Math 403 - Advanced linear Algebra - 01/09

A group $(G, *)$ is ...

A bunch of examples

A field is ...

All of math 221 works over an arbitrary field k (rather than $\mathbb{R}$)

except for notions of:

length, angle, order

$\Downarrow$ topology

$\hookrightarrow a < b$

Def a v.s. V over f

is $(V, +)$ an abelian group

and an action of $f \curvearrowright V$.

$f \times V \longrightarrow V$

$(\alpha, v) \mapsto \alpha v$

(associative, distributive, $1 \cdot v = v \ \forall v \in V$)

Def a homomorphism is a linear map

E.g. if $\mathcal{B}$ is a basis of V, then function

$f: \mathcal{B} \longrightarrow W \Longleftrightarrow$ hom. $\varphi: V \to W$ with

$\varphi|_{\mathcal{B}} = f$

universal property of Bases

$$\mathrm{Hom}(V, W) \cong \mathrm{fun}(\mathcal{B}, W)$$

canonical bij.

Def: A hom. $\varphi: V \to W$ has

- kernel $\ker \varphi = \{v \in V \mid \varphi(v) = 0\} \subseteq V$
- image $\operatorname{im} \varphi = \{\varphi(v) \mid v \in V\} \subseteq W$

Thm (Rank-nullity)

If $\varphi: V \to W$ then

$$\dim \ker \varphi + \dim \operatorname{im} \varphi = \dim V$$

Pf: pick $\mathcal{B}'$ = basis of $\ker\varphi$

$\mathcal{B}'' \subseteq V$

$\mathcal{B}'' \hookrightarrow \varphi(\mathcal{B}'')$ = basis of $\operatorname{im}\varphi$

Then $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$ is a basis of V. ⊕

slight checkage required

Def: A hom. $\varphi: V \to W$ is an <u>isomorphism</u> if φ is injective and surjective

$$\underbrace{\overset{\ker\varphi = 0}{\text{one-to-one}} \qquad \overset{\ker\varphi = W}{\text{onto}}}_{\text{bijective}}$$

e.g. $\dim V = n \iff V \cong F^n$

pf: Basis $v_1, \dots, v_n$ of V $\Rightarrow$

$v_i \mapsto e_i$ induces $\cong$ by universal prop of bases

e.g. $\mathrm{sols}_{\mathbb{R}}(f'' + f = 0) \subseteq C^2(\mathbb{R})$

$\mathbb{R}^2 \cong \mathrm{Span}\{\sin, \cos\}$

$\mathbb{C}^2 = \mathrm{Span}\{e^{ix}, e^{-ix}\}$

Duality

Def. Fix a v.s./F. The dual of V is $V^* = \{\text{homomorphisms } V \to F\}$
$= \hom(V, F)$

lemma: V^* is a v.s./F.

pf: $\mathrm{fun}(S, F)$ is a v.s. for any set S

$(f+g)(s) = \underset{\in F}{f(s)} + \underset{\in F}{g(s)}$

$(\alpha f)(s) = \underset{\in F}{\alpha}\,(\underset{\in F}{f(s)})$

Then we just need to show $\mathrm{Hom} \leq \mathrm{fun}$

$\ell \in V^*$, $\alpha \in F$ $\Rightarrow$ $\underset{\in V^*}{\alpha\ell}: v \longmapsto \alpha\ell(v) = \ell(\alpha v)$

$\ell_1, \ell_2 \in V^* \Rightarrow \ell_1 + \ell_2 \in V^*$

$\Rightarrow \mathrm{Hom}(V, F) \leq \mathrm{fun}(V, F)$ is a subspace.

Ex: 1. $V = F^n_{col} \Rightarrow V^* = F^n_{row}$

$$[y_1 \cdots y_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [\underset{\in F}{y_1 x_1 + \cdots y_n x_n}]$$

2. $V = C^0([0,1]) \Rightarrow \int_0^1 - \, dt \in V^*$

$f \mapsto \int_0^1 f(t)\, dt$ is linear

$ev_{1/2} : f \mapsto ev_{1/2}(f) = f(1/2)$

$ev_{1/2} \in V^*$ linear by definition

3. $V = C^1([0,1]) \Rightarrow ev_{1/2} \circ \frac{d}{dt} \in V^*$

$f \mapsto f'(1/2)$ linear as a function of f

<u>Notation</u>: write $\ell(v) = \langle \ell, v \rangle$

Suppose

$$\begin{tikzcd} W \arrow[d, "\ell"'] & V \arrow[l, "\varphi"'] \arrow[dl, "\ell\circ\varphi"] \\ F & \end{tikzcd}$$

$\Rightarrow$ implies linearity

Then $\ell \in W^* \Rightarrow \ell \circ \varphi \in V^*$ so

$\varphi^* : \ell \mapsto \ell \circ \varphi$

$W^* \longrightarrow V^*$ "transpose of φ"

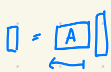
In bracket notation:

$$\langle \varphi^* \ell, v \rangle = \langle \ell \circ \varphi, v \rangle$$
$$= \langle \ell, \varphi v \rangle$$

left mult by matrix A

Eg: $\underset{W}{F^m_{col}} \xleftarrow{\lambda_A} \underset{V}{F^n_{col}}$

F^m_{row}



$y \in (F^m_{col})^*$

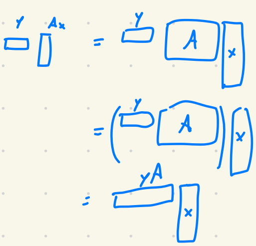
$\langle y, \lambda_A x \rangle$

$= \langle y, Ax \rangle$

$= yAx$

$= \langle yA, x \rangle$

$= \langle \lambda_A^*(y), x \rangle$

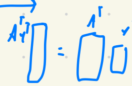


$\Rightarrow \lambda_A^* = \rho_A$

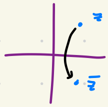
right mult by A

$F^m_{row} \xrightarrow{\rho_A} F^n_{row}$

"Same as"



$\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\}$ v.s$/\mathbb{R}$ with basis $\{1, i\}$



Conjugate of $z = a+bi$ is $\bar{z} = a-bi$

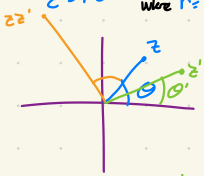
$z\bar{z} = |z|^2 = a^2+b^2$

$\quad \|$

$(a+bi)(a-bi)$

$z = re^{i\theta}$ where $r = |z|$ = modulus of z

θ = argument of z



$zz' = re^{i\theta} r'e^{i\theta'}$
$= rr'e^{i(\theta+\theta')}$

Why?

1. Taylor series

$e^{it} = 1 + it - \frac{t^2}{2} - \frac{it^3}{6} + \frac{t^4}{4!} + \dots$ (sin, cos)

$= 1 - \frac{t^2}{2} + \frac{t^4}{4!} + i\left(t - \frac{t^3}{6} + \frac{t^5}{5!}\right)$

$(e^x)' = e^x \quad e^0 = 1$

2. ODE for e^{it}: $\frac{d}{dt}e^{it} = ie^{it}$

$z = a+bi \quad iz = -b+ai$

e^{it} is curve orthogonal to tangent

$\Rightarrow$ circle around 0 unit circle

Eg: $z = -1 \Rightarrow |z| = 1$

$\Rightarrow \arg z = \pi$

$\Rightarrow e^{i\pi} = -1$

Def: A (hermitian) inner product on v.s. $V/\underbrace{\mathbb{R} \text{ or } \mathbb{C}}_{\mathbb{F}}$ is a function $V \times V \to \mathbb{F}$

$(v,w) \mapsto \langle v, w\rangle$

That is,

1. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$
2. $\langle x,y\rangle = \overline{\langle y,x\rangle}$
3. $\langle x,x\rangle \in \mathbb{R}_{\geq 0} \ \forall x \in V$
4. $\langle x,x\rangle = 0 \iff x = 0$

$V = \mathbb{C}^n \quad \langle z, w\rangle = z_1\bar{w}_1 + \dots + z_n\bar{w}_n$

$\qquad = \bar{w}^T z = w^* z$

<u>Def</u>: for $A \in \mathbb{F}^{m\times n}$, its <u>adjoint</u> is $A^* = \bar{A}^T$ "conjugate transpose"

Note: $\mathbb{F} = \mathbb{R}$, $A^* = A^T$

<u>Check</u>: $W \xleftarrow{A} V \Rightarrow \langle w, Av\rangle = (Av)^* w$

$\qquad = v^* A^* w$

$\qquad = \langle A^* w, v\rangle$

<u>Prop</u>: fix V with $\langle \cdot, \cdot \rangle$. If $\dim V < \infty$, then

$v \mapsto \langle \cdot, v\rangle$

induces a conjugate-linear isomorphism

$V \xrightarrow{\sim} V^*$ ← not a hom of v.s./$\mathbb{C}$

$\alpha v \mapsto \langle \cdot, \alpha v\rangle = \bar{\alpha}\langle \cdot, v\rangle$

<u>pf</u>: any orthonormal basis $e_1, \dots, e_n$ maps to dual basis $e_1^*, \dots, e_n^*$ of V^* such that

$e_j^*(e_i) = \delta_{ij}$

$e_1^*, \dots, e_n^*$ span V^* since $\dim V^* = n$ $\blacksquare$

01/21 <u>Jordan form</u>

fix v.s. $V/\mathbb{F}$, $\dim V = n < \infty$

$\varphi: V \to V$ linear with matrix

$A \in \mathbb{F}^{n\times n}$ with respect to $\mathcal{E} = e_1, \dots, e_n$

basis independent

- <u>Characteristic polynomial</u>: $p_\varphi(t) = \det(\varphi - tI) = \det(A - tI)$

λ eigenvalue of $\varphi \Leftrightarrow \lambda$ root of p_φ

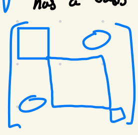
- <u>geometric multiplicity</u> $g(\lambda) = \dim\ker(\varphi - \lambda I)$
- <u>algebraic multiplicity</u> $a(\lambda) = \#$ of times $(t-\lambda)$ divides $p_\varphi(t)$

$g(\lambda) \leq a(\lambda)$

equal $\forall \lambda \Leftrightarrow$ diagonalizable

matrix of φ wrt $\mathcal{B}$

<u>Thm (Jordan form)</u> Assume all $\lambda \in \mathbb{F}$. V has a basis $\mathcal{B}$ s.t. $[\varphi]_\mathcal{B}$ is block-diagonal



with each

block being a Jordan block: $\begin{bmatrix}\lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda\end{bmatrix}$ of same size d for eigenvalue λ

The multiset of blocks (the d's and λ's) depends only on φ, not on $\mathcal{B}$

there are lots of bases in which φ has this form, idea is form is unique up to permutation of blocks

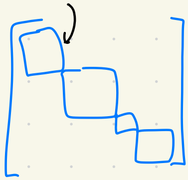
<u>def</u> F is algebraically closed if every polynomial w/ coeffs in F has a root in F

implies every

$$\Rightarrow \quad p_\varphi(t) = \alpha(t-\alpha_1)^{n_1}\cdots(t-\alpha_r)^{n_r}$$

<u>fundamental theorem of algebra:</u> $\mathbb{C}$ is algebraically closed

linear transformation from V_1 to itself



<u>Prop:</u> $[\varphi]_{\mathcal{B}}$ block-diagonal $\iff$ $V = V_1 \oplus \cdots \oplus V_r$ with $\varphi(V_i) \subseteq V_i \ \forall i$

<u>Def:</u> V_i is φ-<u>invariant</u>

<u>Cayley-Hamilton Thm</u> $\quad p_\varphi(\varphi) = 0$

<u>E.g.</u> $A^2 = A \qquad p_A = t^{\sim}(1-t)^{\sim}$

$m(t) = t^2 - t$

$m(A) = A^2 - A = 0$

poly of smallest possible deg s.t. $m(A) = 0$

<u>Pf:</u> fix basis $e_1, \dots, e_n$ of V

so $\varphi e_i = a_{i1}e_1 + \cdots + a_{in}e_n = A_i \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$

$$\Rightarrow \quad A\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \varphi e_1 \\ \vdots \\ \varphi e_n \end{bmatrix}$$

$$\iff \quad (A - \varphi I)\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = 0$$

$$\begin{bmatrix} a_{11} - \varphi & a_{12} & \cdots & a_{1n} \\ \vdots & a_{22} & & \vdots \\ & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

$A - tI$ has cofactor matrix C s.t. $C^T(A - tI) = \begin{bmatrix} \det(A-tI) & & 0 \\ & \ddots & \\ 0 & & \det(A-tI) \end{bmatrix} = \det(A-tI)\, I$

$$c^T\Big|_{t=\varphi}(A-\varphi I) = p_\varphi(\varphi)\, I$$

Want: $\begin{bmatrix} p_\varphi(\varphi) & & \\ & \ddots & \\ & & p_\varphi(\varphi)\end{bmatrix}\begin{bmatrix} e_1 \\ \vdots \\ e_n\end{bmatrix} = 0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ $\in V$

$= c^T(A-\varphi I)I$ $\blacksquare$

Def: A <u>minimal polynomial</u> of φ (or A) is a monic poly $m(t)$ of minimal degree with $m(\varphi)=0$ (or $m(A)=0$)

Prop 1. $\exists!\ m(t)$

2. $f(\varphi)=0 \Rightarrow m \mid f$, $f \in F[t]$

pf: (1 ⇐ 2) $m \mid f$ and $\deg m = \deg f$

$\Rightarrow f = \alpha m \overset{\text{monic}}{\Rightarrow} f = m$

2. Assume $f(\varphi)=0$. Write $f = qm + r$ with $\deg r < \deg m$ (by euclidean algorithm)

Then $0 = f(\varphi) - q(\varphi)m(\varphi) = r(\varphi) \Rightarrow r = 0$ (minimal degree)

$\Rightarrow m \mid f$

<u>Jordan block</u>: $A \in F^{d\times d}$ or $\varphi: V \to V$ with $\dim V = d$ whose minimal poly is $(t-\lambda)^d$

$$A = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} = p_A(t) = (t-\lambda)^d \underset{\text{not obvious}}{=} m_A(t)$$

$d = 1 \Rightarrow A = [\lambda]$ $(\varphi - \lambda)v = 0$

$\Leftrightarrow v \in E(\lambda)$ ← eigenspa

$\Rightarrow \mathcal{B} = \{v\}$ has $[\varphi]_{\mathcal{B}} = [\lambda]$

$d = 2 \Rightarrow (\varphi - \lambda)V \neq 0$ (min poly has deg 2)

$\quad\quad\quad\;\; \neq V$

$\Rightarrow \dim(\varphi-\lambda)V = 1$

$(\varphi-\lambda)V \neq V$ since $(\varphi-\lambda)^d V = 0$

$(\varphi-\lambda)V \geq d-1$ or else $(\varphi-\lambda)^{d-1}V = 0$

		dim
$V_0 :=$	V	d
$V_1 :=$	$(\varphi-\lambda)V$	$d-1$
$V_2 :=$	$(\varphi-\lambda)^2 V$	$d-2$
$\vdots$		
$V_k =$	$(\varphi-\lambda)^k V$	$d-k$
$\vdots$		
$V_d :=$	$(\varphi-\lambda)^d V$	0

Choose $\underset{v_d}{\overset{\shortparallel}{v}} \in V \setminus V_{d-1}$

$(\varphi-\lambda)v_d = v_{d-1} \in V_{d-1} \setminus V_{d-2}$

$\vdots$

$(\varphi-\lambda)^{d-k} = v_k \in V_k \setminus V_{k-1}$

$$(\varphi-\lambda)\left((\varphi-\lambda)^{d-1} v_1 = v_1\right) = 0$$

eigenvector w/ eigenvalue λ

$$(\varphi-\lambda)v_k = v_{k-1} \iff \varphi v_k = \lambda v_k + v_{k-1}$$

<u>Banach Spaces</u>: complete, normed v.s. over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

complete: all Cauchy sequences converge.

e.g. $\mathbb{Q}$ not complete, $\mathbb{R}$ is <u>completion</u> of $\mathbb{Q}$

not complete: $\mathbb{R}\setminus\{0\}$

<u>Def</u>: A <u>norm</u> on $V/\mathbb{F}$ is

$\nu : V \to \mathbb{R}_+$ ← <u>nonnegative</u>

s.t.

positive-definite → • $\nu(x) = 0 \iff x = 0$

homogeneous → • $\nu(\alpha x) = |\alpha|\nu(x)$

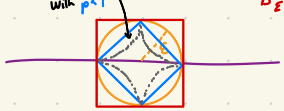
sub-additivity → • $\nu(x+y) \leq \nu(x) + \nu(y)$ (triangle inequality)

$B_\varepsilon^\nu(x) = \{y \in V \mid \nu(x-y) < \varepsilon\}$

default: $B_\varepsilon = B_\varepsilon^{\|\cdot\|_2}$

$B_\varepsilon^{\|\cdot\|_1}(0) \subset B_\varepsilon(0) \subset B_\varepsilon^{\|\cdot\|_\infty}(0)$

with p-norm $p<1$



$V = \mathbb{F}^n$

Eg: $\|x\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{1/2}$ euclidean

$\|x\|_1 = \left(|x_1|^1 + \cdots + |x_n|^1\right)^{1/1}$ taxicab

$= |x_1| + \cdots + |x_n|$

$\|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}$ p-norm $p \geq 1$

$\|x\|_\infty = \max\limits_{i \in \{1,n\}} |x_i|$

<u>Def</u>: A metric space is a set X with a <u>distance</u> $d : X \times X \to \mathbb{R}_+$ s.t.

$\forall x, y$

separates $\cdot$ $d(x,y) = 0 \iff x = y$

symmetric $\cdot$ $d(x,y) = d(y,x)$

triangle inequality $\cdot$ $d(x,y) \le d(x,z) + d(z,y) \quad \forall z$

Ex: norm ν induces $d_\nu(x,y) = \nu(x-y)$

Def: A topology on a set S is a collection $\mathcal{U}$ of subsets of S called open sets s.t.

- Any union of open sets is open
- Any finite intersection of open sets is open
- $\emptyset$ and S are open.

Examples:
- Zariski topology: $D(f) = \{\mathfrak{p} \subseteq R \mid f \notin \mathfrak{p}\}$
- Euclidean Topology: U open if $x \in U$ has $B_\varepsilon(x) \subseteq U$ for some $\varepsilon > 0$
  $\iff B_\varepsilon(x)$ is open $\forall x \in S \quad \varepsilon > 0$
- Discrete Topology: $\mathcal{U} = \mathcal{P}(S)$
- Trivial Topology: $\mathcal{U} = \{\emptyset, S\}$
- metric on $X \rightsquigarrow$ topology on X with
  U open $\iff$ $x \in U$ has $B_\varepsilon(x) \subseteq U$ for some $\varepsilon > 0$

Def: $\mathcal{B} \subseteq \mathcal{U}$ is a base for the topology $\mathcal{U}$ if

$U \in \mathcal{U} \Rightarrow U = \bigcup_{B \in \mathcal{B}'} B$ for some $\mathcal{B}' \subseteq \mathcal{B}$

"any open set is a union of sets in the base"

E.g. $\{B_\varepsilon^\nu(x) \mid x \in V \text{ and } \varepsilon \in \mathbb{R}_+\}$

Def $\{x_k\}_{k \in K} \longrightarrow x$ (partial order on K) if $\{x_k\}$ is eventually in U $\forall$ open $U \ni x$,

meaning $\exists N_U \in K$ with $x_k \in U \ \forall \ k \ge N_U$

$X \subseteq S$ is closed if $x \in X$ whenever
$\{x_k\} \to x$ in S with $\{x_k\} \subseteq X$

"X is closed if X contains its limit points"

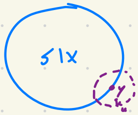
Prop: $X \subseteq S$ is closed $\iff$ $S \setminus X$ is open

Pf: sps $S \setminus X$ open and $\{x_k\} \subseteq X$

$\Rightarrow \lim x_k$ (if $\exists$) can't lie in $S \setminus X$ so it must lie in X

Sps $S\setminus X$ not open $\Rightarrow \exists\, y \in S\setminus X$ s.t. for all open $U \ni y$ $\exists\, x_U \in U \cap X$

Thus $\{x_U\}_{U \ni y} \to y \notin X$. $\boxtimes$



Prop: Any norm ν on $\mathbb{F}^n$ is continuous in $d_{\|\cdot\|_2}$

01/28

pf: Given $\varepsilon > 0$, need δ s.t.

$$|\nu(x) - \nu(y)| < \varepsilon \text{ whenever } \|x-y\|_2 \le \delta$$

Subadditivity $\Rightarrow$ $\nu(x) \le \nu(x-y) + \nu(y)$ and

$\nu(y) \le \nu(y-x) + \nu(x)$

$\Rightarrow$ $\nu(x) - \nu(y) \le \nu(x-y)$

$\nu(y) - \nu(x) \le \nu(x-y) = |-1|\,\nu(x-y) = \nu(y-x)$

So $|\nu(x) - \nu(y)| \le \nu(x-y)$

$$= \nu\left(\sum_{i=1}^n (x_i - y_i) e_i\right)$$

$$\le \sum_{i=1}^n \nu((x_i - y_i)e_i) \quad \leftarrow \text{subadditivity}$$

$$= \sum_{i=1}^n |x_i - y_i|\, \nu(e_i) \quad \leftarrow \text{homogeneity}$$

$$\le \|x-y\|_2 \|v\|_2 \quad \text{Cauchy-Schwarz}$$

where $v = (\nu(e_1), \dots, \nu(e_n))$

So take $\delta = \dfrac{\varepsilon}{\|v\|_2}$ $\boxtimes$

Def: Norms ν and μ on $V = \mathbb{F}^n$ are (topologically) equivalent, written $\nu \sim \mu$, if $\exists\, \alpha, \beta \in \mathbb{R}_{>0}$ with

$\alpha \nu(x) \le \mu(x) \le \beta \nu(x) \quad \forall x \in V$

why is this true?

Interpretation: $\nu \sim \mu \Leftrightarrow B^{\nu}_{\varepsilon/\beta}(x) \subseteq B^{\mu}_{\varepsilon}(x) \subseteq B^{\nu}_{\varepsilon/\alpha}(x)$

$\Leftrightarrow$ every ε-ball base for the μ topology is a base for the ν topology

$$\alpha\, \nu(x-y) \le \mu(x-y) < \varepsilon \Rightarrow \nu(x-y) < \varepsilon/\alpha$$

$$y \in B^{\mu}_{\varepsilon}(x) \subseteq y \in B^{\nu}_{\varepsilon/\alpha}(x)$$

Eg. $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$

$\frac{1}{\sqrt{n}} \|x\|_2 \le \|x\|_\infty \le \|x\|_2$

$\|x\|_\infty \le \|x\|_1 \le n \|x\|_\infty$

Lemma: $\sim$ is an equivalence relation

pf: symmetric: $\frac{1}{\beta}\mu(x) \le \nu(x) \le \frac{1}{\alpha}\nu(x) \iff \alpha\nu(x) \le \mu(x) \le \beta\nu(x)$

transitive: exercise

reflexive: $\alpha = \beta = 1$ ☺

Thm: If μ and ν are norms on $V = \mathbb{F}^n$, then $\mu \sim \nu$

pf: By lemma, only need to check $\nu = \|\cdot\|_2$

Can assume $x \neq 0$.

Recall $|\nu(x) - \nu(y)| \le \|x-y\|_2 \|v\|_2$, where $v = (\nu(e_1), \dots, \nu(e_n))$

If $y = 0$, μ instead of ν,

no abs b/c pos def $\rightarrow$ $\mu(x) \le \|x\|_2 \|v\|_2 \quad v = (\mu(e_1), \dots, \mu(e_n))$

$\Rightarrow$ take $\beta = \|v\|_2$

$\Rightarrow x \neq 0 \Rightarrow \mu(x) > 0$

Set $\alpha = \min\{\mu(x) \mid \|x\|_2 = 1\}$

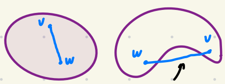
exists by Prop (μ continuous), sphere in V closed and bounded (compact).

Then $\mu(x) = \mu\left(\|x\|_2 \frac{x}{\|x\|_2}\right)$

$\ge \alpha\|x\|_2$ $\boxed{\checkmark}$

Convexity fix $\mathbb{R}$-vector space V.

def $X \subseteq V$ is convex if $\forall v, w \in X \Rightarrow$ line segment $\overline{vw} \subseteq X$.



convex — not convex

$\overline{vw} = \{\alpha v + (1-\alpha)w \mid \alpha \in [0,1]\}$

$= \{\alpha v + \beta w \mid \alpha + \beta = 1, \alpha, \beta \ge 0\}$

$= \{w + \alpha(v-w) \mid \alpha \in [0,1]\}$

E.g. any interval $\subseteq \mathbb{R}$

$(a,b]$ (a,b) $[a,b]$

$[a,\infty)$ $(\infty, b]$

Pf: analysis $a \le v$ $w \le b$

Properties

1) $\varphi: V \to W$ linear $\Rightarrow \varphi^{-1}(\text{convex}) = \text{convex}$

2) **pf:** $\varphi^{-1}(\overline{vw}) = \overline{\varphi(v)\varphi(w)}$ by linearity

3) X convex $\Rightarrow$ $a+X$ is convex $\forall a \in V$.

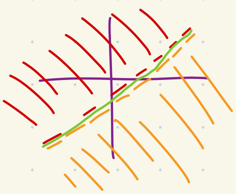
4) $\{X_i\}_{i\in I}$ all convex $\Rightarrow$ $\bigcap_{i\in I} X_i$ convex

Corollary (5): Every affine subspace is convex

pf: 1 and 3

subspace is image of linear map

6. Every ~~open~~ closed half-space $\bar{H}^{\pm} = \{x \in V \mid \ell(x) \gtrless c\}$ for some $\ell \in V^*$ and $c \in \mathbb{R}$ is convex



Pf: 2 + (ray is convex)

E.g. Standard $(n-1)$-simplex

$$\sigma = \{x \in \mathbb{R}^n \mid x_i \geq 0 \ \forall i \text{ and } x_1 + \cdots + x_n = 1\}$$

is convex

Pf: $\sigma = \left(\overset{4}{\bigcap_{i=1}^{n}} \overset{6}{\{x_i \geq 0\}}\right) \overset{4}{\cap} \overset{5}{\left(\underset{x_1+\cdots+x_n=1}{\text{affine hyperplane}}\right)}$

half spaces

Prop: $x_1, \ldots, x_n \in V \Rightarrow$

7. $P = \{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid \alpha_i \geq 0, \ \alpha_1 + \cdots + \alpha_n = 1\}$ is convex

8. $X \supseteq \{x_1, \ldots, x_n\}$ convex $\Rightarrow$ $X \supseteq P$

Def: P = convex hull $\operatorname{conv}\{x_1, \ldots, x_n\}$ is a polytope.

Pf 7. $P = \varphi(\sigma)$ for $\varphi: \mathbb{R}^n \to V$, $e_i \mapsto x_i$

8. We will use induction on n

Trivial if $n=1$

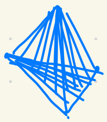
$n \geq 2$: need $\alpha_1 x_1 + \dots + \alpha_n x_n \in X$. Assume $\alpha_n \neq 1$

Then $\alpha_1 x_1 + \dots + \alpha_n x_n = \alpha\left(\boxed{\beta_1 x_1 + \dots + \beta_{n-1} x_{n-1}}\right) + (1-\alpha) x_n$

$\alpha = 1 - \alpha_n = \alpha_1 + \dots + \alpha_{n-1}$ $\qquad\qquad \in X$ by induction

$\beta_i := \frac{\alpha_i}{\alpha}$

☒



1/30

More examples

- $X = \{ f \in \mathbb{R}[t] \mid f(\alpha) \geq 0 \ \forall \alpha \in (0,1) \}$

- $X = \{ A \in \mathbb{R}^{n \times n} \mid A = A^* \text{ and } \lambda > 0 \text{ for all eigenvalues } \lambda \}$

  positive definite   self-adjoint

Def: $v \in X \subseteq V$ is an <u>interior point</u> if $v \in U \subseteq X$ for some open $U \subseteq V$

Def: $v \in \partial X$ (boundary) is a boundary pt. if $U \cap X$ and $U \cap (V \setminus X)$ both not empty for all open $U \ni v$

Prop: $\overline{X} = X \cup \partial X$ is the <u>closure</u> of X

Prop: $\overline{X} = \{ \lim x_k \mid \{x_k\} \subseteq X \text{ converges} \}$

Q: Is  open in V?

A: yes, if $V =$ 

no, if $V = \mathbb{R}^2$

Prop: $X \subseteq V$ convex $\Rightarrow \overline{X}$ convex

pf: $\{v_k\} \to v$, $\{w_k\} \to w$ $\Rightarrow \alpha v_k + \beta w_k \to \alpha v + \beta w$

algebraic limit thm

But $v_k \in X$ and $w_k \in X$ $\forall k \Rightarrow$ ($\alpha+\beta=1$, $\alpha, \beta \geq 0$) $\alpha v_k + \beta w_k \in X$ $\forall k$ by convexity

$\Rightarrow \alpha v + \beta w \in \overline{X}$ by prop ☒

Def: A hyperplane $H \subseteq V$ **separates** v from X if $v \in \bar{H}^{\mp}$ and $X \subseteq \mathring{X}^{\pm}$
for some orientation of H

A linear function $\ell \in V^*$ **separates** v from X iff $\ell(v) < \ell(x)$ $\forall x \in X$

Thm If $X = \overline{X}$ and convex and $v \notin X$, then
$\exists H$ separating v from X → could also say X convex, $v \notin \overline{X}$
not necessarily closed

Pf: For any choice of inner product, the norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$
yields a function $f(x) = \|x - v\| : X \to \mathbb{R}_+$
f is bounded below by 0, continuous, and proper (inverse of compact is compact)
by topological equivalence: $f^{-1}(\text{bounded set})$ is bounded
$f^{-1}([0,r]) = B_r^{\|\cdot\|}(v)$ is bounded
bounded ⇒ contained in here; top. equiv ⇒ contained in ball for any norm

Since X is closed, f attains a minimum on X
say at $y \in X$.

Set $\ell = \langle y - v, \cdot \rangle$.
Note: $\ell \neq 0$ since $y \in X$ and $v \notin X$.
$\forall \alpha \in (0,1]$ and $x \in X$,
$$\cancel{\|y - v\|^2} \leq \|y + \alpha(x-y) - v\|^2 = \|y - v + \alpha(x-y)\|^2$$
y is min.
$$= \cancel{\|y - v\|^2} + 2\alpha\langle y - v, x - y\rangle + \alpha^2 \|(x-y)\|^2$$

⇒ $0 \leq 2\ell(x-y) + \alpha\|x-y\|^2$
⇒ $0 \leq 2\ell(x-y)$ $(\alpha \to 0)$
⇒ $\ell(x) \leq \ell(y)$ linearity of ℓ
But $\ell(v) < \ell(y)$ because $\ell(y-v) = \|y-v\|^2 > 0$
So $\ell(v) < \ell(x)$ $\forall x \in X$. $\boxed{\Sigma}$

Def: H is a support hyperplane of X
at $v \in \overline{X}$ if $v \in H$ and $X \subseteq \bar{H}^+$ or $X \subseteq \bar{H}^-$

Thm: X convex and $v \in \partial X$ ⇒ $\exists$ support H of X at v
pf: $X \subseteq \bar{H}^+ \Rightarrow \overline{X} \subseteq \bar{H}^+$ so assume $X = \overline{X}$.
fix a norm $\|\cdot\|$ on V^*. for $k \in \mathbb{N}$ pick

$v_k \notin X$ and $\ell_k \in S^{n-1} = \{w \in V^* \mid \|w\| = 1\} \subseteq V^*$

separating v_k from X.

Assume $v_k \to v$ as $k \to \infty$. S^{n-1} is compact (closed and bounded) so $\{\ell_k\}$ has a convergent subsequence, replace $\{\ell_k\}$ with that to get $\{\ell_k\} \to \ell$. $\forall x \in X$, $\{\underbrace{\ell_k(x) - \ell_k(v_k)}_{>0}\} \to \underbrace{\ell(x) - \ell(v)}_{\geq 0}$ $\Rightarrow$

so $H = \{y \in V \mid \ell(y) = \ell(v)\}$ suffices. $\boxtimes$

Def: $S \subseteq V$ has <u>convex hull</u> $\text{conv}(S) = \{\text{convex combinations of pts of } S\}$

Prop: $\text{Conv}(S)$ is the smallest convex set $\supseteq S$.

Pf: HW8 + convex combination only involves finitely many pts of S.

Def: for X convex, $v \in X$ is an <u>extreme point</u> if $v \notin \text{conv}(X \setminus \{v\})$

lemma: extreme $\Leftrightarrow$ $v \notin \overline{xy}$ where $v \neq x \in X$, $v \neq y \in X$

Thm: If X is closed, bounded, and convex $\Rightarrow$ each supporting H contains an extreme point of X

Thm (Krein-Milman) $X = \overline{X}$, bounded, convex
$\Rightarrow X = \text{conv}(\text{extreme pts of } X)$

Pf (hyperplane contains extreme pt, assuming $\dim V < \infty$)

fix supporting H. Set $Y = X \cap H$ $\Rightarrow$ closed, bounded, convex

Claim: $v \in Y$ extreme in $Y \Rightarrow v$ extreme in X.

$\Rightarrow$ thm by induction on $\dim V$ via previous thm.

pf (of claim): pick $\ell \perp H$ with $\ell(X) \geq \ell(v)$.

Sps $v = \alpha x + \beta y$, $x, y \in X$, $\alpha + \beta = 1$, $\alpha \neq 0 \neq \beta$ $\to$ v not extreme

$\Rightarrow \ell(v) = \alpha \ell(x) + \beta \ell(y)$

$\Rightarrow \ell(v) = \ell(x)$ or $\alpha = 0$ and $\ell(v) = \ell(y)$

$\Rightarrow \ell(v) = \ell(x) = \ell(y)$

$\Rightarrow x, y \in H$ $\lightning$ $\perp$ $\circledast$

Pf (Krein-Milman, $\dim V < \infty$): Sps $X \supseteq Y$ with $Y = \overline{Y}$ convex.

$X \not\subseteq Y \Rightarrow \exists\, v \in X \setminus Y$ with ℓ separating v from Y.

ℓ attains minimum on X at x. compact $\rightarrow$ $\operatorname{argmin}\{\ell(z) \mid z \in X\} = x$

Note $\ell(x) < \ell(Y)$ by separation.

$H = \{w \in V \mid \ell(w) = \ell(x)\}$ supports X at x by construction.

(thm) $\Rightarrow$ $\exists$ extreme pt of X in H (could be distinct from x)

$\Rightarrow$ Y can't contain all extreme pts of X.

$\Rightarrow$ $X \subseteq \text{Conv}(\text{extreme pts of } X)$

$\supseteq$ by prop. $\square$

Grassmannians

Def: A k-plane in v.s./F W is a subspace of $\dim = k$

Def: The Grassmannian is $G_k(W) = \{k\text{-planes in } W\}$

Q: $V \subseteq F^n$ specified by... a basis $v_1, \dots, v_k$

$F^n = F^n_{row}$

$$k\left[\begin{array}{c} n \\ \hline \text{---} \\ \text{---} \\ \text{---} \end{array}\right]$$

$\Rightarrow$ $F^{k\times n} \overset{?}{\longleftrightarrow} G_k(F^n)$ no (consider $[\,0\,]$). Need rank $= k$ $F_*^{k\times n}$

$F_*^{k\times n} \overset{?}{\not\longleftrightarrow} G_k(F^n)$, $\twoheadrightarrow$

$A \mapsto \text{rowspan}(A)$

E.g. $A = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \longmapsto V$

$A' = \begin{bmatrix} 2 & 0 & 2 & 6 \\ 1 & 1 & 0 & 4 \end{bmatrix} \longmapsto V$

$A' = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} A$

$A = \text{RREF}(A')$

Prop $A, A' \in F_*^{k\times n}$ yield same V

$\Leftrightarrow$ rows of A are lin. combinations of rows of A'

$\Leftrightarrow$ $A' = gA$ for some $g \in F^{k\times k}$

$\Leftrightarrow$ " " " $g \in F_*^{k\times k} \overset{\text{def}}{=} GL_k(F)$

Cor: $G_k(F^n) = GL_k(F) \backslash F_*^{k\times n}$, the quotient of $F_*^{k\times n}$ modulo the action of GL_k on the left

$GL_k(F) \times F_*^{k\times n} \to F_*^{k\times n}$

$\boxed{g}\ \boxed{A} \mapsto \boxed{g}\boxed{A} = A'$

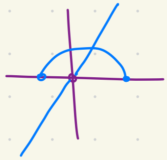
$GL_k(F)\backslash F_*^{k\times n} = \left\{ [A] \subseteq F_*^{k\times n} \;\middle|\; A \in F_*^{k\times n} \right\}$

where $A' \in \underbrace{[A]}_{GL_k(F)A} \iff A' = gA$ for $g \in GL_k(F)$

$GL_k(F)A$, the orbit of A w.r.t. $\circlearrowleft$

E.g. $n=2,\ k=1$.

$[\alpha \;\; \beta] \mapsto \big[[\alpha \;\; \beta]\big]$



$\mathbb{R}^2 \setminus \{0\} \to G_1(\mathbb{R}^2) = \mathbb{RP}^1 = \mathbb{P}^1(\mathbb{R})$

Unit circle $/ \pm 1$ $\quad \cong S^1$

Compare: $V/W = \left\{ [v] \subseteq V \;\middle|\; v \in V \right\}$

$v' \in [v] \iff v' = v + w$ for some $w \in W$.

Sps left k columns of $A \in F^{k\times n}$ are indep

$$k\left[\begin{array}{c|c} g & \end{array}\right]$$
$$\;\; k \qquad n-k$$

Let $g = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \cdots & A_k \\ | & | & & | \end{bmatrix} \in GL_k$

so $\hat{A} = g^{-1}A = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \text{stuff} \\ & & 1 & \end{array}\right] = REF(A)$

$\in F^{k\times n}_{[k]}$, $[k] = \{1,\dots,k\}$

Then $A \mapsto V \Rightarrow \hat{A} \mapsto V$

Lemma: $A' \mapsto V \Rightarrow \hat{A}' = \hat{A}$.

$\Uparrow$

Lemma': $\left\{ A \in F^{k\times n} \;\middle|\; \text{left } k \text{ columns} = I_k \right\} \hookrightarrow G_k(F^n)$

Surjective? No! Consider $\begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 3 & 4 & 5 \end{bmatrix}$

$\operatorname{rank} A = k \Rightarrow$ some set σ of k cols is indep.

Write $F_\sigma^{k\times n} = \left\{ \hat{A} \in F^{k\times n} \;\middle|\; \hat{A} \text{ has } I_k \text{ in cols from } \sigma \right\}$

Lemma'': $F_\sigma^{k\times n} \xhookrightarrow{\pi_\sigma} G_k(F^n) \quad \forall\ \sigma \in \binom{[n]}{k}$

set of k-element subsets $\sigma \subseteq \{1,\dots,n\}$

Let $G_k^\sigma = \operatorname{im}(\pi_\sigma)$

Prop: $G_k(F^n) = \bigcup_{\sigma \in \binom{[n]}{k}} G_k^\sigma$

$\Rightarrow G_k(F^n)$ covered by $\binom{n}{k}$ affine subspaces $F_\sigma^{k\times n} \xhookrightarrow{\pi_\sigma} G_k(F^n)$

each w/ $\dim = k(n-k)$

Def: A *manifold* is a topological space X with an *atlas*: set of maps ("charts") $U_\alpha \xhookrightarrow{\pi_\alpha} X$ s.t.

- $X = \bigcup_\alpha X_\alpha$, where $X_\alpha = \mathrm{im}(\pi_\alpha)$ is open in X.
- $\forall \alpha$, U_α open in an affine space of $\dim = d$. ← dimension of manifold
- $\forall \alpha$, π_α is a homeomorphism $U_\alpha \xrightarrow{\sim} X_\alpha$
  - ↪ $W \subseteq U_\alpha$ open $\Leftrightarrow$ $\pi_\alpha(W) \subseteq X_\alpha$ open in X

02/06 and $\forall \alpha, \beta$: transition function

- $\pi_\beta^{-1} \circ \pi_\alpha : U_{\alpha\beta} \xrightarrow{\sim} U_{\beta\alpha}$ is $\mathbb{R}$ - continuous → topological manifold
  - ⇑
  - differentiable ⇒ differentiable manifold
  - ⇑
  - smooth (C^∞) ⇒ smooth manifold
  - ⇑
  - $\mathbb{R}$ or $\mathbb{C}$ - analytic ⇒ analytic manifold
  - ⇑
  - works over any field - ratio of polynomials ⇒ smooth rational algebraic variety

Thm $G_k(\mathbb{F}^n)$ is a smooth rational algebraic variety of dim $k(n-k)$ with atlas

$$\left\{ \pi_\sigma : \mathbb{F}_\sigma^{k\times n} \to G_k(\mathbb{F}^n) \;\middle|\; \sigma \in \binom{[n]}{k} \right\}$$

row span; $\mathrm{im}\,\pi_\sigma = G_k^\sigma$

Pf: prop ⇒ $X = \bigcup_\alpha X_\alpha$

$G_k(\mathbb{F}^n) = \bigcup G_k^\sigma$

- $G_k^\sigma \simeq \mathbb{F}_\sigma^{k\times n} \simeq \mathbb{F}^d$ for $d = k(n-k)$
- Declare: $U \subseteq G_k(\mathbb{F}^n)$ open $\Leftrightarrow$ $U \cap G_k^\sigma$ open $\forall \sigma$
- Set $\mathbb{F}_{\sigma,\tau}^{k\times n} = \{ A \in \mathbb{F}_\sigma^{k\times n} \mid A_\tau \text{ is invertible} \}$

$A \in \mathbb{F}^{k\times n} \Rightarrow A_\tau = [\text{cols of } A \text{ indexed by } \tau]$

Then $\pi_\tau^{-1} \circ \pi_\sigma : \mathbb{F}_{\sigma,\tau}^{k\times n} \to \mathbb{F}_{\tau,\sigma}^{k\times n}$

$A \mapsto A_\tau^{-1} A$

↳ entries look like $\dfrac{\text{cofactor of } A_\tau}{\det A_\tau}$ ← poly in entries of A_τ $\boxtimes$

Def: A (complete) *flag* in V is a chain

$0 = V_0 < V_1 < \cdots < V_{n-1} < V_n = V$ ← strict containment

of subspaces of V with $\dim V_i = i$

Set $\mathcal{F}\ell_n(\mathbb{F}) = \{\text{complete flags in } \mathbb{F}^n\}$

Ex: express $\mathcal{F}\ell_n$ as a quotient of

$V_1 = \langle v_1 \rangle$

$V_2 = \langle v_1, v_2 \rangle$

$\vdots$

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in GL_n$$

Quotient by what?

$$\underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{\in GL_n} \overset{U}{\begin{bmatrix} * & * & & \\ 0 & * & \ddots & * \\ \vdots & 0 & \ddots & \\ 0 & 0 & 0 & * \end{bmatrix}} \in B_n^+$$

Borel Subgroup

Prop: $\mathcal{F}\ell_n = GL_n / B_n^+$

Q: manifold? variety?

A: • "big" subset that is open subset of an affine space

• enough copies to cover

$$U_n^- = \overset{L}{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}}$$

unipotent subgroup

Prop: $U_n^- \hookrightarrow \mathcal{F}\ell_n = GL_n / B_n^+$

LU - decomposition: A generic $\Rightarrow A = LU$

no column swaps needed in column reduction

$$\left\{ wU_n^- \hookrightarrow GL_n / B_n^+ \;\middle|\; w \text{ is a permutation matrix} \right\}$$

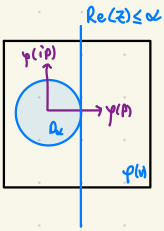
atlas! (prove in HW)

02/11

Aside: hw. HB $\Rightarrow$ Separation

let $p \notin B$, set $V = \mathrm{span}_{\mathbb{C}}(p)$ and $\varphi = \langle \cdot, p \rangle$. Then $\varphi(B \cap V) = D_\alpha$. Extend φ to $\tilde{\varphi} : \mathbb{C}^n \twoheadrightarrow \mathbb{C}$ by HB so that $\tilde{\varphi}(B) = D_\alpha$.

disk of radius α in $\mathbb{C}$.



$\tilde{\varphi}^{-1}(\mathrm{Re}(z) = \alpha + \varepsilon)$ is H.

Orthogonal groups

Q: What is $\{\text{symmetries of } S^2 \subseteq \mathbb{R}^3\}$

maps $S^2 \to S^2$ as metric space!

A: $SO_3 = \{A \in \mathbb{R}^{3\times 3} \mid \underbrace{AA^T = I, \det}_{SO_3} A = 1\}$

Def: An isometry is a bijection $X \to X$ with
$d(\varphi x, \varphi y) = d(x,y)$ for all $x,y \in X$.

don't need to say this; surjection is sufficient
$x \neq y \Rightarrow d(x,y) \neq 0$
$\Rightarrow d(\varphi x, \varphi y) \neq 0$
$\Rightarrow \varphi x \neq \varphi y$

E.g. find (X,d) and φ satisfying all but surjection.

A: $\mathbb{R}_{\geq 0}$, $\varphi: x \mapsto x+1$

Q: Is every isometry of S^2 rotation about some axis?

metric on S^2: $d(x,y) = \angle(x,y)$

No! But if assume preserves orientation then

- φ isometry of $S^2 \Rightarrow$ preserves $\perp$ normality of any basis
- (v_1, v_2, v_3) right-handed if $v_1 \times v_2 = v_3$
  "positively oriented"

in $\mathbb{R}^n$, $\det \begin{bmatrix} | & & | \\ v_1, & \dots, & v_n \\ | & & | \end{bmatrix} > 0$

Def: φ preserves orientation if φ preserves handedness

E.g.
- $\varphi = -I_3$ no! $\det(-I_3) = -1$
- φ = reflection $\Rightarrow$ no!
  " through a line $\Rightarrow$ yes!
- rotation $\Rightarrow$ yes!

Lemma: $\varphi: S^{n-1} \to S^{n-1}$ isometry
$\Rightarrow \{\varphi_\alpha \mid \alpha \in \mathbb{R}_+\}: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of $\mathbb{R}^n = \bigcup_\alpha \alpha S^{n-1}$

Pf: $x,y \in \mathbb{R}^n \Rightarrow \varphi$ preserves
- $\|x\|$ and $\|y\|$
- $\angle(x,y) \overset{\text{def}}{=} \angle\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right)$ by isom of S^{n-1}
- $\|x-y\|$ by law of cosines ☺

Thm: Every isom of $\mathbb{R}^n$ has form

$A \circ T_v$ for some
- $A \in O_n\mathbb{R}$
- T_v = translation by $\mathbb{R}^n$

$\mathbb{R}^n \rtimes O_n\mathbb{R}$

Pf: Assume $\varphi \in isom(\mathbb{R}^n)$ with $\varphi(0)=v$

replace φ with $\varphi - T_v$ to assume $\varphi(0)=0$

need $\varphi \in O_n\mathbb{R}$. Proof is as in lemma +

(preserves inner products $\Rightarrow$ linear)

$\downarrow$

$(Ax)_i = \langle Ax, e_i \rangle$

Note: $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$

fixed by $\varphi \in isom(\mathbb{R}^n)$

$\Rightarrow \varphi(0)=0$

Cor: $Isom(S^{n-1}) \overset{\text{bijection}}{\longleftrightarrow} O_n\mathbb{R}$

Answer to Q:

- $A \in O_3 \Rightarrow A$ has a real eigenvalue

  Pf: 3 is odd $\boxed{Z}$

  $\Rightarrow$ at least one eigenvalue is ± 1

  all eigenvalues λ of O_n have $|\lambda| = 1$

  $\Rightarrow$ eigenline = axis fixed pointwise by A or $-A$

  why? if $\det A$ negative, $\det(-A) = (-1)^3 \det A$ = positive

But $\cdot$ $A \in O_3 \Rightarrow A$ takes orthonormal basis of P = axis$^\perp$

to " " " " "

$\Rightarrow A|_P$ is rotation of P possibly followed by reflection

not if A preserves orientation

Pf: $(e_1, e_2, e_3) \overset{A}{\mapsto} (v_1, v_2, v_3)$

$\Rightarrow \det A = v_1 \cdot (v_2 \times v_3) = v_1 \cdot (\pm v_1)$

But $rot_{axis}(\theta)$ has eigenvalues $1, \lambda, \bar{\lambda}$ $\quad \lambda \in \mathbb{C}$ with $\lambda\bar{\lambda} = |\lambda|^2 = 1$

So $\det(\;) = 1 \cdot \lambda \cdot \bar{\lambda} = 1$

$\Rightarrow v_1 \cdot (\pm v_1) = 1 \Rightarrow +$ $\boxed{Z}$

Prop: let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $\langle \cdot, \cdot \rangle$ standard Hermitian form in $\mathbb{F}^n$ $\quad \langle v, w \rangle = w^* v$

TFAE:

1) $\langle xA, yA \rangle = \langle x, y \rangle$ $\forall$ rows $x, y \in \mathbb{F}^n$ (def: $A \in O_n\mathbb{F}$)

2) Right mult ρ_A preserves $\perp$normal basis ($v_1, \dots, v_n$ $\perp$normal $\Rightarrow v_1A, \dots, v_nA$ $\perp$normal)

3) rows of A are $\perp$normal basis of $\mathbb{F}^n$.

4) $AA^* = I_n$

5) cols of A $\perp$ normal basis of $\mathbb{F}^n$

Pf: exercise, pretty trivial/elementary

Def: $U_n = O_n\mathbb{C}$ unitary group

02/13

Unitary Matrices

$U_n = O_n\mathbb{C}$ standard Hermitian $\langle \cdot , \cdot \rangle$

$\mathbb{C}^n = \mathrm{span}_{\mathbb{C}}(e_1, \dots, e_n)$

$= \mathrm{span}_{\mathbb{R}}(e_1, ie_1, \dots, e_n, ie_n)$

so $\mathbb{C}^n \cong \mathbb{R}^{2n}$

by $(a_1+b_1i, \dots, a_n+b_ni)$
$\updownarrow$
$(a_1, b_1, \dots, a_n, b_n)$

induces: $d_n : \underset{M_n\mathbb{C}}{\mathbb{C}^{n\times n}} \to \underset{M_{2n}\mathbb{R}}{\mathbb{R}^{2n\times 2n}}$

$a+bi \mapsto \begin{bmatrix} a & \\ & a \end{bmatrix} + \begin{bmatrix} & -b \\ b & \end{bmatrix}$

or $1 \mapsto \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$

$i \mapsto \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$

Def: $A \in M_{2n}\mathbb{R}$ is complex linear if $A \in d_n(M_n\mathbb{C})$

Prop $A \in U_n \iff d_n(A) \in O_{2n}\mathbb{R} \cap d_n(M_n\mathbb{C})$

$d_n(U_n) = O_{2n}\mathbb{R} \cap d_n(M_n\mathbb{C})$

Pf: $d_n(A^*) = d_n(A)^* = d_n(A)^T$

$d_n(AB) = d_n(A)d_n(B)$ ring hom

LHS
- View A, B as $\mathbb{C}$-linear maps on $\mathbb{C}^n$.
- Compose
- View $\mathbb{C}^n$ as v.s. $/\mathbb{R}$.

RHS
- View A, B as linear operators on $\mathbb{R}^n$
- Compose

The two sides are equal as functions $\mathbb{C}^n \to \mathbb{C}^n$

$d_n(A)d_n(A)^* = d_n(A)d_n(A^*) = d_n(AA^*)$

$\overset{\shortparallel}{I_{2n}^{\mathbb{R}}} \iff d_n(A) \in O_{2n}\mathbb{R}$ $\qquad \overset{\shortparallel}{I_{2n}^{\mathbb{R}}} \iff AA^* = I_n^{\mathbb{C}} \iff A \in U_n$ $\boxed{2}$

Def: for v.s. $V/\mathbb{F}$ with $\langle \cdot,\cdot \rangle$,

$$O(V) = \{\varphi : V \to V \mid \langle \varphi_x, \varphi_y \rangle = \langle x, y \rangle \ \forall x,y \in V\}$$

Prop: $O(V) = \{\varphi \in GL(V) \mid \|\varphi x\| = \|x\| \ \forall x \in V\}$

pf: $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\,\mathrm{Re}\langle x,y \rangle$

$\Rightarrow \mathrm{Re}\langle x,y\rangle = \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x-y\|^2\right)$

So φ preserves $\|\cdot\| \Rightarrow \varphi$ preserves $\langle \cdot,\cdot \rangle$

Prop: $A \in O_n\mathbb{F} \Rightarrow |\det A| = 1$

Pf: $AA^* = I_n \Rightarrow \det(AA^*) = (\det A)\overline{(\det A)} = 1$

$$(\det A)\overline{(\det A)} = |\det A|^2 = 1$$

Def: $SO_n\mathbb{F} = \{A \in O_n\mathbb{F} \mid \det A = 1\}$

$SU_n = \{A \in U_n \mid \det A = 1\}$

Running assumption: V v.s. $/\mathbb{C}$ with $\langle \cdot,\cdot \rangle$, $\dim_{\mathbb{C}} V = n$

$\varphi : V \to V$ $\mathbb{C}$-linear.

Thm V has orthonormal basis β with $[\varphi]_\beta$ is upper triangular.

$\Updownarrow$ $A \in M_n\mathbb{C} \Rightarrow UAU^*$ upper-triangular for some $U \in U_n$

$\downarrow$ Cols are $\perp$normal $\Rightarrow$ orthogonal

Pf: $n=1$: ✓

$n \geq 2$: pick unit eigenvector u_1, $\perp$normal basis $v_2, \dots, v_n$ for $u_1^\perp$ with $QAQ^* = \begin{bmatrix} \lambda_1 & * \cdots * \\ 0 & \\ \vdots & \boxed{A'} \\ 0 & \end{bmatrix}$

where $Q = \begin{bmatrix} | & | & & | \\ u_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in U_n$.

Choose $W = \begin{bmatrix} | & & | \\ w_2 & \cdots & w_n \\ | & & | \end{bmatrix} \in U_{n-1}$, By induction, $WA'W^*$ upper Δ

Set $U = \begin{bmatrix} 1 & \\ & \boxed{W} \end{bmatrix} Q$

Then $UAU^* = \begin{bmatrix} 1 & \\ & \boxed{W} \end{bmatrix} QAQ^* \begin{bmatrix} 1 & \\ & \boxed{W} \end{bmatrix}$

$= \begin{bmatrix} 1 & \\ & \boxed{W} \end{bmatrix} \begin{bmatrix} \lambda_1 & * \cdots * \\ 0 & \boxed{A'} \end{bmatrix} \begin{bmatrix} 1 & \\ & \boxed{W^*} \end{bmatrix}$

$= \begin{bmatrix} \lambda_1 & * \\ 0 & \boxed{\diagdown *} \end{bmatrix}$

Thm + pf work over $\mathbb{R}$ if eigenvals are real

Cor (Spectral Thm): $\varphi = \varphi^* \Rightarrow V$ has $\perp$normal basis of evectors and all evals $\in \mathbb{R}$

$\Updownarrow$ $A = A^* \Rightarrow A = UDU^*$ for some $U \in \mathcal{U}_n$ and real diagonal D

Pf: Thm $\Rightarrow A = UBU^*$ with $U \in \mathcal{U}_n$, B upper$\triangle$

$\Rightarrow A^* = UB^*U^* \Rightarrow B = B^* \Rightarrow B$ real diagonal $\square$

Def: φ is normal if $\varphi\varphi^* = \varphi^*\varphi$

Cor: φ is normal $\overset{\Leftarrow ?}{\Rightarrow}$ V has $\perp$normal basis of evectors

$\Updownarrow$ A normal $\Rightarrow A = UDU^*$ for some $U \in \mathcal{U}_n$, diagonal D.

Pf: Suffices by thm: normal upper$\triangle$ is diagonal.

Assume N normal upper$\triangle$. Check:

$$N = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ 0 & \\ \vdots & \boxed{\begin{matrix} \diagdown & * \\ 0 & \diagdown \end{matrix}} \\ 0 & \end{bmatrix}$$

$(N^*N)_{11} = \bar{a}_{11} a_{11} = |a_{11}|^2$

$(NN^*)_{11} = |a_{11}|^2 + \underbrace{|a_{12}|^2 + \cdots + |a_{1n}|^2}_{0}$

$\Rightarrow a_{12} = a_{13} = \cdots = a_{1n} = 0$

$\Rightarrow$ diagonal by induction $\square$

Prop: φ normal $\Leftrightarrow \|\varphi x\| = \|\varphi^* x\|$ for all $x \in V$.

Pf: ($\Rightarrow$) $\|\varphi_x\|^2 = \langle \varphi_x, \varphi_x \rangle = \langle \varphi^*\varphi x, x \rangle = \langle \varphi\varphi^* x, x \rangle = \langle \varphi^* x, \varphi^* x \rangle = \|\varphi^* x\|^2$

($\Leftarrow$) Compare $\langle \varphi^*\varphi x, y \rangle$ to $\langle \varphi\varphi^* x, y \rangle$

$\langle \varphi^*\varphi x, y \rangle = \langle \varphi x, \varphi y \rangle$ $\quad$ $\langle \varphi\varphi^* x, y \rangle = \langle \varphi^* x, \varphi^* y \rangle$

by expressing $\langle \cdot, \cdot \rangle$ in terms of $\|\cdot\|$ $\square$

02/18

Positive (semi)definite matrices and singular values

Def: fix V with a Hermitian inner product $\langle \cdot, \cdot \rangle$ and $\dim_{\mathbb{C}} V = n$

A $\mathbb{C}$-linear $\varphi: V \to V$ is self-adjoint (Hermitian) if $\varphi = \varphi^*$ and further

- positive semidefinite (definite) if $\langle \varphi x, x \rangle \geq 0$ (>0) $\forall x \in V$ "$\varphi \geq 0$" ("$\varphi > 0$")

Thm: let $\varphi = \varphi^*$. Then $\varphi \geq 0$ ($>$) iff all eigenvalues of φ are ≥ 0 ($>$).

Pf: pick $\perp$normal $\mathcal{B}$ so $[\varphi]_{\mathcal{B}}$ is diagonal (spectral thm)

Now $[\varphi]_{\mathcal{B}} \geq 0$ ($>$) $\Leftrightarrow$ all diagonal entries ≥ 0 ($>$) $\square$

Def: $B = \sqrt{A}$

Cor $A = A^* \geq 0 \Rightarrow \exists! \; B \geq 0$ with $B^2 = A$

Pf: $\exists$: $A = UDU^* \Rightarrow \sqrt{A} = U\sqrt{D}U^*$

$!$: Exercise

Prop: $A > 0 \Rightarrow \nu(x) = \sqrt{x^* A x}$ is a norm.

Pf: $\nu(x) = \|A^{1/2} x\|_2$, apply HW2 #3 Kernel = eigenvalue 0

$\nu = \mu \circ \varphi$ for $\mu = \|\cdot\|_2$ and $\varphi = A^{1/2}$ $\boxed{}$

Def: $A \in \mathbb{C}^{m \times n}$ has Modulus $|A| = \underbrace{\sqrt{A^* A}}_{\geq 0} \in \mathbb{C}^{n \times n}$

Needs: $(A^*A)^* = A^* A^{**} = A^* A$ ✓

$\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|x\|^2 \geq 0$

Prop: $\||A|x\| = \|Ax\| \quad \forall x \in \mathbb{F}^n$

Pf: $\||A|x\|^2 = \langle |A|x, |A|x\rangle$

$= \langle |A|^*|A|x, x\rangle$

$= \langle |A|^2 x, x\rangle$

$= \langle A^*Ax, x\rangle$

$= \langle Ax, Ax\rangle = \|Ax\|^2$

Cor: $\ker A = \ker |A| = \operatorname{im}(|A|)^\perp$

Pf: $=_1$: $\|Ax\| = 0 \Leftrightarrow \||A|x\| = 0$

$=_2$: $\ker T = (\operatorname{im} T^*)^\perp$

$|A|^* = |A|$ $\boxed{}$

Note: Cor $\Rightarrow$

- $|A| = \perp$ projection followed by ... some $\cong$ of $\operatorname{im}|A|$
- $\operatorname{im} A \xrightarrow[\cong]{|A|} (\ker A)^\perp$

Def: the singular values of A are the eigenvalues of $|A|$

$\Updownarrow$

$\sqrt{\lambda_1} \geq \cdots \geq \sqrt{\lambda_n}$ where A^*A has evals $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$\sigma_1 \geq \cdots \geq \sigma_n$

Running assumptions $Y \leftarrow X$ hom. over hermetian v.s./$\mathbb{F}$

- singular values $\underbrace{\sigma_1, \dots, \overset{r=\text{rank}}{\sigma_r}}_{\neq 0}, \underbrace{\sigma_{r+1}, \dots, \sigma_n}_{0}$
- $v_1, \dots, v_n$ $\perp$normal basis of X so that $|A|$ scales v_i by σ_i

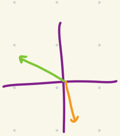
$\rightarrow$ $v_{r+1}, \dots, v_n$ " " " $\ker A$

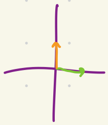
Thm (polar decomposition)

$A \in \mathbb{F}^{n \times n} \Rightarrow A = U|A|$ for some $U \in O_n \mathbb{F}$

Pf: will follow from SVD ☺

Eg: $A = \begin{bmatrix} -2 & \frac{1}{5}(4-\sqrt{6}) \\ 1 & \frac{1}{5}(-2-2\sqrt{6}) \end{bmatrix} \approx \begin{bmatrix} -2 & 0.31 \\ 1 & -1.38 \end{bmatrix}$



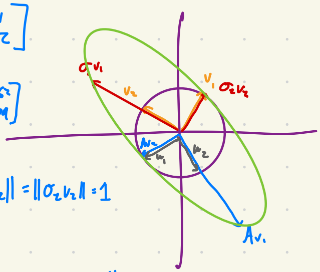


$A^*A = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$ $\quad p_{A^*A} = (5-t)(2-t) - 4$
$= (t-6)(t-1)$

$\sigma_1^2 = 6 \quad \sigma_2^2 = 1$

$v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad v_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$Av_1 \approx \begin{bmatrix} 1.93 \\ -1.51 \end{bmatrix} \quad Av_2 \approx \begin{bmatrix} -0.62 \\ -0.79 \end{bmatrix}$



$\|Av_1\| = \|\sigma_1 v_1\| = \sqrt{6} \qquad \|Av_2\| = \|\sigma_2 v_2\| = 1$

Lemma: $w_k = \frac{1}{\sigma_k} A v_k \Rightarrow w_1, \dots, w_r \perp$ in Y

pf: ex. check $\langle \sigma_i w_i, \sigma_j w_j \rangle = \dots$

Thm: A has a Schmidt decomp:

$A = \sum_{k=1}^{r} \sigma_k w_k v_k^*$ $\qquad v_k^*: x \mapsto \langle x, v_k \rangle$

so coeff on w_k in Ax is $\sigma_k \langle x, v_k \rangle$

Pf: RHS applied to v_i is

$\sigma_i w_i = Av_i$ if $i \leq r$ and 0 if $i > r$. $\boxed{\smile}$

01/25 **Perturbation theory**

Given $A \in \mathbb{C}^{n\times n}$ how do $\Lambda(A)$ = spectrum of A = multiset of roots of p_A and $\Lambda(\tilde{A})$ relate if $\tilde{A} = A + E$ with $\|E\| < \varepsilon$

E.g. $A = 0 \Rightarrow |\lambda| \le \|E\|$

Def: A has operator norm $\|A\| = \max\limits_{x \in B_1(0)} \|Ax\|_2 = \sigma_1$ ← largest singular value

Def: $V = \mathbb{C}^{m\times k}$, $\mathbb{C}^{k\times n}$, $\mathbb{C}^{m\times n}$ norms μ, ν, ρ are consistent

if $\rho(AB) \le \mu(A)\nu(B)$ $\forall A, B$ of appropriate sizes

if $\mu = \nu = \rho$ on $\mathbb{C}^{n\times n}$ then ν is consistent

E.g. $\|\cdot\|_2 = \sqrt{\mathrm{tr}(A^*A)}$ is consistent (HW4)

$\|\cdot\|_\infty = \nu_\infty(\cdot) = \max\limits_{i,j} |a_{ij}|$ norm but

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \nu_\infty(A^2) = 2 > 1 = (\nu_\infty(A))^2$

Prop: $\|\cdot\|$ consistent on $\mathbb{C}^{n\times n} \Leftrightarrow \exists$ norm ν on $\mathbb{C}^n$ consistent with $\|\cdot\|$: $\nu(Ax) \le \|A\|\nu(x)$ $\forall x$

Pf: fix $v \in \mathbb{C}^n \setminus \{0\}$. Set $\nu(x) = \|xv^T\|$

- ν is a norm by HW2 #3

$$\mathbb{C}^n \hookrightarrow \mathbb{C}^{n\times n} \xrightarrow{\|\cdot\|} \mathbb{C}$$
$$x \longmapsto xv^T \longmapsto \|xv^T\|$$

- ν consistent with $\|\cdot\|$: $\nu(Ax) = \|(Ax)v^T\| = \|A(xv^T)\| \overset{\text{consistent}}{\le} \|A\|\,\|xv^T\| = \|A\|\nu(x)$

$\Rightarrow \nu(Ax) \le \|A\|\nu(x)$ ☺

Def: $A \in \mathbb{C}^{n\times n}$ has spectral radius $\rho(A) = \max\{|\lambda| \mid \lambda \in \Lambda(A)\}$.

Thm: $\|\cdot\|$ consistent on $\mathbb{C}^{n\times n} \Rightarrow \rho(A) \le \|A\|$ $\forall A \in \mathbb{C}^{n\times n}$

Pf: Pick ν consistent with $\|\cdot\|$ by proposition. If $\lambda \in \Lambda(A)$ and $Ax = \lambda x$ and $x \ne 0$

then $|\lambda|\nu(x) = \nu(\lambda x) = \nu(Ax) \le \|A\|\nu(x)$

$\Rightarrow |\lambda| \le \|A\|$

E.g. $A=0 \Rightarrow |\lambda| \le \|E\|$ for $\lambda \in \Lambda(\tilde{0})$

$\tilde{0} = 0 + E = E$ so if $\|E\| \sim 10^{-8}$ then $\rho(\tilde{0}) < 10^{-8}$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \\ 0 & & & 0 \end{bmatrix} \quad E = \begin{bmatrix} & & & \\ & & 0 & \\ & & & \\ \varepsilon & & & \end{bmatrix} \quad \text{so } \tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varepsilon & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad p_{\tilde{A}}(t) = t^4 - \varepsilon \Rightarrow \Lambda(\tilde{A}) = \{\pm \varepsilon^{1/4}, \pm i \varepsilon^{1/4}\}$

$\varepsilon \sim 10^{-8} \Rightarrow \rho(\tilde{A}) \sim 10^{-2}$

Thm: locations of eigenvalues are continuous under perturbation:

if $\lambda \in \Lambda(A)$ and $a(\lambda) = m$, $\|\cdot\|$ any norm, $\varepsilon << 1$,

then $\exists\, \delta > 0$ s.t. $\|E\| < \delta \Rightarrow B_\varepsilon(\lambda) \ni$ exactly m eigenvalues of $\tilde{A} = A + E$

Rouché's Thm: Suppose $\Omega \subseteq \mathbb{C}$ open and $\phi, f : \overline{\Omega} \to \mathbb{C}$ assume:

- f is analytic on $\overline{\Omega}$ (Taylor series at $z \to f$ in nbhd of z $\forall z \in \overline{\Omega}$) as is ϕ. (e.g. ϕ, f polynomials)
- $\partial\Omega$ simple closed curve
- $|\phi(z)| < |f(z)| \quad \forall z \in \partial\Omega$

Then f and $f + \phi$ have the same # of roots in $\overline{\Omega}$, counted with multiplicity.

E.g. $f(z) = z^n$ on $\Omega = B_1(0)$

$\phi(z) = \varepsilon z^j$ for any $j \in \mathbb{N}$.

Rouché $\Rightarrow$ $z^n + \varepsilon z^j$ has n roots in Ω whenever $\varepsilon < 1$

General: $z^n + \varepsilon_j z^j + \dots + \varepsilon_1 z + \varepsilon_0$ has exactly n roots in Ω if $\sum_k |\varepsilon_k| < 1$

Lemma: $A \mapsto p_A$ is a continuous function $\mathbb{C}^{n \times n} \to \mathcal{P}_n$.

pf: coeffs are polynomials in entries of A $\boxed{\text{Z}}$

Pf (continuity thm): choose ε s.t. $\overline{\Omega} = \overline{B_\varepsilon(\lambda)}$ has no eigenvalues other than λ.

Lemma $\Rightarrow p_{\tilde{A}} \to p_A$ as $\tilde{A} \to A$

$\Rightarrow \underbrace{p_{\tilde{A}}(z) - p_A(z)}_{\phi_\epsilon(z)} \to 0$ for $z \in \bar{\Omega} \supseteq \partial\Omega$ as $E \to 0$

Set $f(z) = p_A(z)$ so $f + \phi = p_{\tilde{A}}$

∂A compact $\Rightarrow |f(z)|$ bounded away from 0
achieves its min $\alpha \neq 0$
at $z_0 \in \partial\Omega$

∂A compact $\Rightarrow \exists\, \delta > 0$ s.t. $|\phi_\epsilon(z)| < \alpha$ whenever $\|E\| < \delta$

Rouché's thm $\Rightarrow f + \phi_\epsilon = p_{\tilde{A}}$ has same # roots in Ω as does f. $\boxed{\mathbb{Z}}$

2/27

def: $d(\tilde{\lambda}, \Lambda)$ = distance from arbitrary $\tilde{\lambda} \in \mathbb{C}$
to closed (here, finite) set Λ
$= \min_{\lambda \in \Lambda} |\tilde{\lambda} - \lambda|$

fix $A \in \mathbb{C}^{n \times n}$ and $\tilde{A} = A + E$. $\Lambda = \Lambda(A)$
$\tilde{\Lambda} = \Lambda(\tilde{A})$

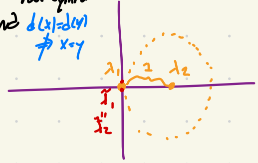
Def 1: spectral variation $sv_A(\tilde{A}) = \max_{\tilde{\lambda} \in \tilde{\Lambda}} d(\tilde{\lambda}, \Lambda)$

geometric interpretation:
$$\tilde{\Lambda} \subseteq \bigcup_{i=1}^{n} D_i, \text{ where } D_i = B_{sv_A(\tilde{A})}(\lambda_i)$$

fatten up pts of Λ, min radius s.t. swallow $\tilde{\Lambda}$

Not a metric: $n = 2$ $\lambda_1 = \tilde{\lambda}_1 = \tilde{\lambda}_2 = 0$
$\lambda_2 = 1$

not symmetric
and $d(x) = d(y)$
$\not\Rightarrow x = y$



don't have to fatten • to swallow •

$sv_A(\tilde{A}) = 0$

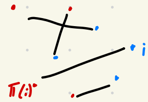
$sv_{\tilde{A}}(A) = 1$

2: Hausdorff distance $hd(A, \tilde{A}) = \max\{sv_A(\tilde{A}), sv_{\tilde{A}}(A)\}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

same hd, not equal $\to$ not a distance

3. matching distance



$$md(A,\tilde{A}) = \min_{\pi\in S_n} \max_i \left|\tilde{\lambda}_{\pi(i)} - \lambda_i\right|$$

= min length of longest edge in perfect matching

Lemma: (Hadamard's inequality):

$$|\det A| \le \prod_{i=1}^{n} \|\text{col}_i \text{ of } A\|_2$$

"pretend A preserves $\perp$, would max volume"

Pf: true for $\begin{bmatrix} \diagdown & \\ & \end{bmatrix} = A$, both sides unchanged by $A \mapsto UA$ for unitary U

use Schur decomp.

Elsner's Thm $\;hd(A,\tilde{A}) \le \underbrace{(\|A\|+\|\tilde{A}\|)^{1-1/n} \|E\|^{1/n}}_{\beta}$

(operator norm)

Pf: β is symmetric in $A, \tilde{A}$. So only need $sv_A(\tilde{A}) \le \beta$.

Sps. $sv_A(\tilde{A}) = d(\tilde{\lambda},\Lambda)$ (choose $\tilde{\lambda}$). Pick $\perp$normal basis $x_1,\dots,x_n$ with $\tilde{A}x_1 = \tilde{\lambda}x_1$. Then

$$sv_A(\tilde{A})^n \le \prod_{\lambda\in\Lambda} |\tilde{\lambda}-\lambda| \qquad \text{since } d(\tilde{\lambda},\Lambda) \le |\tilde{\lambda}-\lambda| \;\forall \lambda\in\Lambda.$$

$$= |\det(A-\tilde{\lambda}I)| \qquad \lambda\in\Lambda(A) \Leftrightarrow \lambda-\tilde{\lambda} \in \Lambda(A-\tilde{\lambda}I)$$

by lemma $\;\le \prod_{i=1}^{n} \|\underbrace{(A-\tilde{\lambda}I)x_i}_{\text{col}_i}\| \qquad A-\hat{\lambda}I = (A-\tilde{\lambda}I)-(\lambda-\hat{\lambda})I$

$\perp$normal basis and $\det = \prod \lambda$

$$\le \|(A-\tilde{\lambda}I)x_1\|_2 \prod_{i=2}^{n} \|(A-\tilde{\lambda}I)x_i\|_2$$

$$\le \|Ex_1\|_2 \prod_{i=2}^{n} \left(\|Ax_i\|_2 + \|\tilde{\lambda}x_i\|_2\right)$$

$$\le \|E\|(\|A\|+\|\tilde{A}\|)^{n-1}$$

nth roots $\;sv_A(\tilde{A}) \le (\|A\|+\|\tilde{A}\|)^{1-1/n}\|E\|^{1/n}$ $\boxtimes$

Example:

$$\underbrace{\begin{bmatrix} 1 & 10^{-8} \\ 10^{-8} & 2 \end{bmatrix}}_{\tilde{A}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 & 10^{-8} \\ 10^{-8} & 0 \end{bmatrix}}_{E}$$

$$\Rightarrow hd(A,\tilde{A}) \le \left(2 + (2+10^{-8})\right)^{1/2}(10^{-8})^{1/2}$$

$$< \left(2+\tfrac{5}{2}\cdot 10^{-9}\right)10^{-2}$$

$$< .021$$

$$\rightarrow \tilde{\Lambda} \subseteq [.979, 1.021] \cup [1.979, 2.021]$$

$$p_{\tilde{A}}(z) = z^2 - 3z + (2 - 10^{-8})$$

$$\tilde{\Lambda} = \{1-\varepsilon, 2+\varepsilon\} \quad \varepsilon = \overbrace{10^{-8} + O(10^{-16})}^{hd(A,\tilde{A})}$$

Thm (Ostrowski; Elsner): $md(A, \hat{A}) \le n\beta$.

Thm (Bauer - Fike): $\|\cdot\|$ consistent on $\mathbb{C}^{n\times n}$, and $\tilde{\lambda} \in \tilde{\Lambda} \setminus \Lambda$

$$\Rightarrow \underbrace{\|\underbrace{\underbrace{(A - \tilde{\lambda} I)}_{\sim 0}{}^{-1}}_{\text{big}}\|^{-1}}_{\text{small}} \le \|E\|$$

that small

Pf: $\underbrace{\tilde{A} - \tilde{\lambda} I}_{\text{singular}} = A - \tilde{\lambda} I + E = \underbrace{(A - \tilde{\lambda} I)}_{\text{nonsingular}} \Rightarrow \underbrace{(I + (A - \tilde{\lambda} I)^{-1} E)}_{\text{singular}}$

$$\Rightarrow 1 \le \|(A - \tilde{\lambda} I)^{-1} E\|$$
$$\le \|(A - \tilde{\lambda} I)^{-1}\| \|E\|$$
$$\Rightarrow \|(A - \tilde{\lambda} I)^{-1}\|^{-1} \le \|E\| \quad \boxed{\square}$$

Thm (Gershgorin): for $A \in \mathbb{C}^{n\times n}$ let $\alpha_i = \overbrace{\sum_{j\neq i} |a_{ij}|}^{\text{sum of off diagonal entries in row } i}$ $\forall i$

and $g_i(A) = B_{\alpha_i}(a_{ii})$

Then $\Lambda \subseteq \bigcup_{i=1}^{n} g_i(A)$

Pf: fix $\lambda \in \Lambda$. If $\lambda = a_{ii}$ $\lambda \in g_i$.

So assume $\lambda \neq a_{ii}$ $\forall i$. In Bauer-Fike thm,

view A as perturbation of $D = \text{diag}(a_{11}, \dots, a_{nn})$

$$A = D + E \quad E = A - D$$

By BF thm, $1 \le \|(D - \lambda I)^{-1} E\|_\infty$

$$= \max_i |a_{ii} - \lambda|^{-1} \sum_{j \neq i} |a_{ij}|$$

where $\|C\|_\infty = \max_i \sum_{j=1}^{n} \|c_{ij}\|$

$$\Rightarrow |a_{ii} - \lambda| \le \underbrace{\sum_{j\neq i} |a_{ij}|}_{= \alpha_i} \text{ for some } i$$

$$\Rightarrow \lambda \in B_{\alpha_i}(a_{ii}) \quad \boxed{\square}$$

Def: g_i is the ith Gershgorin disk

Recall: $\underbrace{\begin{bmatrix} 1 & 10^{-8} \\ 10^{-8} & 2 \end{bmatrix}}_{\tilde{A}} = \underbrace{\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} & 10^{-8} \\ 10^{-8} & \end{bmatrix}}_{E}$

by Gershgorin,

$\tilde{\Lambda} \subseteq [.9999, 1.0001] \cup [1.9999, 2.0001]$

better, but "halfway there"

Trick: $\tilde{A} \sim \begin{bmatrix} \gamma & \\ & 1 \end{bmatrix} \tilde{A} \begin{bmatrix} \gamma^{-1} & \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & 10^{-8}\gamma \\ 10^{-8}\gamma^{-1} & 2 \end{bmatrix}$

choose γ small but with $10^{-8}\gamma + 10^{-8}\gamma^{-1} < 1$

Suffices: $\gamma^{\pm 1} = 10^{-8} + 10^{-11}$

$\Rightarrow \tilde{\Lambda} \subseteq [1-\delta, 1+\delta] \cup [2-\delta, 2+\delta]$

for $\delta = 10^{-8} + 10^{-15}$

03/04

Lie algebras

Def: for $X \subset$ v.s./$\mathbb{R}$, the tangent space at $p \in X$ is

$T_pX = \{\gamma'(0) \mid \gamma : (-\varepsilon, \varepsilon) \to X$ diff'able with $\gamma(0) = p\}$

E.g. $X = S^{n-1} \subseteq \mathbb{R}^n$

$\gamma(t) \subseteq S^{n-1} \iff \gamma(t) \cdot \gamma(t) = 1$

$\iff \gamma_1(t)^2 + \dots + \gamma_n(t)^2 = 1$

$\Rightarrow 2\gamma_1(t)\gamma_1'(t) + \dots 2\gamma_n(t)\gamma_n'(t) = 0$

$\iff \gamma(t) \cdot \gamma'(t) = 0$

$\Rightarrow T_p S^{n-1} \subseteq p^{\perp}$

all tangent vectors of p are orthogonal to p

Now take L line in $p^{\perp} \Rightarrow L \perp p \Rightarrow L = T_p(W \cap X)$ for $W = \text{span}(L, p)$

So $T_p S^{n-1} = p^{\perp}$ $\boxed{L}$

Def: A subgroup $G \in GL_n \mathbb{F}$ that is a manifold

in $M_n\mathbb{F}$ v.s./$\mathbb{R}$

Has Lie Algebra $\mathfrak{g} = T_I G$

e.g. • $G = U_1 \cong O_2\mathbb{R} \Rightarrow \mathfrak{u}_1 = i\mathbb{R} = \text{span}_{\mathbb{R}}(i)$

$\frac{d}{dt}(e^{it}) = ie^{it}$

- $G = GL_n(\mathbb{F}) \Rightarrow \mathfrak{g} = \mathfrak{gl}_n(\mathbb{F}) = M_n\mathbb{F}$

  pf: $\{\det = 0\}$ closed $\Rightarrow GL_n$ open in $M_n\mathbb{F}$ $\blacksquare$

  $I \nearrow^{E}$ $\quad E = \frac{d}{dt}(I + tE)$

Def: $A \in M_n(\mathbb{F})$ has matrix exponential $\exp(A) = e^A := I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots + \frac{A^k}{k!} + \dots$

Convergence issues: dw ab it, converges very fast

Prop: $\gamma: \mathbb{R} \to M_n\mathbb{F}$ via $\gamma(t) = e^{tA}$ is

diffable with $\gamma'(t) = e^{tA}A = Ae^{tA} = A\gamma(t) = \gamma(t)A$

look at def for commutation

pf: termwise and entrywise differentiate

$$e^{tA} = I + tA + t^2\frac{A^2}{2!} + t^3\frac{A^3}{3!} + \dots + t^k\frac{A^k}{k!} + \dots$$

$$(e^{tA})' = 0 + A + t\frac{A^2}{1!} + t^2\frac{A^3}{2!} + \dots + t^k\frac{A^{k+1}}{k!} + \dots$$

$$= Ae^{tA} = e^{tA}A \quad \blacksquare$$

Prop: $AB = BA \Rightarrow e^{A+B} = e^Ae^B = e^Be^A$

Pf: Binomial thm $\Rightarrow \frac{(A+B)^k}{k!} = \sum_{i+j=k}\frac{A^i}{i!}\frac{B^j}{j!}$ deg k terms in e^Ae^B

deg k term in e^{A+B} $\blacksquare$

Cor: $\exp: \mathfrak{gl}_n\mathbb{F} \to GL_n\mathbb{F}$ → exp of arb matrix is invertible

pf: $e^Ae^{-A} = e^0 = I$

$tr \circ \partial = \partial \circ \det$

Thm: $tr = \det'$

$$\frac{d}{dt}\Big|_{t=0}(\det\gamma(t)) = tr(\gamma'(0))$$

if $\gamma(0) = I$.

pf: sps $\gamma(t)$ is a path in $M_n\mathbb{F}$ with distinct eigenvalues

$\lambda_1(t), \dots, \lambda_n(t)$ in $\mathbb{C}$ for all $t \neq 0$.

path is valid b/c of continuity thm

Then $\det\gamma(t) = \lambda_1(t)\cdots\lambda_n(t)$

3/6 Continuing proof

$$(\det\gamma(t))' = \lambda_1'(t)\frac{\det\gamma(t)}{\lambda_1(t)} + \dots \lambda_n'(t)\frac{\det\gamma(t)}{\lambda_n(t)}$$

$$\Rightarrow \quad \left(\det \gamma(0)\right)' = \lambda_1'(0) + \cdots + \lambda_n'(0) \qquad \Rightarrow \quad \begin{array}{l}\gamma(0) = I \\ \det\gamma(0) = 1 \\ \lambda_i(0) = 1\end{array}$$

$$= \operatorname{tr}\gamma'(0)$$

Even if the eigenvalues not all distinct, same argument works by replacing $\gamma(t)$ with

$$\gamma_\varepsilon(t) = \gamma(t) + \varepsilon D \quad \text{for} \quad D = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{bmatrix}$$

and taking $\lim_{\varepsilon\to 0}$

Uses: • $|\Delta(A)| = n \Rightarrow |\Delta(A+E)| = n$ for all $\|E\| \ll 1$

$A = I + \varepsilon D$

holds by continuity then $E = \gamma(t) - I$ $\forall\, |t| \ll 1$

• $(\det \gamma_\varepsilon)'(0) \xrightarrow{\varepsilon \to 0} (\det\gamma)'(0)$

holds because

$$\begin{array}{ccccc} t & \longmapsto & (A, \varepsilon) & \longmapsto & \det(A + \varepsilon D) \\ \mathbb{R} & \longrightarrow & M_n\mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \end{array}$$

is diff'able (polynomial in A, ε)

• $\gamma_\varepsilon'(t) = \gamma'(t)$ $\forall t$ because

$$\gamma_\varepsilon'(t) = \left(\gamma(t) + \cancel{\varepsilon D}\right)' \qquad \boxtimes$$

E.g. $G = SL_n\mathbb{F} = \{A \in GL_n\mathbb{F} \mid \det A = 1\}$

$\rightarrow \quad \mathfrak{g} = \mathfrak{sl}_n\mathbb{F} := \{A \in M_n\mathbb{F} \mid \operatorname{tr} A = 0\}$

Pf: $\det\gamma(t) \equiv 1$ if $\gamma \subseteq SL_n \rightarrow$

$\mathfrak{g} \subseteq \mathfrak{sl}_n$ by $\operatorname{tr} = \det'$

For $\supseteq$, let $A \in \mathfrak{sl}_n\mathbb{F}$. Then

$$\det e^{tA} = e^{\operatorname{tr}(tA)} \quad \text{(HW4 ü)}$$
$$= e^{t \cdot 0} = 1$$

$\rightarrow \quad e^{tA} \in SL_n\mathbb{F}$

So e^{tA} realizes A as $\left(e^{tA}\right)'\Big|_{t=0}$. $\boxtimes$

Lemma: $\left(\gamma(t)^*\right)' = \left(\gamma'(t)\right)^*$ $\boxtimes$

Prop: $\beta, \gamma : (-\varepsilon, \varepsilon) \to M_n$ diff'able

$\Rightarrow \quad (\beta\gamma)' = \beta'\gamma + \beta\gamma'$

Pf: $(x \cdot y)' = x' \cdot y + x \cdot y'$

$x = \beta_i$ row

$y = (\text{col } \gamma)_j^T$ $\qquad \boxtimes$

E.g. $G = O_n\mathbb{F} \rightsquigarrow \mathfrak{g} = \mathfrak{o}_n\mathbb{F} := \{A \in M_n\mathbb{F} \mid ?\}$

$A^* = -A$ skew-hermitian

pf: $\subseteq$: product rule (prop) + lemma:

$$\gamma(t) \subseteq O_n\mathbb{F} \Rightarrow \gamma(t)\gamma(t)^* \equiv I$$

$$\Rightarrow \gamma'(t)\gamma(t)^* + \gamma(t)\gamma'(t)^* = 0$$

$$\overset{t=0}{=} \gamma'(0) I + I \gamma'(0)^* = 0$$

$$\Rightarrow 0 = A + A^*$$

$\supseteq$: $\Leftrightarrow \exists\, \gamma(t) \in O_n\mathbb{F}$ with $\gamma'(0) = A$ whenever $A^* = -A$

use $\gamma(t) = e^{tA}$
$\gamma'(t) = A$ $\Rightarrow e^{tA}(e^{tA})^* = e^{tA}e^{tA^*}$ (by power-series expansion)

$$= e^{tA}e^{-tA}$$

$$= e^0 = I. \ \boxtimes$$

Prop: $\dim X = \dim T_pX$ for any p in a manifold X

"Pf": $\{\gamma : (-\varepsilon, \varepsilon) \to X_\alpha\} \overset{\text{Thm}}{\underset{\text{bij}}{\longleftrightarrow}} \{\gamma : (-\varepsilon,\varepsilon) \to U_\alpha\}$ $\boxtimes$

Cor: $\dim O_n\mathbb{F} = ?$

- $\mathbb{F} = \mathbb{R}$: $A^T = -A = \begin{bmatrix} 0 & & d \\ & 0 & \\ \text{spoken for} & & 0 \end{bmatrix}$

$$n^2 = 2d + n$$

$$d = \frac{n^2 - n}{2} = \boxed{\binom{n}{2}}$$

- $\mathbb{F} = \mathbb{C}$: $A^* = -A \Rightarrow \begin{bmatrix} i\mathbb{R} & & d \\ & i\mathbb{R} & \\ \text{spoken for} & & i\mathbb{R} \end{bmatrix}$

$$\Rightarrow \dim = n + 2d = \boxed{n^2}$$

So $\dim_{\mathbb{R}} U_n = n^2$

$\Rightarrow$ manifold $\to$ we didn't actually use ab manifold condition for

Thm: fix closed subgroup $G \leq GL_n\mathbb{F}$ with Lie algebra $\mathfrak{g}$. Then

- $A \in \mathfrak{g} \Rightarrow e^A \in G$
- $B_\varepsilon = \{A \in M_n \mid \|A\| < \varepsilon\}$

$$\Rightarrow \exp_\varepsilon : \mathfrak{g} \cap B_\varepsilon \overset{\exp}{\hookrightarrow} G \text{ if } \varepsilon \ll 1$$

- G is a manifold with atlas

$$\{g \cdot \exp \mid g \in G\}$$

Thm: closed subgroup $H \leq G \Rightarrow$
G/H, $H\backslash G$ are manifolds.

E.g. • $\mathcal{F}\ell_n$ for $G = GL_n$, $H = B_n^+$

• chains $\{V_d \subseteq V_e\}$ $G = GL_n$ and $H = \begin{bmatrix} d & * & * \\ & e-d & * \\ & & n-e \end{bmatrix}$

• $G_k(\mathbb{F}^n)$ $G = GL_n$ and $H = \begin{bmatrix} \square_k & * \\ & \square_{n-k} \end{bmatrix}$

3/18 **Perron-Frobenius theory**

Def: $A \geq B$ for real A, B of same size if $a_{ij} \geq b_{ij}$ $\forall i,j$

$A \gneq B$ and $a_{ij} > b_{ij}$ for some i,j

E.g. $P \geq 0 \iff P$ entrywise nonnegative ($\neq$ pos semi-def)
$P > 0$ — positive

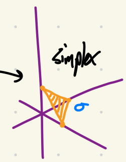
Perron's Thm: $P \in \mathbb{R}^{n \times n}$, $P > 0 \Rightarrow P$ has dominant eigenvalue $\lambda(P)$

1. $\lambda(P) > 0$ and $Pv = \lambda(P)v$ for some $v > 0$; (vector)
2. $a(\lambda(P)) = 1$; and (alg. mult)

for $\kappa \in \Lambda(P) \setminus \{\lambda(P)\}$:

3. $|\kappa| < \lambda(P)$ and
4. $Py = \kappa y$ and $y \neq 0 \Rightarrow y \ngeq 0$ (if real, must have at least one pos + one neg entry)

Pf: set $L(P) = \{\lambda \geq 0 \mid Px \geq \lambda x$ for some $x \gneq 0\}$ ($x \neq 0$ and $\boxed{x \geq 0}$); $\mathbb{1}x = 1$, $\mathbb{1} = [1 \cdots 1]$



Lemma: $L(P)$ is compact and has some $\lambda > 0$

Pf: $x \in \mathbb{R}^n$ and $0 \neq x \geq 0 \overset{P>0}{\Rightarrow} Px > 0$

$\lambda \to 0_+ \Rightarrow \lambda x \to 0$

$\Rightarrow \lambda x < \varepsilon \mathbb{1}^T$ eventually

$\Rightarrow \lambda x < Px$ "

$\Rightarrow \lambda \in L(P)$ "

Bounded: $b = b\mathbb{1}x \geq \mathbb{1}Px \geq \mathbb{1}\lambda x = \lambda \mathbb{1}x = \lambda$

$b = \|\mathbb{1}P\|_\infty \to$ bounded ✓

Closed: $\lambda_k \to \lambda$ with $\lambda_k \in L(P)$ $\forall k \in \mathbb{N}$

$\Rightarrow \exists x_k \in \sigma$ with $Px_k \geq \lambda_k x_k$

σ compact $\Rightarrow \{x_k\}$ has convergent subsequence

So can replace $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{x_k\}_{k \in \mathbb{N}}$ with subsequences to assume $\lambda_k \to \lambda$ and $x_k \to x \in \sigma$

$\Rightarrow \lim(Px_k \geq \lambda_k x_k)$ is $(Px \geq \lambda x) \Rightarrow \lambda \in L(P)$ ☒

1. Set $\lambda(P) = \max L(P)$. Lemma $\Rightarrow$ $\lambda(P) > 0$

<u>Claim</u>: $\lambda(P) \in \Lambda(P)$. In fact, $Pv \geq \lambda(P)v$ for $v \geq 0$ $\Rightarrow$ $Pv = \lambda(P)v$.

<u>Pf (of claim)</u>: Sps $\lambda \in L(P)$, so $Pv \geq \lambda v$ for some $v \in \sigma$

$\left(\begin{array}{l}\text{want: } Pv \neq \lambda v \\ \Uparrow \\ \lambda \neq \lambda(P)\end{array}\right)$ $\Rightarrow 0 \neq Pv - \lambda v = w$

$\overset{P>0}{\Rightarrow} \varepsilon Pw > 0 \ \forall \varepsilon > 0$

$$\Rightarrow P(v+\varepsilon w) = Pv + \varepsilon Pw > Pv = \lambda v + w \geq \lambda v + \varepsilon \lambda w \text{ for } \varepsilon \leq \tfrac{1}{\lambda} = \lambda(v+\varepsilon w)$$

So $x = v + \varepsilon w \Rightarrow Px > \lambda x$

$\Rightarrow Px \geq \lambda' x$ for any $\lambda' > \lambda$ with $\lambda' - \lambda \ll 1$

$\Rightarrow \lambda \neq \lambda(P)$ by maximality $\boxed{\checkmark}$

$Pv > 0$, $\overset{\shortparallel}{\lambda v}$ $\Rightarrow \lambda v > 0 \Rightarrow v > 0$ (*) $\boxed{\checkmark}$

2. $g(\lambda(P)) = 1$: $w \in E(\lambda(P))$ (eigenspace) indep of v.

$\Rightarrow$ line $\overleftrightarrow{vw}$ exits $\mathbb{R}^n_+$ $\Rightarrow E(\lambda(P)) \cap (\partial \mathbb{R}^n_+ \setminus \{0\}) \neq 0$ ⨯ (*)

For $a(\lambda(P)) = 1$, need:

no $y \in \mathbb{R}^n$ with $Py = \lambda(P)y + \alpha v$ (**)

By $y \rightsquigarrow -y$ assume $\alpha > 0$

$y \rightsquigarrow y + \beta v$ assume $y > 0$

(**) and $v > 0 \Rightarrow Py > \lambda(P) Py$

$\Rightarrow Py \geq \lambda' y$ for any $\lambda' \in \lambda(P)$ with $\lambda' - \lambda(P) \ll 1$ ⨯

3. $\kappa \in \Lambda(P)$ with $Py = \kappa y$ both / $\mathbb{C}$

$\Rightarrow P_{i1}y_1 + \cdots + P_{in}y_n = \kappa y_i$

$\Rightarrow P_{i1}|y_1| + \cdots + P_{in}|y_n| \geq |P_{i1}y_1 + \cdots + P_{in}y_n| = |\kappa||y_i|$ (triangle ineq)

$\Rightarrow |\kappa| \in L(P)$ $\quad P\begin{bmatrix}|y_1|\\ \vdots \\ |y_n|\end{bmatrix} \geq |\kappa| \begin{bmatrix}|y_1|\\ \vdots \\ |y_n|\end{bmatrix}$

$|\kappa| \leq \lambda(P)$ but

$|\kappa| = \lambda(P) \Rightarrow x \in E(\lambda(P)) \Rightarrow x = \alpha v$ and "=" in triangle ineq $\Rightarrow y_i$ are parallel

$\Rightarrow y_1, \dots, y_n$ lie along ray in $\mathbb{C}$

$\Rightarrow y_i = \omega |y_i|$ for some $\omega \in U_1$

$\Rightarrow \omega \alpha v \in E(\lambda(P)) \Rightarrow \kappa = \lambda(P)$

03/25 **Multilinear algebra**

Def: fix v.s. $V_1, \dots, V_r$ and W / $\mathbb{F}$

$f: V_1 \times \cdots \times V_r \longrightarrow W$ **multilinear** if

$$f(\dots, v_{i-1}, \alpha v_i + v_i', v_{i+1}, \dots) = \alpha f(v_1, \dots, v_r) + f(\dots, v_{i-1}, v_i', v_{i+1}, \dots)$$

for all i, $v_i, v_i' \in V_i$, $\alpha \in \mathbb{F}$ with v_j fixed for $j \neq i$

E.g.

- $V_i = \mathbb{F}$ for all i
  $f: \mathbb{F}^n \to \mathbb{F}$ multiplication
  $(\lambda_1, \dots, \lambda_r) \mapsto \lambda_1 \cdots \lambda_r$

- $A \in \mathbb{F}^{m \times n} \Rightarrow (v, w) \mapsto vAw$ for $v \in \mathbb{F}^m_{row}$ and $w \in \mathbb{F}^n_{col}$ bilinear

- $V_i = \mathbb{F}^n$ $\forall i = 1, \dots, n$ and $(v_1, \dots, v_n) \mapsto \det \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

Lemma

$$\begin{tikzcd} V_1 \times \cdots \times V_r = \prod V_i \arrow[r, "\text{multilinear}"] \arrow[dr, dashed, "\Rightarrow \text{multilinear}"'] & W \arrow[d, "\text{linear map}"] \\ & W' \end{tikzcd}$$

composition

Interpretation: $\{$v.s. W with multilinear $\prod V_i \to W\}$ form a category

Def: The tensor product of $V_1, \dots, V_r$ is a universal such thing:

multilinear $t: V_1, \dots, V_r \to T$ s.t. for all multilinear f

$$\begin{tikzcd} V_1, \dots, V_r \arrow[r, "t"] \arrow[dr, "f"'] & T \arrow[d, dashed, "\exists ! \tilde{f}"] \\ & W \end{tikzcd}$$

linear with $\tilde{f} \circ t = f$

Thm: t exists. Notation:

$T = \bigotimes_{i=1}^{r} V_i = V_1 \otimes \cdots \otimes V_r$

$t(v_1, \dots, v_r) = v_1 \otimes \cdots \otimes v_r$

Q: Does every element of $V_1, \dots, V_r$ have the form $v_1 \otimes \cdots \otimes v_r$?

A: NO. Key e.g. $w \in W = \mathbb{F}^m_{col}$

$\varphi \in V^* = \mathbb{F}^n_{row}$

$\Rightarrow$ $w \otimes \varphi$ "=" $w\varphi$ has rank ≤ 1 $\to$ a column copied a bunch of times

$$\begin{array}{cc} & \\ i & \end{array}\begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} \otimes \overset{j}{\begin{bmatrix}0 \cdots 0 & 1 & 0 \cdots 0\end{bmatrix}} = i\begin{array}{c} j \\ \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}\end{array}$$

but $W \otimes V^* = \mathrm{Hom}(V, W) = F^{m \times n}$

and lots of matrices have rank > 1

$$\begin{tikzcd} (w, \varphi) \arrow[r, mapsto] \arrow[dr, "\text{bilinear}"'] & w \otimes \varphi \arrow[d, dashed, "\exists! \tilde{f}"] \\ & w\varphi \end{tikzcd}$$

$\Rightarrow$ $\exists! \tilde{f}$ (turns out to be iso)

pf (of thm earlier): want: $\cdots \otimes (v_i + v_i') \otimes \cdots = (\cdots \otimes v_i \otimes \cdots) + (\cdots \otimes v_i' \otimes \cdots)$

$\cdots \otimes \alpha v_i \otimes \cdots = \alpha(\cdots \otimes v_i \otimes \cdots)$

Set $T = M/N$ for $M = \mathrm{span}\{e_{v_1, \ldots, v_r} \mid v_i \in V_i \ \forall i\}$

$N = \mathrm{span}\{e_{v_1, \ldots, \alpha v_i + v_i', \ldots, v_r} - (\alpha e_{v_1, \ldots, v_r} + e_{v_1, \ldots, v_i', \ldots, v_r}) \mid v_i \in V_i \ \forall i, \alpha \in F\}$

$\prod V_i \hookrightarrow M$ map of sets, not (multi) linear

$(v_1, \ldots, v_r) \mapsto e_{(v_1, \ldots, v_r)}$

But
$$\begin{tikzcd} \prod V_i \arrow[r, hook] \arrow[rr, bend right, "+"'] & M \arrow[r, two heads] & M/N \end{tikzcd}$$
is multilinear by construction

Suppose
$$\begin{tikzcd} \prod V_i \arrow[r, hook] \arrow[dr, "f"', "\text{multilinear}"] & M \arrow[d, "\exists!"] \\ & W \end{tikzcd}$$
bc $\prod V_i$ is a basis of M.

But f multilinear $\Rightarrow N \subseteq \ker(\downarrow)$

Universal property of quotients $\Rightarrow$ $\begin{matrix} M \\ \downarrow \\ W \end{matrix}$ induces $\begin{matrix} M/N =: T \\ \downarrow \\ W \end{matrix}$ $\square$

Lemma: $B \subseteq V$ independent iff $\exists$ linear $\{\varphi_b : V \to F\}_{b \in B}$ with $\varphi_b(b') = \delta_{b,b'}$

"you have a basis iff you have a dual basis"

pf: $\sum_{b \in B} \alpha_b b = 0 \Rightarrow 0 = \varphi_{b'}\left(\sum_{b \in B} \alpha_b b\right)$

$= \alpha_{b'} \quad \forall b' \in B$

$\Rightarrow \alpha_{b'} = 0 \quad \forall b' \in B$ $\square$

$f : V_1 \times \cdots \times V_r$

$\downarrow$

Thm: B_i basis for V_i $\forall i$ $\Rightarrow B_1 \times \cdots \times B_r \xrightarrow{\sim}$ basis B for $T := V_1 \otimes \cdots \otimes V_r$

$(b_1, \ldots, b_r) \mapsto b_1 \otimes \cdots \otimes b_r$

E.g. $\mathbb{R}^2 \otimes \mathbb{R}^3$ has basis $\begin{matrix} e_1\otimes e_1 & e_1\otimes e_2 & e_1 \otimes e_3 \\ e_2\otimes e_1 & e_2\otimes e_1 & e_2\otimes e_1 \end{matrix}$

pf: multilinear map f on $\prod V_i$ is determined by values on $\prod B_i$.

[Universal prop redux: $\{\text{multilinear } \prod V_i \to W\} \longleftrightarrow \{\text{liner } \otimes V_i \to W\}$]

$v_1 = \sum_j \alpha_j b_j^1 \Rightarrow f(v_1, \underbrace{\ldots \text{stuff}\ldots}) = \sum_j \alpha_j f(b_j^1, \underbrace{\ldots \text{stuff} \ldots})$

jth basis element of B_1

and similarly for $i \neq 1$

Thus $B = \{b_1 \otimes \cdots \otimes b_r \mid b_i \in B_i \ \forall i\}$ spans T

Suppose $f_i : V_i \to f$ $\forall i$. Set $f : \prod V_i \to f$

$$\begin{matrix} (v_1, \ldots, v_r) \mapsto f_1(v_1)\cdots f_r(v_r) \\ \downarrow \\ v_1 \otimes \cdots \otimes v_r \qquad \tilde{f} = f_1 \otimes \cdots \otimes f_r \end{matrix}$$

f multilinear $\Rightarrow$ induces $\tilde{f} : T \to f$

Take $f_i = b_i^\vee \in B_i^*$ dual basis, so

$b_i^*(b_i) = 1$ but $b_i^*(b_i') = 0$ when $b_i' \in B_i \setminus \{b_i\}$

Then $b \in B \rightsquigarrow \tilde{f}_b : T \to f$

$b \mapsto 1$

$b' \mapsto 0$ for $b' \in B \setminus \{b\}$ $\longrightarrow \{\tilde{f}_b : T \to f\}_{b \in B}$ is dual basis of T

use lemma $\boxed{2}$

Cor: $\dim(V_1 \otimes \cdots \otimes V_r) = (\dim V_1)\cdots(\dim V_r)$ $\boxed{1}$

E.g. $v \in \mathbb{R}^4$ $w \in \mathbb{R}^2 \Rightarrow v \otimes w =$

$$\begin{bmatrix} \pi \\ -e \\ 0 \\ \sqrt{2}^i \end{bmatrix} \otimes \begin{bmatrix} 6\cdot 10^6 \\ i^i \end{bmatrix} = 6\pi \cdot 10^6 e_1 \otimes e_1 + \pi i^i e_1 \otimes e_2 - e \cdot 6 \cdot 10^6 e_2 \otimes e_1 - e i^i e_2 \otimes e_2 + 0 e_3 \otimes (e_1 + e_2) + \sqrt{2}^i \cdot 6 \cdot 10^6 e_4 \otimes e_1 + \sqrt{2}^i i^i e_4 \otimes e_2$$

Exterior Algebra

Def: An alternating operation $\varphi : \underbrace{V \times \cdots \times V}_{r} = V^{\otimes r} \to W$ is a multilinear map such that $v_1, \ldots, v_r$ linearly dep. $\Rightarrow \varphi(v_1, \ldots, v_r) = 0$

E.g. volume of a parallelepiped on $v_1,\dots,v_n$ in $\mathbb{R}^n$

vol = det:
- Volume $= 0$ if dependent (dim of thing $= n-1$ or less)
- $v_i \rightsquigarrow \alpha v_i \Rightarrow \text{vol} \rightsquigarrow \alpha\,\text{vol}$
- $v_i \rightsquigarrow v_i + v_j$ for $j \neq i \Rightarrow$ vol unchanged
- $\text{vol}(e_1,\dots,e_n) = 1$

Def: the r^{th} exterior power of V is a universal such alternating operator:

alternating map $\bigwedge^r : V \to U$ s.t. $\forall\, \varphi$ alternating $\exists!\, \tilde{\varphi}$ linear with $\varphi = \tilde{\varphi} \circ \bigwedge^r$

$$\begin{tikzcd} V \arrow[r, "\bigwedge^r"] \arrow[dr, "\varphi"'] & U \arrow[d, dashed, "\exists!\,\tilde{\varphi}"] \\ & W \end{tikzcd}$$

Thm: $\bigwedge^r$ exists.

pf: set $U = \bigwedge^r V = V^{\otimes r} \Big/ \text{span}\left(v_1 \otimes \cdots \otimes v_r \;\middle|\; \text{two of the } v_i \text{ are equal}\right)$

$$\begin{tikzcd} V^{\times r} \arrow[r, "ml"] \arrow[dr, "ml"'] & V^{\otimes r} \arrow[d, "\text{linear}"] \\ & \bigwedge^r V \end{tikzcd}$$

$V^{\times r} \to \bigwedge^r V$

$(v_1,\dots,v_r) \mapsto v_1 \wedge \cdots \wedge v_r$

- multilinear b/c factors through $V^{\otimes r}$.
- Alternating because $v_1 = \sum_{j\neq 1} \alpha_j v_j \Rightarrow v_1 \wedge \cdots \wedge v_r$

$$= \sum_{j\neq i} (\alpha_j v_j) \wedge v_2 \cdots \wedge v_r$$
$$= \sum_{j \neq i} \alpha_j \underbrace{(v_j \wedge v_2 \wedge \cdots \wedge v_r)}_{\text{at least one repeat by pigeonhole} \Rightarrow = 0}$$
$$= 0$$

$\square$

E.g. $v, w \in \mathbb{R}^3$

$$\begin{bmatrix} i^i \\ \pi \\ e \end{bmatrix} \wedge \begin{bmatrix} g \\ \hbar \\ 0 \end{bmatrix} = i^i \hbar\, e_1 \wedge e_2 + \pi g\, e_2 \wedge e_1 + 0\, e_1 \wedge e_3 + eg\, e_3 \wedge e_1 + 0\, e_2 \wedge e_3 + e\hbar\, e_3 \wedge e_2$$
$$= (i^i \hbar - \pi g)\, e_1 \wedge e_2 + (0 - eg)\, e_1 \wedge e_3 + (0 - e\hbar)\, e_2 \wedge e_3$$

$$0 = (e_i + e_j) \wedge (e_i + e_j)$$
$$= \cancel{e_i \wedge e_i} + e_i \wedge e_j + e_j \wedge e_i + \cancel{e_j \wedge e_j}$$
$$= e_i \wedge e_j + e_j \wedge e_i$$

Prop: $V \xrightarrow{\varphi} W$ linear induces canonical linear map
$$\bigwedge^r V \xrightarrow{\bigwedge^r \varphi} \bigwedge^r W$$
("$\bigwedge^r$ is a functor")

pf: HW5, including the entries of matrix if A is given

<u>Quintessential example</u> $V = W$ and $r = n = \dim V$

<u>determinant</u> of $\varphi : V \to V$ is $\det \varphi = \Lambda^n \varphi$

$$\varphi(e_j) = \sum_{i=1}^n a_{ij} e_i = v_j$$

$$\Rightarrow \Lambda^r \varphi(e_1 \wedge \cdots \wedge e_n)$$

$$= v_1 \wedge \cdots \wedge v_n$$

$$= \left(\sum_{i=1}^n a_{i1} e_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^n a_{in} e_i\right) = \underbrace{}_{\det A} e_1 \wedge \cdots \wedge e_n$$

$$= \sum_{i_1, \dots, i_n} a_{i_1 1} e_{i_1} \wedge \cdots \wedge a_{i_n n} e_{i_n}$$

terms are zero unless $i_1, \dots, i_n$ distinct

so $i_j = \pi(j)$ for some $\pi \in S_n$

$$v_1 \wedge \cdots \wedge v_n = \sum_{\pi \in S_n} a_{\pi(1)1} e_{\pi(1)} \wedge \cdots \wedge a_{\pi(n)n} e_{\pi(n)}$$

$$= \sum_{\pi \in S_n} a_{\pi(1)1} \cdots a_{\pi(n)n} \underbrace{e_{\pi(1)} \wedge \cdots \wedge e_{\pi(n)}}_{(-1)^\pi \underbrace{e_1 \wedge \cdots \wedge e_n}_{e_{[n]}}}$$

$\det A$ $\qquad [n] = \{1, \dots, n\}$

Notation: for $\sigma = \{\sigma_1 < \cdots < \sigma_r\} \subseteq [n]$ and

$E = e_1, \dots, e_n \in V$ set

$e_\sigma = e_{\sigma_1} \wedge \cdots \wedge e_{\sigma_r}$ and $\Lambda^r E = \left\{ e_\sigma \mid \sigma \in \binom{[n]}{r} \right\}$

<u>Thm</u> E basis for $V \Rightarrow \Lambda^r E$ basis for $\Lambda^r V$.

04/01

Notation: for $\sigma = \{\sigma_1 < \cdots < \sigma_r\} \subseteq [n] = \{1, \dots, n\}$

and $E = e_1, \dots, e_n \in V$

set $e_\sigma = e_{\sigma_1} \wedge \cdots \wedge e_{\sigma_r}$

and $\Lambda^r E = \left\{ e_\sigma \mid \sigma \in \binom{[n]}{r} \right\}$

<u>Thm</u>: E is a basis for $V \Rightarrow \Lambda^r E$ is a basis for $\Lambda^r V$.

<u>Prop</u>: E is a basis for $V \Rightarrow \Lambda^r E$ spans $\Lambda^r V$

<u>Pf</u>: $\Lambda^r E = \{$squarefree elements of basis $E^{\otimes r}$ of $V^{\otimes r}\}$

and non squarefree elements $\mapsto 0$ in $\Lambda^r V$ under

$$\begin{tikzcd} V^{\otimes r} \arrow[r] \arrow[d] & \Lambda^r V \\ \operatorname{span} \Lambda^r E \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

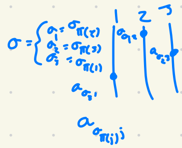
inj., surj.

Q: coeff on e_σ in $v_1 \wedge \cdots \wedge v_r$ = ?

A: $\det A_\sigma$ where $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_r \\ | & & | \end{bmatrix}$

A_σ takes $\boxed{\text{rows}}$ indexed by σ

Pf: In row form σ, get one e_i from each col j with coef a_{ij}



$$\begin{aligned} v_1 \wedge \cdots \wedge v_r &= \sum_{\sigma \in \binom{[n]}{r}} \sum_{\pi \in S_r} a_{\sigma_{\pi(1)} 1} e_{\sigma_{\pi(1)}} \wedge \cdots \wedge a_{\sigma_{\pi(r)} r} e_{\sigma_{\pi(r)}} \\ &= \sum_\sigma \sum_\pi a_{\sigma_{\pi(1)} 1} \cdots a_{\sigma_{\pi(r)} r} \, e_{\sigma_{\pi(1)}} \wedge \cdots \wedge e_{\sigma_{\pi(r)}} \\ &= \sum_\sigma \sum_\pi a_{\sigma_{\pi(1)} 1} \cdots a_{\sigma_{\pi(r)} r} (-1)^\pi e_\sigma \\ &= \sum_\sigma \det A_\sigma e_\sigma \end{aligned}$$

Pf of thm $\overset{r}{\Lambda} \in$ spans by prop.

Indepdt: existence of dets

$\Rightarrow$ $(v_1, \ldots, v_r) \mapsto \det A_\sigma$ is alternating (nice!)

so it induces linear map

$e_\sigma^* : \overset{r}{\Lambda} V \to F$ with

$$e_\sigma^*(e_\tau) = \begin{cases} 1 & \text{if } \sigma = \tau \\ 0 & \text{if } \sigma \neq \tau \end{cases}$$

$= \delta_{\sigma,\tau}$, lemma $\Rightarrow$ indep ☒

Cor: $\dim V = n \Rightarrow \dim \overset{r}{\Lambda} V = \binom{n}{r}$

$\left(= 0 \text{ if } r > n \right)$

Prop: $v \in V \Rightarrow v \wedge : \overset{r}{\Lambda} V \to \overset{r+1}{\Lambda} V$ linear

$\omega \in \overset{j}{\Lambda} V \Rightarrow \omega \wedge : \overset{r}{\Lambda} V \to \overset{r+j}{\Lambda} V$ linear

$$(\omega_1 \wedge) \circ (\omega_2 \wedge) = (\omega_1 \wedge \omega_2 \wedge) : \overset{r}{\Lambda} V \to \overset{r+j+k}{\Lambda} V$$

if $\omega_1 \in \overset{j}{\Lambda} V$ $\omega_2 \in \overset{k}{\Lambda} V$

(Rmk) $\Rightarrow$ $\Lambda V = \bigoplus_r \overset{r}{\Lambda} V$ is a ring

$v, w \in \mathbb{R}^3_{\text{row}}$, $v \times w = u$ whose entries are coeffs on i j k in $\det\begin{pmatrix} i & j & k \\ - & v & - \\ - & w & - \end{pmatrix}$

$u \cdot v = 0 = u \cdot w$

$\|u\|$ = area of parallelogram 

Def: $\mathrm{vol}(v_1,\dots,v_r) = \left|\mathrm{vol}(v_1,\dots,v_r,u_{r+1},\dots,u_n)\right|$

for any orthonormal basis $u_{r+1},\dots,u_n$ of $\{v_1,\dots,v_r\}^\perp$

Def: The cross product u of $v_2,\dots,v_n \in \mathbb{R}^n$

satisfies $v_j \cdot u = 0 \ \forall j = 2,\dots,n$

and $\|u\| = \mathrm{vol}(v_2,\dots,v_n)$

Thm: $u = \sum_{i=1}^n u_i e_i \Rightarrow -v_2 \wedge \cdots \wedge v_n$

omit e_i

has coeff $(-1)^i u_i$ on $e_{\bar{i}} = e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_n$

Pf: $v_2 \wedge \cdots \wedge v_n \in \operatorname{span}(e_{\bar{1}},\dots,e_{\bar{n}}) \Rightarrow$

it is $-\sum_{i=1}^n (-1)^i u_i e_{\bar{i}}$ for some u.

$$w \wedge v_2 \wedge \cdots \wedge v_2 = \left(\sum_{i=1}^n w_i e_i\right) \wedge \left(-\sum_{i=1}^n (-1)^i u_i e_{\bar{i}}\right)$$

$$= \sum_{i=1}^n -w_i u_i (-1)^i \, e_i \wedge e_{\bar{i}}$$

$(-1)^{i-1}$ to put it in place

$$= w \cdot u \ e_{[n]}$$

$w = v_j \Rightarrow w \cdot u \ e_{[n]} = 0 \Rightarrow v_j \cdot u = 0 \ \forall j \geq 2$

let $w \in \{v_2,\dots,v_n\}^\perp$ and $\|w\| = 1$. Then

$$\mathrm{vol}(v_2,\dots,v_n) = \left|\mathrm{vol}(w, v_2,\dots,v_n)\right|$$

$$= \left|\text{coeff on } e_{[n]} \text{ in } w \wedge v_2 \wedge \cdots \wedge v_n\right|$$

$$= |w \cdot u| = \frac{u}{\|u\|} \cdot u = \frac{\|u\|^2}{\|u\|} = \|u\| \quad \boxtimes$$

Laplace expansion

along row 1: 

$A[\leq r]$ = top r rows

$A[> r]$ = bottom $n-r$ rows

$A[\leq r]^\sigma$ = $r \times r$ submatrix in cols $\sigma \in \binom{[n]}{r}$

$A[> r]^{\bar{\sigma}} = (n-r) \times (n-r)$ ...

$\bar{\sigma} = [n] \setminus \sigma$

$(-1)^{\#\text{ swaps to put } \sigma, \bar\sigma \text{ in order}}$ $\quad e_\sigma \wedge e_{\bar\sigma} = (-1)^\sigma e_{[n]}$

Thm $\det A = \sum_{\sigma \in \binom{[n]}{r}} (-1)^\sigma \det A[\le r]^{\sigma} \det[>r]^{\bar\sigma}$

pf: $\det A =$ coeff on $e_{[n]}$ in $v_1 \wedge \cdots \wedge v_n$

where $A = \begin{bmatrix} - v_1 - \\ \vdots \\ - v_n - \end{bmatrix}$

factor $v_1 \wedge \cdots \wedge v_n$: (coeff on e_σ in $v_1 \wedge \cdots \wedge v_r$) $= \det A[\le r]^\sigma$

(" $e_{\bar\sigma}$ " $v_{r+1} \wedge \cdots \wedge v_n$) $= \det A[>r]^{\bar\sigma}$

$\Rightarrow v_1 \wedge \cdots \wedge v_n = (v_1 \wedge \cdots \wedge v_r) \wedge (v_{r+1} \wedge \cdots \wedge v_n)$

$= \left(\sum_\sigma \det A[\le r]^\sigma e_\sigma\right) \wedge \left(\sum_{\bar\sigma} \det A[>r]^{\bar\sigma} e_{\bar\sigma}\right)$ ☒

Thm $\operatorname{rank} A < r \iff$ all $\underbrace{\text{minors}}_{\det(r\times r \text{ submatrix})}$ of size r vanish

Pf: $A \in F^{m\times n}$ represents $\varphi: F^n \to F^m$.

$\operatorname{rank}\varphi < r \iff \dim(\operatorname{im}\varphi) < r$

$\iff \Lambda^r \varphi = 0$ ☒

entries are r-minors of A.

Thm: $v_1 \wedge \cdots \wedge v_r$ identifies $V = \operatorname{span}(v_1, \dots, v_r)$ up to scalar factor

if $v_1, \dots, v_r$ are indep for some $\alpha \neq 0$

$\iff \underset{W}{\operatorname{span}(w_1, \dots, w_r)} = \underset{V}{\operatorname{span}(v_1, \dots, v_r)}$

pf: $W = V \Rightarrow \begin{bmatrix} - w_1 - \\ \vdots \\ - w_r - \end{bmatrix} = A \begin{bmatrix} - v_1 - \\ \vdots \\ - v_r - \end{bmatrix}$ for some $A \in GL_r$.

$\Rightarrow w_1 \wedge \cdots \wedge w_r = \det A \, v_1 \wedge \cdots \wedge v_r$

Now sps $w_1 \wedge \cdots \wedge w_r = \alpha v_1 \wedge \cdots \wedge v_r \neq 0$.

Then $w_i \wedge v_1 \wedge \cdots \wedge v_r = \alpha^{-1} w_i \wedge w_1 \wedge \cdots \wedge w_r = 0 \;\forall i$

$\Rightarrow w_i \in V \;\forall i$ ☒

Def: $\left\{ \underbrace{e_\sigma^*(v_1 \wedge \cdots \wedge v_r)}_{\text{coeff on } e_\sigma \text{ in } v_1 \wedge \cdots \wedge v_r} \;\middle|\; \sigma \in \binom{[n]}{r} \right\}$ Plücker coordinates

Cor: $G_r(F^n) = \left\{ \text{decomposable forms in } \mathbb{P}\Lambda^r F^n = G_1(\Lambda^r F^n) \right\}$

algebraic condition/ closed level set