# Geometric Central Limit Theorems on Singular Spaces

9 lectures - #1 = overview

## Linear CLT

Input: vector space $\mathbb{R}^n$

independent random variables $X_1, X_2, \ldots$

distributed according to measure $\mu$

Compare empirical mean $\bar{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$

to population mean $\bar{\mu} = \int x \, \mu(dx)$

LLN: $\bar{\mu}_n \xrightarrow{n \to \infty} \bar{\mu}$

CLT: $\sqrt{n}\,(\bar{\mu}_n - \bar{\mu}) \xrightarrow{n \to \infty} N(0, \Sigma)$ in distribution

for $N(0, \Sigma)$

- Gaussian
- Centered at 0
- Same covariance $\Sigma$ as $\mu$

Nonlinear data

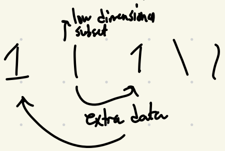
Look into $d \times n$ shape matrices

Initial Rationale: "Big Data" often sampled from nonlinear spaces

Ex. A phylogenetic tree is a rooted metric tree with $n$ labeled leaves

BHV Tree space

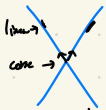
The digit 1 - MNIST dataset = set of handwritten digits



# Motivation & History

Mimic Ordinary Stats
- ↳ averages
- ↳ variance, PCA
- ↳ LLN, confidence intervals/regions
- ↳ CLTs ← focus

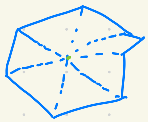
Singular space = not smooth = has at least one nonlinear tangent space




3-spider

open book

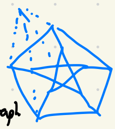
$\mathbb{R}^d \times$ Spider



isolated hyperbolic planar singularity "Kale"

$\Upsilon_3$

Take the cone of a petersen graph



$\Upsilon_4$

# Fréchet Means

Sample Space: Riemannian manifold $M$

Sums and averages fail!

**Def** fréchet function $F_\mu(y) = \frac{1}{2}\int_M d(x,y)^2 \mu(dx)$

Probability distribution $\mu$ on any metric space $M$

fréchet mean $\bar{\mu} = \underset{y \in M}{\operatorname{argmin}} F_\mu(y)$

empirical mean $\bar{\mu}_n$ from empirical measure $\mu_n = \frac{1}{n}(\delta x_1 + \dots + \delta x_n)$

↳ "least square approximation"

LLN unaffected $\bar{\mu}_n \xrightarrow{n\to\infty} \bar{\mu}$ almost certainly define gaussian by reducing to linear case

## Logarithm Maps

**Recall** CLT on manifold $M$

- Variation of rescaled differences $\sqrt{n}(\bar{\mu}_n - \bar{\mu})$

  $\bar{\mu}_n$ changes $\bar{\mu}$ fixed

→ limit is random tangent vector in tangent space $T_{\bar{\mu}}$

**Def** The logarithm map is

$$\log_{\bar{\mu}} : M \to T_{\bar{\mu}}M$$
$$x \mapsto d(\bar{\mu}, x)V$$

where $V$ = unit tangent to geodesic from $\bar{\mu}$ to $x$

Back to linear setting ↗ pushforward of $\mu$ to $T_{\bar{\mu}}M$

- $\mu$ on $M \rightsquigarrow \nu$ on $T_{\bar{\mu}}M$ for $\nu = \mu \circ \log_{\bar{\mu}}^{-1} = (\log_{\bar{\mu}})_{\#}\mu$
- linear CLT: $\sqrt{n}\,\bar{\nu}_n \xrightarrow{n\to\infty} N(0, \Sigma)$ in distribution

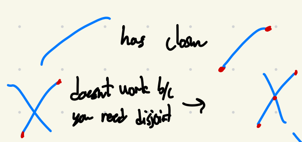
Is this the manifold CLT? Not quite. But it is a Gaussian distribution of $T_{\bar{\mu}}M$

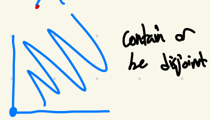
Generalizes to singular space by tangential collapse

## Stratified Spaces

**def** $M$ is smoothly stratified with distance $d$ if

- $M$ complete, locally compact, geodesic space ← distance realized via geodesic
- $M = \bigsqcup_{i=0}^{d} M^i$ has disjoint locally closed strata $M^j$
- Each stratum $M^j$
  - ↳ is a manifold with geodesic distance $d|_{M^j}$
  - ↳ has closure $\overline{M^j}$ which is a union of (not necessarily all) strata $M^k$ with $k \le j$
- Locally well defined exponential maps that are local homeomorphisms
  - ↳ essential for bringing asymptotics of sampling to $T_{\bar{\mu}}M$ and back to $M$.
- Curvature bounded above by $\kappa$ ($M$ is CAT($\kappa$))
  - ↳ only really needed at $\bar{\mu}$ which
  - ↳ morally won't be infinitely curved anyway: Fréchet means would flee



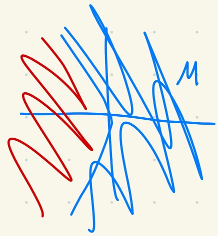

Stratum closures form poset under containment.

Examples

↳ graph → strata are vertices and edges

↳ finite fly wings

↳ tree spaces




tangential collapse = inclusion of $M$ into the plane

## Escape Vectors

**Prop** $M$ smoothly stratified $\Rightarrow$ $T_{\bar\mu}M$ is a smoothly stratified CAT(0) cone

**Def** If $Y \in T_{\bar\mu}M$ is logarithm (based at $\bar\mu$) of $y \in M$ then $Y$ has escape vector

$$\mathcal{E}(Y) = \lim_{t\to 0} \frac{1}{t} \log_{\mu}\big(\underbrace{\overline{\mu + t\delta_y}}\big)$$

← take fréchet mean

start w/ measure $\mu$, add (infinitesimal) point mass

* in a smooth setting this is an influence function

More generally, $\Delta = \lambda_1 \delta_{y_1} + \dots + \lambda_j \delta_{y_j}$ has escape vector

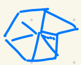
$$\mathcal{E}(\Delta) = \lim_{t\to 0} \frac{1}{t} \log_{\bar\mu}\big(\overline{\mu + t\delta}\big)$$

if $\delta = \lambda \delta_{y_1} + \dots + \lambda_j \delta_{y_j}$ with $Y^i = \log_{\bar\mu} y^i$.

## Example

Isolated hyperbolic planar singularity = "kale"

angle sum at apex is $\alpha > 2\pi$



$\mathcal{E}$ is convex projection to the fluctuating cone


limiting distribution

$$C_\mu = \{ V \in \mathrm{hull}(\mathrm{supp}\, \mu) \mid \nabla_\mu F(V) = 0 \}$$

## Stratified Gaussians

Reduce to linear case:

- Smooth $M$: use (most of) $\log_\mu : M \to T_{\bar\mu}M$
- Singular $M$: need extra step of tangential collapse $\mathcal{L} : T_{\bar\mu}M \to \mathbb{R}^m$, so

$$M \xrightarrow{\log_{\bar\mu}} T_{\bar\mu}M \xrightarrow{\mathcal{L}} \mathbb{R}^m$$

using "limit log" via radial transport away from $\bar\mu$

$\nu = (\mathcal{L} \circ \log_{\bar\mu})_{\#}\mu$ on $\mathbb{R}^m \Rightarrow \sqrt{n}\,\bar\nu_n \xrightarrow{n\to\infty} N(0,\Sigma)$ on $\mathbb{R}^\ell \subseteq \mathbb{R}^m$

**Lemma.** The map $\mathcal{L}$ has a measurable section over $\mathbb{R}^\ell$

$\Delta : \mathbb{R}^\ell \to$ discrete measures on $\overline{\mathbb{R}_+ \operatorname{supp}(\log_{\bar\mu})_{\#}\mu} \subseteq T_{\bar\mu}M$

with $\mathcal{L} \circ \Delta = id_{\mathbb{R}}$ where $\mathcal{L}(\lambda_1 \delta_{y^1} + \dots + \lambda_j \delta_{y^j}) = \lambda_1 \mathcal{L}(Y^1) + \dots + \lambda_j \mathcal{L}(Y^j)$.

**Def** A Gaussian tangent mass $\Gamma_\mu$ is any measurable section of any $\mathbb{R}^\ell$-valued random variable $N \sim N(0,\Sigma)$:

$$\Gamma_\mu = \Delta(N).$$

## Central Limit Theorem

Perturbative CLT: $\lim_{n\to\infty} \sqrt{n} \log_{\bar\mu} \bar\mu_n \xrightarrow{d} \mathcal{E}(\Gamma_\mu)$

**Def** The distortion map is $\mathcal{H} = \xi \circ \Delta : \mathbb{R}^\ell \to T_{\bar\mu}M$

**Prop** Distortion $\mathcal{H}$ does not depend on choice of section $\Delta$.

**Geometric CLT.** $\lim_{n\to\infty} \sqrt{n} \log_{\bar\mu} \bar\mu_n \xrightarrow{d} \mathcal{H}_{\#} N(0,\Sigma)$

**Corollary** Smooth CLT where

$$\mathcal{H} = (\nabla \nabla_{\bar\mu} F_\mu)^{-1}$$

is the inverse Hessian of the Fréchet function

**CLT 3** $\lim_{n\to\infty} \sqrt{n} \log_{\bar\mu} \bar\mu_n = \nabla_\mu b(\Gamma_\mu)$ 
fake barycenter

- directional derivative in the space $\mathcal{P}_2 M$ of $L^2$ measures on $M$
- of the barycenter map $b : \mathcal{P}_2 M \to M$ sending $\mu \mapsto \bar\mu$
- at $\mu$
- along any Gaussian tangent mass $\Gamma_\mu$

View escape as derivative in space of measures via random perturbation

**CLT 4** $\lim_{n\to\infty} \sqrt{n} \log_{\bar\mu} \bar\mu_n \stackrel{d}{=} \nabla_{F_\mu \circ \exp_{\bar\mu}} \mathcal{B}(G)$

- directional derivative in space of continuous maps $C(T_{\bar\mu}M, \mathbb{R})$
- of the minimizer map $\mathcal{B} : C(T_{\bar\mu}M, \mathbb{R}) \to T_{\bar\mu}M$ that sends $f \mapsto \operatorname{argmin}_{x \in C_\mu} f(X)$
- at $F_\mu \circ \exp_{\bar\mu}$
- along the gaussian tangent field $G = G(\cdot) = \langle \Gamma_\mu, \cdot \rangle_{\bar\mu}$ induced by $\mu$

# Applications

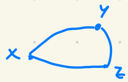
- heat dissipation
- random walks
- infinite divisibility of probability distributions

**Def:** for any $\kappa \in \mathbb{R}$, a <u>model space of curvature $\kappa$</u> is a Riemannian manifold $M_\kappa$ with geodesic distance $d_\kappa$ with constant curvature $\kappa$

The <u>injectivity radius</u> of $M_\kappa$ is $R_\kappa = \frac{\pi}{\sqrt{\kappa}}$ when $\kappa > 0$ and $\infty$ otherwise

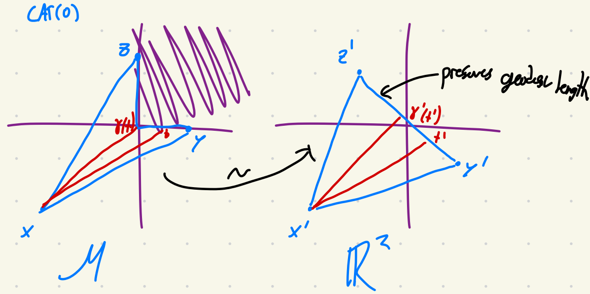
E.g. $\kappa = 1 \Rightarrow M_\kappa =$ sphere

$\kappa = -1 \Rightarrow M_\kappa =$ hyperbolic plane




$\kappa = 1 \Rightarrow R_\kappa = \pi = d$ (north pole, south pole) (on sphere)

Given a triangle $\Delta xyz$ (union of geodesics) in $(M,d)$, a <u>comparison triangle</u> is a triangle $\Delta x'y'z'$ in $M_\kappa$ with $\{x', y', z'\} \underset{\text{isometric}}{\cong} \{x, y, z\}$



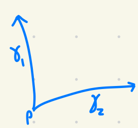

**Def:** A metric space $(M,d)$ is $CAT(\kappa)$ if "curvature bounded above by $\kappa$"

1. $x, y \in M$ with $d(x,y) < R_\kappa$ can be joined by unique geodesic of length $d(x,y)$

2. $\Delta xyz \in M$ with $d(x,y) + d(y,z) + d(z,x) < 2R_\kappa$
   $\Rightarrow$ for any comparison $\Delta x'y'z'$ in $M_\kappa$, $d(x, \gamma(t)) \leq d_\kappa(x', \gamma'(t))$
   $\rightsquigarrow$ "triangles are thinner"

Fix a $CAT(\kappa)$ space $(M,d)$.

**Def:** The <u>angle</u> between geodesics $\gamma_i : [0, \varepsilon_i] \to M$ for $i = 1, 2$

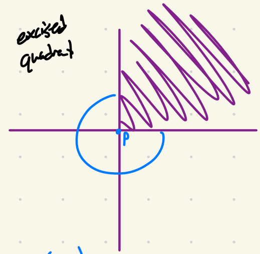


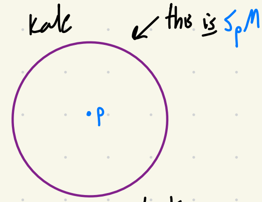

($p \to \mu$) with $\gamma_i(0) = p$ and parametrized by arc length is given by

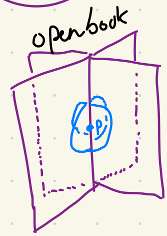
$$\cos(\angle(\gamma_1, \gamma_2)) = \lim_{t,s \to 0} \frac{s^2 + t^2 - d^2(\gamma_1(s), \gamma_2(t))}{2st}$$

**Def** $S_pM := \{\text{equivalence class of geodesics at } p\}$ ← unit tangent sphere; $\angle = 0$

$= $ space of directions at $p$.




**Lemma:** $\angle$ makes $S_pM$ into a length space whose angular metric $d_s$ has $d_s(v,w) = \angle(v,w)$ whenever $v, w \in S_pM$ with $\angle(v,w) < \pi$

**Def: Tangent Cone** $T_pM := \underbrace{S_pM \times [0,\infty)}_{\text{cylinder}} / \underbrace{S_pM \times \{0\}}_{\text{collapse the bottom}}$

**Def:** $v, w \in T_pM$ have inner product $\langle v, w\rangle_p = \|v\| \cdot \|w\| \cos(\angle(v,w))$

**Lemma:** for fixed $p$, $\langle \cdot, \cdot \rangle_p : T_pM \times T_pM \to \mathbb{R}$ is continuous (lipschitz)

**Def:** The conical metric on $T_pM$ is $d_p(v,w) = \sqrt{\|v\|^2 + \|w\|^2 - 2\langle v, w\rangle}$

**Lemma:** any geodesic triangle in $T_pM$ with apex as a vertex is Euclidean

**Def:** the log map $\log_p : M' \to T_pM$ ($M'$ := pts with unique shortest path to $p$)

$v \mapsto d(p,v)V$

where $V$ = tangent to shortest path $p$ to $v$

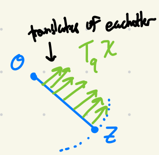
**Def** $M$ is conical with apex $p$ if $M' = M$ and $\log_p$ is an isometry. "conical not canonical"

**Def:** let $M$ be $CAT(\kappa)$ and $X = T_pM$ with apex $O$. The shadow of $z \in T_O X$ is $\mathcal{I}(z) := \{v \in T_O X \mid \angle(v,z) = \pi\}$


**Prop** for points $q, q' \in Oz$ and $z = \exp_O Z$ there is a radial transport $\Pi_{q \to q'}$ that identifies $T_{q'}X$ with $T_qX$.

**Def:** The limit tangent cone along $Z$ is the direct limit $\vec{T}_Z X = \varinjlim_{O \neq q \in Oz} T_q X$

↑ tangent vectors to $X$

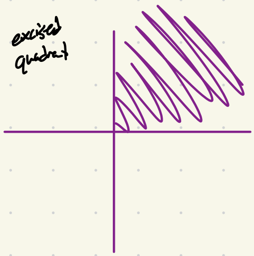


**Def:** $\operatorname{codim}(p) := \dim M - \dim(\text{stratum at } p)$

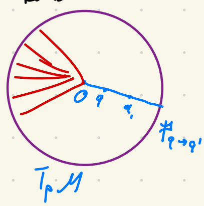
"upper bound for how bad stratum can be"

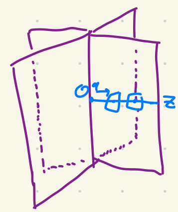
$\operatorname{codim} = 0 \Rightarrow$ smooth

excised quadrant



Kale



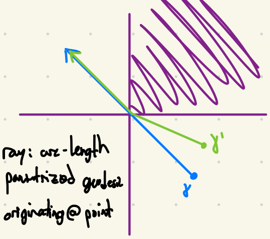




**Lemma:** $M$ $CAT(k)$ and $Z \in T_pM$

$\Rightarrow$ $T_pM$ and $\tau_Z M$ are $CAT(0)$.

**Def:** rays $\gamma \| \gamma'$ (parametrized by arc length) if $d(\gamma_t, \gamma'_t)$ bounded $\forall t$

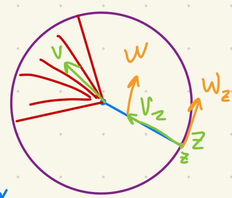
**Prop:** ray $\gamma \subseteq CAT(0)$ and $z \in$ $\Rightarrow \exists! \gamma'$ from $z$ with $\gamma \| \gamma'$




ray: arc-length parametrized geodesic originating @ point

**Def:** parallel vectors $\Leftarrow$ parallel rays

$\tau_{O \to z} : V \mapsto V_z$




$Z = \log z$

**Def** $M$ $CAT(k)$, $X = T_pM$, $Z \in T_O X$ "Kale"

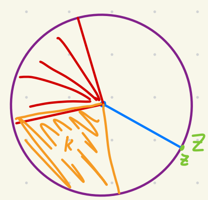
The limit log map along $Z$ is

$\mathcal{L}_Z : T_O X \to \tau_Z X$ $\forall t \geq 0$, where $V_Z$ = image in $\tau_Z X$
$tV \mapsto tV_Z$ of $\tau_{O \to qV}$ for any $q \neq 0$ in $O_Z$

Shadow of $Z$ = set of rays at angle $\geq \pi$ from $Z$

$\Rightarrow$ geodesic goes through $O$

**Theorem:** If $K \subseteq T_pM$ is a geodesically convex subcone containing at most one ray in shadow of $Z$ then $\mathcal{L}_Z|_K$ is an isometry to $\mathcal{L}_Z(K)$.



# Measures!

fix measure $\mu$ on CAT($\kappa$) $M$

**Def** $\mu$ is localized if

- $\mu$ has unique fréchet mean $\bar{\mu}$
- $\mu$ has locally convex fréchet function
  $$F_\mu(y) = \frac{1}{2}\int d(x,y)^2 \, d\mu(x)$$
  in nbd of $\bar{\mu}$
- $\log_{\bar{\mu}} : M \to T_{\bar{\mu}} M$ is $\mu$-almost surely defined

Write $\hat{\mu} = (\log_{\bar{\mu}})_{\#\mu}$

injectivity radius $\frac{\pi}{\sqrt{\kappa}}$

E.g. $\operatorname{supp}\mu \subseteq B(\bar{\mu}, R_\mu)$ [Kuwae]

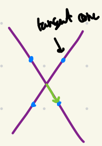
**Def**: $F$ has directional derivative at $\bar{\mu}$
$$\nabla_{\bar{\mu}} F : T_{\bar{\mu}} M \to \mathbb{R}$$
$$v \mapsto \frac{d}{dt} F(\exp_{\bar{\mu}} tv)\Big|_{t=0}$$

**Def**: The escape cone is
$$E_\mu = \{ X \in T_{\bar{\mu}} M \mid \nabla_{\bar{\mu}} F(X) = 0 \}.$$
large, flightless bird

**Prop**: $E_\mu$ is a closed, path-connected, geodesically convex subcone of $T_{\bar{\mu}} M$ and $E_\mu \cap S_{\bar{\mu}} M \subseteq$ single connected component.

think of this as $T_{\bar{\mu}} M$ or $\vec{T}_Z M$.

lost sank requires $E_\mu$ be only one ray!




**Def**: fix CAT(0) cone $X$. $S \subseteq X$ has $\operatorname{hull} S \subseteq X$, the smallest convex cone containing $S$.

**Def**: $\operatorname{Hull}_\mu := \operatorname{Hull} \operatorname{supp}(\hat{\mu})$.

**Def**: $\mu$ has fluctuating cone $C_\mu = E_\mu \cap \operatorname{hull}_\mu$.

**Def**: A tangential collapse of $\mu$ is a map $\mathcal{L} : T_{\bar{\mu}} M \to \mathbb{R}^m$ s.t.

1) $\mathcal{L}_{\#}\hat{\mu} = \mathcal{L}(\log_{\bar{\mu}}(\bar{\mu}))$ → cone pt. (fréchet mean) ↦ origin
2) $\mathcal{L}\big|_{\overline{C_\mu}}$ is injective

3. $v \in T_{\bar{\mu}} M$, $u \in C_\mu \Rightarrow \langle u, v \rangle_{\bar{\mu}} = \langle \mathcal{L}(u), \mathcal{L}(v) \rangle_{\mathcal{L}(\bar{\mu})}$ $\longrightarrow$ metric is preserved (not necessarily everywhere)

4. $\mathcal{L}(tV) = t\mathcal{L}(V)$

5. $\mathcal{L}$ is continuous

## Random Tangent fields

$$f_\mu(p) = \frac{1}{2}\int_M d(p,x)^2 \mu(dx)$$

$\uparrow$ fréchet function $\qquad$ $\uparrow$ CAT(K) stratified

CLT limiting distribution on $T_{\bar{\mu}} M$

rewrite $F_\mu = F$ $\qquad$ $\mu_n$ sampled from $\mu$

$F_{\mu_n} = f_n$ $\qquad$ iid $x_1, \dots, x_n$

### Linear case

$$f_n(p) = \frac{1}{2}\int |p-x|^2 \mu_n(dx)$$

$$= \frac{1}{2}\int \left(|p|^2 - 2p\cdot x + |x|^2\right)\mu_n(dx)$$

$$= \underbrace{\frac{1}{2}|p^2|}_{} - \underbrace{\frac{1}{n}\sum_{i=1}^n p\cdot x_i}_{} + \underbrace{\frac{1}{2n}|x_i|^2}_{}$$

$$= \frac{|p^2|}{2}$$

$$= F(p) - p\cdot\bar{\mu} + \int \frac{|x|^2}{2}\mu(dx) - \overbrace{\frac{1}{n}\sum_{i=1}^n \left(p\cdot x_i - p\bar{\mu}\right)}^{\bar{g}_n(p)} + \frac{1}{n}\sum_{i=1}^n \frac{|x_i|^2}{2} - \int \frac{|x|^2}{2}\mu(dx)$$

$$= \underbrace{F(p) - \bar{g}_n(p)}_{\text{argmin } p} + C_n$$

$\nwarrow$ doesn't depend on $p$

$$\sqrt{n}\,\underbrace{\left(\bar{\mu}_n - \bar{\mu}\right)}_{\substack{\text{vector from population} \\ \text{mean to sample mean}}} \stackrel{d}{=} \lim_{n\to\infty} \sqrt{n}\, \operatorname*{argmin}_p \left(F(p) - \bar{g}_n(p)\right)$$

replace $x_i$ with $X_i$ in definition

$$\rightsquigarrow \lim_{n\to\infty} \sqrt{n}\, \operatorname*{argmin}_{V\in T_{\bar{\mu}}M} \left(F(\exp_{\bar{\mu}} V) - \bar{g}_n(V)\right)$$

Where $\bar{g}_n(V) = \frac{1}{n}\sum_{i=1}^n \left(\langle V, X_i\rangle_{\bar{\mu}} - m(V)\right)$ $\qquad$ $\nearrow \hat{\mu} = (\log_{\bar{\mu}})_\# \mu$

and $m(V) = \int_M \langle V, \log_{\bar{\mu}} x\rangle \mu(dx) = \int_{T_{\bar{\mu}}M} \langle V, U\rangle_{\bar{\mu}}\, \hat{\mu}(dU)$

**Prop:** $\int_{T_{\bar\mu}M} \langle v,u\rangle_{\bar\mu}\,\hat\mu(du) = -\nabla_{\bar\mu} F(v)$

**Def:** (Random tangent field) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $M$ as usual. A stochastic process $f$ indexed by $S_{\bar\mu}M$ at $\bar\mu \in M$, meaning a measurable map

$$f: \Omega \times S_{\bar\mu}M \to \mathbb{R}$$

is a random tangent field on $S_{\bar\mu}M$. $\rightsquigarrow$ homogeneous on $T_{\bar\mu}M$
$f(tv) = tf(v)$

$f$ is thought of as $\{f(V): \Omega \to \mathbb{R} \mid V \in S_{\bar\mu}M\}$
e.g. any $M$-valued random variable $x = X(\omega)$ and $g: M \times S_{\bar\mu}M \to \mathbb{R}$
induce $(\omega, v) \mapsto g(X(\omega), V)$ written $g(x, V)$

e.g. $g(x_i, V) = \langle \log_{\bar\mu} x_i, V\rangle - m(V)$
call it $g_i(V)$, $x_i \sim \mu$
distributed from

**Covariance** $\Sigma: T_{\bar\mu}M \times T_{\bar\mu}M \to \mathbb{R}$ — covariance for a random field
$(u,v) \mapsto \int_{T_{\bar\mu}M} \langle u,y\rangle\langle v,y\rangle \hat\mu\, dy$

s.t.

1.) $(v,w) \mapsto \mathbb{E}[G(v)G(w)]$ is $\Sigma(v,w)$
2.) $\forall V_1, \dots, V_n \in S_{\bar\mu}M$, $(G(V_1), \dots, G(V_n))$ is multivariate Gaussian distributed

$$G_n(V) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i(V) = \sqrt{n}\,\bar g_n(V)$$

**Thm:** fix stratified $CAT(\kappa)$ space $M$ and localized measure $\mu$ on $M$.
$G_n \xrightarrow{d} G$ as $n \to \infty$.
distribution

**Def 6.1** fix a tangential collapse $\mathcal{L}: T_{\bar\mu}M \to \mathbb{R}^m$ — $M$: smoothly stratified space
Set $\hat\mu = (\log_{\bar\mu})_{\#}\mu$ $\qquad \hat\mu \rightsquigarrow \mu^{\mathcal{L}} = \mathcal{L}_{\#}\hat\mu$
$\mu^{\mathcal{L}}$: collapsed $\mu$

$x_i \sim \mu$ iid
$X_i = \log_{\bar\mu} x_i$ vectors in $\mathbb{R}^m$
$\bar X_n^{\mathcal{L}} = \frac{1}{n}\sum_{i=1}^{n} \overbrace{\mathcal{L}(\underbrace{X_i}_{\text{random var - can't sum}})}^{\text{can sum}}$

Gaussian distributed vector on $\mathbb{R}^m$:
$N_{\mu^{\mathcal{L}}} \sim \mathcal{N}(0, \Sigma)$ for $\Sigma = \mathrm{cov}(\mu^{\mathcal{L}})$

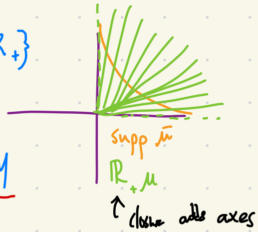
$$\sqrt{n}\,\bar{X}_n^{\mathcal{L}} \to \underbrace{\mathcal{N}(0,\Sigma)}_{\text{supp} = \mathbb{R}^{\ell} = \text{hull}_{\mu}\mathcal{L}}$$

$\to$ smallest convex cone containing $\text{supp}(\mu^{\mathcal{L}})$

$$\mathcal{L} : M = T_{\bar{\mu}} M \hookrightarrow \mathbb{R}^2$$

**Def 4.4** Set $\mathbb{R}_+\mu = \mathbb{R}_+(\text{supp}\,\hat{\mu})$
$= \{\alpha X \mid X \in \text{supp}\,\hat{\mu},\ \alpha \in \mathbb{R}_+\}$

with closure $\overline{\mathbb{R}_+\mu}$




**Def 4.5** measure $\delta$ on $M$ is sampled from $M$ if it is $\delta = \lambda_1 \delta_{y^1} + \dots + \lambda_j \delta_{y^j}$

**Def 5.9** for $\Delta = \lambda_1 \delta_{y^1} + \dots + \lambda_j \delta_{y^j}$ on $T_{\bar{\mu}} M$

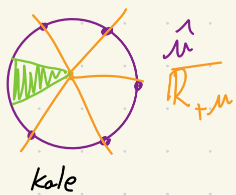
Set $\mathcal{L}(\Delta) = \lambda_1 \mathcal{L}(y^1) + \dots + \lambda_j \mathcal{L}(y^j)$

not the pushforward

"selection" "choice" $\quad \mathcal{L} : T_{\bar{\mu}} M \to \mathbb{R}^m$, $\mathbb{R}^{\ell}$

**Def 6.5** A section of an $\mathbb{R}^{\ell}$-valued variable $Y$ is a (random) variable $\Delta$ valued in $\text{Disc}(\overline{\mathbb{R}_+\mu})$ s.t. $\mathcal{L}(\Delta) = Y$ $\to \Delta \in \mathcal{L}^{-1}_{|\mu|}(Y)$

**Lemma 5.11** $\mathcal{L} : \text{Disc}_m(\text{supp}\,\hat{\mu}) \twoheadrightarrow \mathbb{R}^{\ell}$




**Remark 6.6**: Sections can be chosen measurably

**Proof**: $\mathcal{L}$ is proper followed by convex geodesic projection

**Def 6.7** A Gaussian Mass induced by $\mu$ is a section

$$\Gamma_{\mu} \in \mathcal{L}^{-1}_{|\mu|}(N_{\mu\mathcal{L}})$$

a random variable valued in $\text{Disc}_m(\overline{\mathbb{R}_+\bar{\mu}})$

**Def 4.6** $\Delta = \lambda_1 \delta_{y^1} + \cdots + \lambda_j \delta_{y^j}$

Set $\langle \Delta, \cdot \rangle = \lambda_1 \langle y^1, \cdot \rangle + \cdots + \lambda_j \langle y^j, \cdot \rangle$

**Lemma 6.9:** $\mathcal{L}(\Delta) = \mathcal{L}(\Delta') \Rightarrow \langle \Delta, X \rangle = \langle \Delta', X \rangle \ \forall X \in \overline{C}_\mu$

↳ fluctuating cone

**Lemma 6.10:** $\mu$ localized $\Rightarrow$ $\overline{g}_n(X) = \langle \overline{X}_n^{\mathcal{L}}, \mathcal{L}(X) \rangle_{\mathcal{L}(\bar{\mu})}$ for all $X \in \overline{C}_\mu$

**Thm 6.12:** $\mu$ localized $\Rightarrow$ $G(X) = \langle \Gamma_\mu, X \rangle_{\bar{\mu}} \ \forall X \in \overline{C}_\mu$.

Gaussian random tangent field

Recall: in distribution

$$\sqrt{n}\left(\bar{\mu}_n - \bar{\mu}\right) \overset{d}{=} \underbrace{\sqrt{n}\, \underset{p}{\arg\min} \left(F(p) - \overline{g}_n(p)\right)}_{\text{gaussian thingy}}$$

minimizer not necessarily unique, but results don't depend on choice, so choose any

$$\sqrt{n} \log_{\bar{\mu}} \bar{\mu}_n \overset{d}{=} \lim_{n\to\infty} \sqrt{n} \boxed{\underset{V \in T_{\bar{\mu}} M}{\arg\min} \left(F(\exp_{\bar{\mu}} V) - \overline{g}_n(V)\right)} = H_n$$

$$\overline{g}_n(V) = \frac{1}{n}\sum_{i=1}^{n} \left(\langle X_i, V \rangle - m(V)\right) = 0 \iff V \in E_\mu$$

Expected value

$x_1, \dots, x_n \in M$

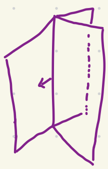
$X_1, \dots, X_n \in T_{\bar{\mu}} M$ $\quad X_i = \log_{\bar{\mu}} x_i$

barycenter, sample mean

$b(\mu_n) = \bar{\mu}_n = \underset{p}{\arg\min}\, F_{\mu_n}(p)$ $\qquad \sqrt{n}\, \overline{g}_n \xrightarrow{d} G$

$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$

Set $H(t) \in \underset{X \in \overline{C}_\mu}{\arg\min} \left(F(\exp_{\bar{\mu}} X) - tG(X)\right)$



$C_\mu = E_\mu \cap \operatorname{hull} \mu$

**Thm (perturbative CLT):** fix a localized, immured, amenable measure $\mu$ on $M$.

Then

$$\lim_{n\to\infty} \sqrt{n} \log_{\bar{\mu}} \bar{\mu}_n \overset{d}{=} \lim_{n\to\infty} \sqrt{n}\, H_n \overset{d}{=} \lim_{t\to 0} \frac{1}{t} H(t) = \xi(\Gamma_\mu)$$

escape vector of gaussian tangent mass $\Gamma_\mu$

$$\begin{aligned} \iota : T_{\bar\mu} M &\to \mathbb{R}^m \\ \Gamma_\mu &\mapsto N(0, \Sigma) \end{aligned}$$

<u>Def</u>: $\mu$ is <u>amenable</u> if $\rho_w = \frac{1}{2} d^2(w, \cdot)$ has $2^{nd}$-order directional derivatives.

$\rightsquigarrow$ Taylor expansion: $\exists$ cts. $\Lambda_{\bar\mu} : S_{\bar\mu} M \to \mathbb{R}$ s.t.

$$F(x) = F(\bar\mu) + r_x \nabla_{\bar\mu}(\theta_x) + r_x^2 \Lambda_{\bar\mu}(\theta_x) + o(r_x^2)$$

Recall: $\hat\mu = (\log_{\bar\mu})_{\#} \mu$

$\hat\mu + t\Delta$ $\quad$ $\Delta$ sampled from $T_{\bar\mu} M$

$\Rightarrow$ finitely supported

<u>Def</u>: For $\Delta = \lambda_1 \delta_{y_1} + \dots + \lambda_j \delta_{y_j}$ and $r \geq 0$

set $\Delta r = \lambda_1 \delta_{r y_1} + \dots + \lambda_j \delta_{r y_j}$

<u>Def (4.9)</u>

1. $\delta \overset{\text{\Subset}}{\in} M$ has <u>escape vector</u> $\xi(\delta) = \lim_{t \to 0^+} \frac{1}{t} \log_{\bar\mu}\left(\overline{\mu + t\delta}\right)$
2. If $\Delta \in T_{\bar\mu} M$ with exponential $\delta = \exp_{\bar\mu} \Delta \in M$ then the <u>escape vector</u> of $\Delta$ is $\xi(\Delta) = \xi(\delta)$
3. Any $\Delta \in T_{\bar\mu} M$ set $\xi(\Delta) = \frac{1}{r} \xi(\Delta r)$ for any $r > 0$ s.t. $\Delta r$ is exponentiable.

---

argmin we're interested in:

$$\underset{x \in M}{\operatorname{argmin}} \Big( \underbrace{F(x) + t F_\delta(x)}_{F_{\mu + t\delta}(x)} \Big) = \tilde{a}_n$$

fix $\{t_n\} \to 0^+$

$\delta_n \to \delta$

<u>Prop (4.32)</u>: $\log_{\bar\mu} \tilde{a}_n \in E_\mu$ $\quad \forall n \gg 0$

$$\underbrace{\nabla_{\bar\mu} F(v) = 0 \text{ for } v \in E_\mu}_{\text{definition of } E_\mu}$$

Thm (4.37)

$\mu$ amenable

$\Delta_n \to \Delta \in T_{\bar{\mu}} M$

$\{t_n\} \to 0^+$

then

$$\xi(\Delta) = \lim_{n\to\infty} \frac{1}{t_n} \operatorname*{argmin}_{X \in E_\mu} \left( f(\exp_{\bar{\mu}} X) - t_n \langle \Delta_n, X \rangle \right)$$

$$= \lim_{t\to 0^+} \frac{1}{t} \operatorname*{argmin}_{X \in E_\mu} \left( f(\exp_{\bar{\mu}} X) - t \langle \Delta, X \rangle \right)$$

If $\Delta_n = \log_{\bar{\mu}} \delta_n \ \forall n$, then $\xi(\Delta) = \lim_{n\to\infty} \frac{1}{t_n} \tilde{a}_n$.

Last one!

Let's summarize last time's lecture:

Escape Vector: $\xi(\Delta) = \lim_{t\to 0^+} \frac{1}{t} \operatorname*{argmin}_{X \in T_{\bar{\mu}} M} \left( \underbrace{F_\mu(\exp_{\bar{\mu}} X)}_{\text{log of frechet mean} \to \text{apex of cone}} + f_{t\delta}(\exp_{\bar{\mu}} X) \right)$

$\Delta = \log_{\bar{\mu}} \delta \qquad \delta \in M$

$\delta = \lambda_1 \delta_{y_1} + \dots + \lambda_j \delta_{y_j}$

$\Delta = \lambda_1 \delta_{y_1} + \dots + \lambda_j \delta_{y_j}$

Thm 4.37

$$\xi(\Delta) = \lim_{t\to 0^+} \frac{1}{t} \operatorname*{argmin}_{X \in E_\mu} \left( f_\mu(\exp_{\bar{\mu}} X) - t \langle \Delta, X \rangle \right)$$

$$= \lim_{n\to\infty} \frac{1}{t_n} \operatorname*{argmin}_{X \in E_\mu} \left( f_\mu(\exp_{\bar{\mu}} X) - t_n \langle \Delta_n, X \rangle \right) \qquad \begin{array}{l} \Delta_n \to \Delta \\ t_n \to 0 \end{array}$$

---

New stuff: Define $\xi_n(\Delta) = \frac{1}{t_n} \operatorname*{argmin}_{X \in E_\mu} \left( F_\mu(\exp_{\bar{\mu}} X) - t_n \langle \Delta, X \rangle \right)$

Thm $\Rightarrow \xi_n(\Delta_n) \to \xi(\Delta)$

**Def:** $\mu$ is immured if $\bar{\mu}$ has a neighborhood $U \subseteq M$ s.t. $\log_{\bar{\mu}} \bar{\mu}_n \in \underbrace{\text{hull}}_{}\,\mu$

Convex cone generated by $\text{supp}\,\hat{\mu}$

$\hat{\mu} = (\log_{\bar{\mu}})_{\#}\mu$

Whenever $\bar{\mu}_n \in U$ and $\mu_n \in \text{supp}\,\mu$.

**Lemma:** $\mu$ immured $\Rightarrow$ $\log_{\bar{\mu}}\left(\overline{\mu + t\delta}\right) \subseteq \text{hull}\,\mu \quad \forall\, t \ll 1$

if $\delta \in \text{supp}\,\mu$

**Corollary:** fix $\mu$ immured and amenable. If $\Delta \in \overline{\mathbb{R}_+\mu}$ then $\xi(\Delta) \in \overline{C_\mu}$

**Pf:** immured $\Rightarrow \xi(\Delta) \in \text{hull}_\mu$

Thm $\Rightarrow \xi(\Delta) \in E_\mu$

**Thm 6.14**

$$\lim_{n\to\infty} \sqrt{n}\,\log_{\bar{\mu}} \bar{\mu}_n = \lim_{n\to\infty} \sqrt{n}\,\underset{X\in T_\mu M}{\text{argmin}} \left(F(\exp_\mu X) - \bar{g}_n(X)\right)$$

$$= \lim_{t\to 0} \frac{1}{t}\,\underset{X\in C_\mu}{\text{argmin}} \left(F(\exp_\mu X) - tG(X)\right)$$

$$= \xi(\Gamma_\mu) \qquad \Gamma_\mu = \mathcal{L}^{-1}(N(0,\Sigma))$$

So
$$\xi(\Gamma_\mu) = \lim_{t\to 0^+} \frac{1}{t}\,\underset{X\in C_\mu}{\text{argmin}} \left(F(\exp_{\bar{\mu}} X) - t\underbrace{\langle \Gamma_\mu, X\rangle}_{G(X)}\right)$$

$$\bar{g}_n(V) = \frac{1}{n}\sum_{i=1}^{n} \left(\langle X_i, V\rangle - \underset{\underset{0 \text{ if } X\in E_\mu}{\parallel}}{m(V)}\right)$$

So when we take $\underset{X\in E_\mu}{\text{argmin}}$,

$$\bar{g}_n(V) = \frac{1}{n}\sum_{i=1}^{n} \langle X_i, V\rangle$$

Continuous mapping theorem: $f$ continuous and $X_n \to X$ then $f(X_n) \to f(X)$

Souped-up version if $f_n(x_n) \to f_0(x)$ whenever $x_n \to x$, then $f_n(X_n) \to f_0(X)$

for $x_n \in \mathbb{D}_n \quad x_0 \in \mathbb{D}_0$.

$$\underset{\underset{G}{\in}}{\bar{g}_n} \in C(T_{\bar{\mu}}M, \mathbb{R})$$

Need: $\underset{\underset{R}{\downarrow}\ \in}{R_n} \in C(T_{\bar{\mu}}M, \mathbb{R}) \Rightarrow$ "$\xi_n(R_n)$" $\to$ "$\xi(R)$"

**Def 5.6** $R \in C(T_{\bar\mu}M, \mathbb{R})$ is representable if $\exists \Delta \in \overline{\mathbb{R}_{+\mu}}$ with $R(X) = \langle \Delta, X\rangle$ $\forall X \in E_\mu$

A limit $R_n \to R$ in $C(T_{\bar\mu}M, \mathbb{R})$ is representable if all $R_n$ are.

$R$ need not be representable on $E_\mu$, but for $\sqrt{n}\,\bar{g}_n$ linear $G$ is rep. on $C_n \Rightarrow$ Thm.

**Thm:** $\sqrt{n} \lim_{n\to\infty} \log_{\bar\mu} \bar{\mu}_n \sim \xi(\Gamma_\mu)$

$$\underbrace{\boxed{\mathcal{H}_\#}}_{\text{inverse hessian "distortion"}} \underbrace{\mathcal{N}(0, \Sigma)}_{\text{Gaussian on } \mathbb{R}^l}$$

**Prop 6.17:** $\mathcal{L}(\Delta) = \mathcal{L}(\Delta')$ and $\Delta, \Delta' \in \overline{\mathbb{R}_{+\mu}} \Rightarrow \xi(\Delta) = \xi(\Delta')$

**Def 6.19:** fix $\mathcal{L}: T_{\bar\mu}M \to \mathbb{R}^m$ for an immured, amenable $\mu$, and set $\mathbb{R}^l = \mathrm{hull}_\mu \mathcal{L} \subseteq \mathbb{R}^m$. The distortion map is $\mathcal{H}: \mathbb{R}^l \to \mathbb{R}^l$
$v \mapsto \xi \circ \mathcal{L}^{-1}_{|\mu|}(v)$

$M$ smooth $\Rightarrow C_\mu = \mathbb{R}^l = T_{\bar\mu}M$

**Remark:** $\xi$ is "influence function" in statistics.