Physics 622: General relativity

Today: Newtonian Gravity

$$m_a \ddot{\vec{r}}_a(t) = -G \sum_{b\neq a} \frac{m_a m_b (\vec{r}_b - \vec{r}_a)}{|\vec{r}_b - \vec{r}_a|^3}$$

$a = 1, \dots, N$ particles

G = Newton's constant $\quad 10^{-11} \frac{Nm^2}{kg^2} = 1$ ← pick units to make it 1

Particle = point in space $\quad \vec{r}_a(t)$

Newton: Space is 3D, Euclidean, immutable, homogeneous, isotropic ("featureless properties")

Time t equal in all terms of equation ("scorekeeper")

Choose $t=0$: time is $t \in \mathbb{R}$

Choose origin O fixed point: position is displacement $\vec{r}$ from O $\quad \vec{r}_a(t)$ → now makes sense

Choose $\hat{e}_i$ basis right-handed orthonormal

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

$$\hat{e}_i \times \hat{e}_j = \sum_k \varepsilon_{ij}{}^k \hat{e}_k = \varepsilon_{ij}{}^k \hat{e}_k$$

3-tensor? ← einstein summation convention

$$\vec{r} = \sum r^i \hat{e}_i = (\vec{r}\cdot\hat{e}_i)\hat{e}_i$$ ← implicit summation

$y = Ax \longleftrightarrow y^i = A^i{}_j x^j$

$y = A^T x \longleftrightarrow y^i = A_j{}^i x^j$ ← switch what you sum over first

$x^i = y^j$ ↳ meaningless! index imbalance!

$x^i = y^i$ ↕ $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$

$$m_a \ddot{\vec{r}}_a(t) = -G \sum_{b\neq a} \frac{m_a m_b (\vec{r}_b - \vec{r}_a)}{|\vec{r}_b - \vec{r}_a|^3}$$

↳ $\underset{\text{3 dim, N particles}}{3N}$ ODEs $\quad r_a{}^i(t)$

$\exists!$ solution given $6N$ initial conditions $\quad \{\vec{r}_a(0)\} = \vec{r}_{a0}$

$\{\dot{\vec{r}}_a(0)\} = \vec{v}_{a0}$

Euclidean space: $|\vec{r}| = \sqrt{r^i r_i}$ pythagoras

Immutable: Laws of Newton don't change with time

Set $\vec{r}_a{}'(T) = \vec{r}_{a0}$, $\dot{\vec{r}}_a{}'(T) = \vec{v}_{a0}$ $\Rightarrow$ $\vec{r}_a{}'(t) = \vec{r}_a(t-T)$

↑ would fail if Newton's Constant was time dependent!

If we chose $t=T$ to be $t'=0$ so $t' = t - T$, everything still works!

Same motion because $\vec{r}_a{}'(t') = \vec{r}_a(t'+T)$

$$\underline{\frac{d}{dt}\left(\sum_a \frac{1}{2} m_a \dot{\vec{r}}_a^{\,2}\right)} = \sum_a m_a \dot{\vec{r}}_a \cdot \ddot{\vec{r}}_a$$

$$= -G \sum_{a\neq b} \dot{\vec{r}}_a \cdot \frac{m_a m_b (\vec{r}_b - \vec{r}_a)}{|\vec{r}_b - \vec{r}_a|^3} = -\sum_a \dot{\vec{r}}_a \vec{\nabla}_a \phi$$

$$= \underline{-\frac{d\phi}{dt}}$$

Conservation of energy!

energy is conserved because of time invariance!

Consider $\phi(\{\vec{r}_a\}) = -\underline{G} \sum_{a\neq b} \frac{m_a m_b}{\underline{|\vec{r}_b - \vec{r}_a|}}$

$$-\vec{\nabla}_a \phi = -G m_a \sum_{b\neq a} \frac{m_b (\vec{r}_b - \vec{r}_a)}{|\vec{r}_b - \vec{r}_a|^3}$$

violates homogeneity

$$\boxed{m_a \ddot{\vec{r}}_a(t) = -G \sum_{b\neq a} \frac{m_a m_b (\vec{r}_b - \vec{r}_a)}{|\vec{r}_b - \vec{r}_a|^3}} + F_a(\vec{r}_a)$$

"Homogeneous, like homogenized milk, means it's the same everywhere"

= laws of nature same everywhere

Active: move experiment, new initial conditions

$\vec{r}_a'(0) = \vec{r}_{a0} + \vec{\Delta}$ constant shift $\rightarrow$ $\vec{r}_a'(t) = \vec{r}_a(t) + \vec{\Delta}$ "deep shit, but true"

$\dot{\vec{r}}_a'(0) = \vec{v}_{a0}$

Passive: choice of origin O immaterial

Choose O' displaced by $-\vec{\Delta}$

Same motion described by $\vec{r}_a' = \vec{r}_a + \vec{\Delta}$

LHS of eqn above, antisymmetric sum

$$\frac{d}{dt}\left(\sum_a m_a \dot{\vec{r}}_a\right) = \sum_a m_a \ddot{\vec{r}}_a = 0 \rightarrow \text{Newton's 3rd law}$$

no gravitational momentum in Newtonian physics $\rightarrow$ problem "spooky action at a distance"

Isotropic $\rightarrow$ laws of nature same in all directions

rotate experiment $\rightarrow$ same results!

<u>Rotations</u>: R linear action on vectors

$$(Rv)^i = R^i{}_j v^j$$

Preserve lengths & angles

$$(Rv)^T(Ru) = v^T R^T R u = v^T u$$

$$R^T R = I \qquad R \in O(3)$$

$$(Rv)^i (Ru)_i = v_i u_i = \underline{v^j u^k \delta_{jk}}$$

$$= (R^i{}_j v^j)(R_{ik} u^k)$$

$$= \underline{R^i{}_j R_{ik} v^j u^k} \qquad R^i{}_j R_{ik} = \delta_{jk}$$

$$\det R^T = \det R = \pm 1$$

Rotations: $\det R = 1$ $\qquad SO(3)$

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ orthogonal but not rotation

Active rotation: $r_a^{i\prime}(0) = R^i{}_j\, r_a^j(0)$

$\dot{r}_a^{i\prime}(0) = R^i{}_j\, v_{a0}^j$

$\Rightarrow\ r_a^{\prime i}(t) = R^i{}_j\, r_a^i(t)$

Passive: $\hat{e}_i{}' = R^{-1}\hat{e}_i$

Consequence: conservation of angular momentum

$$\hat{L} = \sum_a m_a \vec{r}_a \times \dot{\vec{r}}_a \qquad \frac{d\hat{L}}{dt} = 0$$

$$\underbrace{m_a \ddot{\vec{r}}_a(t)}_{\text{vector}} = \underbrace{-G\sum_{b\neq a} \frac{m_a m_b(\vec{r}_b - \vec{r}_a)}{|\vec{r}_b - \vec{r}_a|^3}}_{\text{vector}}$$

Change of basis: transform both sides by same R

$$\vec{A} = \vec{B} \qquad \underbrace{\vec{A} - \vec{B}}_{\text{Stays zero if you rotate!}} = 0$$

Galilean Relativity

$\vec{r}_a(t) \to r_a'(t) = \vec{r}_a(t) - \vec{\delta}(t)$

$\dot{\vec{r}}'(t) = \dot{\vec{r}}_a(t) - \dot{\vec{\delta}}(t)$

$\underline{\ddot{\vec{r}}(t)} = \ddot{\vec{r}}_a(t) - \cancel{\ddot{\vec{\delta}}(t)}\ 0$

does not change

same same

$$m_a \ddot{\vec{r}}_a(t) = -G\sum_{b\neq a} \frac{m_a m_b(\vec{r}_b - \vec{r}_a)}{|\vec{r}_b - \vec{r}_a|^3}$$

same

Active: experiment can move at constant $\vec{v}$ i.e. in relativistic physics, it means nothing to be at rest

Passive: can measure from O' moving at constant $\vec{v}$ (relative to O)

10 variables!
$$\begin{cases} t' = t + T \\ \vec{r}' = R\vec{r} + \vec{\delta} - \vec{v}t \end{cases}$$

Consequence? Homework, I guess...

$$\cancel{m}_a \ddot{\vec r}_a(t) = -G \underbrace{\sum_{b\neq a} \frac{\cancel{m_a} m_b (\vec r_b - \vec r_a)}{|\vec r_b - \vec r_a|^3}}_{\text{diverges when particles get very close!}}$$

Gravity: pick t, $\vec r_a = \vec r$ → can find $\ddot{\vec r}_a(t)$

If you change m_a, leave m_b, $\{\vec r_a\}$ unchanged

↳ will have same acceleration

$\vec g(\vec r, t)$: any probe accelerates at same $\vec g$

↳ small mass (relative)

Divergences → replace particles with smooth distribution of mass

$$m_a\, \vec r_a(t) \to \rho(\vec r, t) = \sum m_a\, \delta^{(3)}(\vec r - \vec r_a(t))$$
$$N \to \infty$$
$$m_a \to 0$$

Newtonian gravity invariant under Galilean group ⊃ Euclidean Group

$G(T, \vec\Delta, R, \vec v)$:
$$t \to t' = t - T$$
$$\vec r \to \vec r' = R\vec r - \vec\Delta - \vec v t$$

10-dimensional group

identity element → $G(0, \vec 0, I, 0)$: $t \to t$, $\vec r \to \vec r$

Introduce a probe $m \ll m_a$ $\forall a$ at $(t, \vec r)$ it will accelerate at

$$\cancel{m}\, \vec g(t, \vec r) = G \sum_a \frac{\cancel{m}\, m_a (\vec r_a(t) - \vec r)}{|\vec r_a(t) - \vec r|^3}$$

☆ Principle of Equivalence: if you move at $(t, \vec r)$ with acceleration $\vec g(t, \vec r)$, you are weightless

$$\boxed{m_a \ddot{\vec{r}}_a(t) = -G \sum_{b \neq a} \frac{m_a m_b (\vec{r}_b - \vec{r}_a)}{|\vec{r}_b - \vec{r}_a|^3}}$$

Continuum limit $\begin{matrix} N \to \infty \\ m_a \to 0 \end{matrix}$

$$\rho(t, \vec{r}) = \sum_a m_a \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$ ~ smooth function of space/time

$$\int_V d^3\vec{r}\, \rho(t,\vec{r}) = \sum_a m_a \int_V d^3\vec{r}\, \delta^{(3)}(r - r_a(t)) = \sum_{a \in V_t} m_a$$

$$V_t = \{ a \,|\, \vec{r}_a(t) \in V \}$$

$$\frac{d}{dt} \rho(t, \vec{r}) = \sum_a m_a \left(-\dot{\vec{r}}_a(t)\right) \cdot \vec{\nabla} \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$= -\vec{\nabla} \cdot \left(\sum_a m_a \dot{\vec{r}}(t) \delta^{(3)}(\vec{r} - \vec{r}_a(t)) \right)$$

$$\vec{\pi}(t, \vec{r}) = \sum_a m_a \dot{\vec{r}}_a(t) \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

mass current

Momentum density = mass current

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{\pi} = 0$$ continuity equation = local mass conservation

$$\frac{\partial}{\partial t} \int_V d^3\vec{r}\, \rho(t,\vec{r}) = -\int_V d^3\vec{r}\, \vec{\nabla} \cdot \vec{\pi}(t, \vec{r})$$

$$= -\int_{\partial V} \vec{\pi} \cdot d\vec{a}$$ mass locally conserved

Is $\vec{\pi} = \sum_a m_a \dot{\vec{r}}_a(t) \delta^{(3)}(\vec{r} - \vec{r}_a(t))$ (locally) conserved?

Of course: (translation invariant) $\frac{d}{dt}\left(\int \vec{\pi}\, d^3\vec{r} \right) = \frac{d\vec{p}}{dt} = 0$

$$m_a \ddot{\vec{r}}_a(t) = \vec{F}_a(\{\vec{r}_i\}) = -\vec{\nabla}_a U(\{\vec{r}_i\})$$

$$\frac{\partial \pi^i}{\partial t} = \frac{\partial}{\partial t}\left(\sum_a m_a \dot{r}_a^i(t) \delta^{(3)}(\vec{r} - \vec{r}_a(t)) \right)$$

annoying! $\left[= \sum_a m_a \ddot{r}_a^i(t) \delta^{(3)}(\vec{r} - \vec{r}_a(t)) \right.$

divergence $\left[- \sum_a m_a \dot{r}_a^i(t) \dot{\vec{r}}_a(t) \cdot \vec{\nabla} \delta^{(3)}(\vec{r} - \vec{r}_a(t)) \right.$

$$\sum_a \vec{F}_a \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$\frac{\partial}{\partial x^i}$ $\frac{\partial}{\partial r^i}$

$$- \partial_j \sum_a m_a \dot{r}_a^i \dot{r}_a^j \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

Momentum conserved... but not locally conserved

Einstein: "Spooky"

$$\frac{\partial \pi^i(t, \vec{r})}{\partial t} + \partial_j t^{ij}(t, \vec{r}) = f^i(t, \vec{r})$$

$$f^i = \sum_a \vec{F}_a \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$ force density

$t^{ji} = t^{ij} = \sum_a m_a \dot{r}_a^i \dot{r}_a^j \delta^{(3)}(\vec{r} - \vec{r}_a(t))$

Maxwell tensor

$f(\vec{r})$ $\quad \frac{d}{dt} f(\vec{r}(t))$

$= \frac{d}{dt} f(x(t), y(t), z(t))$

Under rotation $\quad \vec{r} \to \vec{r}\,' = R\vec{r} \qquad r'^i = R^i{}_j r^j$

$$t^{ij\prime}(t, \vec{r}\,') = R^i{}_k R^j{}_\ell t^{k\ell}(t, R^{-1}\vec{r}\,')$$

Given $\rho(t, \vec{r})$ determines

$$\vec{g}(t, \vec{r}) = G \int d^3\vec{r}\,' \cdot \frac{\rho(t, \vec{r}\,')(\vec{r}\,' - \vec{r})}{|\vec{r}\,' - \vec{r}|^3}$$

$\vec{f}$ on m is $\vec{f} = m\vec{g} = -m\vec{\nabla}\phi$

$$\vec{\nabla}\left(\frac{1}{|\vec{r}|}\right) = -\frac{\vec{r}}{|\vec{r}|^3}$$

$$\vec{g}(t, \vec{r}) = -\vec{\nabla}\phi(t, \vec{r})$$

$$\phi(t, \vec{r}) = -G \int d^3\vec{r}\,' \frac{\rho(t, \vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

$$\vec{\nabla} \cdot \left(\frac{\vec{r}}{|\vec{r}|^3}\right) = -\nabla^2\left(\frac{1}{|\vec{r}|}\right) = 4\pi\delta^{(3)}(\vec{r})$$

$$\nabla^2\phi(t, \vec{r}) = 4\pi G\rho = -\vec{\nabla} \cdot g(t, \vec{r})$$

Tensors

Vectors: $\vec{v}$ has magnitude / direction $\to$ physical

rotate: $\vec{v} = v^i \hat{e}_i$

using $\hat{e}'_i = R^{-1}\hat{e}_i = R^T\hat{e}_i$

$\vec{r} = r'^i \hat{e}'_i \qquad r'^i = \vec{r} \cdot \hat{e}'_i$

$= (R^{-1}\hat{e}_i) \cdot \vec{r}$

$= \hat{e}_i \cdot R\vec{r}$

$= R^i{}_j r^j$ 3 physical quantities

$\vec{A} = \vec{B} \leftrightarrow \vec{A} - \vec{B} = 0$

9 quantities

$T^{ij} \to R^i{}_k R^j{}_l T^{kl}$ (rank) 2-tensor representations of the rotation group

$T^{i_1 \cdots i_n} \to R^{i_1}{}_{j_1} \cdots R^{i_n}{}_{j_n} T^{j_1 \cdots j_n}$ n-tensor

We already know some tensors!

δ_{ij} is a 2-tensor $\delta'^{ij} = R^i{}_k R^j{}_l \delta^{kl} = \delta^{ij}$

$\varepsilon'^{ijk} = R^i{}_l R^j{}_m R^k{}_n \varepsilon^{lmn} = \varepsilon^{ijk}$

2) Doing with tensors

- S, T rank n $(a \cdot S + b \cdot T)'^{i_1 \cdots i_n} = a S^{i_1 \cdots i_n} + b T^{i_1 \cdots i_n}$

S rank-n
T rank-m $(S \otimes T)^{i_1 \cdots i_n j_1 \cdots j_m} = S^{i_1 \cdots i_n} T^{j_1 \cdots j_m}$ rank-$n+m$

Example: u^i v^i

$(u \otimes v)^{ij} = u^i v^j$

$(u \otimes v)^{ij} \delta_{ij} = u \cdot v$

$T^{ij} \nearrow \underbrace{S^{ij}}_{\text{symmetrized}} = \frac{1}{2}(T^{ij} + T^{ji}) = T^{(ij)} = S^{ji}$

$\searrow A^{ij} = \frac{1}{2}(T^{ij} - T^{ji}) = T^{[ij]} = -A^{ji}$

~ playing with representations of rotation group

Check hw: 8 conserved quantities in Newtonian gravitation

Newtonian Gravity + Galilean Relativity: Nice! ... but wrong :(

Maxwell's electrodynamics is NOT invariant

$\vec{\nabla} \cdot \vec{B} = 0$ $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$

$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ $\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{J}$

speed of light does not satisfy Galilean Relativity

✻ Read up on Special Relativity

Lorentz transformations: Position + time $(t, \vec{r})$

of some event measured by two inertial observers

O, O' with relative velocity $\vec{v} = v\hat{x} = v\hat{x}'$

By shifts, rotations align axes, chosen $t = t' = 0$

$$t' = \frac{t - vx}{\sqrt{1-v^2}} \qquad t = \frac{t + vx'}{\sqrt{1+v^2}}$$

$$x' = \frac{x - vt}{\sqrt{1-v^2}} \qquad x = \frac{x' + vt'}{\sqrt{1+v^2}}$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

speed of light (chosen units)

$v^2 \ll 1 \longrightarrow$ timespans become relatively (two) large

$\rightarrow$ recover Galileo at slow velocities (relative to speed of light)

Two inertial frames: Standard Configuration (SC)

$$\boxed{\begin{aligned} t' &= \frac{t - vx}{\sqrt{1-v^2}} & t &= \frac{t + vx'}{\sqrt{1+v^2}} \\ x' &= \frac{x - vt}{\sqrt{1-v^2}} & x &= \frac{x' + vt'}{\sqrt{1+v^2}} \end{aligned}}$$

1. Simultaneity is relative. $t' \neq t$

   Space: not a physical (invariant) concept

   Solution: spacetime – 4D space

free (inertial) motion is linear world line

important! this means this shift preserves straight lines

2. Since Lorentz Transformations (LT) are linear transformations (in x, y, z, t),

   Spacetime is affine space (pick an origin, O becomes V.S.)

   Event: position @ time

3. Choose inertial frame

   Coordinates: $(t, \vec{r})$ some people write $x^0 = ct$ (but we are working in units s.t. $c = 1$)

   X: $x^\mu = (x^0, x^i) = (t, \vec{r})$ $\mu = 0, 1, 2, 3$

   Greek: μ, ν, α, β go $0, 1, 2, 3$

   Latin: i, j, k, l, m go $1, 2, 3$

$$\boxed{\begin{array}{ll} t' = \dfrac{t - vx}{\sqrt{1-v^2}} & t = \dfrac{t' + vx'}{\sqrt{1+v^2}} \\[2ex] x' = \dfrac{x - vt}{\sqrt{1-v^2}} & x = \dfrac{x' + vt'}{\sqrt{1+v^2}} \end{array}}$$

4. LT

label multiple equations $\sum_{\nu=0}^{3}$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

sum over 4-terms

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \cdots & \\ \Lambda^1{}_0 & \ddots & & \\ \vdots & & & \Lambda^3{}_3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = (1-v^2)^{-\frac{1}{2}}$$

5. This is SC.

Any two inertial frames related by <u>affine</u> (shift + linear)

Bring to SC by shifts + rotation

linear structure is preserved.

6. By shift, $\vec{v}$ genl $\quad \vec{r} = \overbrace{\dfrac{\vec{r}\cdot\vec{v}}{v^2}\vec{v}}^{\vec{r}_{\parallel}} + \vec{r}_{\perp}$

$$t' = \frac{t - \vec{v}\cdot\vec{r}}{\sqrt{1-v^2}} \qquad \vec{r}_{\perp}{}' = \vec{r}_{\perp}$$

$$\vec{r}_{\parallel}{}' = \frac{1}{\sqrt{1-v^2}}\left(\vec{r}_{\parallel} - \vec{v}t\right)$$

7. More structure: Check

$$x'^2 - t'^2 + y'^2 + z'^2 = s^2 = x^2 - t^2 + y^2 + z^2$$

$$\vec{r}'^2 - t'^2 = s^2 = \vec{r}^2 - t^2$$

Preserved by LT $\quad \mathcal{O} = \mathcal{O}'$

↳ Linear preserving $\vec{r}^2 - t^2$

(cf: $\quad (R\vec{v})^2 = \vec{v}^2 \qquad R \in O(3)$

$$v^2 = v^i v^j \delta_{ij}$$

With shifts: Poincaré - preserves

$$(\vec{r}_1 - \vec{r}_2)^2 - (t_1 - t_2)^2 = \Delta s^2 \qquad A_1 = (t_1, \vec{r}_1)$$

invariant interval $\qquad A_2 = (t_2, \vec{r}_2)$

Note: s^2 is just notation – it need not be anything squared ... or even positive!

8. Invariance ⇝ physical. What is it?

Consider $A_1 = (t_1, \vec{r}_1)$ WLOG, let $t_2 > t_1$

$A_2 = (t_2, \vec{r}_2)$

if $\Delta s^2 = (\vec{r}_2 - \vec{r}_1)^2 - (t_2 - t_1)^2 < 0$, two events are separated by a timelike interval

$\Delta \vec{r} = \vec{r}_2 - \vec{r}_1$

$\Delta t = t_2 - t_1 > 0$

$\Delta t > |\Delta \vec{r}|$ i.e. events separated more by time than space

✲ $t_2' - t_1' \overset{\text{"}}{=} \Delta t' = \gamma(v)\left(\Delta t - \vec{v} \cdot \Delta \vec{r}\right)$ $\quad |\vec{v}| < 1$

$\geq \gamma(v)\left(\Delta t - |\cancel{\vec{v}} \cdot \Delta \vec{r}|\right) > 0$

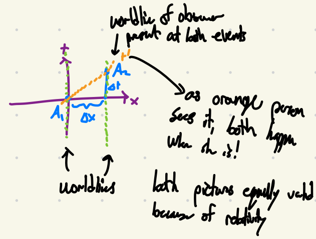
order is invariant: "all observers agree which event happens first"

✲ A probe from A_1 toward $\vec{r}_2$ @ $\vec{v} = \frac{\Delta \vec{r}}{\Delta t}$ $\quad |\vec{v}| < 1$

arrives @ A_2

✲ Probe - clock time it measures is:

$\Delta t'^2 - |\Delta \vec{r}'|^2 = -\Delta s^2 = (\Delta t)^2 - |\Delta \vec{r}|^2$



Proper time $\Delta \tau := \sqrt{-\Delta s^2}$ = time on inertial clock present at both events

= physical property

✲ In any inertial frame

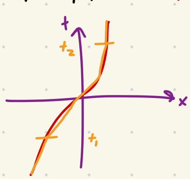
$\Delta t^2 - \Delta \vec{r}^{\,2} = -\Delta s^2 = \Delta \tau^2$

$\Rightarrow \Delta t > \Delta \tau$

↑ in any other frame

Proper time is shortest time measured by inertial observers.

✲ Noninertial clock? $c = \vec{r}(t)$ in inertial frame.



$\Delta \tau_c = \sum_{i=1}^{N}$

$t_1, t_1 + \frac{\Delta t}{N}, t_1 + \frac{2\Delta t}{N}, \ldots, t_2$

$\vec{r}(t_1), \vec{r}\left(t_1 + \frac{\Delta t}{N}\right), \ldots$

$\Delta \tau \approx \sum_{i=1}^{N} \sqrt{\left(\frac{\Delta t}{N}\right)^2 - \left(\vec{r}\left(t + \frac{i}{N}\Delta t\right) - \vec{r}\left(t + \frac{i-1}{N}\Delta t\right)\right)^2}$

$= \sum_{i=1}^{N} \frac{\Delta t}{N} \sqrt{1 - \left(\frac{\vec{r}(\ldots) - \vec{r}(\ldots)}{\Delta t / N}\right)^2}$

$\rightarrow \int_{t_1}^{t_2} dt \sqrt{1 - \vec{v}^{\,2}(t)}$

9. If $\Delta s^2 > 0$ $|\Delta \vec{r}| > \Delta t$

* Choose $\vec{v}$ $\vec{v} \cdot \Delta \vec{r} > \Delta t$ $|\vec{v}| < 1$

$\Delta t' = \gamma(v)(\Delta t - \vec{v} \cdot \Delta \vec{r}) < 0$

order of events is reversed! (if you're moving fast enough in right direction)

↳ order is observer dependent

Causal theory: information moves @ $|\vec{v}| \leq 1$

* Choose $\vec{v} \cdot \Delta \vec{r} = \Delta t$ $\Delta t' = 0$

$(\Delta \vec{r}')^2 = \Delta s^2$

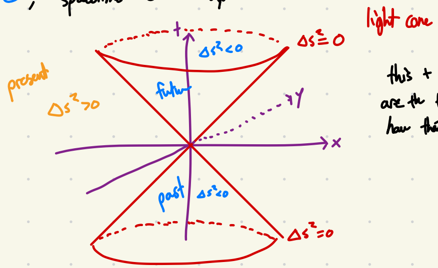
$\sqrt{\Delta s^2}$ proper distance (simultaneous event)

10. $\Delta s^2 = 0$ (boundary) $\Delta t = |\Delta \vec{r}|$ light-like separated.

Not simultaneous or colocated unless moving at speed of light (limit case)

11. Invariant interval endows spacetime with causal structure.

Given $\mathcal{O}$, spacetime breaks up:



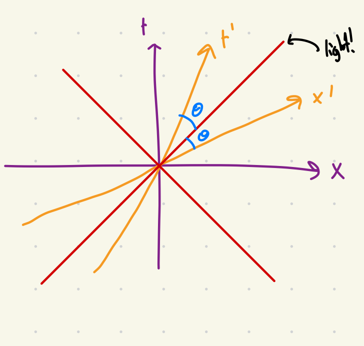
$\boxed{\text{Minkowski space}}$

this + linear structure are the two things spacetime have that tell us what physics is

Causal theory: you can't impact any events farther away from you than the distance light would travel in the time it takes

$\Delta s^2 = \Delta \vec{r}^2 - \Delta t^2$

$= \Delta x^\mu \Delta x^\nu \eta_{\mu\nu}$ $\eta = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$



$d = \sqrt{x^2 + t^2} \neq \sqrt{x'^2 + t'^2}$

↑ not invariant, so it means nothing!

The Lorentz group

Galilean $\xrightarrow{SR}$ Poincaré

$\mu = 0, \dots, 3$
$i = 1, \dots, 3$

$$x^\mu = (x^0, x^i) = (t, \vec{r})$$

- Translations $\quad x^\mu \to x'^\mu = x^\mu - a^\mu$
- Rotations $\quad x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$\Lambda = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R_{3\times 3} \end{array}\right)$$

- Boosts $\quad x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

Lorentz

Lorentz transformation preserve

$$x^2 = \vec{r}^2 - t^2 = x^\mu x^\nu \eta_{\mu\nu} \qquad \eta_{\mu\nu} = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & \mathbb{1} \end{array}\right)$$

Lorentz preserve $\quad x^\mu y^\nu \eta_{\mu\nu} = x \cdot y$

$$\underbrace{(x \pm y)^2}_{\text{preserved}} = \underline{x^2} \pm 2x\cdot y\ \underline{+y^2}$$

so too must be

$$(\Lambda x)\cdot(\Lambda y) = x\cdot y = x^\mu y^\nu \eta_{\mu\nu}$$

$$= (\Lambda x)^\rho (\Lambda y)^\sigma \eta_{\rho\sigma} = x^\mu y^\nu \eta_{\mu\nu}$$

$$= (\Lambda^\rho{}_\mu x^\mu)(\Lambda^\sigma{}_\nu y^\nu)\eta_{\rho\sigma}$$

$$= \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} \cancel{x^\mu y^\nu} = \eta_{\mu\nu} \cancel{x^\mu y^\nu}$$

$$\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} = \eta_{\mu\nu}$$

$$\Lambda^T \eta \Lambda = \eta \qquad R^T \cdot \mathbb{1} \cdot R = \mathbb{1}$$

$\Lambda \in O(1,3)$, $\quad R \in O(3) = O(0,3)$

$$\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1_3 \end{pmatrix}$$

$$P = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right)$$

Lorentz $\neq O(1,3)$

- $\Lambda^T \eta \Lambda = \eta \Rightarrow \det(\Lambda)^2 \det(\eta) = \det(\eta) = -1$ $\qquad L \subset SO(1,3)$
  $\Rightarrow \det(\Lambda) = \pm 1$ $\qquad \det(\Lambda) = 1 \ \forall \Lambda \in L$
  want $=1$ b/c need subgroup

- $-1 = \eta_{00} = (\Lambda^T \eta \Lambda)_{00} = \Lambda^\mu{}_0 \eta_{\mu\nu} \Lambda^\nu{}_0$
  $= -\underset{\mu=\nu=0}{\Lambda^0{}_0 \cdot \Lambda^0{}_0} + \Lambda^i{}_0 \Lambda^i{}_0$
  $= -(\Lambda^0{}_0)^2 + (\Lambda^i{}_0)(\Lambda^i{}_0) \geq -(\Lambda^0{}_0)^2$

  $(\Lambda^0{}_0)^2 \geq 1$ $\qquad \Lambda^0{}_0 \geq 1$ time-preserving
  or $\quad \Lambda^0{}_0 \leq -1$ time-reversing
  $\Rightarrow$ disconnected

$$T = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & \mathbb{1} \end{array}\right)$$

$L = SO(1,3)^\uparrow$ "up"

Lorentz tensors

Any additive physical property not dependent on the observer should be a tensor

Vectors: $u^\mu \to u'^\mu = \Lambda^\mu{}_\nu u^\nu$

all indices up!

Tensor $\quad T^{\mu_1 \cdots \mu_n} \to T'^{\mu_1 \cdots \mu_n} = \Lambda^{\mu_1}{}_{\nu_1} \Lambda^{\mu_2}{}_{\nu_2} \cdots \Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1 \cdots \nu_n}$

$u^\mu, v^\nu \longrightarrow (u \otimes v)^{\mu\nu}$

Contraction used invariant tensor δ_{ij} $\qquad$ "I hate myself even more than usual"

$u \cdot v = u^i v^j \delta_{ij} = (u \otimes v)^{ij} \delta_{ij}$

$u^\mu \quad v^\nu$

$$\boxed{u \cdot v = u^\mu v^\nu \eta_{\mu\nu}} \qquad \boxed{\Lambda^T \eta \Lambda = \eta}$$

$$S^{\mu_1 \cdots \mu_{n-2}} = T^{\mu\nu\mu_1 \cdots \mu_{n-2}} \eta_{\mu\nu}$$

$$S' = T' \cdots \eta = \overbrace{\Lambda \cdots \Lambda}^{n-2} T$$

$$T'^{\mu_1 \cdots \mu_n} = \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1 \cdots \nu_n}$$

$$S^{\mu_1 \cdots \mu_{n-2}} = T^{\mu\nu\mu_1 \cdots \mu_{n-2}} \eta_{\mu\nu}$$

$$S'^{\mu_1 \cdots \mu_{n-2}} = T'^{\mu\nu\mu_1 \cdots \mu_{n-2}} \eta_{\mu\nu}$$

$$= \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_{n-2}}{}_{\nu_{n-2}} T^{\rho\sigma\nu_1 \cdots \nu_{n-2}} \eta_{\mu\nu}$$

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$$

$$= \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_{n-2}}{}_{\nu_{n-2}} T^{\rho\sigma\nu_1 \cdots \nu_{n-2}} \eta_{\rho\sigma}$$

In $O(3)$, T two-tensor

$$T(v_1, v_2) = T^{ij} v_1^k v_2^l \delta_{ik} \delta_{jl}$$

In $O(1,3)$,

$$T^{\mu\nu} u_1^\lambda u_2^\rho \eta_{\mu\lambda} \eta_{\nu\rho}$$

However: $x^\mu x^\mu = x^{0^2} + x^i x^i = t^2 + r^2$

New ESC: Contractions use $x^\mu y^\nu \eta_{\mu\nu}$

$$u^\mu \to u'^\mu = \Lambda^\mu{}_\nu u^\nu$$

$$u_\mu = \eta_{\mu\nu} u^\nu \to \eta_{\mu\nu} \Lambda^\nu{}_\sigma u^\sigma$$

not a vector!

$$u' = \underbrace{\eta \Lambda} u \qquad \Lambda^T \underbrace{\eta \Lambda} = \eta$$

$$\eta \Lambda = \eta (\Lambda^T)^{-1}$$

$$u' = \underbrace{\eta (\Lambda^T)^{-1}} u$$

$$((\Lambda^T)^{-1})^T \Lambda = \mathbb{1}$$

u_μ is a <u>covector</u>

$$u'_\mu \to (\Lambda^T)^{-1}{}_\mu{}^\nu \, u_\nu$$

$$u_\mu v^\mu = u^\lambda v^\mu \eta_{\lambda\mu} = u \cdot v$$

u^λ vector $\qquad x^\mu = (\overset{x^0}{t}, \overset{x^i}{\vec{r}})^T$

$u_\mu = \eta_{\mu\nu} u^\nu$ covector $\qquad x_\mu = (-t, \vec{r})$

$$\eta_{\mu\nu} \to \eta'_{\mu\nu} = (\Lambda^T)^{-1}{}_\mu{}^\rho \, (\Lambda^T)^{-1}{}_\nu{}^\sigma \, \eta_{\rho\sigma}$$
$$= ((\Lambda^{-1})^T \eta \Lambda^{-1})_{\mu\nu} = \eta_{\mu\nu}$$

<u>Notation:</u>

$$(\eta^{-1})^{\mu\lambda} \eta_{\lambda\sigma} = \delta^\mu{}_\sigma$$
$$= \eta^{\mu\lambda} \eta_{\lambda\sigma}$$

$$\eta^{\mu\lambda}$$

$$(\Lambda^{-1})^T{}_\mu{}^\sigma \, \Lambda^\mu{}_\nu = \delta^\sigma{}_\nu \qquad \Lambda_\mu{}^\sigma \, \Lambda^\mu{}_\nu$$

<u>Consistent?</u>

$$\eta_{\mu\nu} \Lambda^\nu{}_\rho \, \eta^{\rho\sigma}$$
$$= (\eta \Lambda \eta^{-1})_\mu{}^\sigma = (\Lambda^T)^{-1}{}_\mu{}^\sigma$$

$$\Lambda^T \eta \Lambda = \eta$$
$$\eta \Lambda = (\Lambda^T)^{-1} \eta$$
$$\eta \Lambda \eta^{-1} = (\Lambda^T)^{-1}$$

$T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m}$ (n,m)-tensor

$$T \to T'^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} = \Lambda^{\mu_1}{}_{\rho_1} \cdots \Lambda^{\mu_n}{}_{\rho_n} \Lambda_{\nu_1}{}^{\sigma_1} \cdots \Lambda_{\nu_m}{}^{\sigma_n} T^{\rho_1 \cdots \rho_n}{}_{\sigma_1 \cdots \sigma_m}$$

E.S.C.

- Every summand is a (n,m)-tensor
- Unpaired indices $(n+m)$ agree
- Paired indices (up + down) are dummy indices to sum over
- Indices lowered: $u_\mu = \eta_{\mu\nu} u^\nu$, $u^\mu = \eta^{\mu\nu} u_\nu$ $\quad \eta^{-1}$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \longleftrightarrow x' = \Lambda x$$

$$x^\sigma = \Lambda_\mu{}^\sigma x'^\mu \longleftrightarrow x = \Lambda^{-1} x'$$

$$\underline{\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}}$$

$$\underline{\Lambda_\mu{}^\sigma = \frac{\partial x^\sigma}{\partial x'^\mu}}$$

$f(x)$ $\quad$ $\underline{\partial'_\mu f} = \frac{\partial f}{\partial x'^\mu} = \frac{\partial f}{\partial x^\nu} \cdot \frac{\partial x^\nu}{\partial x'^\mu}$

$= \underline{\Lambda_\mu{}^\nu \partial_\nu f}$

$$\partial_\mu f = \frac{\partial f}{\partial x^\mu}$$

$$\partial_\mu f \cdot \Delta x^\mu$$

$\eta^{-1} = \eta$ $\quad$ $\eta^{\mu\nu} = \eta_{\mu\nu}$

$$\partial_\mu = (\partial_t, \vec{\nabla}) \quad \partial^\mu = \eta^{\mu\nu} \partial_\nu = (-\partial_t, \vec{\nabla})^T$$

Tensors – a summary

Tensors of $SO(1,3)\uparrow$

4^n physical quantities: $T^{\mu_1 \cdots \mu_n} \to \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1 \cdots \nu_n}$

$u^\mu \to \Lambda^\mu{}_\nu u^\nu \qquad u \to \Lambda u$

"Some tensor w.r.t. the Dow Jones"

T is like outer product: $u_1 \otimes u_2 \otimes \cdots \otimes u_n$

$S^{\mu_1 \cdots \mu_n} = u_1^{\mu_1} u_2^{\mu_2} \cdots u_n^{\mu_n}$

Add dual $u_\mu = \eta_{\mu\nu} u^\nu$

$u^* = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$u^* \to (\Lambda^T)^{-1} u^*$

$u^{*T} v \to \left[(\Lambda^T)^{-1} u^*\right]^T \Lambda v$

$= u^{*T} \Lambda^{-1} \Lambda v = u^{*T} v$ ↗ invariant 0-tensor

ESC Notation:

- $u^* \to (\Lambda^T)^{-1} u^* \rightsquigarrow u_\mu \to \overbrace{\Lambda_\mu{}^\nu}^{(\Lambda^T)^{-1} \text{ written as matrix}} u_\nu = \eta_{\mu\rho} \Lambda^\rho{}_\sigma \eta^{\sigma\nu} u_\nu$

  $(\Lambda^T)^{-1} = (\eta \Lambda \eta)^T \leftrightarrow \Lambda^T \eta \Lambda = \eta$

- $\eta^{-1\,\mu\nu} \to \eta^{\mu\nu} \qquad \eta^{\mu\nu} \eta_{\nu\rho} \eta^{\rho\sigma} = \eta^{\mu\sigma}$

- $A^\mu = B^\mu \Leftrightarrow A_\mu = B_\mu \qquad \eta_{\mu\nu} A^\nu = \eta_{\mu\nu} B^\nu$

- $A_\mu B^\mu = A^\mu B_\mu = A^\mu B^\nu \eta_{\mu\nu}$

$T^\mu{}_\nu \qquad T_\mu{}^\nu = \eta_{\mu\lambda} \eta^{\nu\sigma} T^\lambda{}_\sigma$

numerically, not equal, but describe same physical thing

$x^\mu = (t, \vec{r})^T \qquad x_\mu = \eta_{\mu\nu} x^\nu = (-t, \vec{r})$

Use the math for physics?

What does SR do to Newton?

- ☆ What replaces $\vec{F} = m\vec{a}$
- ☆ What about conservation laws? Symmetry?
- ☆ What does it all mean for gravity? (try Maxwell first)

Particles: $\vec{r}(t)$ $\quad |\dot{\vec{r}}|<1 \quad \forall t$ worldline

Can find proper time τ along worldline

$$\Delta\tau^2 = -\Delta s^2 = \Delta t^2 - \Delta\vec{r}^{\,2}$$

$$d\tau = \sqrt{dt^2 - d\vec{r}^{\,2}}$$

$$\tau = \int_0^t dt' \sqrt{1-\dot{\vec{r}}(t)^2} \qquad \frac{d\tau}{dt} > 0$$

$$= \int_0^t dt' \frac{1}{\gamma(v(t))}$$

invariant

Can invert $t(\tau)$

describe $x^\mu(\tau) = (t(\tau), \vec{r}(t(\tau)))$

$$u^\mu = \frac{dx^\mu}{d\tau} \quad \text{4-velocity}$$

$$= \left(\frac{dt}{d\tau}, \frac{d\vec{r}}{dt}\right) = (\gamma, \gamma\vec{v}) = \gamma(1, \vec{v})$$

$$\frac{dt}{d\tau} = \frac{1}{\frac{d\tau}{dt}} = \frac{1}{\gamma^{-1}} = \gamma(v(t)) \qquad \gamma = (1-v^2)^{-1/2}$$

physical invariant but may change numerically dependent on observer

$$\frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{dt}\frac{dt}{d\tau} = \vec{v}\cdot\gamma$$

An observer related by $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

measures $u'^\mu = \Lambda^\mu{}_\nu u^\nu$

Note:  can be any parameter

$$s(\tau) \qquad \frac{ds}{d\tau} > 0$$

$$\frac{dx^\mu}{ds} = \frac{dx^\mu}{d\tau}\frac{d\tau}{ds} = u^\mu\left(\frac{ds}{d\tau}\right)^{-1}$$

u is not arbitrary

- $u^0 > 0 \quad u^0 = \frac{dx^0}{d\tau} = \frac{dt}{d\tau} = 1/\frac{d\tau}{dt}$

  $\rightarrow$ timelike, forward-pointing

- $u^2 = u\cdot u = \eta_{\mu\nu}u^\mu u^\nu = \vec{u}^{\,2} - (u^0)^2 = \gamma^2\vec{v}^{\,2} - \gamma^2$

  $= \gamma^2(v^2-1) = -1$ because we used τ

Inertial worldlines have $\vec{v}$ constant $\Leftrightarrow$ u constant

$\Rightarrow$ N1: a free particle has a constant $u^\mu(\tau)$ $\forall\tau$

$$a^\mu = \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} = \gamma(v(t))\frac{d}{dt}(\gamma, \gamma\vec{v})$$

tensor!

N1: free particle has $a^\mu = 0$ in any frame

Instantaneous rest frame $\quad u^\mu = (1, \vec{0}) \quad a^\mu = (0, \frac{d\vec{v}}{dt})$

$u\cdot a = 0$

Light: a wave: plane wave

$$\underline{\phi_\epsilon}(x) = \phi_0 \sin(\underline{\vec{k}\cdot\vec{r} - \omega t}) \qquad k^2 = \omega^2$$

$$\vec{k}\cdot\vec{r} - \omega t \text{ invariant} \Rightarrow k^\mu = (\omega, \vec{k}) \quad \text{4-vector}$$

$$= k\cdot x = k'\cdot x'$$

So $\omega = k^0$ frequency in a frame

An observer moving @ $\vec{v}$ measures $\omega^0 = k^{0'}$

$$k' = \Lambda^\mu{}_\nu k^\nu$$

Let $u^\mu = (\gamma, \gamma\vec{v})$ $\qquad \eta_{\mu\nu} u'^\nu = (-1, \vec{0})$

in observer's rest frame $u'^\mu = (1, \vec{0})^T$

$$\omega' = k^{0'} = -k'\cdot u' = -k\cdot u = \gamma(\omega - \vec{k}\cdot\vec{v})$$

* $\vec{k}\cdot\vec{v} = 0$ transverse $\qquad \omega' = \gamma(v)\,\omega = \dfrac{\omega}{\sqrt{1-v^2}}$ $\nearrow$ moving observer moves fast $\Leftarrow$ clock moves slow $\to$ higher frequency

* $\vec{k}\cdot\vec{v} = |\vec{k}|\cdot|\vec{v}| = \omega v$ longitudinal

$$\omega' = \frac{1}{\sqrt{1-v^2}}(\omega - \omega v) = \omega\frac{1-v}{\sqrt{1-v^2}} = \omega\sqrt{\frac{1-v}{1+v}}$$

Interactions? force law $\vec{F}(\vec{r}_1(t), \vec{r}_2(t))$

Collision: instantaneous, ultralocal interactions

Initial state: m_a^i $\quad \dot{\vec{r}}^i$ $\quad a = 1, \dots, N$ $\quad t = 0 - \varepsilon$ $\quad \vec{r}_a{}^i = 0$

Final state: m_b^f $\quad \dot{\vec{r}}_b^f$ $\quad b = 1, \dots, N'$ $\quad t = 0 + \varepsilon$ $\quad \vec{r}_b^f = 0$

Newton:
$$\begin{cases} \vec{p}: \quad \vec{p}^i = \sum_a m_a^i \dot{\vec{r}}_a^i = \sum_b m_b^f \dot{\vec{r}}^f = \vec{p}^f \\ E^i = \frac{1}{2}\sum_a m_a^i (\dot{\vec{r}}_a^i)^2 = \frac{1}{2}\sum_b m_b^f (\dot{\vec{r}}_b^f)^2 = E^f \\ \sum_a m_a^i = \sum_b m_b^f \end{cases}$$

Not a tensor! Natural candidate: $u^\mu = \gamma(1, \vec{v})$

$$p = p^\mu = m u^\mu \quad \text{4-momentum}$$

$$p^i = \sum_a p_a = \sum_b p_b^f = p^f$$

for $|v| \ll 1$ $\quad \gamma(v) = (1-v^2)^{-1/2} \approx 1 + \frac{1}{2}v^2 + O(v^4)$

$$p = m(\gamma, \gamma\vec{v}) \approx (m + \tfrac{1}{2}mv^2, m\vec{v}) + O(v^3)$$

if $|v_a| \ll 1 \quad \forall a$ $\qquad m \to \frac{1}{2} m v^2$

$|v_b'| \ll 1 \quad \forall b$

$$\sum m_a^i = \sum m_b f \qquad O(v^0)$$

$$\sum m_a \vec{v}_a^{\,i} = \sum m_b \vec{v}_b^{\,f} \qquad O(\vec{v})$$

$$\frac{1}{2} \sum m_a' v_a^{i\,2} = \frac{1}{2} \sum_b m_b^f v_b^{f2} \qquad O(v^2)$$

$$p = mu = m(\gamma, \gamma \vec{v})$$

$E = p^0 = m\gamma$ energy

$\vec{p} = \gamma m \vec{v}$ momentum

$$p^2 = m^2 u^2 = -m^2 = p^2 - E^2$$

$$E^2 = \vec{p}^{\,2} + m^2$$

$$E^2 = \vec{p}^{\,2} c^2 + m^2 c^4$$

$m = 0 \qquad E = |\vec{p}| \qquad p^\mu = (|\vec{k}|, \vec{k})$

$p^2 = 0$ massless

p^μ conserved quantity

massive: $\quad p^\mu = m u^\mu \qquad u^\mu = \gamma(v)(1, \vec{v})$

$$\vec{p} = m\vec{v}\,\gamma(v)$$

$$E = m\gamma(v)$$

$$E^2 = m^2 + \vec{p}^{\,2}$$

massless: $\quad E = |\vec{p}|$

A gas of charged particles

N particles $\quad m_a \qquad q_a$

Continuous limit $\quad m_a \to 0 \qquad N \to \infty$

smooth distribution $\quad q_a \to 0$

$\rho_E = \sum_a q_a \, \delta^{(3)}(\vec{r} - \vec{r}_a(t))$ charge density

$$\vec{J}_E = \sum_a q_a \, \dot{\vec{r}}_a(t) \, \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$\frac{\partial \rho_E(t, \vec{r})}{\partial t} = \sum_a q_a \, \dot{\vec{r}}_a(t) \cdot \nabla \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$= -\vec{\nabla}\cdot\vec{J}_E$$

$$\frac{\partial\rho_E(t,\vec{r})}{\partial t} + \nabla\cdot\vec{J}_E = 0$$ continuity, follows from Maxwell's equations

Should be observer independent –
Can we write this as a tensor or scalar?

$$\frac{\partial\rho_E}{\partial t} + \vec{\nabla}\cdot\vec{J}_E = 0 \qquad \partial_0\rho_E + \partial_i J_E^i = 0$$

$x = (t,\vec{r})$

$$j_E^\mu(x) = (\rho_E, \vec{J}_E)$$

lump this together to make ρ and $\vec{J}$ mix under boosts

$$\partial_\mu J_E^\mu = 0$$

$$j_E^\mu(t,\vec{r}) = \sum_a q_a \frac{\partial x_a^\mu}{\partial t}\delta^{(3)}(\vec{r}-\vec{r}_a(t))$$

$$j_E^0(t,\vec{r}) = \sum_a q_a\,\delta^{(3)}(\vec{r}-\vec{r}_a(t)) = \rho_E(t,\vec{r})$$

$$j_E^i(t,\vec{r}) = \sum_a q_a \frac{\partial\vec{r}_a}{\partial t}\delta^{(3)}(\vec{r}-\vec{r}_a(t)) = J_E^i(t,\vec{r})$$

$$j'^\mu = \Lambda^\mu{}_\nu J^\nu$$

volume not Lorentz-invariant

$$= \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}\rho \\ \vec{J}\end{pmatrix}$$

<u>Claim</u>: J_E^μ <u>is</u> a vector

<u>Pf</u>: $$j_E^\mu(x) = \int dt'\,\delta(t-t')\,j_E^\mu(t,\vec{r})$$

$\int dx\,\delta(x) = 1$ $\qquad$ $\delta(x-y)f(x) = \delta(x-y)f(y)$

$$= \int dt' \sum_a q_a \delta^{(3)}(\vec{r}-\vec{r}_a(t'))\frac{dx_a^\mu(t')}{dt'}\delta(t-t')$$

$$= \sum_a \int dt' \cancel{\sum_a} q_a\,\underline{\delta^{(4)}(x-x_a(t'))}\,\frac{dx_a^\mu}{\underline{dt'}}$$ ← invariant under reparameterizations

$$= \sum_a \int d\tau_a\, q_a\,\delta^{(4)}(x-x_a(\tau_a))\frac{dx_a^\mu(\tau_a)}{d\tau_a}$$ ← this is a 4-vector!

$$Q(t) = \int d^3r\, j_E^0(t,\vec{r}) \qquad \frac{dQ}{dt} = \int d^3r\,\partial_0 j^0 = -\int d^3r\,\partial_i j^i = 0$$

↑ divergence theorem

We made continuity covariant!

$$Q \text{ conserved} \rightsquigarrow \partial_\mu j^\mu = 0$$

$$p^\mu \text{ conserved} \rightsquigarrow \partial_\nu T^{\mu\nu} = 0$$

$$T^{\mu\nu}(x) = \sum_a p_a^\mu(t)\frac{dx^\nu(t)}{dt}\delta^{(3)}(\vec{r}-\vec{r}_a(t))$$

* $T^{\mu 0}(x) = \sum_a p_a^\mu(t)\, \underline{\delta^{(3)}(\vec{r} - \vec{r}_a(t))}$ density of p^μ!

* Tensor — we did similar above!
* conserved?

$$T^{\mu\nu} = \int dt'\, \delta(t-t') \sum_a p_a^\mu(t') \frac{dx^\nu}{dt'} \delta^{(3)}(\vec{r} - \vec{r}_a(t'))$$

$$= \sum_a \int d\tau\, \delta^{(4)}(x_a(\tau) - x)\, \underbrace{p_a^\mu(\tau) \frac{dx_a^\nu}{d\tau}}_{= m_a u_a^\mu u_a^\nu} = T^{\nu\mu}$$

covariant… and symmetric!

$T^{\mu\nu} = T^{\nu\mu}$ = energy-momentum-stress tensor

$T^{\mu 0} = T^{0\mu}$ = density of p^μ

$T^{00} = \rho$ = density of p^0 = density of energy!

$T^{0i} = T^{i0}$ density of p^i = i current of energy

$T^{ij} = T^{ji}$ i flux of p^j = j flux of p^i

in i direction

Conservation:

$$\partial_i T^{\mu i}(t, \vec{r}) = \partial_i \sum_a p_a^\mu(t) \frac{dx_a^i(t)}{dt} \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$= \sum_a p_a^\mu(t) \frac{dx_a^i(t)}{dt}\, \partial_i \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$\frac{\partial \delta(\ )}{dx^i} \cdot \frac{dx_a^i(t)}{dt}$$

$$\frac{dx_a^i}{dt} \partial_i \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$= -\sum_a p_a^\mu(t)\, \partial_0 \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$= -\partial_0 \sum_a p_a^\mu(t)\, \delta^{(3)}(\vec{r} - \vec{r}_a(t)) + \sum_a \frac{dp_a^\mu}{dt} \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$\partial_\nu T^{\mu\nu} = \underbrace{\partial_0 T^{\mu 0}}_{\text{time change of total } p^\mu \text{ in volume}} + \underbrace{\partial_i T^{\mu i}}_{\text{advection of } p^\mu \sim \text{amount leaving volume}} = \underbrace{\sum_a \frac{dp_a^\mu}{dt} \delta^{(3)}(\vec{r} - \vec{r}_a(t))}$$

→ particles not free — they interact!

4-vector

$f^\mu = \frac{dp^\mu}{d\tau}$ → looks like force in rest frame

$= m \frac{du^\mu}{d\tau} = m a^\mu$

rest frame: $u^\mu = (1, \vec{0})$ $a^\mu = (0, \vec{a})$ $f^\mu = (0, m\vec{a})$

Conserved for: ① free $\frac{dp^\mu}{dt} = 0$

② collisions that conserve p^μ

$$\partial_\mu T^{\mu\nu} = \sum_{\text{collisions}} \delta(\vec{r} - \vec{r}_c) \frac{d}{dt} \sum_{a \in c} p_a^\mu = 0$$

Charged: $f^\mu = m \frac{dx^\mu}{d\tau} = q \underline{F^\mu{}_\nu} u^\nu$ lorentz force

field strength tensor

$$F^\mu{}_0 = (0, \vec{E})$$

rest frame: $u^\nu = (1, \vec{0})$

$$f = q(0, \vec{E}) = (0, F^\mu{}_0)$$

$$\partial_\mu T^{\mu\nu} = \sum_a q_a F^\mu{}_\nu (x_a) \underline{\frac{dx_a^\mu}{dt}} \underline{\delta}^{(3)}(\vec{r} - \vec{r}_a(t))$$

doesn't look like tensor but can use to make it look like one

$$= F^\mu{}_\nu(x) \sum_a q_a \frac{dx_a^\nu}{d\tau} \delta^{(3)}(\vec{r} - \vec{r}_a)$$
$$= F^\mu{}_\nu j_E^\nu(x)$$

together, conserved

Poynting vector $\frac{\vec{E} \times \vec{B}}{4\pi}$ - momentum density of EM field

Energy density $\frac{E^2 + B^2}{8\pi}$

ϕ_G $\nabla^2 \phi_G = 4\pi\rho$ ← scalar

$\nabla^2 \phi_E = 4\pi\rho_E$ ← 0-component of vector

Today: gravity

Relativistic field theory of E&M ✓

Gravity ?

EM couples to j_E^μ (current) for conserved scalar charge $Q = \int d^3r\, j^0$

Can gravity couple to mass? Not conserved

not conserved!

energy

$$p^0 = m\gamma \approx m + \tfrac{1}{2}mv^2 + \cdots$$

Couple to $\underset{\text{current}}{T^{0\mu}}$ for conserved $p^0 = E$. not invariant

Couple to $T^{\mu\nu}$ for conserved p^μ

Special relativity: inertial frame

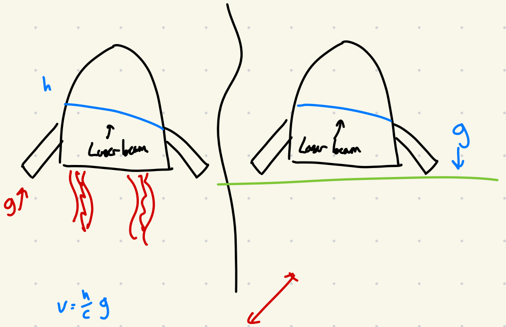
Universality of gravity: problem

Also a solution:

In freefall – no gravity S.R works

$\forall x \quad \exists \vec{g}(x)$ $\qquad g \mapsto \frac{Gm}{R^2}$ $\qquad$ tidal effects are real!

equivalence principle



$v = \frac{h}{c} g$

$\lambda = \lambda_0 \left(1 + \frac{v}{c}\right) = \lambda_0 \left(1 + \frac{hg}{c^2}\right)$

gravitational redshift

Gravity is geometry

can make inertial frame locally

$x = (t, \vec{r})$
$x' = (t', \vec{r}')$

NOT Poincaré!! inertial differently

$x'(x)$ worse – invariants under this?

No global linear structure

what do we have? Differential geometry

Spacetime is a globally hyperbolic[1] smooth[2] Lorentzian 4-manifold

A manifold is like $\mathbb{R}^n$ locally ... like enough to do calculus

not like enough for global linear structure

A smooth manifold is a set $\mathcal{M}$ together with an atlas or cover $\{\mathcal{O}_\alpha, \psi_\alpha\}_{\alpha\in A}$

$\hat{\mathcal{M}}$

* $\mathcal{O}_\alpha \subset M \quad \forall\alpha$
* $\bigcup_{\alpha\in A} \mathcal{O}_\alpha = M$
* $\psi_\alpha : \mathcal{O}_\alpha \to \mathbb{R}^n \qquad Im\,\psi_\alpha = V_\alpha \subset \mathbb{R}^n$ open

$$\left[\forall y\in V_\alpha \ \exists \varepsilon>0 \mid B_{y,\varepsilon} = \{ z\in V_\alpha \mid \sum_1^n (z^n-y^n)^2 < \varepsilon^2\} \subset V_\alpha \right]$$

$\exists\, \psi_\alpha^{-1} : V_\alpha \to \mathcal{O}_\alpha$ we will identify them

$\psi_\alpha^{-1}(B_{y,\varepsilon}) \subset \mathcal{O}_\alpha$ declare it open

$p \in \mathcal{O}_\alpha \subset M$ a point

Speak of "point" $\psi_\alpha(p) \in V_\alpha \subset \mathbb{R}^n$

used interchangeably

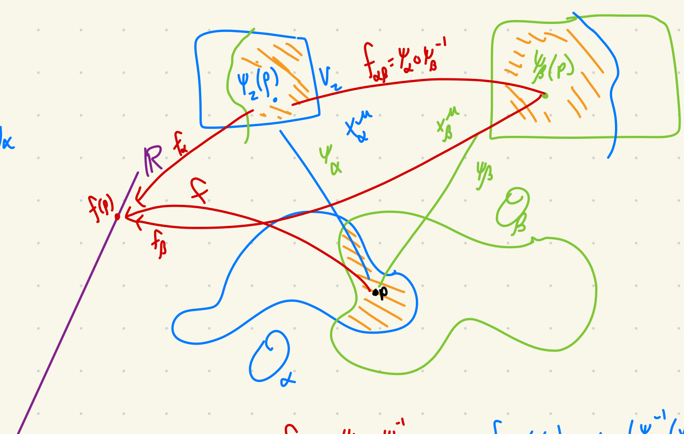
Denote this $x_\alpha^\mu(p)$

A function $f: M \to \mathbb{R}$ produces $\forall\alpha$

$$f_\alpha = f\circ\psi_\alpha^{-1} : V_\alpha \to \mathbb{R}$$
$$f_\alpha(x_\alpha)$$
$$\shortparallel$$
$$f(\psi_\alpha^{-1}(x_\alpha))$$

f is called smooth if f_α smooth $\forall\alpha$

* On overlaps $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \phi$



$f_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1} \qquad f_{\alpha\beta}(x_\beta) = x_\alpha(\psi_\beta^{-1}(x_\beta))$

* Demand $f_{\alpha\beta}$ smooth

$$f_{\beta\alpha} = f^{-1}_{\alpha\beta} \qquad x^{\mu}_{\beta}(x_\alpha)$$

$f: M \to \mathbb{R}$ smooth if f_α smooth $\forall \alpha$

$$f(x_\alpha) = f(\psi^{-1}_\alpha(x_\alpha)) \quad \text{if } p \in O_\alpha \cap O_\beta$$
$$f_\beta(x_\beta) = f(\psi^{-1}_\beta(x_\beta))$$

$$f_{\alpha\beta} = \psi_\alpha \circ \psi^{-1}_\beta \quad \text{smooth}$$

$$f_\beta(x_\beta) = f_\alpha(f_{\alpha\beta}(x_\beta))$$

Examples

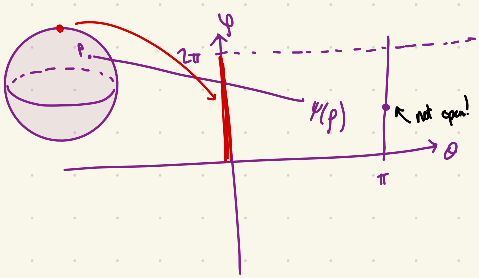
1) $\mathbb{R}^n \quad O = M = \mathbb{R}^n \quad \psi = \mathrm{id}$

2) Consider $S^2 = \{\vec{r} \in \mathbb{R}^3 \mid \vec{r}^2 = 1\}$ unit sphere

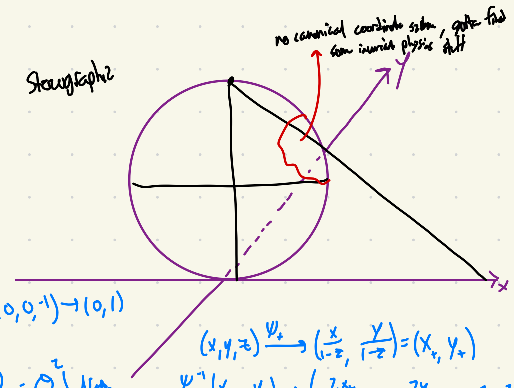
Not open

Parts of S_2 look like $\mathbb{R}^2$ (maps)

traditionally: $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, $z = \cos\theta$ $\qquad \varphi = \tan^{-1}(y/x)$, $\theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right)$ $\qquad \psi$



Steregraphic



better: $(0,0,-1) \to (0,1)$

$$(x,y,z) \xrightarrow{\psi_+} \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (X_+, Y_+)$$

$O_+ = O^2 \setminus$ North pole

$V_+ = \mathbb{R}^2$

$$\psi_+^{-1}(X_+, Y_+) \to \left(\frac{2X_+}{1+X_+^2+Y_+^2}, \frac{2Y_+}{1+X_+^2+Y_+^2}, \frac{X_+^2+Y_+^2-1}{1+X_+^2+Y_+^2}\right)$$

analogously define: $O_- = S^2 \setminus$ South pole

$V_- = \mathbb{R}^2$

$\psi_-(NP) = (0,0)$

$$f_{+-}: (X_+, Y_+) = \left(\frac{X_-}{X_-^2+Y_-^2} \quad \frac{Y_-}{X_-^2+Y_-^2}\right)$$
$$f_{-+} = f^{-1}_{+-}$$

Spacetime: different inertial observers generate different coordinates

Do not share linear structure. Share smooth maps

Temporarily, $\mathcal{M}$ a space in which particles move → ignore spacetime (for now)

take advantage of inclusion $S^2 \subset \mathbb{R}^3$

velocity ≃ a way to take derivatives

$f: M \to \mathbb{R}$, move through p velocity determines $\dot{f}$

A tangent vector V_p to $\mathcal{M}$ at $p \in \mathcal{M}$
takes $f: \mathcal{M} \to \mathbb{R}$ to $V_p(f) \in \mathbb{R}$

* Linear: $V_p(af + bg) = aV_p(f) + bV_p(g)$ $\quad a, b \in \mathbb{R}$, f, g maps

* Leibnitz: $V_p(fg) = f(p)V_p(g) + V_p(f) \cdot g(p)$

show: $f(p) = 3 \quad \forall p \Rightarrow V_p(f) = 0 \quad \forall V_p$

Set of tangent vectors $T_p\mathcal{M}$ is a linear space

$(aV_p + bW_p)(f) = aV_p(f) + bW_p(f)$ satisfies conditions

If $\mathcal{M}$ is n dimensional, $\dim T_p\mathcal{M} = n$

Proof: use coordinates $p \in \mathcal{O}_\alpha$

$\forall f: \mathcal{M} \to \mathbb{R}$

$f_\alpha : V_\alpha \to \mathbb{R} \quad f_\alpha(x_\alpha) = f(\varphi_\alpha^{-1}(x_\alpha))$

Define $X_\mu^\alpha \in T_p\mathcal{M} : f \mapsto \left.\frac{\partial f_\alpha}{\partial x_\alpha^\mu}\right|_{x(p)}$

- Show $X_\mu^\alpha \in T_p\mathcal{M} \qquad x_\alpha^\nu : \mathcal{O}_\alpha \to \mathbb{R}$
- $X_\mu^\alpha(x_\alpha^\nu) = \delta_\mu^\nu$ independent
- Span:

Lemma: $h: V_\alpha \to \mathbb{R}$

$$h(x) = h(x_0) + \sum_{\mu=1}^{n} (x^\mu - x_0^\mu) \cdot \underbrace{H_\mu(x)}_{\text{smooth}}$$

$x_0 \in V_\alpha$

$$\left.\frac{\partial h}{\partial x^\mu}\right|_{x_0} = H_\mu(x_0)$$

$$f_\alpha(x) = f_\alpha(x(p)) + \sum_{\mu=1} (x^\mu - x^\mu(p)) \cdot H_\mu(x)$$

$$H_\mu = \left.\frac{\partial f_\alpha}{\partial x^\mu}\right|_p = X_\mu(f)$$

$$V_p(f) = \sum_\mu V_p(x^\mu) \cdot H_\mu(x(p))$$

$$V_p = \sum_\mu V_p(x_\alpha^\mu) \cdot X_\mu^\alpha$$

$T_p\mathcal{M} \leftrightarrow T_p^*\mathcal{M}$ dual vector

$\omega \in T_p^*\mathcal{M}$ cotangent vector

$\omega(v): \mathcal{M} \to \mathbb{R}$

$f: \mathcal{M} \to \mathbb{R} \rightsquigarrow df(v)(p) = V_p(f)$

choose $f = x^\mu{}_\alpha \quad dx^\mu{}_\alpha(\partial^\alpha{}_\nu) = \delta^\mu{}_\nu$

$\omega = \underbrace{\omega^\alpha{}_\mu}_{\text{components}} \underbrace{dx^\mu{}_\alpha}_{\text{basis}} \quad / \quad v = v^\mu{}_\alpha \partial_\mu{}^\alpha$

$$dx^{\mu}_{\beta} = \frac{\partial x^{\mu}_{\beta}}{\partial x^{\nu}_{\alpha}} dx^{\nu}_{\alpha} \qquad v^{\mu}_{\beta} = \frac{\partial x^{\mu}_{\beta}}{\partial x^{\nu}_{\alpha}} v^{\nu}_{\alpha}$$

$$\omega^{\beta}_{\mu} = \frac{\partial x^{\nu}_{\alpha}}{\partial x^{\mu}_{\beta}} \omega^{\alpha}_{\nu} \qquad \partial^{\beta}_{\mu} = \frac{\partial x^{\nu}_{\alpha}}{\partial x^{\mu}_{\beta}} \partial^{\alpha}_{\nu}$$

$$dx^{\mu}_{\beta}(\partial^{\beta}_{\nu}) = \delta^{\mu}{}_{\nu}$$

define an (l,m)-tensor field T at p is an element

$$T_p \in (T_p\mathcal{M})^{\otimes l} \times (T^*_p\mathcal{M})^{\otimes m}$$

$$T = T_{\alpha}{}^{\mu_1\mu_2\cdots\mu_l}{}_{\nu_1\cdots\nu_m}\, \partial^{\alpha}_{\mu_1}\partial^{\alpha}_{\mu_2}\cdots\partial^{\alpha}_{\mu_l}\, dx^{\nu_1}_{\alpha}\cdots dx^{\nu_m}_{\alpha}$$

* Space of tensors is a vector space
* Can form tensor products
* Can (anti) symmetrize
* Can <u>contract</u>

$$T^{\mu_1\mu_2\cdots\mu_l}{}_{\nu_1\cdots\nu_n} \qquad (l-1, m-1)\text{-tensor}$$

$$\underset{(0,1)}{\omega_{\mu}} \otimes \underset{(1,0)}{u^{\mu}} \longrightarrow (0,0)$$

$$\delta^{\mu}{}_{\nu} \rightarrow \underset{\text{Jacobian}}{J} \cdot \mathbb{I} \cdot J^{-1} = \mathbb{I}$$

$T = 0$ ← want to find some invariant equation

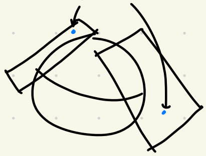
coordinate-independent

Calculus? $v(f)$ directional derivative?

$T_p\mathcal{M}$ vector space → at p $\quad v_p - w_p$

$T_{p'}\mathcal{M}$ vector space

$v_{p'} - w_p$ ← what???



Indeed: $\quad v = v^{\mu}_{\alpha}\partial^{\alpha}_{\mu}$

so: $\quad \partial_{\mu}(v^{\nu}_{\alpha})dx^{\mu}\partial^{\alpha}_{\nu} \quad$ is this $(1,1)$-tensor?

$$\frac{\partial}{\partial x^{\mu}_{\beta}}(v^{\nu}_{\beta}) = \frac{\partial x^{\rho}_{\alpha}}{\partial x^{\mu}_{\beta}} \partial^{\alpha}_{\rho}\left(\frac{\partial x^{\nu}_{\beta}}{\partial x^{\sigma}_{\alpha}} \cdot v^{\sigma}_{\alpha}\right)$$

$$= \frac{\partial x^\rho_\alpha}{\partial x^\mu_\beta} \frac{\partial x^\nu_\beta}{\partial x^\sigma_\alpha} \left(\partial^\alpha_\rho V^\sigma_\alpha\right)$$

$$+ \underbrace{\frac{\partial x^\rho_\alpha}{\partial x^\mu_\beta} \left(\frac{\partial^2 x^\nu_\beta}{\partial x^\rho_\alpha \partial x^\sigma_\alpha} \right) V^\sigma_\alpha}$$

womp womp

$\partial_\mu V^\nu$ <u>not</u> a tensor!

$\partial^\alpha_\mu \Big|_p$ - a coordinate basis for $T_p \mathcal{M}$

$$\partial^\alpha_\mu (f) = \frac{\partial f_\alpha}{\partial x^\mu_\alpha} = \frac{\partial}{\partial x^\mu_\alpha} \left(f \circ \psi_\alpha^{-1}\right)$$

$$\mathcal{M} \xrightarrow{f} \mathcal{N}$$

f: smooth

1-1

f diffeomorphism

preserve differential structure

$\underbrace{\text{cube \& sphere diffeomorphic}}$

→ topology, not geometry

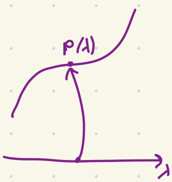
In Minkowski space we had linear structure.

To a worldline:

$$C: \mathbb{R} \to \mathcal{M}$$

$$\lambda \to p(\lambda)$$

$$\lambda \mapsto x^\mu(\lambda) \text{ in some coordinates}$$



$$V_\lambda(f) = \frac{d}{d\lambda}\left(f(p(\lambda))\right)$$

$$= \frac{dx^\mu}{d\lambda} \cdot \frac{\partial f_\alpha}{\partial x^\mu}\Big|_{x(\lambda)}$$

$$V_\lambda = \frac{dx^\mu_\alpha}{d\lambda} \cdot \partial^\alpha_\mu$$

The equivalence principle (SEP) ⇒ we can choose x^μ_α to be locally inertial coordinates of $p(\lambda)$ associated to freely falling observer @ $p(\lambda)$

Then

$$ds^2 = \eta_{\mu\nu} dx^\mu_\alpha dx^\nu_\alpha \quad \text{invariant interval}$$

A different local inertial observer at $p(\lambda)$

$$x^\mu_\beta = \Lambda^\mu{}_\nu x^\nu_\alpha \qquad \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$$

$$\eta_{\mu\nu} \dot{x}^\mu_\alpha \dot{x}^\nu_\alpha = \eta_{\mu\nu} \dot{x}^\mu_\beta \dot{x}^\nu_\beta$$

other coordinates not so simply related to x^μ_α

Interval invariant because physical

if $ds^2 < 0$ timelike

$d\tau = \sqrt{-ds^2}$ is proper time

A physical property of $p(\lambda)$ worldline

Proper time measured by a clock - physical property of $p(\lambda)$.

In general coordinates $x'(x)$ nonlinear

$$\frac{dx^\mu}{d\lambda} = \frac{\partial x^\mu}{\partial x'^\nu} \frac{dx'^\nu}{d\lambda} \quad \text{chain rule}$$

$$d\tau = \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

$$= \sqrt{-\eta_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} \frac{dx'^\rho}{d\lambda} \frac{dx'^\sigma}{d\lambda}}\, d\lambda$$

$$= \sqrt{-g_{\rho\sigma}(x) \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda}}\, d\lambda$$

$$\underline{\underline{g_{\rho\sigma}}}(x) = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial x^\rho} \frac{\partial x^\nu}{\partial x^\sigma} \qquad \underline{\underline{\text{Not invariant}}}$$

(0, 2) - tensor. The <u>metric tensor</u>

Newton $\vec{g}(x)$ acceleration physical

not invariant

$g_{\mu\nu}(x)$ breaks coordinate invariance

property of universe <u>encodes gravitational fields</u>

question: how does g encode gravity?

The Metric

1. What kind of matrix (tensor) is $g_{\mu\nu}(x)$?

$$g_{\mu\nu}(x) = \eta_{\rho\sigma} \frac{dx^\rho}{dx^\mu} \underbrace{\frac{dx^\sigma}{dx^\nu}}_{J_{\sigma b}}$$

$$g = J^T \eta J$$

$$J^T_{\sigma b} \eta J_{\sigma b}$$

$\det J \neq 0$ nonsingular

$\det J > 0$ oriented

$J^0{}_0 > 0$ time-preserving

* $g_{\mu\nu} = g_{\nu\mu}$ symmetric $(0,2)$-tensor

* $\det(g) = \det(J)^2 \cdot \det(\eta) = \det(J)^2 \cdot (-1) < 0$ nonsingular

time direction

* g has signature $(1,3)$

neg. eigenvalue pos. eigenvalue

* Lorentzian Metric

Riemannian metric: signature $(0,n)$

$\mathbb{R}^2$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$ds^2 = dx^2 + dy^2 \quad \text{euclidean plane}$$

Recall Spacetime is a smooth, orientable, time oriented 4-manifold

last time: equipped with a Lorentzian metric

- Equivalence principle (E.P.) @ $x \in \mathcal{M}$ $\exists$ choice of coords s.t.

$\forall\ \gamma: \mathbb{R} \to \mathcal{M}$ with $\gamma(0) = x$ $\leftarrow$ curve in spacetime starting at x

Worldline of observer then

$$d\tau = \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

any small region of spacetime: special relativity applies

Let x^μ be coords of such an observer (Local Inertial Observer LIO) @ x

If $x^\mu(\lambda)$ trajectory of a clock

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} < 0 \quad \text{timelike}$$

Then $d\tau = \sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\, d\lambda$

rate clock is ticking locally

Then in $x'(x)$

non zero eigenvalues, positive def,

$$J^\mu{}_\nu(x) = \frac{\partial x'^\mu}{\partial x^\nu} \qquad J^0{}_0 > 0$$

arbitrary

$$\gamma(\lambda) \leftrightarrow x'^\mu(x(\lambda)) \qquad x(\lambda) = x(x'(\lambda))$$

chain rule

$$d\tau = \sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\, d\lambda$$

$$= \sqrt{-\underbrace{\eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}}\dot{x}'^\rho\dot{x}'^\sigma}\, d\lambda$$

$$= \sqrt{-g^{(x')}_{\rho\sigma}\dot{x}'^\rho\dot{x}'^\sigma}\, d\lambda$$

- Metric $g'_{\mu\nu}(x')$ is a $(0,2)$-tensor.
- encodes gravitational field

$$g'_{\mu\nu}(x') = \eta_{\rho\sigma}\underbrace{\frac{\partial x^\rho}{\partial x'^\mu}}_{J \text{ Jacobian}}\frac{\partial x^\sigma}{\partial x'^\nu} \quad \text{for some } x^\mu(x')$$

1)

$$g = J\eta J^T$$

$$\det(g) = (\det J)^2 \det\eta = (-1)(\det J)^2 < 0$$

in geometry, this becomes pythagoras

g has signature $(1,3) \longrightarrow$ Lorentzian

2) Can use g to map (l,m)-tensors to $(l-1,m+1)$-tensor

$$T^{\mu_1\cdots\mu_{l-1}\mu}{}_{\nu_1\cdots\nu_m}g_{\mu\nu} = T^{\mu_1\cdots\mu_{l-1}}{}_{\nu_1\cdots\nu_m\nu}$$

$$g_{\mu\nu}V^\nu = V_\mu \qquad g_{\mu\nu}V^\mu U^\nu \Big\} \text{inner product}$$

$$g_{\rho\sigma} = g^{\mu\nu}g_{\nu\rho}g_{\mu\sigma}$$

$$(g^{\mu\nu}) = (g^{-1})^{\mu\nu}$$

Examples $(0,n)$ Riemannian

"Describe shape via distance"

$\ast$ $\quad ds^2 = dx^2 + dy^2 = g_{\mu\nu}(x)\,dx^\mu dx^\nu \qquad g = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

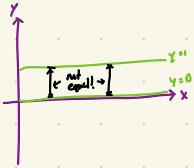
$$dl = \sqrt{g_{\mu\nu}(x(\lambda))\dot{x}^\mu \dot{x}^\nu}\, d\lambda$$
$$= (\dot{x}^2 + \dot{y}^2)^{1/2}\, d\lambda$$

(2nd simplest example) *

$$ds^2 = dx^2 + x^2 dy^2 \quad (x > 0)$$

otherwise could have $\det = 0$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}$$



$$dl = \sqrt{dx^2 + x\,dy^2} = x\,dy$$

$$y = y + 2\pi$$

→ vertical lines are circles

→ circumference $2\pi x$

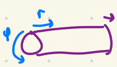
→ plane! This is just polar coordinates

* $ds^2 = dr^2 + f(r)^2 d\varphi^2$

$$\varphi = \varphi + 2\pi$$

→ all directions in plane are same

- $f(r) = 1$ → cylinder



- $f(r) = \alpha r$ → cone

rate of opening (related to angle)

$$ds^2 = dr^2 + \alpha^2 r^2 d\varphi^2$$
$$= dr^2 + r^2 d\theta^2$$

$$\theta = \alpha\varphi$$
$$\theta \sim \theta + 2\pi\alpha$$

Invariant tensors: $\delta^\mu{}_\nu = g^{\mu\rho} g_{\rho\nu}$

$\epsilon \longleftrightarrow dt\, d^3\vec{r}$ <u>not invariant</u>

$\int d^4x\, f(x)$ not invariant

upper indices

$$\epsilon^{\cdot\cdot} \to \det J\, \epsilon^{\cdot\cdot}$$

$$g = |g_{\mu\nu}|$$

$$g \to J^{-1} g J^{-1}$$
$$\det g \to (\det J)^2 \det g$$

$$g_{\mu\nu}(x) = \eta_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}$$

Claim $\forall$ $g_{\mu\nu}$ $\exists$ $x'(x)$ s.t. $g'_{\mu\nu} = \eta_{\mu\nu}$

$x'(x)$ s.t. $g'(x'(x=0)) = \eta_{\mu\nu}$

$x'(0) = 0$ $\quad g'(0) = \eta_{\mu\nu}$

$$x'^\mu(x) = \underbrace{\frac{\partial x'^\mu}{\partial x^\lambda}\Big|_{x=0}}_{\text{16 params}} x^\lambda + \frac{1}{2} \underbrace{\frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\sigma}\Big|_{x=0}}_{\text{40 of these (mixed partials)}} x^\lambda x^\sigma + \dots$$

$$\underbrace{g'_{\mu\nu}\Big|_{x=0} = \eta_{\mu\nu}}_{\text{10 params}} \quad = \quad + \underbrace{6 \text{ params}}_{\text{Lorentz}}$$

$$g'_{\mu\nu}(x') = \underbrace{\eta_{\mu\nu}}_{\text{10 functions}} + \underbrace{\partial_\lambda g'_{\mu\nu}\Big|_{x=0}}_{\to \text{ 40 first derivatives}} x'^\lambda + \dots$$

So can set equal and find $x'(x)$ accordingly

Riemannian Normal Coordinates

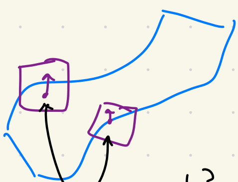
for us $\perp$

Local Inertial Coordinates

Connections

$V(f)$ smooth function if $f: \mathcal{M} \to \mathbb{R}$ smooth

$V_p \in T_p\mathcal{M} \quad \forall p \in \mathcal{M}$



Need: rate of change of a tensor

Def. A connection ∇ on $T\mathcal{M}$ is an action on vector fields s.t.

- ∇u is a $(1,1)$-tensor — recall: $\partial_\mu u^\nu$ is not a $(1,1)$-tensor
- $\nabla(u+v) = \nabla u + \nabla v$ ← linear
- $\nabla(fu) = f\nabla u + df \otimes u$ ← Leibnitz

corrected version of this

A way to relate $T_p\mathcal{M}$ to $T_{p'}\mathcal{M}$

In coordinates:

most general thing satisfying Linear + Leibniz

$$u^\mu \to (\nabla u)_\nu{}^\mu = \partial_\nu u^\mu + A_\nu{}^\mu u^\lambda$$

not coeffs of a tensor

$$[\nabla(fu)]_\nu{}^\mu = \partial_\nu(fu^\mu) + A_{\nu\lambda}{}^\mu f u^\lambda$$
$$= f\cdot(\nabla u)_\nu{}^\mu + (\partial_\nu f)u^\mu$$

$A_{\nu\lambda}{}^\mu(x)$ coeffs of a connection

In coords $x'(x)$ $\quad J^{\mu'}{}_\lambda = \frac{\partial x'^{\mu'}}{\partial x^\lambda} \quad J^{-1}{}^\lambda{}_{\mu'} = \frac{\partial x^\lambda}{\partial x'^{\mu'}}$

$$(\nabla u')_{\nu'}{}^{\mu'} = J_{\nu'}{}^\lambda J^{\mu'}{}_\rho (\nabla u)_\lambda{}^\rho = J^\lambda_{\nu'} J^{\mu'}_\rho(\partial_\lambda u^\rho + A^\rho_{\lambda\sigma} u^\sigma)$$

$$= \frac{\partial}{\partial x'^{\nu'}} u'^{\mu'} + A'_{\nu'\lambda'}{}^{\mu'} u'^{\lambda'}$$

tensor object

tensor

$$= J^\lambda_{\nu'}\partial_\lambda(J^{\mu'}_\rho u^\rho) + A'_{\nu'\lambda'}{}^{\mu'} J^{\lambda'}_\rho u^\rho$$

$$= \cancel{J^\lambda_{\nu'} J^{\mu'}_\rho \partial_\lambda u^\rho} + u^\rho J^\lambda_{\nu'}(\partial_\lambda J^{\mu'}_\rho) + A'_{\nu'\lambda'}{}^{\mu'} J^{\lambda'}_\rho u^\rho$$

need equal

$$J^\lambda_{\nu'} J^{\mu'}_\rho(\cancel{\partial_\lambda u^\rho} + A^\rho_{\lambda\sigma} u^\sigma)$$

$$J^\rho_{\tau'} J^{\lambda'}_\rho A^{\mu'}_{\nu'\lambda'} \cancel{u^\rho} = J^\rho_{\tau'} J^\lambda_{\nu'} J^{\mu'}_\sigma A^\sigma_{\lambda\rho} \cancel{u^\rho} - \cancel{u^\rho} J^\lambda_{\nu'}(\partial_\lambda J^{\mu'}_\rho) J^\rho_{\tau'}$$

$$A^{\mu'}_{\nu'\tau'} = J^\rho_{\tau'} J^\lambda_{\nu'} J^{\mu'}_\sigma A^\sigma_{\lambda\rho} - J^\lambda_{\nu'} J^\rho_{\tau'}(\partial_\lambda J^{\mu'}_\rho)$$

If we find such an A, ∇ becomes connection/covariant derivative

we write $(\nabla u)_\nu{}^\mu = \nabla_\nu u^\mu$

misleading notation, interpret as

"Connections are like pringles. You can't have just one"

If ∇ is a connection,

$(\nabla' u)^\mu_\nu = (\nabla u)^\mu_\nu + S^\mu_{\nu\lambda} u^\lambda \qquad S$ any $(1,2)$-tensor

is also a connection

∇, ∇' both connections: $\nabla u - \nabla' u = Su$ for some S

Only need to check once:

$$d(\omega(v)) = \nabla(\omega \cdot v) := (\nabla\omega)\cdot v + \omega\cdot(\nabla v)$$

$(0,1)$ $(0,1)$ $(1,0)$

$$\partial_\mu(\omega(v)) = (\nabla\omega)_{\mu\lambda} v^\lambda + \omega_\lambda (\nabla v)_\mu{}^\lambda$$

$\Rightarrow$

$$(\nabla\omega)_{\mu\lambda} v^\lambda = \partial_\mu(\omega(v)) - \omega_\lambda(\nabla v)_\mu{}^\lambda$$

$$(\partial_\mu\omega_\lambda)v^\lambda + \omega_\lambda\partial_\mu v^\lambda = \partial_\mu(\omega(v)) = (\nabla\omega)_{\mu\lambda}v^\lambda \qquad \omega_\lambda(\nabla v)_\mu{}^\lambda = (\nabla\omega)_{\mu\lambda}v^\lambda + \omega_{\cancel{\lambda}\,\sigma}\left(\partial_\mu v^\lambda + A_{\mu\cancel{\sigma}\,\lambda}^{\cancel{\lambda}\,\sigma} v^{\cancel{\sigma}\,\lambda}\right)$$

$$(\nabla\omega)_{\mu\lambda} = \partial_\mu\omega_\lambda - A_{\mu\lambda}^\sigma\omega_\sigma$$

Let S be (l,m)-tensor

∇S $(l, m+1)$-tensor

$$(\nabla S)^{\mu_1\cdots\mu_l}{}_{\mu,\nu_1\cdots\nu_m} = \partial_\mu S^{\mu_1\cdots\mu_l}{}_{\nu_1\cdots\nu_m}$$
$$+A^{\mu_1}_{\mu\sigma}S^{\sigma\mu_2\cdots\mu_l}$$

l terms $\rightarrow$ $+A^{\mu_2}_{\mu\sigma}S^{\mu\sigma\cdots}$

m terms $\rightarrow$ $-A^\sigma_{\mu\nu_1}S^{\mu\cdots\mu_l}{}_{\sigma\nu_2\cdots\nu_m}$

Use ∇ to perform parallel transport a tensor $p \to p'$

Let $\gamma : \mathbb{R} \to \mathcal{M}$ be a curve

$\gamma(0) = p$ $\gamma(1) = p'$

T parallel transported along γ if

$\dot\gamma \cdot \nabla T = 0$

$$\frac{d\gamma^\mu}{d\lambda}\cdot(\nabla T)^{\mu_1\cdots\mu_l}{}_{\mu\nu_1\cdots\nu_m} = \frac{\partial\gamma^\mu}{\partial\lambda}\,\partial_\mu T^{\mu\cdots\mu_\lambda}{}_{\nu_1\cdots\nu_m}(\gamma(\lambda)) + [\text{connection terms}]$$

u vector PT

$u^\mu(\gamma(\lambda))$ constant

$$\frac{\partial\gamma^\mu}{\partial\lambda}\left(\partial_\mu u^\nu + A^\nu_{\mu\sigma}\cdot u^\sigma\right) = 0$$

For any connection ∇, can define

$$T(\omega, u, v) = \omega\left((\nabla v)\cdot u - (\nabla u)\cdot v - [u,v]\right)$$

$$\underbrace{T_\mu{}^\sigma{}_\nu}_{\text{torsion tensor of } \nabla.} = \underbrace{A^\sigma_{\mu\nu} - A^\sigma_{\nu\mu}}_{\text{antisymmetrizing } A \text{ turns this into tensor}}$$

Metric connection

Have $\begin{matrix}\text{non-degenerate}\\\text{symmetric}\\\text{signature } (1,3)\end{matrix}$ $(0,2)$ tensor $g_{\mu\nu}$

Produce ∇:

- Torsion free $A^\lambda_{\mu\nu} = A^\lambda_{\nu\mu}$
- $\nabla g = 0$ $(0,3)$-tensor

replaces A

(1) $(\nabla g)_{\mu\nu\lambda} = 0 = \partial_\mu g_{\nu\lambda} - \Gamma^\rho_{\mu\nu} g_{\rho\lambda} - \cancel{\Gamma^\rho_{\mu\lambda}} g_{\nu\rho}$

(2) $\overset{+}{(\nabla g)}_{\nu\lambda\mu} = 0 = \partial_\nu g_{\lambda\mu} - \cancel{\Gamma^\rho_{\nu\lambda} g_{\rho\mu}} - \Gamma^\rho_{\nu\mu} g_{\lambda\rho}$

(3) $\overset{-}{(\nabla g)}_{\lambda\mu\nu} = 0 = \partial_\lambda g_{\mu\nu} - \cancel{\Gamma^\rho_{\lambda\mu}} g_{\rho\nu} - \cancel{\Gamma^\rho_{\lambda\nu}} g_{\mu\rho}$

$$\Rightarrow \frac{g^{\rho\sigma}}{2}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right) = \left(2\Gamma^\rho_{\mu\nu} g_{\rho\lambda}\right) g^{\rho\sigma}$$

$$= \Gamma^\sigma_{\mu\nu}$$

Levi-Civita connection

Γ - christoffel symbols

Spacetime:

- 4-manifold, oriented, time-oriented
- Lorentzian metric $g_{\mu\nu}$
- Levi-Civita (metric) connection

$$(Dv)_\mu{}^\nu = \partial_\mu v^\nu + \Gamma_{\mu\lambda}^\nu v^\lambda$$

Christoffel symbols

$$\Gamma_{\mu\lambda}^\nu = \frac{1}{2} g^{\nu\rho}\left(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}\right) = \Gamma_{\lambda\mu}^\nu$$

This is a connection:

- $(Dv)_\mu{}^\nu$ is a $(1,1)$-tensor $\to$ right jacobians, no derivatives of jacobians
- $(Dg)_{\mu\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma_{\mu\nu}^\rho g_{\rho\lambda} - \Gamma_{\mu\lambda}^\rho g_{\nu\rho} = 0$

Γ disappears

(1) $\partial_\lambda (u_\mu v^\mu) = \partial_\lambda (u^\mu v^\nu g_{\mu\nu}) = D_\lambda (u^\mu v^\nu g_{\mu\nu})$
$= (D_\lambda u^\mu) \cdot v_\nu + u_\mu (D_\lambda v^\nu) + \cancel{u^\mu v^\nu D_\lambda g_{\mu\nu}}$

normal partial equal to covariant derivative
__for functions__ $\partial_\mu = D_\mu$

$$D_\mu v^\mu = D^\mu v_\mu$$

- At any $x \in M$ $\exists$ L.I.C. ← riemann normal coords (physics version)
  - $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$
  - $\partial_\lambda \hat{g}_{\mu\nu} = 0$

  $\Big\} \; \hat{\Gamma}_\mu^{\nu\lambda} = 0$

__Parallel Transport:__

$\gamma : [0,1] \to M \quad x^\mu(t)$

$T^{\cdots}_{\cdots}$ is parallel transported along γ if

$$\dot{x}^\mu (\nabla T)_\mu{}^{\cdots}{}_{\cdots} = 0$$

flat space → $\underbrace{(\vec{\dot{x}} \cdot \vec{\nabla}) f}_{f \text{ constant along curve}} = 0$

$\frac{d}{dt} f(\vec{x}(t)) = (\vec{\dot{x}} \cdot \vec{\nabla}) f(\vec{x}(t))$

If T tangent vector:

$$\dot{x}^\mu (\nabla v)_\mu{}^\nu = \dot{x}^\mu \partial_\mu v^\nu + \Gamma_{\mu\lambda}^\nu \dot{x}^\mu v^\lambda$$
$$= \frac{d}{dt} v^\nu(x(t)) + \Gamma_{\mu\lambda}^\nu(x(t))\, \dot{x}(t)\, v^\lambda(x(t))$$

"Is the velocity p.t.'d along the curve itself"

Special curves

$\gamma \leftrightarrow x^\mu(t)$ is a geodesic of g if

$\underset{T_{x(t)}M}{\underset{\cap}{u^\mu(t)}} = \dot{x}^\mu$ is P.T. along γ

$$\frac{d}{d\lambda}\left(u^\mu(\lambda)\right) + \Gamma^\mu_{\nu\rho}\dot{x}^\nu(\lambda)u^\rho(\lambda) = \frac{d}{d\lambda}u^\mu(\lambda) + \Gamma^\mu_{\nu\rho}u^\nu u^\rho$$

$$= \frac{d^2}{d\lambda^2}x^\mu(\lambda) + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho$$

$$= \left(\hat{u}^\nu D_\nu \hat{u}^\mu\right)(x(\lambda)) \qquad \hat{u}^\mu(x) \quad \hat{u}^\mu(x(t)) = u^\mu(t)$$

In L.I.C. @ X,

$\hat{g}_{\mu\nu} = \eta_{\mu\nu}$

$\hat{\Gamma}^\lambda_{\mu\nu} = 0$

$$\frac{d^2}{dt^2}(\hat{x}^\mu) + \hat{\Gamma}^\mu_{\nu\lambda}\dot{\hat{x}}^\nu\dot{\hat{x}}^\lambda = 0$$

$\hat{\Gamma}^\mu_{\nu\lambda}\dot{\hat{x}}^\nu\dot{\hat{x}}^\lambda \to 0$

constant velocity = inertial

trajectory of freely falling objects

Inertial: freely-falling (ff) massive particles' worldlines are timelike geodesics

Note: $\frac{d}{d\lambda}\left(g_{\mu\nu}(x(\lambda))\, u^\mu(\lambda)\, u^\nu(\lambda)\right)$

$= \hat{u}^\rho D_\rho\left(g_{\mu\nu}\hat{u}^\mu\hat{u}^\nu\right) = g_{\mu\nu}\left[\hat{u}^\rho(D_\rho\hat{u}^\mu)\cdot\hat{u}^\nu + \hat{u}^\mu\hat{u}^\rho D_\rho\hat{u}^\nu\right]$

$$\frac{d^2}{d\lambda^2}x^\mu + \Gamma^\mu_{\nu\sigma}(x(\lambda))\dot{x}^\nu(\lambda)\dot{x}^\sigma(\lambda) = 0$$

$\lambda = a\tau + b$

nonlinear second order ode

Initial conditions: $x^\mu(0)$

$\dot{x}^\mu(0) \qquad \dot{x}^\mu\dot{x}^\nu g_{\mu\nu}(x(0)) = -1 \quad \dot{x}^0 > 0$

$\exists!$ Solution $x^\mu(\tau)$

Fails:

(1) Geodesics cross :(
(2) Geodesics end :(
$\Big\}$ singularities

Massless: $\frac{d}{d\lambda}k^\mu(\lambda) + \Gamma^\mu_{\nu\sigma}k^\nu(\lambda)k^\sigma(\lambda) = 0$

$k^\mu = \dot{x}^\mu$

λ - affine parameter $\quad k^2 = 0$

$k^\mu =:$ 4-momentum

Curvature

For any $u^\mu(x)$

$$D_\lambda D_\sigma u^\mu = \partial_\lambda (D_\sigma u^\mu) - \Gamma^\rho_{\lambda\sigma} D_\rho u^\mu + \Gamma^\mu_{\lambda\rho} D_\sigma u^\rho \qquad \lambda \leftrightarrow \sigma$$

$$= \partial_\lambda \partial_\sigma u^\mu + (\partial_\lambda \Gamma^\mu_{\sigma\rho}) u^\rho + \Gamma^\mu_{\sigma\rho} \partial_\lambda u^\rho - \Gamma^\rho_{\lambda\sigma} \partial_\rho u^\mu - \Gamma^\rho_{\lambda\sigma} \Gamma^\mu_{\rho\tau} u^\tau + \Gamma^\mu_{\lambda\rho} \partial_\sigma u^\rho + \Gamma^\mu_{\lambda\rho} \Gamma^\rho_{\sigma\tau} u^\tau$$

(1,2)-tensor

$$[D_\lambda, D_\sigma] u^\mu = \underbrace{\left(\partial_\lambda \Gamma^\mu_{\sigma\rho} - \partial_\sigma \Gamma^\mu_{\lambda\rho} + \Gamma^\mu_{\lambda\tau}\Gamma^\tau_{\sigma\rho} - \Gamma^\mu_{\sigma\tau}\Gamma^\tau_{\lambda\rho}\right]}_{\text{must be (1,3)-tensor}} u^\rho$$

commutator

$$= R^\mu{}_{\rho\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\sigma\rho} - \partial_\sigma \Gamma^\mu_{\lambda\rho} + \Gamma^\mu_{\lambda\tau}\Gamma^\tau_{\sigma\rho} - \Gamma^\mu_{\sigma\tau}\Gamma^\tau_{\lambda\rho}$$

Riemann curvature tensor

$$F_{\lambda\sigma} = \partial_\lambda A_\sigma - \partial_\sigma A_\lambda + \cancel{A_\lambda A_\sigma}$$

analog

Spacetime has: $\quad ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$

invariant interval

- A metric $g_{\mu\nu}$
- A connection $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$
- Geodesics $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda}$ – inertial worldlines
- Curvature $[D_\lambda, D_\sigma] V^\mu = R^\mu{}_{\nu\lambda\sigma} V^\nu$

$$R^\mu{}_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\lambda\nu} + \Gamma^\mu_{\lambda\alpha}\Gamma^\alpha_{\sigma\nu} - \Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\lambda\nu}$$

↳ is a tensor!

How many independent components does $R^\mu{}_{\nu\lambda\sigma}$ have in d-dimensional space?

Naively: $d^4 \underset{d=4}{=} 256$

but $R^\mu{}_{\nu\lambda\sigma} = -R^\mu{}_{\nu\sigma\lambda}$

So $\cancel{d^4}$ $\quad d^2 \cdot \frac{d(d-1)}{2} \underset{d=4}{=} 96$

$$R_{\mu\nu\lambda\sigma} = g_{\mu\rho} R^\rho{}_{\nu\lambda\sigma}$$

In L.I.C. @ x

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu}$$

$$\partial_\lambda \hat{g}_{\mu\nu} = 0 \Rightarrow \hat{\Gamma}^\lambda_{\mu\nu} = 0$$

$$\hat{R}_{\mu\nu\lambda\sigma} = \eta_{\mu\rho}(\partial_\lambda \hat{\Gamma}^\rho_{\sigma\nu} - \partial_\sigma \hat{\Gamma}^\rho_{\lambda\nu})$$

$$\hat{\Gamma}^{\lambda}_{\mu\nu}(y) = \frac{1}{2}\hat{g}^{\lambda\sigma}\left(\partial_\mu \hat{g}_{\sigma\nu} + \partial_\nu \hat{g}_{\mu\sigma} - \partial_\sigma \hat{g}_{\mu\nu}\right) \qquad \hat{g} = \eta + \mathcal{O}(y-x)^2$$

$$\hat{R}_{\mu\nu\lambda\sigma} = \frac{1}{2}\Big(\partial_\lambda \partial_\nu \hat{g}_{\mu\sigma} + \cancel{\partial_\lambda \partial_\sigma \hat{g}_{\mu\nu}} - \partial_\lambda \partial_\mu \hat{g}_{\nu\sigma}$$
$$- \partial_\sigma \partial_\nu \hat{g}_{\mu\lambda} - \cancel{\partial_\sigma \partial_\lambda \hat{g}_{\mu\nu}} + \partial_\sigma \partial_\mu \hat{g}_{\nu\lambda}\Big)$$

Symmetry: antisymmetric in $\mu \leftrightarrow \nu$

$$R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma}$$

$$d^2 \frac{d\cancel{(d-1)}}{2} \to \frac{(d(d-1))^2}{4} \underset{d=4}{=} 36$$

$(\mu\nu) \leftrightarrow (\lambda\sigma)$

$$R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu}$$

$$\frac{1}{4}\cancel{(d(d-1))^2} \to \frac{d(d-1)}{4}\left[\frac{d(d-1)}{2} + 1\right] \underset{d=4}{=} 21$$

More symmetry:

$$R_{\mu\nu\lambda\sigma} + R_{\mu\sigma\nu\lambda} + R_{\mu\lambda\sigma\nu} = 0$$

$$\Leftrightarrow R_{\mu[\nu\lambda\sigma]} = 0$$

$$\Leftrightarrow R_{[\mu\nu\lambda\sigma]} = 0$$

$$\frac{d(d-1)}{4}\left(\cancel{\frac{d(d-1)}{2}} + 1\right) \to \frac{d(d-1)}{4}\left[\frac{d(d-1)}{2} + 1\right] - \binom{d}{4} \underset{d=4}{=} 20$$

$$= \frac{1}{12} d^2 (d^2 - 1)$$

$$\hat{D}_\rho \hat{R}^\mu{}_{\nu\lambda\sigma} = \partial_\rho \hat{R}^\mu{}_{\nu\lambda\sigma}$$
$$= \partial_\rho \partial_\lambda \hat{\Gamma}^\rho_{\sigma\nu} - \partial_\rho \partial_\sigma \hat{\Gamma}^\rho_{\lambda\nu}$$

$$\hat{D}_\rho \hat{R}^\mu{}_{\nu\lambda\sigma} + \hat{D}_\lambda \hat{R}^\mu{}_{\nu\sigma\rho} + \hat{D}_\sigma \hat{R}^\mu{}_{\nu\rho\lambda} = 0$$

tensor, antisymmetry

$$D_{[\rho} R_{\mu\nu]\lambda\sigma} = 0$$ Bianchi identity

$$R^\mu{}_{\nu\mu\sigma} = R_{\nu\sigma} = R_{\sigma\nu}$$ Ricci tensor

$$R_{\nu\sigma} = g^{\mu\lambda} R_{\mu\nu\lambda\sigma}$$

$$g^{\mu\nu} R_{\mu\nu} = R$$ Ricci tensor

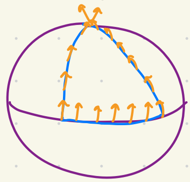
Bianchi: $0 = g^{\nu\sigma} g^{\mu\lambda}\left(D_\lambda R_{\rho\sigma\mu\nu} + D_\rho R_{\sigma\lambda\mu\nu} + D_\sigma R_{\lambda\rho\mu\nu}\right)$

$= D^\mu R_{\rho\mu} + D^\nu R_{\rho\nu} - D_\rho R \Rightarrow D_\rho R = \frac{1}{2} D^\mu R_{\mu\rho}$

R ⇝ measure of how much covariant derivatives commute

R measures path-dependence of P.T. ↙ parallel transport

parallel transport results in different vector, this is $R^\nu{}_\mu$



$\gamma : [0,1] \to M$

$\gamma(0) = \gamma(1) = p$

$V(0) \in T_p M$

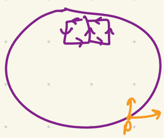
$$\frac{dv^\mu(\lambda)}{d\lambda} + \Gamma^\mu_{\nu\sigma}(x(\lambda))\, v^\nu(\lambda) \cdot \frac{dx^\sigma(\lambda)}{d\lambda} = 0$$

Plug in IC + solve $\Rightarrow V(1) = V(0) + R^\mu_{(\gamma)\nu} V^\nu(0)$

$ds = \gamma$

$= \sum \gamma_A$

$R_{(\gamma_A)} \ll 1 \qquad R_{(\gamma)} = \sum R_{(\gamma_A)}$



$$v^\mu(\lambda) = v^\mu(0) - \int_0^\lambda \Gamma^\mu_{\nu\sigma}(x(s)) \frac{dx^\nu(s)}{ds} v^\sigma(s)\, ds$$

$$v^\sigma(s) \approx v^\sigma(0) - \Gamma^\sigma_{\rho\lambda}(x(0))\, v^\lambda(0)\left(x^\rho(s) - x^\rho(0)\right) + O(s^2)$$

$$\Gamma^\mu_{\nu\sigma}(x(s)) = \Gamma^\mu_{\nu\sigma}(x(0)) + \partial_\rho \Gamma^\mu_{\nu\sigma}(x(0))\left(x^\rho(s) - x^\rho(0)\right) + O(s^2)$$

$$v^\mu(\lambda) \approx v^\mu(0) - \Gamma^\mu_{\nu\sigma}(p)\, v^\sigma(0) \oint_0^\lambda \frac{dx^\nu(s)}{ds} ds + \left(\partial_\rho \Gamma^\mu_{\nu\sigma}(p) - \Gamma^\mu_{\nu\lambda}(p)\Gamma^\lambda_{\rho\sigma}(p)\right) v^\sigma(0) \int_0^\lambda \left(x^\rho(s) - x^\rho(0)\right)\frac{dx^\nu}{ds} ds$$

$$\oint_{\gamma_A} \left(x^\rho(s) - x^\rho(0)\right)\frac{dx^\nu}{ds} ds \quad \left(\partial_\rho \Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma}\right)$$

$R_{(\gamma_A)}{}^\mu{}_\nu \approx R^\mu{}_{\nu\lambda\sigma}(p) \cdot d\sigma^{\lambda\sigma}$

$R_{(\gamma)}{}^\mu{}_\nu = \int_S R^\mu{}_{\nu\lambda\sigma}\, d\sigma^{\lambda\sigma}$

Cloud of <u>inertial</u> probes

$u^\mu(x)$

$u^\nu D_\nu u^\mu(x) = 0$

$\Theta \qquad x^\mu(\tau)$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}(x(\tau)) \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

$y^\mu(\lambda)$ nearby worldline

$$\frac{dx^\mu(\tau)}{d\tau} = u^\mu(\tau)$$

$$y^\mu(\tau) = x^\mu(\tau) + y^\mu(\tau)$$

$$u_\mu y^\mu = 0$$

$\tau \approx$ proper time $\quad \frac{dy^\mu}{d\tau}\frac{dy^\nu}{d\tau} g_{\mu\nu}(y(\tau))$

$$\frac{dy^\mu}{d\tau} = u^\mu(y(t))$$

$\forall\, v(x)$ define $D_{(u)} v^\mu = u^\nu D_\nu v^\mu$

@ $x(\tau)$ $\quad D_{(u)} v^\mu \big|_{y(\tau)} = \frac{dx^\nu}{d\tau}\left(\partial_\nu v^\mu + \Gamma^\mu_{\nu\sigma} v^\sigma\right)$

$$= \frac{dv^\mu(x(\tau))}{d\tau} + \Gamma^\mu_{\nu\sigma} v^\sigma \frac{dx^\nu}{d\tau}$$

$$0 = D_{(u)}(u_\mu y^\mu) = u_\mu D_{(u)} y^\mu$$

$$= \left[\cancel{D_{(u)} u_\mu}\right] y^\mu$$

$$u^\mu(y(\tau)) = \frac{dy^\mu}{d\tau} = \frac{dx^\mu}{d\tau} + \frac{dy^\mu}{d\tau} = u^\mu(\tau) + \frac{dy^\mu}{d\tau}$$

$$u^\mu = (1, \vec{0})$$

$$y^\mu = (0, \vec{y})$$

$$D_{(u)} y^\mu \frac{dy^\mu}{d\tau} = (0, \vec{v})$$

$$D^2_{(u)} y^\mu = (0, \vec{a})$$

$$a^i = R^i{}_{00j}\, y^j$$

$$u^\mu(y) - u^\mu(x) = \frac{dy^\mu}{d\tau} = y^\nu \partial_\nu u^\mu(x(\tau))$$

$$D_{(u)} y^\mu = \frac{dy^\mu}{d\tau} + \Gamma^\mu_{\nu\sigma} u^\nu y^\sigma$$

$$y^\nu \partial_\nu + \Gamma^\mu_{\nu\sigma} u^\sigma y^\nu$$

$$y^\nu\left(\partial_\nu u^\mu + \Gamma^\mu_{\nu\sigma} u^\sigma\right) = y^\nu D_\nu u^\mu$$

$$D^2_{(u)} y^\mu = D_{(u)}(y^\nu D_\nu u^\mu)$$

$$= (D_{(u)} y^\nu)(D_\nu u^\mu) + y^\nu (D_{(u)} D_\nu u^\mu)$$

$$(y^\lambda D_\lambda u^\nu)(D_\nu u^\mu) + y^\nu u^\lambda D_\lambda D_\nu u^\mu$$

$$+ y^\lambda D_\lambda u^\nu \cdot D_\nu u^\mu + y^\nu u^\lambda \left(D_\nu D_\lambda u^\mu + R^\mu{}_{\sigma\lambda\nu} u^\sigma\right)$$

$$u^\lambda D_\nu D_\lambda u^\mu = D_\nu(u^\lambda D_\lambda u^\mu) - (D_\nu u^\lambda) D_\lambda u^\mu$$

$$D^2_{(u)} y^\mu = R^\mu{}_{\sigma\lambda\nu} u^\sigma u^\lambda y^\nu$$

Geodesics: $\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$

In L.I.C. : $\frac{d^2 \hat{x}^\mu}{d\tau^2} = 0$

More generally: Lorentz-invariant equation

$\partial_\mu \to D_\mu$ Minimal Coupling

replace

- Tensor
- Reduces to S.R. in L.I.C.

Maxwell:

contains inhomogeneous $\to \partial_\mu F^{\mu\nu} = -4\pi j^\nu$

$\partial_{[\mu} F_{\nu\sigma]} = 0$ $\Rightarrow$ $\partial_\mu j^\mu = 0$ continuity equation for charge

contains homogeneous maxwell

$= \frac{1}{3!} \varepsilon^{\mu\nu\sigma} \partial_\mu F_{\nu\sigma}$

$Q(t) = \int d^3\vec{r}\, j^0(t,\vec{r})$

- Lorentz-invariant
- Conserved with suitable boundary condition $\frac{dQ}{dt} = 0$

Minimally couple:

$D_\mu F^{\mu\nu} = -4\pi j^\nu$

$D_{[\mu} F_{\nu\sigma]} = 0$ $\Rightarrow$ $D_\mu j^\mu = 0$

$Q(t) = \int d^3\vec{r}\, j^0(t,\vec{r})$

- not transformation invariant
- not conserved

General coordinate invariant $Q(t) = \int d^3\vec{r}\, \sqrt{-|g|}\, j^0(t,\vec{r})$

$0 = \partial_\mu\left(\sqrt{-|g|}\, j^\mu\right) = \sqrt{-|g|}\, D_\mu j^\mu$

- conserved
- invariant

$m \frac{d^2 \hat{x}^\mu}{d\tau^2} = -q F^\mu{}_\sigma(x(\tau)) \frac{dx^\sigma}{d\tau}$

$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = -\frac{q}{m} F^\mu{}_\sigma \frac{dx^\sigma}{d\tau}$

$T^{\mu\nu}_{EM} = \frac{1}{4\pi}\left(F^{\mu\sigma} F^{\nu\rho} \eta_{\sigma\rho} - \frac{1}{4}\eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}\right)$

$\Rightarrow \cancel{\partial_\mu} T^{\mu\nu}_{EM} = -F^\nu{}_\sigma j^\sigma$

$D_\mu T^{\mu\nu}_{EM} = -F^\nu{}_\sigma j^\sigma$

Note $0 = D_\mu T^{\mu\nu} = \underbrace{\partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma} T^{\sigma\nu}}_{\frac{1}{\sqrt{-|g|}} \partial_\mu\left(\sqrt{-|g|}\, T^{\mu\nu}\right)} + \overbrace{\Gamma^\nu_{\mu\sigma} T^{\mu\sigma}}^{\text{gravity does work}}$

$$p^{\mu}(t) = \int d^3\vec{r}\, \overset{\text{makes it a tensor}}{\sqrt{-|g|}}\, T^{\mu 0}(t,\vec{r})$$

- tensor
- not conserved

"No shit gravity does not conserve energy"

Free particle

$$T^{\mu\nu}_{(t,\vec{r})} = p^{\mu}(t) \frac{dx^{\nu}}{dt} \delta^{(3)}(\vec{r} - \vec{r}(t))$$

$$= \frac{m}{\underset{\text{minimal coupling}}{\sqrt{-|g|}}} \int d\tau \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \underbrace{\delta^{(4)}(x - x^{\mu}(\tau))}_{\text{not a tensor, messes stuff up}}$$

$$1 = \int d^4x\, \delta^{(4)}(x-y) = \int d^4x \sqrt{-|g|} \left[\frac{1}{\sqrt{-|g|}} \delta^{(4)}(x-y)\right]$$

Tensor yay!

Is it conserved?

$$D_{\mu} T^{\mu\nu} = \frac{m}{\sqrt{-|g|}} \left[\int d\tau \frac{dx^{\nu}}{d\tau} \underbrace{\frac{dx^{\mu}}{d\tau} \partial_{\mu} \delta^{(4)}(x - x^{\mu}(\tau))}_{-\frac{d}{d\tau}\delta^{(4)}(x-x(\tau))} + \Gamma^{\nu}_{\mu\sigma} \int d\tau \frac{dx^{\mu}}{d\tau} \frac{dx^{\sigma}}{d\tau} \delta^{(4)}(x - x(\tau)) \right.$$

$$= \frac{m}{\sqrt{-|g|}} \int d\tau \left[\frac{d^2x^{\nu}}{d\tau^2} + \Gamma^{\mu}_{\nu\sigma}(x(\tau)) \frac{dx^{\mu}}{d\tau} \frac{dx^{\sigma}}{d\tau} \right] \delta^{(4)}(x - x(\tau))$$

A free particle conserves energy (covariantly)

$\iff$ it follows a geodesic

$\iff$ it is in freefall

Dust: $T^{\mu\nu}(x) = \rho(x) u^{\mu}(x) u^{\nu}(x)$

$u^{\mu} D_{\mu} u^{\nu} = 0$ geodesic

$\rho u^{\mu} D_{\mu} u^{\nu} = -f^{\nu}{}_{\sigma} j^{\sigma}$ charged

$\Rightarrow D_{\mu} T^{\mu\nu} = f^{\nu}{}_{\sigma} j^{\sigma}$

Takeaways:

1. $D_{\mu} T^{\mu\nu}$ is equation of motion
2. $Q(t) = \int d^3\vec{r} \sqrt{-|g|}\, j^0(t,\vec{r})$
   - ~~tensor~~ scalar
   - conserved

   $p^{\mu}(t) = \int d^3\vec{r} \sqrt{-|g|}\, T^{\mu\nu}(t,\vec{r})$
   - vector
   - not conserved

- An observer w/ 4-velocity ξ^{μ} measures E-M density $T^{\mu\nu}\xi_{\nu} = j^{\mu}_{(\xi)}$

Is $D_\mu j^\mu_{(\xi)} = 0$?

$$D_\mu j^\mu_{(\xi)} = \cancel{D_\mu T^{\mu\nu}}\,\xi_\nu + T^{\mu\nu} D_\mu \xi_\nu$$

$D_\mu j^\mu_{(\xi)} = 0$ iff $D_\mu \xi_\nu + D_\nu \xi_\mu = 0$ Killing's equation

metric $\to$ connection

E.O.M. $D_\mu T^{\mu\nu} = 0$

Local (tidal) effects $R^\mu{}_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\lambda\nu} + \Gamma^\mu_{\lambda\alpha}\Gamma^\alpha_{\sigma\nu} - \Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\lambda\nu}$

Geodesic Deviation

$$D^2 y^\mu = -R^\mu{}_{\sigma\nu\lambda} u^\sigma u^\lambda y^\nu \xrightarrow[\text{limit, weak static fields}]{T} \partial^\mu \partial_\nu \phi\, y^\nu$$

L.I.C.: $a^i = -R^i{}_{00j} y^j$

Newtonian: $a^i = y^j \partial_j(-\partial_i \phi) = -\partial_i \partial_j \phi\, y^j$

$$\partial^\mu \partial_\mu \phi \sim \partial_i \partial^i \phi = 4\pi G \overset{T^{00}}{\rho} = 4\pi G u^\mu u^\nu T_{\mu\nu}$$

$$R^\mu{}_{\sigma\mu\lambda} u^\sigma u^\lambda$$

$$\boxed{R_{\mu\nu} \overset{?}{\cancel{=}} 4\pi G T_{\mu\nu}}$$ Einstein!

$R = 4\pi G T$

$T^\mu{}_\mu = T$

Bianchi: $D^\mu R_{\mu\nu} = \frac{1}{2} D_\nu R = \frac{1}{2}\partial_\nu R$

$$G_{\mu\nu} = R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$$

$$D^\mu G_{\mu\nu} = D^\mu R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} D^\nu R = 0$$

$$G_{\mu\nu} = k\, T_{\mu\nu} \Rightarrow D_\mu T^{\mu\nu} = 0$$

$G = kT$

$= R - 2R = -R$

$$R_{\mu\nu} = k\left(T^{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) = k\left(\rho + \frac{1}{2} u^2 (-\rho)\right) = \frac{k}{2}\rho = 4\pi G \rho$$

If $T_{\mu\nu} = \rho u^\mu u^\nu = g_{00} \rho (u^0)^2$
$\sim -\rho$

$T = \rho u^2 = -\rho$

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}}$$ Einstein's equation(s)

Einstein's equation: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

$\Rightarrow D_\mu T^{\mu\nu} = 0$

1) 10 2nd-order nonlinear PDEs for $g_{\mu\nu}$ given $T^{\mu\nu}$

4 constraints: $D_\mu G^{\mu\nu} \equiv 0$

coordinate transformations!

$\Rightarrow$ 6 eqns for 10 $g_{\mu\nu}$ $\Rightarrow$ determine $g_{\mu\nu}$ up to 4 functions $\rightarrow x'(x)$

2) In vacuum $T_{\mu\nu} = 0$

independently,

$$\left[\begin{aligned} &g^{\mu\nu} G_{\mu\nu} = R - 2R = 8\pi G T \\ &R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \end{aligned}\right]$$

Spacetimes with $R_{\mu\nu}$: Einstein Manifolds

Ricci-flat

Still nonlinear!

3) Traceless part of $R^\mu{}_{\nu\lambda\sigma}$

Weyl:

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} - \frac{1}{d-2}\left[g_{\mu\lambda} R_{\nu\sigma} - g_{\mu\sigma} R_{\nu\lambda} + g_{\nu\sigma} R_{\mu\lambda} - g_{\nu\lambda} R_{\mu\sigma}\right] + \frac{1}{(d-1)(d-2)}\left(g_{\mu\lambda} g_{\sigma\nu} - g_{\mu\sigma} g_{\lambda\nu}\right) R$$

$$\frac{d^2(d^2-1)}{12} - \frac{1}{2} d(d+1) = \frac{1}{12} d(d+1)(d+2)(d-3) \qquad d \geq 3$$

$d=4 \nearrow = 10$

$C^\mu{}_{\lambda\mu\sigma} = 0$

4) Read it backwards (left-to-right)

10 (-4) algebraic equations for $T_{\mu\nu}$

<u>Any</u> $g_{\mu\nu}$ "solves" Einstein's Equations

<u>most</u> require "<u>nonphysical</u>" $T_{\mu\nu}$

Restrictions on $T_{\mu\nu}$:

- Energy density positive, Weak Energy Condition

$\forall u^\mu$ <u>timelike</u>, $T_{\mu\nu} u^\mu u^\nu \geq 0$ coordinate-independent

$\rho \geq 0 \quad \rho + p \geq 0$

- Null energy condition:

  $\forall\, k^\mu$ null $\quad T_{\mu\nu} k^\mu k^\nu \geq 0$

  $\rho + p \geq 0$

- Dominant energy condition:

  weak + $\forall$ future-directed causal y^μ (timelike or null)

  $T^\mu{}_\nu y^\nu$ future-directed causal

  $\rho \geq |p|$

  $y^\mu = (y^0, \vec{y})$

  ↳ causal: $g_{\mu\nu} y^\mu y^\nu \leq 0$ (equality = speed of light)

  ↳ future-directed: $y^0 > 0$

- Strong E.C.:

  $\forall$ timelike u^μ, $\quad R_{\mu\nu} u^\mu u^\nu \propto \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) u^\mu u^\nu \geq 0$

  $T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$ $\quad \rho + p \geq 0$, $\quad \rho + 3p \geq 0$

5) Hope: specify suitable initial condition on T, g (@ $t = t_0$, spacelike hypersurface) (+ boundary conditions)

→ $\exists!$ solution for some time

Singularity Theorems (1970 ...):

generic initial conditions evolve singularities

in finite proper time in future and/or past

GR is incomplete!

Quantum Mechanical effects significant

@ $m_p c^2 \sim 1.2 \times 10^{19}\, GeV$

6) $\forall$ Tensor: Symmetric, Conserved, $g, \partial g, \partial^2 g$ $\iff$ $G_{\mu\nu} + \Lambda g_{\mu\nu} = k T_{\mu\nu}$

$$\frac{1}{2} \nabla^2 h_{00} = R_{00} \sim R_{\mu\nu} u^\mu u^\nu$$

$$R^i{}_{0j0} = \frac{1}{2} \partial_i \partial_j h_{00} = \partial_i \partial_j \phi$$

$k = 8\pi G$ $\quad \nabla^2 \phi = \frac{k}{2} \rho - \Lambda$

$$-\vec{\nabla} \phi = \vec{g} = -\frac{GM}{r^2} \hat{r} + \Lambda \vec{r}$$

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}\right)$$

$\rho = +\frac{\Lambda}{8\pi G} \quad p = -\rho$ $\qquad \frac{\Lambda}{8\pi G} \sim 10^{-8}\, erg/cm^3$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

* Vacuum Solutions $T_{\mu\nu} = 0$

  $R_{\mu\nu} = 0$

  $g_{\mu\nu} = \eta_{\mu\nu}$ ✓ more?

* Use symmetries $\longleftrightarrow$ isometries $\longleftrightarrow$ Killing vectors

  $D_\mu \xi_\nu + D_\nu \xi_\mu = 0$

* Perturbation / linearization

  $j^\mu_\xi = T^{\mu\nu} \xi_\nu \Rightarrow D_\mu j^\mu_\xi = 0$

* Numerical

@ x, basis for $T_x M$

$$u = u^\mu \frac{\partial}{\partial x^\mu} = u^\mu \partial^{(x)}_{(\mu)}$$

∂_t, ∂_x, ∂_y, ∂_z

$$\partial_{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\partial^{(x)}_{(\mu)} \cdot \partial^{(x)}_{(\nu)} = g_{\mu\nu}(x)$$

<u>Start</u>: vacuum outside a <u>spherical</u> <u>static</u> distribution

<u>Newton</u>: $\phi_g = -\frac{GM}{r}$

<u>Stationary</u> spacetime: symmetry under time translations

$\exists$ coords $(t, \vec{r})$ s.t. $ds^2 = g_{\mu\nu}(\vec{r}) dx^\mu dx^\nu$

we distinguish time by

$(\partial_t)^2 < 0 \quad (\partial_i)^2 > 0 \ \forall x$

invariant under time translations

$$\xi = \partial_0 \quad \xi^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi_\nu = g_{0\nu}$$

$$D_\mu \xi_\nu + D_\nu \xi_\mu = \partial_\mu g_{0\nu} - \Gamma^\lambda_{\mu\nu} g_{0\lambda} + \partial_\nu g_{0\mu} - \Gamma^\lambda_{\nu\mu} g_{0\lambda}$$

$$= \cancel{\partial_\mu g_{0\nu}} + \cancel{\partial_\nu g_{0\mu}} - \cancel{2} \underbrace{\cancel{g_{0\lambda}} \cancel{\tfrac{1}{2}} \cancel{g^{\lambda\sigma}}}_{\delta^\sigma_0} \left(\cancel{\partial_\mu g_{\sigma\nu}} + \cancel{\partial_\nu g_{\mu\sigma}} - \partial_\sigma g_{\mu\nu} \right)$$

$$= -\partial_0 g_{\mu\nu} \to \text{Metric} \Rightarrow \text{no time dependency}$$

$$= 0$$

<u>Static Spacetime</u>

$\exists$ coords $(t, \vec{r})$ in which

$$ds^2 = g_{00}(\vec{r}) dt^2 + \underbrace{g_{ij}(\vec{r}) dx^i dx^j}_{\text{Space}}$$

Vacuum outside spherical static distribution

$$ds^2 = g_{00}(\vec{r}) dt^2 + g_{ij}(\vec{r}) dx^i dx^j$$

Spherical symmetry

$SO(3)$ acts on M

$R \in SO(3)$ $\quad x \in M \to Rx \in M$ $\qquad x \in C_t \quad T_x C = \underbrace{\left[\langle \partial/\partial x^i \rangle\right]}_{\text{Spanned by these}}$

$\mathcal{O}_x = \{Rx \mid R \in SO(3)\}$ orbit of x by $SO(3)$

$\mathcal{O}_x \cap C_t = S^2$ for most x

On each orbit $ds^2 = r^2 \underbrace{(d\theta^2 + \sin^2\theta \, d\varphi^2)}_{d\Omega}$

can line up such that $g_{ij}(\vec{r}) dx^i dx^j = e^{2\beta(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$

So metric of the form

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega \qquad g = \begin{pmatrix} -e^{2\alpha} & & & \\ & e^{2\beta} & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}$$

$\to$ everything is one variable $\to$ ODE's!

$$\Gamma^t_{tr} = \frac{1}{2} g^{tt} \left(\cancel{\partial_t g_{tr}} + \partial_r g_{tt} - \cancel{\partial_t g_{tt}} \right)$$
$$= \frac{1}{2} (-e^{-2\alpha}) \partial_r (-e^{2\alpha})$$
$$= \alpha'$$

$$\Gamma^r_{tt} = \frac{1}{2} g^{rr} \left(\cancel{\partial_t g_{tr}} + \cancel{\partial_t g_{tr}} - \partial_r g_{tt} \right)$$
$$= \frac{1}{2} e^{-2\beta} \cdot (-\partial_r e^{2\alpha}) = -\alpha' e^{2(\alpha - \beta)}$$

$$R_{tt} = -e^{2(\alpha-\beta)} \left[\alpha'' + (\alpha')^2 - \alpha'\beta' + \tfrac{2}{r}\alpha' \right]$$
$$R_{rr} = \left[\alpha'' + (\alpha')^2 - 2\beta' - \tfrac{2}{r}\beta' \right]$$
$$R_{\theta\theta} = e^{-2\beta} \left[r(\beta' - \alpha') - 1 \right] + 1$$
$$R_{\varphi\varphi} = R_{\theta\theta} \sin^2\theta$$

$$e^{2(\beta - \alpha)} R_{tt} + R_{rr} = 0 \Rightarrow \alpha' + \beta' = 0$$
$$\Rightarrow \alpha = C - \beta \leftarrow t \to e^{c/2} t$$
$$\Rightarrow C = 0$$

$$-R_{\theta\theta} = e^{2\alpha}(2r\alpha' + 1) - 1 = 0$$
$$\frac{d}{dr}(re^{2\alpha}) = e^{2\alpha}(1 + 2r\alpha') = 1$$
$$\Rightarrow re^{2\alpha} = r - \underset{\text{constant of integration}}{R}$$

$$ds^2 = -\left(1 - \frac{R}{r}\right) dt^2 + \left(1 - \frac{R}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \quad \text{Swarzchild}$$

* $e^{2\alpha} > 0 \iff r > R$

* $r \to \infty$

$ds^2 = -dt^2 + dr^2 + r^2 d\Omega$ $\quad R^{\mu}{}_{\nu\rho\sigma} \to 0$

weak static gravity

$r \to \infty$

$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu$

$h_{00} = R/r \ll 1$

$h_{rr} = \frac{1}{1 - R/r} - 1 = \frac{r}{r-R} - \frac{r-R}{r-R}$

$= \frac{R}{r-R} \ll 1$

Compare to Newton:

$R_s = 2GM/c^2$

Swarzchild radius

$= \frac{M}{M_\odot} = 2.95\,km$

Nontrivial solution to Einstein

Best known: Swarzchild

$ds^2 = -(1 - R/r)dt^2 + (1 - R/r)^{-1} dr^2 + r^2 d\Omega$

* $r \to \infty$ $\quad ds^2 \sim \eta_{\mu\nu}\, dx^\mu dx^\nu$ Minkowski: $\quad R = \frac{2GM}{c^2} = \left(\frac{M}{M_\odot}\right) \cdot 2.95\,km$

* $R_{\mu\nu} = 0 \iff T_{\mu\nu} = 0$ & $r > R$

* r? $\quad 2\pi r$ circumference $\quad (d\Omega = d\theta^2 + \sin^2 d\varphi^2)$

πr^2 area

r is <u>NOT</u> distance

$\int_{r_1}^{r_2} (1 - R/r)^{-1/2} dr$

What is t?

$\Gamma^r_{tt} \neq 0$ static such. observers r, θ, φ constant

$u^\mu = \frac{1}{\sqrt{1 - R/r}} (1, \vec{0})$

$x^\mu(\lambda) \quad u^\mu = \frac{dx^\mu}{dt} = (u^0, 0) = \left((1 - R/r)^{-1/2}, \vec{0}\right)$

$u^0 > 0$ timelike

$$g_{\mu\nu} u^\mu u^\nu = -1$$

$$d\tau = (1 - R/r)^{1/2} dt$$ gravitational redshift

Birkhoff's thm: any spherically symmetric vacuum spacetime is Schwarzschild

* Irrelevant of spherical shell, possibly moving → Vaidya spacetime

Geodesics:

$$ds^2 = -(1 - R/r)\, dt^2 + (1 - R/r)^{-1} dr^2 + r^2 d\Omega$$

4 Killing vectors

$$d\Omega = d\theta^2 + \sin^2\theta\, d\varphi^2$$

- $\xi_t = \partial_t$ ← Symmetry: time translation $\quad \xi_t^\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $\xi_\varphi = \partial_\varphi \quad \xi_\varphi^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
- ξ_θ
- ξ_r

$$[\xi^i, \xi^j] = \epsilon^{ijk} \xi^k$$

$$p^\mu = m \frac{dx^\mu}{d\tau} \qquad u^\mu = \frac{dx^\mu}{d\lambda}$$

τ or σ affine

Conserved quantity

$$K_t = g_{\mu\nu}(x(\lambda))\, p^\mu(\lambda)\, \xi_t^\nu(x(\lambda))$$
$$= -(1 - R/r)\, p^0 = -E$$

$$K_\varphi = g_{\mu\nu} p^\mu \xi_\varphi^\nu = r^2 \sin^2\theta\, p^\varphi = L$$

$$E = (1 - R/r) \frac{dt}{d\lambda}$$

energy (per unit mass)

$$L = r^2 \sin^2\theta \frac{d\varphi}{d\lambda}$$

angular momentum (per unit mass)

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

(t) $$\frac{d^2t}{d\lambda^2} + \frac{R}{r(r-R)} \frac{dr}{d\lambda} \frac{dt}{d\lambda} = 0$$

(r) $$\frac{d^2r}{d\lambda^2} + \frac{R}{2r(r-R)} \left[\left(1 - \frac{r}{R}\right)^2 \left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dr}{d\lambda}\right)^2 \right] - (r-R)\left[\cancel{\left(\frac{d\theta}{d\lambda}\right)^2} + \sin^2\theta \left(\frac{d\varphi}{d\lambda}\right)^2 \right] = 0$$

(θ) $$\cancel{\frac{d^2\theta}{d\lambda^2} + \frac{2}{r} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda}} - \sin\theta\cos\theta \left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

$$(\varphi)\quad \frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + \cancel{2\cot\theta\left(\frac{d\theta}{d\lambda}\right)\left(\frac{d\varphi}{d\lambda}\right)}^{\to 0} = 0$$

$$(t) = \left(1-\frac{R}{r}\right)^{-1}\frac{d}{d\lambda}\left[\left(1-\frac{R}{r}\right)\frac{dt}{d\lambda}\right] = 0$$

$$= \left(1-\frac{R}{r}\right)^{-1}\frac{dE}{d\lambda} = 0$$

$$\theta(0) = \pi/2 \quad \frac{d\theta}{d\lambda}(0) = 0 \Rightarrow \theta(0) = \pi/2 \quad \forall\lambda$$

orbits are planar!

$$(\varphi)\quad \frac{1}{r^2\sin\theta}\frac{d}{d\lambda}\left(r^2\sin\theta\frac{d\varphi}{d\lambda}\right) = 0$$

L conserved

$$E = \left(1-R/r\right)\frac{dt}{d\lambda}$$

$$L = r^2\underbrace{\cancel{\sin^2\theta}}_{\text{planar}}\frac{d\varphi}{d\lambda}$$

Choose λ proper (timelike) / affine (null)

$$g_{\mu\nu}(x(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = -\varepsilon$$

what is this? $\varepsilon = \begin{cases} 1 & \text{matter} \\ 0 & \text{light}\end{cases}$

$$-\varepsilon = -\left(1-R/r\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-R/r\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2\right)$$

$$-\frac{\varepsilon}{2} = \frac{1}{2}\left(1-R/r\right)^{-1}\left(-E^2 + \dot{r}^2\right) + \frac{L^2}{2r^2}$$

$\dot{r} = \frac{dr}{d\lambda}$

$$\frac{1}{2}\dot{r}^2 = \frac{1}{2}E^2 - \overbrace{\frac{1}{2}\left(1-R/r\right)\left(\varepsilon + L^2/r^2\right)}^{V(r)}$$

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2 = \mathcal{E}$$

recall $R = \frac{2GM}{c^2}$

$$V(r) = \frac{1}{2}\left(1-R/r\right)\left(\varepsilon + L^2/r^2\right)$$

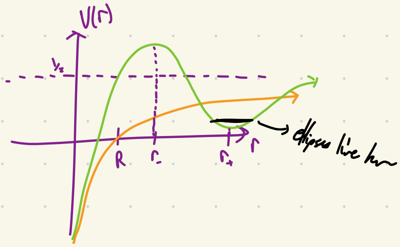
$(\varepsilon=1)$

$$= \underbrace{\frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2}}_{\text{ellipses}} + \underbrace{\frac{GML^2}{r^3}}_{\text{precession}}$$

$\boxed{\mathcal{E}=1}$

$$V' = \frac{M}{r^2} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4}$$

$$V'=0 \quad r = \pm\frac{L^2}{2M}\left(1 \pm \sqrt{1-12\frac{M^2}{L^2}}\right)$$



for $L < 2\sqrt{3}M$

$L > 2\sqrt{3}M$

$$0 < x = \frac{12M^2}{L^2} < 1$$

$$3R = 6M \le r_+ = \frac{6M}{x}\left(1+\sqrt{1-x}\right) \xrightarrow{x\to 0} \frac{12M}{x} = \frac{L^2}{M} \to \infty$$

closest stable orbit

$$\frac{3R}{2} = 3M \xleftarrow{x\to 0} r_- = \frac{6M}{x}\left(1-\sqrt{1-x}\right) \le 3R$$

minimal unstable orbit

$$\frac{1}{2}\epsilon^2 = \mathcal{E}_c = V(r_c) = \frac{(r-2M)^2}{2r(r-3M)} \overset{r=3R=6M}{=} \frac{4}{9} \Rightarrow E = \frac{2\sqrt{2}}{3} = 0.94$$

6% of mass turns to energy. This is a lot

$$V'=0 = Mr^2 = L^2(r-3M)$$

$$\omega_\varphi^2 = \frac{L^2}{r_0^4} = \frac{Mr_0^2}{r_0^4(r_0-3M)} = \frac{M}{r_0^2(r_0-3M)} \overset{r_0 \gg M}{\sim} \frac{M}{r_0^3}$$

Kepler's 3rd Law... corrected!

$$\omega_r^2 = V'' = \frac{M}{r_0^3}\,\frac{r_0-6M}{r_0-3M}$$

$r_0 \gg M \quad \omega_r^2 < \omega_\varphi^2$

precession

$$\frac{\omega_\varphi - \omega_r}{\omega_\varphi} = 1-\sqrt{\frac{r_0-6M}{r_0}}$$

$\approx 3M/r_0 \longrightarrow$ result of $\frac{GML^2}{r^3}$ from earlier

Stars n shit

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega$$

$$T_{\mu\nu} = \begin{pmatrix} \rho(r) & & & \\ & p(r) & & \\ & & p(r) & \\ & & & p(r) \end{pmatrix} \qquad u^\mu = e^{-\alpha}\begin{pmatrix}1\\0\end{pmatrix}$$

$$G_{tt} = r^{-2}e^{2(\alpha-\beta)}\left(2r\beta' - 1 + e^{2\beta}\right) = 8\pi e^{2\alpha}\rho(r) \quad \checkmark$$

$$G_{rr} = r^{-2}\left(2r\alpha' + 1 - e^{-2\beta}\right) = 8\pi e^{2\beta}p(r) \quad \checkmark$$

$$G_{\theta\theta} = r^2 e^{-2\beta}\left(\alpha'' + (\alpha')^2 - \alpha'\beta' + \tfrac{1}{r}(\alpha' - \beta')\right) = 8\pi r^2 p(r)$$

$$G_{\varphi\varphi} = \sin^2\theta\, G_{\theta\theta} = 8\pi T_{\varphi\varphi} = 8\pi \sin^2\theta\, T_{\varphi\varphi}$$

Equation of state: $\rho(p)$

$$e^{2\beta(r)} = \left(1 - \frac{2m(r)}{r}\right)^{-1}$$

$$m(r) = 4\pi\int_0^r dr' r'^2 \rho(r') \Big\} \text{ total energy in sphere of radius } r$$

At $r = r_0$ $\quad p \to 0$, $\rho \to 0$ $\qquad m(r_0) = M$

$$\alpha' = \frac{4\pi r^3 p(r) + m(r)}{r(r - 2m(r))}$$

Check (nonrelativistic, static, weak field)

$$r^3 p(r) \ll m(r) \ll r$$

$$\alpha' \sim \frac{m(r)}{r^2} = \phi'$$

$$g_{00} \sim -(1 + 2\alpha) \qquad \alpha = \phi$$

$$0 = D_\mu T^{\mu\nu} = e^{-2\beta}\left(p' - \underbrace{\alpha'(p + \rho)}_{=0}\right)$$

$$p'(r) = -(p+\rho)\frac{4\pi r^3 p(r) + m(r)}{r(r - 2m(r))}$$ Tolman Oppenheimer Volkoff

nonrelativistic

$$p' = -\rho\phi' = -\frac{G\rho(r)m(r)}{r^2}$$

$$p'(r) = \frac{Gm(r)}{r^2}\rho\left(1 + \frac{p}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 p(r)}{m(r)c^2}\right)\left(1 - \frac{2m(r)}{rc^2}\right)$$

Assume $\rho(r=0) \xrightarrow{\text{equation of state relates the two}} p(r=0)$

$\to p(r) \to r_0 \ @ \ {}^{p}_{\rho} \to 0$

Ex: $\rho(r) = \begin{cases} \rho_0 & r<r_0 \\ 0 & r>r_0 \end{cases}$

$$p(r) = \rho_0 \frac{\left(1-\frac{2m(r)}{r}\right)^{1/2} - \left(1-\frac{2m}{r_0}\right)^{1/2}}{3\left(1-\frac{2m}{r_0}\right)^{1/2} - \left(1-\frac{2m(r)}{r}\right)^{1/2}}$$

$$m(r) = \begin{cases} \frac{4\pi}{3}\rho_0 & r<r_0 \\ M & r>r_0 \end{cases} \qquad m < \frac{r_0}{2}$$

$$P_N(0) = \frac{2\pi}{3}\rho_0^2 r_0^2 \sim M^{2/3}\rho_0^{4/3}$$

$$P_{TOV}(0) = \rho_0 \frac{1-\sqrt{1-\frac{2M}{r_0}}}{3\sqrt{1-\frac{2M}{r_0}}-1}$$

$\downarrow$

$\infty \qquad r_0 = \frac{9M}{4} > 2M$

Chandrasekhar Degenerate e^- gas

$M \lesssim M_{ch} \simeq 1.4 M_\odot$

TOV Degenerate n gas

$M \lesssim M_{NS} \simeq 3\text{-}5 M_\odot$

Schönberg-Chandrasekhar:

Isothermal core $M_{core} \lesssim 0.37\, M$

Near $r \sim R$?

$r < R$? $\qquad R_{\mu\nu} = 0$

Check: Radial infalling matter ($\theta = \pi/2$ $\dot\theta = 0$ $\underset{L=0}{\dot\varphi = 0}$)

$$E = (1 - R/r)\dot t \qquad \dot{} = \frac{d}{d\tau}$$

$$-1 = -(1-R/r)\dot t^2 + (1-R/r)^{-1}\dot r^2 = \frac{\dot r^2 - E^2}{1-R/r}$$

$$\dot r^2 = E^2 - 1 + R/r$$

$$\dot r = \pm\sqrt{E^2 - 1 + R/r}$$

Choose $E=1$

$E \sim 1 + \frac{v^2}{2} - R/r$

$$d\tau = -\sqrt{\frac{r}{R}}\, dr$$

$$\Delta\tau = -\frac{1}{\sqrt R}\int_{r_i}^{R} \sqrt r\, dr \sim -\frac{R}{3\sqrt R}\left(r_f^{3/2} - r_i^{3/2}\right) \qquad r_f = R$$

$$\Delta t = -\sqrt{R}\int^{R+t}\frac{dr\, r^{3/2}}{r-R} \sim -R^{2}\log t \to \infty$$

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \underset{r\to 0}{\sim} \frac{6R^2}{r^6}$$

$r=0 \leftrightarrow$ curvature singularity

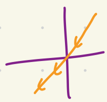
$r=R \leftrightarrow$ No " "

$$ds^2 = dr^2 + r^2 d\varphi^2$$

problems:

1) $g_{\varphi\varphi} = r^2 \underset{r\to\infty}{\to} \infty$ → fine, not a problem, can never reach ∞.

2) $g_{\varphi\varphi} = r^2 \underset{r\to 0}{\to} 0$ → bad choice of coordinate



$t=\infty \longrightarrow t' =$

$t \geq 0$, $x \in \mathbb{R}$

$$ds^2 = -\frac{dt^2}{t^4} + dx^2 = -dt'^2 + dx^2$$

$g_{tt} \to 0$, $t \to \infty$

$$t' = -\frac{1}{t} < 0$$

$$dt' = dt/t^2$$

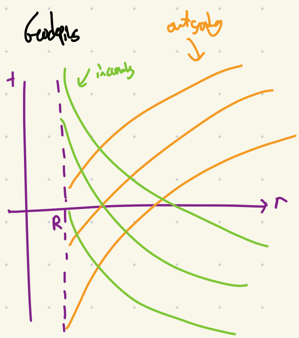
Null radial Geodesics

$$\left(1-\tfrac{R}{r}\right)\dot{t}^2 = \left(1-\tfrac{R}{r}\right)^{-1}\dot{r}^2$$

$$dt = \pm\left(1-\tfrac{R}{r}\right)dr$$

$$t = \pm r \pm R\log(r-R) + C$$

$$= C \pm r_*(r)$$



$$r_*(r) = r + R\log\left(\tfrac{r}{R}-1\right)$$

$$\frac{dr_*}{dr} = \left(1-\tfrac{R}{r}\right)^{-1}$$

Regge-Wheeler-Tortoise

Constant along:

outgoing $u = t - r_*(r)$

incoming $v = t + r_*(r)$

$$t = \tfrac{1}{2}(u+v)$$

$$\tfrac{1}{2}(v-u) = r_*$$

$$e^{\frac{1}{2R}(v-u)} = e^{r_*/R} = \left(\tfrac{r}{R}-1\right)e^{r/R}$$

$$du = dt - (1-R/r)^{-1}dr \qquad r=R: \quad u\to-\infty$$

$$dv = dt + (1-R/r)^{-1}dr \qquad\qquad v\to\infty$$

$$ds^2 = -(1-R/r)dt^2 + (1-R/r)dr^2 + r^2 d\Omega$$

$$= -(1-R/r)\,du\,dv + r^2(u,v)\,d\Omega = \underbrace{-\frac{4R^3}{r}e^{-r/R}\,dU\,dV}_{\text{gets rid of singularity}} + r^2(U,V)\,d\Omega$$

$$U = -e^{-u/2R} \qquad u\to\infty,\ U\to 0$$

$$V = e^{v/2R} \qquad v\to-\infty,\ V\to 0$$

"bring infinity in so we can see it"

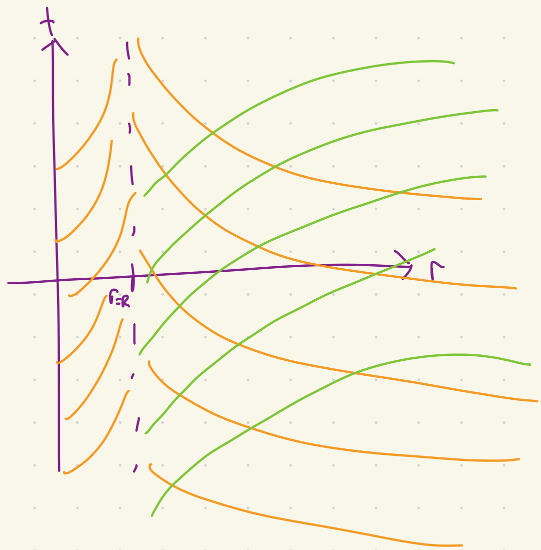
$$ds^2 = -(1-R/r)dt^2 + (1-R/r)^{-1}dr^2 + r^2 d\Omega$$

Infalling null geodesics

$$(1-R/r)\dot t^2 = (1-R/r)^{-1}\dot r^2$$

$$dt = \pm(1-R/r)^{-1}dr$$

$$t = \pm r \pm R\log(r-R) + C$$



replace t $\leftarrow$ $\overset{\text{null coordinate}}{v} = t + r + R\log(r-R)$ constant on incoming null geodesics

$$dv = dt + dr\left(1 + \frac{R}{r-R}\right) \qquad \text{Eddington-Finkelstein}$$

$$ds^2 = -(1-R/r)dv^2 + 2\,dv\,dr + r^2 d\Omega$$

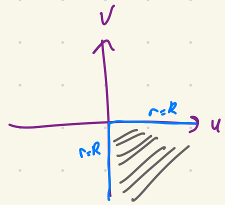
$$\sqrt{-ds^2} = d\tau$$

$\underline{r<R}$

$$0 > ds^2 > (1-R/r)^{-1}dr^2 = -\frac{r}{R}\frac{dr^2}{1-r/R}$$

$$\Delta\tau = \int\sqrt{-dr^2} \le \frac{1}{\sqrt{R}}\int_\varepsilon^R \frac{\sqrt{r}\,dr}{\sqrt{1-r/R}} \xrightarrow[\varepsilon\to 0]{} \frac{R\pi}{2}$$

new coord: $u = t - \underbrace{r - R\log|r-R|}_{r_*(r)\ \text{Regge-Wheeler}}$ outgoing



$$t = \tfrac{1}{2}(u+v)$$
$$r_* = \tfrac{1}{2}(v-u) \qquad e^{r_*/R} = (r/R - 1)\,e^{r/R}$$

$$ds^2 = -(1 - R/r)\,du\,dv + r^2(u,v)\,d\Omega$$

$r \to R \quad v \to -\infty \quad (V \to 0)$

$\qquad\quad u \to \infty \quad (U \to 0)$

Kruskal

$$U = -e^{-u/2R} \le 0$$
$$V = e^{v/2R} \ge 0$$

$$UV = (1 - r/R)\,e^{r/R} = T^2 - X^2$$
$$R\log(V/-U) = t = R\log\frac{T+X}{|T-X|}$$

nonsingular at $r=R$!

$$ds^2 = -\frac{4R^3}{r}e^{-r/R}\,dU\,dV + r^2(U,V)\,d\Omega$$

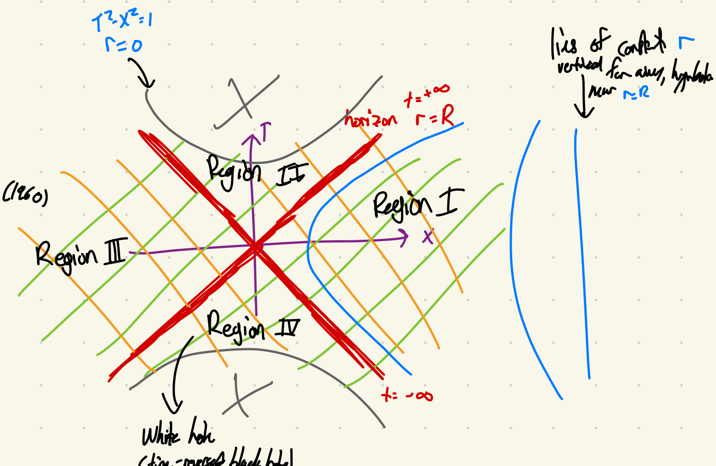
One last time!

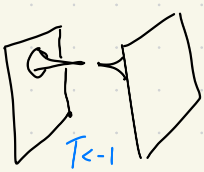
$T = \tfrac{1}{2}(V+U)$ ← everywhere timelike

$X = \tfrac{1}{2}(V-U)$ ← everywhere spacelike

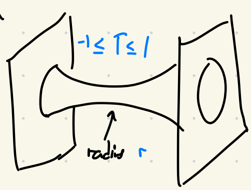
Kruskal (1960)

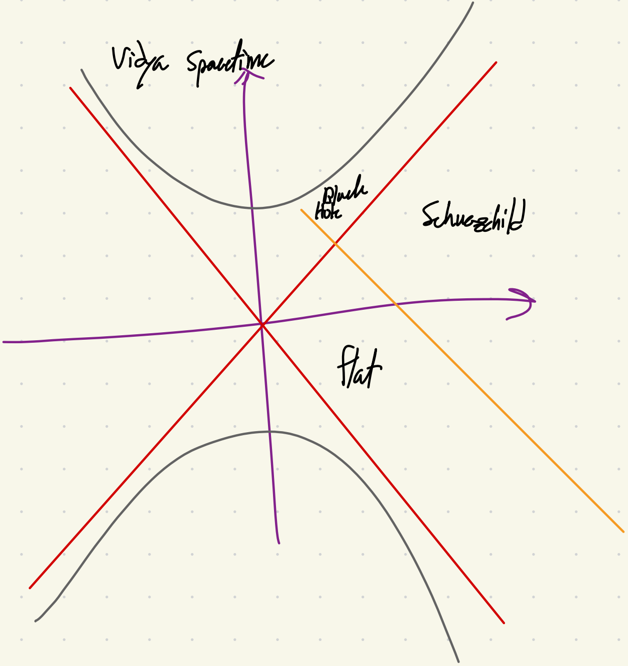
$$ds^2 = -\frac{4R^2}{r}e^{-r/R}\left(dT^2 - dX^2\right) + r^2 d\Omega$$





Non-traversable wormhole – not enough time to cross it at speed of light





Yummy metric:

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}\,dt\,d\varphi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left[r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\rho^2}\right]\sin^2\theta\, d\varphi^2$$

Neutral spinning black hole

angular momentum $\rightarrow J = aM$

$$\rho^2 = r^2 + a^2\cos^2\theta$$
$$\Delta = r^2 + a^2 - 2Mr$$

$$\Sigma_+ = \{r^2 + a^2\cos^2\theta = 2Mr\}$$

surface (not sum)

1) $g_{tt} = 0$

2) Causal propagation ($u^2 \leq 0$) has $\dot{\varphi} \geq 0$

$$\omega_- < \omega = \frac{d\varphi}{dt} < \omega_+$$

$$\omega_\pm = \underbrace{-\frac{g_{t\varphi}}{g_{\varphi\varphi}}}_{=\,\omega(r,\theta)} \pm \sqrt{\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}$$

(outer) <u>Horizon</u> $S_\chi = r = r_+ = M + \sqrt{M^2 - a^2}$

1) ∂_t timelike (null on S_χ)

2) $\partial_t + \Omega_H \partial_\varphi = \chi$

$$\Omega_H = \omega(r_+, \theta) = \frac{a}{r_+^2 + a^2} = \frac{a}{2Mr_+}$$

$$= \frac{-\cancel{2M}a\cancel{r_+}\cancel{\sin^2\theta}/\rho_+^2}{\Big[\underbrace{r_+^2 + a^2}_{\cancel{2Mr_+}} + \cancel{2M}a^2\cancel{r_+}\sin^2\theta/\rho_+^2\Big]\cancel{\sin^2\theta}}$$

$$= \frac{2a\sin^2\theta}{1 + a^2\sin^2\theta/\rho_+^2}$$

$$= \frac{a}{r_+^2 + a^2}$$

S_χ : Killing Horizon

$f = g_{\mu\nu}\chi^\mu\chi^\nu = 0$ $\qquad T_p(S_\chi) \subset T_pM$

$\qquad\qquad\qquad\qquad v \Leftrightarrow v^\mu \partial_\mu f = 0$

<u>Penrose</u>: Throw in a probe with $E < 0$.

$E = -g_{\mu\nu} p^\mu \xi^\nu_{(t)}$ conserved

$p^0 = m u^0 > 0$ $\forall$ timelike trajectories

In ergoregion ∂_t spacelike

① Start with $E_0 = p_0^{\,0} > 0$ @ $r \gg M, a$

② Carefully blow it up $E_1 + E_2 = E_0$

can arrange $E_2 < 0 \Rightarrow E_1 > E_0$

① ejected back to $r \gg M$

② crossed $r = r_+ \to$ absorbed

$\chi = \partial_t + \Omega_H \partial_\varphi$ timelike

$-p \cdot \chi > 0$

$= -p(\partial_t + \Omega_H \partial_\varphi)$

$= E - \Omega_H L > 0$

$E_2 < 0 \Rightarrow L_2 < 0$

$$\delta J = L < \frac{E}{\Omega_H} = \frac{\delta M}{\Omega_H} < 0$$

$$M_{irr}^2(M,J) = \frac{1}{2}\left(M^2+\sqrt{M^4-J^2}\right)$$
irreducible
$$= \frac{M}{2}\left(M+\sqrt{M^2-a^2}\right) = \frac{Mr_+}{2}$$

$$M^2(M_{irr},J) = M_{irr}^2 + J^2/4M_{irr}^2$$

$$2M_{irr}\,\delta M_{irr} = M\delta M + \frac{1}{4\sqrt{M^4-J^2}}\left(4M^3\delta M - 2J\delta J\right)$$
$$= \frac{M}{\sqrt{M^4-J^2}}\left(2M_{irr}^2\delta M - \frac{J}{2M}\delta J\right)$$
$$= \frac{2MM_{irr}^2}{\sqrt{M^4-J^2}}\left(\delta M - \Omega_H\delta J\right) \geq 0$$

$$\Omega_H = \frac{a}{r_+^2+a^2} = \frac{J/M}{2Mr_+} = \frac{J}{4MM_{irr}^2}$$

$$A_H = 8\pi Mr_+ = 16\pi M_{irr}^2$$
area of horizon

Perturbative GR

Weak fields: $\exists x$ s.t. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ("small" $h_{\mu\nu}$; x: fix the coords)

Work to order h

x fixed to order $h^0=1$ up to poincaré $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

Linearized gravity is CFT (classical field theory) of symmetric $(0,2)$-tensor in Minkowski

To order h: $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

$$h^{\mu\nu} = g^{\mu\rho}g^{\nu\lambda}h_{\rho\lambda} \approx \eta^{\mu\rho}\eta^{\nu\lambda}h_{\rho\lambda}$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\eta^{\lambda\sigma}\left(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right)$$

$$R^\mu{}_{\nu\lambda\sigma} = \partial_\lambda\Gamma^\mu_{\sigma\nu} - \partial_\sigma\Gamma^\mu_{\lambda\nu}$$
$$= \eta^{\mu\rho}\left(\partial_\nu\partial_{[\lambda}h_{\sigma]\rho} - \partial_\rho\partial_{[\lambda}h_{\sigma]\nu}\right)$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \qquad h = \eta^{\mu\nu}h_{\mu\nu}$$
(h: trace)

$$R_{\mu\nu} = \frac{1}{2}\left[\partial^\lambda \partial_\mu \bar{h}_{\mu\nu} + \partial^\lambda \partial_\nu \bar{h}_{\mu\nu} - \partial^2 \bar{h}_{\mu\nu}\right] + \frac{1}{4}\eta_{\mu\nu}\partial^2 \bar{h}$$

$$R = \partial^\lambda \partial^\mu \bar{h}_{\mu\lambda} + \frac{1}{2}\partial^2 \bar{h}$$

$$G_{\mu\nu} \approx \frac{1}{2}\left[\partial^\lambda \partial_\mu \bar{h}_{\mu\nu} + \partial^\lambda \partial_\nu \bar{h}_{\mu\lambda} - \partial^2 \bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\lambda \partial_\rho \bar{h}^{\lambda\sigma}\right]$$

$$= 8\pi \cancel{G} T_{\mu\nu}$$

Gaussian ord. h.

Coordinate change

$g'_{\mu\nu}$ $\quad g_{\mu\nu}$ = same physics if $\exists$ $x(x')$ s.t.

$$g'_{\mu\nu} = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} g_{\lambda\rho}$$

If $x'^\mu = x^\mu + \xi^\mu(x)$ $\quad$ ξ small

$x^\mu = x'^\mu - \xi^\mu(x')$

$$g' \sim g + \underbrace{\mathcal{O}(\xi)}_{\text{order } \xi}$$

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) - D_\mu \xi_\nu - D_\nu \xi_\mu$$

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h'$$

$$= h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}(h - 2\partial^\lambda \xi_\lambda)$$

$$= \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \partial^\lambda \xi_\lambda$$

$$\partial_\mu \bar{h}'^{\mu\nu} = \underline{\partial_\mu \bar{h}^{\mu\nu} - \partial^2 \xi^\nu} - \cancel{\partial^\nu(\partial_\mu \xi^\mu)} + \cancel{\partial_\mu \partial^\lambda \xi_\lambda}$$

Solve "Poisson" wave: $\partial^2 \xi^\nu = \partial_\mu \bar{h}^{\mu\nu}$

$\Rightarrow \partial_\mu \bar{h}'^{\mu\nu} = 0$ $\quad$ Lorentz gauge

$(\partial_\mu A^\mu = 0)$

$$G_{\mu\nu} = \frac{1}{2}\left[\cancel{\partial^\lambda \partial_\mu \bar{h}_{\nu\lambda}} + \cancel{\partial^\lambda \partial_\nu h_{\mu\lambda}} - \partial^2 \bar{h}_{\mu\nu} - \cancel{\eta_{\mu\nu}\partial_\lambda \partial_\rho \bar{h}^{\lambda\rho}}\right]$$

Einstein: $\partial^2 \bar{h}_{\mu\nu} - 16\pi T_{\mu\nu}$

$\partial_\mu \bar{h}^{\mu\nu} = 0 \Rightarrow \partial_\mu T^{\mu\nu} = 0$

Waves:

$$\partial^2 u = -4\pi f(x)$$

Construct $G(x, x')$ s.t. $\partial^2_x G = -4\pi \delta^{(4)}(x-x')$

Green's Function

1) Translation invariant $G(x, x') = G(x-x')$

2) Lorentz $G(\Lambda x) = G(x)$

$$G^{\pm}(x) = \frac{1}{r}\delta(t \mp r)$$

$x = (t, \vec{r})$ $r = |\vec{r}|$

$$u(x) = \int d^4x' \, G(x, x') f(x')$$

$$\bar{h}^{\mu\nu}(t, \vec{r}) = 4\int d^3\vec{r}' \frac{T^{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|}$$

Lorentz gauge if $\partial_\mu T^{\mu\nu} = 0$

Waves: $\partial^2 \bar{h}^{\mu\nu} = 0$ $\partial_\mu \bar{h}^{\mu\nu} = 0$

$$\bar{h}^{\mu\nu} \to \bar{h}^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\nu \xi^\mu + \eta^{\mu\nu} \partial_\lambda \xi^\lambda$$

$$\partial^2 \xi^\mu = 0$$

Can achieve: Transverse Traceless (TT)

$$\bar{h}_{TT} = h_{TT} = 0 \quad h^{0i}_{TT} = 0$$

$$0 = \partial_0 h^{00} + \partial_i h^{i0} = 0 \Rightarrow \partial_0 h^{00} = 0$$

$$\partial^2 h^{00} = 0 \Rightarrow \vec{\nabla}^2 h^{00} = 0$$

Plane Waves

$$h^{\mu\nu}_{TT} = A^{\mu\nu}_{(t)} e^{ik \cdot x} \qquad \partial_\lambda \bar{h}^{\mu\nu} = A^{\mu\nu} \cdot ik_\lambda e^{ik\cdot x}$$

Einstein: $\partial^2 h^{\mu\nu} = 0 \Leftrightarrow k^2 = 0$ lightlike

Lorentz: $\partial_{\mu\nu} \bar{h}^{\mu\nu} = 0 \Leftrightarrow A^{\mu}{}_{\mu} = 0$

$A^{0i} = 0$

Chosen $k = (\omega, 0, 0, \omega)$

$$A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = a\, e_{+}^{\mu\nu}(k) + b\, e_{\times}^{\mu\nu}$$