Linear Algebra

Theorem 2

Let $(V; F; +; \cdot)$ be a vector space over the field F. Let n $\in \mathbb{N}^*$ and U$u_1, \vec{u_2}...u_n$ be a set of vectors $\in$ V. Then span(U) is a subspace of V.

Proof. Let span(U) be the set of vectors which can be written as a linear combination of the vectors in U

- $\vec{0} \in$ span(u) because $\vec{0} = \overbrace{\lambda_1}^{0} \cdot \vec{u_1} + \overbrace{\lambda_2}^{0} \cdot \vec{u_2} + \overbrace{\lambda_n}^{0} \cdot \vec{u_n}$
- Let $\vec{u}, \vec{w} \in$ span(U) and let $\mu \in$F
  We want to show that $\vec{u} + \mu \cdot \vec{w} \in$ span(U)
  Let $\vec{u}$=$\alpha_1 \vec{u_1} + ...\alpha_n \vec{u_n}$ $\quad \vec{w} = \beta_1 u_1 ... \vec{\beta_n} \vec{u_n}$
  $\vec{u} + \mu\vec{w} = \alpha_1 \vec{u_1} + ... + \alpha_n \vec{u_n} + \mu \cdot (\beta_1 \vec{u_1} + ... + \beta_n \vec{u_n})$=
  =$(\alpha_1 + \mu\beta_1) \cdot \vec{u_1}$+...+$(\alpha_n + \mu\beta_n) \cdot \vec{u_n}$ $\quad \in$ span(U) $\Rightarrow$ span(U) is a subset of V.

□

Theorem 3

Uniqueness of representation. *Let $(V; F; +; \cdot)$ be a vector space over the field F. Let $B = \{\vec{b_1}, \vec{b_2}, ..., \vec{b_n}\}$ be a set of vectors $\in$ V. Then B is a basis of V if and only if every vector $\vec{v} \in$ V can be written uniquely as a linear combination of the basis vectors of B.*

Proof. Let B = $\{\vec{b_1}, \vec{b_2}, ..., \vec{b_n}$ be a set of vectors in V.
$\rightarrow$Let B be a basis of V.
We want to show that every vector $\vec{v} \in$ V can be written uniquely as a linear combination of the vectors of B.
Assume that $\vec{v}$ can't be written uniquely as a linear combination of the vectors of B:
Since B is a basis of V, then $\vec{v}$ can be written as a linear combination of the vectors of B.
$\vec{v}$ = $\lambda_1 \cdot \vec{b_1}$ +...+ $\lambda_n \cdot \vec{b_n}$ and $\vec{v}$ = $\alpha_1 \cdot \vec{b_1}$+...+$\alpha_n \cdot \vec{b_n}$
$\vec{v}$ - $\vec{v}$=$(\lambda_1 \cdot \vec{b_1}$+...+$\lambda_n \cdot \vec{b_n})$-$(\alpha_1 \cdot \vec{b_1}$+...+$\alpha_n \cdot \vec{b_n})$=$\vec{0}$
$\iff$ $\vec{0}$=$(\lambda_1 - \alpha_1) \cdot \vec{b_1}$+...+$(\lambda_1 - \alpha_n) \cdot \vec{b_n}$ // Since B is a basis of V, them $\vec{b_1}...\vec{b_1}$ are linearly independent which implies:
$\lambda_1 - \alpha_1 = 0, ..., \lambda_n - alpha_n = 0 \iff \lambda_1 = \alpha_1, ..., \lambda_n = \alpha_n$
which is a contradiction, since it proves that any vector $\vec{v} \in$ V is written uniquely as a linear combination of the basis vectors of B.

$\leftarrow$ Assume that every vector $\vec{v} \in$ V can be written uniquely as a linear combination of the vectors of B.
We want to show that B is a basis of V
Let $\vec{v} \in$ span(B): $\vec{v} = \alpha_1 \cdot \vec{b_1} + ... + \alpha_n \cdot \vec{b_n} \in$ V since V is a vector space and closed under vector addition and scalar multiplication. Hence span(B)=V since every vector $\vec{v} \in$ V can be written as a linear combination of the vectors of B by hypothesis.

$$\begin{cases} \vec{0} = \lambda_1 \cdot \vec{b_1} + ... + \lambda_n \cdot \vec{b_n} \\ \vec{0} = 0 \cdot \vec{b_1} + ... + 0 \cdot \vec{b_n} \end{cases}$$

Since we assumed that there is only one way to write every vector $\vec{v} \in$V, then $\lambda_1 = 0 ... \lambda_n = 0$ It proves that the vectors of the set B are linearly independent. $\Rightarrow$ B is a basis of V. □

Theorem 5

Kernel. *Let f: V→W be a linear map. Then ker(f) is a subspace of V.*

Proof. Let f: V→W be a linear map.
f($\vec{u}+\lambda\vec{v}) = f(\vec{u}) + \lambda \cdot f(\vec{v}) \qquad \forall \vec{u}, \vec{v} \in V$
Since V is a vector space,then V $\neq \emptyset$ by definition.
$\Rightarrow \exists \vec{u} \in V$ and since V is a vector space, there is closure under scalar multiplication, ie $\lambda \cdot \vec{u} \in V$ $\forall \lambda \in$
$\Rightarrow 0 \cdot \vec{u} \in V \iff \vec{0} \in V$
1) We want to show that Ker(f) $\neq \emptyset$ by showing that $\vec{0} \in Ker(f)$

$$\text{f}(\underbrace{\vec{0}}_{\in \mathbf{V}})=\text{f}(0 \cdot\vec{u}) \overbrace{=}^{\mathbf{f = lin.\ map}} 0 \cdot f(\vec{u})=\underbrace{\vec{0}}_{\in \mathbf{W}} \Rightarrow \vec{0} \in ker(f)$$

2) Let $\vec{u}, \vec{v} \in ker(f), \lambda \in \mathbb{R}$
Since $\vec{u}$, $\vec{v} \in$ ker(f) $\Rightarrow$ f($\vec{u}$)=$\vec{0}$ and f($\vec{v}$)=0

$$\text{f}(\vec{u}+\lambda\vec{v}) \underbrace{=}_{\mathbf{f = lin.\ map}} \text{f}(\vec{u})+\lambda \cdot f(\vec{v})=\vec{0}+\lambda\vec{0}=\vec{0} \Rightarrow \vec{u}+\lambda v \in ker(f)$$

Since $\vec{0} \in$ ker(f) and $\vec{u}+\lambda\vec{v} \in$ ker(f) $\forall \vec{u}, \vec{v} \in$ ker(f)
$\Rightarrow$ ker(f) is a subspace of V

□

Theorem 6

Range. *Let $f: V \to W$ be a linear map. Then Im(f) is a subspace of W.*

Proof.
1) We have show that $\text{f}(\vec{0})=\underbrace{\vec{0}}_{\in W}$

$\vec{0} \in$ Im(f) since $\exists \vec{0} \in V$ such that f($\vec{0}$)=$\vec{0}$

2) Let $\vec{w_1}, \vec{w_2} \in Im(f)$
$\exists \vec{u_1}, \vec{u_2} \in$V such that f($\vec{u_1}$)=$\vec{w_1}$ and f($\vec{u_2}$)=$\vec{w_2}$
Let $\lambda \in$R: We want to show that $\vec{w_1}+\lambda\vec{w_2} \in$ Im(f)

$$\vec{w_1}+\lambda \cdot \vec{w_2}=\text{f}(\vec{u_1})+\lambda \cdot f(\vec{u_2}) \overbrace{=}^{f=lin.map} \underbrace{f(\vec{u_1}+\lambda \cdot \vec{u_2})}_{\in\text{V(since V is a vec. space)}}$$

Since $\exists \vec{u_1}+\vec{u_2} \in$ V such that
f($\vec{u_1}+\lambda\vec{v_2}$)=$\vec{w_1}+\lambda\vec{w_2}$, then $\vec{w_1}+\lambda\vec{w_2} \in$Im(f)
$\Rightarrow$Im(f) is a subspace of W

□