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THE CALCULI OF

LAMBDA-CONVERSION

BY

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CONTENTS

Chapter I

INTRODUCTORY

1. THE CONCEPT OF A FUNCTION. Underlying the formal cal-
culi which we shall develop is the concept of a function, as it
appears in various branches of mathematics, either under that
name or under one of the synonymous names, "operation" or "trans-
formation." The study of the general properties of functions,
independently of their appearance in any particular mathematical
(or other) domain, belongs to formal logic or lies on the boun-
dary line between logic and mathematics. This study is the orig-
inal motivation for the calculi — but they are so formulated
that it is possible to abstract from the intended meaning and
regard them merely as formal systems.

A <u>function</u> is a rule of correspondence by which when any-
thing is given (as <u>argument</u>) another thing (the <u>value</u> of the
function for that argument) may be obtained. That is, a func-
tion is an operation which may be applied on one thing (the ar-
gument) to yield another thing (the value of the function). It
is not, however, required that the operation shall necessarily
be applicable to everything whatsoever; but for each function
there is a class, or range, of possible arguments -- the class
of things to which the operation is significantly applicable --
and this we shall call the <u>range of arguments</u>, or <u>range of the
independent variable</u>, for that function. The class of all values
of the function, obtained by taking all possible arguments, will
be called the <u>range of values</u>, or <u>range of the dependent variable</u>.

If f denotes a particular function, we shall use the nota-
tion $(f\alpha)$ for the value of the function f for the argument
α. If α does not belong to the range of arguments of f, the
notation $(f\alpha)$ shall be meaningless.

It is, of course, not excluded that the range of arguments
or range of values of a function should consist wholly or partly
of functions. The derivative, as this notion appears in the el-

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ementary differential calculus, is a familiar mathematical exam-
ple of a function for which both ranges consist of functions.
Or, turning to the integral calculus, if in the expression
$\int_0^1 (fx)dx$ we take the function f as independent variable, we
are led to a function for which the range of arguments consists
of functions and the range of values, of numbers. Formal logic
provides other examples; thus the existential quantifier, accord-
ing to the present account, is a function for which the range of
arguments consists of propositional functions, and the range of
values consists of truth-values.

In particular it is not excluded that one of the elements of
the range of arguments of a function f should be the function
f itself. This possibility has frequently been denied, and in-
deed, if a function is defined as a correspondence between two
previously given ranges, the reason for the denial is clear.
Here, however, we regard the operation or rule of correspondence,
which constitutes the function, as being first given, and the
range of arguments then determined as consisting of the things to
which the operation is applicable. This is a departure from the
point of view usual in mathematics, but it is a departure which
is natural in passing from consideration of functions in a spec-
ial domain to the consideration of function in general, and it
finds support in consistency theorems which will be proved below.

The identity function I is defined by the rule that (Ix)
is x, whatever x may be; then in particular (II) is I. If
a function H is defined by the rule that (Hx) is I, what-
ever x may be, then in particular (HH) is I. If Σ is the
existential quantifier, then $(\Sigma\Sigma)$ is the truth-value truth.

The functions I and H may also be cited as examples of
functions for which the range of arguments consists of all things
whatsoever.

2. EXTENSION AND INTENSION. The foregoing discussion
leaves it undetermined under what circumstances two functions
shall be considered the same.

The most immediate and, from some points of view, the best
way to settle this question is to specify that two functions f
and g are the same if they have the same range of arguments and,
for every element a that belongs to this range, (fa) is the

same as $(g\alpha)$. When this is done we shall say that we are deal-
ing with <u>functions in extension.</u>

It is possible, however, to allow two functions to be dif-
ferent on the ground that the rule of correspondence is differ-
ent in meaning in the two cases although always yielding the same
result when applied to any particular argument. When this is
done we shall say that we are dealing with <u>functions in inten-
sion.</u> The notion of difference in meaning between two rules of
correspondence is a vague one, but, in terms of some system of
notation, it can be made exact in various ways. We shall not at-
tempt to decide what is the true notion of difference in meaning
but shall speak of functions in intension in any case where a
more severe criterion of identity is adopted than for functions
in extension. There is thus not one notion of function in inten-
sion, but many notions, involving various degrees of intensional-
ity.

In the calculus of λ-conversion and the calculus of re-
stricted λ-*K*-conversion, as developed below, it is possible, if
desired, to interpret the expressions of the calculus as denoting
functions in extension. However, in the calculus of λ-δ-conver-
sion, where the notion of identity of functions is introduced in-
to the system by the symbol   δ,   it is necessary, in order to
preserve the finitary character of the transformation rules, so
to formulate these rules that an interpretation by functions in
extension becomes impossible. The expressions which appear in
the calculus of λ-δ-conversion are interpretable as denoting
functions in intension of an appropriate kind.

3. FUNCTIONS OF SEVERAL VARIABLES. So far we have tacitly
restricted the term "function" to functions of one variable (or,
of one argument). It is desirable, however, for each positive
integer n, to have the notion of a function of n variables.
And, in order to avoid the introduction of a separate primitive
idea for each n, it is desirable to find a means of explaining
functions of n variables as particular cases of functions of
one variable. For our present purpose, the most convenient and
natural method of doing this is to adopt an idea of Schönfinkel
[49], according to which a function of two variables is regarded
as a function of one variable whose values are functions of one

variable, a function of three variables as a function of one va-
riable whose values are functions of two variables, and so on.

Thus if f denotes a particular function of two variables,
the notation $((fa)b)$ -- which we shall frequently abbreviate as
(fab) or fab -- represents the value of f for the arguments
a,b . The notation (fa) -- which we shall frequently abbreviate
as fa -- represents a function of one variable, whose value for
any argument x is fax . The function f has a range of argu-
ments, and the notation fa is meaningful only when a belongs
to that range; the function fa again has a range of arguments,
which is, in general, different for different elements a , and
the notation fab is meaningful only when b belongs to the
range of arguments of fa .

Similarly, if f denotes a function of three variables,
$(((fa)b)c)$ or $fabc$ denotes the value of f for the arguments
a,b,c , fa denoting a certain function of two variables, and .
$((fa)b)$ or fab denoting a certain function of one variable --
and so on.

(According to another scheme, which is the better one for
certain purposes, a function of two variables is regarded as a
function (of one variable) whose arguments are ordered pairs, a
function of three variables as a function whose arguments are
ordered triads, and so on. This other concept of a function of
several variables is not however, excluded here. For, as will
appear below, the notions of ordered pair, ordered triad, etc.,
are definable by means of abstraction (§4) and the Schönfinkel
concept of a function of severable variables; and thus functions
of several variables in the other sense are also provided for.)

An example of a function of two variables (in the sense of
Schönfinkel) is the underline{constancy function K , defined by the rule
that Kxy is x , whatever x and y may be. We have, for in-
stance that KII is I , KHI is H , and so on. Also KI is
H (where H is the function defined above in §1). Similarly
KK is a function whose value is constant and equal to K .

Another example of a function of two variables is the func-
tion whose value for the arguments f , x is (fx) ; for rea-
sons which will appear later we designate this function by the
symbol 1 . The function 1 , regarded as a function of one vari-
able, is a kind of identity function, since the notation $(1f)$

whenever significant, denotes the same function as f; the func-
tions I and 1 are not, however, the same function, since the
range of arguments consists in one case of all things whatever,
in the other case merely of all functions.

Other examples of functions of two or more variables are the
function H, already defined, and the functions T, J, B, C, W,
S, defined respectively by the rules that Txf is (fx), $Jfxyz$
is $fx(fzy)$, $Bfgx$ is $f(gx)$, $Cfxy$ is (fyx), Wfx is (fxx),
$Snfx$ is $f(nfx)$.

Of these, B and C may be more familiar to the reader un-
der other names, as the product or resultant of two transforma-
tions f and g, and as the converse of a function of two vari-
ables f. To say that BII is I is to say that the product
of the identity transformation by the identity transformation is
the identity transformation, whatever the domain within which
transformations are being considered; to say that $B11$ is 1
is to say that within any domain consisting entirely of functions
the product of the identity transformation by itself is the iden-
tity transformation. BI is 1, since it is the operation of
composition with the identity transformation, and thus an iden-
tity operation, but one applicable only to transformations.

The reader may further verify that CK is H, CT is 1,
$C1$ is T, CI is T -- that 1 and I have the same converse
is explained by the fact that, while not the same function, they
have the same effect in all cases where they can significantly
be applied to two arguments. The function BCC, the converse
of the converse, has the effect of an identity when applied to a
function of two variables, but when applied to a function of one
variable it has the effect of so restricting the range of argu-
ments as to transform the function into a function of two vari-
ables (if possible); thus $BCCI$ is 1.

There are many similar relations between these functions,
some of them quite complicated.

4. ABSTRACTION. For our present purpose it is necessary
to distinguish carefully between a symbol or expression which
denotes a function and an expression which contains a variable
and denotes ambiguously some value of the function -- a distinc-
tion which is more or less obscured in the usual language of

mathematical function theory.

To take an example from the theory of functions of natural numbers, consider the expression $(x^2+x)^2$. If we say, "$(x^2+x)^2$ is greater than 1,000," we make a statement which depends on x and actually has no meaning unless x is determined as some particular natural number. On the other hand, if we say, "$(x^2+x)^2$ is a primitive recursive function," we make a definite statement whose meaning in no way depends on a determination of the variable x (so that in this case x plays the rôle of an apparent, or bound, variable). The difference between the two cases is that in the first case the expression $(x^2+x)^2$ serves as an ambiguous, or variable, denotation of a natural number, while in the second case it serves as the denotation of a particular function. We shall hereafter distinguish by using $(x^2+x)^2$ when we intend an ambiguous denotation of a natural number, but $(\lambda x(x^2+x)^2)$ as the denotation of the corresponding function -- and likewise in other cases.

(It is, of course, irrelevant here that the notation $(x^2+x)^2$ is commonly used also for a certain function of real numbers, a certain function of complex numbers, etc. In a logically exact notation the functions, addition of natural numbers, addition of real numbers, addition of complex numbers, would be denoted by different symbols, say $+_n$, $+_r$, $+_c$; and the three functions, square of a natural number, square of a real number, square of a complex number, would be similarly distinguished. The uncertainty as to the exact meaning of the notation $(x^2+x)^2$, and the consequent uncertainty as to the range of arguments of the function $(\lambda x(x^2+x)^2)$, would then disappear.)

In general, if M is an expression containing a variable x (as a free variable, i.e., in such a way that the meaning of M depends on a determination of x), then $(\lambda x M)$ denotes a function whose value, for an argument a, is denoted by the result of substituting (a symbol denoting) a for x in M. The range of arguments of the function $(\lambda x M)$ consists of all objects a such that the expression M has a meaning when (a symbol denoting) a is substituted for x.

If M does not contain the variable x (as a free variable), then $(\lambda x M)$ might be used to denote a function whose value is constant and equal to (the thing denoted by) M, and whose

range of arguments consists of all things. This usage is contemplated below in connection with the calculi of λ-K-conversion, but is excluded from the calculi of λ-conversion and λ-δ-conversion -- for technical reasons which will appear.

Notice that, although x occur as a free variable in M, nevertheless, in the expression $(\lambda x M)$, x is a bound, or apparent, variable. Example: the equation $(x^2+x)^2 = (y^2+y)^2$ expresses a relation between the natural numbers denoted by x and y and its truth depends on a determination of x and of y (in fact, it is true if and only if x and y are determined as denoting the same natural number); but the equation $(\lambda x(x^2+x)^2) = (\lambda y(y^2+y)^2)$ expresses a particular proposition -- namely that $(\lambda x(x^2+x)^2)$ is the same function as $(\lambda y(y^2+y)^2)$ -- and it is true (there is no question of a determination of x and y).

Notice also that λ, or λx, is not the name of any function or other abstract object, but is an <u>incomplete symbol</u> -- i.e., the symbol has no meaning alone, but appropriately formed expressions containing the symbol have a meaning. We call the symbol λx an <u>abstraction operator</u>, and speak of the function which is denoted by $(\lambda x M)$ as obtained from the expression M by <u>abstraction</u>.

The expression $(\lambda x(\lambda y M))$, which we shall often abbreviate as $(\lambda xy.M)$, denotes a function whose value, for an argument denoted by x, is denoted by $(\lambda y M)$ -- thus a function whose values are functions, or a function of two variables. The expression $(\lambda y(\lambda x M))$, abbreviated as $(\lambda yx.M)$, denotes the converse function to that denoted by $(\lambda xy.M)$. Similarly $(\lambda x(\lambda y(\lambda z M)))$, abbreviated as $(\lambda xyz.M)$, denotes a function of three variables, and so on.

Functions introduced in previous sections as examples can now be expressed, if desired, by means of abstraction operators. For instance, I is (λxx); J is $(\lambda fxyz.fx(fzy))$; S is $(\lambda nfx.f(nfx))$; H is (λxI), or $(\lambda x(\lambda yy))$, or $(\lambda xy.y)$; K is $(\lambda xy.x)$; 1 is $(\lambda fx.fx)$.

Chapter II

LAMBDA-CONVERSION

5. PRIMITIVE SYMBOLS, AND FORMULAS. We turn now to the development of a formal system, which we shall call the calculus of λ-conversion, and which shall have as a possible interpretation or application the system of ideas about functions described in Chapter I.

The primitive symbols of this calculus are three symbols,

$$\lambda, \ (\ , \ ),$$

which we shall call improper symbols, and an infinite list of symbols,

$$a, \ b, \ c, \ \ldots \ , \ x, \ y, \ z, \ \bar{a}, \ \bar{b}, \ \ldots \ , \ \bar{z}, \ \bar{\bar{a}}, \ \ldots \ ,$$

which we shall call variables. The order in which the variables appear in this originally given infinite list shall be called their alphabetical order.

A formula is any finite sequence of primitive symbols. Certain formulas are distinguished as well-formed formulas, and each occurrence of a variable in a well-formed formula is distinguished as free or bound, in accordance with the following rules (1-4), which constitute a definition of these terms by recursion:

1. A variable x is a well-formed formula, and the occurrence of the variable x in this formula is free.

2. If F and A are well-formed, (FA) is well-formed, and an occurrence of a variable y in F is free or bound in (FA) according as it is free or bound in F, and an occurrence of a variable y in A is free or bound in (FA) according as it is free or bound in A.

3. If M is well-formed and contains at least one free occurrence of x, then $(\lambda x M)$ is well-formed, and an occurrence

of a variable y, other than x, in (λxM) is free or bound
in (λxM) according as it is free or bound in M. All occur-
rences of x in (λxM) are bound.

4. A formula is well-formed, and an occurrence of a vari-
able in it is free, or is bound, only when this follows from
1-3.

The <u>free variables</u> of a formula are the variables which
have at least one free occurrence in the formula. The <u>bound va-</u>
<u>riables</u> of a formula are the variables which have at least one
bound occurrence in the formula.

Hereafter (as was just done in the statement of the rules
1-4) we shall use bold capital letters to stand for variable or
undetermined <u>formulas</u>, and bold small letters to stand for vari-
able or undetermined <u>variables</u>. Unless otherwise indicated in a
particular case, it is to be understood that formulas represent-
ed by bold capital letters are well-formed formulas. Bold let-
ters are thus not part of the calculus which we are developing
but are a device for use in <u>talking about</u> the calculus: they be-
long, not to the system itself, but to the <u>metamathematics</u> or
<u>syntax</u> of the system.

Another syntactical notation which we shall use is the no-
tation,

$$S_N^x M$$

which shall stand for the formula which results by substitution
of N for x throughout M. This formula is well-formed, ex-
cept in the case that x is a bound variable of M and N is
other than a single variable -- see §7. (In the special case
that x does not occur in M, it is the same formula as M.)

For brevity and perspicuity in dealing with particular well-
formed formulas, we often do not write them in full but employ
various abbreviations.

One method of abbreviation is by means of a <u>nominal defini-</u>
<u>tion,</u> which introduces a particular new symbol to replace or
stand for a particular well-formed formula. We indicate such a
nominal definition by an arrow, pointing from the new symbol
which is being introduced to the well-formed formula which it is
to replace (the arrow may be read "stands for"). As an example

we make at once the nominal definition:

$$I \longrightarrow (\lambda a a).$$

This means that I will be used as an abbreviation for $(\lambda a a)$ -- and consequently that (II) will be used as an abbreviation for $((\lambda a a)(\lambda a a))$, $(\lambda a(aI))$ as an abbreviation for $(\lambda a(a (\lambda a a)))$, etc.

Another method of abbreviation is by means of a <u>schematic definition</u>, which introduces a class of new expressions of a certain form, specifying a scheme according to which each of the new expressions stands for a corresponding well-formed formula. Such a schematic definition is indicated in a similar fashion by an arrow, but the expressions on each side of the arrow contain bold letters. When a bold small letter -- one or several -- occurs in the expression following the arrow (the definiens) but not in the expression preceding the arrow (the definiendum), the following convention is to be understood:

   a stands for the first variable in alphabetical order not otherwise appearing in the definiens, *b* stands for the second such variable in alphabetical order, *c* the third, and so on.

As examples, we make at once the following schematic definitions:

$$[\textit{M}+\textit{N}] \longrightarrow (\lambda \textit{a}(\lambda \textit{b}((\textit{M}\textit{a})((\textit{N}\textit{a})\textit{b})))).$$
$$[\textit{M}\times\textit{N}] \longrightarrow (\lambda \textit{a}(\textit{M}(\textit{N}\textit{a}))).$$
$$[\textit{M}^{\textit{N}}] \longrightarrow (\textit{N}\textit{M}).$$

The first of these definitions means that, for instance, $[x+y]$ will be used as an abbreviation for $(\lambda a(\lambda b((xa)((ya)b))))$, and $[a+c]$ will be used as an abbreviation for $(\lambda b(\lambda d((ab)((cb)d))))$, and $[I+I]$ as an abbreviation for $(\lambda b(\lambda c((Ib)((Ib)c))))$, etc.

As a further device of abbreviation, we shall allow the omission of the parentheses ( ) in $(\textit{F}\textit{A})$ when this may be done without ambiguity, whether $(\textit{F}\textit{A})$ is the entire formula being written or merely some part of it. In restoring such omitted parentheses, the convention is to be followed that association

is to the left (cf. Schönfinkel [49], Curry [17]). For example,
fxy is an abbreviation of ((*fx*)*y*), *f*(*xy*) is an abbreviation
of (*f*(*xy*)), *fxyz* is an abbreviation of (((*fx*)*y*)*z*), *f*(*xy*)*z*
is an abbreviation of ((*f*(*xy*))*z*), *f*(λ*xx*)*y* is an abbreviation
of ((*f*(λ*xx*))*y*), etc.

In expressions which (in consequence of schematic defini-
tions) contain brackets [ ], we allow a similar omission of
brackets, subject to a similar convention of association to the
left; thus *x+y+z* is an abbreviation for [[*x+y*]+*z*], which ex-
pression is in turn an abbreviation for a certain well-formed
formula in accordance with the schematic definition already in-
troduced. Moreover we allow, as an abbreviation, omitting a
pair of brackets and at the same time putting a dot or period
in the place of the initial bracket [; in this case the conven-
tion, instead of association to the left, is that the omitted
bracket extends from the bold period as far to the right as pos-
sible, consistently with the formula's being well-formed -- so
that, for instance, *x+y+z* is an abbreviation for [*x*+[*y+z*]],
and *x+.y+z+t* is an abbreviation for [*x*+[*y*+[*z+t*]]], and (λ*x.*
x+x) is an abbreviation for (λ*x*[*x+x*]).

We also introduce the following schematic definitions:

$$(\lambda\, x.FA) \longrightarrow (\lambda x(FA)),$$
$$(\lambda\, xy.FA) \longrightarrow (\lambda x(\lambda y(FA))),$$
$$(\lambda\, xyz.FA) \longrightarrow (\lambda x(\lambda y(\lambda z(FA)))),$$

and so on for any number of variables *x*, *y*, *z*, ... (which must
be all different). And we allow similar omissions of λ's, pre-
ceding a bold period which represents an omitted bracket in the
way described in the preceding paragraph -- using, e.g., λ*xyz.x*
+y+z as an abbreviation for (λ*x*(λ*y*(λ*z*[[*x+y*]+*z*]))).

Finally, we allow omission of the outside parentheses in
(λ*xM*), or in (λ*x.FA*), or (λ*xy.FA*), or (λ*xyz.FA*), etc.,
when this is the entire formula being written -- but not when one
of these expressions appears as a proper part of a formula.

Hereafter, in writing definitions, we shall abbreviate the
definiens in accordance with previously introduced abbreviations
and definitions. Thus the definition of [*M+N*] would now be
written:

$$[M+N] \longrightarrow \lambda ab.Ma(Nab).$$

Definitions and other abbreviations are introduced merely
as matters of convenience and are not properly part of the
formal system at all. When we speak of the free variables of
a formula, the bound variables of a formula, the length (num-
ber of symbols) of a formula, the occurrences of one formula
as a part of another, etc., the reference is always to the
unabbreviated form of the formulas in question.

The introduction and use of definitions and other abbrevia-
tions is, of course, subject to the restriction that there shall
never be any ambiguity as to what formula a given abbreviated
form stands for. In practice certain further restrictions are
also desirable, e.g., that all free variables of the definiens
be represented explicitly in the definiendum. Exact formulation
of these restrictions is unnecessary for our present purpose,
since all definitions and abbreviations are extraneous to the
formal system, as just explained, and in principle dispensable.

6. CONVERSION. We introduce now the three following oper-
ations, or transformation rules, on well-formed formulas:

I.   To replace any part M of a formula by $S_y^x M|$, pro-
     vided that x is not a free variable of M and y
     does not occur in M.

II. To replace any part $((\lambda x M)N)$ of a formula by
     $S_N^x M|$, provided that the bound variables of M are
     distinct both from x and from the free variables
     of N.

III. To replace any part $S_N^x M|$ of a formula by $((\lambda x M)N)$,
     provided that $((\lambda x M)N)$ is well-formed and the bound
     variables of M are distinct both from x and from
     the free variables of N.

In the statement of these rules -- and hereafter gener-
ally -- it is to be understood that the word part (of a formu-

1a) means consecutive well-formed part not immediately follow-ing an occurrence of the symbol λ.

When the same formula occurs several times as such a part of another formula, each occurrence is to be counted as a dif-ferent part. Thus, for instance, Rule I may be used to trans-form $ab(\lambda aa)(\lambda aa)$ into $ab(\lambda bb)(\lambda aa)$. Rule III may be used to transform λaa into $\lambda a.(\lambda aa)a$. But Rule III may not be used to transform (λaa) into $(\lambda((\lambda aa)a)a)$ -- the latter formula is, in fact, not even well-formed.

Rules I-III have the important property that they are ef-fective or "definite," i.e., there is a means of always deter-mining of any two formulas **A** and **B** whether **A** can be trans-formed into **B** by an application of one of the rules (and, if so, of which one).

If **A** can be transformed into **B** by an application of one of the Rules I-III, we shall say that **A** is immediately convert-ible into **B** (abbreviation, " **A** imc **B** "). If there is a finite sequence of formulas, in which **A** is the first formula and **B** the last, and in which each formula except the last is immediately convertible into the next one, we shall say that **A** is convert-ible into **B** (abbreviation, " **A** conv **B** "); and the process of ob-taining **B** from **A** by a particular finite sequence of applica-tions of Rules I-III will be called a conversion of **A**   into **B** (no reference is intended to conversion in the sense of forming the converse -- for the corresponding noun we use, not "converse," but "convert"). It is not excluded that the number of applica-tions of Rules I-III in a conversion of **A**   into **B**   should be zero, **B**   being then the same formula as **A** .

The relation which holds between **A** and **B**   when **A** conv **B** will be called interconvertibility, and we shall use the expres-sion " **A** and **B** are interconvertible" as synonymous with " **A** conv **B** ." The relation of interconvertibility is transitive, symmet-ric, and reflexive -- symmetric because Rules II and III are in-verses of each other and Rule I is its own inverse.

If there is a conversion of **A**   into **B**   which contains no application of Rule II or Rule III, we shall say that **A**   is convertible-I into **B** (**A** conv-I **B**). Similarly we define " **A** conv-I-II **B** " and " **A** conv-I-III **B** ."

A conversion which contains no application of Rule II and

exactly one application of Rule III will be called an expansion.
A conversion which contains no application of Rule III and ex-
actly one application of Rule II will be called a reduction. If
there is a reduction of A into B, we shall say that A is
immediately reducible to B (A imr B). If there is a conversion
of A into B which consists of one or more successive reduc-
tions, we shall say that A is reducible to B (A red B). (The
meaning of "A red B" thus differs from that of "A conv-I-II B"
only in that the former implies the presence of at least one ap-
plication of Rule II in the conversion of A into B.)

An application of Rule II to a formula will be called a
contraction of the part $((\lambda x M)N)$ which is affected.

A well-formed formula will be said to be in normal form if
it contains no part of the form $((\lambda x M)N)$. We shall call B a
normal form of A if B is in normal form and A conv B. We
shall say that A has a normal form if there is a formula B
which is a normal form of A.

A well-formed formula will be said to be in principal nor-
mal form if it is in normal form, and no variable is both a bound
variable and free variable of it, and the first bound variable
occurring in it (in the left-to-right order of the symbols which
compose the formula) is the same as the first variable in alpha-
betical order which is not a free variable of it, and the vari-
ables which occur in it immediately following the symbol λ are,
when taken in the order in which they occur in the formula, in
alphabetical order, without repetitions, and without omissions
except of variables which are free variables of the formula. For
example, $\lambda ab.ba$, and $\lambda a.a(\lambda c.bc)$, and $\lambda b.ba$ are in principal
normal form; and $\lambda ac.ca$, and $\lambda bc.cb$, and $\lambda a.a(\lambda a.ba)$ are
in normal form but not in principal normal form.

We shall call B a principal normal form of A if B is
in principal normal form and A conv B. A formula in normal form
is always convertible-I into a corresponding formula in principal
normal form, and hence every formula which has a normal form has
a principal normal form. We shall show in the next section that
the principal normal form of a formula, if it exists, is unique.

An example of a formula which has no normal form (and there-
fore no principal normal form) is $(\lambda x.xxx)(\lambda x.xxx)$.

It is intended that, in any interpretation of the formal

calculus, only those well-formed formulas which have a normal
form shall be meaningful, and, among these, interconvertible for-
mulas shall have the same meaning. The condition of being well-
formed is thus a necessary condition for meaningfulness but not
a sufficient condition.

It is important that the condition of being well-formed is
effective in the sense explained at the beginning of this sec-
tion, whereas the condition of being well-formed and having a
normal form is not effective.

7. FUNDAMENTAL THEOREMS ON WELL-FORMED FORMULAS AND ON THE
NORMAL FORM. The following theorems are taken from Kleene [34]
(with non-essential changes to adapt them to the present modified
notation). Their proof is left to the reader; or an outline of
the proof may be found in Kleene, loc. cit.

7 I.   In a well-formed formula K there exists a unique pair-
         ing of the occurrences of the symbol (, each with a
         corresponding occurrence of the symbol), in such a
         way that two portions of K, each lying between an oc-
         currence of (and the corresponding occurrence of)
         inclusively, either are non-overlapping or else are con-
         tained one entirely within the other. Moreover, if such
         a pairing exists in the portion of K lying between the
         nth and the (n+r)th symbol of K inclusively, it is a
         part of the pairing in K. .

7 II.   A necessary and sufficient condition that the portion N
         of a well-formed formula K which lies between a given
         occurrence of (in K and a given occurrence of)
         in K inclusively be well-formed is that the given oc-
         currence of (and the given occurrence of) corre-
         spond.

7 III.   Every well-formed formula has one of the three forms,
         x, where x is a variable, or (FA), where F and
         A are well-formed, or $(\lambda x M)$, where M is well-formed
         and x is a free variable of M.

7 IV.   If (FA) and either F or A is well-formed, then both
         F and A are well-formed.

7 V.   If $(\lambda x M)$ is well-formed, x being a variable, then M is well-formed and x is a free variable of M.

7 VI.   A well-formed formula can be of the form (FA), where F (or A) is well-formed, in only one way.

7 VII.   A well-formed formula can be of the form $(\lambda x M)$, where x is a variable, in only one way.

7 VIII.   If P and Q are well-formed parts of a well-formed formula K, then either P is a part of Q, or Q is a part of P, or P and Q are non-overlapping.

7 IX.   Two distinct occurrences of the same well-formed formula P as a part of a well-formed formula K must be non-overlapping.

7 X.   If P, F, and A are well-formed and P is a part of (FA), then P is (FA) or P is a part of F or P is a part of A.

7 XI.   If P and M are well-formed and x is a variable and P is a part of $(\lambda x M)$, then P is $(\lambda x M)$ [or P is x] or P is a part of M. (The clause in brackets is superfluous because of the meaning we give to the word part of a formula -- see §6).

7 XII.   An occurrence of a variable x in a well-formed formula K is bound or free according as it is or is not an occurrence in a well-formed part of K of the form $(\lambda x M)$. (Hence, in particular, no occurrence of a variable in a well-formed formula is both bound and free.)

7 XIII.   If M is well-formed and the variable x is not a free variable of M and the variable y does not occur in M, then $S_y^x M|$ is well-formed and has the same free variables as M.

7 XIV.   If M and N are well-formed and the variable x occurs in M and the bound variables of M are distinct both from x and from the free variables of N, then $S_N^x M|$ and $((\lambda x M)N)$ are well-formed and have the same free variables.

7 XV.   If K, P, Q are well-formed and all free variables of P are also free variables of Q, the formula obtained

by substituting Q for a particular occurrence of P
in K , not immediately following an occurrence of λ,
is well-formed.

7 XVI. If A is well-formed and A conv B, then B is well-
formed.

7 XVII. If A is well-formed and A conv B, then A and B
have the same free variables.

7 XVIII.If K, P, Q are well-formed, and P conv Q, and L
is obtained by substituting Q for a particular occur-
rence of P in K, not immediately following an occur-
rence of λ, then K conv L.

We shall call a well-formed part P of a well-formed for-
mula K a <u>free</u> occurrence of P in K if every free occurrence
of a variable in P is also a free occurrence of that variable
in K; in the contrary case (if some free occurrence of a vari-
able in P is at the same time a bound occurrence of that vari-
able in K) we shall call the part P of K a <u>bound</u> occurrence
of P in K. If P is an occurrence of a variable in K, not
immediately following an occurrence of λ, this definition is
in agreement with our previous definition of free and bound oc-
currences of variables.

Moreover we shall extend the notation $S_{N}^{x}M|$ introduced in
§5 by allowing $S_{N}^{P}M|$ to stand for the result of substituting
N for P throughout M, where N, P, M are any well-formed
formulas. This is possible without ambiguity, by 7 IX.

7 XIX. A well-formed part P of a well-formed formula K is a
bound or free occurrence of P in K according as it
is or is not an occurrence in a well-formed part of K
of the form $(\lambda x M)$ where x is a free variable of P.

7 XX.   If K, P, Q are well-formed, the formula obtained by
substituting Q for a particular free occurrence of P
in K is well-formed.

7 XXI. If K, P, Q are well-formed and there is no bound oc-
currence of P in K, then $S_{Q}^{P}K|$ is well-formed.

7 XXII. Let x be a free variable of the well-formed formula

M and let *P* be the formula obtained by substituting *N* for the free occurrences of *x* in *M*. If the resulting occurrences of *N* in *P* are free, $((\lambda x M)N)$ conv *P*.

In what follows we shall frequently make tacit assumption of these theorems.

In stating these theorems, it has been necessary to hold in abeyance the convention that formulas represented by bold capital letters are well-formed. Hereafter this convention will be restored, and formulas so represented are to be taken always as well-formed.

We turn now to a group of theorems on conversion taken from Church and Rosser [16]. In order to state these, it is necessary first to define the notion of the <u>residuals</u> of a set of parts $((\lambda x_j M_j)N_j)$ of a formula *A* after a sequence of applications of Rules I and II to *A* (§6).

We assume that, if $p \neq q$, then $((\lambda x_p M_p)N_p)$ is not the same part of *A* as $((\lambda x_q M_q)N_q)$ -- though it may be the same formula. The parts $((\lambda x_j M_j)N_j)$ of *A* need not be all the parts of *A* which have the form $((\lambda y P)Q)$. The <u>residuals</u> of the $((\lambda x_j M_j)N_j)$ after a particular sequence of applications of Rules I and II to *A* are then certain parts, of the form $((\lambda y P)Q)$, of the formula into which *A* is converted by this sequence of applications of Rules I and II. They are defined as follows:

If the sequence of applications of Rules I and II in question is vacuous, each part $((\lambda x_j M_j)N_j)$ is its own residual.

If the sequence consists of a single application of Rule I, each part $((\lambda x_j M_j)N_j)$ is changed into a part $((\lambda y_j M_j')N_j')$ of the resulting formula, and this part $((\lambda y_j M_j')N_j')$ is the residual of $((\lambda x_j M_j)N_j)$.

If the sequence consists of a single application of Rule II, let $((\lambda x M)N)$ be the part of *A* which is contracted (§6), and let *A'* be the resulting formula into which *A* is converted. Let $((\lambda x_p M_p)N_p)$ be a particular one of the $((\lambda x_j M_j)N_j)$, and distinguish the six following cases.

Case 1: $((\lambda x M)N)$ and $((\lambda x_p M_p)N_p)$ do not overlap. Under

the reduction of A to A', $((\lambda x_p M_p) N_p)$ goes into a definite part of A', which is the same formula as $((\lambda x_p M_p) N_p)$. This part of A' is the residual of $((\lambda x_p M_p) N_p)$.

Case 2: $((\lambda x M) N)$ is a part of M_p. Under the reduction of A to A', M_p goes into a definite part M_p' of A, which arises from M_p by contraction of $((\lambda x M) N)$, and $((\lambda x_p M_p) N_p)$ goes into the part $((\lambda x_p M_p') N_p)$ of A'. This part $((\lambda x_p M_p') N_p)$ of A' is the residual of $((\lambda x_p M_p) N_p)$.

Case 3: $((\lambda x M) N)$ is a part of N_p. Under the reduction of A to A', N_p goes into a definite part N_p' of A', which arises from N_p by contraction of $((\lambda x M) N)$, and $((\lambda x_p M_p) N_p)$ goes into the part $((\lambda x_p M_p) N_p')$ of A'. This part $((\lambda x_p M_p) N_p')$ of A' is the residual of $((\lambda x_p M_p) N_p)$.

Case 4: $((\lambda x M) N)$ is $((\lambda x_p M_p) N_p)$. In this case $((\lambda x_p M_p) N_p)$ has no residual in A'.

Case 5: $((\lambda x_p M_p) N_p)$ is a part of M. Let M' be the result of replacing all x's of M except those occurring in $((\lambda x_p M_p) N_p)$ by N. Under these changes the part $((\lambda x_p M_p) N_p)$ of M goes into a definite part of M' which we shall denote also by $((\lambda x_p M_p) N_p)$, since it is the same formula. If now we replace $((\lambda x_p M_p) N_p)$ in M' by $S_N^x((\lambda x_p M_p) N_p)|$, M' becomes $S_N^x M|$ and we denote by $S_N^x((\lambda x_p M_p) N_p)|$ the particular occurrence of $S_N^x((\lambda x_p M_p) N_p)|$ in $S_N^x M|$ that resulted from replacing $((\lambda x_p M_p) N_p)$ in M' by the formula $S_N^x((\lambda x_p M_p) N_p)|$. Then the residual in A' of $((\lambda x_p M_p) N_p)$ in A is defined to be the part $S_N^x((\lambda x_p M_p) N_p)|$ in the particular occurrence of $S_N^x M|$ in A' that resulted from replacing $((\lambda x M) N)$ in A by $S_N^x M|$.

Case 6: $((\lambda x_p M_p) N_p)$ is a part of N. Let $((\lambda y_1 P_1) Q_1)$ respectively stand for the particular occurrences of the formula $((\lambda x_p M_p) N_p)$ in $S_N^x M|$ which are the part $((\lambda x_p M_p) N_p)$ in each of those particular occurrences of the formula N in $S_N^x M|$ that resulted from replacing the x's of M by N. Then the residuals in A' of $((\lambda x_p M_p) N_p)$ in A are the parts $((\lambda y_1 P_1) Q_1)$ in the particular occurrence of the formula $S_N^x M|$ in A' that resulted from replacing $((\lambda x M) N)$ in A by $S_N^x M|$.

Finally, in the case of a sequence of two or more succes-
sive applications of Rules I, II to A, say A imc A' imc A''
imc ... , we define the residuals in A' of the parts $((\lambda x_j M_j)$
$N_j)$ of A in the way just described, and we define the resid-
uals in A'' of the parts $((\lambda x_j M_j) N_j)$ of A to be the resid-
uals of the residuals in A', and so on.

7 XXIII. After a sequence of applications of Rules I and II to
          A, under which A is converted into B, the resid-
          uals of the parts $((\lambda x_j M_j) N_j)$ of A are a set (pos-
          sibly vacuous) of parts of B which each have the
          form $((\lambda y P) Q)$.

7 XXIV.   After a sequence of applications of Rules I and II to
          A, no residual of the part $((\lambda x M) N)$ of A can co-
          incide with a residual of the part $((\lambda x' M') N')$ of A
          unless $((\lambda x M) N)$ coincides with $((\lambda x' M') N')$.

We say that a sequence of reductions on A_1, say A_1 imr
A_2 imr A_3 ... imr A_{n+1}, is a <u>sequence of contractions on the</u>
<u>parts</u> $((\lambda x_j M_j) N_j)$ <u>of</u> A_1 if the reduction from A_i to A_{i+1}
($i = 1, ..., n$) involves a contraction of a residual of the
$((\lambda x_j M_j) N_j)$. Moreover, if no residuals of the $((\lambda x_j M_j) N_j)$ oc-
cur in A_{n+1} we say that the sequence of contractions on the
$((\lambda x_j M_j) N_j)$ <u>terminates</u> and that A_{n+1} is the result.
    In some cases we wish to speak of a sequence of contractions
on the parts $((\lambda x_j M_j) N_j)$ of A where the set $((\lambda x_j M_j) N_j)$ may
be vacuous. To handle this we agree that, if the set $((\lambda x_j M_j) N_j)$
is vacuous, the sequence of contractions shall be a vacuous se-
quence of reductions.

7 XXV.   If $((\lambda x_j M_j) N_j)$ are parts of A, then a number m
          can be found such that any sequence of contractions on
          the $((\lambda x_j M_j) N_j)$ will terminate after at most m con-
          tractions, and if A' and A'' are two results of ter-
          minating sequences of contractions on the $((\lambda x_j M_j) N_j)$,
          then A' conv-I A''.

This is proved by induction on the length of A. It is
trivially true if the length of A is 1 (i.e., if A consists

of a single symbol), the number m being then 0. As hypothe-
sis of induction, assume that the proposition is true of every
formula A of length less than n. On this hypothesis we have
to prove that the proposition is true of an arbitrary given for-
mula A of length n. This we proceed to do, by means of a
proof involving three cases.

Case 1: A has the form $\lambda x M$. All the parts $((\lambda x_j M_j) N_j)$
of A must be parts of M. Since M is of length less than
n, we apply the hypothesis of induction to M.

Case 2: A has the form FX, where FX is not one of
the $((\lambda x_j M_j) N_j)$. All the parts $((\lambda x_j M_j) N_j)$ of A must be
parts either of F or of X. Since F and X are each of
length less than n, we apply the hypothesis of induction.

Case 3: A is $((\lambda x_p M_p) N_p)$, where $((\lambda x_p M_p) N_p)$ is one
of the $((\lambda x_j M_j) N_j)$. By the hypothesis of induction, there is
a number a such that any sequence of contractions on those
$((\lambda x_j M_j) N_j)$ which are parts of M_p terminates after at most a
contractions, and there is a number b such that any sequence
of contractions on those $((\lambda x_j M_j) N_j)$ which are parts of N_p
terminates after at most b contractions; moreover, if we
start with the formula M_p and perform a terminating sequence
of contractions on those $((\lambda x_j M_j) N_j)$ which are parts of M_p,
the result is a formula M, which is unique to within applica-
tions of Rule I, and which contains a certain number c, ≥ 1,
of free occurrences of the variable x_p.

Now one way of performing a terminating sequence of con-
tractions on the parts $((\lambda x_j M_j) N_j)$ of A is as follows. First
perform a terminating sequence of contractions on those $((\lambda x_j M_j)
N_j)$ which are parts of M_p, so converting A into $((\lambda t M) N_p)$.
Then there is one and only one residual of $((\lambda x_p M_p) N_p)$, name-
ly the entire formula $((\lambda x M) N_p)$. Perform a contraction of this,
so obtaining

$$S_{N_p}^t M' \mid$$

where M' differs from M at most by applications of Rule I.
Then in this formula there are c occurrences of N_p resulting

from the substitution of N_p for t. Take each of these occur-
rences of N_p in order and perform a terminating sequence of
contractions on the residuals of the $((\lambda x_j M_j)N_j)$ occurring in
it.

Let us call such a terminating sequence of contractions on
the parts $((\lambda x_j M_j)N_j)$ of A a special terminating sequence
of contractions on the parts $((\lambda x_j M_j)N_j)$ of A. Clearly such
a special terminating sequence of contractions contains at most
$a+1+cb$ contractions.

Consider now any sequence of contractions, μ, on the
parts $((\lambda x_j M_j)N_j)$ of A. The part $((\lambda x_p M_p)N_p)$ of A will
have just one residual (which will always be the entire formula)
up to the point that a contraction of its residual occurs, and
thereafter will have no residual; moreover, if the sequence of
contractions is continued, a contraction of the residual of
$((\lambda x_p M_p)N_p)$ must occur within at most $a+b+1$ contractions.
Hence we may suppose, without loss of generality, that μ con-
sists of a sequence of contractions, ϕ, on the $((\lambda x_j M_j)N_j)$
which are different from $((\lambda x_p M_p)N_p)$, followed by a contrac-
tion β_0 of the residual of $((\lambda x_p M_p)N_p)$, followed by a se-
quence of contractions, ϑ, on the then remaining residuals of
the $((\lambda x_j M_j)N_j)$. Clearly, ϕ can be replaced by a sequence
of contractions, α_0, on the $((\lambda x_j M_j)N_j)$ which are parts of
M_p, followed by a sequence of contractions, η, on the $((\lambda x_j$
$M_j)N_j)$ which are parts of N_p -- in the sense that α_0 fol-
lowed by η gives the same end formula as ϕ and the same set
of residuals for each of the $((\lambda x_j M_j)N_j)$. Moreover, replacing
ϕ by α_0 followed by η does not change the total number of
contractions of residuals of parts of M_p or of residuals of
parts of N_p. Next, η followed by β_0 can be replaced by a
contraction β' of the residual $((\lambda y P)N_p)$ of $((\lambda x_p M_p)N_p)$
followed by a set of applications of η on each of those oc-
currences of N_p in the resulting formula

$$S_{N_p}^{y} P' \mid$$

which arose by substituting N_p for y in P'. (Here P' dif-
fers from P at most by applications of Rule I. Since η may
be thought of as a transformation of the formula N_p, the con-

vention will be understood which we use when we speak of the se-
quence of reductions of a given formula which results from ap-
plying η to a particular occurrence of N_p in that formula.)

By this means the sequence of contractions, μ , is re-
placed by a sequence of contractions, μ' , which consists of a
sequence of contractions, α_0 , on the $((\lambda x_j M_j)N_j)$ which are
parts of M_p , followed by a contraction β' of the residual
of $((\lambda x_p M_p)N_p)$, followed by further contractions on the then
remaining residuals of the $((\lambda x_j M_j)N_j)$.

Consider now the part ζ of μ' , consisting of β' and
the contractions that follow it, up to and including the first
contraction of a residual of a part of M_p . Denoting the for-
mula on which ζ acts by $((\lambda y P)N_p)$, we see that ζ can be
considered as the act of first replacing the free y 's of P'
by various formulas N_{pk} , got from N_p by various sequences
of reductions (which may be vacuous), and then (possibly after
some applications of Rule I) contracting a residual $((\lambda z R)S)$
of one of the $((\lambda x_j M_j)N_j)$ which are parts of M_p , say $((\lambda x_q$
 $M_q)N_q)$. From this point of view, we see that none of the free
 z 's of R are parts of any N_{pk} , and hence ζ can be re-
placed by a contraction (possibly after some applications of
Rule I) of that residual in P of $((\lambda x_q M_q)N_q)$ of which $((\lambda z R)$
 $S)$ is a residual, followed by a contraction (possibly after
some applications of Rule I) of the residual of $((\lambda x_p M_p)N_p)$,
followed by a sequence of contractions on residuals of parts of
 N_p .

If μ' is altered by replacing ζ in this way, the result
is a sequence of contractions, μ'' , having the same form as μ' ,
but having the property that after the contraction of the resid-
ual of $((\lambda x_p M_p)N_p)$ one less contraction of residuals of parts
of M_p occurs.

By repetitions of this process, μ is finally replaced by
a sequence of contractions ν , which consists of a sequence of
contractions, α , on the $((\lambda x_j M_j)N_j)$ which are parts of M_p ,
followed by a contraction β of the residual of $((\lambda x_p M_p)N_p)$,
followed by a sequence of contractions γ on residuals of the
 $((\lambda x_j M_j)N_j)$ which are parts of N_p . Moreover, ν contains at
least as many contractions as μ -- for in the process of ob-
taining ν from μ there is no step which can decrease the
number of contractions. The sequence of contractions, α , con-

tains at most a   contractions, and γ contains at most cb
contractions. Thus ν, and consequently μ, contains at most
a+1+cb contractions.

    Thus we have proved that any sequence of contractions on
the parts $((\lambda x_j M_j) N_j)$ of A will terminate after at most
a+1+cb contractions.

    Now suppose that μ is a _terminating_ sequence of contrac-
tions. Then ν either is a _special_ terminating sequence of
contractions (see above) or can be made so by some evident
changes in the order in which the contractions in γ are per-
formed. By the hypothesis of induction, applied to M_p and N_p,
the result of a special terminating sequence of contractions is
unique to within possible applications of Rule I. Therefore
the result of any terminating sequence of contractions, μ, is
unique to within possible applications of Rule I.

7 XXVI. If A imr B by a contraction of the part $((\lambda x M) N)$ of
      A, and A_1 is A, and A_1 imr A_2, A_2 imr A_3, ...,
      and, for all k, B_k is the result of a terminating
      sequence of contractions on the residuals in A_k of
      $((\lambda x M) N)$, then:
         (1) B_1 is B.
         (2) For all k, B_k conv-I-II B_{k+1}:
         (3) Even if the sequence A_1, A_2, ... can be
      continued to infinity, there is a number u_m, depend-
      ing on the formula A, the part $((\lambda x M) N)$ of A, and
      the number m, such that, starting with B_m, at most
      u_m consecutive B_k's occur for which it is not true
      that B_k red B_{k+1}.

    (1) is obvious.
    To prove (2), let $((\lambda y_1 P_1) Q_1)$ be the residuals in A_k of
$((\lambda x M) N)$ and let the reduction of A_k into A_{k+1} involve a
contraction of (a residual of) the part $((\lambda z R) S)$ of A_k. Then
B_{k+1} is the result of a terminating sequence of contractions
on $((\lambda z R) S)$ and the parts $((\lambda y_1 P_1) Q_1)$ of A_k. If $((\lambda z R) S)$
is one of the $((\lambda y_1 P_1) Q_1)$, no residuals of $((\lambda z R) S)$ occur in
B_k, and B_k conv-I B_{k+1} by 7 XXV. If, however, $((\lambda z R) S)$ is
not one of the $((\lambda y_1 P_1) Q_1)$, a set of residuals of $((\lambda z R) S)$

does occur in B_k and a terminating sequence of contractions on these residuals in B_k gives B_{k+1} by 7 XXV.

Thus B_k red B_{k+1} unless the reduction of A_k into A_{k+1} involves a contraction of a residual of $((\lambda x M) N)$; but if we start with any particular A_k this can be the case only a finite number of successive times by 7 XXV. Hence (3) is proved, u_m being defined as follows:

Perform m successive reductions on A in all possible ways. This gives a finite set of formulas (since, for this purpose, we need not distinguish formulas differing only by applications of Rule I). In each formula find the largest number of reductions that can occur in a terminating sequence of contractions on the residuals of $((\lambda x M) N)$. Then let u_m be the largest of these.

7 XXVII. If A conv B, there is a conversion of A into B
          in which no expansion precedes any reduction.

In the given conversion of A into B, let the last expansion which precedes any reduction be an expansion of B_1 into A_1. This expansion is followed by a sequence of one or more reductions, say A_1 imr A_2, A_2 imr A_3, ..., A_{n-1} imr A_n, and A_n conv-I-III B. The inverse of the expansion of B_1 into A_1 is a reduction of A_1 into B_1; let $((\lambda x M) N)$ be the part of A which is contracted in this reduction, and let B_k $(k = 2,$ $3, ..., n)$ be the result of a terminating sequence of contractions on the residuals in A_k of $((\lambda x M) N)$. By 7 XXVI, B_1 conv-I-II B_2, B_2 conv-I-II B_3, ..., B_{n-1} conv-I-II B_n, B_n conv-I-III A_n, A_n conv-I-III B. This provides an alternative conversion of B_1 into B in which no expansion precedes any reduction. The given conversion of A into B may be altered by employing this alternative conversion of B_1 into B instead of the one originally involved, with the result that the number of expansions which are out of place (precede reductions) in the conversion of A into B is decreased by one. Repetitions of this process lead to a conversion of A into B in which no expansion precedes reductions.

7 XXVIII. If B is a normal form of A, then A conv-I-II B.

This is a corollary of 7 XXVII, since no reductions are possible of a formula in normal form.

7 XXIX.   If **A** has a normal form, its normal form is unique to within applications of Rule I.

For if **B** and **B'** are both normal forms of **A**, then **B'** is a normal form of **B**. Hence **B** conv-I-II **B'**. Hence **B** conv-I **B**, since no reductions are possible of the normal form **B**.

Note that 7 XXIX ensures a kind of consistency of the calculus of λ-conversion, in that certain formulas for which different interpretations are intended are shown not to be interconvertible.

7 XXX.   If **A** has a normal form, it has a unique principal normal form.

7 XXXI.   If **B** is a normal form of **A**, then there is a number m   such that any sequence of reductions starting from **A** will lead to **B** (to within applications of Rule I) after at most m   reductions.

In order to prove 7 XXXI, we first prove the following lemma by induction on   n:

If **B** is a normal form of **A** and there is a sequence of n   reductions leading from **A** to **B**, then there is a number $v_{A,n}$ such that any sequence of reductions starting from **A** will lead to a normal form of **A** in at most $v_{A,n}$ reductions.

If n = 0, we take $v_{A,0}$ to be 0.

Assume, as hypothesis of induction, that the lemma is true when n = k. Suppose **A** imr **C**, **C** imr C_1, C_1 imr C_2, C_2 imr C_3, ..., C_{k-1} imr **B**. Also, where A_1 is the same as **A**, suppose A_1 imr A_2, A_2 imr A_3, ... By 7 XXVI there is a sequence (D_1 the same as **C**), D_1 conv-I-II D_2, D_2 conv-I-II D_3, ..., such that A_j conv-I-II D_j for all j's for which A_j exists; and, if the reduction from **A** to **C** involves a contraction of ((λ**xM**)**N**), then, starting with D_m, at most u_m consecutive

D_j's occur for which it is not true that D_j red D_{j+1}.

Since the sequence C imr C_1, C_1 imr C_2, ... leads to B
in k reductions, there is, by hypothesis of induction, a num-
ber $v_{C,k}$ such that any sequence of reductions starting from
C leads to a normal form (and thus terminates) after at most
$v_{C,k}$ reductions. Hence there are at most $v_{C,k}$ reductions in
the sequence D_1 conv-I-II D_2, D_2 conv-I-II D_3, ..., and this
sequence must terminate after at most $f(v_{C,k})$ steps, $f(x)$
being defined as follows:

$$f(0) = u_1,$$

$$f(x+1) = f(x)+M+1,$$

where M is the greatest of the numbers u_1, u_2, ..., $u_{f(x)+1}$.
(Of course $f(x)$ depends on the formula A and the part
$((\lambda x M) N)$ of A, as well as on x, because u_m depends on A
and $((\lambda x M) N)$.

Since the sequence of D_j's continues as long as there are
A_j's on which reductions can be performed, it follows that after
at most $f(v_{C,k})$ reductions an A_j is reached on which no re-
ductions are possible. But this is equivalent to saying that
this A_j is in normal form. Thus any reductionsof A to a
formula C, such that there is a sequence of k reductions
leading from C to a normal form of A, determines an upper
bound, $f(v_{C,k})$, which holds for all sequences of reductions
starting from A. Since the number of possible reductions of
A to such formulas C is finite (reductions, or formulas C,
which differ only by applications of Rule I need not be distin-
guished as different), we take $v_{A,k+1}$ to be the least of the
numbers $f(v_{C,k})$.

This completes the proof of the lemma. Hence 7 XXXI fol-
lows by 7 XXVIII.

7 XXXII. If A has a normal form, every [well-formed] part of
          A has a normal form.

This follows from 7 XXXI, since any sequence of reductions
on a part of A implies a sequence of reductions on A and
therefore must terminate.

Chapter III

LAMBDA-DEFINABILITY

8. LAMBDA-DEFINABILITY OF FUNCTIONS OF POSITIVE INTEGERS.
We define,

$$1 \longrightarrow \lambda ab.ab,$$

$$2 \longrightarrow \lambda ab.a(ab),$$

$$3 \longrightarrow \lambda ab.a(a(ab)),$$

and so on, each numeral (in the Arabic decimal notation) being
introduced as an abbreviation for a corresponding formula of the
indicated form. But where a numeral consists of more than one
digit, a bar is used over it, in order to avoid confusion with
other notations; thus,

$$\overline{11} \longrightarrow \lambda ab.a(a(a(a(a(a(a(a(a(a(ab))))))))))),$$

but 11, without the bar, is an abbreviation for

$$(\lambda ab.ab)(\lambda ab.ab).$$

In connection with these definitions an interpretation of
the calculus of λ-conversion is contemplated under which each
of the formulas abbreviated as a numeral is interpreted as de-
noting the corresponding positive integer. Since it is intended
at the same time to retain the interpretation of the formulas of
the calculus (which have a normal form) as denoting certain func-
tions in accordance with the ideas of Chapter I, this means that
the positive integers are identified with certain functions.
For example, the number 2 is identified with the function which,
when applied to the function f as argument, yields the product
of f by itself (product in the sense of the product, or resul-

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tant, of two transformations); similarly the number 14 is identi-
fied with the function which, when applied to the function f
as argument, yields the fourteenth power of f (power in the
sense of power of a transformation). This is allowable on the
ground that abstract number theory requires of the positive in-
tegers only that they form a progression and, subject to this
condition, the integers may be identified with any entities
whatever; as a matter of fact, logical constructions of the pos-
itive integers by identifying them with entities thought to be
logically more fundamental are possible in many different ways
(the present method should be compared with that familiar in the
works of Frege and Russell, according to which the non-negative
integers are identified with classes of similar finite classes).

A function F of positive integers -- i.e., a function of
one variable for which the range of arguments and the range of
values each consist of positive integers -- is said to be λ-de-
finable if there is a formula F such that (1) whenever m and
n are positive integers, and $Fm = n$, and M and N are the
formulas which represent (denote) the integers m and n re-
spectively, then FM conv N, and (2) whenever the function F
has no value for the positive integer m as argument, and M
represents m, then FM has no normal form. Similarly the
function F of two integer variables is said to be λ-definable
if there is a formula F such that (1) if l, m, n are posi-
tive integers, and $Flm = n$, and L, M, N represent the inte-
gers l, m, n respectively, then FLM conv N, and (2) if the
function F has no value for the positive integers l, m as ar-
guments, and L, M represent l, m respectively, then FLM has
no normal form. And so on, for functions of any number of vari-
ables.

We shall say also, under the circumstances described, that
the formula F λ-defines the function F (we use the word "λ-
defines rather than "denotes" or "represents" only because the
function which F denotes, in general has other elements than
positive integers in its range -- or ranges -- of arguments).

The successor function of positive integers (i.e., the func-
tion $x + 1$) is λ-defined by the formula S, where

$$S \longrightarrow \lambda abc. b(ab c).$$

It is left to the reader to verify this, and also to verify that addition, and multiplication, and exponentiation of positive integers are λ-defined by the formulas $\lambda mn.m+n$, and $\lambda mn.m \times n$, and $\lambda mn.m^n$ respectively (see definitions in §5).

These λ-definitions of addition, multiplication, and exponentiation are due to Rosser (see Kleene [35]). The definition of multiplication depends on the observation that the product of two positive integers in the sense of the product of transformations is the same as their product in the arithmetic sense, and the definition of exponentiation then follows because, when the positive integer n is taken of any function f as argument, there results the nth power of f in the sense of the product of transformations.

The reader may also verify that, for any formulas L, M, N (whether representing positive integers or not):

$$[L+M]+N \text{ conv } L+[M+N],$$

$$[L \times M] \times N \text{ conv } L \times [M \times N],$$

$$[L+M] \times N \text{ conv } [L \times N]+[M \times N],$$

$$L^{M+N} \text{ conv } L^M \times L^N,$$

$$L^{M \times N} \text{ conv } [L^N]^M,$$

$$SM \text{ conv } 1+M.$$

9. ORDERED PAIRS AND TRIADS, THE PREDECESSOR FUNCTION. We now introduce formulas which may be thought of as representing ordered pairs and ordered triads, as follows:

$$[M, N] \longrightarrow \lambda a.aMN,$$

$$[L, M, N] \longrightarrow \lambda a.aLMN,$$

$$2_1 \longrightarrow \lambda a.a(\lambda bc.c\,Ib),$$

$$2_2 \longrightarrow \lambda a.a(\lambda bc.b\,Ic),$$

$$3_1 \longrightarrow \lambda a.a(\lambda bcd.c\,Id\,Ib),$$

$$3_2 \longrightarrow \lambda a.a(\lambda bcd.b\,Id\,Ic),$$

$$3_3 \longrightarrow \lambda a.a(\lambda bcd.b\,Ic\,Id).$$

If L, M, N are formulas representing positive integers, then $2_1[M, N]$ conv M, $2_2[M, N]$ conv N, $3_1[L, M, N]$ conv L, $3_2[L, M, N]$ conv M, and $3_3[L, M, N]$ conv N.

Verification of this depends on the observation that, if M is a formula representing a positive integer, MI conv I (the mth power of the identity is the identity).

By the predecessor function of positive integers we mean the function whose value for the argument 1 is 1 and whose value for any other positive integer argument x is $x-1$. This function is λ-defined by

$$P \longrightarrow \lambda a. 3_3(a(\lambda b[S(3_1 b), \ 3_1 b, \ 3_2 b]))[1, \ 1, \ 1]).$$

For if K, L, M represent positive integers,

$$(\lambda b[S(3_1 b), \ 3_1 b, \ 3_2 b])[K, \ L, \ M] \ \text{conv} \ [SK, \ K, \ L],$$

and hence if A represents a positive integer,

$$A(\lambda b[S(3_1 b), \ 3_1 b, \ 3_2 b])[1, \ 1, \ 1] \ \text{conv} \ [SA, \ A, \ B],$$

where B represents the predecessor of the positive integer represented by A. (The method of λ-definition of the predecessor function due to Kleene [35] is here modified by employment of a different formal representation of ordered triads.)

A kind of subtraction of positive integers, which we distinguish by placing a dot above the sign of subtraction, and which differs from the usual kind in that $x \overset{.}{-} y = 1$ if $x \leqq y$, may now be shown to be λ-definable:

$$[M \overset{.}{-} N] \longrightarrow NPM.$$

The functions the lesser of the two positive integers x and y and the greater of the two positive integers x and y are λ-definable respectively by

$$\text{min} \longrightarrow \lambda a b \ . \ Sb \overset{.}{-} \ . \ Sb \overset{.}{-} a,$$

$$\text{max} \longrightarrow \lambda a b \ . \ [a+b] \overset{.}{-} \text{min} \ ab.$$

The parity of a positive integer, i.e., the function whose value is 1 for an odd positive integer and 2 for an even positive integer, is λ-defined by

$$par \longrightarrow \lambda a.a(\lambda b. 3 \overset{.}{-} b)2.$$

Using ordered pairs in a way similar to that in which ordered triads were used to obtain a λ-definition of the predecessor function, we give a λ-definition of the function <u>the least integer not less than half of</u> x -- or, in other words, the quotient upon dividing $x+1$ by 2, in the sense of division with a remainder:

$$H \longrightarrow \lambda a.P(2_1(a(\lambda b[P[2_1 b + 2_2 b], 3 \overset{.}{-} 2_2 b])[1, 2])).$$

Of course this H is unrelated to the -- entirely different -- function H which was introduced for illustration in §1.
If we let

$$\mathfrak{L} \longrightarrow \lambda b.b(\lambda c \lambda d[dPc(\lambda e.e1 I)(\lambda fg.fgS)c,$$
$$dPc(\lambda h.h1 IS)(\lambda ij k.kij(\lambda l.l1))d]),$$

$$\mathfrak{U} \longrightarrow \lambda a.a\mathfrak{L}[1, 1],$$

$$Z \longrightarrow \lambda a.2_2(\mathfrak{U}a),$$

$$Z' \longrightarrow \lambda a.\mathfrak{U}a(\lambda bc.b \overset{.}{-} c),$$

then, if M, N represent the positive integers m, n respectively, $\mathfrak{L}[M, N]$ conv $[SM, 1]$ if $m \overset{.}{-} n = 1$ and conv $[M, SN]$ if $m \overset{.}{-} n > 1$; hence $\mathfrak{U}1, \mathfrak{U}2, \ldots$ are convertible respectively into

$[2, 1], [3, 1], [3, 2], [4, 1], [4, 2], [4, 3], [5, 1], \ldots$;

hence $Z1, Z2, \ldots$ are convertible respectively into

$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$,

and $Z'1$, $Z'2$, ... are convertible respectively into

$$1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 5, 4, 3, 2, 1, \ldots .$$

Thus the infinite sequence of ordered pairs,

$$[Z1, Z'1], [Z2, Z'2], [Z3, Z'3], \ldots,$$

contains all ordered pairs of positive integers, with no repetitions. The function whose value for the arguments x, y is the number of the ordered pair $[x, y]$ in this enumeration is λ-defined by

$$nr \longrightarrow \lambda ab . S(H[[a+b] \times P[a+b]]) \doteq b.$$

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10. PROPOSITIONAL FUNCTIONS; THE KLEENE p-FUNCTION. By a propositional function we shall mean a function (of one or more variables) whose values are truth values — i.e., truth and falsehood. A property is a propositional function of one variable; a relation is a propositional function of two variables. The characteristic function associated with a propositional function is the function whose value is 2 when (i.e., for an argument or arguments for which) the value of the propositional function is truth, whose value is 1 when the value of the propositional function is falsehood, and which has no value otherwise.
  A propositional function of positive integers will be said to be λ-definable if the associated characteristic function is a λ-definable function. (It can readily be shown that the choice of the particular integers 2 and 1 in the definition of characteristic function is here non-essential; the class of λ-definable propositional functions of positive integers remains unaltered if any other pair of distinct positive integers is substituted.)
  In particular, the relations $>$ and $=$ between positive integers are λ-definable, as is shown by giving λ-definitions of the associated characteristic functions:

$$exc \rightarrow \lambda ab . \min 2 \ [S a \dot{-} b].$$

$$eq \rightarrow \lambda ab . \ 4 \dot{-} . \ exc \ ab + exc \ ba.$$

From this follows the λ-definability of a great variety of prop-
erties and relations of positive integers which are expressible
by means of equations and inequalities; conjunction, disjunction,
and negation of equations and inequalities can be provided for
by using min, max, and $\lambda a . 3 \dot{-} a$ respectively.

　　　We prove also the two following theorems from Kleene [35],
and a third closely related theorem:

10 I.　If R is a λ-definable propositional function of
　　　　$n+1$ positive integer arguments, then the function
　　　　F is λ-definable (1) whose value for the positive
　　　　integer arguments x_1, x_2, ..., x_n is the least pos-
　　　　itive integer y such that $Rx_1 x_2 ... x_n y$ holds (i.e.,
　　　　has the value truth), provided that there is such a
　　　　least positive integer y and that, for every posi-
　　　　tive integer z less than this y, $Rx_1 x_2 ... x_n z$ has
　　　　a value, truth or falsehood, and (2) which has no value
　　　　otherwise.

　　　In the case that R has a value for every set of $n+1$ pos-
itive integer arguments, F may be described simply by saying
that $Fx_1 x_2 ... x_n$ is the least positive integer y such that
$Rx_1 x_2 ... x_n y$ holds.
　　　Let

$$\mathfrak{G} \rightarrow \lambda n . n(\lambda r . r(\lambda s . s \ 1 \ II(\lambda xgt . g \ 1(t x) I x)))$$

$$(\lambda f . f \ I \ 1 II)(\lambda xgt . g(t(Sx))(Sx)gt).$$

Then

$$\mathfrak{G}1 \ red \ \lambda xgt . g(t(Sx))(Sx)gt,$$

$$\mathfrak{G}2 \ red \ \lambda xgt . g \ 1(t x) I x.$$

Hence if N represents a positive integer and TN conv either
1 or 2, we have (using 7 XXVIII to show that TN red 1 or 2),

$$\mathfrak{G}1 N \mathfrak{G} T \ red \ \mathfrak{G}(T(SN))(SN)\mathfrak{G} T,$$

$$\mathfrak{G}2N\mathfrak{G}T \text{ red } N.$$

Hence if we let

$$\mathsf{p} \longrightarrow \lambda tx.\mathfrak{G}(t\,x)x\mathfrak{G}t,$$

we have   ꝓ*TN* red *N*   if   *TN* conv 2,   and   ꝓ*TN* conv ꝓ*T*(*SN*)   if
TN conv 1,   and (by 7 XXXI, 7 XXXII)   ꝓ*TN*   has no normal form
if   *TN*   has no normal form.

   If   *N*   represents the positive integer   *n*   and   *T* λ-defines
the characteristic function associated with the property   *T*   of
positive integers, it follows that   ꝓ*TN*   is convertible into the
formula which represents the least positive integer   *y*,   not
less than   *n*,   for which   *Ty*   holds, provided that there is such
a least positive integer   *y*   and that, for every positive inte-
ger   *z*   less than this   *y*   and not less than   *n*,   *Tz*   has a val-
ue, truth or falsehood; and that in any other case ꝓ*TN*   has no
normal form (in the case that   *Ty*   has the value falsehood for
all positive integers   *y*   not less than   *n*,   we have

ꝓ*TN* red $\mathfrak{G}(T N)N\mathfrak{G}T$ red $\mathfrak{G}(T(SN))(SN)\mathfrak{G}T$ red $\mathfrak{G}(T(S(SN)))(S(SN))\mathfrak{G}T$

$$\text{red } \ldots$$

to infinity, and hence no normal form by 7 XXXI).

   Let   *R*   be a formula which   λ-defines the characteristic
function associated with the propositional function   *R*   referred
to in 10 I. Then   *F*   is   λ-defined by

$$\lambda x_1 x_2 \ldots x_n.\mathsf{p}(Rx_1 x_2 \ldots x_n)1.$$

10 II.   If   *T*   is a   λ-definable property of positive inte-
     gers, the function   *F*   is   λ-definable (1) whose val-
     ue for the positive integer argument   *x*   is the   *x*th
     positive integer   *y*   (in the order of magnitude of
     the positive integers) such that   *Ty*   holds, provided
     that there is such a positive integer   *y*   and that,
     for every positive integer   *z*   less than   *y*,   *Tz*   has
     a value, truth or falsehood, and (2) which has no
     value otherwise.

For let T be a formula which λ-defines the characteristic function associated with T, and let

$$\wp \longrightarrow \lambda t x.P(x(\lambda n.S(\wp\, t\, n))1).$$

Then $\wp T$ λ-defines F.

10 III. If R_1 and R_2 are λ-definable propositional functions each of $n+1$ positive integer arguments, then the propositional function R is λ-definable

(1) whose value for the positive integer arguments x_1, x_2, ..., x_n is falsehood if there is a positive integer y such that $R_1 x_1 x_2 \ldots x_n y$ holds and $R_1 x_1 x_2 \ldots x_n z$ and $R_2 x_1 x_2 \ldots x_n z$ both have the value falsehood for every positive integer z less than y, and

(2) whose value for the positive integer arguments x_1, x_2. ..., x_n is truth if there is a positive integer y such that $R_2 x_1 x_2 \ldots x_n y$ holds and $R_1 x_1 x_2 \ldots x_n y$ has the value falsehood and $R_1 x_1 x_2 \ldots x_n z$ and $R_2 x_1 x_2 \ldots x_n z$ both have the value falsehood for every positive integer z less than y, and

(3) which has no value otherwise.

Let

$$\text{alt} \longrightarrow \lambda x y n.\text{par}n(\lambda a.a(\lambda b.b1Iy))(\lambda c.c(\lambda def.fde))x(Hn).$$

$$\pi \longrightarrow \lambda x y.\text{par}(\wp(\text{alt } xy)1).$$

.If F and G are functions of positive integers, each being a function of one argument and including the integer 1 in its range of arguments, and if F and G λ-define F and G respectively, then alt FG λ-defines the function whose value for the odd integer $2x-1$ is Fx and whose value for the even integer $2x$ is Gx.

If R_1 and R_2 λ-define the characteristic functions associated with R_1 and R_2 respectively, then the characteristic function associated with R is λ-defined by

$$\lambda x_1 x_2 \ldots x_n \cdot \pi (R_1 x_1 x_2 \ldots x_n)(R_2 x_1 x_2 \ldots x_n)$$

-- this completes the proof of 10 III.

Formulas having the essential properties of p and P were first obtained by Kleene. These formulas λ-define (in a sense which will be readily understood without explicit definition) certain functions of functions of positive integers, as already indicated.

As a further application of the formula p, we give λ-definitions of subtraction of positive integers in the ordinary sense (so that $x-y$ has no value if $x \leq y$) and exact division (so that $x \div y$ has no value unless x is a multiple of y):

$$[M-N] \longrightarrow p(\lambda \alpha \cdot \text{eq } M [N+\alpha])1.$$

$$[M \div N] \longrightarrow p(\lambda \alpha \cdot \text{eq } M [N \times \alpha])1.$$

11. DEFINITION BY RECURSION. A function F of n positive integer arguments is said to be defined by <u>composition</u> in terms of the functions G and H_1, H_2, ..., H_m of positive integers (of the indicated numbers of arguments) by the equation,

$$F x_1 x_2 \ldots x_n = G(H_1 x_1 x_2 \ldots x_n)(H_2 x_1 x_2 \ldots x_n) \ldots (H_m x_1 x_2 \ldots x_n).$$

(The case is not excluded that m or n or both are 1.)

A function F of n+1 positive integer arguments is said to be defined by <u>primitive recursion</u> in terms of the functions G_1 and G_2 of positive integers (of the indicated numbers of arguments) by the pair of equations:

$$F x_1 x_2 \ldots x_n 1 = G_1 x_1 x_2 \ldots x_n,$$
$$F x_1 x_2 \ldots x_n (y+1) = G_2 x_1 x_2 \ldots x_n y (F x_1 x_2 \ldots x_n y).$$

(The case is not excluded that n = 0, the function G_1 being replaced in that case by a given positive integer a.)

The class of <u>primitive recursive functions</u> of positive integers is defined by the three following rules, a function being primitive recursive if and only if it is determined as such by these rules:

(1) The function C such that $Cx = 1$ for every posi-
tive integer x, the successor function of positive integers,
and the functions U_1^n (where n is any positive integer and
1 is any positive integer not greater than n) such that
$U_1^n x_1 x_2 \ldots x_n = x_1$, are primitive recursive.

(2) If the function F of n arguments is defined by
composition in terms of the functions G and H_1, H_2, ..., H_m
and if G, H_1, H_2, ..., H_m are primitive recursive, then F is
primitive recursive.

(3) If the function F of n+1 arguments is defined by
primitive recursion in terms of the functions G_1 and G_2 and
if G_1 and G_2 are primitive recursive, then F is primitive
recursive; or in the case that n = 0, if F is defined by
primitive recursion in terms of the integer a and the function
G_2 and if G_2 is primitive recursive, then F is primitive re-
cursive.

In order to show that every primitive recursive function
of positive integers is λ-definable, we must show that all the
functions mentioned in (1) are λ-definable; that if F is de-
fined by composition in terms of G and H_1, H_2, ..., H_m and
G, H_1, H_2, ..., H_m are λ-definable, then F is λ-definable;
and that if F is defined by primitive recursion in terms of
G_1 and G_2(or, in the case n = 0, in terms of a and G_2)
and if G_1 and G_2 are λ-definable (or, in the case n = 0,
if G_2 is λ-definable), then F is λ-definable.

Only the last of these three things makes any difficulty.
Suppose that F is defined by primitive recursion in terms of
G_1 and G_2, and that G_1 and G_2 are λ-defined respectively
by $\mathbf{G}_1$ and $\mathbf{G}_2$. Then in order to obtain a formula $\mathbf{F}$ which
λ-defines F we employ ordered triads:

$$\mathbf{F} \longrightarrow \lambda x_1 x_2 \ldots x_n y . \, 3_3(y(\lambda z[S(3_1 z),$$
$$\mathbf{G}_2 x_1 x_2 \ldots x_n (3_1 z)(3_2 z), \; 3_2 z])[1, \mathbf{G}_1 x_1 x_2 \ldots x_n, \; 1])$$

(x_1, x_2, ..., x_n, y, z being any n+2 distinct variables). In
the case n = 0, this reduces to:

$$F \longrightarrow \lambda y.\, 3_3(y(\lambda z[S(3_1 z), \; G_2(3_1 z)(3_2 z), \; 3_2 z])[1, \; A, \; 1]),$$

where A represents the positive integer a.

(The λ-definition of the predecessor function given in §9 may be regarded as a special case of the foregoing in which a is 1 and G_2 is U_1^2. The extension of the method used for the predecessor function to the general case of definition by primitive recursion is due to Paul Bernays, in a letter of May 27th, 1935 -- where, however, the matter is stated within the context of the calculus of λ-K-conversion and ordered pairs are consequently used instead of ordered triads. As remarked by Bernays, this method of dealing with definition by primitive recursion has the advantage that it shows also, for each n, the λ-definability of the function ρ of functions of positive integers whose value for the arguments G_1 and G_2 is the function F defined by primitive recursion in terms of G_1 and G_2 -- i.e., essentially, the function ρ of Hilbert [31].)

Thus we have:

11 I.   Every primitive recursive function of positive integers
        is λ-definable.

The class of primitive recursive functions is known to in-clude substantially all the ordinarily used numerical functions -- cf., e.g., Skolem [50], Gödel [27], Péter [41] (it is readily seen to be a non-essential difference that some of these authors deal with primitive recursive functions of non-negative integers rather than of positive integers). Primitive recursive, in par-ticular, are functions corresponding to the quotient and remain-der in division, the greatest common divisor, the xth prime num-ber, and many related functions; λ-definitions of these func-tions can consequently be obtained by the method just given.
    The two schemata, of definition by composition and by prim-itive recursion, have this property in common, that -- on the hypothesis that all particular values are known of the functions in terms of which F is defined -- the given equations make pos-sible the calculation of any required particular value of F by

a series of steps each consisting of a substitution, either of
a (symbol for a) particular number for (all occurrences of) a
variable, or of one thing for another known to be equal to it.
By allowing additional, or more general, schemata having this
property, various more extensive notions of recursiveness are
obtainable (cf. Hilbert [31], Ackermann [1], Péter [41, 42, 43,
44]). If the definition of primitive recursiveness is modified
by allowing, in place of (2) and (3), any definition by a set of
equations having this property, the functions obtained are called
general recursive -- if it is required of all functions defined
that they have a value for every set of the relevant number of
positive integer arguments -- or partial recursive if this is
not required. For a more exact statement (which may be made in
any one of several equivalent ways), the reader is referred to
Gödel [28], Church [9], Kleene [36, 39], Hilbert and Bernays
[33].

   That every general recursive function of positive integers
is λ-definable can be proved in consequence of 10 I and 11 I
by using the result of Kleene [36], that every general recursive
function of n positive integer arguments x_1, x_2, ..., x_n can
be expressed in the form $F(\epsilon y(R x_1 x_2 ... x_n y))$, where F is a
primitive recursive function of positive integers, R is a prop-
ositional function of positive integers whose associated char-
acteristic function is primitive recursive, and "ϵy" is to be
read "the least positive integer y such that."   (Cf. Kleene
[37]). The converse proposition, that every λ-definable func-
tion of positive integers, having a value for every set of the
relevant number of positive integer arguments, is general recur-
sive, is proved by the method of Church [9] or Kleene [37] (the
proof makes use of the fact that, by 7 XXXI, the process of re-
duction to normal form provides a method of calculating explic-
itly any required particular value of a function whose λ-defi-
nition is given, and proceeds by setting up a set of recursion
equations which in effect describe this process of calculation).

   These proofs may be extended to the case of partial recur-
sive functions without major modifications (cf. Kleene [39]).
Hence are obtained the following theorems (proofs omitted here):

11 II. Every partial recursive function of positive integers

is λ-definable.

11 III.   Every λ-definable function of positive integers
          is partial recursive.

The notion of a method of effective calculation of the val-
ues of a function, or the notion of a function for which such a
method of calculation exists, is of not uncommon occurrence in
connection with mathematical questions, but it is ordinarily left
on the intuitive level, without attempt at explicit definition.
The known theorems concerning λ-definability, or recursiveness,
strongly suggest that the notion of an effectively calculable
function of positive integers be given an exact definition by
identifying it with that of a λ-definable function, or equiva-
lently of a partial recursive function. As in all cases where a
formal definition is offered of what was previously an intuitive
or empirical idea, no complete proof is possible; but the writer
has little doubt of the finality of the identification. (Con-
cerning the origin of this proposal, see Church [9],.footnotes
3, 18.)
        An equivalent definition of effective calculability is to
identify it with calculability within a formalized system of
logic whose postulates and rules have appropriate properties of
recursiveness -- cf. Church [9], §7, Hilbert and Bernays [33],
Supplement II.
        Another equivalent definition, having a more immediate in-
tuitive appeal is that of Turing [55], who calls a function com-
putable if (roughly speaking) it is possible to make a finite
calculating machine capable of computing any required value of
the function. The machine is supplied with a tape on which com-
putations are printed (the analogue of the paper used by a human
calculator), and no upper limit is placed on the length of tape
or on the time required for computation of a particular value of
the function, except that it be finite in each case. Further re-
strictions imposed on the character of the machine are more or
less clearly either non-essential or necessarily contained in the
requirement of finiteness. The equivalence of computability to
λ-definability and general recursiveness (attention being confined
to functions of one argument for which the range of arguments con-

sists of all positive integers) is proved in Turing [57].

Mention should also be made of the notion of a *finite com-
binatory process* introduced by Post [46]. This again is equiva-
lent to the other concepts of effective calculability.

Examples of functions which are not effectively calculable
can now be given in various ways. In particular, it is proved
in Church [9] that if the set of well-formed formulas of the cal-
culus of λ-conversion be enumerated in a straightforward way
(any one of the particular enumerations which immediately suggest
themselves may be employed), and if F is the function such that
F is 2 or 1 according as the xth formula in this enumeration
has or has not a normal form, then F is not λ-definable. This
may be taken as the exact meaning of the somewhat vague statement
made at the end of §6, that the condition of having a normal form
is not effective.

In the explicit proofs of many of the theorems which have
been stated without proof in this section, use is made of the
notion of the Gödel number of a formula or formal expression.
In the published papers referred to, this notion is introduced
by a method closely similar to that employed by Gödel [27]. In
the case of well-formed formulas of the calculus of λ-conversion,
however, it would be equally possible to use the somewhat differ-
ent method of our next chapter.

COMBINATIONS, GÖDEL NUMBERS

12. COMBINATIONS. If s is any set of well-formed formulas, the class of s-combinations is defined by the two following rules, a formula being an s-combination if and only if it is determined as such by these rules:

(1) Any formula of the set s, and any variable standing alone, is an s-combination.

(2) If *A* and *B* are s-combinations, *AB* is an s-combination.

In the cases in which we shall be interested the formulas of s will contain no free variables and will none of them be of the form *AB*. In such a case it is possible to distinguish the terms of an s-combination, each occurrence of a free variable or of one of the formulas of s being a term.

If s is the null set, the s-combinations will be called combinations of variables.

If s consists of the two formulas *I*, *J*, where

$$I \longrightarrow \lambda a a,$$
$$J \longrightarrow \lambda a b c d . a b (a d c),$$

the s-combinations will be called simply combinations.

We shall prove that every well-formed formula is convertible into a combination. This theorem is taken from Rosser [47], the present proof of it from Church [8]; the ideas involved go back to Schönfinkel [49] and Curry [18, 21].

Let:

$$\tau \longrightarrow JII.$$

Then τ conv $\lambda a b . b a$, and hence τ*AB* conv *BA*.

If M is any combination containing x as a free variable, we define an associated combination $\lambda_x M|$, which does not contain x as a free variable but otherwise contains the same free variables as M. This definition is by recursion, according to the following rules:

(1) $\lambda_x x|$ is I.

(2) If B contains x as a free variable and A does not, $\lambda_x AB|$ is $J\tau\lambda_x B|(JIA)$.

(3) If A contains x as a free variable and B does not, $\lambda_x AB|$ is $J\tau B\lambda_x A|$.

(4) If both A and B contain x as a free variable, $\lambda_x AB|$ is $J\tau\tau(JI(J\tau\tau(J\tau\lambda_x B|(J\tau\lambda_x A|J))))$.

12 I. If M is a combination containing x as a free variable, $\lambda_x M|$ conv $\lambda x M$.

We prove this by induction with respect to the number of terms of M.

If M has one term, then M is x, and $\lambda_x M|$ is I, which is convertible into $\lambda x x$.

If M is AB and B contains x as a free variable and A does not, then $\lambda_x M|$ is $J\tau\lambda_x B|(JIA)$, which (see definitions of I, J, τ) is convertible into $\lambda d.A(\lambda_x B|d)$, which, by hypothesis of induction, is convertible into $\lambda d.A((\lambda x B)d)$ which finally is convertible into $\lambda x.AB$.

If M is AB and A contains x as a free variable and B does not, then $\lambda_x M|$ is $J\tau B\lambda_x A|$, which is convertible into $\lambda d.\lambda_x A|dB$, which, by hypothesis of induction is convertible into $\lambda d.(\lambda x A)dB$, which finally is convertible into $\lambda x.AB$.

If M is AB and both A and B contain x as a free variable, then $\lambda_x M|$ is $J\tau\tau(JI(J\tau\tau(J\tau\lambda_x B|(J\tau\lambda_x A|J))))$, which is convertible into $\lambda d.\lambda_x A|d(\lambda_x B|d)$, which, by hypothesis of induction, is convertible into $\lambda d.(\lambda x A)d((\lambda x B)d)$, which finally is convertible into $\lambda x.AB$.

The foregoing tacitly assumes that A and B do not contain d as a free variable. The modification necessary for the contrary case is, however, obvious.

This completes the proof of 12 I. We define the combination belonging to a well-formed formula, by recursion as follows:

    (1) The combination belonging to x is x (where x
        is any variable).

    (2) The combination belonging to FA is $F'A'$, where
        F' and A' are the combinations belonging to F
        and A respectively.

    (3) The combination belonging to $\lambda x M$ is $\lambda_x M' |$, where
        M' is the combination belonging to M.

12 II.   Every well-formed formula is convertible into the com-
       bination belonging to it.

    Using 12 I, this is proved by induction with respect to the
length of the formula. The proof is straightforward and details
are left to the reader.

12 III. The combination belonging to X and the combination
       belonging to Y are identical if and only if X
       conv-I Y.

    13. PRIMITIVE SETS OF FORMULAS. A set s of well-formed
formulas is called a **primitive set**, if the formulas of s con-
tain no free variables and are none of them of the form AB,
and every well-formed formula is convertible into an s-combi-
nation. (When necessary to distinguish this idea from the analo-
gous idea in the calculus of λ-K-conversion, the calculus of
λ-δ-conversion, etc. -- see Chapter V -- we may speak of primi-
tive sets of λ-formulas, primitive sets of λ-K-formulas, prim-
itive sets of λ-δ-formulas, etc.)
    It was proved in §12 that the formulas I, J are a primi-
tive set. Another primitive set of formulas, suggested by the
work of Curry, consists of the four formulas B, C, W, I, where:

$$B \longrightarrow \lambda abc.a(bc).$$

$$C \longrightarrow \lambda abc.acb.$$

$$W \longrightarrow \lambda ab.abb.$$

    In order to prove this it is sufficient to express J as a
$\{B, C, W, I\}$-combination, as follows:

$$J \text{ conv } B(BC(BC))(B(W(BBB))C).$$

Still another primitive set of formulas consists of the four formulas B, T, D, I, where:

$$T \longrightarrow \lambda ab.ba.$$

$$D \longrightarrow \lambda a.aa.$$

In order to prove this it is sufficient to express C and W as $\{B, T, D, I\}$-combinations, as follows:

$$C \text{ conv } B(T(BBT))(BBT).$$

$$W \text{ conv } B(B(T(BD(B(TT)(B(BBB)T))))(BBT))(B(T(B(TI)(TI)))B).$$

A primitive set of formulas is said to be <u>independent</u> if it ceases to be a primitive set upon omission of any one of the formulas. It seems plausible that each of the three primitive sets which have been named is independent. -- In the case of the set $\{I, J\}$, the independence of J follows (using 7 XVII) from the fact that any combination all of whose terms are I is convertible into I; and the independence of I follows (using 7 XXVIII) from the fact that if A imr B and B contains a (well-formed) part convertible-I into I then A must contain a (well-formed) part convertible-I into I.

14. AN APPLICATION OF THE THEORY OF COMBINATIONS. We prove now the following theorems, due to Kleene [34, 35, 37]:

14 I. If A_1 and A_2 contain no free variables, a formula L can be found such that $L1 \text{ conv } A_1$ and $L2 \text{ conv } A_2$.

For, by 12 II, A_1 and A_2 are convertible into combinations A_1' and A_2' respectively. We take A_1' to be the combination belonging to A_1, unless that combination fails to contain an occurrence of J, in which case we take A_1' to be $JIIII$; and A_2' is similarly determined relatively to A_2. Let A_1'' and A_2'' be the result of replacing all occurrences of J by the variable j in A_1' and A_2' respectively, and let B_1 and

B_2 be $\lambda j A_1''$ and $\lambda j A_2''$ respectively. Then $B_1 J$ conv A_1, and $B_2 J$ conv A_2, and $B_1 I$ conv I, and $B_2 I$ conv I. Consequently a formula L having the required property is:

$$\lambda n . n(\lambda x . x(\lambda y . y I B_2))(\lambda z . z I I) B_1 J.$$

14 II.   If A_1, A_2, ..., A_n contain no free variables, a
         formula L can be found such that $L1$ conv A_1,
         $L2$ conv A_2, ..., LN conv A_n(N being the formula
         which represents n).

For the case that n is 1 or 2, this follows from 14 I.
For larger values of n, we prove it by induction.
    Let L_2 be a formula such that $L_2 1$ conv A_1, and let L_1
be a formula such that $L_1 1$ conv A_2, $L_1 2$ conv A_3, ..., $L_1 M$ conv
A_n (where M represents n-1). Also let G be a formula such
that $G1$ conv L_1 and $G2$ conv L_2. Then a formula L having
the required property is:

$$\lambda i . G[\overset{\centerdot}{3} i](P i).$$

14 III.   If A_1, A_2, ..., A_n, F_1, F_2, ..., F_m contain no
         free variables, a formula E can be found which
         represents an enumeration of the least set of formu-
         las which contains A_1, A_2, ..., A_n and is closed
         under each of the operations of forming $F_\alpha XY$ from the
         formulas X, Y (α = 1, 2, ..., n), in the sense that
         every formula of this set is convertible into one of
         the formulas in the infinite sequence

$$E1, \ E2, \ ...,$$

         and every formula in this infinite sequence is con-
         vertible into one of the formulas of the set.

We prove this first for the case $m = 1$, using a device
due to Kleene for obtaining formulas satisfying arbitrary con-
version conditions of the general kind illustrated in (1) below.
    Using 14 II, let U be a formula such that

$$U1 \text{ conv } I,$$

$$U2 \text{ conv } \lambda xy. F_1 (y(S[N'\dot{-}Zx])[Zx\dot{-}N]y)(y(S[N'\dot{-}Z'x])[Z'x\dot{-}N]y),$$

$$U3 \text{ conv } \lambda xy. yxA_1,$$

$$U4 \text{ conv } \lambda xy. yxA_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$UN' \text{ conv } \lambda xy. yxA_n,$$

where N represents n and N' represents n+2, and Z and Z' are the formulas introduced in §9. Let E be the formula,

$$\lambda i. U(S[N' \dot{-} i])[i \dot{-} N]U.$$

Then we have:

$$E1 \text{ conv } A_n,$$

$$E2 \text{ conv } A_{n-1},$$

(1) $\cdots\cdots\cdots\cdots\cdots\cdots$

$$EN \text{ conv } A_1,$$

$$EK \text{ conv } F_1 (E(Z[K\dot{-}N]))(E(Z'[K\dot{-}N])),$$

K being any formula which represents an integer greater than n. From this it follows that E is a formula of the kind required.

Consider now the case m $>$ 1. Let M represent m and let F be a formula such that $F1$ conv F_1, $E2$ conv F_2, ..., FM conv F_m. By the preceding proof for the case m = 1, a formula E' can be found which represents an enumeration of the least set of formulas which contains [1, A_1], [2, A_1], ..., [M, A_1], [1, A_2], [2, A_2], ..., [M, A_2], ..., [1, A_n], [2, A_n], ..., [M, A_n] and is closed under the operation of forming $V(\lambda xy[x, XFy])$ from the formulas X, Y. Then a formula E of the kind required is:

$$\lambda i. 2_2(E'i).$$

It is immaterial that the enumeration so obtained contains repetitions. (Notice that $2_2[B, C]$ conv C if B is any formula such that BI conv I, in particular if B is any formula

representing a positive integer; the case considered in §9 that
B and **C** both represent positive integers is thus only a spe-
cial case.)

14 IV. If $A_1, A_2, \ldots, A_n, F_1, F_2, \ldots, F_m, F_{m+1}, F_{m+2}, \ldots,$
F_{m+r} contain no free variables, a formula **E** can be
found which represents an enumeration of the least set
of formulas which contains $A_1, A_2, \ldots, A_n$ and is
closed under each of the operations of forming $F_\alpha XY$
from the formulas X, Y ($\alpha = 1, 2, \ldots, m$) and of
forming $F_{m+\beta} X$ from the formula X ($\beta = 1, 2, \ldots, r$)
-- in the sense that every formula of this set is con-
vertible into one of the formulas in the infinite se-
quence
$$E1, E2, \ldots,$$

and every formula in this infinite sequence is convert-
ible into one of the formulas of the set.

(The case is not excluded that m = 0 or that r = 0, provided
that m and r are not both 0.)
    By the method used in the proof of 14 I, find formulas B_1,
$B_2, \ldots, B_n, C_1, C_2, \ldots, C_{m+r}$ such that $B_1 J$ conv A_1, $B_2 J$ conv
$A_2, \ldots, B_n J$ conv A_n, $C_1 J$ conv F_1, $C_2 J$ conv F_2, $\ldots, C_{m+r} J$ conv
F_{m+r}, and $B_1 I$ conv I, $B_2 I$ conv I, $\ldots, B_n I$ conv I, $C_1 I$ conv I,
$C_2 I$ conv I, $\ldots, C_{m+r} I$ conv I. By 14 III, a formula E' can be
found which represents an enumeration of the least set of formu-
las which contains $B_1, B_2, \ldots, B_n$ and is closed under each of
the operations of forming $\lambda x . C_\alpha x (Xx)(Yx)$ from the formulas X, Y
($\alpha = 1, 2, \ldots, m$) and of forming $\lambda x . Y I C_{m+\beta} x (Xx)$ from the for-
mulas X, Y ($\beta = 1, 2, \ldots, r$). Then a formula **E** of the kind
required is:
$$\lambda i . E' i J.$$

    15. A COMBINATORY EQUIVALENT OF CONVERSION. It is desir-
able to have a set of operations (upon combinations) which have
the property that they always change a combination into a com-
bination and which constitute an equivalent of conversion in the
sense that a combination X can be changed into a combination

Y by a sequence of (0 or more of) these operations if and only if *X* conv *Y*. Such a set of operations is the following (OI - OXXXVIII) -- where *F*, *A*, *B*, *C*, *D* are arbitrary combinations, β, γ, ω are defined as indicated below, and the sign ⊢ is used to mean that the combination which precedes ⊢ is changed by the operation into the combination which follows:

OI. *IA* ⊢ *A*.

OII. *A* ⊢ *IA*.

OIII. *F*(*IA*) ⊢ *FA*.

OIV. *FA* ⊢ *F*(*IA*).

OV. *F*(*IAB*) ⊢ *F*(*AB*).

OVI. *F*(*AB*) ⊢ *F*(*IAB*).

OVII. *F*(*JABCD*) ⊢ *F*(*AB*(*ADC*)).

OVIII. *F*(*AB*(*ADC*)) ⊢ *F*(*JABCD*).

OIX. *FJ* ⊢ *F*(ω(βγ(β(β(βγ))(β(β(βββ)))*I*))))).

OX. *F*(ω(βγ(β(β(βγ))(β(β(βββ)))*I*)))) ⊢ *FJ*.

OXI. *F*β ⊢ *F*(β(β(β*I*))β).

OXII. *F*(β(β(β*I*))β) ⊢ *F*β.

OXIII. *F*γ ⊢ *F*(β(β(β*I*))γ).

OXIV. *F*(β(β(β*I*))γ) ⊢ *F*γ.

OXV. *FI* ⊢ *F*(β*II*).

OXVI. *F*(β*II*) ⊢ *FI*.

OXVII. *F*(γ(ββ(βββ))β) ⊢ *F*(β(ββ)β).

OXVIII. *F*(β(ββ)β) ⊢ *F*(γ(ββ(βββ))β).

OXIX. *F*(γ(ββ(βββ))γ) ⊢ *F*(β(βγ)(βββ)).

OXX. *F*(β(βγ)(βββ)) ⊢ *F*(γ(ββ(βββ))γ).

OXXI. *F*(γ(βββ)ω) ⊢ *F*(β(βω)(βββ)).

OXXII. *F*(β(βω)(βββ)) ⊢ *F*(γ(βββ)ω).

OXXIII. *F*(γβ*I*) ⊢ *F*(β(β*I*)*I*).

OXXIV. *F*(β(β*I*)*I*) ⊢ *F*(γβ*I*).

OXXV. *F*(ββγ) ⊢ *F*(β(β(βγ)γ)(ββ)).

OXXVI. *F*(β(β(βγ)γ)(ββ)) ⊢ *F*(ββγ).

OXXVII. *F*(ββω) ⊢ *F*(β(β(β(β(βω)ω)(βγ))(β(ββ)))β).

OXXVIII. *F*(β(β(β(β(βω)ω)(βγ))(β(ββ)))β) ⊢ *F*(ββω).

OXXIX. *F*(βγγ) ⊢ *F*(β(β*I*)).

OXXX. *F*(β(β*I*)) ⊢ *F*(βγγ).

OXXXI. *F*(β(β(βγ)γ)(βγ)) ⊢ *F*(β(βγ(βγ))γ).

OXXXII. *F*(β(βγ(βγ))γ) ⊢ *F*(β(β(βγ)γ)(βγ)).

OXXXIII. $F(\beta\gamma\omega) \vdash F(\beta(\beta(\beta\omega)\gamma)(\beta\gamma))$.

OXXXIV. $F(\beta(\beta(\beta\omega)\gamma)(\beta\gamma)) \vdash F(\beta\gamma\omega)$.

OXXXV. $F(\beta\omega\gamma) \vdash F\omega$.

OXXXVI. $F\omega \vdash F(\beta\omega\gamma)$.

OXXXVII. $F(\beta\omega\omega) \vdash F(\beta\omega(\beta\omega))$.

OXXXVIII. $F(\beta\omega(\beta\omega)) \vdash F(\beta\omega\omega)$.

$$\gamma \longrightarrow J\tau(J\tau)(J\tau).$$

$$\beta \longrightarrow \gamma(JI\gamma)(JI).$$

$$\omega \longrightarrow \gamma(\gamma(\beta\gamma(\gamma(\beta J\tau)\tau))\tau).$$

(Note that τ, γ, β, ω are convertible respectively into *T*, *C*, ,
B, *W*.)

These thirty-eight operations have characteristics of sim-
plicity not possessed by the operations I, II, III of §6, name-
ly: (1) they are one-valued, i.e., given the combination op-
erated on and the particular one of the thirty-eight operations
which is applied, the combination resulting is uniquely deter-
mined; (2) they do not involve the idea of substitution at an
arbitrary place, but only that of substitution at a specified
place. This has the effect of rendering some of the develop-
ments in §16 much simpler than they otherwise might be.

The proof of the equivalence of OI-OXXXVIII to conversion
is too long to be included here. It may be found in Rosser's
dissertation [47] (cf. Section H therein). Many of the import-
ant ideas and methods involved derive from Curry [17, 18, 20,
21]; in fact, Curry has results which may be thought of as con-
stituting an approximate equivalent to the one in question here
but which are nevertheless sufficiently different so that we are
unable to use them directly.

16. GÖDEL NUMBERS. The <u>Gödel number of</u> a combination is
defined by induction as follows:

(1) The Gödel number of *I* is 1.

(2) The Gödel number of *J* is 3.

(3) The Gödel number of the *n*th variable in alphabetical order
     (see §5) is $2n+5$.

(4) If *m* and *n* are the Gödel numbers of *A* and *B* respec-

tively, the Gödel number of AB is $(m+n)(m+n-1)-2n+2$.

The <u>Gödel number belonging to</u> a formula is defined to be
the Gödel number of the combination belonging to the formula.
(Notice that the Gödel number belonging to a combination is thus
in general not the same as the Gödel number of the combination.)
It is left to the reader to verify that the Gödel numbers
of two combinations A and B are the same if and only if A
and B are the same; and that the Gödel numbers belonging to
two formulas A and B are the same if and only if A conv-I B
(cf. 12 III). (Notice that the Gödel number of AB, according
to (4), is twice the number of the ordered pair $[m, n]$ in the
enumeration of ordered pairs described at the end of §9.)
The usefulness of Gödel numbers arises from the fact that
our formalism contains no notations for formulas -- i.e., for
sequences of symbols. (It is not possible to use formulas as
notations for themselves, because interconvertible formulas must
denote the same thing although they are not the same formula, and
because formulas containing free variables cannot denote any
[fixed] thing.) The Gödel number belonging to a formula serves
in many situations as a substitute for a notation for the formu-
la and often enables us to accomplish things which might have
been thought to be impossible without a formal notation for for-
mulas.
This use of Gödel numbers is facilitated by the existence of
a formula, form, such that, if N represents the Gödel number be-
longing to A, and A contains no free variables, then, form N
conv A.   In order to obtain this formula, first notice that par N
conv 2   if N represents the Gödel number of a combination hav-
ing more than one term, and par N conv 1 if N represents the
Gödel number of a combination having only one term; also that if
 N represents the Gödel number of a combination AB, then $Z(HN)$
is convertible into the formula representing the Gödel number of
 A, and $Z'(HN)$ is convertible into the formula representing the
Gödel number of B (see §9). We introduce the abbreviations:

$$N_1 \longrightarrow Z(HN).$$

$$N_2 \longrightarrow Z'(HN).$$

Subscripts used in this way may be iterated, so that, for instance,

$$N_{122} \longrightarrow Z'(H(Z'(H(Z(HN))))).$$

By the method of §14, find a formula $\mathfrak{D}$ such that

$$\mathfrak{D}1 \text{ conv } \lambda x. x12.$$

$$\mathfrak{D}2 \text{ conv } I,$$

$$\mathfrak{D}3 \text{ conv } \lambda x. x12J,$$

and a formula $\mathfrak{U}$ such that

$$\mathfrak{U}1 \text{ conv } \mathfrak{D},$$

$$\mathfrak{U}2 \text{ conv } \lambda xy. y(\text{par } x_1)x_1 y(y(\text{par } x_2)x_2 y),$$

(these formulas $\mathfrak{D}$ and $\mathfrak{U}$ can be explicitly written down by referring to the proofs of 14 I and 14 II).

Let

$$\text{form} \longrightarrow \lambda n. \mathfrak{U}(\text{par } n)n\mathfrak{U}.$$

Then

$$\text{form } 1 \text{ conv } I,$$

$$\text{form } 3 \text{ conv } J, \text{ and}$$

$$\text{form } N \text{ conv form } N_1(\text{form } N_2)$$

if N represents an even positive integer. From this it follows that form has the property ascribed to it above; for if N represents the Gödel number of a combination A' belonging to a formula A, containing no free variables, then form N conv A', and A' conv A.

Let:

$$\sigma \rightarrow \lambda n \ . \ [\text{par}n + \text{par}n_1 + \text{eq}\overline{24812}n_{11} + [3 \doteq \text{eq}\overline{156}n_{12}] + \text{par}n_2 + \text{eq}\overline{12}n_{21} \doteq \overline{10}]$$

$$+ \ [2 \times [\text{par}n + \text{par}n_1 + \text{eq}\overline{24812}n_{11} + [3 \doteq \min(\text{par}n_2)(\text{eq}\overline{12}n_{21})] \doteq 6]]$$

$$+ \ [3 \times [\text{par}n + \text{eq}\overline{623375746}n_1 + \text{par}n_2 + \text{eq}\overline{12} \ n_{21} + \text{par}n_{22}$$

$$+ \text{eq}\overline{623375746}n_{221} + \text{par}n_{222} + \text{par}n_{2221} + \text{eq}\overline{24812}n_{22211}$$

$$+ \text{par}n_{2222} + \text{par}n_{22221} + \text{eq}\overline{24812}n_{222211} + \text{eq}3n_{22222} \doteq \overline{24}]]$$

$$\doteq 5.$$

Noting that the Gödel numbers of JI, τ, $J\tau$, $J\tau\tau$ are respectively 12, 156, 24812, 623375746, the reader may verify that:

$$\sigma N \text{ conv } 1, \ 2, \ 3, \text{ or } 4 \text{ if } N \text{ represents a positive integer;}$$

σN conv 2 if N represents the Gödel number of a combination of the form $J\tau B(JIA)$, with B different from τ;

σN conv 3 if N represents the Gödel number of a combination of the form $J\tau BA$ but not of the form $J\tau B(JIA)$;

σN conv 4 if N represents the Gödel number of a combination of the form $J\tau\tau(JI(J\tau\tau(J\tau B(J\tau AJ))))$;

σN conv 1 if N represents the Gödel number of a combination not of one of these three forms.

Again using §14, we find a formula u such that

u1 conv $\lambda xy.y5x$,

u2 conv $\lambda xy.y(\sigma x_{12})x_{12}y$,

u3 conv $\lambda xy.y(\sigma x_2)x_2 y$,

u4 conv $\lambda xy.\min(y(\sigma x_{22212})x_{22212}y)(y(\sigma x_{222212})x_{222212}y)$,

u5 conv $\lambda x.3^{\pm}x$,

and we let

$$o \longrightarrow \lambda n.u(\sigma n)nu.$$

Then o λ-defines a function of positive integers whose value is 2 for an argument which is the Gödel number of a combination of the form $\lambda_x M|$, and 1 for an argument which is the Gödel number of a combination not of this form -- or, as we shall say briefly, o λ-defines the property of a combination of being of the form $\lambda_x M$.

By similar constructions, involving lengthy detail but nothing new in principle, the following formulas may be obtained:

1) A formula, occ; such that, if N represents a positive integer n, we have that occ N λ-defines the property of a combination of containing the nth variable in alphabetical order, as a free variable (i.e., as a term).

2) A formula ϵ, such that, N representing a positive integer n, if G represents the Gödel number of a combination not of the form $\lambda_x M|$, then ϵNG conv G, and if G represents the Gödel number of a combination $\lambda_x M|$, then ϵNG is convertible into the formula representing the Gödel number of the combination obtained from M by substituting for all free occurrences of x in M the nth variable in alphabetical order.

3) A formula G, such that, if G represents the Gödel

number of a combination not of the form $\lambda_x M |$, then $\mathfrak{C}G$ conv G,
and if G represents the Gödel number of a combination $\lambda_x M |$,
then $\mathfrak{C}G$ is convertible into the formula representing the Gödel
number of the combination obtained from M by substituting for
all free occurrences of x in M the first variable in alpha-
betical order which does not occur in M as a free variable.

    4) A formula r which λ-defines the property of a com-
bination, that there is a formula to which it belongs.

    5) A formula $\wedge$ which λ-defines the property of a com-
bination of belonging to a formula of the form $\lambda x M$.

    6) A formula, prim, which λ-defines the property of a
combination of containing no free variables.

    7) A formula, norm, which λ-defines the property of a
combination of belonging to a formula which is in normal form.

    8) A formula 0_1 which corresponds to the operation OI
of §15, in the sense that, if G represents the Gödel number of
a combination of such a form that OI is not applicable to it,
then $0_1 G$ conv G, and if G represents the Gödel number of a
combination M to which OI is applicable, then $0_1 G$ is convert-
ible into the formula representing the Gödel number of the com-
bination obtained from M by applying OI.

    9) Formulas 0_2, 0_3, ..., 0_{38} which correspond respective-
ly to the operations OII, OIII, ..., OXXXVIII of §15, in the same
sense.

    By 14 III, a formula, cb, can be found which represents
an enumeration of the least set of formulas which contains 1 and
3 and is closed under the operation of forming $(\lambda ab \, . \, 2 \, \times$
nr $ab) X V$ from the formulas X, V. But if X, V represent the
Gödel numbers of combinations A, B respectively, then $(\lambda ab \, . \, 2$
$\times$ nr $ab) X V$ is convertible into the formula which represents the
Gödel number of AB. Hence the formula, cb, enumerates the
Gödel numbers of combinations containing no free variables, in
the sense that every formula representing such a Gödel number is
convertible into one of the formulas in the infinite sequence

        cb 1, cb 2, ...,

and every formula in this infinite sequence is convertible into
a formula representing such a Gödel number.

If now we let

$$\text{ncb} \longrightarrow \lambda n \;.\; \text{cb}\; (\Phi(\lambda x \;.\; \text{norm}\; (\text{cb}\; x))n),$$

then ncb enumerates, in the same sense, the Gödel numbers of combinations which belong to formulas in normal form and contain no free variables (cf. 10 II).

By 14 IV, a formula O can be found which represents an enumeration of the least set of formulas which contains I and is closed under each of the thirty-eight operations of forming $(\lambda ab.O_\beta(ab))X$ from the formula X ($\beta = 1, 2, \ldots, 38$). Let

$$\text{cnvt} \longrightarrow \lambda ab.Oba.$$

Then if G represents the Gödel number of a combination M, the formula, cnvt G, enumerates (again in the same sense as in the two preceding paragraphs) the Gödel numbers of combinations obtainable from M by conversion -- cf. §15.

Let

$$\text{nf} \longrightarrow \lambda n \;.\; \text{cnvt}\; n(\wp(\lambda x \;.\; \text{norm}\; (\text{cnvt}\; nx))1).$$

Then nf λ-defines the operation <u>normal form of</u> a formula, in the sense that (1) if G represents the Gödel number of a combination M, then nf G is convertible into the formula representing the Gödel number belonging to the normal form of M; and hence (2) if G represents the Gödel number belonging to a formula M, then nf G is convertible into the formula representing the Gödel number belonging to the normal form of M. If G represents the Gödel number of a combination (or belonging to a formula) which has no normal form, then nf G has no normal form (cf. 10 I).

Let i and s be the formulas representing the Gödel numbers belonging 1 and S respectively. Then the formulas

$$Z'(H(1(\lambda x \;.\; 2 \times \text{nr}\,sx)i)),\; Z'(H(2(\lambda x \;.\; 2 \times \text{nr}\; sx)i)),$$

$$Z'(H(3(\lambda x \;.\; 2 \times \text{nr}\; sx)i)),\; \ldots,$$

are convertible respectively into formulas representing Gödel numbers belonging to

$$1,\; S1,\; S(S1),\; \ldots \;.$$

Hence a formula ν which λ-defines the property of a combination of belonging to a formula in normal form which represents a

positive integer, may be obtained by defining:

$\nu \longrightarrow \lambda n \ . \ \pi(\text{eq } n)(\lambda m \ . \ \text{eq } n(\text{nf}(Z'(H(m(\lambda x \ . \ 2 \ ^{\times} \ \text{nr} \ \mathbf{s}x)\mathbf{i}))))))$.

(It is necessary, in order to see this, to refer to 10 III, and
to observe that the Gödel number belonging to a formula in nor-
mal form representing a positive integer is always greater than
that positive integer.)

Chapter V

THE CALCULI OF λ-*K*-CONVERSION AND λ-δ-CONVERSION

17. THE CALCULUS OF λ-*K*-CONVERSION. The calculus of λ-*K*-conversion is obtained if a single change is made in the construction of the calculus of λ-conversion which appears in §§ 5,6: namely, in the definition of well-formed formula (§5) deleting the words "and contains at least one free occurrence of **x**" from the rule 3. The rules of conversion, I, II, III, in §6 remain unchanged, except that well-formed is understood in the new sense.

Typical of the difference between the calculi of λ-conversion and λ-*K*-conversion is the possibility of defining in the latter the constancy function,

$$K \longrightarrow \lambda a(\lambda ba),$$

and the integer zero, by analogy with definitions of the positive integers in §8,

$$0 \longrightarrow \lambda a(\lambda bb).$$

Many of the theorems of §7 hold also in the calculus of λ-*K*-conversion. But obvious minor modifications must be made in 7 III and 7 V, and the following theorems fail:· 7 XVII, clause (3) of 7 XXVI, and 7 XXXI, and 7 XXXII. Instead of 7 XXXI, the following weaker theorem can be proved, which is sufficient for certain purposes, in particular for the definition of $\mathfrak{p}$ (see §10):

17 I. Let a reduction be called of order one if the application
   of Rule II involved is a contraction of the initial
   $(\lambda \mathbf{x} M) N_1$ in a formula of the form

$$(\lambda \mathbf{x} M) N_1 N_2 \ldots N_r \qquad (r = 1,\ 2,\ \ldots).$$

Then if A has a normal form, there is a number m
such that at most m reductions of order one can oc-
cur in a sequence of reductions on A.

A notion of λ-K-definability of functions of non-negative
integers may be introduced, analogous to that of λ-definability
of functions of positive integers, and the developments of Chap-
ter III may then be completely paralleled in the calculus of λ-
K-conversion. The same definitions may be employed for the suc-
cessor function and for addition and multiplication as in Chap-
ter III. Many of the developments are simplified by the pres-
ence of the zero: in particular, ordered pairs may be employed
instead of ordered triads in the definition of the predecessor
function, and the definition of ℘ may be simplified as in Tur-
ing [58].

It can be proved (see Kleene [37], Turing [57]) that a
function F of one non-negative integer argument is λ-K-defin-
able if and only if $\lambda x . F(x-1)+1$ is λ-definable -- and sim-
ilarly for functions of more than one argument.

The calculus of λ-K-conversion has obvious advantages over
the calculus of λ-conversion, including the possibility of de-
fining the constancy function and of introducing the integer
zero in a simpler and more natural way. However, for many pur-
poses -- in particular for the development of a system of sym-
bolic logic such as that sketched in §21 below -- these advan-
tages are more than offset by the failure of 7 XXXII. Indeed
if we regard those and only those formulas as meaningful which
have a normal form, it becomes clearly unreasonable that FN
should have a normal form and N have no normal form (as may
happen in the calculus of λ-K-conversion); or even if we impose
a more stringent condition of meaningfulness, Rule III of the
calculus of λ-K-conversion can be objected to on the ground that
if M is a meaningful formula containing no free variables, the
substitution of $(\lambda x M)N$ for M ought not to be possible unless
N is meaningful. This way of putting the matter involves the
meanings of the formulas, and thus an appeal to intuition, but
corresponding difficulties do appear in the formal developments
in certain directions.

§18. THE CALCULUS OF RESTRICTED λ-K-CONVERSION. In order
to avoid the difficulty just described, Bernays [4] has proposed
a modification of the calculus of λ-K-conversion which consists
in adding to Rules II and III the proviso that *N* shall be in
normal form (notice that the condition of being in normal form
is effective, although that of having a normal form is not). We
shall call the calculus so obtained the calculus of restricted
λ-K-conversion. In it, as follows by the methods of §7, a for-
mula which in the calculus of λ-K-conversion had a normal form
and had no parts without normal form will continue to have the
same normal form; in particular, no possibility of conversion
into a normal form is lost which existed in the calculus of λ-
conversion. On the other hand, all of the theorems 7 XXVIII -
7 XXXII remain valid in the calculus of restricted λ-K-conver-
sion -- and are much more simply proved than in the calculus of
λ-conversion. (It should be added that the content of the the-
orems 7 XXVIII - 7XXXXII for the calculus of restricted λ-K-con-
version is in a certain sense much less than the content of these
theorems for the calculus of λ-conversion, and in fact cannot be
regarded as sufficient to establish the satisfactoriness of the
calculus of restricted λ-K-conversion from an intuitive viewpoint
without addition of such a theorem as that asserting the equiva-
lence to the calculus of (unrestricted) λ-K-conversion in the
case of formulas all of whose parts have normal forms.)

The development of the calculus of restricted λ-K-conversion
may follow closely that of the calculus of λ-conversion (as in
Chapters II-IV), with such modifications as are indicated in §17
for the calculus of λ-K-conversion. Many of the theorems must
have added hypotheses asserting that certain of the formulas in-
volved have normal forms.

§19. TRANSFINITE ORDINALS. Church and Kleene [15] have ex-
tended the concept of λ-definability to ordinal numbers of the
second number class and functions of such ordinal numbers.
There results from this on the one hand an extension of the no-
tion of effective calculability to the second number class (cf.
Church [13], Kleene [39], Turing [59]), and on the other hand a
method of introducing some theory of ordinal numbers into the
system of symbolic logic of §21 below.

Instead of reproducing here this development within the calculus of λ-conversion, we sketch briefly an analogous development within the calculus of restricted λ-K-conversion.

According to the idea underlying the definitions of §8, the positive integers (or the non-negative integers) are certain functions of functions, namely the finite powers of a function in the sense of iteration. This idea might be extended to the ordinal numbers of the second number class by allowing them to correspond in the same way to the transfinite powers of a function, provided that we first fixed upon a limiting process relative to which the transfinite powers should be taken. Thus the ordinal ω could be taken as the function whose value for a function f as argument is the function g such that gx is the limit of the sequence, $x,\ fx,\ f(fx),\ \dots$. Then $\omega+1$ would be $\lambda x.f(\omega f x)$, and so on.

Or, instead of fixing upon a limiting process, we may introduce the limiting process as an additional argument a (for instance taking the ordinal ω to be the function whose value for a and f as arguments is the function g such that gx is the limit of the sequence $x,\ fx,\ f(fx),\ \dots$, relative to the limiting process a). This leads to the following definitions in the calculus of restricted λ-K-conversion, the subscript o being used to distinguish these notations from similar notations used in other connections:

$$0_o \;\rightarrow\; \lambda a(\lambda b(\lambda cc)),$$

$$1_o \;\rightarrow\; \lambda abc.bc,$$

$$2_o \;\rightarrow\; \lambda abc.b(bc), \quad \text{and so on.}$$

$$S_o \;\rightarrow\; \lambda dabc.b(dabc).$$

$$L_o \;\rightarrow\; \lambda rabc.a(\lambda d.rdabc).$$

$$\omega_o \;\rightarrow\; \lambda abc.a(\lambda d.dabc).$$

We prescribe that 0_o shall represent the ordinal 0; if N represents the ordinal n, the principal normal form of $S_o N$ shall represent the ordinal $n+1$; if R represents the monotone increasing infinite sequence of ordinals, $n_0,\ n_1,\ n_2,\ \dots$, in the sense that $R0_o,\ R1_o,\ R2_o,\ \dots$ are convertible into formulas representing $n_0,\ n_1,\ n_2,\ \dots$, respectively, then the

principal normal form of L_oR shall represent the upper limit
of this infinite sequence of ordinals. The transfinite ordinals
which are represented by formulas then turn out to constitute
a certain segment of the second number class, which may be de-
scribed as consisting of those ordinals which can be effectively
built up to from below (in a sense which we do not make explicit
here).

    The formula representing a given ordinal of the second num-
ber class is not unique: for example, the ordinal ω is repre-
sented not only by $ω_o$ but also by the principal normal form of
L_oS_o, and by many other formulas. Hence the formulas represent-
ing ordinals are not to be taken as denoting ordinals but rather
as denoting certain things which are in many-one correspondence
with ordinals.

    A function F of ordinal numbers is said to be λ-K-defined
by a formula F if (1) whenever $Fm = n$ and M represents m,
the formula FM is convertible into a formula representing n,
and (2) whenever an ordinal m is not in the range of F and
M represents m, the formula FM has no normal form.

    The foregoing account presupposes the classical second num-
ber class. By suitable modifications (cf. Church [13]), this
presupposition may be eliminated, with the result that the cal-
culus of restricted λ-K-conversion is used to obtain a defini-
tion of a (non-classical) constructive second number class, in
which each classical ordinal is represented, if at all, by an
infinity of elements.

    20. THE CALCULUS OF λ-δ-CONVERSION. The calculus of λ-δ-
conversion is obtained by making the following changes in the
construction of the calculus of λ-conversion which appears in
§§5, 6: adding to the list of primitive symbols a symbol δ,
which is neither an improper symbol nor a variable, but is classed
with the variables as a proper symbol; adding to the rule 1 in
the definition of well-formed formula that the symbol δ is a
well-formed formula; and adding to the rules of conversion in
§6 four additional rules, as follows:

 IV. To replace any part δMN of a formula by 1, provided
      that M and N are in δ-normal form and contain no

free variables and M is not convertible-I into N.

V.   To replace any part 1 of a formula by δMN, provided
that M and N are in δ-normal form and contain no
free variables and M is not convertible-I into N.

VI.   To replace any part δMM of a formula by 2, provided
that M is in δ-normal form and contains no free va-
riables.

VII.   To replace any part 2 of a formula by δMM, provided
that M is in δ-normal form and contains no free va-
riables.

Here a formula is said to be in δ-normal form if it con-
tains no part of the form $(\lambda x P)Q$ and contains no part of the
form δRS with R and S containing no free variables. It is
necessary to observe that both the condition of being in δ-nor-
mal form and the condition that M is not convertible-I into
N are effective.

A conversion (or a λ-δ-conversion) is a finite sequence of
applications of Rules I-VII. A λ-δ-conversion is called a re-
duction (or a λ-δ-reduction) if it contains no application of
Rules III, V, VII and exactly one application of one of the Rules
II, IV, VI. A is said to be immediately reducible to B if
there is a reduction of A into B, and A is said to be re-
ducible to B if there is a conversion of A into B which
consists of one or more successive reductions.

All the theorems of §7 hold also in the calculus of λ-δ-
conversion, if some appropriate modifications are made (see Church
and Rosser [16]). The residuals of $(\lambda x_p M_p)N_p$ after an appli-
cation of Rule I or II are defined in the same way as before, and
after an application of IV or VI they are defined as what $(\lambda x_p$
$M_p)N_p$ becomes (this is always something of the form $(\lambda x_p M_p')N_p'$).
The residuals of $\delta M_p N_p$ after an application of I, II, IV, or
VI are defined only in the case that M_p and N_p are in δ-
normal form and contain no free variables. In that case the re-
siduals of $\delta M_p N_p$ are whatever part or parts of the entire re-
sulting formula $\delta M_p N_p$ becomes, except that after an application
of IV or VI in which $\delta M_p N_p$ itself is contracted (i.e., replaced
by 1 or 2), $\delta M_p N_p$ has no residual. Thus residuals of $\delta M_p N_p$

are always of the form $\delta M N$, where M and N are in δ-normal form and contain no free variables. A sequence of contractions on a set of parts $(\lambda x_j M_j) N_j$ and $\delta R_1 S_1$ of A_1, where R_1 and S_1 are in δ-normal form and contain no free variables, is defined by analogy with the definition in §7. Similarly a terminating sequence of such contractions. In 7 XXV, the set of parts of A on which a sequence of contractions is taken is allowed to include not only parts of the form $(\lambda x_j M_j) N_j$, but also parts of the form $\delta R_1 S_1$ in which R_1 and S_1 are in δ-normal form and contain no free variables. The modified 7 XXV may then be proved by an obvious extension of the proof given in §7, and thereupon 7 XXVI - 7 XXXII follow as before. In 7 XXVI - 7 XXXII "conv-I-II" must be replaced throughout by "conv-I-II-IV-VI" and in 7 XXVI the case must also be considered that A imr B by a contraction of the part $\delta M N$ of A. For 7 XXX, there must be supplied a definition of principal δ-normal form of a formula, analogous to the definition in §6 of the principal (λ-)normal form.

In connection with the calculus of λ-δ-conversion we shall use both of the terms λ-conversion and λ-δ-conversion, the former meaning a finite sequence of applications of Rules I-III, the latter a finite sequence of applications of Rules I-VII. The term conversion will be used to mean a λ-δ-conversion, as already explained.

Similarly we shall use both of the terms λ-normal form of a formula and δ-normal form of a formula. A formula will be called a λ-normal form of another if it is in λ-normal form and can be obtained from the other by λ-conversion. A formula will be called a δ-normal form of another if it is in δ-normal form and can be obtained from the other by λ-δ-conversion. By 7 XXIX applied to the calculus of λ-conversion, the λ-normal form of a formula (in the calculus of λ-δ-conversion), if it exists, is unique to within applications of Rule I. By the analogue of 7 XXIX for the calculus of λ-δ-conversion, the δ-normal form of a formula, if it exists, is unique to within applications of Rule I.

In order to see that the calculus of λ-δ-conversion requires an intensional interpretation (cf. §2), it is sufficient to observe that, for example, although 1 and $\lambda ab.\delta ab1ab$ correspond

to the same function in extension, they are nevertheless not in-
terchangeable, since δ11 conv 2 but δ1(λab.δab1ab) conv 1.

A constancy function κ may be defined:

$$\kappa \longrightarrow \lambda ab.\delta bb\,Ia.$$

Then κAB conv A, if B has a δ-normal form and contains no
free variables, and in that case only (the conversion properties
of κ are thus weaker than those of the formula K in either
of the calculi of λ-K-conversion).

The entire theory of λ-definability of functions of posi-
tive integers carries over into the calculus of λ-δ-conversion,
since the calculus of λ-conversion is contained in that of λ-
δ-conversion as a part. It only requires proof that the notion
of λ-δ-definability of functions of positive integers is not
more general than that of λ-definability, and this can be sup-
plied by known methods (e.g., those of Kleene [37]).

The theory of combinations carries over into the calculus
of λ-δ-conversion, provided that we redefine a <u>combination</u> to
mean an {I, J, δ}-combination. In defining the <u>combination be-
longing to</u> a formula, it is necessary to add the provision that
the combination belonging to δ is δ.

If A_1 is a well-formed formula of the calculus of λ-δ-
conversion and contains no free variables, a formula B_1 can
be found such that B_1J conv A_1 and B_1I conv I. For let A_1'
be the combination belonging to A_1, unless that combination
fails to contain an occurrence of either J or δ, in which
case let A_1' be JIII. Let A_1'' be obtained from A_1' be re-
placing J and δ throughout by j and δIj(λx.x(λy.yIII))
(λz.zI)δ respectively. Then B_1 may be taken as $\lambda jA_1''$.

Hence 14 I, and the remaining theorems of §14, may be proved
for the calculus of λ-δ-conversion in the same way as for the
calculus of λ-conversion.

In order to obtain a combinatory equivalent of λ-δ-conver-
sion, analogous to the combinatory equivalent of λ-conversion
given in §15, it is necessary to add to OI-OXXXVIII the follow-
ing four additional operations -- where F, A, B, C are combi-
nations, and A and B belong to formulas in δ-normal form,
contain no free variables, and are not the same, and C belongs
to the formula which represents the Gödel number of A:

OXXXIX. $F(\delta AB) \vdash F(\beta I)$.

OXL. $FAB(\beta I) \vdash FAB(\delta AB)$.

OXLI. $F(\delta AA) \vdash F(\omega\beta)$.

OXLII. $FC(\omega\beta) \vdash FC(\delta AA)$.

The reader should verify that the conditions on **A, B, C** -- al-
though complex in character -- are effective (§6).

    In order to see that these four operations are equivalent,
in the presence of OI - OXXXVIII, to the rules of conversion IV
- VII, it is necessary to observe that βI and $\omega\beta$ are λ-con-
vertible into 1 and 2 respectively.

    To show that OXLII provides an equivalent to Rule VII, we
must show that it enables us to change $C(\omega\beta)$ into $C(\delta AA)$.
Since OI - OXXXVIII are equivalent to λ-conversion, this can be
done as follows: $C(\omega\beta)$ is λ-convertible into $\gamma(\tau I)CC(\omega\beta)$,
and this becomes, by OXLII, $\gamma(\tau I)CC(\delta AA)$, and this in turn is
λ-convertible into $C(\delta AA)$.

    Similarly, to show that OXL provides an equivalent of Rule
V, we must show that it enables us to change $C(\beta I)$ into $C(\delta AB)$.
This can be done as follows: $C(\beta I)$ is λ-convertible into
$\gamma(\gamma(\tau I)C)(\beta I)(\omega\beta)$; and this can be changed by the method of the
preceding paragraph into $\gamma(\gamma(\tau I)C)(\beta I)(\delta BB)$; and this is λ-
convertible into $\gamma(\gamma(\tau I)(\gamma(\gamma(\tau I)C)(\beta I)))(\delta BB)(\omega\beta)$; and this can
be changed by the method of the preceding paragraph into $\gamma(\gamma(\tau I)$
$(\gamma(\gamma(\tau I)C)(\beta I)))(\delta BB)(\delta AA)$; and this is λ-convertible into
$\gamma(\beta(\gamma(\beta\gamma(\gamma(\gamma(\beta(\beta\beta)(\omega\delta))I)(\gamma(\gamma(\tau I)C)))))(\omega\delta))AB(\beta I)$; and this be-
comes, by OXL, $\gamma(\beta(\gamma(\beta\gamma(\gamma(\gamma(\beta(\beta\beta)(\omega\delta))I)(\gamma(\gamma(\tau I)C)))))(\omega\delta))AB$
(δAB); and this is λ-convertible into $\gamma(\gamma(\tau I)(\gamma(\gamma(\tau I)C)(\delta AB)))$
$(\delta BB)(\delta AA)$; and this becomes, by OXLI, $\gamma(\gamma(\tau I)(\gamma(\gamma(\tau I)C)(\delta AB)))$
$(\delta BB)(\omega\beta)$; and this is λ-convertible into $\gamma(\gamma(\tau I)C)(\delta AB)(\delta BB)$;
and this becomes, by OXLI, $\gamma(\gamma(\tau I)C)(\delta AB)(\omega\beta)$; and this, fi-
nally, is λ-convertible into $C(\delta AB)$.

    Only minor modifications are necessary in §16 in order to
carry over its results to the calculus of λ-δ-conversion. In
the definition of the Gödel number of a combination the clause
must be added: (2a) The Gödel number of δ is 5. In the con-
struction of the formula, form, it is only necessary to impose
on η the further condition that ϐ 5 conv λx.x12δ, so insuring
that form 5 conv δ. The construction of o remains unchanged.
The formulas occ, ε, ℭ, r, Λ, prim, norm, and O_1 - O_{38} may then

be obtained, having the properties described in §16 (norm λ-de-
fines the property of a combination of belonging to a formula
which is in λ-normal form). The formulas cb, ncb, O, cnvt, nf
(the λ-normal form of), and ν may then also be obtained as be-
fore. The formula, cb, represents an enumeration of the least
set of formulas which contains 1, 3, and 5 and is closed under
the operation of forming $(\lambda ab \cdot 2 \times nr\ ab)XY$ from the formulas
X, Y.

Besides norm it is also possible to obtain a formula, dnorm,
which λ-defines the property of a combination of belonging to a
formula in δ-normal form. Details of this are left to the read-
er.

Formulas O_{39}- O_{42} may be obtained, related to the operations
OXXXIX - OXLII in the same way that O_1 - O_{38} are related to OI -
OXXXVIII. We give details in the case of O_{40} and O_{42}. Let $\mathcal{F}_{40}$
be a formula such that $\mathcal{F}_{40}1$ conv *I* and $\mathcal{F}_{40}2$ conv $\lambda x \cdot 2 \times$
nr $x_1[2 \times nr[2 \times nr\ 5x_{112}]x_{12}]$; then let

$$O_{40} \longrightarrow \lambda x \cdot \mathcal{F}_{40}[par\ x + par\ x_1 + par\ x_{11} + prim\ x_{112}$$

$$+\ dnorm\ x_{112} + prim\ x_{12} + dnorm\ x_{12}$$

$$+\ eq\ \eta x_2 \doteq eq\ x_{112}x_{12} \doteq \overline{13}]x,$$

η being the formula representing the Gödel number of β*I*. Let
$\mathcal{F}_{42}$ be a formula such that $\mathcal{F}_{42}1$ conv *I* and $\mathcal{F}_{42}2$ conv $\lambda x \cdot 2$
$\times$ nr $x_1[2 \times nr\ [2 \times nr\ 5(form\ x_{12})](form\ x_{12})]$; then let

$$O_{42} \longrightarrow \lambda x \cdot \mathcal{F}_{42}[par\ x + par\ x_1 + h(\nu x_{12})x_{12} + eq\ \zeta x_2 \doteq 6]x,$$

where ζ is the formula representing the Gödel number of ωβ,
and h is such a formula that h1 conv $\lambda x.x1$ and h2 conv
$\lambda x.min\ (prim\ (form\ x))(dnorm\ (form\ x))$.

Then a formula, do, may be obtained, analogous to O but
involving all of O_1 - O_{42} instead of only O_1 - O_{38}. Let

$$dcnvt \longrightarrow \lambda ab \cdot do\ ba.$$

Then, if *G* represents the Gödel number of a combination *M*,
the formula, dcnvt *G,* enumerates the Gödel numbers of combina-
tions obtainable from *M* by λ-δ-conversion (whereas cnvt *G*

enumerates merely the Gödel numbers of combinations obtainable
from *M* by λ-conversion).

It is also possible, by using the formula, dnorm, to obtain
a formula, dnf, which λ-defines the operation δ-normal form of
a formula, and a formula, dncb, which enumerates the Gödel num-
bers of combinations which belong to formulas in δ-normal form
and contain no free variables. The definitions parallel those
of nf and ncb.

Finally, in the calculus of λ-δ-conversion, a formula, met,
may be obtained which provides a kind of inverse of the function,
form: if *M* is a formula which contains no free variables and
has a δ-normal form, then met *M* is convertible into the for-
mula representing the Gödel number belonging to the δ-normal
form of *M*. The definition is as follows:

$$met \longrightarrow \lambda x \ . \ dncb \ (\wp(\lambda n \ . \ \delta(form \ (dncb \ n))x)1).$$

21. A SYSTEM OF SYMBOLIC LOGIC. If we identify the truth
values, truth and falsehood, with the positive integers 2 and 1
respectively, we may base a system of symbolic logic on the cal-
culus of λ-δ-conversion. This system has one primitive formula
or axiom, namely the formula 2, and seven rules of inference,
namely the rules I - VII of λ-δ-conversion; the provable for-
mulas, or theses, of the system are the formulas which can be
derived from the formula 2 by sequences of applications of the
rules of inference. (As a matter of fact, the rules of inference
II, IV, VI are superfluous, in the sense that their omission
would not decrease the class of provable formulas, as follows
from 7 XXVII, or rather from the analogue of this theorem for
the calculus of λ-δ-conversion.)

The identification of the truth values, truth and falsehood,
with the positive integers 2 and 1 is, of course, artificial,
but apparently it gives rise to no actual formal difficulty. If
it be thought objectionable, the artificiality may be avoided by
a minor modification in the system, which consists in introduc-
ing a symbol ⊢ and writing ⊢2, instead of 2, as the primitive
formula; all the theses of the system will then be preceded by
the sign ⊢, which may be interpreted as asserting that that
which follows is equal to 2.

In this system of symbolic logic the fundamental operations of the propositional calculus -- negation, conjunction, disjunction -- may be introduced by the following definitions:

$$[\sim A] \longrightarrow \pi(\lambda a.a I(\delta 2A))(\lambda a.a I(\delta 1A)).$$

$$[A \& B] \longrightarrow 4 \doteq . \; [\sim A] + [\sim B].$$

$$[A \vee B] \longrightarrow \sim . \; [\sim A] \& [\sim B].$$

It follows from these definitions that $A \vee B$ cannot be a thesis unless either A or B is a thesis -- and this situation apparently cannot be altered by any suitable change in the definitions. Since this property is known to fail for classical systems of logic, e.g., that of Whitehead and Russell's Principia Mathematica, it is clear that the present system therefore differs from the classical systems in a direction which may be regarded as finitistic in character.

Functions of positive integers are of course represented in the system by the formulas λ-defining these functions, and properties of and relations between positive integers are represented by the formulas λ-defining the corresponding characteristic functions. The propositional function to be a positive integer is represented in the system as a formula N, defined as follows (referring to §§16, 20):

$$N \longrightarrow \lambda x.\nu(\text{met } x).$$

The general relation of equality or identity (in intension) is represented by δ.

An existential quantifier Σ may be introduced:

$$\iota \longrightarrow \lambda f \; . \; \text{form } (Z'(H(\text{dcnvt } \alpha(\mathfrak{p}(\lambda n \; . \; \delta f$$
$$(\text{form } (Z(H(\text{dcnvt } \alpha n))))))1)))),$$

where α is the formula representing the Gödel number belonging to the formula 2;

$$\Sigma \longrightarrow \lambda f.f(\iota f).$$

Here ι represents a general selection operator. Given a formula F; if there is any formula A such that FA conv 2, then ιF is one of the formulas A having this property; and in the contrary case ιF has no normal form. Consequently Σ repre-

sents an existential quantifier without a negation: ΣF conv 2
if there is a formula A such that FA conv 2, and in the con-
trary case ΣF has no normal form.

The operator ι should be compared with Hilbert's opera-
tor ϵ [31 and elsewhere], or, perhaps better, the η-operator
of Hilbert and Bernays [33]. The ι should be used with the
caution that the equivalence of propositional functions repre-
sented in the system by F and G need not imply the equality
of ιF and ιG.

The interpretation of ι as a selection operator and of
Σ as an existential quantifier depends on an identification of
formal provability in the system with truth. But this is justi-
fied by a completeness property which the system possesses: a
formula which is not provable, unless it is convertible into a
principal normal form other than 2 and hence is disprovable,
must have no normal form, and hence be meaningless.

For convenience in the further development of the system,
or for the sake of comparison with more usual notations, we may
introduce the abbreviations:

$$[\iota x M] \longrightarrow \iota(\lambda x M).$$

$$[\exists x M] \longrightarrow \Sigma(\lambda x M).$$

The problem of introducing universal quantifiers into the
system, or, equivalently, of introducing existential quantifiers
having a negation, is beyond the scope of the present treatise.
It follows by the methods of Gödel [27] that any universal quan-
tifier introduced by definition will have a certain character of
incompleteness; this is in effect the same incompleteness proper-
ty which, in accordance with the results of Gödel, almost any
consistent and satisfactorily adequate system of formal logic
must have, except that it here appears transferred from the realm
of provability to the realm of meaning of the quantifiers.

The consistency of the system of symbolic logic just out-
lined is a corollary of 7 XXX, or rather of the analogue of this
theorem for the calculus of λ-δ-conversion. This consistency
proof is of a strictly constructive or finitary nature.

(The failure in this system of the known paradoxes of set
theory depends, in some of the simpler cases, merely on the fact
that the formula which would otherwise lead to the paradox fails

to have a normal form. Thus, in the case of Russell's paradox,
we find that $(\lambda x.\sim(xx))(\lambda x.\sim(xx))$ has no normal form; and in
the case of Grelling's paradox concerning heterological words,
or, as we shall put it, concerning heterological Gödel numbers,
we find that $(\lambda x.\sim(\text{form } xx))(\text{met } (\lambda x.\sim(\text{form } xx)))$ has no nor-
mal form. In more complicated cases, where the expression of
the paradox requires a universal quantifier, the failure may de-
pend on the above indicated incompleteness property of the quan-
tifier.)

INDEX OF THE PRINCIPAL FORMULAS INTRODUCED BY DEFINITION

§5. I, $[M+N]$, $[M \times N]$, $[M^N]$.

§8.   1, 2, 3, ..., S, $[M+N]$, $[M \times N]$, $[M^N]$.

§9. $[M, N]$, $[L, M, N]$, 2_1, 2_2, 3_1, 3_2, 3_3, P, $[M \dot- N]$,
       min, max, par, K, Z, Z', nr.

§10.   exc, eq, ƥ, Ƥ, π.

§12. I, J, τ.

§13. B, C, W, T, D.

§15.   γ, β, ω.

§16.   form, o, occ, ϵ, ℭ, r, Λ, prim, norm, 0_1 - 0_{38}, cb,
       ncb, 0, cnvt, nf, ν.

§17. K, o.

§19. 0_0, 1_0, 2_0, ..., S_0, L_0, $ω_0$.

§20.   κ, form, o, occ, ϵ, ℭ, r, Λ, prim, norm, 0_1 - 0_{38},
       cb, ncb, 0, cnvt, nf, ν, dnorm, 0_{39} - 0_{42}, do, dcnvt,
       dnf, dncb, met.

§21. $[\sim A]$, $[A \& B]$, $[A \vee B]$, N, ι, Σ, $[\iota x M]$, $[\exists x M]$.

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CORRECTION AND ADDITIONS

Page 75, line 12. For "Jaques," read "Jacques."

On page 46 the amendment should also be taken into account
which is suggested by Rosser [109]. The following simpler ex-
pression for W is available:

$$W \text{conv } B(T(B(BDB)T))(BBT).$$

Hence replace line 9 on page 46 by this.

In §15, pages 49-51, the combinatory equivalent of conver-
sion which is given can be simplified by the method of Rosser
[110], and in particular the proof of the equivalence to con-
version can be greatly shortened. Details of this, including
the proof of equivalence, may be obtained from Rosser's paper;
and the formula 0 of §16, and the formula do of §20, may then
be modified correspondingly.

For a combinatory equivalent of λ-K-conversion, and also
of λ-K-conversion with the addition of a rule by which BI and
I are interchangeable, see [70] -- where Curry employs Rosser's
method in order to simplify his earlier treatments of the theory
of combinators (which are referred to at the end of §15).