The Navier-Stokes Equations

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Abstract

The **Navier-Stokes Equations** are a set of **partial differential equations** that describe **fluid flow**. In this paper, we review the history behind the equations, look at a derivation of the most general form of the equations, apply the general form to applications in **physics**, and discuss the **millennium problem** proposed by the Clay Mathematics Institute regarding solutions to these equations.

Contents

1 History and Background

Developed by **Claude-Louis Navier** and **George Gabriel Stokes**, the **Navier-Stokes equations** were developed between 1822 and 1850. The equations express key physical concepts relating to fluid flow including **conservation of momentum** and, for Newtonian fluids, **conservation of mass**.

The equations stem from combining Newton's second law with fluid motion to describe **viscous fluid flow**. They are closely related to **Euler's equations** for fluid flow; however, the Navier-Stokes equations provide a better model as they take into account viscosity while Euler's equations "model only inviscid flow" [Navier-Stokes].

The wide range of variables that the Navier-Stokes equations take into account make them useful in a variety of practical uses including "weather, ocean currents, water flow in a pipe, and air flow around a wing" [Navier-Stokes]. These equations have many applications in science and engineering and are often used in conjunction with other equations in a diverse range of fields.

2 Derivation

When studying fluids, there are two primary ways to measure fluid flow: either we fix a reference point and study particles as they pass by or we fix a bit of fluid and follow it as it flows [Derivation].

Definition 2.1 (Eulerian Derivative)**.** The **Eulerian derivative** of a field of fluids is defined as the operator $\frac{\partial}{\partial t}$. This operator measures the rate of change of an arbitrary fluid property at a *fixed point* over time [Brennen].

Definition 2.2 (Lagrangian Derivative)**.** The **Lagrangian derivative**, also referred to as the **material derivative** is defined as the operator $\frac{D}{Dt} := \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$, where $\mathbf{u}$ is the **flow velocity** of a fluid [Derivation].

Remark 2.3 — Note that the Lagrangian derivative *includes* the Eulerian derivative in its definition—this represents change at a point with respect to time whereas the second term, $\mathbf{u} \cdot \nabla$, represents change in a property with respect to position [Derivation].

We will now shift to discuss an idea central to physics: **continuity**. Given an **intensive property** φ (meaning the property is independent of the amount of material one has) defined on a fluid volume Ω, we can write a general **continuity equation** representing a **conservation law** [Gibanksy]:

$$\frac{d}{dt}\int_{\Omega} \varphi \, d\Omega = -\int_{\partial\Omega} \varphi \mathbf{v} \cdot \mathbf{n} - \int_{\Omega} s \, d\Omega.$$

The equation is made up of several integral terms, including

- The left hand side, $\frac{d}{dt}\int_{\Omega} \varphi \, d\Omega$, which represents the change in the property φ over the volume Ω,
- $\int_{\partial\Omega} \varphi \mathbf{v} \cdot \mathbf{n}$, which represents **flux** and describes how the property φ flows over the **boundary** of the region, $\partial\Omega$.
- $\int_{\Omega} s \, d\Omega$, which describes how the property φ leaves the volume due to **sinks** and/or **sources** within the volume [Gibanksy].

This equation, a consequence of **Reynold's Transport Theorem**, emphasizes a key idea:

> **"The change in the total amount of a property is due to how much flows out through the volume boundary as well as how much is lost or gained through sources or sinks inside the boundary" [Gibanksy].**

We can apply the **divergence theorem** to the surface integral on the right hand side, allowing us to express it as a volume integral [Derivation]:

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \varphi \, d\omega = -\int_\Omega \nabla \cdot (\varphi \mathbf{u}) \, d\Omega - \int_\Omega s \, d\Omega. \\
\int_\Omega \frac{\partial \varphi}{\partial t} \, d\Omega = -\int_\Omega (\nabla \cdot (\varphi \mathbf{u}) + s) \, d\Omega \\
\implies \int_\Omega \left(\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{u}) + s \right) \, d\Omega = 0 \\
\implies \frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{u}) + s = 0.
\end{aligned}$$

Example 2.4 (Conservation of Mass)

Beginning with our general conservation equation $\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{u}) + s = 0$, we can substitute in $\varphi = \rho$, where ρ represents density, to get $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) + s = 0$. Since our volume Ω is constant, $s = 0$, so we get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

which represents **conservation of mass**. This equation ususally accompanies the Navier-Stokes equations as context [Derivation].

We now have all the ingredients necessary to derive the Navier-Stokes equations. Beginning with Newton's Second Law, $\sum \mathbf{F} = m\mathbf{a}$, we replace $\mathbf{F} = \mathbf{b}$ to represent the **body force** on a particle and substitute density for mass because we are dealing with individual particles

[Gibanksy]:

$$\begin{aligned}
\mathbf{F} &= m\mathbf{a} \\
\mathbf{b} &= \rho \frac{d}{dt}\mathbf{v}(x, y, z, t) \\
&= \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathbf{v}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \mathbf{v}}{\partial z}\frac{\partial z}{\partial t}\right) \\
&= \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) \\
&= \rho\frac{D\mathbf{v}}{Dt}.
\end{aligned}$$

We can then look at the body force on fluid particles, which we notice is made up of two components: **fluid stresses** that act on the individual particles and **external forces** that act on the fluid as a whole, so $\mathbf{b} = \nabla \cdot \sigma + \mathbf{f}$, where σ represents the **stress tensor** of the fluid and $\mathbf{f}$ represents external forces [Gibanksy].

Abuse of Notation 2.5 (Tensors)**.** **Tensors** are a type of **generalized matrix** in n dimensions; however, for our use, we can use a 3×3 matrix to represent the stress tensor.

We write

$$\sigma = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \sigma_{xx}+p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}+p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}+p \end{pmatrix} = -p\mathbf{I} + \tau$$

to represent the stress tensor, where σ are the **normal stresses** on our fluid particle and τ are the **shear stresses**.

Combining equations from earlier, we get

$$\begin{aligned}\rho\frac{D\mathbf{v}}{Dt} &= \mathbf{b} \\ &= \nabla\cdot\sigma + \mathbf{f} \\ &= -\nabla p + \nabla\cdot\tau + \mathbf{f}.\end{aligned}$$

This gives us the most general form of the Navier-Stokes Equation [Derivation]:

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \nabla\cdot\tau + \mathbf{f}.$$

Where

- $-\nabla p$ is the **volumetric stress tensor**; it represents pressure and prevents motion due to normal stresses;
- $\nabla\cdot\tau$ is the **stress deviator tensor** which causes motion due to horizontal friction and shear stress—equivalently turbulence within the fluid; and
- $\mathbf{f}$ is the **external force** acting on the fluid as a whole, usually gravity [Gibanksy].

3 Applications

Though it looks fancy, our general form of the Navier-Stokes Equation doesn't have any specific uses. Through specified values and expressions for the various tensors and forces, we can adapt our equation to model specific varieties of fluids.

3.1 Newtonian Fluid

Newton studied a certain set of fluids with which he observed the defining charactaristic that $\tau \propto \frac{\partial u}{\partial y}$, stated in **Newton's Law of Viscosity**.

Corollary 3.1

Stokes, when formalizing the Navier-Stokes Equations, made the following three assumptions as a consequence of Newton's observation:

- The stress tensor is a linear function of the **velocity gradient**;
- The fluid is **isotropic**—essentially meaning it acts symmetrically in all orientations; and
- $\nabla \cdot \tau = 0$ for fluids at rest [Derivation].

These assumptions led Stokes to derive an expression $\tau = \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}$, where μ represents **shear viscosity** and λ represents **volume viscosity**.

We now begin with the general form of the Navier-Stokes Equation, replacing $\mathbf{f} = \rho \mathbf{g}$ to indicate that gravity is the only external force acting on our fluid:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \tau + \rho \mathbf{g}$$

Substituting in Stokes' expression for τ, we get

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \left[\mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \right] + \nabla \cdot \left[\lambda (\nabla \cdot \mathbf{u}) \mathbf{I} \right] + \rho \mathbf{g}.$$

The associated mass continuity equation from earlier holds: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$.

3.1.1 Incompressible Newtonian Fluid

Now we make another assumption that our fluids are **incompressible**, meaning

- shear viscosity μ will be constant,
- volume viscosity $\lambda = 0$, and

- the mass continuity equation simplifies to $\nabla \cdot \mathbf{u} = 0$.

This allows us to simplify our equation to

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \left[\mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \right] + \rho \mathbf{g}.$$

However, the stress deviator tensor term simplifies nicely to a vector Laplacian: $\nabla \cdot \tau = \mu \nabla^2 \mathbf{u}$, giving us a final equation

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}.$$

3.2 Parallel and Radial Flow

The Navier-Stokes equations are easily applied to real-world problems regarding **fluid flow**. Two such examples are parallel and radial flow.

3.2.1 Parallel Flow

Given fluid flowing between parallel plates, we can use describe a **boundary value problem** which models the situation: $\frac{d^2u}{dy^2} = -1;\ u(0) = u(1) = 0$. This can be easily solved to find $u(y) = \frac{y-y^2}{2}$. This expression for $u(y)$ allows us to then use the Navier-Stokes equations to analytically obtain other measurements such as "viscous drag force or net flow rate" [Navier-Stokes].

3.2.2 Radial Flow

A more complex yet still feasible problem is radial flow between parallel plates, which can be represented by the **ordinary differential eqation** $\frac{d^2f}{dz^2}+Rf^2 = -1;\ f(-1) = f(1) = 0$, where R is a chosen **Reynolds number** (a constant relating to laminar fluid flow) [Navier-Stokes].

4 Millennium Problem

Perhaps the biggest testimony to the importance of the Navier-Stokes Equations is their inclusion in the Clay Mathematics Institute's **Millennium Prize Problems**. This set of seven problems were established in 2000 as an outline for mathematics for the coming century. According to the Institute itself, "Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations" [Navier-Stokes Equation]. The challenge set forth by the Institute is to prove or disprove the **existence** and **smoothness** of solutions to the Navier-Stokes equations—that is, to prove "whether smooth solutions always exist in three dimensions—i.e. they are infinitely differentiable (or even just bounded) at all points in the domain" [Navier-Stokes].

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