Algebraic Geometry

Def: A category $\mathcal{C}$ consists of the following:

- A collection of objects $Ob(\mathcal{C})$
- For all objects $A, B \in Ob(\mathcal{C})$, a set of morphisms $Hom_{\mathcal{C}}(A,B)$
- Maps $\circ_{A,B,C}: Hom(A,B) \times Hom(B,C) \to Hom(A,C)$

Subject to:

- for all $A \in Ob(\mathcal{C})$, we have $1_A \in Hom(A,A)$ s.t. $f \circ 1_A = f$ $1_A \circ g = g$
- composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$

Examples:

1) Category of sets → Objects: sets
   morphisms: set maps

   groups → ob: groups
   morph: homomorphisms

Constructing new cats from other cats:

1) Subcategories $\mathcal{C}$ subset of $\mathcal{D}$ if

- $Ob(\mathcal{C}) \subseteq Ob(\mathcal{D})$
- $Hom_{\mathcal{C}}(A,B) \subseteq Hom_{\mathcal{D}}(A,B)$
- Composition in $\mathcal{C}$ = composition in $\mathcal{D}$
- Identities in $\mathcal{C}$ = Ids in $\mathcal{D}$

(counter) example:

$Ob(\mathcal{C}) = *$

$Hom_{\mathcal{C}}(*,*) = \mathbb{Z}/10\mathbb{Z}$ ← but not in here

$Ob(\mathcal{D}) = *$

acts as id

$Hom_{\mathcal{D}}(*,*) = 2 \cdot \mathbb{Z}/10\mathbb{Z} = \{0, 2, 4, 6, 8\}$

If $\mathcal{C} \subseteq \mathcal{D}$ and for any $A, B \in Ob(\mathcal{C})$

$Hom_{\mathcal{C}}(A,B) = Hom_{\mathcal{D}}(A,B)$

Then $\mathcal{C}$ is said to be a <u>full subcategory</u> (restrict obs, not Homs)

Opposite category: Let $\mathcal{C}$ be a category. Define $\mathcal{C}^{op}$ as follows:

- $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$
- $Hom_{\mathcal{C}^{op}}(A,B) = Hom_{\mathcal{C}}(B,A)$ (flip arrows)

$A \xrightarrow{f^{op}} B \xrightarrow{g^{op}} C$

$A \xleftarrow{f \circ g} C$

$A \xrightarrow{(f\circ g)^{op}} C$

Initial and final objects

$* \in Ob(\mathcal{C})$

Initial object: exactly one map for each $A \in Ob(\mathcal{C})$

$Hom_{\mathcal{C}}(*, A) = \{*\}$

singleton set

$$\begin{tikzcd} * \arrow[r] \arrow[dr] \arrow[d] \arrow[ddr] & A \\ {} & B \\ {} & C \\ {} & D \end{tikzcd}$$

$*$ final iff $Hom_{\mathcal{C}}(A, *) = \{*\}$ for all $A \in Ob(\mathcal{C})$

Grp: initial & final both trivial group

Rng: initial is $\mathbb{Z}$, final is 0

Products

$\mathcal{C}$ category, $A, B \in Ob(\mathcal{C})$

the product $A \times B \in Ob(\mathcal{C})$ maps

$$\begin{tikzcd} A \times B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \\ A & \end{tikzcd}$$

this data is part of the product

s.t. for any
$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \\ A & \end{tikzcd}$$
, $\exists!$ $C \to A \times B$ s.t.

$$\begin{tikzcd} C \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & A \times B \arrow[r] \arrow[d] & B \\ & A & \end{tikzcd}$$

commutes

unique up to unique isomorphism

$\to$ justifies calling it "the" product

Coproducts

$$\begin{tikzcd} A \amalg B & B \arrow[l, "i_B"'] \\ A \arrow[u, "i_A"] & \end{tikzcd}$$

s.t. for any
$$\begin{tikzcd} C & B \arrow[l] \\ A \arrow[u] & \end{tikzcd}$$
, $\exists!$ $A \amalg B \leftarrow C$ s.t.

$$\begin{tikzcd} & B \arrow[d] \arrow[ddr, bend left] & \\ A \arrow[r] \arrow[drr, bend right] & A \amalg B \arrow[dr, dashed, "\exists!"'] & \\ & & C \end{tikzcd}$$

Coproduct in Rng $= \otimes$

Coproduct in $\mathcal{C}$ = product in $\mathcal{C}^{op}$

Functors $\mathcal{C}, \mathcal{D}$ cats

A covariant functor $F: \mathcal{C} \to \mathcal{D}$ consists of the following:

- $F: Ob(\mathcal{C}) \to Ob(\mathcal{D})$
- $F_{A,B}: Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{D}}(F(A), F(B))$ for all $A, B \in Ob(\mathcal{C})$

s.t. • $F(g \circ h) = F(g) \circ F(h)$ (when that makes sense)

• $F(1_A) = 1_{F(A)}$

A contravariant (presheaf) functor $F : \mathcal{C} \to \mathcal{D}$

is just a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$

$\text{Grps} \xrightarrow{\text{Forgets}} \text{Sets}$

$(G, \cdot) \to G$ (forgets operation/structure)

$\mathfrak{p} \subseteq R$

$R\text{-mod} \to R_{\mathfrak{p}}\text{-mod}$

$M \mapsto M \otimes_R R_{\mathfrak{p}}$

$\text{Top}_* \xrightarrow{\pi_1} \text{Grps}$

Natural Transformations

$F, G : \mathcal{C} \longrightarrow \mathcal{D}$ functors

A natural trans. $\alpha : F \to G$ is the following data:

• for all $A \in \text{Ob}(\mathcal{C})$, a morphism in $\mathcal{D}$ $\alpha(A) : F(A) \to G(A)$

s.t. given a morphism $A \xrightarrow{f} B$, the following commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\alpha(B)"'] & G(B)
\end{tikzcd}$$

$\mathcal{C}$ and $\mathcal{D}$ categories, define functor cat $\text{Fun}(\mathcal{C}, \mathcal{D})$:

• $\text{Ob}(\text{Fun}(\mathcal{C}, \mathcal{D}))$ = functors from $\mathcal{C} \to \mathcal{D}$

• $\text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F, G)$ = Natural transformations $F \to G$

(co) Limits

Let I be a category

Define I_+ as the following:

$\text{Ob}(I_+) = \text{Ob}(I) \cup \{*\}$

$\text{Hom}_{I_+}(A, *) = \{*\}$ → add/replace final object

def I^+ analogously.

Limit: $F : I \to \mathcal{C}$. Limit F^+ of F is the final object in the full subcategory of $\text{Fun}(I^+, \mathcal{C})$ consisting of functors that restrict to F on I

$$\begin{tikzcd}
I^+ \arrow[r, dashed, "F^+"] & \mathcal{C} \\
I \arrow[u, hook] \arrow[ur, "F"'] \arrow[ur, phantom, bend left, "\Downarrow"]
\end{tikzcd}$$

natural transformation

$$\begin{tikzcd}
G \arrow[r, dotted, "\exists!"] & F^+
\end{tikzcd}$$

$$\begin{tikzcd}
G \circ i \arrow[r] \arrow[dr] & F^+ \circ i \arrow[d] \\
& F
\end{tikzcd}$$

for colimits just reverse all the arrows!

Products are limits!

$$I = \left\{ \begin{tikzcd} & \dagger \arrow[loop right] \\ * \arrow[loop left] & \end{tikzcd} \right\} \qquad I^{+} = \left\{ \begin{tikzcd} \# \arrow[r] \arrow[d] & \dagger \arrow[loop right] \\ * \arrow[loop left] & \end{tikzcd} \right\}$$

$A, B \in Ob(\mathcal{C})$

$F : I \to \mathcal{C}$ $\quad F^{+}:$

$* \to A$
$\dagger \to B$

Cones

$F : I \to \mathcal{C}$

$cocone(F)$ is the following data:

- an object $c \in Ob(\mathcal{C})$
- for all $i \in I$, a morphism in $\mathcal{C}$

$F(i) \to c$ s.t. for all $f : A \to B$

$$\begin{tikzcd} F(A) \arrow[rr, "F(f)"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & F(B) \arrow[dl] \\ & c & \end{tikzcd}$$

colimit: initial cocone

limit: final cone

Recall:

$F : I \to \mathcal{C}$

colimit of F is an object in $\mathcal{C}$ w/ maps

$F(i) \to colim\, F$ for all i

s.t. given $i_1 \xrightarrow{f} i_2$ in I, the following commutes:

$$\begin{tikzcd} F(i_1) \arrow[rr, "F(f)"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & F(i_2) \arrow[dl] \\ & colim\, F & \end{tikzcd}$$

and $colim\, F$ is initial wrt this property, i.e.

for any $c \in Ob(\mathcal{C})$ with maps $F(i) \to c$ for all i

s.t.

$$\begin{tikzcd} F(i_1) \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & F(i_2) \arrow[dl] \arrow[ddl, bend left] \\ & c & \\ & colim\, F \arrow[u, dotted, "\exists!"'] & \end{tikzcd}$$

Cocone def of colimit

for limit, just "co" everything

+ initial $\leftrightarrow$ final

Let $F : I \to \mathcal{C}$

$$Ob(cones(F)) = \left\{ \begin{array}{l} c \in Ob(\mathcal{C}) \\ \text{with maps } c \to F(i) \text{ for all } i \in I \end{array} \right\}$$

Example:

$$\mathrm{Ob}(I) = \mathbb{N}$$

$$\mathrm{Hom}_I(n,m) = \begin{cases} * & \text{if } n \le m \\ \emptyset & \text{if } n > m \end{cases}$$

$$0 \to 1 \to 2 \to 3 \to \cdots$$

$$F : I^{op} \to \mathbb{Z}\text{-mod}$$
$$n \mapsto \mathbb{Z}/p^n\mathbb{Z}$$

$$\mathbb{Z}/p^n\mathbb{Z} \leftarrow \mathbb{Z}/p^{n+1}\mathbb{Z}$$

$\lim F = \mathbb{Z}_p$ (p-adic integers)

Equalizers

$$\begin{tikzcd} C \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ C' \arrow[u, dotted, "\exists!"] \arrow[ur, "e'"'] & & \end{tikzcd} \quad \text{in } \mathcal{C}$$

s.t. $f \circ e = g \circ e$

ex.

$$\begin{tikzcd} p\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r, shift left, "\pi"] \arrow[r, shift right, "0"'] & \mathbb{Z}/p\mathbb{Z} \end{tikzcd} \quad \text{in } \mathbb{Z}\text{-mod}$$

$p\mathbb{Z} = \ker(\pi)$

equalizer

Let $\mathcal{C}$ be a cat, $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$

"injection"

we say f is $\binom{\text{monic}}{\text{epic}}$ iff it has $\binom{\text{left}}{\text{right}}$ cancellation

"surjection"

(left) $f \circ g = f \circ g' \Leftrightarrow g = g'$

(right) $h \circ f = h' \circ f \Leftrightarrow h = h'$

Warning! monic + epic $\neq$ isomorphic

$$\mathbb{Z} \to \mathbb{Q} \quad \text{in Ring}$$

$$\mathbb{Z} \xrightarrow{i} \mathbb{Q} \underset{g}{\overset{f}{\rightrightarrows}} R$$

$f \circ i = g \circ i$ ($\mathbb{Z}$ initial object)

Adjoint functors

Let $\mathcal{C}, \mathcal{D}$ cats and $L : \mathcal{C} \to \mathcal{D}$
$R : \mathcal{D} \to \mathcal{C}$

$$\mathcal{C}^{op} \times \mathcal{D} \to Set$$

$Hom_{\mathcal{D}}(L(-), T)$
$Hom_{\mathcal{C}}(-, R(T))$

$$\begin{tikzcd}
L(-) \arrow[r] \arrow[d] & L(-)' \arrow[d] \\
Hom_{\mathcal{D}}(L(-), D) & Hom_{\mathcal{D}}(L(-)', D) \arrow[l]
\end{tikzcd}$$

Why we use $\mathcal{C}^{op}$

we say L is left adjoint to R
(equivalently R is right adjoint to L)

if $\exists\ \eta : Hom_{\mathcal{D}}(L(-), T) \xrightarrow{\sim} Hom_{\mathcal{C}}(-, R(T))$
"natural bijection"

$$\begin{tikzcd}
Grp \arrow[r, bend left, "\text{abelianization } (-)^{ab}"] & Ab \arrow[l, bend left, "\text{inclusion } i"]
\end{tikzcd}$$

Claim: abelianization is left adjoint to inclusion

$\eta_{G,A} : Hom_{Grp}(G, A) \to Hom_{Ab}(G^{ab}, A)$

$G \xrightarrow{f} A \Rightarrow ghg^{-1}h^{-1} \in \ker(f)$

$ghg^{-1}h^{-1} \in G$
$f(g)f(h)f(g)^{-1}f(h)^{-1} = 1$

$G \to G/\langle ghg^{-1}h^{-1} \mid g, h \in G \rangle \to A$
$\qquad\quad \|$
$G \to G^{ab} \to A$

$$\begin{tikzcd}
 & G^{ab} \arrow[r, "h"] & A \\
G \arrow[r, "\pi"] \arrow[rr, bend right] & G/\langle - \rangle \arrow[u, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
Grp \arrow[r, bend left, "\text{forgetful } F"] & Set \arrow[l, bend left, "\text{Words } W \text{ (free group)}"]
\end{tikzcd}$$

$\tilde{h}(s_1 s_2 s_3) = h(s_1)h(s_2) \cdots h(s_n)$

$$\begin{tikzcd}
S \arrow[r, "h"] \arrow[d] & G \\
W(S) \arrow[ur, dashed, "\tilde{h}"'] &
\end{tikzcd}$$

$\uparrow$
$S \to G$

$$\begin{tikzcd}
Hom_{Grp}(W(S), G) \arrow[d] \\
Hom_{Set}(S, G)
\end{tikzcd}$$

$W(S) \xrightarrow{f} G$
$\downarrow$
$S \to W(S) \to G$

$f : R \to S$ CRing morphism

$$\begin{tikzcd} R\text{-mod} \arrow[r, bend left, "- \otimes_R S"] & S\text{-mod} \arrow[l, bend left, "(-)|_R"] \end{tikzcd}$$

restricting, forget S-structure

$$\mathrm{Hom}_S(M \otimes_R S, N) \cong \mathrm{Hom}_R(M, N|_R)$$

$$M \otimes_R S \to N \qquad M \xrightarrow{f} N|_R$$

$$\begin{matrix} M \otimes_R S & & m \otimes 1 \\ \uparrow & & \uparrow \\ M & & m \end{matrix} \qquad \begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N|_R \\ M \otimes_R S \arrow[r] \arrow[dr] & N|_R \otimes S \arrow[d] \\ & N \end{tikzcd}$$

def: (Preadditive category)

A preadditive category is a cat $\mathcal{C}$ equipped with an abelian group structure on $\mathrm{Hom}_{\mathcal{C}}(A,B)$ $\forall A, B \in \mathrm{Ob}(\mathcal{C})$

s.t. $\circ : \mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$

is bilinear

$$f \circ (g + g') = (f \circ g) + (f \circ g) \qquad (h + h') \circ f = (h \circ f) + (h' \circ f)$$

full subcat preserves preadditivity

Examples: $R\text{-mod} \supseteq \mathrm{Ab}$, Vect_k, Rng

Non-examples: : Grp, Set, Ring

def: (Additive category)

An additive cat is a preadditive cat with a 0 object

i.e. an object that is both final and initial + has all binary biproducts, i.e. given $A, B \in \mathrm{Ob}(\mathcal{C})$, $A \times B \cong A \amalg B$ and both exist

$A \oplus B$

$0 \to A$

$0 \to B$

$A \to A \times B$

$B \to B \times A$

$\Rightarrow A \amalg B \xrightarrow{\sim} A \times B$

Today: sheaves & presheaves

Presheaves: Let $\mathcal{C}$ be a category.

Def A presheaf of sets is a functor $\mathcal{C}^{op} \to \mathrm{Set}$

$\boxed{\text{☆ review precomposition}}$

Yoneda's Lemma $X \in \mathcal{C}$ $h_X : \mathcal{C}^{op} \to \mathrm{Set}$

$h_X(Y) = \mathrm{hom}_{\mathcal{C}}(Y, X)$

Natural transformations from $h_X \to h_Y$ are in a canonical bijection with $\mathrm{hom}_{\mathcal{C}}(X, Y)$

In other words, $\mathcal{C} \xhookrightarrow{h} \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ is fully faithful

pf: More generally, for $\mathcal{F} \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$, we claim

$$\mathrm{Nat'l\ Trans}(h_X, \mathcal{F}) \underset{\substack{\text{canonical}\\\text{bijection}}}{=} \mathcal{F}(X)$$

Let $\theta : h_X \to \mathcal{F}$ be a natural transformation

Thus $\theta(X) : h_X(X) \to \mathcal{F}(X)$

$$\begin{array}{ccc} h_X(X) & \to & \mathcal{F}(X) \\ \| & & \\ \mathrm{map}_{\mathcal{C}}(X,X) & & \\ \cup & & \cup \\ 1 & \mapsto & \theta_X(1_X) \end{array}$$

Associating $\theta_X(1_X)$ to θ defines a map $\mathrm{Nat'l\ Trans}(h_X, \mathcal{F}) \to \mathcal{F}(X)$

In the other direction, choose $\sigma \in \mathcal{F}(X)$. Define $\theta_\sigma : h_X \to \mathcal{F}$ to have associated to $Y \in \mathcal{C}^{op}$

$$\theta_\sigma(Y) : \hom(Y, X) \to \mathcal{F}(Y)$$
$$f \longmapsto \mathcal{F}(f)(\sigma)$$

$$\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$$

"modulo thinking through, these are inverse to each other" $\square$

Moving any category into a category of presheaves

We can study any category by studying presheaves

Presheaves of sets "or heck, a topological space"

ex: Let X be a C^∞ manifold, let $\mathrm{Opens}(X)$ be the category whose objects are open subsets $U \subset X$

morphisms: $\exists!$ morphism $U_1 \to U_2$ precisely if $U_1 \subset U_2$

Terminology: a presheaf of sets (objects of A) on $\mathrm{Opens}(X)$ is abbreviated by "a presheaf of sets on X"

Rmk:

Let $\mathcal{F} : \mathrm{Opens}(X)^{op} \to \mathrm{Set}$ (A)

be a (functor) presheaf of sets

This is the same as the data: an object of A

- for each open $U \subset X$, we have a set $\mathcal{F}(U)$
- for each inclusion $U_1 \subset U_2$, we have a restriction map in A

$res_{U_2,U_1} : \mathcal{F}(U_2) \to \mathcal{F}(U_1)$

such that $res_{U_3,U_1} = res_{U_2,U_1} \circ res_{U_3,U_2}$

$res_{U_1,U_1} = Id$

ex. $A = AbGrp$ (Abelian Category)

$\rightsquigarrow$ Chain Complexes (can invert quasi-isomorphisms)

Set $\rightsquigarrow$ Topological Spaces

Example?: X a C^∞-manifold, (or call it $\mathbb{R}^n$)

"Structure Sheaf" Let $\mathcal{O}_X : \mathrm{Opens}(X)^{op} \to Set$ ((Comm.) Rings (with 1), Abelian Groups)

be the presheaf

$$\mathcal{O}_X(U) = \{ f : U \to \mathbb{R} \mid f \text{ is a } C^\infty\text{-function} \}$$

$res_{U_2,U_1}(f) = f|_{U_1}$ for $U_1 \subset U_2$

"Sheaf" indicates that a presheaf on (a category that behaves like) $\mathrm{Opens}(X)$

Satisfies the further properties

(1) "Separation axiom" Suppose $U = \bigcup_{i \in I} U_i$

Then if $\sigma, \sigma' \in \underbrace{\mathcal{F}(U)}$

the object of A associated to U

"the sections over U"

satisfy $Res_{U,U_i}(\sigma) = Res_{U,U_i}(\sigma')$

for all $i \in I$, then $\sigma = \sigma'$

(2) "Glueability" If $\sigma_i \in \mathcal{F}(U_i)$

Satisfy $Res_{U_i, U_i \cap U_j} \sigma_i = Res_{U_j, U_i \cap U_j} \sigma_j$

Then there exists $\sigma \in \mathcal{F}(U)$ s.t. $Res_{U,U_i} \sigma = \sigma_i \; \forall i \in I$

- for A the category of Abelian Groups,

"Gluability" and "Separation" are the same as

$\forall U = \bigcup_{i \in I} U_i$

(✱) $$0 \to \mathcal{F}(U) \xrightarrow{\prod Res_{U,U_i}} \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{\Theta} \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

and Θ is the unique map s.t.

$$\begin{tikzcd}
\prod_i \mathcal{F}(U_i) \arrow[r, "\theta"] \arrow[d] \arrow[dr, dashed] & \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j) \arrow[d] \\
\mathcal{F}(U_i) & \mathcal{F}(U_i \cap U_j)
\end{tikzcd}$$

$Res_{U_i, U_i \cap U_j} \mathcal{F}(\pi_i) - Res_{U_j, U_i\cap U_j} \mathcal{F}(\pi_j)$

(⊠) is "exact" in the sense that

$\ker \theta = Im \prod Res_{U, U_i}$

$\prod Res_{U,U_i}$ is injective

<u>Remark</u>: A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of abelian groups is "exact" if $\ker g = Im f$ → can be in any category with images + kernels

[separation → injective
glueability → surjective]

$$0 \to \underbrace{\mathcal{F}(U_i) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)}_{(⊠⊠)} \ \substack{\to\\\to\\\to}\ \mathcal{F}(U_i \cap U_j \cap U_k) \ \substack{\to\\\to\\\to\\\to} \ \cdots$$

In $\mathbf{Set}$, we can't subtract maps to define θ but we can summarize the sheaf conditions categorically by saying (⊠⊠) is an "equalizer" i.e. the largest subset where the two maps are equal.

<u>Rmk</u> continuing (⊠⊠) further becomes important when working with presheaves of topological spaces and chain complexes

<u>Sheaves and presheaves form categories:</u>

- The $\mathcal{A}$-valued presheaves on $\mathcal{C}$ form the category $Fun(\mathcal{C}^{op}, \mathcal{A})$

  $mor(\mathcal{F}_1, \mathcal{F}_2)$ are the natural transformations $\mathcal{F}_1 \xrightarrow{\theta} \mathcal{F}_2$

  i.e. for any $c \in \mathcal{C}$ we have $\mathcal{F}_1(c) \xrightarrow{\theta(c)} \mathcal{F}_2(c)$ in $\mathcal{A}$ s.t. for any $c_1 \to c_2$

$$\begin{tikzcd}
\mathcal{F}_1(c_2) \arrow[r, "\theta(c_2)"] \arrow[d, "\mathcal{F}_1(c_1\to c_2)"'] & \mathcal{F}_2(c_2) \arrow[d, "\mathcal{F}_2(c_1\to c_2)"] \\
\mathcal{F}_1(c_1) \arrow[r] & \mathcal{F}_2(c_1)
\end{tikzcd}$$

commutes.

- Let $(\mathrm{Pre})\mathrm{Sh}(X)$ denote the category:

  Ob: (Pre)Sheaves: $\mathrm{Opens}(X) \to A$

  Hom: $\hom_{\mathrm{Sh}(X)}(\mathcal{F}_1, \mathcal{F}_2) = \hom_{\mathrm{PreSh}(X)}(\mathcal{F}_1, \mathcal{F}_2)$

  $= \mathrm{Nat'l\,Trans}(\mathcal{F}_1, \mathcal{F}_2)$

  i.e. for every open $U \subset X$ we have

  $\mathcal{F}_1(U) \xrightarrow{\theta(U)} \mathcal{F}_2(U)$ in A

  which satisfies:

$$\begin{tikzcd}
\mathcal{F}_1(U_1) \arrow[r, "\theta(U_1)"] \arrow[d, "\mathrm{Res}^{\mathcal{F}_1}_{U_1,U_2}"] & \mathcal{F}_2(U_1) \arrow[d, "\mathrm{Res}^{\mathcal{F}_2}_{U_1,U_2}"] \\
\mathcal{F}_2(U_1) \arrow[r, "\theta(U_2)"'] & \mathcal{F}_2(U_2)
\end{tikzcd}$$

"Let's grab some stalks on our board"

$\mathcal{C}$ cat admitting colimits $\mathcal{C} = $ Ring, Set, Ab

~ initial object of indexing cat

X top space $\mathrm{Opens}(X)$: $\mathrm{Ob}(\mathrm{Opens}(X)) = \{U \overset{\text{open}}{\subset} X\}$

unique morphism $U_1 \to U_2$

when $U_1 \subseteq U_2$

$\mathcal{F}$ is a presheaf means

$\mathcal{F} : \mathrm{Opens}(X)^{op} \to \mathcal{C}$

$\mathcal{F}$ is a sheaf if for all $U = \bigcup_{i \in I} U_i$

elements called sections over U

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer.

So $\mathcal{F}$ satisfies "separability"

"Gluability"

$\mathrm{Sh}(\mathcal{F})$ category of sheaves on X

fully faithful $\cap\ \iota$

$\mathrm{PreSh}(\mathcal{F})$ " presheaves on X

$\mathrm{Fun}(\mathrm{Open}^{op}, \mathcal{C})$

ι admits "left adjoint"

$L : \mathcal{D} \to \mathcal{E} : R$ are "adjoint"

"left adjoint"

$\forall d \in \mathcal{D},\ e \in \mathcal{E}$ we have natural bijections

means $\quad mor_{\mathcal{D}}(d, Re) = mor_{\mathcal{E}}(Ld, e)$

$\mathcal{F} \quad sh(\mathcal{F}) \qquad \alpha \qquad \tilde{\alpha}$

ι has a left adjoint, "sheafification"

defined follows:

$\mathcal{F} \to Sh\mathcal{F}$ ← this map

any presheaf $\quad$ sheafification

$$map_{Sh(X)}(Sh(\mathcal{F}), Sh(\mathcal{F})) \ni 1_{Sh(\mathcal{F})}$$
$$\|$$
$$map_{PreSh(X)}(\mathcal{F}, \cancel{\iota}\, Sh(\mathcal{F}))$$

ι implied

satisfying the universal property that

for any sheaf G and map $\alpha : \mathcal{F} \to G$

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr, "\alpha"'] & Sh(\mathcal{F}) \arrow[d, dashed, "\tilde{\alpha}"] \\ & G \end{tikzcd}$$

there is a unique $\tilde{\alpha} : Sh\mathcal{F} \to G$ s.t. commutes.

So Thm ι admits a left adjoint as described above.

To prove Thm, we construct $Sh\mathcal{F}$.

To do this, let's introduce the stalk of $\mathcal{F}$ at a point $p \in X$.

Ex: for $\mathcal{F}$ sheaf of smooth functions, the stalk $\mathcal{F}_p$ are "germs" of functions.

Def: The stalk $\mathcal{F}_p$ is the colimit:

$$\mathcal{F}_p := \operatorname*{colim}_{X \overset{open}{\supseteq} U \ni p} \mathcal{F}(U)$$

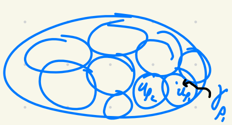
in Set, Ab, Ring, Modules

claim $= \{(\gamma, U) \mid \gamma \in \mathcal{F}(U) \text{ for } p \in U \underset{open}{\subset} X\} \Big/ (\gamma_1, U_1) \sim (\gamma_2, U_2)$ if $\gamma_1|_U = \gamma_2|_U$ for some $U \subset U_1 \cap U_2$

Pf of thm: Define for $U \underset{open}{\subseteq} X$

$$(Sh\mathcal{F})(U) = \{(\gamma_p \in \mathcal{F}_p)_{p \in U} \mid \text{for each } p \in U\ \exists\, U_p \text{ and representative } \gamma_p \in \mathcal{F}(U_p) \text{ s.t. } \forall p_1, p_2 \quad \gamma_{p_1}|_{U_{p_1} \cap U_{p_2}} = \gamma_{p_2}|_{U_{p_1} \cap U_{p_2}}\}$$

$\mathcal{F}(U) = \gamma = (\gamma_p)_{p \in U} \qquad \gamma_{p_1}, p_2$



Define $\mathcal{F} \to Sh\,\mathcal{F}$ by $\mathcal{F}(U) \to Sh\,\mathcal{F}(U)$
$$\gamma \mapsto (\gamma, U)_{p \in X}$$

Recall "separability" for $Sh\,\mathcal{F}$ is the requirement that for all $U = \bigcup_{i \in I} U_i$

$$Sh\,\mathcal{F}(U) \to \prod Sh\,\mathcal{F}(U_i)$$
is injective

by def, every stalk there is also in

This holds for $Sh\,\mathcal{F}$ because
$Sh\,\mathcal{F}(U) \underset{\text{injects}}{\hookrightarrow} \prod_{p \in U} \mathcal{F}_p$ and all stalks

appear in $\prod Sh(U_i)$ because every p is in some U_i

Recall that "gluability" for $Sh\,\mathcal{F}$ is the requirement that given $U = \bigcup_{i \in I} U_i$ and $\gamma_i \in Sh\,\mathcal{F}(U_i)$ s.t. $\forall i \in I$
$$\gamma_i|_{U_i \cap U_j} = \gamma_j|_{U_i \cap U_j}$$
then there is $\gamma \in Sh\,\mathcal{F}(U)$ s.t.
$$\gamma|_{U_i} = \gamma_i \quad \forall i$$

$\gamma_i = (\gamma_{p,i})_{p \in U_i}$ by definition of $Sh\,\mathcal{F}$

The condition $\gamma_i|_{U_i \cap U_j} = \gamma_i|_{U_i \cap U_j}$
$\Rightarrow \gamma_{p,i} = \gamma_{p,j}$ when $p \in U_i \cap U_j$

Thus, for every p, $\exists U_i$ s.t. $p \in U_i$ take γ_p to be represented by (γ_i, U_i)

Then $(\gamma_p)_{p \in U} \in Sh\,\mathcal{F}$

and shows gluability holds.

exercise: $Sh\,\mathcal{F}$ satisfies the claimed universal property.

Hints: - a morphism $f : \mathcal{F} \to \mathcal{G}$ between <u>sheaves</u> is determined by
$$(F_p : \mathcal{F}_p \to \mathcal{G}_p)_{p \in X},$$
the induced map on stalks

- f is $\begin{pmatrix}\text{injective}\\\text{surjective}\\\text{isomorphism}\end{pmatrix}$ iff it is such on all stalks. $\longrightarrow$ think localizing a module at p.

Ex: Constant (pre)sheaves

let $A \in \mathcal{C}$, define $\underline{A} \in \mathrm{Presh}(X)$

by letting $\underline{A}(U) = A$ with constant restriction maps

Usually not a sheaf because for $\mathcal{F}$ a sheaf

$\mathcal{F}(\emptyset)$ = terminal object (i.e. a point)

$\Rightarrow \mathcal{F}(U_1 \amalg U_2) \cong \mathcal{F}(U_1) \times \mathcal{F}(U_2)$

$Sh(\underline{A})$ is called "constant sheaf"

Stalks

$(Sh\,\underline{A})_p \cong A$

Let $\mathcal{C}$ = Set, Ab, R-Mod, Ring

Then $\underline{Sh}\,A(U) = \left\{\begin{matrix}\text{locally constant functions}\\ \gamma_i : U \to A\end{matrix}\right\}$

$\qquad \parallel$

$\left\{ (\gamma_p)_{p\in U} \,\middle|\, \forall p\ \gamma_p \in \underline{A}_p \text{ s.t. } \exists\, U_p \text{ and } \gamma_p \in \underline{A}(U_p) = A \text{ s.t. when } U_{p_1} \cap U_{p_2} \neq 0,\ \gamma_{p_1} = \gamma_{p_2} \right\}$

Ex: X C^∞ manifold or $X \underset{\text{open}}{\subset} \mathbb{R}^n$

$\mathcal{O}_X(U) = \{\text{smooth functions } U \to \mathbb{R}\}$

cnly $\mathcal{F}$

$\mathcal{O}_{X,p}$: germs

Ex: Define $\mathcal{O}_X^*$ ← ab groups s.t.

$\mathcal{O}_X^*(U) = \mathcal{O}_X(U)^* = \{\text{units in ring } \mathcal{O}_X(U)\}$

$= \{\text{smooth function } \gamma_i : U \to \mathbb{R} - \{0\}\}$

$\mathcal{O}_{X,p}$ = germs of non-vanishing functions at p

X complex manifold or $X \underset{\text{open}}{\subset} \mathbb{C}^n$

$\mathbb{C} \to \mathbb{C}$, $x \mapsto e^x$

Ex: $0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$

is an exact sequence of sheaves of abelian groups

Pushforward and pullback of sheaves

$f : X \to Y$ map of top spaces

Then $\mathrm{Opens}(Y) \to \mathrm{Opens}(X)$

$U \mapsto f^{-1}(U)$

induces a map

$$\begin{matrix} Sh(X) & \to & Sh(Y) \\ \cap & & \cap \\ f_* : \mathrm{Presh}(X) & \to & \mathrm{Presh}(Y) \end{matrix}$$

$\mathcal{F} \mapsto f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$

Thm: f_* admits a left adjoint

$$f^{pre}: PreSh(Y) \rightleftarrows PreSh(X) : f_*$$

And $f^* := Sh\, f^{pre}$ is the left adjoint on sheaves

Construction: $f^*\mathcal{F}(\underset{\subseteq X}{V}) = \underset{\substack{U \supset f(V) \\ \cap \\ Y}}{\operatorname{colim}} \mathcal{F}(U)$

Recall: $f: X \to Y$ map of topological spaces

$Sh(X)$ = cat of sheaves of sets on X

Thm: There is an adjunction $f^*: Sh(Y) \rightleftarrows Sh(X) : f_*$

pf:

$$f_* G(\underset{\substack{\cap\, open \\ Y}}{U}) = G(f^{-1}(U))$$

One approach: Adjoint functor thm: a functor is a right adjoint
$\iff$ the functor commutes with limits
Then check f_* commutes with limits

Alternatively,

$$f^*\mathcal{F}(\underset{\substack{\cap\, open \\ X}}{V}) = \underset{U \supset f(V)}{\operatorname{colim}} \mathcal{F}(U)$$

$$\boxed{Hom_{Sh(X)}(f^*\mathcal{F}, G) \overset{\equiv}{\to} Hom_{Sh(Y)}(\mathcal{F}, f_* G)}$$

$f: \underset{\substack{U_{open} \\ V}}{X} \longrightarrow \underset{\substack{U_{open} \\ U, W}}{Y}$

$$\left(f^*\mathcal{F} \xrightarrow{\phi} G\right) \longmapsto \left(\mathcal{F} \xrightarrow{\psi} f_* G\right)$$

$$\phi(f^{-1}(U)): f^*\mathcal{F}(f^{-1}(U)) \to G(f^{-1}(U)) \longmapsto \psi(U): \mathcal{F}(U) \to G(f^{-1}(U))$$

$$f^*\mathcal{F}(f^{-1}(U)) = \underset{\substack{W \supset f(f^{-1}(U)) \\ U \supset f(f^{-1}(U))}}{\operatorname{colim}} \mathcal{F}(W)$$

$$\mathcal{F}(U) \to \underset{W \supset f(f^{-1}(U))}{\operatorname{colim}} \mathcal{F}(W) \to G(f^{-1}(U))$$

$\square$

Prop: Let $i_p: p \to X$ be the inclusion of a point of X. Then

$$(i_p)^*\mathcal{F} = \mathcal{F}_p \quad \swarrow \text{stalk}$$

pf: definition of *. $\square$

Ex: $Sh(\{p\}) = \begin{matrix}\text{Set}\\ \text{Ab}\\ \text{Ring}\end{matrix}$

Given $A \in Sh(\{p\})$, the skyscraper sheaf on X for A at p is

$$(i_p)_*(A)(U) = \begin{cases} A & \text{for } p \in U \\ \text{terminal object} & \text{for } p \notin U \end{cases}$$

$$\| \qquad\qquad \text{i.e. } 0 \text{ in Ab}, \ \{*\} \text{ in Set}$$

$$A(i_p^{-1}(U))$$

$$\| $$

$$\{*\}$$

Def: A ringed space is $(X, \mathcal{O}_X)$ where X is a top space, and $\mathcal{O}_X$ is a sheaf of rings.

A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces is $f: X \to Y$ a continuous map and

$$\mathcal{O}_Y \to f_* \mathcal{O}_X$$

or equivalently $f^* \mathcal{O}_Y \to \mathcal{O}_X$

ex: X top space, $\mathcal{O}_X(U) = \{\text{continuous functions } U \to \mathbb{R}\}$

Then $f: X \to Y$ a map of top. spaces induces a map of ringed spaces

$$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

$$\mathcal{O}_Y \to f_* \mathcal{O}_X \quad \text{defined by}$$

$$\mathcal{O}_Y(U) \to f_* \mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$$

$$g: U \to \mathbb{R} \ \mapsto f^{-1}(U) \xrightarrow{f} U \xrightarrow{g} \mathbb{R}$$

Let $Sh(X, \mathcal{O}_X)$ denote the category of "Sheaves of $\mathcal{O}_X$-modules" whose objects are sheaves $\mathcal{F}$ on X of abelian groups together with a multiplication $\mathcal{O}_X \otimes \mathcal{F} \xrightarrow{\mu_\mathcal{F}} \mathcal{F}$ s.t.

$$\begin{array}{ccc} \mathcal{O}_X \otimes \mathcal{O}_X \otimes \mathcal{F} & \xrightarrow{\mu_{\mathcal{O}_X} \otimes Id} & \mathcal{O}_X \otimes \mathcal{F} \\ \downarrow Id_{\mathcal{O}_X} \otimes \mu_\mathcal{F} & & \downarrow \mu_\mathcal{F} \\ \mathcal{O}_X \otimes \mathcal{F} & \xrightarrow[\mu_\mathcal{F}]{} & \mathcal{F} \end{array}$$

In other words, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module so that restriction maps give module homomorphisms

$$\mathcal{O}_X(U) \xrightarrow{Res_{V,U}} \mathcal{O}_X(V)$$

Sidebar: what is $\mathcal{O}_X \otimes \mathcal{F}$?

def: for sheaves $\mathcal{F}, G$ of abelian groups, there is a sheaf $\mathcal{F} \otimes G$ defined as the sheafification of $U \mapsto \mathcal{F}(U) \otimes G(U)$

- Given $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a map of ringed spaces, there is an adjunction

$$f^*_{\substack{\text{ringed}\\\text{space}}} : Sh(Y, \mathcal{O}_Y) \longleftarrow Sh(X, \mathcal{O}_X) : f_*$$

where $f_* \mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$

$$\mathcal{O}_Y(U) \to \underbrace{\mathcal{O}_X(f^{-1}(U))}_{\text{ring}}\text{-module}$$

note: $\mathcal{F}(f^{-1}(U))$ is naturally an $\mathcal{O}_Y(U)$-module

$$f^*_{\substack{\text{ringed}\\\text{space}}} G = f^*_{\text{sheaf}} G \otimes_{f^*\mathcal{O}_Y} \mathcal{O}_X$$

where $\mathcal{O}_X$ is an $f^*\mathcal{O}_Y$-module via the map of ringed spaces $f^*\mathcal{O}_Y \to \mathcal{O}_X$

G is an $\mathcal{O}_Y$-module

$f^*_{\text{sheaf}} G$ is an $f^*_{\text{sheaf}} \mathcal{O}_Y$-module

Gluing Morphisms of Sheaves:

$X = \bigcup_{i \in I} U_i$

let $\mathcal{F}, G$ be sheaves on X

Given $\phi_i : \mathcal{F}|_{U_i} \to G|_{U_i}$ morphisms of sheaves s.t.

$$\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$$

then we can "glue" the morphisms ϕ_i to form a sheaf morphism $\phi : \mathcal{F} \to G$

i.e. $\phi|_{U_i} = \phi_i$

$\leftrightsquigarrow$ there is a sheaf $Hom(\mathcal{F}, G)$ on X given by $Hom(\mathcal{F}, G)(U) = Hom(\mathcal{F}|_U, G|_U)$

+Sheaf homs determined on stalks

Thm: (exercise in Vakil)

Gluing Sheaves:

Given sheaves $\mathcal{F}_i$ on U_i and isomorphisms $\mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\phi_{ij}} \mathcal{F}_j|_{U_i \cap U_j}$

Satisfying on $U_i \cap U_j \cap U_k$

$$\phi_{jk} \circ \phi_{ij} = \phi_{ik}$$

then, there is a sheaf $\mathcal{F}$ on X and isomorphisms $\mathcal{F}|_{U_i} \overset{\gamma_i}{\cong} \mathcal{F}_i$ inducing ϕ_{ij} by $\gamma_j \gamma_i^{-1} = \phi_{ij}$

unique up to unique isomorphism.

Sketch of pf. $U \subset X$

$$\mathcal{F}(U) \to \prod_i \mathcal{F}_i(U \cap U_i) \underset{\phi_{ij}^{-1} \circ Res_j}{\overset{Res_i}{\rightrightarrows}} \prod_{i,j} \mathcal{F}_i(U \cap U_i \cap U_j)$$

(or can put j)

is an equalizer defines $\mathcal{F}(U)$

restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ defined by induced map of equalizer product by

$$\mathcal{F}_j(U \cap U_j) \to \mathcal{F}_j(V \cap U_j)$$

$$\mathcal{F}_i(U \cap U_i \cap U_j) \to \mathcal{F}_i(V \cap U_i \cap U_j)$$

This defines a presheaf.

We need to check

1) Separability
2) Gluability

Check 1: sps $U = \bigcup_j W_j$

and $\gamma_1|_{W_j} = \gamma_2|_{W_j}$ $\forall j$

$\Rightarrow$ $\gamma_1|_{W_j \cap U_i} = \gamma_2|_{W_j \cap U_j}$ $\overset{\text{separability for } \mathcal{F}_i}{\Longrightarrow}$

$\gamma_1 = \gamma_2$ in $\mathcal{F}_i(U_i \cap U)$

equalizer def of $\mathcal{F}$ $\Rightarrow$

$\gamma_1 = \gamma_2$ in $\mathcal{F}(U)$

Schemes

Gluing Sheaves:

X topological space
$\| $
$\bigcup_{i \in I} U_i$ ← glued

$\mathcal{F}_i$ sheaf on U_i

$\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\cong} \mathcal{F}_j|_{U_i \cap U_j}$

s.t. $\phi_{jk} \circ \phi_{ij}|_{U_i \cap U_j \cap U_k} = \phi_{ik}|_{U_i \cap U_j \cap U_k}$ ↗ "cocycle condition"

Then there is a unique sheaf $\mathcal{F}$ up to unique isom.

and isoms. $\mathcal{F}|_{U_i} \underset{\psi_i}{\cong} \mathcal{F}_i$ s.t. $\psi_j \circ \psi_i^{-1} = \phi_{ij}$

$\mathcal{F}_{i,p} = \operatorname{colim} \mathcal{F}(U_{p,i})$
↑ nbhd of p in U_i

Construction: $\mathcal{F}(U) = \left\{ (\gamma_{p,i})_{\substack{p \in U \cap U_i \\ I \in I}} \in \mathcal{F}_{i,p} \;\middle|\; \begin{array}{l} \gamma_{p,i} \text{ represented by } \tilde{\gamma}_{p,i} \in \mathcal{F}_i(U_{p,i}) \text{ s.t. for all } q \in U_{p,i} \text{ and} \\ \text{all } j \text{ st. } q \in U_j \text{ we have } \phi_{ij}(\tilde{\gamma}_{p,i}|_{U_{q,j} \cap U_{p,i}}) = \tilde{\gamma}_{q,j}|_{U_{q,j} \cap U_{p,i}} \end{array} \right\}$

remark: for $p \in U_i \cap U_j \cap U_k$, we have $U_{p,i} \cap U_{p,j} \cap U_{p,k} =: U_{p,ijk}$

$$\phi_{ij}(\tilde{\gamma}_{p,i}|_{U_{p,ijk}}) = \tilde{\gamma}_{p,j}|_{U_{p,ijk}}$$

$$\phi_{ik}(\tilde{\gamma}_{p,i}|_{U_{p,ijk}}) = \tilde{\gamma}_{p,k}|_{U_{p,ijk}}$$

$$\phi_{jk}(\tilde{\gamma}_{p,j}|_{U_{p,ijk}}) = \tilde{\gamma}_{p,k}|_{U_{p,ijk}}$$

$$\Big(\Rightarrow \phi_{jk}\phi_{ij}(\tilde{\gamma}_{p,i}|_{U_{p,ijk}}) = \tilde{\gamma}_{p,k}|_{U_{p,ijk}} = \phi_{ik}(\tilde{\gamma}_{p,i}|_{U_{p,ijk}})$$

↳ cocycle condition $\phi_{jk}\phi_{ij} = \phi_{ik}$ shows this holds

This allow us to show $\mathcal{F}_i \to \mathcal{F}|_{U_i}$ is an isomorphism.

Take $\gamma \in \mathcal{F}_i(U)$

Then $\gamma_{p,i} = \gamma$ for all p

$\gamma_{p,j} = \phi_{ij}\gamma$ for all $p \in U_j \cap U$ for all $j \in I$

is an element of $\mathcal{F}(U)$, i.e.

satisfies $*$ because of cocycle condition.

Sheaves on a base

Recall X topological space, a basis B for the topology is a subset of the opens

s.t. every open is a union of
union of elements of the base

Prop: If $\tilde{B}$ is a set of subsets of X s.t.

(1) $\bigcup_{u \in \tilde{B}} u = X$

(2) $\forall U_i, U_j \in \tilde{B}$, $\forall p \in U_i \cap U_j$, there is $U \in \tilde{B}$ with $p \in U \subset U_i \cap U_j$

Then the set of unions of elements of B is a topology on X with basis $\tilde{B}$

A basis B is a category with
$\mathrm{Ob}\, B$ = basis. elements = open subsets of X in B

There is a unique morphism $V \to U$ when $V \subset U$

Def: a (pre) sheaf on a basis B is

$$B^{op} \longrightarrow \mathcal{C}$$

sheaf: satisfying

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer for any $U, U_i \in B$ with $U = \bigcup_{i \in I} U_i$.

Thm: Restriction

$$PSh(X) \longrightarrow PSh(X, B)$$

is an equivalence

"Lets make schemes!"

Schemes will be ringed spaces which look locally like a ringed space called Spec A associated to a (commutative) ring, A

Defining Spec A

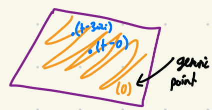
- Set $\quad$ $\operatorname{Spec} A = \{\mathfrak{p} \mid \mathfrak{p} \subset A \text{ prime ideal}\}$
- Topology
- Sheaf of rings

ex. k field $\quad$ point: $\operatorname{Spec} k = \{(0)\}$

$\operatorname{Spec} \frac{k[t]}{(t^2)} = \{(t)\}$

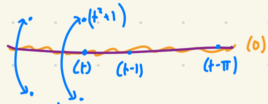
$k = \bar{k}$ $\quad$ $\operatorname{Spec} k[t] = \{(t-a) \mid a \in k\} \cup \{(0)\}$

$k = \mathbb{C}$



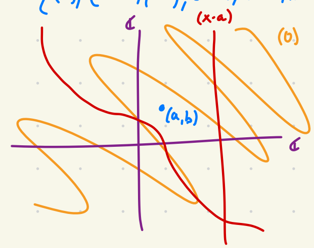
$\operatorname{Spec} k[t] = \{(0), (f(t)) \mid f(t) \text{ irreducible poly}\}$

$\operatorname{Spec} \mathbb{R}[t]$



$\operatorname{Spec} \mathbb{Z} = \{(0), (p) \mid p \text{ prime number}\}$

$\operatorname{Spec} \mathbb{C}[x,y] = \{(0), (x-a, y-b), (x-a), (x^2-y), \dots\}$



The Zariski Topology of Spec A

Closed sets are those of the form

$$V(I) = \{\mathfrak{p} \mid \mathfrak{p} \supset I\} \quad \text{prime, } \in \operatorname{Spec} A$$

for some ideal I.

Prop: this is a topology

$\{V(I) \mid I \text{ ideal of } A \ (\text{or } I = A)\}$

defines closed subsets of a topology

must check:

(1) $\emptyset$, X closed

(2) $\bigcap_{\alpha \in J} V(I_\alpha)$ is closed

(3) $V(I_1) \cup \dots \cup V(I_n)$ is closed

Pf:

(1) $X = V(0) \qquad \emptyset = V(A)$

(2) $\bigcap_{\alpha \in J} V(I_\alpha) \overset{\text{exercise}}{=} V\left(\sum_{\alpha \in J} I_\alpha\right)$ ← smallest containing ideal

(3) $V(I_1) \cup V(I_2) \overset{\text{exercise}}{=} V(I_1 \cap I_2)$ (induction)

(2) $\mathfrak{p} \in \bigcap_{\alpha \in J} V(I_\alpha) \Rightarrow \mathfrak{p} \supset I_\alpha \ \forall \alpha \in J$

$\Rightarrow \mathfrak{p} \supset \sum_\alpha I_\alpha \iff \mathfrak{p} \in V\left(\sum_{\alpha \in J} I_\alpha\right)$

(3) $\mathfrak{p} \in V(I_1 \cap I_2)$ sps $x \in I_1$, $x \notin \mathfrak{p}$.

then for all $y \in I_2 \qquad xy \in I_1 \cap I_2$

$\Rightarrow xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p} \Rightarrow y \in \mathfrak{p} \Rightarrow I_2 \subset \mathfrak{p}$

Rmk: open subsets of $\operatorname{Spec} A$ are
$\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \not\supset I\}$ for
some I.

$f \in A$

Def: $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \not\ni f\}$

Prop: $\{D(f) \mid f \in A\}$ is a base for the Zariski topology on $\operatorname{Spec} A$

Pf: $D(I) = \bigcup_{f \in I} D(f)$

Prop: A ring homomorphism $f: A \to B$ determines a continuous map $\operatorname{Spec} f : \operatorname{Spec} B \to \operatorname{Spec} A$

Pf: $\operatorname{Spec}(f)(\underset{B}{\underset{\cap}{\mathfrak{p}}}) = f^{-1}(\mathfrak{p})$

Note that $f^{-1}(\mathfrak{p})$ is indeed prime because

$$A/f^{-1}(\mathfrak{p}) \hookrightarrow B/\mathfrak{p}$$

and since $B/\mathfrak{p}$ is an integral domain, so is $A/f^{-1}(\mathfrak{p})$

$\operatorname{Spec}(f)$ is continuous because $\operatorname{Spec}(f)^{-1}(V(I)) = V(f(I))$

to see this, note $\operatorname{Spec}(f)(\mathfrak{p}) \in V(I) \iff f^{-1}(\mathfrak{p}) \supset I$

$\iff \mathfrak{p} \supset f(I)$

$f(f^{-1}(\mathfrak{p})) \subset \mathfrak{p}$
$\cup$
$f(I)$

$\mathfrak{p} \supset f(I)$
$\Rightarrow f^{-1}(\mathfrak{p}) \supset f^{-1}(f(I))$
$\supset I$

Topological properties of (affine) varieties

- Connectedness, irreducibility, quasi-compactness

Connectedness

Def a topological space X is connected if it cannot be written as a disjoint union two of nonempty open sets in X

Example/exercise: if $A = A_1 \times \cdots \times A_n$ ← rings
then there is a homeomorphism

$$\bigsqcup \operatorname{Spec}(A_i) \longrightarrow \operatorname{Spec}(A)$$

$$\mathfrak{p}_i \in \operatorname{Spec} A_i \longmapsto \operatorname{Spec} A_1 \times \cdots \times \mathfrak{p}_i \times \cdots \times \operatorname{Spec} A_n$$

[**Lemma:** if $A = A_1 \times A_2$, then the prime ideals of A are $\mathfrak{p} \times A_2$ $\quad A_1 \times \mathfrak{q}$]

Rmk $\operatorname{Spec} A$ is disconnected

$$\Updownarrow$$

iff $A = A_1 \times A_2 \qquad A_i \neq 0$

[or equivalently $\exists\, a_1, a_2 \in A$ s.t. $a_1^2 = a_1,\ a_2^2 = a_2\quad a_1 + a_2 = 1 \iff a_1 a_2 = 0$]

Def: A topological space X is irreducible if it is nonempty and not the union of two proper closed subsets of X.

In other words, cannot write $X = Y \cup Z$, $Y, Z \subset X$ closed

Easier characterization: any two nonempty open subsets of X must intersect nontrivially

$\mathbb{A}^1_{\mathbb{C}}$ " $\operatorname{Spec} \mathbb{C}[x]$ is irreducible

Exercise: (i) Let X be an irreducible top. space. Then any nonempty open set is dense.

(ii) Let A be an integral domain. Then $\operatorname{spec} A$ is irreducible.

Lemma: an irreducible topological space X is connected

Rmk: connected $\not\Rightarrow$ irreducibility

Example: $A = \mathbb{C}[x,y]/(x,y)$ $\qquad \operatorname{Spec} A =$ 

$V(x) \cup V(y)$

Exercise: 3.6F in Vakil

Def A topological space is quasicompact if given any cover $X = \bigcup_{i \in I} U_i$ $\quad U_i$ open sets, there is a finite subset $S \subset I$ s.t. $X = \bigcup_{i \in S} U_i$

→ quasi = don't require Hausdorff

"every open cover has a finite subcover"

$\mathbb{A}^1_{\mathbb{C}}$ is quasicompact – we don't want compactness

Points of Spec A

points are not necessarily closed.

Spec $\mathbb{Z}$

(0) not closed in Spec $\mathbb{Z}$

$\overline{\{(0)\}} = \operatorname{Spec}\mathbb{Z}$

Def X a top space $p \in X$ is closed if $\{p\}$ is a closed subset of X

Eg Spec $\mathbb{Z}$ - prime ideals $\mathfrak{p} = (p)$ $p \neq 0$ are closed

Spec $\mathbb{C}[t]$ prime ideals $\mathfrak{p} = (t-a)$ are closed

Hilbert's Nullstellensatz

Thm: Let k be any field. Then every maximal ideal of $k[x_1, \dots, x_n]$ has residue field = finite extension of k

$\mathfrak{m} \subset A$ residue field $A/\mathfrak{m}$

Exercise: Let k be a field and A = f.g. k-algebra.
Show that the closed points of Spec A are dense.

Hint: take $f \in A$ and sps $D(f) \neq \emptyset$ then $D(f)$ contains a closed point

A_f = localization at f
is also a f.g. k-algebra

Spec A_f

$\downarrow$

primes of A s.t. $f \notin \mathfrak{p}$

closed pts of A (or A_f) $\longleftrightarrow$ points for which the residue field is a finite extension

finish by applying Nullstellensatz to A and A_f

Lemma/Exercise:

$A = k[x_1, \dots, x_n]/I$ reduced

k = algebraically closed field

and assume nilradical of A is (0) $\quad \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$ i.e. $\exists n$ s.t. $f^n = 0$

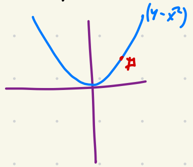
Consider $X = \operatorname{Spec} A \subseteq \mathbb{A}^n_k$

Show that the functions on $X = \operatorname{Spec} A$ are determined by their values at the closed points of X

Specialization / generization

Def given two points x, y of top. space X
we say that x is a specialization of y or y generization of x
if $x \in \overline{\{y\}}$

Ex $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$
$(y-x^2) \in \operatorname{Spec} \mathbb{C}[x,y]$
$\mathfrak{p} = (x-a)(y-a^2) \in \operatorname{Spec} \mathbb{C}[x,y]$
$\mathfrak{p}$ is a specialization of $(y-x^2)$



Exercise: if $X = \operatorname{Spec} A$, q is a specialization of p
$\iff$ $p \subset q$ (in A)

Def: we say a point $\mathfrak{p}_k \in X$ is a generic point
for a closed subset $k \subset X$ if $k \subset \overline{\{\mathfrak{p}_k\}}$

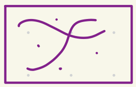
Exercise: Let $\mathfrak{p}$ be a generic point for the closed subset k
Show that $\mathfrak{p}$ is "near every point of k" i.e. every
open nbhd of $q \in k$ contains $\mathfrak{p}$

Irreducible and connected components

Def An irreducible component of a top space X
is a maximal irreducible subset; i.e., not contained
in any larger irreducible subset.

Def A subset $Y \subset X$ is a connected component
if it is a maximally connected set

Exercise: show every point $p \in X$ is contained in an irreducible component (Hint: use Zorn's lemma)



$\to$ 6 irreducible components

Def a topological space is Noetherian if it satisfies the descending chain condition for closed sets.

$$Z_1 \supsetneq Z_2 \supsetneq \cdots \supsetneq Z_n$$

is finite.

Exercise: Show $\mathbb{A}^2_{\mathbb{C}}$ ($= \operatorname{Spec} \mathbb{C}[x,y]$) is Noetherian (note $\mathbb{C}^2$ is not Noetherian)

Thm: Suppose X is a Noetherian top. space. Then every nonempty closed subset $Z \subset X$ can be expressed uniquely as a finite union $Z = Z_1 \cup \cdots \cup Z_r$ where Z_i is an irreducible, closed subset of X

"There is a finite number of pieces of X"

Recall: top space X is Noetherian if every descending sequence of closed subsets

$$X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots$$

stabilizes, i.e. $\exists N$ s.t. $X_i = X_{i+1}$ for all $i \geq N$

Def top space X is irreducible if

$$X = Z_1 \cup Z_2$$

with Z_i closed $\Rightarrow$

$$X = Z_1 \text{ or } X = Z_2$$

Thm Let X be a Noetherian top space then $X = Z_1 \cup \cdots \cup Z_n$ for unique up to reordering irreducible Z_i

Rmk X can be replaced by a (usually closed) subset with the subset topology.

Pf: (existence) "Noetherian Induction"

Let $\mathcal{F}$ be the collection of closed subsets that cannot be as a finite union of irreducibles.

Suppose, FTSOC, that $\mathcal{F}$ is nonempty. Then

Choose $Y_1 \in \mathcal{F}$. Recursively suppose

$Y_1 \supset Y_2 \supset \cdots \supset Y_i$ in $\mathcal{F}$.

If Y_i does not contain $Y_{i+1} \subsetneq Y_1$ with

$Y_{i+1} \in \mathcal{F}$, stop. Otherwise, choose such an Y_{i+1}

(can do so b/c X is Noetherian)

Thus have $Y_N \in \mathcal{F}$ minimal.

Since Y_N reducible, have $Y_N = W_1 \cup W_2$

with $W_i \neq Y_N$, W_i closed. But since Y_N minimal

$W_i \notin \mathcal{F}$ thus $W_i = Z_{1i} \cup \cdots \cup Z_{n_i i}$

with Z_{jl} irreducible

$\Rightarrow Y_N$ is a union of irreducibles. Since $Y_N \in \mathcal{F}$, this

is a contradiction.

Rmk for $Y = Z_1 \cup \cdots \cup Z_n$ with

Z_i irreducible, we may assume $Z_i \not\subset Z_j$ $\forall i,j$

by throwing out redundancies

Claim: If $Z_1 \cup \cdots \cup Z_n = Z_1' \cup \cdots \cup Z_{n'}'$

with $Z_i \not\subset Z_j$ $Z_i' \not\subset Z_j'$ for all i,j

and Z_i, Z_i' irreducible

then $n' = n$ and the Z_i' can be obtained from Z_i

by reordering.

Pf: $Z_1 = (\underbrace{Z_1' \cap Z_1}) \cup \cdots \cup (\underbrace{Z_{n}' \cap Z_1})$

closed

Since Z_1 is irreducible, $\exists i$ s.t. $Z_1 = Z_i' \cap Z_1 \subset Z_i'$

By the same argument, $\exists j$ s.t. $Z_i' \subset Z_j$

$\Rightarrow Z_1 \subset Z_j$ since we through out redundancies

$\Rightarrow j = 1$

$\Rightarrow Z_1 = Z_i'$

repeating for $Z_2, \ldots, Z_n$ proves uniqueness. $\boxtimes$

Let A be a commutative ring with 1, M an A-module

<u>Def</u> M is Noetherian if an increasing sequence of submodules

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_i \subseteq \cdots \subseteq M$$

stabilizes.

A is <u>Noetherian</u> if A is Noetherian as a module over itself. $\longleftrightarrow$ every ideal is f.g.

This means $\forall\, I_1 \subseteq I_2 \subseteq \cdots \subseteq \cdots$ increasing chain of ideals, the chain stabilizes

By def of Zariski topology on $\operatorname{Spec} A$, a decreasing sequence of closed subsets is of the form $V(I_1) \supset V(I_2) \supset \cdots \supset V(I_i) \supset \cdots$ stabilizes

$\Rightarrow$ $\operatorname{Spec} A$ is Noetherian as a top space if A Noetherian (converse not true: $\operatorname{Spec} k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots) \cong \operatorname{Spec} k$ as top spaces

but $(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$ shows $k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$ non Noetherian)

<u>Ex</u>: PID's are Noetherian

<u>Ex</u>: $\operatorname{Spec} A$ for A artinian is Noetherian
↳ all max ideals

<u>Ex</u>:

$$\begin{array}{c} \operatorname{Spec} \overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \underset{\text{set bij}}{\cong} \operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \\ \downarrow \\ \operatorname{Spec} \mathbb{F}_q \end{array}$$

All primes in $A = \overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ are maximal, but A is not Noetherian

<u>Thm</u> (Hilbert Basis Thm)

If A is a Noetherian ring, then so is $A[x]$.

<u>Pf</u>: consider $I_1 \subset I_2 \subset \cdots$ in $A[x]$

then $I = \bigcup I_i$ is an ideal.

Want: I f.g.

We construct $f_1, f_2, \ldots \in I$ in the following manner.

Choose f_1 of minimal degree in I.

If $I \neq (f_1, \ldots, f_n)$, then let f_{n+1} be any element of

$I=(f_1,\dots,f_n)$ of minimal degree. We claim this process terminates. FTSOC, suppose we construct an infinite sequence $f_1, f_2, \dots, f_n, \dots$ then

$$f_i = \sum_{j \le \deg f_i} a_{ji} x^j$$

Let $I = (a_{j\,\deg f_j} \mid j = 1,2,\dots) \overset{\text{ideal}}{\subset} A$

B/c A is Noetherian, $\exists N$ s.t.

$$I = (a_{1\,\deg f_1}, \dots, a_{N\,\deg f_N})$$

Then $f_{N+1} - \sum^{N}_{j} b_j f_j\, x^{\deg f_{N+1} - \deg f_j}$ has lower degree than f_{N+1} but is not in $(f_1,\dots,f_N)$ and is in I.

This contradicts the construction of f_{N+1}. $\boxed{\lightning}$

$\operatorname{Spec} A$
- Set ✓
- top space ✓
- ringed space

we now construct a sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} A$.

- Can construct $\mathcal{O}$ on the basis $D(f)$ (thm that said sheaves on a basis $\{B_i\} = X \cong Sh(X)$)

$$\mathcal{O}(D(f)) = S^{-1}A$$

where $S \subset A$ is defined to be

$$S = \{ g \in A \text{ s.t. } g \notin \mathfrak{p}\ \forall \mathfrak{p} \in D(f) \}$$

S is multiplicatively closed in the sense that

$$g_1, g_2 \in S \Rightarrow g_1 g_2 \in S$$

Aside on <u>localization</u>

<u>Def</u> for any $S \subset A$ multiplicatively closed,

$\exists\ A \xrightarrow{\iota} S^{-1}A$

s.t. for all $A \xrightarrow{\psi} B$ with $\psi(g) \in B^{\times}$ (units) for all $g \in S$, $\exists!$ map $\bar{\psi}$ s.t.

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "\psi"'] & S^{-1}A \arrow[d, "\bar{\psi}"] \\ & B \end{tikzcd}$$

commutes.

Construction $S^{-1}A = \{(n,d) \mid n \in A, d \in S\} / (n_1,d_1) \sim (n_2,d_2)$ if $d(n_1 d_2 - n_2 d_1) = 0$

ex. $\mathbb{Z}[\frac{1}{p}] = \{ \frac{a}{b} \mid \underset{a \in \mathbb{Z}}{b \in (p)} \}$
$\cap$
$\mathbb{Q}$

---

anyways... $f \in S = \{ g \mid g \notin \mathfrak{p} \ \forall \mathfrak{p} \in D(f) \}$

$$\Rightarrow \begin{tikzcd} A \arrow[r] \arrow[dr] & A[\frac{1}{f}] \arrow[d, "\exists ! \Theta"] \\ & S^{-1}A \end{tikzcd}$$

Claim: $A[\frac{1}{f}] \xrightarrow{\Theta} S^{-1}A$
is an isomorphism

Pf: by universal prop of localization, it suffices to show $\forall g \in S$ g is invertible in $A[\frac{1}{f}]$
thus suffices to show $f^n \in (g)$ for some n, then $\frac{a}{f^n}$ is an inverse for g.

$D(g) \supset D(f)$ b/c $g \in S \Leftrightarrow V(g) \subset V(f)$

Exercise: given A ring, I_1, I_2, ideals
$V(I_1) \supset V(I_2)$
$\Rightarrow \sqrt{I_1} \supset \sqrt{I_2}$
$\Rightarrow \sqrt{(g)} \supset \sqrt{(f)} \supset (f)$
$\Rightarrow \exists n$ s.t. $f^n \in (g)$ $\boxed{\text{Z}}$

Last time: Vakil 3.6 - topological + Noetherian properties

A = commutative ring
$\operatorname{Spec} A$ = set of prime ideals of A with Zariski topology
↳ closed sets are $V(S) := \{ [\mathfrak{p}] \in \operatorname{Spec} A \mid S \subseteq \mathfrak{p} \}$
i.e. set of pts on which elements of S are 0.

Given $\varphi : B \to A$ in Ring
get $\operatorname{Spec} \varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ in Top

3.6: connectedness, irreducibility, quasicompactness

Connectedness: non ex: $\operatorname{Spec}\left(\prod_{i=1}^{n} A_i\right) = \coprod_{i=1}^{n} \operatorname{Spec} A_i$

Irreducibility: can't be written as union of nonempty disjoint closed subsets

irreducible $\Rightarrow$ connected
$\not\Leftarrow$

$V(xy) = V(x) \cup V(y)$ in $\mathbb{A}^2_{\mathbb{R}}$

connected but reducible

quasicompactness: open cover $\Rightarrow$ finite subcover covers X

3.7: ideal vanishing correspondence

Start with subsets of $\operatorname{Spec} A \rightsquigarrow$ ideal in A

Def: given $S \subseteq \operatorname{Spec} A$, the ideal of S
is $I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p} \subseteq A$
"set of functions vanishing on S"

Facts:
- $I(S) \overset{\text{ideal}}{\subseteq} A$
- If $S_1 \subseteq S_2$, then $I(S_2) \subseteq I(S_1)$
- $I(\bar{S}) = I(S)$

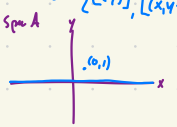
Example: Let $S \subseteq \mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y,z]$
be the union of the three axes.

x-axis $\rightsquigarrow V(y,z)$
y-axis $\rightsquigarrow V(x,z)$
z-axis $\rightsquigarrow V(x,y)$

$$S = \{[(y,z)], [(x,z)], [(x,y)]\}$$

Can check: $I(S) = (yz, xz, xy) \subseteq \mathbb{C}[x,y,z]$

exercise: $A = k[x,y]$
$S = \{[(y)], [(x, y-1)]\} \subseteq \operatorname{Spec} A$



$I(S)$: set of poly's in $k[x,y]$ that vanish @ $(0,1)$ and on the x-axis

Q: what are the generators of $I(S)$?

Would like: $V(-)$ and $I(-)$ to be inverse to each other

claim/exercise: $V(I(S)) = \overline{S}$ for $S \subseteq \operatorname{Spec} A$

A integral domain

Note: $I(S)$ always a radical ideal for $S \subseteq \operatorname{Spec} A$

$f \in \sqrt{I(S)} \Rightarrow f^n$ vanishes on S for some $n > 0$

$\Rightarrow f$ vanishes on S

$\Rightarrow f \in I(S)$

In particular:

$V(I(S)) = S$ if $S = \overline{S}$

claim: $I(V(J)) = \sqrt{J}$ for J ideal of A.

Thm: $V(-)$ and $I(-)$ give an inclusion-reversing bijection between closed subsets of $\operatorname{Spec} A$ and radical ideals of A

Thm: $V(-)$ and $I(-)$ give an inclusion reversing bijection between irreducible closed subsets of $\operatorname{Spec} A$ and prime ideals of A

pf (sketch) If $S \subseteq \operatorname{Spec} A$ closed, can write $S \subseteq V(I)$ for I a radical ideal of A by thm. 1. W.t.s. I is prime. Let $ab \in I$

$$V(I,(a)) \cup V(I,(b)) = V((I,a)\cdot(I,b))$$
$$= V(I^2, aI, bI, ab)$$
$$= V(I) \text{ since } I = \sqrt{I}.$$

we've written

$$S = V(I) = \underbrace{V(I,(a)) \cup V(I,(b))}_{\text{two nonempty closed subsets}}$$

Since S irreducible, $V(I) = V(I,(a))$ or $V(I) = V(I,(b))$

$(\ ab \in I\ ) \Rightarrow a \in I$ or $b \in I$ $\boxtimes$

<u>Def</u> A prime ideal $\mathfrak{p} \subseteq A$ is a <u>minimal prime ideal</u> if it's minimal w.r.t. inclusion

e.g. (0) minimal in $k[x,y]$ (or any integral domain) exercise: if A = ring,

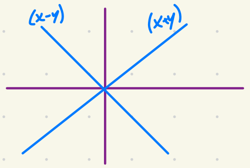
$$\left\{\begin{matrix}\text{irred. components} \\ \text{of } \operatorname{Spec} A\end{matrix}\right\} \xleftrightarrow{\text{bij.}} \left\{\begin{matrix}\text{minimal} \\ \text{primes of } A\end{matrix}\right\}$$

reality check / why this makes sense:

if $\mathbb{A}^n_k = \operatorname{Spec} k[x_1,\dots,x_n]$, subset cut out by $f(x_1,\dots,x_n)$ has irreducible components corresponding to irreducible factors of f

e.g. $\mathbb{A}^2_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x,y]$

$f(x,y) = x^2 - y^2 = (x+y)(x-y)$

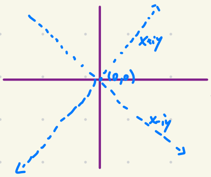


reducible, 2 components $\rightsquigarrow$ 2 factors of f: $(x+y)$, $(x-y)$

e.g. $\mathbb{A}^2_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x,y]$

$f(x,y) = x^2 + y^2 = (x+iy)(x-iy)$

<u>not</u> minimal primes of $\mathbb{R}[x,y]$



reducible, 2 components defined over $\bar{k}$ $\rightsquigarrow$ 2 factors of f defined over $\bar{k}$

* minimal primes/irreducible components depend on field of definition

to deal with this: base change (tensor coeffs w/ field extension)

"Grand Dictionary b/w algebra & geometry":

Algebraic objects	Geometric objects
ring A	affine scheme $\operatorname{Spec} A$
$\mathfrak{p} \subseteq A$ prime ideal	point $[\mathfrak{p}] \in \operatorname{Spec} A$
elements $f \in A$	functions f on $\operatorname{Spec} A$
$f \pmod{\mathfrak{p}}$	$f(\mathfrak{p})$
maximal ideals of A	irreducible closed subsets of $\operatorname{Spec} A$ (closed points)
prime ideals $\mathfrak{p} = I(Z)$	irreducible closed subsets $Z = V(\mathfrak{p}) \subseteq \operatorname{Spec} A$
minimal primes of A	irreducible components of $\operatorname{Spec} A$
radical ideals $I = \sqrt{I} = I(S)$	closed subsets $S = V(I) \subseteq \operatorname{Spec} A$

Today: • 4.1: Structure sheaf

• Digression on modules

• 4.2: Visualizing nilpotents

4.1: the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$

A ring $\operatorname{Spec} A$

$\mathcal{O}_{\operatorname{Spec} A}$ will denote the structure sheaf of $\operatorname{Spec} A$

↳ "sheaf of algebraic functions on $\operatorname{Spec} A$"

[Reminder $D(f) = \{ \mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \}$ form a base of distinguished opens on $\operatorname{Spec} A$]

Def: let $f \in A$. Then define $\mathcal{O}_{\operatorname{Spec} A}(D(f)) := A$ localized at $\{ g \in A \mid V(g) \subseteq V(f) \}$

"functions that only vanish on $V(f)$"

$\{ g \in A \mid D(f) \subseteq D(g) \}$

we've shown:

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) \cong A_f = A\left[\tfrac{1}{f}\right]$$

Given $D(f') \subseteq D(f)$

$\operatorname{res}_{D(f), D(f')} : \mathcal{O}_{\operatorname{Spec} A}(D(f)) \to \mathcal{O}_{\operatorname{Spec} A}(D(f'))$

by $A_f \longmapsto (A_f)_{f'}$

$\mathcal{O}_{\operatorname{Spec} A}$ is the structure sheaf of $\operatorname{Spec} A$

Notation: an affine scheme refers to both $\operatorname{Spec} A$ & $\mathcal{O}_{\operatorname{Spec} A}$ together

$(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$

Thm: $\mathcal{O}_{\operatorname{Spec} A}$ is a sheaf.

Need: identity + gluability

Recall: if $\mathcal{F}$ is a sheaf on $\mathcal{B}$,

• identity: if $B = \cup B_i$ + $f, g \in \mathcal{F}(B)$ s.t.

$\operatorname{res}_{B, B_i} f = \operatorname{res}_{B, B_i} g \;\; \forall i$, then $f = g$

• gluability: if $B = \cup B_i$ + $f_i \in \mathcal{F}(B_i)$ s.t.

$\operatorname{res}_{B_i, B_k} f_i = \operatorname{res}_{B_j, B_k} f_j$ for any $B_k \subseteq B_i \cap B_j$

then $\exists f \in \mathcal{F}(B)$ s.t. $\operatorname{res}_{B, B_i} f = f_i$

pf (identity) Sps $X := \operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$

i.e. $A = (f_i)_{i \in I}$. Since $\operatorname{Spec} A$ is quasicompact, it is covered by finite f_i, so $\operatorname{Spec} A = \bigcup_{i=1}^{n} D(f_i)$

Sps $\exists s \in A$ s.t. $\operatorname{res}_{X, D(f_i)} s = 0$ w.t.s. $s = 0$.

$\operatorname{res}_{X, D(f_i)} s = 0$ in $\mathcal{O}(D(f_i)) \cong A_{f_i}$ means

$\exists m$ s.t. $f_i^m s = 0$ $\forall i$.

But $X = \bigcup_{i=1}^{n} D(f_i) = \bigcup_{i=1}^{n} D(f_i^m)$, so

in particular, $A = (f_1^m, \dots, f_n^m)$ so

can write $1 = \sum_{i=1}^{n} r_i f_i^m$ for some $r_i \in A$.

Thus $s = s \cdot 1 = s \cdot \sum_{i=1}^{n} r_i f_i^m$

$= \sum_{i=1}^{n} r_i \cancel{(f_i^m s)}$ 0

$= 0$ in A

pf (gluability) $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$ and sps we have elements $a_i / f_i^{l_i} \in A_{f_i} \in \mathcal{O}(D(f_i))$ $\forall i$

that agree on overlaps. Assume I finite, $\{1, \dots, n\}$

write $g_i = f_i^{l_i}$ so that the elements are $a_i / g_i \in A_{g_i}$.

Since $D(f_i) = D(g_i)$, $\operatorname{Spec} A = \bigcup_{i=1}^{n} D(g_i)$, and

a_i / g_i, a_j / g_j agreeing on overlaps means $\exists m_{ij} \in A_{g_i g_j}$ s.t.

$(g_i g_j)^{m_{ij}} (g_j a_i - g_i a_j) = 0$

Since I finite, can find m s.t. $(g_i g_j)^m / (g_j a_i - g_i a_j) = 0$ $\forall i, j$

write $b_i := a_i g_i^m$ and $h_i = g_i^{m+1}$. Then on

$D(g_i) = D(h_i)$ we have h_i's and $h_j b_i = h_i b_j$.

$\operatorname{Spec} A = \bigcup_{i=1}^{n} D(h_i) \Rightarrow 1 = \sum_{i=1}^{n} r_i h_i$ for some $r_i \in A$.

Define $r := \sum r_i b_i \in A$.

Claim $\operatorname{res}_{X, D(h_i)} r = h_i / b_i$ for any i.

$\boxed{\text{Z}}$

Def: Let $(X, \mathcal{O}_X)$ be an affine scheme. Say, $X \cong \operatorname{Spec} A$.

A <u>sheaf of $\mathcal{O}_X$-modules</u> is a sheaf $\mathcal{F}$ on X s.t.

for all $U \subseteq X$ open, the abelian group $\underbrace{\mathcal{F}(U)}_{\text{group}}$ is an $\underbrace{\mathcal{O}_X(U)}_{\text{ring}}$-module

- $V \subseteq U \rightsquigarrow \operatorname{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$

  has to be compatible with $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$

- A morphism $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a morphism of sheaves s.t. for all $U \subseteq X$ open,

$\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules

- an example: $\exists$ sheaf of $\mathcal{O}_X$ modules for any M an A-module.

Def Let M be an A-module. Define the sheaf $\tilde{M}$ on the distinguished basis of $\operatorname{Spec} A$ by $\tilde{M}(D(f)) := M$ localized at functions that only vanish on $V(f)$

Can show: $\tilde{M}(D(f)) \cong M_f$

Given $D(f') \subseteq D(f)$

$\operatorname{res}_{D(f), D(f')} : M_f \to (M_f)_{f'}$

Exercise
- $\tilde{M}$ is a sheaf on $\operatorname{Spec} A$ (analogous to earlier proof)
- This is an $\mathcal{O}_{\operatorname{Spec} A}$-Module

Stalks $\tilde{M}_p = M_{\mathfrak{p}}$ for $\mathfrak{p} \subseteq A$ prime

Def the support of M is $\operatorname{Supp} M := \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}} \neq 0\} \subseteq \operatorname{Spec} A$

☆ read 4.1 for caution: not distinguished

given arbitrary open $U \subseteq \operatorname{Spec} A$

$\mathcal{O}_{\operatorname{Spec} A}(U)$ is <u>not</u> A localized at multiplicative set of fns that don't vanish on any pt of U

4.2: Visualizing Nilpotents (motivation)

Algebra		Geometry
A	$\longleftrightarrow$	$\operatorname{Spec} A$
max idls	$\longleftrightarrow$	closed pts
prime idls	$\longleftrightarrow$	irreducible closed subsets
radical ideals	$\longleftrightarrow$	closed subsets
non-radical ideals	$\longleftrightarrow$	???

e.g. $\operatorname{Spec} \mathbb{C}[x]/(x(x-1)(x-2))$

corresponds to closed set $\{0,1,2\} \subseteq \mathbb{A}^1_{\mathbb{C}}$

e.g. $\operatorname{Spec} \mathbb{C}[x]/(x^2)$

as a set: $\{0\} = [(x)]$

but, derivative of x^2 at zero also zero.

hard to formalize. Informally, "fuzzy point"

$\operatorname{Spec} \mathbb{C}[x]/(x)$

$\operatorname{Spec} \mathbb{C}[x]/(x^2)$

$\operatorname{Spec} \mathbb{C}[x]$

e.g.

$\operatorname{Spec} \mathbb{C}[x,y]/(x,y)^2$

just $\operatorname{Spec} \mathbb{C}[x,y]/(x,y)$

$\operatorname{Spec} \mathbb{C}[x,y]/(x^2,y)$

$\operatorname{Spec} \mathbb{C}[x,y]/(x,y^2)$

$\operatorname{Spec} \mathbb{C}[x,y]/(x,y)^2$

Last time: $\operatorname{Spec} A$:

- Set
- Topological space
- Structure sheaf $\mathcal{O}_X$ $\quad X = \operatorname{Spec} A$

$\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \cong A$

maximal ideals A $\leftrightarrow$ closed pts X

prime ideals A $\leftrightarrow$ pts X

$f \in \mathfrak{p}$ $\Updownarrow$

radical ideals $\sqrt{I} = I$ $\leftrightarrow$ $\{f \in A \mid f(\mathfrak{p}) = 0 \ \forall \mathfrak{p} \in V(I)\}$

Ideals $\leftrightarrow$???

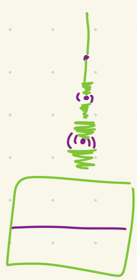
$\operatorname{Spec} \mathbb{C}[x][y]/(x)$

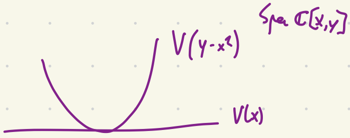
$\operatorname{Spec} \mathbb{C}[x][y]/(x^2)$

$\operatorname{Spec} \mathbb{C}[x][y]/(x^3)$

$\vdots$

$\operatorname{Spec} \mathbb{C}[x][y]$





$V(y-x^2)\cap V(y) = V(y,y-x^2)$

$\cong \operatorname{Spec} \mathbb{C}[x,y]/(y,y-x^2)$

$\cong \operatorname{Spec} \mathbb{C}[x]/(x^2)$

<u>Def:</u> an isomorphism in a category $\mathcal{C}$ is a morphism $f: c_1 \to c_2$ s.t. $\exists$ $g: c_2 \to c_1$ s.t. $g\circ f = id_{c_1}$, $f\circ g = id_{c_2}$

<u>Def:</u> The category of ringed spaces has objects $(X, \mathcal{O}_X)$ where X top space and $\mathcal{O}_X$ sheaf of rings on X.

Morphisms $(X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ are pairs (f,θ), where $f: X\to Y$ is a continuous map of topological spaces and $\theta: \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves of rings.

<u>Def:</u> A <u>scheme</u> is a ringed space $(X,\mathcal{O}_X)$ s.t. $\forall p \in X$, there exists an open $p\in U \subset X$ and an isomorphism $(U, \mathcal{O}_X|_U) \cong \operatorname{Spec} A_U$ for some ring A_U

<u>Def:</u> An <u>affine scheme</u> is a scheme which is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some A.

<u>Def:</u> a (continuous, smooth, C_n) $\mathbb{R}$-manifold is a ringed space $(X,\mathcal{O}_X)$ s.t. $\forall p\in X$ $\exists$ open $U\subset X$, $p\in X$ s.t. $(U, \mathcal{O}_X|_U) \cong (\mathbb{R}^n, \mathcal{O})$ where $\mathcal{O}$ is the sheaf of (continuous, smooth, C_n) functions to $\mathbb{R}$.

n continuous derivatives

Rmk: We can take disjoint union of schemes

Def: an open subscheme of $(X, \mathcal{O}_X)$ is $(U, \mathcal{O}_X|_U)$ for some open $U \subset X$

$D(f) \subset \operatorname{Spec} A$, $f \in A$ $\quad D(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$

Prop: $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong (\underbrace{\operatorname{Spec} A[\tfrac{1}{f}]}_{= \operatorname{Spec} A_f}, \mathcal{O}_{\operatorname{Spec} A_f})$

Pf: $A \xrightarrow[\text{localization}]{L} A[\tfrac{1}{f}]$

$\rightsquigarrow \operatorname{Spec} A_f \xrightarrow{\phi} \operatorname{Spec} A$

we claim the image of ϕ is precisely $D(f)$

$\operatorname{im} \phi \subset D(f)$: any prime $\mathfrak{p} \subset A_f$ does not contain $f \in A_f^\times$

Thus $L^{-1}(\mathfrak{p})$ does not contain f.

Any prime $\mathfrak{q}$ in $D(f)$ is contained in the image of ϕ because:

$L(\mathfrak{q})$ generates an ideal: $L(\mathfrak{q})^e = \{\frac{a}{f^n} \in A_f \mid a \in \mathfrak{q}, n \in \mathbb{Z}\}$

$$L^{-1}(L(\mathfrak{q})^e) = \{a \in A \mid \exists\, b \in \mathfrak{q}, n \in \mathbb{Z} \text{ s.t. } b/f^n = a \text{ in } A_f\}$$

$b/f^n = a$ in A_f

$\Leftrightarrow f^m(b - af^n) = 0$ in A

$\Leftrightarrow f^m b = af^n$ in A

$b \in \mathfrak{q} \Rightarrow af^n \in \mathfrak{q}$

$f \notin \mathfrak{q} \Rightarrow a \in \mathfrak{q}$ (prime)

$\Rightarrow L^{-1}(L(\mathfrak{q})^e) = \mathfrak{q}$ so $\mathfrak{q} \in \operatorname{im} \phi$

We want to see $\mathcal{O}_{\operatorname{Spec} A}|_{D(f)} = \phi_* \mathcal{O}_{\operatorname{Spec} A_f}$

on distinguished basis $D(g) \subset D(f)$

$\mathcal{O}_{\operatorname{Spec} A}(D(g)) = A_g = S^{-1}A \quad S = \{h \in A \mid h(\mathfrak{p}) \neq 0\ \forall \mathfrak{p} \in D(g)\}$

$\cong (\bar{S})^{-1} A_f$ where $\bar{S}$ is image of S under localization

$\phi_* \mathcal{O}_{\operatorname{Spec} A_f}(D(g)) = (S')^{-1} A_f$

Note $S' = \{\frac{s}{f^n} \mid s \in \bar{S}, n \in \mathbb{N}\}$

$\Rightarrow (\bar{S})^{-1} A_f \cong (S')^{-1} A_f$ $\quad \boxtimes$

Let $(X, \mathcal{O}_X)$ be a scheme

Rmk: given $Y \underset{\text{closed}}{\subset} X$

There is more than one natural choice for scheme structure on Y:

$\cdots V(I^n) = V(I^2) = V(I) \subset \operatorname{Spec} A$

$(V(I), \mathcal{O}_{\operatorname{Spec} A/I^n})$ are all schemes

Recall: $\mathcal{F}$ sheaf on X $\quad p \in X \quad i_p : \{p\} \to X$

stalk: $\mathcal{F}_p := i_p^* \mathcal{F} = \underset{U \ni p}{\operatorname{colim}} \mathcal{F}(U)$

Hw: $(\mathcal{O}_{\operatorname{Spec} A})_{\mathfrak{p}} \cong A_{\mathfrak{p}}$

Rmk: in particular, $\mathcal{O}_{\mathfrak{p}}$ is a local ring which means it has a unique maximal ideal

Def: A locally ringed space is $(X, \mathcal{O}_X)$ a ringed space s.t. $\mathcal{O}_{X,\mathfrak{p}}$ are local rings

ex Schemes, manifolds are locally ringed spaces

for $U \subset X$ and $\gamma \in \mathcal{O}_U$ view γ as "function" on U whose "value" at $p \in U$ is defined $\bar{\gamma} \in \underbrace{\mathcal{O}_p / \mathfrak{m}}_{\text{def of residue field}}$ ← unique max ideal

$\gamma(p) = \bar{\gamma}$ $\quad\quad K(p)$

$(\operatorname{Spec} \mathbb{C}[x,y], \mathcal{O})$

for example: $f = \dfrac{x^3+y^2}{x(y^2-x)} \in \mathcal{O}(D(x(y^2-x)))$

$\mathfrak{p} = (x-3, y-5) \in D(x(y^2-x))$

$f(\mathfrak{p}) = \dfrac{3^3+5^2}{3(5^2-3)} \in \dfrac{\mathbb{C}[x,y]_{(x-3,y-5)}}{(x-3,y-5)} \cong \mathbb{C}$ $\quad\quad$ tip: $A_{\mathfrak{p}}/\mathfrak{p}e = (A/\mathfrak{p})_{\bar{\mathfrak{p}}}$

generic pt → $(0) \in D(x(y^2-x))$

Residue field $= \mathbb{C}(x,y)$

$\in$

$f(0) = \dfrac{x^3+y^2}{x(y^2-x)}$

$\operatorname{Spec} \dfrac{\mathbb{Q}[x,y]}{(y-x^2)} \ni (y-3)$

$\quad\quad \Psi$

$(x-2, y-4)$

Residue field $(y-3) \cong \dfrac{\mathbb{Q}[x,y]}{(y-3, y-x^2)} \cong \dfrac{\mathbb{Q}[x]}{(x^2-3)} \cong \mathbb{Q}[\sqrt{3}]$

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on a locally ringed space $\mathcal{F}_{(p)} = \mathcal{F}_p \otimes_{\mathcal{O}_X} \mathcal{O}_X/p$ ← residue field

<u>Def</u>: the <u>fiber</u> of $\mathcal{F}$ at p,

This is a vector space over $\mathcal{O}_X/p$

its dimension is the <u>rank</u>

Examples of schemes

Create by gluing:

can glue top spaces

can glue sheaves

⇝ <u>Prop</u>: given schemes $\{X_i\}_{i \in I}$ and open $X_{ij} \subset X_i$ with $X_{ii} = X_i$, $(X_{ij}, \mathcal{O}_{X_{ij}}) \subset (X_i, \mathcal{O}_{X_i})$

and isomorphisms $f_{ij} : X_{ij} \to X_{ji}$ of ringed spaces

with $f_{ii} = \mathrm{id}_{X_i}$ and cocycle condition satisfied:

$$f_{jk}\big|_{X_{jk} \cap X_{ji}} \circ f_{ij}\big|_{X_{ik} \cap X_{ij}} = f_{ik}\big|_{X_{ik} \cap X_{ij}}$$

In particular, we need $f_{ij}(X_{ik}) \subset X_{jk}$

Then there is a unique scheme X with open cover $X = \bigcup_{i \in I} X_i$

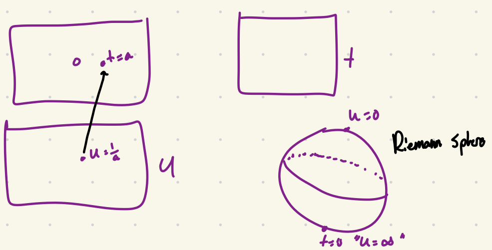
inducing the f_{ij}

<u>Ex</u>: projective space $\mathbb{P}^1$ is formed by gluing $\operatorname{Spec} A[t]$

to $\operatorname{Spec} A[u]$ along $D(t) \cong D(u)$

$$t \leftrightarrow \frac{1}{u}$$

For simplicity, let $A = \mathbb{C}$



Points of $\mathbb{P}^1_{\mathbb{C}}$: generic pt.

$$\begin{tikzcd} (0) \in \mathbb{C}[t] \arrow[d, leftrightarrow] \arrow[r, phantom] & \mathbb{C}[t] \arrow[d] \\ (0) \arrow[d, leftrightarrow] & \mathbb{C}[t, \frac{1}{t}] \arrow[d, phantom, "\cong"] \\ & \mathbb{C}[u, \frac{1}{u}] \\ (0) & \mathbb{C}[u] \arrow[u] \end{tikzcd}$$

Thus $\mathbb{P}^1_{\mathbb{C}}$ has one point y which is identified with the generic pt in both $\operatorname{Spec} \mathbb{C}[t]$ and $\operatorname{Spec} \mathbb{C}[u]$

closed pts $(t-a) = (u - \frac{1}{a})$ $a \in \mathbb{C}^\times$

(t), (u)

V a k-vector space:

$$\mathbb{P}V := V \setminus \{0\} \big/ {\scriptstyle v \sim \lambda v,\ \lambda \in \mathbb{R}^\times}$$

closed pts $\mathbb{P}^1_{\mathbb{C}} \cong \mathbb{P}(\mathbb{C}^2)$

canonical bij

<u>Def</u> A <u>scheme</u> X is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to $\operatorname{Spec} A$ for some A.

<u>Def</u> an <u>affine scheme</u> is a scheme isomorphic to $\operatorname{Spec} A$ for some ring A.

<u>ex.</u> $\operatorname{Spec} \mathbb{Q}[x,y] \big/ (y^2 - x(x-1)(x-2))$

<u>Non-example</u> $\operatorname{Spec} \mathbb{C}[x,y] \setminus \{(0,0)\}$ is a scheme, but <u>not</u> an affine scheme

$(x,y) \subseteq \mathbb{C}[x,y]$

Recall: In $\operatorname{Spec} A$, any closed subset $V(I)$ has the property

$$I(V(I)) := \{ f \mid f(x) = 0 \ \forall x \in V(I) \}$$
$$= \{ f \mid f \in \mathfrak{p} \ \forall \cancel{\mathfrak{p} \in V(I)} \}$$
$$\mathfrak{p} \supseteq I$$

$$\| \qquad \sqrt{I}$$

Now, $\mathcal{O}(\overbrace{\operatorname{Spec}\mathbb{C}[x,y]\setminus 0}^{U})$ is computed

kernel $\Rightarrow f-g=0$

$$0 \to \mathcal{O}(U) \to \begin{matrix} f\in\mathcal{O}(D(x)) = \mathbb{C}[x,y,\tfrac{1}{x}] \\ \oplus \\ g\in\mathcal{O}(D(y)) \end{matrix} \to \mathcal{O}(D(xy)) = \mathbb{C}[x,y,\tfrac{1}{x},\tfrac{1}{y}]$$

Because $U = D(x)\cup D(y)$ $= \mathbb{C}[x,y,\frac{1}{y}]$

x nonzero or y nonzero

denominators from need to agree

so no x, y in denominators

$$\Rightarrow \mathcal{O}(U) \cong \mathbb{C}[x,y]$$
$$= \mathcal{O}(\mathbb{A}^2)$$
$$\mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[x,y]$$

Hartog's thm (this is close to a proof, we avoid using this)

$$V_U(\underbrace{(x,y)}_{I}) = \varnothing$$

$$I(V_U(I)) = \mathbb{C}[x,y] \neq \sqrt{(x,y)} = (x,y)$$

So U not affine scheme.

<u>Recall</u>: X_i schemes $i\in I$

and $X_{ij} \subset X_i$ (open) for $i,j\in I$

and isomorphisms $f_{ij}: X_{ij} \xrightarrow{\sim} X_{ji}$

satisfying the cocycle condition

$$f_{ik}\big|_{X_{ij}\cap X_{ik}} = f_{jk}\circ f_{ij}\big|_{X_{ij}\cap X_{ik}}$$

$$f_{ij}(X_{ij}\cap X_{ik}) \subset X_{jk}$$

Then there is a unique scheme $X = \bigcup X_i$

s.t. $X_{ij} = X_i \cap X_j$

and f_{ij} is the composition

$$X_{ij} \cong X\big|_{X_{ij}} \cong X_{ji}$$

<u>Ex</u>: $\mathbb{A}^n_A \cong \operatorname{Spec} A[x_1,\dots,x_n]$

<u>Ex</u>: $\mathbb{G}_{m,A} = \operatorname{Spec} A[x,\frac{1}{x}]$
$= \mathbb{A}^1_A - \{0\}$

<u>Ex</u>: affine line w/ doubled origin (scheme but not affine)

Projective Schemes

$$\mathbb{RP}^n := \mathbb{P}(\mathbb{R}^{n+1}) := \frac{\mathbb{R}^{n+1} - \{0\}}{\substack{v \sim \lambda v \\ \lambda \in \mathbb{R}^*}}$$

$= \{$ lines through origin in $\mathbb{R}^{n+1}\}$

$= \{[x_0, \dots, x_n] \in \mathbb{R}^{n+1} \mid$ not all $x_i = 0$, $[\lambda x_0, \dots, \lambda x_n] = [x_0, \dots, x_n]\}$

$\mathbb{R}^n \cong \{[1, x_1, \dots, x_n] \mid x_i \in \mathbb{R}\}$

$\sqcup_{set}$

$\mathbb{R}^{n-1} \cong \{[0, x_1, \dots, x_n] \mid$ not all x_i are 0 and $[0, \lambda x_1, \dots, \lambda x_n] = [0, x_1, \dots, x_n]\}$

Ex: $\mathbb{RP}^3 = \mathbb{R}^3 \sqcup_{set} \mathbb{RP}^2$

advantages over $\mathbb{R}^3$:

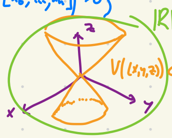
- any two lines meet
- compactified

Rmk: given a homogeneous polynomial f of degree d, $f((\lambda x_0, \dots, \lambda x_n)) = \lambda^d f(x_1, \dots, x_n)$

So well-defined to define

$$V(f) = \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid f([x_0, \dots, x_n]) = 0\}$$

Ex: $f(x,y,z) = x^2 + y^2 + z^2$



$V(x^2 + y^2 - z^2) \subset \mathbb{RP}^2_{x,y,z}$

$= \mathbb{R}^2_{\frac{y}{x}, \frac{z}{x}} \cup \mathbb{RP}^1$

$\{[1, \frac{y}{x}, \frac{z}{x}]\} \quad [0, y, z]$

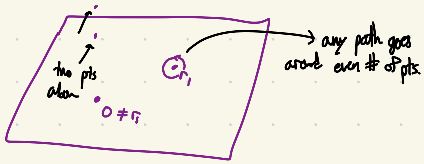
$V(x^2 + y^2 - z^2) \cap \mathbb{R}^2 = \{(\frac{y}{x}, \frac{z}{x}) \mid 1 + (\frac{y}{x})^2 = (\frac{z}{x})^2\}$

$V(x^2 + y^2 - z^2) \cap \mathbb{RP}^1 = \{[0,1,-1], [0,1,1]\}$

Picture: $\mathrm{Spec}\, \mathbb{C}[x,y] / (y^2 - (x - r_1)(x - r_2) \cdots (x - r_n))$

e.g. rational curves, elliptic curves, hyperelliptic top space ⇝ $X = \{(x,y) \in \mathbb{C}^2 \mid y^2 = (x-r_1)\cdots(x-r_n)\}$

$(x,y) \mapsto x \qquad X \to \mathbb{C}$

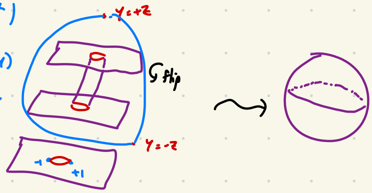


Ex: $V(f) \cap \mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[x,y]/(x^2+y^2-1)$

or $\operatorname{Spec} \mathbb{C}[y,z]/(1+y^2-z^2)$

$\operatorname{Spec} \mathbb{C}[x,y]/(x^2+y^2-1) \qquad (x,y) \mapsto x$

$y^2 = 1 - x^2$



curve X has genus

genus = g

rank $H^1(\mathcal{O}_X) = g$

Def: a scheme is a locally ringed space locally isomorphic to $\operatorname{Spec} A$

requirement that $\mathcal{O}_{X,p}$ be a local ring (unique max ideal)

$(X, \mathcal{O}_X)$ top space + sheaf of rings

Recall: For $\mathcal{O}_A$ structure sheaf of $\operatorname{Spec} A$,

$\mathcal{O}_{A,p} \cong A_p = [(A \setminus p)^{-1}]A$

canonical isomorphisms

Today: morphism day

Remark: Chinese Remainder Theorem

$I_1, \dots, I_j$ comaximal ($I_i + I_j = R$)

$R/(I_1 \cap \cdots \cap I_j) \xrightarrow{\sim} \prod R/I_j$

e.g. $\mathbb{Z}/(24) \cong \mathbb{Z}/2^3 \times \mathbb{Z}/3$

$\mathbb{Z}/(120) \cong \mathbb{Z}/2^3 \times \mathbb{Z}/3 \times \mathbb{Z}/5$ product

Pset 5: affine schemes are opposite category to Ring

$\operatorname{Spec} \mathbb{Z}/(120) = \operatorname{Spec} \mathbb{Z}/2^3 \sqcup \operatorname{Spec} \mathbb{Z}/3 \sqcup \operatorname{Spec} \mathbb{Z}/5$

coprod/disjoint union

A = ring $\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \dots, x_n]$

Ex: $\mathbb{P}^n_A$ is the scheme formed by gluing $n+1$ copies of $\mathbb{A}^n_A$ as follows:

ith copy: $\mathbb{A}^n_i = \operatorname{Spec} A[x_{0/i}, x_{1/i}, \dots, x_{n/i}]/(x_{i/i} - 1)$

Glue $D(x_{j/i})$ to $D(x_{i/j})$

$$D(x_{j/i}) \subset \mathbb{A}^n_i \qquad D(x_{i/j}) \subset \mathbb{A}^n_j$$

$$\operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}][1/x_{j/i}]/(x_{i/i}-1) \cong \operatorname{Spec} A[x_{0/j}, \dots, x_{n/j}][1/x_{i/j}]/(x_{j/j}-1)$$

$$x_{k/i} \mapsto \frac{x_{k/j}}{x_{i/j}}$$

$$\frac{x_{k/i}}{x_{j/i}} \mapsfrom x_{k/j}$$

cocycle condition → cancelling denominators ✓

Proj construction

Def: a $\mathbb{Z}$-graded ring $S_\bullet$ is

$S_\bullet \cong \bigoplus_{n \in \mathbb{Z}} S_n$ of abelian groups

and commutative mult → tensor product

$$S_n \otimes S_m \to S_{n+m}$$

Makes $S_\bullet$ into a commutative ring

- S_n is called the group of homogeneous elements of degree n
- An ideal $I \subset S_\bullet$ is <u>homogeneous</u> if it is generated by homogeneous elements $I = (f_\alpha, \dots \alpha \in J)$, $f_\alpha \in S_{n_\alpha}$

$\Updownarrow$ PS4

$\forall f \in I$ $f = \sum_{n \in \mathbb{Z}} f_n$ is the decomposition of f into its homogeneous parts $\Rightarrow f_n \in I$

PS4: A homogeneous ideal $I \subset S$ is prime $\iff \forall$ homogeneous f, g with $fg \in I$ we have $f \in I$ or $g \in I$

Ex: $S_\bullet = A[x_1,\dots,x_n]$

is a $\mathbb{Z}$-graded ring by declaring each x_i to have degree 1

$\deg a = 0$ for $a \in A$

ex: $(xy+y^3)$ not homogeneous

$(xy+x^2, xy, y^3)$ homogeneous

Def: a $\mathbb{Z}_{\geq 0}$ graded ring is

a $\mathbb{Z}$-graded ring $S_\bullet$ such that

$S_n = 0$ for all $n<0$

Let $S_\bullet$ be a $\mathbb{Z}_{\geq 0}$-graded ring.

We will define $\operatorname{Proj} S_\bullet$ a scheme.

Ex: $\operatorname{Proj} A[x_0,\dots,x_n] \cong \mathbb{P}^n_A$

HW4: $V(f) \subset \mathbb{P}^n_A$

scheme $f \in A[x_0,\dots,x_n]$

homogeneous deg d

$V(f) \cong \operatorname{Proj} A[x_0,\dots,x_n]/(f)$

Define $S_+ = \bigoplus_{n \geq 1} S_n \underset{\text{ideal}}{\subseteq} S_\bullet$

to be the "irrelevant ideal"

$\operatorname{Proj} S_\bullet =$ • set

• top space

• (local) sheaf of rings

Ex! $S_\bullet = \mathbb{C}[x_0,\dots,x_n]$

$S_+ = (x_0,\dots,x_n)$

idea: avoid the origin

The underlying set of $\operatorname{Proj} S_\bullet = \{\mathfrak{p} \subset S \mid \mathfrak{p} \text{ homogeneous prime ideal}, \mathfrak{p} \not\supset S_+\}$

$f \in S_n \quad n \geq 1$

Let $D_+(f) \underset{\text{subset}}{\subseteq} \operatorname{Proj} S_\bullet$

denote $D_+(f) = \{\mathfrak{p} \subset S_\bullet \mid \mathfrak{p} \text{ homogeneous prime ideal of } S_\bullet, \mathfrak{p} \not\ni f, \mathfrak{p} \not\supset S_+\}$

Rmk: $S_\bullet[\frac{1}{f}]$ is a $\mathbb{Z}$-graded ring w/ $\deg \frac{1}{f} := -\deg f$

Prop: There is a canonical bijection between $D_+(f)$ and $\operatorname{Spec}\left(S_\bullet[\frac{1}{f}]\right)_0$

homogeneous deg 0 elements

Pf: follows from lemma

Lemma: let $A_\cdot$ be a $\mathbb{Z}$-graded ring w/ homogeneous invertible element f in positive $\deg n > 0$

Then $A_0 \hookrightarrow A$ induces

$$\operatorname{Spec} A_0 \leftarrow \{\mathfrak{p} \subset A \text{ homogeneous prime}\}$$

bijection

pf: given a prime $\mathfrak{p} \subset A_0$ we will construct a homogeneous prime in A

Let $Q = \bigoplus_m Q_m$

$$Q_n = \left\{ a \in A_n \,\middle|\, \frac{a^n}{f^m} \in \mathfrak{p} \right\}, \quad \frac{a^n}{f^m} = \frac{a^{\deg f}}{f^{\deg a}}$$

- $a \in Q_m \Leftrightarrow a^2 \in Q_{2m}$

$(\Leftarrow)$: $\dfrac{(a^2)^n}{f^{2m}} \in \mathfrak{p}$ since $a^2 \in Q_{2m}$

$= \left(\dfrac{a^n}{f^m}\right)^2$

$\overset{\mathfrak{p} \text{ prime}}{\Rightarrow} \dfrac{a^n}{f^m} \in \mathfrak{p} \Rightarrow a \in Q_m$ ✓

$(\Rightarrow)$: $a, b \in Q_m \Rightarrow a^2 + 2ab + b^2 \in Q_{2m}$

$$\frac{(a+b)^2}{f^{2m}} = \frac{a^{2n}}{f^{2m}} + \frac{\binom{2n}{1} a^{2n-1} b}{f^{2m}} + \cdots + \frac{b^{2n}}{f^{2m}}$$

divisible by $\dfrac{a^n}{f^m}$ divisible by $\dfrac{a^n}{f^m}$ or b^n/f^m

- $a, b \in Q_m \Rightarrow (a+b)^2 \in Q_{2m}$

$\Rightarrow a+b \in Q_m$

$\Rightarrow Q = \bigoplus_m Q_m$ abelian group

$Q = \bigoplus_m Q_m$ ideal:

suffices: for $b \in A_i$ $a \in Q_m$,

$ab \in Q_{i+m}$ $\dfrac{(ab)^n}{f^{i+m}} = \dfrac{a^n}{f^m} \dfrac{b^n}{f^i}$ $\in \mathfrak{p}$

Q prime b/c for homogeneous elements $f_i \in A_i$ $g_2 \in A_j$ s.t. $g_1 g_2 \in Q$, we have

$$\frac{(g_1 g_2)^n}{f^{i+j}} = \left(\frac{g_1^n}{f^i}\right)\left(\frac{g_2^n}{f^j}\right)$$

$\in \mathfrak{p}$ $\Rightarrow$ one of in Q

So Q a homogeneous prime ideal of A

Refresh: $\mathfrak{p} \underset{\text{prime}}{\subset} A_0$

$Q_m = \{ a \in A_m \mid \frac{a^n}{f^m} \in \mathfrak{p} \}$ $\quad Q_0 = \{ a \in A_0 \mid \frac{a^n}{f^0} \in \mathfrak{p} \} = \mathfrak{p}$

$\operatorname{Spec}(S_\bullet[\tfrac{1}{f}])_0 \cong D_+(f) \subset \operatorname{Proj} S_\bullet$

$S_\bullet$ is $\mathbb{Z}_{\geq 0}$-graded ring.

Let T be a set of homogeneous elements of $S_\bullet$

define $V(T) \subset \operatorname{Proj} S_\bullet$ to be

$V(T) = \{ \mathfrak{p} \underset{\text{prime}}{\subset} S_\bullet \mid \mathfrak{p}$ homogeneous prime, $t \in \mathfrak{p}\ \forall t \in T,\ \mathfrak{p} \not\supset S_+ \}$

Lemma: $\{V(T)\}$ forms closed subsets of Zariski topology

$\{ D_+(f) \mid f \in S_i,\ i > 0 \}$ form basis

The map of sets

$\operatorname{Spec} S_\bullet[\tfrac{1}{f}]_0 \to D_+(f)$

induces a homeomorphism of topological spaces.

Now we have $\operatorname{Proj} S_\bullet$ with a Zariski topology

form a structure sheaf $\mathcal{O}_{\operatorname{Proj} S_\bullet}$ by gluing

$(\mathcal{O}_{\operatorname{Spec} S_\bullet[\frac{1}{f}]_0})$ and

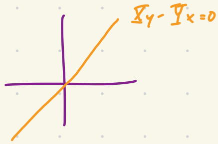
$\mathcal{O}_{\operatorname{Proj} S_\bullet}(D_+(f)) = S_\bullet[\tfrac{1}{f}]_0 \rightsquigarrow$ Scheme

Blowing-up

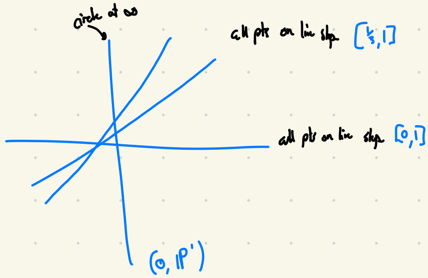
$Bl_0 \mathbb{A}^2 = \{ (p, [\ell]) \mid \ell$ is a line through 0, $p \in \ell \}$

$\subset \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{[X,Y]}$ (exceptional)

$\swarrow$ $= \operatorname{Spec} k[x,y]$ $\qquad \searrow$ $\mathbb{P}^1_{[X,Y]} = \operatorname{Proj} \mathbb{R}[X,Y]$



$Bl_0 \mathbb{A}^2 = V(Xy - Yx) \subset \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{[X,Y]}$



E = exceptional divisor

$$Bl_0 \mathbb{A}^2 = \mathrm{Proj}\left(k[x,y] \oplus (x,y) \oplus (x,y)^2 \oplus \cdots\right)$$

$$Bl_Y X = \mathrm{Proj}\left(\mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots\right)$$

$\mathcal{I}(Y) = \mathcal{I}$

$$\begin{tikzcd} \mathcal{I}^i \otimes \mathcal{I}^j \arrow[r] \arrow[d] & \mathcal{I}^{i+j} \arrow[d] \\ \mathcal{O}_X(x) \times \mathcal{O}_X(x) \arrow[r] & \mathcal{O}_X(x) \end{tikzcd}$$

We have $\mathbb{Z}_{\geq 0}$ graded rings:

$$K[x,y] \oplus (x,y) \oplus (x,y)^2 \oplus \cdots \oplus (x,y)^n \oplus \cdots$$

$$\uparrow \qquad \underset{\mathbb{X}}{\overset{x}{\uparrow}} \quad \underset{\mathbb{Y}}{\overset{y}{\uparrow}}$$

$$\cong \underbrace{K[x,y]}_{\deg 0}\underbrace{[\mathbb{X}, \mathbb{Y}]}_{\deg 1} \Big/ (y\mathbb{X} - x\mathbb{Y})$$

need to understand: products of schemes

$$\mathrm{Proj}\, k[x,y][\mathbb{X}, \mathbb{Y}] \cong \mathbb{P}^1_{k[x,y]} \cong \mathrm{Spec}\, k[x,y] \times \mathbb{P}^1$$
$$\cong \mathbb{A}^2 \times \mathbb{P}^1 \supset V(y\mathbb{X} - x\mathbb{Y})$$

$Y = V(I) \subset X = \mathrm{Spec}\, A$

$I \subset A$ ideal

$Bl_Y X := \mathrm{Proj}(A \oplus I \oplus I^2 \oplus \cdots)$

IOU: the exceptional divisor is $\mathbb{P} N_Y X$

<u>Def</u>: a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces is a continuous map $X \xrightarrow{f} Y$ and $\mathcal{O}_Y \to f_* \mathcal{O}_X$

- morphisms of ringed spaces glue:

  Given $X = \bigcup X_i$ an open cover

  and morphisms $(X_i, \mathcal{O}_{X_i}) \xrightarrow{\varphi_i} (Y, \mathcal{O}_Y)$

  unique restriction of $\mathcal{O}_X$ to $\mathcal{O}_{X_i}$

  s.t.

  $$\varphi_i\big|_{X_i \cap X_j} = \varphi_j\big|_{X_i \cap X_j}$$

there is a unique

$$\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

s.t. $\varphi|_{X_i} = \varphi_i$

Ex: Let $A \xrightarrow{f^\#} B$ be a ring map.

Define $(\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$

by $f : \operatorname{Spec} B \to \operatorname{Spec} A$

$$f(\mathfrak{p}) = (f^\#)^{-1}(\mathfrak{p})$$

Then for all $g \in A$, $f^{-1}(D(g)) = D(f^\#(g))$

Thus

$$\begin{tikzcd}
\mathcal{O}_A(D(g)) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & f_* \mathcal{O}_B(D(g)) \arrow[d, phantom, "\cong" rotate=90] \\
A[\frac{1}{g}] \arrow[rdd, dashed] & \mathcal{O}_B(D(f^\#(g))) \arrow[d, phantom, "\cong" rotate=90] \\
 & \\
 & B[1/f^\#(g)]
\end{tikzcd}$$

universal property

determines a map of sheaves of rings

Possible: define a morphism of schemes $X \xrightarrow{f} Y$ to be a map of ringed spaces s.t. $\exists$ cover of Y by open affines $\operatorname{Spec} A$ and a cover of $f^{-1}(\operatorname{Spec} A)$ by open affines $\operatorname{Spec} B$ s.t. $f|_{\operatorname{Spec} B} : \operatorname{Spec} B \to \operatorname{Spec} A$ is induced by a ring map $A \to B$

Say $(A, \mathfrak{m})$ and $(R, \mathfrak{p})$ are local rings

A map $A \xrightarrow{f^\#} R$ is a map of local rings

if $f^\#(\mathfrak{m}) \subset \mathfrak{p}$

$\cancel{f^\#(A \setminus \mathfrak{m}) \subset R \setminus \mathfrak{p}}$ ✓

Non-example: $K[x]_{(x)} \longrightarrow K(x)$

Def: a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is morphism of ringed spaces s.t. $\forall p \in X$ the induced map $\underbrace{\mathcal{O}_{Y, f(p)} \to (f_* \mathcal{O}_X)_{f(p)}}_{} \to \mathcal{O}_{X,p}$ is a map of local rings?

Def: A map of schemes is a map of locally ringed spaces

Prop: A morphism of locally ringed spaces $\operatorname{Spec} A \to \operatorname{Spec} B$ is the morphism induced by $f^{\#}: \underset{\substack{\| \\ B}}{\mathcal{O}_B(\operatorname{Spec} B)} \to f_* \mathcal{O}_A(\operatorname{Spec} B) = \underset{\substack{\| \\ A}}{\mathcal{O}_A(\operatorname{Spec} A)}$

Rmk: this implies our possible def of morphism of schemes is equivalent to the official def.

Lemma: A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ determines
$k(q) \subseteq k(p)$ $\forall p \mapsto q$ where $\forall p \in X$ $k(p) = \mathcal{O}_{X,p}/\text{max ideal}$

pf: $\mathcal{O}_{Y,p} \to \mathcal{O}_{X,p}$ is a map of local rings

Thus $\underset{\substack{\| \\ k(q)}}{\mathcal{O}_{Y,q}/m_q} \to \underset{\substack{\| \\ k(p)}}{\mathcal{O}_{X,p}/m_p}$

and any map of fields is injective.

pf: (prop) we want to show $\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} B$ $f^{\#}: B \to A$ $\operatorname{Spec}(f^{\#}) = f$

Take $p \in \operatorname{Spec} B$. Then p is the kernel
$p \subset B \to k(p)$
q is the kernel of $A \to k(q)$

sps $f(q) = p$
we have a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f^{\#}"] & k(p) \arrow[d, hook, "f_p^*"] \\ A \arrow[r] & k(q) \end{tikzcd}$$

injection

Thus $q = \ker(B \to k(p)) =$
$\ker(B \xrightarrow{f^{\#}} A \to k(q))$
$= (f^{\#})^{-1}(p)$ thus on topological spaces,

$\operatorname{Spec} f^{\#} = f$

Thus for all $g \in B$,
$f^{-1}(D(g)) = (\operatorname{Spec} f^{\#})^{-1}(D(g))$
$= D(f^{\#}(g))$

Thus $f: \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ B[\frac{1}{g}]}}{\mathcal{O}_B(D(f))} \to \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ A[\frac{1}{f^{\#}g}]}}{f_* \mathcal{O}_A(D(f^{\#}g))}$

Because f is a map of sheaves, restriction induces a commutative diagram from global sections

$$\begin{tikzcd} \mathcal{O}_B(\operatorname{Spec} B) \cong B \arrow[r, "f^\#"] \arrow[d] & A \cong f_* \mathcal{O}_A(f^\# \operatorname{Spec} B) \arrow[d] \\ \mathcal{O}_B(D(g)) \arrow[r] & f_* \mathcal{O}_A(D(g)) \end{tikzcd}$$

$\mathcal{O}_B(D(g)) \cong B[\frac{1}{g}]$ only one such map $f(D(g))$ $\qquad f_*\mathcal{O}_A(D(g)) \cong A[\frac{1}{f^\# g}]$

Thus $f(D(g)) = (\operatorname{Spec} f^\#)(D(g))$

$\Rightarrow f = \operatorname{Spec} f^\#$ as desired $\boxtimes$

X scheme $\quad X(\operatorname{Spec} A) = \operatorname{Mor}_{\text{schemes}}(\operatorname{Spec} A, X)$

$X(\operatorname{Spec} A) = X(A)$

ex: $X = \operatorname{Spec} \mathbb{F}_p[x,y] / (y^2 - x(x-1)(x-2))$

Then $X(\mathbb{F}_p) = \{(x,y) \in \mathbb{F}_p^2 \mid y^2 - x(x-1)(x-2) = 0\}$

$X(A) = \{(x,y) \in A^2 \mid y^2 - x(x-1)(x-2) = 0\}$

$X(\mathbb{F}_{p^2}) = \{(x,y) \in \mathbb{F}_{p^2} \mid y^2 - x(x-1)(x-2) = 0\}$

$\{q \in X \mid k(q) = \mathbb{F}_{p^2}\}$

$\sqcup$

1:1 $\quad \{q \in X \mid k(q) = \mathbb{F}_p\}$

Exercise 4.1.D

A ring

M A-module

$\rightsquigarrow \tilde{M}$ sheaf on $\operatorname{Spec} A$

$\tilde{M}(D(f)) = S^{-1}M \otimes_R M \cong S^{-1}M$

where $S \subset A$ consists of functions not vanishing on $D(f)$

Lemma: $S^{-1}M \cong M_f = M \otimes_R A_f$

This is a sheaf because for any cover $\operatorname{Spec} A = \bigcup_{i=1}^n D(f_i)$

$0 \to M \to M_{f_1} \times \cdots \times M_{f_n} \to \prod_{i \neq j} M_{f_i f_j}$

Def (6.1.1) A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a scheme X is quasi-coherent if for any $\operatorname{Spec} A \underset{\substack{\text{open}\\\text{subscheme}}}{\subset} X$, the restriction $\mathcal{F}|_{\operatorname{Spec} A}$

map $\mathcal{O}_X \times \mathcal{F} \xrightarrow{\mu} \mathcal{F}$ bilinear

$$\begin{tikzcd} \mathcal{O}_X \times \mathcal{O}_X \times \mathcal{F} \arrow[r, "1\times\mu"] \arrow[d, "\mu\otimes 1"] & \mathcal{O}_X \times \mathcal{F} \arrow[d, "\mu"] \\ \mathcal{O}_X \times \mathcal{F} \arrow[r, "\mu"] & \mathcal{F} \end{tikzcd}$$

$\Longleftrightarrow$ $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and restriction maps are Module maps is isomorphic to $\tilde{M}_A$ for some A-module M_A.

Rmk $\mathrm{QCoh}(X)$ is the (abelian) category whose objects are quasi-coherent sheaves and whose morphisms are sheaves of $\mathcal{O}_X$-modules

Goal: make the algebro-geometric analogue of the following line bundles as quasi-coherent sheaves:

$$\begin{array}{ccc} \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C} & \supset & U \times \mathbb{C} \\ \downarrow & & \\ \mathbb{P}^n_{\mathbb{C}} & \supset & U \ \text{line bundle} \end{array}$$

$V(x^2+y^2-z^2) \subset \mathbb{P}^2_{x,y,z} = \mathrm{Proj}\ \mathbb{C}[x,y,z]$

this should ~~be a~~ function

is a section of q.c. sheaf locally isomorphic to $\mathcal{O}$ → invertible q.c. sheaves = locally free of rank 1

→ aka a line bundle

$$\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^{n+1} - \{0\} \Big/ (x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n) \ \forall \lambda \in \mathbb{C}^*$$

$$\mathcal{O}(m) = (\mathbb{C}^{n+1} - \{0\}) \times \mathbb{C} \Big/ (x_0, \dots, x_n) \times y \sim (\lambda x_0, \dots, \lambda x_n) \times \lambda^m y \quad \forall \lambda \in \mathbb{C}^*$$

$$\downarrow$$

$$\mathbb{P}^n(\mathbb{C})$$

sending $(x_0, \dots, x_n) \mapsto (x_0, \dots, x_n) \times f(x_0, \dots, x_n)$

$$\begin{aligned} (\lambda x_0, \dots, \lambda x_n) &\mapsto (\lambda x_0, \dots, \lambda x_n) \times f(\lambda x_0, \dots, \lambda x_n) \\ &= \lambda^m f(x_0, \dots, x_n) \end{aligned}$$

gives a well-defined map $\mathbb{P}^n(\mathbb{C}) \to \mathcal{O}(m)$ for any homogeneous poly f.

Thm (6.1.2) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X is q.c. if there exists a cover of X by open subschemes $\mathrm{Spec}\, A \subset X$ s.t. $\mathcal{F}|_{\mathrm{Spec}\, A} \cong \tilde{M}$ for some A-module M.

Lemma: (5.3.1) Sps $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, B$ are open subschemes of a scheme X. Then $\mathrm{Spec}\, A \cap \mathrm{Spec}\, B$ can be covered by open affines $\mathrm{Spec}\, C$, which are distinguished open affines in both.

$$\begin{array}{c} \mathrm{Spec}\, C = D_A(f) \cong \mathrm{Spec}\, A_f \subset \mathrm{Spec}\, A \\ \| \\ D_B(f) \\ \wr\wr \\ \mathrm{Spec}\, B_g \\ \cap \\ \mathrm{Spec}\, B \end{array}$$

pf: Let $p \in \mathrm{Spec}\, A \cap \mathrm{Spec}\, B$. Since distinguished affine opens form a basis in the Zariski topology, we can find

$p \in \mathrm{Spec}\, A_f \subset \mathrm{Spec}\, A \cap \mathrm{Spec}\, B$

and likewise

$p \in \mathrm{Spec}\, B_g \subset \mathrm{Spec}\, A_f$

$$g \in \mathcal{O}_X(\mathrm{Spec}\, B) \longrightarrow \mathcal{O}_X(\mathrm{Spec}\, A_f) = \frac{g'}{f^n}$$

with $g' \in A$

Thus $\mathrm{Spec}\, B_g = D_B(g) = D_{A_f}\left(\frac{g'}{f^n}\right)$
$= D_{A_f}(g') = D_A(fg')$

Thus $\mathrm{Spec}\, B_g$ is a distinguished open affine of both $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, B$ ⊠

Affine Communication Lemma (5.3.2) Let P be a property enjoyed by some subset of the open affines $\mathrm{Spec}\, A \subset X$.

Sps P satisfies:

(1) If $\mathrm{Spec}\, A$ satisfies P, so do all the distinguished open affines $D_A(f) = \mathrm{Spec}\, A_f \subset \mathrm{Spec}\, A$

(2) If $\mathrm{Spec}\, A = \bigcup_{i=1}^n \mathrm{Spec}\, A_{f_i}$ and $\mathrm{Spec}\, A_{f_i}$ satisfies P, then $\mathrm{Spec}\, A$ satisfies P.

Then sps $X = \bigcup_{i \in I} \mathrm{Spec}\, A_i$ and $\mathrm{Spec}\, A_i$ satisfies P.

Then $\forall$ open $\mathrm{Spec}\, B \subset X$, $\mathrm{Spec}\, B$ satisfies P.

Pf: By lemma 5.3.1, $\exists$ cover $\mathrm{Spec}\, B = \bigcup_{i \in J} \mathrm{Spec}\, B_{f_i}$ with $\mathrm{Spec}\, B_{f_i}$ simultaneously distinguished in both $\mathrm{Spec}\, A_{i'(i)}$ and $\mathrm{Spec}\, B$. We can assume J finite:

Lemma: for any rings A, $\mathrm{Spec}\, A$ is quasi-compact.

pf: $U_i = \mathrm{Spec}\, A - V(I_i)$

Then $\bigcap_{i\in J} V(I_i) = \emptyset$

$\iff \sum_{i\in J} I_i = (1)$

$\iff \exists\, f_1,\dots,f_n$ with f_i in some I_{j_i}

and $\sum_{i=1}^{n} f_i = 1 \Rightarrow U_{j_1}\cup\dots\cup U_{j_n} = \operatorname{Spec} A$ $\boxtimes$

So $\operatorname{Spec} B_{f_i}$ has property p by (1), and by (2), $\operatorname{Spec} B$ has property p $\boxtimes$

To prove the thm that if $\mathcal{F}|_{\operatorname{Spec} A_i}$ for some cover $X = \bigcup_{i\in I}\operatorname{Spec} A_i$ is q.c., then $\mathcal{F}$ is q.c., it remains to show that the property p of open $\operatorname{Spec} B\subset X$ having $\mathcal{F}|_{\operatorname{Spec} B}\cong \widetilde{M}_B$ satisfies (1) and (2).

(1): $\mathcal{F}|_{\operatorname{Spec} B} \cong \widehat{M}_B$

So $\mathcal{F}|_{\operatorname{Spec} B_f} \cong \widetilde{M}_B|_{\operatorname{Spec} B_f}$

$\cong \widetilde{M_B[\frac{1}{f}]}$

$D_B(g) = D_{B_f}(fg')$

$\widetilde{M}_B(D_B(g)) = \widetilde{M}_B[\frac{1}{gf}]$

(2): $\mathcal{F}|_{\operatorname{Spec} B_{f_i}} \cong \widetilde{M}_i$

where M_i is a B_{f_i}-module.

Define the B-module M as in the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & \prod_{i=1}^{n} M_i \arrow[r] \arrow[dr, "\pi_i-\phi_{ij}^{-1}\pi_j"'] & \prod_{i\neq j} M_i[\frac{1}{f_j}] \arrow[d] \\ & & & M_i[\frac{1}{f_j}] \end{tikzcd}$$

$$\begin{tikzcd} & \mathcal{F}|_{\operatorname{Spec} B_{f_i}\cap \operatorname{Spec} B_{f_j}} \arrow[dl] \arrow[dr] & \\ (M_i)[\frac{1}{f_j}] \cong (\mathcal{F}|_{\operatorname{Spec} B_{f_i}})|_{\operatorname{Spec} B_{f_j}} \arrow[rr, "\cong"', "\phi_{ij}"] & & M_j[\frac{1}{f_i}] \end{tikzcd}$$

Show that there is a canonical map

$\widetilde{M} \to \mathcal{F}|_{\operatorname{Spec} B}$

and that the canonical map is an isomorphism $\boxtimes$

$\mathbb{P}^n_A$ covered by the open affines

$\operatorname{Spec} A[x_{0/i},\dots,x_{n/i}]/(x_{i/i}-1) = D_+(x_i)$

We construct a q.c. sheaf $\mathcal{O}(m)$ on $\mathbb{P}^n_A$

$\mathcal{O}(m)|_{D_+(x_i)} \cong \mathcal{O}|_{D_+(x_i)}$

on $D_+(x_i) \cap D_+(x_j)$ we glue via isomorphisms ϕ_{ij} satisfying cocycle condition

"polys of degree m in homogeneous coordinates
$= \underbrace{\text{poly of degree } 0}_{f} \cdot (x_i)^m = f\, x_j^m \left(\frac{x_i^m}{x_j^m}\right) = (x_j^m)\left(f \left(\frac{x_i}{x_j}\right)^m\right)$

$f \longmapsto f \cdot (x_{i/j})^m$

$$\begin{tikzcd} A[x_{0/i}, \dots, x_{n/i}]/(x_{j/i}-1)\left[\frac{1}{x_{j/i}}\right] \arrow[r, bend left, "\phi_{ij}"] & A[x_{0/j}, \dots, x_{n/j}]/(x_{i/j}-1)\left[\frac{1}{x_{i/j}}\right] \arrow[l, bend left, "\phi_{ji}"] \end{tikzcd}$$

Last time: constructed a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules on $\mathbb{P}^n$, which is <u>quasi-coherent</u> in the sense that when we have $\operatorname{Spec} A \subset \mathbb{P}^n$, $\mathcal{O}(m)|_{\operatorname{Spec} A} \cong \widetilde{M}_A$ for some A-module M_A. In fact, $\mathcal{O}(m)$ is locally free of rank 1

<u>Def</u> A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a scheme X is locally free of rank n if there exists an open cover of open affines $\operatorname{Spec} A \subset X$ s.t. $\mathcal{F}|_{\operatorname{Spec} A} \cong \widetilde{A}^n$

- $\mathcal{O}(m)$ constructed using open cover $\{D_+(x_i) \mid i=0,\dots,n\}$ of $\operatorname{Proj} \underset{\text{ring}}{k}[x_0,\dots,x_n]$

  $D_+(x_i) \cong \operatorname{Spec} k[x_{0/i},\dots,x_{n/i}]/(x_{i/i}-1)$

- $\mathcal{O}(m)|_{D_+(x_i)} \xleftarrow[\varphi_i]{\cong} \mathcal{O}_{D_+(x_i)}$

$$\begin{tikzcd} \mathcal{O}_{D_+(x_i)\cap D_+(x_j)} \arrow[r, "\varphi_i"] \arrow[dr, "\varphi_j^{-1}\circ\varphi_i|_{D_+(x_i)\cap D_+(x_j)}", "(x_{i/j})^m"'] & \mathcal{O}(m)|_{D_+(x_i)\cap D_+(x_j)} \arrow[d, "\varphi_j^{-1}"] \\ & \mathcal{O}_{D_+(x_i)\cap D_+(x_j)} \end{tikzcd}$$

automorphism

morphism of free modules of rank 1

Note! b/c $\mathcal{O}$ is q.c., we have $\mathcal{O}_{D_+(x_i)\cap D_+(x_j)} = \widetilde{\mathcal{O}(D_+(x_i)\cap D_+(x_j))}$

$\mathcal{O}(D_+(x_i)\cap D_+(x_j)) =: A_{ij}$

$\Rightarrow$ $\varphi_j^{-1}|\circ\varphi_i|_{D_+(x_i)\cap D_+(x_j)}$ is determined by the map on sections $D_+(x_i)\cap D_+(x_j)$ which is a map $A_{ij}\to A_{ij}$ which is an A_{ij}-module homomorphism

- Similarly, constructing locally free sheaves of rank n on a cover $\{U_i \mid i\in I\}$ is to give $n\times n$ invertible matrices
$$\varphi_{ij}\in GL_n(\mathcal{O}_X(U_i\cap U_j))$$
s.t.
$$\varphi_{jk}|\circ\varphi_{ij}| = \varphi_{ik}|_{U_i\cap U_j\cap U_k} \text{ in } GL_n(\mathcal{O}_X(U_i\cap U_j\cap U_k))$$

$\mathcal{O}(m)(D_+(x_i)\cap D_+(x_j))$

is a free module of rank 1 over

$\mathcal{O}(D_+(x_i)\cap D_+(x_j))$

A "gluing-free" description:
$$\mathcal{O}(m)(D_+(x_i)\cap D_+(x_j)) \cong \left(k[x_0,\dots,x_n]\left[\frac{1}{x_ix_j}\right]\right)_m \text{ degree } m \text{ elements}$$
$\varphi_i(1) = x_i^m$ ← choice of basis

- Given sheaves $\mathcal{F}, \mathcal{G}$ of $\mathcal{O}_X$-modules on a scheme X,
$$\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$$
is the sheaf of $\mathcal{O}_X$-modules given by sheafification of
$U\mapsto \mathcal{F}(U)\otimes_{\mathcal{O}(U)}\mathcal{G}(U)$

If $\mathcal{F}$ and $\mathcal{G}$ are q.c. and $\operatorname{Spec} A\subset X$ is an open affine, then
$$\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}|_{\operatorname{Spec} A} = \widetilde{M\otimes_A N}$$
where $\mathcal{F}|_{\operatorname{Spec} A}\cong\widetilde{M}$
and $\mathcal{G}|_{\operatorname{Spec} A}\cong\widetilde{N}$

pf sketch: Define product on distinguished affine basis by
$$U\mapsto \mathcal{F}(U)\otimes_{\mathcal{O}(U)}\mathcal{G}(U)$$
This is a sheaf because
$$(M\otimes_A N)\left[\tfrac{1}{f}\right]\cong M\left[\tfrac{1}{f}\right]\otimes_{A[\frac{1}{f}]}N\left[\tfrac{1}{f}\right]$$
Invoke equivalence of sheaves on distinguished affine base and sheaves on X $\blacksquare$

- Claim $\mathcal{O}_{\mathbb{P}^n}(m) \otimes \mathcal{O}_{\mathbb{P}^n}(n) \cong \mathcal{O}_{\mathbb{P}^n}(m+n)$

Pf: transition functions for $\mathcal{O}(m) \otimes \mathcal{O}(n)$ is

$$(x_i/x_j)^m (x_i/x_j)^n = (x_i/x_j)^{m+n} \quad \boxed{}$$

Closed Immersions

Def: A morphism $f: X \to Y$ of schemes is called a closed immersion (or closed embedding) if for every $\operatorname{Spec} A \subseteq_{\text{open}} Y$, $f^{-1}(\operatorname{Spec} A)$ is affine $f^{-1}(\operatorname{Spec} A) \cong \operatorname{Spec} B$ and the induced map $A \to B$ is surjective

Lemma: It suffices to check that Y can be covered by open affine $\operatorname{Spec} A$ with the given property, p.

pf: By the affine communication lemma, it suffices to check:

(1) if p holds for $\operatorname{Spec} A$, then p holds for $\operatorname{Spec} A[\frac{1}{f}] \cong D_A(f)$

(2) If $\operatorname{Spec} A$ is covered by $D_A(f_1), \dots, D_A(f_n)$ $f_1, \dots, f_n \in A$, and $D_A(f_i)$ has property p, then $\operatorname{Spec} A$ has property p.

(1): By assumption, $i|_{i^{-1}(\operatorname{Spec} A)}$ is the map $\operatorname{Spec} B \to \operatorname{Spec} A$ induced by a surjection of rings $A \twoheadrightarrow B$

Since localization is exact,

$$A[\tfrac{1}{f}] \longrightarrow B[\tfrac{1}{i^*(f)}] \cong B \otimes_A A[\tfrac{1}{f}]$$

is surjective. Then $i^{-1}(D_A(f)) = \{\mathfrak{p} \subset B \mid \text{prime s.t. } (i^*)^{-1}(\mathfrak{p}) \not\ni f\}$

$$\underset{\text{canonical bijection}}{\leadsto} = \{\mathfrak{p} \subset B[\tfrac{1}{i^* f}]\} = D_B(i^* f)$$

commutative algebra!

1) given $0 \to M \to M' \to M'' \to 0$ exact sequence of A-modules
$0 \to M_f \to M'_f \to M''_f \to 0$ also exact

2) $B \xrightarrow{i} B[\frac{1}{g}]$ induces a bijection $i^{-1}(\mathfrak{p}) \leftarrow \mathfrak{p}$

$D_B(g) = \operatorname{Spec} B[\frac{1}{g}]$

2) Now we suppose $i^{-1}(D_A(f_i)) \xrightarrow{i} D_A(f_i)$ By hypothesis the map is identified with Spec of a surjection of rings, $B_j \overset{i_j}{\twoheadleftarrow} A[\frac{1}{f_i}]$

Then we have an exact sequence

$$\begin{tikzcd} A \arrow[r] & \prod_{j=1}^n A_{f_j} \arrow[r] \arrow[d] & \prod_{j<l} A_{f_j f_l} \arrow[d] \\ & \prod_{j=1}^n B_j \arrow[r, "\rho"] & \prod_{j<l} B_j\left[\frac{1}{i_j^* f_l}\right] \end{tikzcd}$$

$$i^{-1}(D_A(f_j f_l)) \cong \operatorname{Spec} B_j\left[\frac{1}{i_j^* f_l}\right]$$
$$\cong \operatorname{Spec} B_l\left[\frac{1}{i_l^* f_j}\right]$$

$\Rightarrow$ have isos

$$B_j\left[\frac{1}{i_j^* f_l}\right] \cong B_l\left[\frac{1}{i_l^* f_j}\right]$$

Satisfying cocycle cond.

Let $B = \ker \rho$. Then

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, dashed, "g"] & \prod_{j=1}^n A_{f_j} \arrow[r] \arrow[d] & \prod_{j<l} A_{f_j f_l} \arrow[d] \\ 0 \arrow[r] & B \arrow[r] & \prod_{j=1}^n B_j \arrow[r, "\rho"] & \prod_{j<l} B_j\left[\frac{1}{i_j^* f_l}\right] \end{tikzcd}$$

snake lemma (ish)

We have $g: A \to B$. Claim g surjective

6OZ aside:

M module over ring A,

Sps. cover $\bigcup_{i=1}^n D(f_i) = \operatorname{Spec} A$

Then $M = 0 \iff M_{f_i} = 0 \quad \forall i = 1, \dots, n$

Pf: $(\Rightarrow)$ ✓

$(\Leftarrow)$ Take $m \in M$

Since $M_{f_i} = 0$, $\exists\, n_i$ s.t. $f_i^{n_i} m = 0$

$D(f_i^{n_i}) = D(f_i)$, so $\bigcup_{i=1}^n D(f_i^{n_i}) = \operatorname{Spec} A$

Thus $\emptyset = \bigcap_{i=1}^n V(f_i^{n_i}) = V(f_1^{n_1}, \dots, f_n^{n_n})$

$\Rightarrow (f_1^{n_1}, \dots, f_n^{n_n}) = (1)$

$\Rightarrow \exists\, a_i \in A$ s.t. $\sum a_i f_i^{n_i} = 1$

$\Rightarrow$ In M, $\quad m = \sum a_i f_i^{n_i} \cdot 1$
$= \sum a_i (f_i^{n_i} m)$
$= 0$ ☒

So if $g: M \to N$ morphism of modules,

g injective $\iff g_{f_i}$ $\binom{\text{injective}}{\text{surjective}}$ for all i

Pf: $M \xrightarrow{g} N \to \operatorname{coker}(g) \to 0$

Since localization exact,

$\operatorname{coker}(g_{f_i}) = (\operatorname{coker} g)_{f_i}$

By previous lemma, $\operatorname{coker} g = 0 \iff \operatorname{coker}(g_{f_i}) = 0 \ \forall i$

For injectivity, replace $0 \to \ker(g) \to M \xrightarrow{g} N$ $\boxtimes$

Def A morphism of schemes $f: X \to Y$ is a closed immersion if for every affine open $\operatorname{Spec} B \subset Y$, property P holds, where

P: $f^{-1}(\operatorname{Spec} B)$ is an open affine $\operatorname{Spec} A$ and the induced map of rings

$B \to A$

is surjective.

Thm: it suffices to check property P on an open affine cover of Y.

Pf (finished): What remains to show was that for $Y \supset \operatorname{Spec} B = \bigcup_{i=1}^{n} \operatorname{Spec} D_B(f_i)$ such that $\operatorname{Spec} D_B(f_i)$ satisfies P then $\operatorname{Spec} B$ satisfies P

By assumption, $f^{-1}(D_B(f_i)) \cong \operatorname{Spec} A_i$. Since $(A_i)_{f_j} \cong \mathcal{O}_X(f^{-1}(D_B(f_i f_j))) = (A_j)_{f_i}$ we have a canonical iso $(A_i)_{f_j} \cong (A_j)_{f_i}$

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, dashed, "\varphi"] & \prod B_i \arrow[r] \arrow[d] & \prod_{i<j} B_{f_i f_j} \arrow[d] \\
0 \arrow[r] & A \arrow[r] & \prod A_i \arrow[r, "g"] & \prod_{i<j} (A_i)_{f_j}
\end{tikzcd}$$

$A = \ker g$

we claim φ surjective. It suffices to see φ_{f_i} is surjective $\forall i$ by lemma.

For this, it suffices to see $A_{f_i} \xrightarrow{\sim} A_i$ is an iso

Since $\mathcal{O}_Y$ is a q.c. sheaf and $f^{-1}(D(f_i))$ is a cover of $f^{-1}(\operatorname{Spec} B)$

$A \cong \mathcal{O}_X(f^{-1}(\operatorname{Spec} B))$

Since $\mathcal{O}_X$ q.c., canonical map

$$\begin{tikzcd}
\mathcal{O}_X(f^{-1}(\operatorname{Spec} B)) \arrow[r] \arrow[dr] & \mathcal{O}_X(f^{-1}(\operatorname{Spec} B)) \cap \{f^* f_i \neq 0\} \\
& \mathcal{O}_X(f^{-1}(\operatorname{Spec} B))\left[\frac{1}{f_i}\right] \arrow[u, "\sim"']
\end{tikzcd}$$

$f : X \to Y$

$f^{-1}(\operatorname{Spec} B) = \bigcup_{i=1}^{n} \operatorname{Spec} A_i$

and $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$

$= f^{-1}(D_B(f_i f_j))$

$= \operatorname{Spec} A_i\left[\frac{1}{f_j}\right] \cong \operatorname{Spec} A_j\left[\frac{1}{f_i}\right]$

Take any $g \in B$.

We claim that the canonical map

$$\begin{tikzcd}
\mathcal{O}_X(f^{-1}(\operatorname{Spec} B)) \arrow[r] \arrow[dr] & \mathcal{O}_X(f^{-1}(\operatorname{Spec} B)) \cap \{f^* g \neq 0\} \\
& \mathcal{O}_X(f^{-1}(\operatorname{Spec} B))\left[\frac{1}{f^* g}\right] \arrow[u, "\cong"']
\end{tikzcd}$$

Taking $g = f_i$ finishes proof. $\mathcal{O}_X(f^{-1}(\operatorname{Spec} B[\frac{1}{f_i}])) = \mathcal{O}_X(f^{-1}(\operatorname{Spec} B)) \cap \{f_i \neq 0\}$

Still exact b/c localization is

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(f^{-1}(\operatorname{Spec} B))\left[\frac{1}{f^* g}\right] \arrow[r] \arrow[d, "\cong"] & \prod A_i\left[\frac{1}{f^* g}\right] \arrow[r] \arrow[d, "\cong"] \arrow[l, bend right, "\text{implies}"'] & \prod (A_i)_j\left[\frac{1}{f^* g}\right] \arrow[d] \\
& \mathcal{O}_X(f^{-1}(\operatorname{Spec} B)) \cap \{f^* g \neq 0\} \arrow[r] & \prod \mathcal{O}_X(f^{-1}(\operatorname{Spec} B[\frac{1}{f_i}])) \cap \{f^* g \neq 0\} \arrow[r] & \prod \mathcal{O}_X(f^{-1}(\operatorname{Spec} B[\frac{1}{f_i f_j}])) \cap \{f^* g \neq 0\}
\end{tikzcd}$$

6.2.D b/c $\operatorname{Spec} A_i$ is affine and $\mathcal{O}_X$ q.c.

Thus we have $B \to A$ surjection. We need to show that $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$.

We have $\mathcal{O}_X(f^{-1}(\operatorname{Spec} B)) \cong A$. The canonical map $\mathcal{O}_X(f^{-1}(\operatorname{Spec} B)) \to A$ produces (cf. psets)

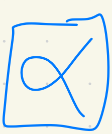
$f^{-1}(\operatorname{Spec} B) \to \operatorname{Spec} A$

When restricted to $f^{-1}(\operatorname{Spec} D_B(f_i))$ this is an iso. Since it is an iso on an open cover, it is an isomorphism.

<u>ex</u>: $\operatorname{Proj} k[x,y,z]/(y^2 z - x^3 - x^2 z)$

$\hookrightarrow \mathbb{P}^2_k$

$\operatorname{Spec} k[x,y]/(y^2 - x^3 - x^2) \hookrightarrow \mathbb{A}^2_k$



Lemma: if $Z \xrightarrow{f} X$ and $X \xrightarrow{g} Y$ are closed immersions, then $Z \xrightarrow{g\circ f} Y$ is a closed immersion.

Pf: Let $\operatorname{Spec} B \subset Y$ be an open affine. $g^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} C$ with $B \twoheadrightarrow C$. $f^{-1}(\operatorname{Spec} C) \cong \operatorname{Spec} A$ with $C \twoheadrightarrow A$ thus $(g\circ f)^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ with $B \twoheadrightarrow A$ $\boxtimes$

Let $f: X \hookrightarrow Y$ be a closed immersion

$$0 \to \underset{\substack{\parallel \\ \ker f^\#}}{\mathcal{I}} \to \mathcal{O}_Y \overset{f^\#}{\twoheadrightarrow} f_*\mathcal{O}_X \to 0$$

Lemma: $\mathcal{I}$ is a sheaf of ideals of $\mathcal{O}_Y$.

Pf: We are supposed to check that for every open affine $\operatorname{Spec} B$ of Y and $g \in B$, the canonical map

$$\begin{tikzcd} \mathcal{I}(\operatorname{Spec} B) \arrow[r] \arrow[dr] & \mathcal{I}(D_B(g)) \\ & \mathcal{I}(\operatorname{Spec} B)[\tfrac{1}{g}] \arrow[u, dashed, "?"] \end{tikzcd}$$

By exact sequence diagram,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}(\operatorname{Spec} B)[\tfrac{1}{g}] \arrow[r] \arrow[d, "f_0"] & \mathcal{O}_Y(\operatorname{Spec} B)[\tfrac{1}{g}] \arrow[r] \arrow[d, "f_1"] & f_*\mathcal{O}_X(\operatorname{Spec} B)[\tfrac{1}{g}] \arrow[d, "f_2"] \\ 0 \arrow[r] & \mathcal{I}(D_B(g)) \arrow[r] & \mathcal{O}_Y(D_B(g)) \arrow[r] & f_*\mathcal{O}_X(D_g(B)) \end{tikzcd}$$

remains exact, commutative b/c localization is

Since $f_*\mathcal{O}_X$ and $\mathcal{O}_Y$ are q.c., f_1 and f_2 are isos, thus f_0 is also an iso.

So $\mathcal{I}$ is quasi-coherent. $\boxtimes$

Ex: Let $f \in k[x_0, \dots, x_n]$ be homogeneous of degree d.

Let $X = \operatorname{Proj} k[x_0,\dots,x_n]/(f)$

and $\iota: X \hookrightarrow \mathbb{P}^n$

the corresponding closed immersion is

$$0 \to \underset{\substack{\cong \\ \mathcal{O}(-d)}}{\mathcal{I}} \xrightarrow{f} \mathcal{O}_{\mathbb{P}^n} \to \iota_*\mathcal{O}_X \to 0$$

multiplication by poly f

Exercise 9.1.f: Let Y be a scheme. Given a q.c. sheaf of ideals $\mathcal{I}$ on Y, there is a canonical closed immersion

$$X \overset{i}{\hookrightarrow} Y$$

with $0 \to \mathcal{I} \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0$

rmk on proof: for $\operatorname{Spec} B \subset Y$ open affine,

Let $A = B/\mathcal{I}(\operatorname{Spec} B)$.

Glue $\operatorname{Spec} A$'s together..

$X = \operatorname{Spec} \mathcal{O}_Y/\mathcal{I}$

Let $\mathcal{L}$ be a q.c. sheaf on a scheme X and let $s: \mathcal{O} \to \mathcal{L}$ be a map.

Then there is $\mathcal{L}^\vee = \underline{\mathcal{H}om}(\mathcal{L}, \mathcal{O})$ ← sheaf hom; view as sheaf

sheaf on X defined

$\underline{\mathcal{H}om}(\mathcal{L}, \mathcal{O})(\operatorname{Spec} B) = \operatorname{Hom}_B(\mathcal{L}(\operatorname{Spec} B), B)$

$\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}$

(the gluing functions are inverse)

$s^\vee : \mathcal{L}^\vee \to \mathcal{O}^\vee \cong \mathcal{O}$

Let $\mathcal{I}$ be the image sheaf.

Then $V(s) \hookrightarrow X$ is the corresponding closed immersion

"vanishing locus of section s of a line bundle $\mathcal{L}$"

Let X scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ a quasi-coherent sheaf of ideals.

Exercise 9.1.F: There is a canonical closed immersion

$$Y \overset{f}{\hookrightarrow} X$$

s.t.

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[dr] & f_* \mathcal{O}_Y \arrow[d, equal, "\wr"] \\ & \mathcal{O}_X/\mathcal{I} \end{tikzcd}$$

Pf: Let the underlying top space of Y be the subspace of X consisting of $x \in X$ s.t.

$\forall U \ni x$ open and all $g \in \mathcal{I}(U)$ we have

$g(x) = 0$

(equivalently, for any open affine nbhd $U = \operatorname{Spec} A$ of X, we have $\mathcal{I}(U) \subset \mathfrak{p}_x$, where $\mathfrak{p}_x \subset A$ is the prime ideal corresponding to X)

In particular, $Y \cap \operatorname{Spec} A = V(I(U)) \overset{\text{homeomorphism}}{\cong} \operatorname{Spec} A/I_A \Rightarrow Y$ is closed.

Define $\mathcal{O}_Y$ by gluing sheaves on open cover $\operatorname{Spec} A \cap Y$ as $\operatorname{Spec} Y$ ranges over open affines of X.

On $(\operatorname{Spec} A \cap Y)$ we have sheaf

$I_A := I(\operatorname{Spec} A)$ $\quad \widetilde{A/I}$ on $\operatorname{Spec} A/I_A \cong Y \cap \operatorname{Spec} A$

We claim the $\widetilde{A/I_A}$ glue. For this, we cover $\operatorname{Spec} A \cap \operatorname{Spec} B$ by opens $\operatorname{Spec} C = D_A(g_A) = D_B(g_B)$ which are distinguished open affines in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$ (proved possible right before proving affine communication lemma)

<u>Recall</u>: given maps $\mathcal{F}|_{U_i} \xrightarrow{\varphi_i} \mathcal{G}|_{U_i}$

s.t. $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$

there is a unique $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$

So we will 1) construct isos $\widetilde{A/I_A}|_{\operatorname{Spec} C} \xrightarrow{\varphi_C} \widetilde{B/I_B}|_{\operatorname{Spec} C}$

s.t. $\varphi_C|_{\operatorname{Spec} C \cap \operatorname{Spec} C'} = \varphi_{C'}|_{\operatorname{Spec} C \cap \operatorname{Spec} C'}$

and this will determine a unique isomorphism

$$\widetilde{A/I_A}\Big|_{\operatorname{Spec} A \cap \operatorname{Spec} B} \xrightarrow{\varphi_{AB}} \widetilde{B/I_B}\Big|_{\operatorname{Spec} A \cap \operatorname{Spec} B}$$

2) show $\varphi_{BE}| \circ \varphi_{AB}|_{\operatorname{Spec} A \cap \operatorname{Spec} B \cap \operatorname{Spec} E} = \varphi_{AE}|$

$$\widetilde{A/I_A}\Big|_{\operatorname{Spec} C = D_A(g_A)} = \widetilde{A[\tfrac{1}{g_A}] / I_A[\tfrac{1}{g_A}]}$$

Since I and $\mathcal{O}_X$ are quasicoherent,

$$\widetilde{\frac{A[\frac{1}{g_A}]}{I[\frac{1}{g_A}]}} \cong \widetilde{\frac{\mathcal{O}_X(\operatorname{Spec} C)}{I(\operatorname{Spec} C)}} \qquad \operatorname{Spec} A[\tfrac{1}{g_A}] \cong \operatorname{Spec} \mathcal{O}_X(\operatorname{Spec} C) \cong \operatorname{Spec} B[\tfrac{1}{g_B}]$$

$$\widetilde{\frac{B[\frac{1}{g_B}]}{I[\frac{1}{g_B}]}} \cong \widetilde{B/I_B}\Big|_{\operatorname{Spec} C}$$

φ_C

To see that

$$\varphi_C\big|_{\operatorname{Spec} C \cap \operatorname{Spec} C'} = \varphi_{C'}\big|_{\operatorname{Spec} C \cap \operatorname{Spec} C'}$$

note that φ_C is the map on sheaves on $\operatorname{Spec} \mathcal{O}_X(\operatorname{Spec} C) \cong \operatorname{Spec} C$

Induced by module isomorphism

$$\begin{tikzcd} \mathcal{O}_X(\operatorname{Spec} A) \arrow[r, "\text{res}"] \arrow[dr] & \mathcal{O}_X(\operatorname{Spec} C) \cong \mathcal{O}_A(g_A) \\ & \mathcal{O}_X(\operatorname{Spec} A)\left[\frac{1}{g_A}\right] = A\left[\frac{1}{g_A}\right] \arrow[u, dashed, "\cong"'] \end{tikzcd}$$

iso

$$\widetilde{\varphi_B^{-1}\varphi_A} = \varphi_C$$

$\Rightarrow$ $\varphi_C\big|_{\operatorname{Spec} C \cap \operatorname{Spec} C'} = \varphi_{C'}\big|_{\operatorname{Spec} C \cap \operatorname{Spec} C'}$

+ cocycle.

Rmk: Same procedure produces $\underline{\operatorname{Spec}}\, \mathcal{F} \to X$ for any q.c. sheaf of $\mathcal{O}_X$-algebras $\mathcal{F}$

Thm: $\operatorname{Qcoh}(X)$ is an abelian category

Thm: $\operatorname{Qcoh}(\operatorname{Spec} A) \cong A\text{-mod}$

$\longrightarrow$

Γ global sections over $\operatorname{Spec} A$

(co) normal sheaves to closed immersions (chapter 21)

Picture: $Y \hookrightarrow X$ embedding of smooth manifolds,

then $0 \to TY \to TX \to N_Y X \to 0$ exact

If $Y = (s=0)$ of a section $s: X \to V$ of a vector bundle which is transverse to 0, then

$N_X Y \cong V$

Rmk: Given $f: Y \to X$ we have $f^*: \operatorname{Qcoh}(X) \rightleftarrows \operatorname{Qcoh}(Y) : f_*$

where for
$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ Y & \longrightarrow & X \end{array}$$

$$f^* \mathcal{F} = \widetilde{B \otimes_A \mathcal{F}(\operatorname{Spec} A)}$$

$\cong$

$\mathcal{O}_Y \otimes_{\mathcal{O}_X} f^{-1}$

Let $Y \overset{f}{\hookrightarrow} X$ be a closed immersion corresponding to the q.coh sheaf of ideals $\mathcal{I}$.
Then $\mathcal{I}/\mathcal{I}^2$ is an $\mathcal{O}_X/\mathcal{I}$-module determining the conormal sheaf $(N_Y X)^\vee$ on Y.

<u>Ex</u>: Let $\mathcal{L}$ be locally free sheaf of rank 1 on X
(e.g. $\mathcal{O}(m)$ on $\mathbb{P}^n$)
Let $s: \mathcal{O}_X \to \mathcal{L}$ be a homomorphism
$\mathcal{L}(X) \overset{bij}{\hookrightarrow}$ ("section of $\mathcal{L}$") which is not a zero divisor in the sense that there is an affine cover $\operatorname{Spec} A$ of X on which $\mathcal{L}$ is trivial with $s|_{\operatorname{Spec} A}: A \longrightarrow \mathcal{L}(\operatorname{Spec} A)$

$$1 \mapsto s_A \qquad \mathcal{L}(\operatorname{Spec} A) \cong A$$

s_A not zero divisor.

$$\Rightarrow \quad 0 \to \mathcal{L}^\vee \overset{s^\vee}{\hookrightarrow} \mathcal{O}_X \to f_* \mathcal{O}_{\{s=0\}}$$

$$\mathcal{L}^\vee \xrightarrow{=} \mathcal{I} \subseteq \mathcal{O}_X$$

$\Rightarrow$ $\mathcal{L}^\vee$ is canonically isomorphic to sheaf of ideals $\{s=0\} \hookrightarrow X$

$$N_{Y/X}^\vee = \mathcal{I}/\mathcal{I}^2 \overset{\text{we'll prove}}{\cong} \mathcal{I} \otimes \mathcal{O}_X/\mathcal{I}$$
$$= f^* \mathcal{L}^\vee$$

<u>Ex</u>: $N_{\{s=0\}/X} = f^* \mathcal{L}^\vee$
for $s: \mathcal{O}_X \to \mathcal{L}$ not zero divisor

For $\mathcal{F}, \mathcal{G}$ sheaves on X,
$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf on X
$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$
- Also sheaves of $\mathcal{O}_X$-modules, Qcoh, etc...
- for a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules let
$\mathcal{F}^\vee := \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{O}_X)$

The <u>normal</u> sheaf $N_Y X$ of closed immersion
$f: Y \hookrightarrow X$ is $\big(\underbrace{N_Y^\vee X}_{\mathcal{I}/\mathcal{I}^2}\big)^\vee$

<u>Warning</u>: In general, $(\mathcal{F}^\vee)^\vee \neq \mathcal{F}$
but if $\mathcal{F}$ is a locally free sheaf of rank n, then $(\mathcal{F}^\vee)^\vee \to \mathcal{F}$

Lemma: $I \subset A$ ideal, A ring

$\Rightarrow I \otimes A/I \cong I/I^2$

Pf:

I/I^2 is an A/I-module inducing a canonical map

$I \otimes_A A/I \to I/I^2$

$i \otimes \bar{a} \mapsto ia$

Map is visibly surjective

$I \to I \otimes_A A/I$ is a ring map

$i \mapsto i \otimes \underline{1}$

$i \otimes \bar{a}$ is in image. It is image of ai

$\Rightarrow I \to I \otimes A/I$ is surjective.

$I^2 \subset I$ is the kernel

$ii' \mapsto i \otimes i' = 0$ this defines

$I/I^2 \to I \otimes A/I$ inverse to canonical map

$I \otimes A/I \to I/I^2$

This defines $I/I^2 \to I \otimes A/I$

inverse to canonical map $I \otimes A/I \to I/I^2$

Prop: Sps $f \hookrightarrow X$ is a closed immersion corresponding to

$\mathcal{I}$ q coh sheaf of ideals then $N_Y X \cong f^* \mathcal{I}$

Def: $f_1, \dots, f_n \in A$ is <u>regular</u> if f_1 is not a zero divisor

and f_r is not a zero divisor in $A/(f_1, \dots, f_{r-1})$.

(21.2.16)

Prop: Suppose $f_1, \dots, f_n \in A$ is a regular sequence and $I = (f_1, \dots, f_n)$.

Then $(A/I)^n \xrightarrow{\cong} I/I^2$ ← conormal sheaf

$(\bar{a}_1, \dots, \bar{a}_n) \mapsto a_1 f_1 + \dots + a_n f_n$

Corollary: Suppose $\mathcal{E}$ is locally free of rank n on a scheme X. ($\mathcal{E}$: q coh sheaf of $\mathcal{O}_X$-modules)

Let $s: \mathcal{O}_X \to \mathcal{E}$ ($\leftrightarrow s \in \mathcal{E}(X)$) be a section of

$\mathcal{E}$ that corresponds to a regular sequence under the local trivializations

of $\mathcal{E}$ corresponding to some cover of X.

Let $\mathcal{I}$ be the image

$$\begin{tikzcd} \mathcal{E}^\vee \arrow[r] \arrow[d, two heads, "s^\vee"'] & \mathcal{O}_X^\vee = \mathcal{O}_X \\ \mathcal{I} \arrow[ur, phantom, "\subset"] & \end{tikzcd}$$

$\mathcal{I}$ corresponds to a closed subscheme $\{s=0\} \hookrightarrow X$

Then $N_{\{s=0\}} X = \mathcal{E}$

pf (cor): $N^\vee_{\{s=0\}} X = \mathcal{I}/\mathcal{I}^2$ on $\{s=0\}$

s determines a map $i^*\mathcal{E}^\vee \xrightarrow{i^*(s^\vee)} \mathcal{I}/\mathcal{I}^2 \cong i^*(\mathcal{I})$

which on an open $U = \operatorname{Spec} A$ is given

$(\mathcal{O}_X/\mathcal{I} \otimes \mathcal{E}^\vee)|_{U \cap \{s=0\}} \cong \mathcal{O}_0^n (\bar{a}_1, \dots, \bar{a}_n) \longmapsto a_1 f_1 + \dots + a_n f_n$

where $s = (f_1, \dots, f_n)$ in a trivialization over U

By prop. $i^*(s^\vee)|_U$ is an iso, thus $i^*(s^\vee)$ is an iso b/c the U's cover. $\blacksquare$

Pf (prop): f is surjective b/c $I = (f_1, \dots, f_n)$.

We show injectivity:

Sps $(\bar{a}_1, \dots, \bar{a}_n) \mapsto 0$

then $a_1 f_1 + \dots + a_n f_n \in I^2$

mod $(f_1, \dots, f_{n-1})$, $a_n f_n \in (f_n)^2$

Since $(f_1, \dots, f_n)$ is a regular sequence,

f_n is not a zero divisor in

$A/(f_1, \dots, f_{n-1})$

So $a_n f_n \in (f_n)^2 \Rightarrow (a_n - f_n c) f_n = 0$

$\Rightarrow a_n \in (f_n) \Rightarrow a_n \in I$,

so $\bar{a}_n = 0$. Repeating arg on $a_{n-1}, \dots, a_1$

shows injectivity $\blacksquare$

<u>Def</u>: an <u>effective Cartier divisor</u> is a locally free sheaf of rank 1, and a section $s: \mathcal{O}_X \to \mathcal{L}$ which is not a zero divisor in a cover of trivializations. • $\{s=0\}$ is the zero locus

(22.2) <u>Blowing up by universal property</u>

Let $X \hookrightarrow Y$ be a closed immersion. The blowup, if it exists is $\beta: Bl_X Y \to Y$

$$\begin{tikzcd} Bl_X Y \arrow[r] & Y \\ E_X Y \arrow[u, hook] \arrow[r] & X \arrow[u, hook] \end{tikzcd}$$

Such that $E_X Y$ is the zero locus of an effective Cartier divisor and for any

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & W \arrow[d] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

where Z is the zero locus of the effective Cartier divisor, there is a unique

$$\begin{tikzcd} & W \arrow[d] \arrow[r, hook] & Bl_X Y \arrow[dl] \\ Z \arrow[ur] \arrow[d] \arrow[r, hook] & Y & E_X Y \arrow[u] \arrow[dl] \\ X \arrow[ur, hook] & & \end{tikzcd}$$

codimension 1 = cut out by 1 equation

<u>Rmk 1</u>: If Blowup exists, it is unique up to unique isomorphism

<u>Rmk 2</u>: $\emptyset = \{1=0\}$ is the zeros of an effective cartier divisor

<u>Rmk 3</u>: If $X \hookrightarrow Y$ is zeros of an effective Cartier divisor, the blowup exists and is itself:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, equal] & Y \arrow[d, equal] \\ E_X Y \arrow[r, hook] & Bl_X Y \end{tikzcd}$$

<u>Claim</u>: Sps
$$\begin{tikzcd} E_X Y \arrow[r, hook] \arrow[d] & Bl_X Y \arrow[d] \\ X \arrow[r, hook] & Y \end{tikzcd}$$
is a blowup.

If $U \subset Y$ is (any) open. Then

$$\begin{tikzcd} E_X Y \cap \beta^{-1}(U) = \beta^{-1}(X \cap U) \arrow[r, hook] \arrow[d] & \beta^{-1}(U) \arrow[d] \\ X \cap U \arrow[r, hook] & U \end{tikzcd}$$

is also a blowup.

<u>Pf</u>: Sps $Z \hookrightarrow W$ is the zero locus of an effective Cartier divisor and we have

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & W \arrow[d] \\ X \cap U \arrow[r] & Y \cap U \end{tikzcd}$$

Then extend:

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d] \arrow[rr, dashed, bend left] \arrow[drr] & W \arrow[d] \arrow[rr, dashed] \arrow[drr, bend left] & \beta^{-1}(X\cap U) \arrow[r, hook] \arrow[d] & \beta^{-1}(U) \arrow[d] \\
X\cap U \arrow[r, hook] \arrow[d] & Y\cap U \arrow[d] & E_X Y \arrow[r, hook] \arrow[dll] & Bl_X Y \arrow[dll] \\
X \arrow[r, hook] & Y & &
\end{tikzcd}$$

Rmk: previous prop combines with $Bl_\emptyset Y = Y$ to show that for any $X \hookrightarrow Y$ s.t. Blowup

$$\begin{tikzcd}
E_X Y \arrow[r, hook] \arrow[d] & Bl_X Y \arrow[d, "\beta"] \\
X \arrow[r, hook] & Y
\end{tikzcd}$$

exists,

then $\beta^{-1}(Y-X) \xrightarrow[\cong]{\beta} Y-X$

$\qquad \| $

$Bl_\emptyset(Y-X)$

Prop: Let $X \hookrightarrow Y$ be a closed immersion.

Sps can cover Y by opens U s.t.

$$\begin{tikzcd}
E_{X\cap U} U \arrow[r, hook] \arrow[d] & Bl_{X\cap U} U \arrow[d] \\
X\cap U \arrow[r, hook] & U
\end{tikzcd}$$

exists, then the Blowup of $X \hookrightarrow Y$ exists.

Rmk: We must have for $\begin{matrix} Bl_X Y \\ \downarrow \beta \\ Y \end{matrix}$ $\beta^{-1}(U) = Bl_{X\cap U} U$

Pf: We will glue the schemes $Bl_{(X\cap U_\alpha)} U_\alpha$ together along $V_{\alpha\beta} \subset Bl_{(X\cap U_\alpha)} U_\alpha$ when $V_{\alpha\beta} = \beta_\alpha^{-1}(U_\alpha \cap U_\beta)$. To do this, we need to construct isos

$$\varphi_{\alpha\beta} : V_{\alpha\beta} \xrightarrow{\cong} V_{\beta\alpha}$$

satisfying the cocycle condition

$$\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} \big|_{V_{\alpha\beta} \cap V_{\alpha\gamma}} = \varphi_{\alpha\gamma} \big|$$

we showed that $\begin{matrix} V_{\alpha\beta} \\ \downarrow \\ U_\alpha \cap U_\beta \end{matrix}$ was the blowup of $X \cap U_\alpha \cap U_\beta \to U_\alpha \cap U_\beta$

Same for $\begin{matrix} V_{\beta\alpha} \\ \downarrow \\ U_\alpha \cap U_\beta \end{matrix}$ so there is a canonical iso by universal prop $\varphi_{\alpha\beta}$

must satisfy cocycle condition by uniqueness arguments
so we glue.

universal property: exercise

Thm: Blow-up exists for any closed immersion

$$X \hookrightarrow Y.$$

Pf: suffices for any $V(I) \to \operatorname{Spec} A$,

$$\begin{tikzcd} V(I \oplus I^2 \oplus \cdots) \arrow[r, hook] \arrow[d] & \operatorname{Proj}(A \oplus I \oplus I^2 \oplus \cdots) \arrow[d] \\ V(I) \arrow[r, hook] & \operatorname{Spec} A \end{tikzcd}$$

satisfies universal property.

Universal property of $Bl_X Y$: the blow-up of a closed immersion $X \hookrightarrow Y$

cartier divisor

terminal: given, $\exists!$ factors

$$\begin{tikzcd} E_X Y \arrow[r, hook] \arrow[d] & Bl_X Y \arrow[d, "\beta"] & Z \arrow[l, dashed] \arrow[r, hook, "\text{cartier}"] \arrow[dll] & W \arrow[dll] \arrow[lll, dashed, bend right] \\ X \arrow[r, hook] & Y & & \end{tikzcd}$$

- $\beta^{-1}(X) \underset{\text{set}}{=} E_X Y$ and $\beta^* \mathcal{I}_X = \mathcal{I}_{E_X Y}$

- $E_X Y$ is the scheme-theoretic inverse image of the closed subscheme, we'll see next time that this is a special case of the fiber product

- Also $g^{-1}(X) = Z$

Last time: we saw that if there is a cover $\{U_\alpha\}_{\alpha \in I}$ of Y s.t. Blow-up of
$$U_\alpha \cap X \hookrightarrow U_\alpha$$
exists, then the blow-up of $X \hookrightarrow Y$ exists.

It thus suffices to show

Thm: Let $V(I) \hookrightarrow \operatorname{Spec} B$ be the closed immersion corresponding to an ideal $I < B$. Then

$$\begin{tikzcd} V(I \oplus I^2 \oplus I^3 \oplus \cdots) \arrow[r, hook] \arrow[d] & \operatorname{Proj}(B \oplus I \oplus I^2 \oplus I^3 \oplus \cdots) \arrow[d] \\ V(I) \arrow[r, hook] & \operatorname{Spec}(B) \end{tikzcd}$$

is a blow-up.

$I \oplus I^2 \oplus I^3 = I(B \oplus I \oplus \cdots)$

is the shift of the graded ring

$B \oplus I \oplus I^2 \oplus \cdots$

down by 1.

Recall: On $\mathrm{Proj}\, S_\bullet$, we defined a line bundle

$\mathcal{O}(m)$ s.t.

$$\mathcal{O}(m)|_{D_+(f)} = \widetilde{\left((S_\bullet)\left[\tfrac{1}{f}\right]\right)_m} \leftarrow \text{degree } m$$

$\Rightarrow$ the ideal sheaf associated to $I(B \oplus I \oplus I^2 \oplus \cdots)$ is isomorphic to $\mathcal{O}(1)$

$$\begin{array}{c} \mathcal{I}_{V(I \oplus I^2 \oplus \cdots)} \subset \mathcal{O}_{\mathrm{Proj}} \\ \| \\ \mathcal{O}_{\mathrm{Proj}}(1) \end{array}$$

$I_\bullet \subset S_\bullet$

$\mathrm{Proj}\, S_\bullet / I_\bullet \cong V(I_\bullet)$

Recall: we defined a cartier divisor on a scheme X to be a line bundle (locally free sheaf of rank 1) on X and $s: \mathcal{O}_X \to \mathcal{L}$ s.t. for an open affine cover U_α of X s.t. $\mathcal{L}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}$, the section s corresponds to s_α not a zero divisor in $\Gamma(\mathcal{O}_{U_\alpha}, U_\alpha)$

Rmk: By dualizing, $\mathcal{L}^\vee \subset \mathcal{O}_X$ is an ideal sheaf which is locally free of rank 1.

Rmk: S_1 not a ring

Rmk: $$\frac{B \oplus I \oplus I^2 \oplus I^3 \oplus \cdots}{I \oplus I^2 \oplus I^3 \oplus \cdots} \cong B/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

$$(b, i_1, i_2, \ldots) \longmapsto (\bar{b}, \bar{i}_1, \bar{i}_2, \ldots)$$

$\Rightarrow E_X Y = V(I \oplus I^2 \oplus \cdots)$

$\cong \mathrm{Proj}\,(B/I \oplus I/I^2 \oplus \cdots)$

Thm 22.3.8 (proof omitted)

$I \subset B$ ideal with $I = (b_1, \ldots, b_n)$ with $b_1, \ldots, b_n$ a regular sequence. Then

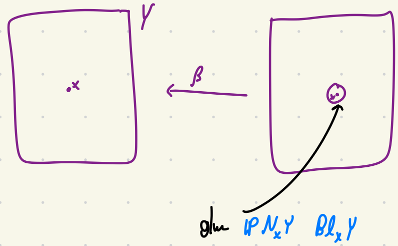
$$I^n/I^{n+1} \cong \mathrm{Sym}^n(I/I^2)$$

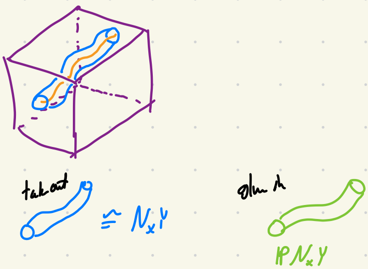
$$i_1 \cdot i_2 \cdots i_n \leftarrow i_1 \otimes \cdots \otimes i_n$$

Also I/I^2 is locally free of rank n
$\overset{\shortparallel}{N_xY^\vee}$

$$\Rightarrow E_xY = \mathrm{Proj} \bigoplus \mathrm{Sym}^n (N_xY)^\vee$$
$$= \mathbb{P} N_xY$$
pset

This equation as a picture:





Thm: $I \subset B$ ideal. Then

$$\begin{tikzcd} \overset{V(I\oplus\cdots)}{\overset{\shortparallel}{E_xY}} \arrow[r, hook, "i_{E_xY}"'] \arrow[d] & \mathrm{Proj}\, B\oplus I \oplus I^2 \oplus \cdots \arrow[d] \\ V(I) \arrow[r, hook] & \mathrm{Spec}\, B \end{tikzcd}$$

is the blowup

Pf: we saw i_{E_xY} is a cartier divisor

Since $I(B\oplus I \oplus I^2 \oplus \cdots)$, it follows that
$\beta^{-1}(V(I)) \subseteq E_xY$
scheme theoretic

$$\mathcal{O}(1) \subseteq B \oplus I \oplus I^2 \oplus \cdots$$
$$\mathcal{O}_{proj} \xrightarrow{s} \mathcal{O}(-1)$$

we want to show that for all

$$\begin{tikzcd} Z \arrow[r, hook, "\text{cartier div}"] \arrow[dr, dashed, "\exists!"] \arrow[ddr, bend right] & W \arrow[dr, dashed, "\exists!"] \arrow[ddr, bend right] & \\ & E_xY \arrow[r, hook] \arrow[d] & Bl_xY \arrow[d] \\ & X \arrow[r, hook] & Y = \mathrm{Spec}\, B \end{tikzcd}$$

(assuming I is locally f.g.)

$(b_1, \ldots, b_n) = I \rightsquigarrow X$

Since we can glue maps together on a cover of W, we may assume $W = \operatorname{Spec} R$. By shrinking W further if necessary, we may assume the Cartier divisor corresponds to a trivial line bundle $\mathcal{O}_{\operatorname{Spec} R} \xrightarrow{s} \mathcal{L} = \mathcal{O}_{\operatorname{Spec} R}$
i.e., $Z = V(s) \hookrightarrow \operatorname{Spec} R = W$ for
$s \in R$ not a zero divisor.

$$\phi : B \to R$$

Since $Z = \phi^{-1}(V(s))$, we have
$(s) = (\phi(b_1), \dots, \phi(b_n))$ in R
$\Rightarrow (1) = \left(\frac{\phi(b_1)}{s}, \dots, \frac{\phi(b_n)}{s}\right)$
Thus $\operatorname{Spec} R = \bigcup_{i=1}^{n} D\left(\frac{\phi(b_i)}{s}\right)$

$Bl_X Y = \operatorname{Proj}(B \oplus I \oplus \cdots)$
Let x_i be the deg 1 element
corresponding to b_i

$$D_+(x_i) \subset \operatorname{Proj} B \oplus I \oplus I^2 \oplus \cdots$$

$$D_R\left(\frac{\phi(b_i)}{s}\right) \subset W$$

We define $D_R\left(\frac{\phi(b_i)}{s}\right) \to D_+(x_i)$
as associated to the map of rings.

$$(B \oplus I \oplus I^2 \oplus \cdots)\left[\frac{1}{x_i}\right]_0 \longrightarrow R\left[\frac{1}{\phi(b_i)/s}\right]$$

induced from the map
$B \oplus I \oplus I^2 \oplus \cdots \longrightarrow R$
deg d part of $f \mapsto \frac{\phi(f)}{s^d}$

This constructs

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[dr, dashed, "g"] \arrow[ddr, bend right] & W \arrow[r, bend left, "g"] \arrow[dr] & Bl_X Y \arrow[dd] \\
& E_X Y \arrow[ur, hook] \arrow[d] & \\
& X \arrow[r, hook] & Y
\end{tikzcd}$$

$\phi^{-1}(X) = Z$
$\Rightarrow g^{-1}\beta^{-1}(X) = Z$
$\Rightarrow g^{-1}(\beta^{-1}(X)) = Z \Rightarrow g^{-1}(E_X Y) = Z$ ☒

Given
$$\begin{tikzcd} & X \arrow[d] \\ W \arrow[r] & Y \end{tikzcd}$$
in a category $\mathcal{C}$

the $\varprojlim$ is also called the <u>fiber product</u> and denoted $W \times_Y X$

In other words, $W \times_Y X$ is defined by the universal property

$$\begin{tikzcd} Z \arrow[r, dashed, "\exists!"'] \arrow[rr, bend left, "h_X"] \arrow[dr, bend right, "h_W"'] & W \times_Y X \arrow[r, "\pi_X"'] \arrow[d, "\pi_W"'] & X \arrow[d, "f"] \\ & W \arrow[r] & Y \end{tikzcd}$$

<u>ex</u>: for $\mathcal{C} = Set$, $W \times_Y X = \{(w,x) \in W \times X \mid g(w) = f(x)\}$

<u>Thm</u>: fiber products exist in schemes (tensor product on affines, then glue)

<u>Thm</u>: let
$$\begin{tikzcd} & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$
be a diagram in Schemes. Then the fibered product $X \times_Z Y$ ($= \varprojlim$) exists, which means there exists
$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$
s.t. for all diagrams
$$\begin{tikzcd} W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & Z \end{tikzcd}$$

A-points of the ring

<u>Rmk</u>: for A ring, $X(A) = \{\operatorname{Spec} A \to X\}$

$$(X \times_Z Y)(A) \xrightarrow[(\pi_X, \pi_Y)]{\overset{\text{bij}}{=}} X(A) \times_{Z(A)} (Y(A))$$

fiber product in sets

Rmk taking Z to be the terminal object defines the product

ex: let k be a field, Schemes_k category of Schemes over k

This has objects $X \to \operatorname{Spec} k$ and morphisms
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd}$$

$X \times_{\operatorname{Spec} k} Y$ is the categorical

Pf: Step 1: thm is true for α an open subscheme (aka open immersion)

Let $X \times_Z Y = \beta^{-1}(X)$

Step 2: True for X, Y, Z affine schemes $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$

Then the diagram
$$\begin{tikzcd} A \otimes_C B & A \arrow[l] \\ B \arrow[u] & C \arrow[l] \arrow[u] \end{tikzcd}$$
in rings is a colimit diagram:

$$\begin{tikzcd} A \times B \arrow[r, "a \times b \mapsto a \otimes b"] \arrow[d, "\text{bilinear map}"'] & A \otimes_C B \arrow[dl, dashed, "\exists!\text{ map of } C\text{-modules}"] \\ M \end{tikzcd}$$
M C-module

$A \otimes_C B$ inherits a C-algebra structure by $C \to A \to A \otimes_C B$, $a \mapsto a \otimes 1$

and giving $A \otimes_C B$ the mult determined by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$

So for any
$$\begin{tikzcd} R & & \\ & A \otimes_C B & A \arrow[l, "g"'] \arrow[ull, bend right, "g"'] \\ & B \arrow[u] \arrow[uul, bend left, "f"] & C \arrow[l, "\alpha"] \arrow[u, "\beta"'] \end{tikzcd}$$

The bilinear map $A \times B \to R$, $(a,b) \mapsto f(a)g(b)$

is C-bilinear b/c $f\alpha = g\beta$

$\Rightarrow \exists!$ C-module map $A \otimes_C B \to R$, $a \otimes b \mapsto f(a)f(g)$

This is a ring homomorphism by def of ring structure.

The Affine schemes is opposite category of rings, so universal prop of fiber product holds for $W = \operatorname{Spec} R$.

$$\begin{tikzcd}
\operatorname{Spec} R_\alpha \subset W \arrow[r, dashed] \arrow[rr, bend left, "f"] \arrow[dr, bend right, "g"'] & \operatorname{Spec} A \otimes_C B \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d, "\beta"] \\
& \operatorname{Spec} B \arrow[r, "\alpha"'] & \operatorname{Spec} C
\end{tikzcd}$$

we claim that for any scheme W, the universal property holds.

$W = \bigcup_{\alpha \in I} \operatorname{Spec} R_\alpha$

We obtain unique $h_\alpha : \operatorname{Spec} R_\alpha \to \operatorname{Spec} A \otimes_C B$ s.t.

$$\begin{tikzcd}
\operatorname{Spec} R_\alpha \arrow[dr, "h_\alpha"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& \operatorname{Spec} A \otimes_C B \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d, "\beta"] \\
& \operatorname{Spec} B \arrow[r, "\alpha"'] & \operatorname{Spec} C
\end{tikzcd}$$

Commutes $\Rightarrow$

$$h_\alpha|_{\operatorname{Spec} R_\alpha \cap \operatorname{Spec} R_\beta} = h_\beta|_{\operatorname{Spec} R_\alpha \cap \operatorname{Spec} R_\beta}$$

B/c $\operatorname{Spec} R_\alpha \cap \operatorname{Spec} R_\beta$ is covered by affines

$\Rightarrow$ The h_α glue to a map $h : W \to \operatorname{Spec} A \otimes_C B$ s.t.

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& \operatorname{Spec} A \otimes B \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d, "\beta"] \\
& \operatorname{Spec} B \arrow[r, "\alpha"'] & \operatorname{Spec} C
\end{tikzcd}$$

commutes. Thus h is unique b/c an h & h' would have to agree on a cover

Rmk this shows $\mathrm{AffSch} \to \mathrm{Sch}$ preserves limits
$\mathrm{AffSch} = \mathrm{Rings}^{op}$

Step 3: sps β factors as affine map

$$Y \overset{i}{\hookrightarrow} Y' \xrightarrow[\text{affine}]{} Z$$

where i is an open immersion and β' is affine in the sense that for any open affine,

$\operatorname{Spec} C_\alpha$ of Z, $(\beta')^{-1}(\operatorname{Spec} C_\alpha)$ is open affine.

Sps $X = \operatorname{Spec} B$ and $Z = \operatorname{Spec} C$ are affine. Then $X \times_Z Y$ exists. Then by step 2 + step 1

$$\begin{tikzcd}
(X \times_Z Y') \times_{Y'} Y \arrow[r] \arrow[d] & Y \arrow[d, hook, "i"] \\
X \times_Z Y' \arrow[r] \arrow[d] & Y' \arrow[d, "\beta'"] \\
X \arrow[r] & Z
\end{tikzcd}$$

It is formal that

$$(X \times_Z Y') \times_{Y'} Y \xrightarrow{\cong} X \times_Z Y$$

(universal prop chasing) ☑

Step 4: Sps $X = \operatorname{Spec} B$ and $Z = \operatorname{Spec} C$ are affine and $Y \xrightarrow{\beta} Z$. We will show $X \times_Z Y$ exists. Since Y is a scheme, Y has a cover $\{U_\alpha\}_{\alpha \in I}$ by open affines. By step 2, the fiber product $X \times_Z U_\alpha$ exists. By step 3, the fiber product $X \times_Z (U_\alpha \cap U_\beta)$ also exists. And since $U_\alpha \cap U_\beta \hookrightarrow U_\alpha$ is an open immersion, by step 1,

$$X \times_Z (U_\alpha \cap U_\beta) \hookrightarrow X \times_Z U_\alpha$$

is an open immersion.

Moreover, $X \times_Z (U_\alpha \cap U_\beta \cap U_\gamma)$ exists and

$$\begin{tikzcd}
U_\alpha \cap U_\beta \cap U_\gamma \arrow[r, hook] \arrow[d, hook] & U_\alpha \cap U_\gamma \arrow[d, hook] \\
U_\alpha \cap U_\beta \arrow[r, hook] & U_\alpha
\end{tikzcd}$$

is a limit, where (limits commute)

$$\begin{tikzcd}
X \times_Z (U_\alpha \cap U_\beta \cap U_\gamma) \arrow[r] \arrow[d] & X \times_Z (U_\alpha \cap U_\gamma) \arrow[d] \\
X \times_Z (U_\alpha \cap U_\beta) \arrow[r] & X \times_Z U_\alpha
\end{tikzcd}$$

is a limit. $\Rightarrow$ $X \times_Z (U_\alpha \cap U_\beta \cap U_\gamma) = \big(X \times_Z (U_\alpha \cap U_\beta)\big) \cap \big(X \times_Z (U_\alpha \cap U_\gamma)\big)$

We will glue $X \times_Z Y$ from $\{\underbrace{X \times_Z U_\alpha}_{=S_\alpha}\}_{\alpha \in I}$

$X \times_Z (U_\alpha \cap U_\beta) \subset X \times_Z U$

and isos $\varphi_{\alpha\beta} : S_{\alpha\beta} \xrightarrow{\sim} S_{\beta\alpha}$

st

$$\varphi_{\beta\gamma}\big|_{S_{\beta\alpha} \cap S_{\beta\gamma}} \circ \varphi_{\alpha\beta}\big|_{S_{\alpha\beta} \cap S_{\alpha\gamma}} = \varphi_{\alpha\gamma}\big|_{S_{\alpha\beta} \cap S_{\alpha\gamma}}$$

$$S_{\alpha\beta} \cap S_{\alpha\gamma} = X \times_Z (U_\alpha \cap U_\beta \cap U_\gamma)$$

and $\varphi_{\alpha\beta}$'s correspond to identities under these isos

This gluing produces a scheme S

We have $\pi_X : S \longrightarrow X$ (by gluing $X \times_Z U_\alpha \xrightarrow{\pi_{X\alpha}} X$)

the maps $U_\alpha \to Y$ glue to a map $\pi_Y : S \to Y$

We claim $S = X \times_Z Y$

$S \supset S_\alpha = (\pi_Y^{-1})(U_\alpha)$

$$\begin{tikzcd}
f^{-1}(U_\alpha) \subset W \arrow[dr, "{(f,g)|_{f^{-1}(U_\alpha)}}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& S_\alpha \subset S \arrow[r] \arrow[d, "\pi_Y"] & X \arrow[d] \\
& U_\alpha \subset Y \arrow[r] & Y
\end{tikzcd}$$

$$(\ast)\quad (f|_{f^{-1}(U_\alpha)}, g)\big|_{f^{-1}(U_\alpha \cap U_\beta)} = (f|_{f^{-1}(U_\alpha \cap U_\beta)}, g)$$

because $S_\alpha \cap S_\beta = X \times_Z (U_\alpha \cap U_\beta)$

Thus can exchange the roles of α and β in $(\ast)$

So the maps

$$(f|_{f^{-1}(U_\alpha)}, g) : f^{-1}(U_\alpha) \to S$$

agree on the overlaps of the open cover

$f^{-1}(U_\alpha)$ of W, thus defining $(f,g) : W \to S$

If we had (f,g), (f',g') the maps would agree

on the cover $f^{-1}(U_\alpha)$ and so would be equal $\boxed{}$

Step 5: $Z = \operatorname{Spec} C$ is affine and $Y \xrightarrow{\beta} Z$ and $X \xrightarrow{\alpha} Z$

are arbitrary.

Again cover Y by affine opens U_α, define $S_\alpha = X \times_Z U_\alpha$ by step 4, repeat argument from step 4.

Step 6: Let $Z \hookrightarrow Z' = \operatorname{Spec} C$ be an open subset of an affine scheme and $X \xrightarrow{\alpha} Z$, $Y \xrightarrow{\beta} Z$ arbitrary.

Then
$$\begin{tikzcd} X \times_{Z'} Y \arrow[r] \arrow[dd] & Y \arrow[dl] \arrow[dd] \\ & Z \arrow[dr] & \\ X \arrow[ur] \arrow[r] & Z' \end{tikzcd}$$
gives $X \times_{Z'} Y = X \times_Z Y$

Given $h, h' : W \to X \times_{Z'} Y$

s.t.
$$\begin{tikzcd} W \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] \arrow[rr, bend left, "g"] \arrow[dr, bend right, "f"'] \arrow[ddrr, bend right=60] \arrow[ddrr, bend left=70] & X \times_{Z'} Y \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] \\ & X \arrow[r] \arrow[dr, bend right] & Z \arrow[d, "i"] \\ & & Z' \end{tikzcd}$$

we have $h = h'$

i monomorphism $\Rightarrow$ green stuff commutes, can "pull back" to Z from Z'

<u>Thm</u> fiber products exist in schemes

<u>pf</u>: We've shown:

1) if
$$\begin{tikzcd} Y \times_Z X \arrow[r] \arrow[d, "\pi_Y"'] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$
exists and $U_\alpha \subset Y$, then

$$\begin{tikzcd} \pi_Y^{-1}(U_\alpha) \arrow[r, hook] \arrow[d] & Y \times_Z X \arrow[r] \arrow[d] & X \arrow[d] \\ U_\alpha \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

has all 3 rectangles fiber products

2) If $Z' \subset Z$ is an open subset of an affine scheme Z and $X \xrightarrow{\alpha} Z'$ and $Y \xrightarrow{\beta} Z'$ are maps of schemes.

Then $X \times_Z Y$ exists and $X \times_{Z'} Y \xrightarrow{\cong} X \times_Z Y$ is a canonical isomorphism.

General case (gluing!): Let $X \xrightarrow{\alpha} Z$, $Y \xrightarrow{\beta} Z$ be any morphisms of schemes.

Let $\{Z_i\}_{i \in I}$ be an open affine cover of Z. Let $Z_{ij} = Z_i \cap Z_j$.

Let $X_i = \alpha^{-1}(Z_i)$ $\quad X_{ij} = \alpha^{-1}(Z_i \cap Z_j)$

$Y_i = \beta^{-1}(Z_i)$ $\quad Y_{ij} = \beta^{-1}(Z_i \cap Z_j)$

By (2) we have fibre products

open immersion

$$W_i = X_i \times_{Z_i} Y_i \overset{\supset}{\longleftarrow} X_{ij} \times_{Z_{ij}} Y_{ij} = W_{ij}$$

and the canonical map is the open immersion

$$X_{ij} \times_{Z_{ij}} Y_{ij} = (\pi_{i,Z})^{-1}(Z_{ij}) \hookrightarrow X_i \times_{Z_i} Y_i$$

by step 1, where $\pi_{i,Z} : X_i \times_{Z_i} Y_i \longrightarrow Z_i$ is the canonical map.

We will glue the W_i along

$$W_{ij} \subset W_i$$

via the canonical iso

$$\varphi_{ij} : W_{ij} \xrightarrow{\sim} W_{ji}$$

we can do this because

$$\varphi_{jk}| \circ \varphi_{ij}\big|_{W_{ij} \cap W_{ik}} = \varphi_{ik}\big|$$

because both correspond to

$$\begin{array}{ccc} & & \alpha^{-1}(Z_i \cap Z_j \cap Z_k) \times_{Z_i \cap Z_j \cap Z_k} \beta^{-1}(Z_i \cap Z_j \cap Z_k) \\ & \rotatebox{45}{$=$} & \parallel \\ \pi_{i,Z}^{-1}(Z_i \cap Z_j \cap Z_k) & \xrightarrow{\text{canonical iso}} & \pi_{k,Z}^{-1}(Z_i \cap Z_j \cap Z_k) \\ \wr\parallel & & \parallel \\ W_{ij} \cap W_{ik} & & W_{ki} \cap W_{kj} \end{array}$$

there is only one.

Then let W be the result of the gluing.

The maps $W_i = X_i \times_{Z_i} Y_i \xrightarrow{\pi_{X,i}} X_i$

$W_i = X_i \times_{Z_i} Y_i \xrightarrow{\pi_{Y,i}} Y_i$

Satisfy $\pi_{X,i}\big|_{W_{ij}} = \pi_{X,j}\big|_{W_{ji}}$

thus we have maps

$$W \xrightarrow{\pi_X} X$$

$$W \xrightarrow{\pi_Y} Y$$

and

$$\begin{tikzcd} W \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd}$$

we claim this exhibits $W = X \times_Z Y$.

Let R be a scheme and

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

be a commutative diagram. Let $R_i = h^{-1}(Z_i)$. The maps $f|_{R_i}$ and $g|_{R_i}$ determine a unique map

$$R_i \xrightarrow{(f|_{R_i}, g|_{R_i})} W_i.$$

The uniqueness shows $(f|_{R_i}, g|_{R_i})\big|_{R_i \cap R_j} = (f|_{R_j}, g|_{R_j})\big|_{R_j \cap R_i}$

thus the $(f|_{R_i}, g|_{R_i})$ glue to

$(f,g) : R \to W$ s.t.

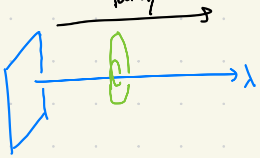
$$\begin{tikzcd} R \arrow[dr, dashed, "{(f,g)}"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\ & W \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

s.t. (f,g) is unique b/c any two would have to agree on the cover R_i. $\boxed{}$

ex:

$$\underset{X}{\operatorname{Spec} \frac{k[x,y,\lambda]}{(y^2 - x(x-1)(x-\lambda))}} \longrightarrow \underset{Z}{\operatorname{Spec} k[\lambda]}$$

family of curves



$$\frac{k[x,y,\lambda]}{y^2 - x(x-1)(x-\lambda)} \otimes_{k[\lambda]} \underset{\lambda \mapsto \frac{1}{2}}{k}$$

$$\cong \frac{k[x,y]}{y^2 - x(x-1)(x-\frac{1}{2})}$$

ex:

$$\begin{tikzcd} \operatorname{Spec} B \arrow[d] & \operatorname{Spec} B[t] \arrow[l] \arrow[d] \\ \operatorname{Spec} A & \operatorname{Spec} A[t] \arrow[l] \end{tikzcd}$$

ex: Sps $Y \hookrightarrow Z$ is the closed immersion corresponding to the qcoh sheaf of ideals $\mathcal{I}$

Then
$$\begin{tikzcd} X \arrow[d, "\alpha"'] & X \times_Z Y \arrow[l, "i'"'] \arrow[d] \\ Z & Y \arrow[l, hook', "i"'] \end{tikzcd}$$

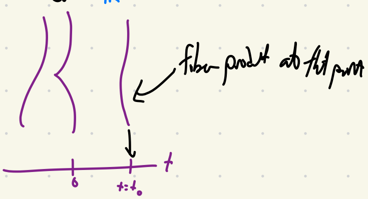
has i' the closed immersion corresponding to $\alpha^* I$.

Ex: we can combine the previous 2 examples

$$\begin{tikzcd} \operatorname{Spec} \frac{k[x,y]}{y^2 - x(x-1)(x-2)} \arrow[d] & \operatorname{Spec} \frac{k[x,y,x',y']}{\left(y^2 - x(x-1)(x-2),\ (y')^2 - x'(x'-1)(x'-2)(x'-3)(x'-\lambda)\right)} \arrow[l] \arrow[d] \\ \operatorname{Spec} k & \operatorname{Spec} \frac{k[x,y]}{y^2 - x(x-1)(x-2)(x-3)(x-\lambda)} \arrow[l] \end{tikzcd}$$

Ex:
$$\begin{tikzcd} \operatorname{Spec} \frac{k[x,y,t]}{y^2 - x^3 + tx} \arrow[d] \\ \operatorname{Spec} k[t] \end{tikzcd}$$

over $\mathbb{R}$



$$\begin{tikzcd} \operatorname{Spec} \frac{k[x,y,t]}{y^2 - x^3 + tx} \arrow[r] \arrow[d] & \operatorname{Spec} \frac{[x,y,t,\lambda]}{y^2 - x^3 - \lambda x^2 + tx} \arrow[d] \\ \operatorname{Spec} k[t] \arrow[r, "0 \mapsto \lambda"] & \operatorname{Spec} k[\lambda, t] \end{tikzcd}$$

Ex: fiber products of connected are not necessarily connected

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C} \times \mathbb{C} = \operatorname{Spec} \frac{\mathbb{C}[x]}{(x^2+1)} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{C} \cong \operatorname{Spec} \frac{\mathbb{R}[x]}{(x^2+1)} \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{R} \end{tikzcd}$$
$= \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C}$

ex:
$$\begin{tikzcd} S^1 \sqcup S^1 \arrow[r] \arrow[d] & \{z \mid |z| = 1\} = S^1 \subset \mathbb{C} \arrow[d, "z \mapsto z^2"] \\ S^1 \arrow[r, "z \mapsto z^2"] & S^1 \end{tikzcd}$$

is a fiber product in Top

Vakil 15.2 functor of points to $\mathbb{P}^n$

Recall: Given a line bundle $\mathcal{L}$ on a scheme X and sections $s_0, \ldots, s_n \in \mathcal{L}(X)$ s.t.

$V(s_0, \ldots, s_n) = \emptyset$ (This is equivalent to saying that for all $x \in X$ $\exists i$ s.t. $s_i(x) \in \mathcal{L}_x/\mathfrak{m}_x \overset{\cong}{=} \mathcal{O}_{X,x}/\mathfrak{m}_x$ is nonzero

$\iff D(s_i) = X - V(f_i)$ cover X.

We constructed

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^n_B = \operatorname{Proj} B[x_0, \ldots, x_n] \\ \cup & & \cup \\ D(s_i) & \longrightarrow & D_+(x_i) = \operatorname{Spec} B[x_{0/i}, x_{1/i}, \ldots, x_{n/i}]/(x_{i/i}-1) \end{array}$$

$$\mathcal{O}_X(D(s_i)) \longleftarrow B[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i}-1)$$

defined by $\quad s_j/s_i \longmapsfrom x_{j/i}$

These glue by problem set.

Thm: This (fix B) correspondence induces a bijection between maps $X \to \mathbb{P}^n$ (over B) and $\{(\mathcal{L}, s_0, \ldots, s_n) \mid \mathcal{L} \to X$ is a line bundle, $s_i \in \mathcal{L}(X)$, $V(s_0, \ldots, s_n) = \emptyset\}/\text{iso}$

e.g. for $\lambda \in \mathcal{O}_X^*(X)$, $\lambda : \mathcal{L} \to \mathcal{L}$ is an iso, so $(\mathcal{L}, s_0, \ldots, s_n) \sim (\mathcal{L}, \lambda s_0, \ldots, \lambda s_n)$ correspond to the same $X \to \mathbb{P}^n$

pf: we construct an inverse bijection: given

$X \xrightarrow{f} \mathbb{P}^n_B = \operatorname{Proj} B[x_0, \ldots, x_n]$

let $(\mathcal{L}, s_0, \ldots, s_n) = (f^*\mathcal{O}(1), f^*x_0, \ldots, f^*x_n)$

$\mathcal{O}(1)(X) = \bigoplus_{i=0}^{n} x_i$

vector space.

We show the map $f' : X \to \mathbb{P}^n$ that we construct with $(\mathcal{L}, s_0, \ldots, s_n)$ is f.

$(f')^{-1}(D_+(x_i)) = D(f^*(x_i)) = f^{-1}(D_+(x_i))$

f' and f restricted to $D(f^* s_i)$:

$$\mathcal{O}_X(D(s_i)) \ni s_j/s_i \xleftarrow{f'} x_{j/i}$$

$$s_j/s_i = \frac{f^* x_j}{f^* x_i} \xleftarrow{f} x_{j/i}$$

Thus $f = f'$

Now let $f: X \to \mathbb{P}^n$ be the map we constructed from $(\mathcal{L}, s_0, \dots, s_n)$. Then $f^*(\mathcal{O}(1))$ is trivialized over $f^{-1}(D_+(x_i)) = D(s_i)$ with transition functions $\frac{f^*x_i}{f^*x_j} = s_i/s_j$ which give transition functions of $\mathcal{L}$. Since the $D(s_i)$ cover, we have $\mathcal{L} \cong f^*\mathcal{O}(1)$ and under this iso

$$s_i \leftrightarrow f^* x_i \qquad \boxtimes$$

Ex (Veronese embedding)

$$\mathcal{O}_{\mathbb{P}^n}(d)(\mathbb{P}^n) = \bigoplus_{\substack{\text{monomials in } x_0, \dots, x_n \\ \text{homogeneous degree } d}} B$$

Let $s_0, \dots, s_m$ $m = \binom{d+n}{d}$ correspond to these monomials. Then the resulting map is

$$\left(\mathcal{O}_{\mathbb{P}^n}(d), s_0, \dots, s_{\binom{n+d}{d}-1}\right)$$
$$\mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$$

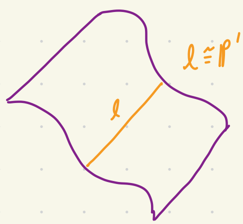
"dth Veronese"

$n=1$ "rational normal curve"

PS6 posted – please assume the cubic surface is smooth, in the sense $X = \{f=0\} \subset \mathbb{P}^3_k$

$f \in k[x_0, x_1, x_2, x_3]$ homogeneous deg 3

$V(\partial_{x_0}f, \partial_{x_1}f, \partial_{x_2}f, \partial_{x_3}f) = \emptyset$



$$0 \to T\ell \to TX \to N_{\ell/X} \to 0$$

$$0 \to TX \to T\mathbb{P}^3 \to N_{X/\mathbb{P}^3} \to 0 \qquad \text{pull back } i^*_{X \hookrightarrow \mathbb{P}^3}\mathcal{I}$$

$\mathcal{I} = (f)$ we defined $N^\vee_{X/\mathbb{P}^3} = \mathcal{I}/\mathcal{I}^2$ ← you can compute this!

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^3}(0) \to i_*\mathcal{O}_X \to 0, \qquad \mathcal{I} \cong \mathcal{O}_{\mathbb{P}^3}(-3)$$

Thm: There is a canonical bijection between maps of schemes $\mathrm{maps}(X, \mathbb{P}^n) = \{(\mathcal{L}, s_0, \dots, s_n) \mid \mathcal{L}$ line bundle on X and $s_i \in \mathcal{L}(X)$ have no common vanishing$\}/\text{iso}$

$$\mathbb{P}^n \xleftarrow{\pi_1} \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\pi_2} \mathbb{P}^m$$

Ex: on $\mathbb{P}^n \times \mathbb{P}^m$ let $\mathcal{O}(a) \boxtimes \mathcal{O}(b) = \pi_1^*\mathcal{O}_{\mathbb{P}^n}(a) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^m}(b)$

$$\Gamma(\mathcal{O}(a) \boxtimes \mathcal{O}(b)) \cong \{f \in k[x_0, \dots, x_n, y_0, \dots, y_m] \text{ bihomogeneous of deg } (a,b) \text{ in } x\text{'s}, y\text{'s}\}$$

canonical bij

This is a free module of rank $\binom{a+n}{n}\binom{b+m}{m} = N$

Using $(\mathcal{O}(a) \boxtimes \mathcal{O}(b), s_0, \dots, s_{N-1})$ and then defines $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{N-1}$

Segre embedding $a=b=1$

$$\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{(n+1)(m+1)-1} = \mathbb{P}^{nm+n+m}$$

$$[x_0,\dots,x_n] \times [y_0,\dots,y_m] \longrightarrow [x_0y_0, x_0y_1, \dots, x_0y_m, x_1y_0, x_1y_1, \dots, \dots x_ny_m]$$

<u>Ex</u>: $\mathbb{Z}[\sqrt{-5}] \supset (2, 1+\sqrt{-5}) = I$

$= \mathcal{O}_{\mathbb{Q}(\sqrt{-5})}$

I is prime

Pf:

$$\mathbb{Z}/2[x] \big/ x^2+5 = \frac{\mathbb{Z}/2[x]}{(x+1)^2}$$

$$\mathbb{Z}[\sqrt{-5}] \big/ (2, 1+\sqrt{-5}) \cong \mathbb{Z}/2$$

ring of integers in

<u>Fact</u>: Localizations of dedekind domains $\left(\mathcal{O}_{\mathbb{Q}[\sqrt{-5}]}\right)$ are P.I.D.s

Thus $\tilde{I}$ is locally free rank 1 on $\operatorname{Spec} \mathbb{Z}[\sqrt{-5}]$

The global sections $s_0 = 2$ $s_1 = 1+\sqrt{-5}$ "generate the ideal" in the sense that s_0, s_1 generate every stalk $\tilde{I}_p$.

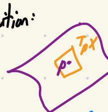
In particular, they do not simultaneously vanish in $\tilde{I}$.

i.e., both belong to $\mathfrak{p}I$ in $\mathbb{Z}[\sqrt{-5}]_{\mathfrak{p}}$

Thus we have $\operatorname{Spec} \mathbb{Z}[\sqrt{-5}] \xrightarrow{(s_0,s_1)} \mathbb{P}^1_{\mathbb{Z}}$

<u>Chapter 21</u> Differentials/Tangent Space/etc.

Intuition: (for closed points p) T_pX is the best linear approx to X at p

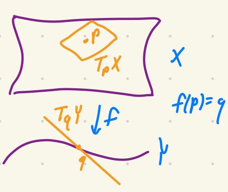


$X = \{f=0\} \subset \mathbb{A}^1_{x_1,\dots,x_n}$

$T_pX = \{\nabla f(x_1,\dots,x_n) = 0\}$

$X = V(f_1,\dots,f_m) \subset \mathbb{A}^n_{x_1,\dots,x_n}$

$T_p = \ker\left[\frac{\partial f_i}{\partial x_j}\right]$
jacobian



relative tangent

$T_p(X/Y) = \ker\left(T_pX \xrightarrow{T_pf} T_q Y\right)$

The dual of a k-vector space V

is $V^\vee = \mathrm{Hom}(V, k)$

cotangent space $T_p^* X = (T_p X)^\vee$

Algebraically, cotangent space works better,

Cotangent space will be a q.coh sheaf

associated to modules of "relative differentials" $\Omega_{A/B}$

$f: \mathrm{Spec}\, A \to \mathrm{Spec}\, B$

$\phi: B \longrightarrow A$

def: we define the sheaf of relative differentials $\Omega_{A/B}$ and

a map $d: A \longrightarrow \Omega_{A/B}$, $a \mapsto d(a)$ — not an A-module map

so that $\Omega_{A/B}$ is generated as an

A-module by symbols $d(a)$

subject to relations

1) $d(a_1 + a_2) = d(a_1) + d(a_2)$

2) $db = 0$ for $b \in \mathrm{Im}\,\phi$

3) "Leibniz" $d(aa') = d(a)a' + a\,d(a')$

Rmk: d is a map of B-modules

b/c it is a group homomorphism and

$d(ba) = d(b)a + b\,d(a)$

$= b\,d(a)$

$\widetilde{\Omega}_{A/B}$

relative cot sheaf of

$\mathrm{Spec}\, A \to \mathrm{Spec}\, B$

Ex: $\Omega_{B[x]/B} \cong B_x$

Ex: $k \subset L$ separable extension. Then $\forall$

$l \in L$ $\exists$ $f \in k[x]$ s.t. $f(l) = 0$

Key fact: sps $A = \dfrac{B[x_i \mid i \in I]}{(r_j \mid j \in J)}$

Then $\Omega_{A/B} = \dfrac{\bigoplus_{i \in I} A dx_i}{\bigoplus_{j \in J} A dr_j} = \mathrm{coker}\left(A^J \xrightarrow{\left[\frac{dr_j}{dx_i}\right]} A^I\right)$

pf: $\{dx_i \mid i \in I\}$ generate by Leibniz rule

and $\{dr_j \mid j \in J\}$ generate by additivity and Leibniz rule

kernel $\bigoplus A dx_i \to \Omega_{A/B}$

Ex: $B \to B/I = A$ then $\Omega_{(B/I)/B} = 0$

but $f'(\ell) \neq 0$ in L

$\Omega_{L/k}$ b/c $f(\ell) \in k$

calc $dF(\ell) = 0$

$\quad\; \| $

$\quad F'(\ell)\, d(\ell) = 0$

$\Rightarrow d(\ell) = 0$

$\Rightarrow \Omega_{L/k} = 0$

Ex: $\Omega_{\mathbb{F}_p(\sqrt[p]{t})/\mathbb{F}_p(t)} = \dfrac{\mathbb{F}_p(\sqrt[p]{t})\, d(\sqrt[p]{t})}{d((\sqrt[p]{t})^p - t)}$

$\quad\; \| $

$\mathbb{F}_p(\sqrt[p]{t})\, d(\sqrt[p]{t}) = \dfrac{\mathbb{F}_p(\sqrt[p]{t})\, d(\sqrt[p]{t})}{\cancel{p(\sqrt[p]{t})^{p-1} d(\sqrt[p]{t})}\; 0}$

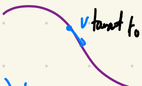
$\underline{\text{AW}}$ k field

$$A = k[x_1, \dots, x_n]/(f_1, \dots, f_n)$$

let p be a k-point of Spec A

$$\underset{\uparrow \text{ fiber at } p}{\Omega_{A/k} \otimes_{A_p} k} \cong \frac{\mathfrak{m}_p}{(\mathfrak{m}_p)^2}$$

$$\overset{T_pX}{\{ \vec{v} \}} \overset{\text{bij}}{\sim} \left\{ \frac{\partial}{\partial v} : \underset{\uparrow \text{ cotangent space}}{\frac{\mathfrak{m}}{\mathfrak{m}^2}} \to k \right\}$$



v tangent to $V(I)$

$V(I) = Y$

$\dfrac{\partial}{\partial v}\Big|_I = 0$

$$TY : \frac{\mathfrak{m}/I}{(\mathfrak{m}/I)^2} \to k$$

inclusion of lin functions 0 on I

$\quad\; \|$

$$TY \longrightarrow TX \longrightarrow \operatorname{Hom}(I/I^2, k)$$

$\uparrow$ should be conormal sheaf

$\underline{\text{Thm}}$ (21.2.9) Sps $C \to B \to A$ are ring homomorphisms.

$$\begin{tikzcd} \Omega_{A/k} \arrow[d] & & \Omega_{B/k} \arrow[d] \\ \operatorname{Spec} A \arrow[rr] \arrow[dr] & & \operatorname{Spec} B \arrow[dl] \\ & \operatorname{Spec} C & \end{tikzcd}$$

Then $\quad A \otimes_B \Omega_{B/C} \xrightarrow{a \otimes db \mapsto adb} \Omega_{A/C} \to \Omega_{A/B} \to 0$ is exact

$da \longmapsto da$

pf $\quad \bigoplus A da \twoheadrightarrow \Omega_{A/B} \Rightarrow$ exact at $\Omega_{A/B}$

Since $db = 0$ in relative differentials over B,

$db = 0 \in \Omega_{A/B}$ $\qquad A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B}$ (composite 0)

The kernel of $\Omega_{A/C} \to \Omega_{A/B}$ is generated by db's

Recall: $\Omega_{A/C}$ is the A-module generated by $\{da \mid a \in A\}$ subject to

1) Additivity $\quad d(a+a') = da + da'$
2) $\quad d(b) = 0 \quad \forall b \in B$
3) Leibniz $\quad d(aa') = ad(a') + d(a)a'$

$\square$

thm: given above + $B \twoheadrightarrow A$ giving $A \cong B/I$. Then

$$I/I^2 \to A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to 0$$

$I/I^2 = (N_{\operatorname{Spec} B/C} \operatorname{Spec} B)^\vee$

is exact

pf:

$$\begin{tikzcd}
I/I^2 \arrow[r, phantom, "\cong"] \arrow[dd, "i"'] & B/I \otimes I \arrow[d] \\
 & B/I \otimes_B B \arrow[d, "1 \otimes d"] \\
1 \otimes di & A \otimes \Omega_{B/C}
\end{tikzcd}$$

Since $i = 0$ in A, composition is 0.

By previous thm, only need to show that the di generate

$\operatorname{Ker}(A \otimes_B \Omega_{B/C} \to \Omega_{A/C})$

as an A-module

generated by db $\qquad$ generated by da, $a \in B/I$

$\square$

for $X \overset{i}{\underset{\text{closed}}{\hookrightarrow}} U \underset{\text{open}}{\hookrightarrow} Y$

corresponding to I qcoh sheaf of ideals

$$N_X Y^\vee := I/I^2 \cong i^* I$$

Is called the conormal sheaf.

Globalize sheaf of differentials

$X \longrightarrow Y$ map of schemes

we define $\Omega_{X/Y}$ qcoh sheaf on X of relative differentials or Kähler Differentials

1) To do this, we could cover Y with $\operatorname{Spec} B$'s cover $f^{-1}(\operatorname{Spec} B)$ with $\operatorname{Spec} A$'s and glue the $\Omega_{A/B}$ on $\operatorname{Spec} A$'s together on distinguished open affines

$$\begin{array}{l} D_A(f) \hookrightarrow \operatorname{Spec} A \\ \quad \wr\| \\ D_{A'}(f') \hookrightarrow \operatorname{Spec} A' \end{array}$$

and showing we can construct compatible isos on $D_A(f) \cong D_{A'}(f')$ this is possible b/c $\Omega_{A/B}$ behaves well under localization

Instead:

2) Use "geometric fact"

$$X \xrightarrow{\Delta} X \times_Y X$$

$$N_X(X \times_Y X) \cong T_Y X$$

To do this:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rr, bend left, "\text{diagonal map } (1,1)"] & U \arrow[r] & X \times_Y X \end{tikzcd}$$

Cover Y with $\operatorname{Spec} B$ s and $f^{-1}(\operatorname{Spec} B)$ with $\operatorname{Spec} A$'s. Then $A \otimes_B A \xrightarrow{\text{mult}} A$ induces $\Delta|_{\operatorname{Spec} A}$. Let $U = \bigcup \operatorname{Spec} A \otimes_B A$.

Then Δ factors as

$$X \underset{\text{closed immersion}}{\overset{\tilde{\Delta}}{\hookrightarrow}} U \subset X \times_Y X$$

warning: Δ not always a closed immersion

When it is, f is called separated.

Non-ex  $\mathbb{A}^1_k \cup \mathbb{A}^1_k$

$\mathbb{A}^1_k - \{0\}$

$$f \downarrow$$

$$\operatorname{Spec} k$$

f not separated.

(sheaf of relative differentials)

Def: (global) $\Omega_{X/Y} := \mathcal{I}/\mathcal{I}^2$

$\cong \delta^* \mathcal{I}$

We show the restriction of $\Omega_{X/Y}|_{\operatorname{Spec} A}$ is iso to our previous def of

$$\begin{tikzcd} \operatorname{Spec} A \arrow[d] & \subset X \arrow[d] \\ \operatorname{Spec} B & \subset Y \end{tikzcd}$$

$\Omega_{A/B}$ as A-module generated by da

1) Additivity 2) $db = 0$ 3) Leibniz

By definition (global)

$\Omega_{X/Y}|_{\operatorname{Spec} A} = \widetilde{\mathcal{I}/\mathcal{I}^2}$

Where $\mathcal{I} = \ker(A \otimes_B A \to A)$

$\overset{\shortparallel}{\mathcal{I}(\operatorname{Spec} A \otimes_B A)}$

Thm (21.2.28): there is a canonical iso $\Omega_{A/B} \cong \mathcal{I}/\mathcal{I}^2$

pf: map of sheaves of sets $d : \mathcal{O}_X \to \mathcal{I}/\mathcal{I}^2$

$df = [\pi_1^* f - \pi_2^* f] \in \mathcal{I}/\mathcal{I}^2$

$$\begin{tikzcd} & X \times_Y X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

This makes sense b/c $df \in \mathcal{I}$

b/c $\overset{\ker}{\mathcal{I}} \to A \otimes A \to A$

$f \otimes 1 - 1 \otimes f \in \mathcal{I}$

$\overset{\shortparallel}{\pi_1|_{\operatorname{Spec} A}^* f} \quad \overset{\shortparallel}{\pi_2|_{\operatorname{Spec} A}^* f}$

The canonical map $\Omega_{A/B} \to \mathcal{I}/\mathcal{I}^2$

$da \mapsto a \otimes 1 - 1 \otimes a$

this is well defined: $db \mapsto b \otimes 1 - 1 \otimes b$

tensor is over B $= 1 \otimes b - 1 \otimes b$

$= 0$

additivity ✓ by bilinearity

$$\begin{array}{c} a \leftarrow a \otimes 1 \\ a \leftarrow 1 \otimes a \end{array}$$

$\mathcal{I} \to A \otimes_B A \twoheadrightarrow A \cong \dfrac{A \otimes_B A}{\mathcal{I}}$

Leibniz: $d(a_1 a_2) = a_1 a_2 \otimes 1 - 1 \otimes a_1 a_2$

$a_1 d(a_2) + a_2 d(a_1)$

$= (1 \otimes a_1) d(a_2) + (a_2 \otimes a_1) d(a_1)$

$= (1 \otimes a_1)(a_2 \otimes 1 - 1 \otimes a_2) + (a_2 \otimes a_1)(a_1 \otimes 1 - 1 \otimes a_1)$

$= \cancel{a_2 \otimes a_1} - 1 \otimes a_1 a_2 + a_2 a_1 \otimes 1 - \cancel{a_2 \otimes a_1} = d(a_1 a_2)$

We claim map is surjective:

suffices to show that I

is generated by $a \otimes 1 - 1 \otimes a$

Take $\sum x_i y_i = 0$ with $x_i, y_i \in A$

an arbitrary element of I is of the form

$$\sum (x_i \otimes y_i) = \sum (x_i \otimes y_i) - \underbrace{\left(\sum x_i y_i\right)}_{0 \text{ from above}} \otimes 1$$

$$= \sum (x_i \otimes y_i - x_i y_i \otimes 1)$$

$$= \sum (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1)$$

Thus the canonical map is surjective.

Define $I/I^2 \longrightarrow \Omega_{A/B}$ by restriction to I

of $A \otimes_B A \longrightarrow \Omega_{A/B}$

$a_1 \otimes a_2 \longmapsto a_1 d(a_2)$

we show the restriction to I^2 is 0 so well-defined.

Let $\sum x_i y_i = 0$ and $\sum x_i' y_i' = 0$.

We wish to show $\left(\sum x_i \otimes y_i\right)\left(\sum x_i' \otimes y_i'\right) \overset{g}{\longmapsto} 0$

$$\| $$

$$\sum_{ij} x_i x_j' \otimes y_i y_j' \longmapsto \sum_{ij} x_i x_j' d(y_i y_j')$$

Leibniz

$$= \sum_{ij} \left(x_i x_j' y_i d(y_j') + x_i x_j' y_j' d(y_i)\right)$$

$$= \sum_j x_j' d(y_j') \cancel{\left(\sum_i x_i y_i\right)}^{0} + \sum_i x_i d(y_i) \cancel{\sum_j (x_j' y_j')}^{0}$$

$$= 0.$$

Suffices to show: $\Omega_{A/B} \twoheadrightarrow I/I^2 \xrightarrow{g} \Omega_{A/B}$

is the minus identity $da \mapsto a \otimes 1 - 1 \otimes a \mapsto \cancel{a\,d1}^{0} - 1\,da$

So the surjection $\Omega_{A/B} \twoheadrightarrow I/I^2$ is also injective

$f : X \to Y$ map of schemes

we define $\Omega_{X/Y} = \tilde{\Delta}^* \mathcal{I}_{\tilde{\Delta}}$

$$\begin{tikzcd} X \arrow[r] \arrow[rrrr, bend left, "\Delta \text{ diagonal map}"] & \underset{\operatorname{Spec} B \subset Y}{\bigcup} \ \underset{\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)}{\bigcup} & \operatorname{Spec}(A \otimes_B A) \arrow[rr] & & X \times_Y X \end{tikzcd}$$

We saw: for any $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$

$\Omega_{X/Y}|_{\operatorname{Spec} A} = \tilde{M}$ where M is the A-module with generators da, $a \in A$ $\qquad \Omega_{A/B} = M = \Omega_{X/Y}(\operatorname{Spec} A)$

+ 3 properties from before

Thm (21.2.29) Let

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & B \end{tikzcd}$$

be a morphism of B-schemes.

Then

(1) $f^*\Omega_{Y/B} \to \Omega_{X/B} \to \Omega_{X/Y} \to 0$

(2) If f^* is a closed immersion, then

$$\mathcal{N}^\vee_{X/Y} \to f^*\Omega_{Y/B} \to \Omega_{X/B} \to 0$$

$$0 \leftarrow N_X Y \leftarrow TY \leftarrow TX \leftarrow 0 \text{ : dual}$$

Pf: exactness can be checked on the level of stalks, and we proved them in affine case. $\boxed{B}$

Def: $f: X \to Y$ is smooth, locally of finite type in the sense that $\exists$ a cover of Y of affines $\operatorname{Spec} B$ and a cover of $f^{-1}(\operatorname{Spec} B)$ by open affines $\operatorname{Spec} A$ s.t. A is a f.g. B-algebra and $\Omega_{X/Y}$ is locally free of rank n.

(More general: 2.2.31)

Thm: Let $X \overset{i}{\hookrightarrow} Y$ be a closed immersion determined by $X = V(s)$, where s is a section of a locally free sheaf of rank $n-m$ corresponding to a regular sequence. Let Y (over k) be smooth of dimension n, let X be smooth of dimension m. Then

$$0 \to \mathcal{N}^\vee_{X/Y} \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to 0$$

is exact.

Pf: we saw, $\mathcal{N}^\vee_{X/Y}$ is locally free of rank $n-m$.

By def of smoothness $\Omega_{X/k}$ is locally free of rank m

$\Omega_{Y/k}$ is locally free of rank n

By previous thm, we have exactness except at left.

Since a surjection of rank $n-m$ vector spaces is an injection, the maps on the fibers of

$$\mathcal{N}_{X/Y}^{\vee} \to f^*\Omega_{Y/k}$$

are injective. It is a result of commutative algebra that if

$$R^{n-m} \xrightarrow{f} R^{n-m}$$ is an iso mod every

maximal ideal, it is an iso. $\boxed{}$

Cohomology of Quasi-Coherent sheaves

Let X be a scheme. There are functors "derived functors" of $\mathcal{F} \mapsto \mathcal{F}(X)$ "global sections"

$H^i(X,-): \; QCoh(X) \longrightarrow AbGrp \qquad i = 0,1,\ldots$

satisfying:

(1) $H^0(X,\mathcal{F}) = \mathcal{F}(X)$

(2) Given $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$ short exact sequence, we have

$$0 \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{Q}) \to H^1(X,\mathcal{F}) \to \cdots$$

$$\to H^i(X,\mathcal{F}) \to H^i(X,\mathcal{G}) \to H^i(X,\mathcal{Q}) \to H^{i+1}(X,\mathcal{F})$$

natural w.r.t. morphisms
$$\begin{array}{ccccccccc} 0 & \to & \mathcal{F} & \to & \mathcal{G} & \to & \mathcal{Q} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{F}' & \to & \mathcal{G}' & \to & \mathcal{Q}' & \to & 0 \end{array}$$

Lemma: Suppose $\mathcal{F}: X \to \operatorname{Spec} R$ is separated. Then for any open affines $\operatorname{Spec} A, \operatorname{Spec} B \subset X$, their intersection $\operatorname{Spec} A \cap \operatorname{Spec} B$ is affine.

Pf:
$$\begin{array}{ccc} \operatorname{Spec} A \cap \operatorname{Spec} B & \xrightarrow{\Delta'} & \operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times_{\operatorname{Spec} R} X \end{array}$$

is a fiber product. $\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B \cong \operatorname{Spec}(A \otimes_R B)$ is affine. Δ is a closed immersion by def. of separated.

Thus Δ' is a closed immersion and any closed immersion into an affine is affine. $\boxtimes$

Assume $X \to \mathrm{Spec}\, R$ separated, X quasicompact.

Let $\mathcal{U} = \{U_1, \dots, U_n\}$ cover X by open affines.

For $I \subset \{1, \dots, n\}$ let $U_I = \bigcap_{i \in I} U_i$

Let $C^*(\mathcal{U}, \mathcal{F})$ be the complex

$$0 \to \prod_{|I|=1} \mathcal{F}(U_I) \to \prod_{|I|=2} \mathcal{F}(U_I) \to \cdots \to \prod_{|I|=i} \mathcal{F}(U_I) \to \prod \mathcal{F}\Big(\bigcap_{i=1}^{n} U_i\Big) \to 0$$

Where the map

$$\begin{tikzcd}
\prod_{|I|=i} \mathcal{F}(U_I) \arrow[r] & \prod_{|J|=i+1} \mathcal{F}(U_J) \arrow[d, "\pi"] \\
\bigoplus_{|I|=i} \mathcal{F}(U_I) \arrow[u] & \mathcal{F}(U_J) \\
\mathcal{F}(U_I) \arrow[u] &
\end{tikzcd}$$

is 0 unless $J = I \cup \{j_0\}$ in which case it is the restriction

$$\mathcal{F}\Big(\bigcap_{i \in I} U_i\Big) \to \mathcal{F}\Big(U_{j_0} \cap \bigcap_{i \in I} U_i\Big)$$

times $(-1)^{k-1}$ where j_0 is the kth element of J

__Claim__: this really is a complex

__pf__: i.e. $\delta_{i+1} \circ \delta_i = 0$

It suffices to show $\pi_J \delta_{i+1} \circ \delta_i \overset{\text{inclusion}}{\iota_I} = 0$ for all I, $|I| = i$ and J with $|J| = i+2$

We may assume $J = I \cup \{j_0, j_1\}$. Then

$\pi_J \circ \delta_{i+1} \circ \delta_i \circ \iota_I$ has two summands correspondingly

$f : \mathcal{F}(U_I) \to \mathcal{F}(U_{I \cup \{j_0\}}) \to \mathcal{F}(U_J)$

$g : \mathcal{F}(U_I) \to \mathcal{F}(U_{I \cup \{j_1\}}) \to \mathcal{F}(U_J)$

WLOG, assume $j_0 < j_1$

$J = \{a_1, \dots, a_{k_1}, j_0, a_{k_1+1}, \dots, a_{k_2}, j_1, a_{k_2+1}, \dots, a_i\}$

f is the restriction $\mathcal{F}(U_I) \to \mathcal{F}(U_J)$ times $(-1)^{k_1+k_2+1}$

g " " " " " times $(-1)^{k_2+k_1}$

thus $f + g = 0$ as desired. So $\pi_J \circ \delta_{i+1} \circ \delta_i \circ \iota_I = 0$ $\boxtimes$

Recall that given a complex

C_* $\quad M_i \xrightarrow{\delta_i} M_{i+1} \xrightarrow{\delta_{i+1}} M_{i+2}$

the $(i+1)$ st cohomology is

$H^{i+1}(C_*) = \ker \delta_{i+1} / \operatorname{Im} \delta_i$

Moreover the snake lemma showed

Lemma: given SES $0 \to C_\bullet \to M_\bullet \to Q_\bullet \to 0$

of complexes meaning

$$\begin{tikzcd} 0 \arrow[r] & C_i \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d] & Q_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_{i+1} \arrow[r] \arrow[d] & M_{i+1} \arrow[r] \arrow[d] & Q_{i+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_{i+2} \arrow[r] \arrow[d] & M_{i+2} \arrow[r] \arrow[d] & Q_{i+2} \arrow[r] \arrow[d] & 0 \\ & \, & \, & \, & \end{tikzcd}$$

has short exact rows we have a long exact sequence

$$H^i(C_*) \to H^i(M_*) \to H^i(Q_*) \to H^{i+1}(C_*) \to \cdots$$

Def: $H^i_{\mathcal{U}}(X, \mathcal{F})$ to be the ith cohomology of the complex above

$$\to \prod_{|I|=1} \mathcal{F}(U_I) \to \prod_{|I|=2} \mathcal{F}(U_I) \to \cdots$$

Rmk: this def satisfies the claimed LES.

Rmk: Since $\mathcal{F}$ is a sheaf,

$$0 \to \mathcal{F}(X) \to \prod_{i=1}^n \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j) \text{ is exact.}$$

Identifying with the first terms of our complex, $H^0_{\mathcal{U}}(X, \mathcal{F}) \cong \mathcal{F}(X)$ for any cover $\mathcal{U}$.

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$$\left\{ x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0 \right\}$$

n coeffs