+k} = \{x_2\} \cup D^{(2)}_{n+k} = \{x_2, z_2\} \cup Z^{(2)}$.

(3) π_3 is the evolving interactive process

$$\begin{array}{cccccccccccc} C_0 & C_1 & \ldots & C_j & C_{j+1} & C_{j+2} & \ldots & C_n & C^{(3)}_{n+1} & C^{(3)}_{n+2} & \ldots & C^{(3)}_{n+k} \\ D_0 & D_1 & \ldots & D_j & D_{j+1} & D_{j+2} & \ldots & D_n & D^{(3)}_{n+1} & D^{(3)}_{n+2} & \ldots & D^{(3)}_{n+k} \\ W_0 & W_1 & \ldots & W_j & W_{j+1} & W_{j+2} & \ldots & W_n & W^{(3)}_{n+1} & W^{(3)}_{n+2} & \ldots & W^{(3)}_{n+k} \\ A_0 & A_1 & \ldots & A_j & A_{j+1} & A_{j+2} & \ldots & A_n & A^{(3)}_{n+1} & A^{(3)}_{n+2} & \ldots & A^{(3)}_{n+k} \\ q_0 & q_1 & \ldots & q_j & q_{j+1} & q_{j+2} & \ldots & q_n & q^{(3)}_{n+1} & q^{(3)}_{n+2} & \ldots & q^{(3)}_{n+k} \end{array}$$

where

- $C^{(3)}_{n+1} = C^{(3)}_{n+2} = \ldots = C^{(3)}_{n+k} = \{x_3\}$,
- $A^{(3)}_{n+1} = A^{(3)}_{n+2} = \ldots = A^{(3)}_{n+k} = A''$, and
- $q^{(3)}_{n+1}, q^{(3)}_{n+2}, \ldots, q^{(3)}_{n+k}$ are trivial rules.

Consequently, because x_3 enables reaction $(\{x_3\}, \{y_3\}, \{z_3\})$ from B' and x_3 inhibits the reactions from $H^{(1)}$ and $H^{(2)}$,

- $D^{(3)}_{n+2} = D^{(3)}_{n+3} = \ldots = D^{(3)}_{n+k} = \{z_3\} \cup Z^{(3)}$, and
- $W^{(3)}_{n+2} = W^{(3)}_{n+3} = \ldots = W^{(3)}_{n+k} = \{x_3\} \cup D^{(3)}_{n+k} = \{x_3, z_3\} \cup Z^{(3)}$.

We note that the three sets $\{x_1, z_1\} \cup Z^{(1)}$, $\{x_2, z_2\} \cup Z^{(2)}$, and $\{x_3, z_3\} \cup Z^{(3)}$ are pairwise disjoint, and so

- the set of states $\{W^{(1)}_{n+2}, \ldots, W^{(1)}_{n+k}\}$ (where each state equals $\{x_1, z_1\} \cup Z^{(1)}$),
- the set of states $\{W^{(2)}_{n+2}, \ldots, W^{(2)}_{n+k}\}$ (where each state equals $\{x_2, z_2\} \cup Z^{(2)}$), and
- the set of states $\{W^{(3)}_{n+2}, \ldots, W^{(3)}_{n+k}\}$ (where each state equals $\{x_3, z_3\} \cup Z^{(3)}$)

are pairwise disjoint.

Consider now stationary processes π'_1, π'_2, π'_3 which differ from interactive processes π_1, π_2, π_3 by the fact that also q_j and q_{j+1} are trivial transformation rules (while in π_1, π_2, π_3, we have $q_j = \overline{q}_1$ and $q_{j+1} = \overline{q}_2$). This means that in all three interactive processes the set of reactions in each instantaneous description equals $A = B \cup H$. We will have then:
(1) in the interactive process π'_1:

- $C^{(1')}_{n+1} = C^{(1')}_{n+2} = C^{(1')}_{n+3} = \ldots = C^{(1')}_{n+k} = \{x_1\}$,
- $D^{(1')}_{n+1} = \{z_1, z_2, z_3\}$, $D^{(1')}_{n+2} = Z^{(1)}$, $D^{(1')}_{n+3} = \ldots = D^{(1')}_{n+k} = \varnothing$,
- $W^{(1')}_{n+1} = \{x_1, z_1, z_2, z_3\}$, $W^{(1')}_{n+2} = \{x_1\} \cup D^{(1')}_{n+2}$, $W^{(1')}_{n+3} = \ldots = W^{(1')}_{n+k} = \{x_1\}$.

(2) in the interactive process π'_2:

- $C^{(2')}_{n+1} = C^{(2')}_{n+2} = C^{(2')}_{n+3} = \ldots = C^{(2')}_{n+k} = \{x_2\}$,
- $D^{(2')}_{n+1} = \{z_1, z_2, z_3\}$, $D^{(2')}_{n+2} = Z^{(2)}$, $D^{(2')}_{n+3} = \ldots = D^{(2')}_{n+k} = \varnothing$,
- $W^{(2')}_{n+1} = \{x_2, z_1, z_2, z_3\}$, $W^{(2')}_{n+2} = \{x_2\} \cup D^{(2')}_{n+2}$, $W^{(2')}_{n+3} = \ldots = W^{(2')}_{n+k} = \{x_2\}$.

(3) in the interactive process π'_3:

- $C^{(3')}_{n+1} = C^{(3')}_{n+2} = C^{(3')}_{n+3} = \ldots = C^{(3')}_{n+k} = \{x_3\}$,
- $D^{(3')}_{n+1} = \{z_1, z_2, z_3\}$, $D^{(3')}_{n+2} = Z^{(3)}$, $D^{(3')}_{n+3} = \ldots = D^{(3')}_{n+k} = \varnothing$,
- $W^{(3')}_{n+1} = \{x_3, z_1, z_2, z_3\}$, $W^{(3')}_{n+2} = \{x_3\} \cup D^{(3')}_{n+2}$, $W^{(3')}_{n+3} = \ldots = W^{(3')}_{n+k} = \{x_3\}$.

There are various ways of interpreting organisms and species within the framework of evolving reaction systems. This topic is more suitable for a publication in a biology-related journal, but we will give now one such interpretation here.

From a chemical point of view, a class of organisms $\mathcal{F}$ may be represented by a set of reactions G taking place within organisms in $\mathcal{F}$. Given an evolving interactive process ρ with $st(\rho) = W_0, \ldots, W_n$ for some $n \geq 1$, we say that G (and hence $\mathcal{F}$) *lives in* ρ if G is enabled in each W_i, $0 \leq i \leq n$.

We note that in our example a sequential development pattern represented by π changes into a branching pattern of three processes π_1, π_2, π_3 resulting from the branching of the environment/context from C_n into three different contexts $C^{(1)}_{n+1}$, $C^{(2)}_{n+1}$, and $C^{(3)}_{n+1}$. Then very soon (immediately afterwards, beginning with $W^{(1)}_{n+2}$, $W^{(2)}_{n+2}$, and $W^{(3)}_{n+2}$, respectively) the three state sequences $st(\pi_1)$, $st(\pi_2)$, and $st(\pi_3)$ become disjoint, hence the groups of organisms living in π_1, π_2, π_3 are disjoint. It is important to notice here that the three result sequences $res(\pi_1)$, $res(\pi_2)$, and $res(\pi_3)$ consist of nonempty sets only. Thus a speciation (a formation of new species) into three new species has happened.

The fact that this has happened and then so quickly after the step $n + 1$ is due to the fact that *invisible* changes (not observable in state sequences) were happening in the past in π (and these changes in general could have been happening over a long period of time). In our example these were changes from A_j to A_{j+1} and from A_{j+1} to A_{j+2}.

On the other hand in interactive stationary processes, π'_1, π'_2, π'_3 no reactions from the given constant set of reactions $A_0 = A$ are enabled from state $n + 2$ onwards, and so no organisms can live in π'_1, π'_2, and π'_3 from this state onwards.

Hence we got here an extinction of species. The difference results from the fact that there was no silent/invisible evolution present in the past (i.e., in transitions from state j to state $j+1$, and from state $j+1$ to state $j+2$).

Presenting the theory of evolution, see e.g., [19], in a very simplified form, one can say that the Darwinian evolution is based on *gradual* changes: small changes of environment over time cause small changes in organisms. Therefore speciation happens gradually, very *slowly*. In the theory of punctuated evolution proposed by N. Eldredge and S.J. Gould, see e.g., [13, 19], evolution can happen *rapidly* when the environment branches into many possible environments (niches). This rapid evolution seems to contradict the principle of (slow) gradualism.

Within the framework of evolving reaction systems the principle of punctuated evolution may be reconciled with the principle of gradualism through the Invisibility Theorem. The rapid evolutionary changes following branchings of the environment could have been prepared for a very long time through "silent evolution" — changes which are not observable in phenotype (which is the set of observable characteristics).

8. Discussion

In this paper we have introduced and investigated evolving interactive processes which generalize standard interactive processes of reaction systems by allowing the set of available reactions to evolve as a process progresses from state to state. The main technical focus of the paper is the Invisibility Theorem which allows for an evolution of a system which is not externally observable. We have also indicated a possible relationship between the Invisibility Theorem and the notion of punctuated evolution from evolution theory.

Obviously, this is only a beginning of developing the framework of evolving reaction systems. A systematic investigation of this new framework could begin by investigating central research themes concerning reaction systems in the framework of evolving reaction systems. The topics/themes that come to mind include

- properties of state sequences, see e.g., [5,14,16–18],
- modularity, see e.g., [10],
- duration, see e.g., [3],
- minimizing resources, see e.g., [6,18],
- use of evolving reaction systems in investigating biological processes, see e.g., [1,2,8].

The notion of enabling equivalence introduced in this paper deserves a thorough systematic investigation. Results presented in Section 3 form a good starting point for such an investigation.

An important topic, especially suited for evolving reaction systems, is concerned with *control sequences*: how properties of the sequence of available sets of reactions $(A_0, A_1, \ldots, A_n)$, such as e.g., periodicity, are reflected in the state sequence of evolving interactive processes.

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