MATH2370: Homework #2

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Solution

Homework 2

Problem 1

To show $\{e_1, e_2, ..., e_n\}$ is a basis of X', we must prove linear independence of the elements in this set and show these vectors span X'.

Assume we have
$\sum_{i=0}^{n} a_i e_i = 0 \implies (\sum_{i=0}^{n} a_i e_i)x_j = 0 \implies \sum_{i=0}^{n} a_i e_i x_j = 0 = a_j, \ \forall j.$

Note this is true for an arbitrary selection of j, and that the last expression equates to a_j due to the definition of e_i. (when $i = j$ within the sum, $e_j x_j = 1$ so $a_j(e_j x_j) = a_j$). Because this coefficient is forced to be 0, the basis has linear independence.

Take arbitrary map e where $ex_i = a_i \ \forall i$. Then $e = \sum_{j=1}^{n} a_j e_j$ since $e(x_i) = \sum_{j=1}^{n} a_j e_j(x_i) = a_i$. Since we can construct an arbitrary map from our basis elements, they span X'.

Problem 2

Take dim$X = n < \infty$. Let $S,\ T \in X'$. We must show $N_S = N_T \iff S = \lambda T,\ \lambda \neq 0$.
Assume $S = \lambda T$ for $\lambda \neq 0$.
$x \in N_S \iff Sx = 0 \iff \lambda Tx = 0 \iff Tx = 0 \iff x \in N_T$.

Now assume $N_S = N_T = N$. We find that dim$N_S =$ dim$N_T = n - 1$ by the Rank-Nullity Theorem. This is because S, T are non-zero linear functionals so the dimension of their rank is 1.
Let the basis for X be $\{x_1, x_2, ..., x_{n-1}, x_n\}$ where $x_1, x_2, ..., x_{n-1} \in N$ and $x_n \notin N$. Now take $Sx_n = s \neq 0$ and $Tx_n = t \neq 0$. Note that for any $x \in X$, $Sx = \lambda Tx$ where $\lambda = \frac{s}{t}$.

Problem 3

Let $T : X \to U$ be a linear map. Assume that T is injective; hence $\dim(N_T) = 0$. Then $\dim U$=$\dim X$=$\dim(N_T)$+$\dim(R_T)$=$\dim(R_T)$, which forces surjectivity of T. Assume now that T is surjective. Then $\dim U = \dim X = \dim(R_T) + \dim(N_T) = \dim U + \dim(N_T)$, which forces $N_T = 0$ and hence injectivity of T.

Problem 4

Observe first that if $y \in R_{T^2}$ then $y \in R_T$, since $y = T^2(x)$ implies that $y = T(T(x))$; hence $R_{T^2} \leq R_T$. From the dimensional assumption we thus conclude that $R_{T^2} = R_T$. It follows that T is a surjective map from R_T to $R_{T.}$ By Problem 3 above $T : R_T \to R_T$ is also necessarily injective. Hence $R_T \cap N_T = 0$.