Hydrostatic Stability Analysis of 2D Shapes

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1 Summary

I analysed the hydrostatic stability of 2D, uniformly-dense lamina using three different methods: minimum potential energy, GZ righting arm curves and metacentric loci generated from a buoyancy locus. All three of these methods provided predictions of the true equilibrium angle that the lamina would float at, points of vanishing stability and somehow characterised the behaviour of the lamina at angles in between. The software I developed defines the lamina as a composition of many triangles, so any (straight-edged) 2D lamina could be constructed and analysed.

2 Method

For all three methods of analysis, the relative locations of the object's centre of mass, centre of buoyancy and the water level need to be known for a given float angle. Below is a short description of how my code achieves this:

The centre of mass of the shape is defined as the origin. The water level is represented as a line tilted at (opposite to) the given angle. For any height of the centre of mass below the water, the area of the submerged section, as well its centroid (i.e. the centre of buoyancy), is calculated. The shape will float at the height where the weight of water it displaces matches its own weight. Thus the height the object floats at (at the given angle) is found using the newton method by minimising

$$\frac{A_{\text{shape submerged}}}{A_{\text{shape total}}} - \rho_{\text{relative}}$$

where ρ_{relative} is the relative density of the shape to the water and is between 0 and 1 (not inclusive).

2.1 Minimum Energy

The height of the centre of mass and centre of buoyancy for a given float angle are known. The potential energy of the system is calculated using

$$PE_{\text{total}} = PE_{\text{lamina}} - PE_{\text{displaced water}}$$

The potential energy can then be found for all angles, and the minimum of these will correspond to the angle the lamina floats at.

2.2 GZ Righting Arm Curves

The horizontal distance between the centre of mass and centre of buoyancy for a given angle is calculated:

$$x_{GZ} = x_{\text{centre of bouyancy}} - x_{\text{centre of mass of lamina}}$$

This is plotted against angle. Using this sign convention, stable points are at angles where the curve crosses upwards throught the x axis, and points of vanishing stability are at angles where the curve crosses downwards through the x axis.

2.3 Catastrophe Theory

The buoyancy locus is the line joining the centres of buoyancy for all angles. by drawing perpendicular lines to the buoyancy locus and finshing their intersection, the evolute of the buoyancy locus (also called metacentric locus) is found: this describes the stability of the lamina. If the centre of mass (here by definition at (0,0)) is below the evolute for a given angle, it remains 'stable': it will return to the equilibrium position within that region. Once the evolute passes below the centre of mass, the stable equilibrium position changes.

3 Results and Discussion

3.1 Minimum Energy

To demonstrate, a square with side lengths 2 was analysed.

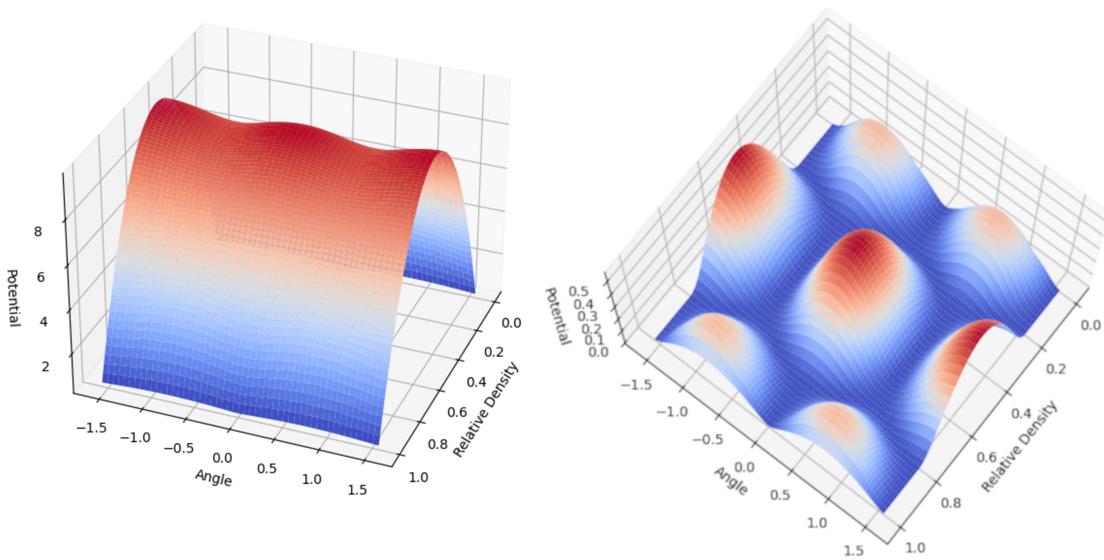

Figure 1: Left: The floating potential energy of the square for varying angles and relative densities. Right: The same data, but in order to aid visualisation, for each relative density the minimum potential energy value was subtracted from all the other values at that relative density. This pattern repeats every 90 degrees; between relative densities of 0 to approx 0.27 and approx 0.73 to 1, the square has a total of 4 stable angles (0,90,180,270 degrees relative to flat) and elswhere has 8 stable angles.

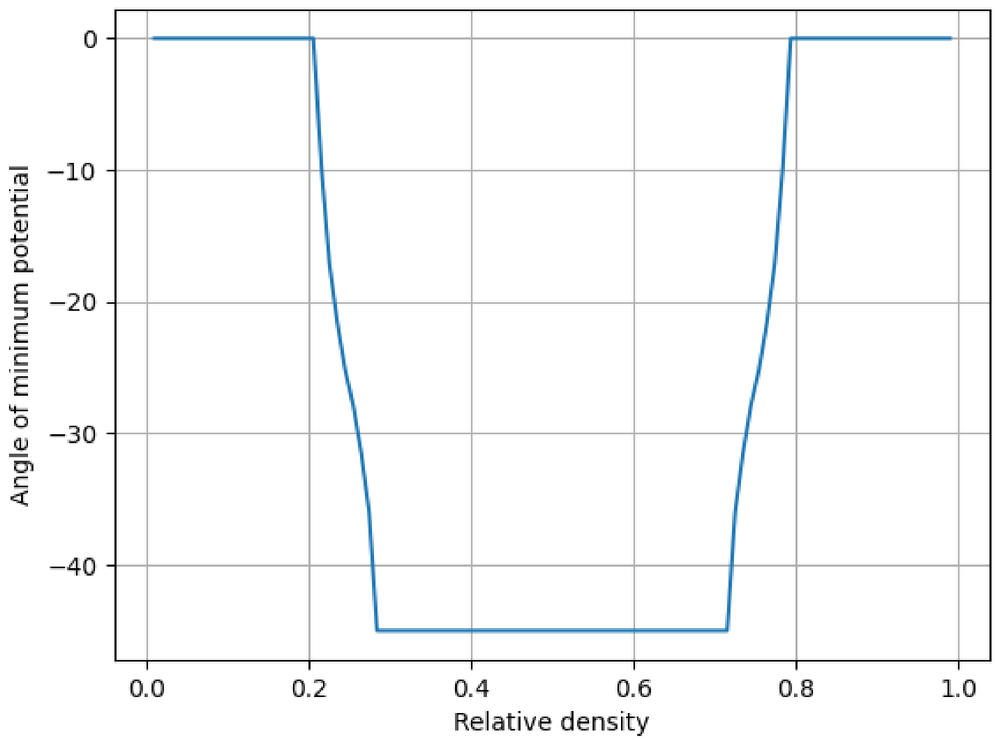

Figure 2: The stable equilibrium angle of the square for varying relative densities (i.e. the minima of 1).

I also tested the algorithm using a slightly more complicated shape, an anchor. 0 degrees was defined as the base (claw) of the anchor facing down.

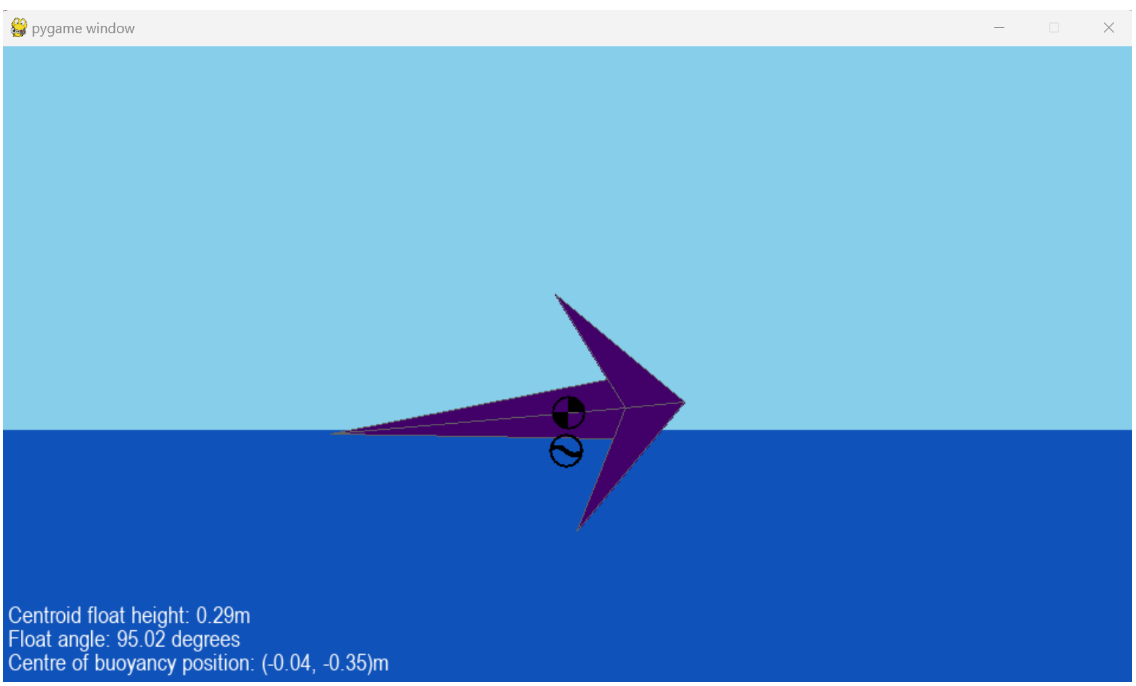

Figure 3: One of the stable equilibria of the anchor, with relative density 0.25.

3.2 GZ Righting Arm Curves

The square with side lengths 2 and anchor were also analysed using GZ righting arms.

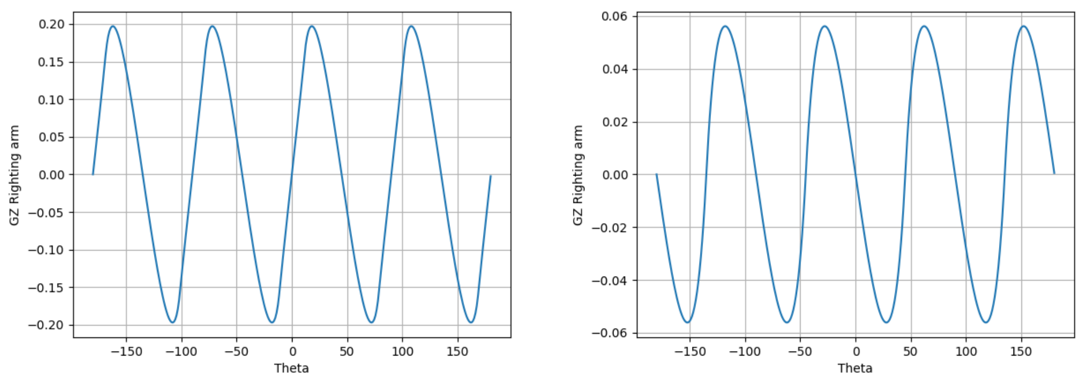

Figure 4: GZ curves of the square. Left: relative density 0.1. Notice stable equilibria at multiples of 90 degrees from zero, and points of vanishing stability at multiples of 90 degrees from 45 dgrees. Right: relative density 0.5. The points of stable equilibrium and vanishing stability have flipped.

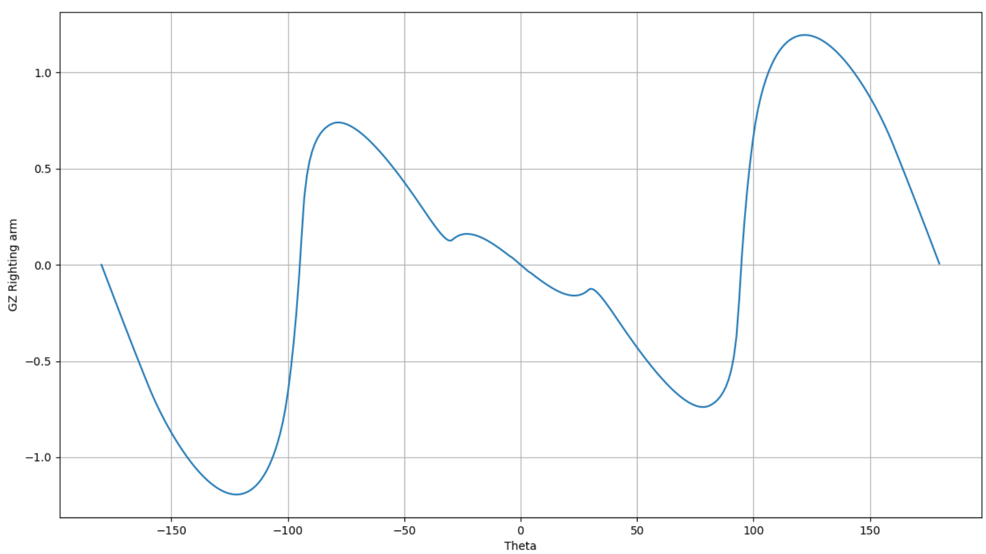

Figure 5: A GZ curve of the anchor, with relative density 0.25. Notice how the stable equilibrium in 3 is reflected here with an upward crossing of the x-axis at approx 95 degrees.

3.3 Catastrophe Theory

The square with side lengths 2 was analysed using catastrophe theory. Compare 6 with 2. At a relative density of 0.1, the square is well within a region whose equilibrium angle is at 0 degrees, which is reflected by the metacentric locus being well above the centre of mass at this angle. Note also how the 'tip' of the locus curve corresponds (by following the cyan line) to the equilibrium floating angle (0 degrees in this case), and following the two dark blue locus curves down from a tip until the next discontinuity describes the region of angles within which the square will return to that equilibrium angle (0 degrees here). At a relative density of 0.2, the equilibrium angle is still 0 degrees but the metacentric locus is much closer to the centre of mass. At relative density 0.25, the equilibrium angle has changed. Between -90 and 90 degrees you can see 4 different equilibrium states, each at about 25 degrees from horizontal and vertical. At relative densities of 0.3 and 0.5, the square has entered the region where the equilibrium angle is multiples of 90 degrees from 45 degrees.

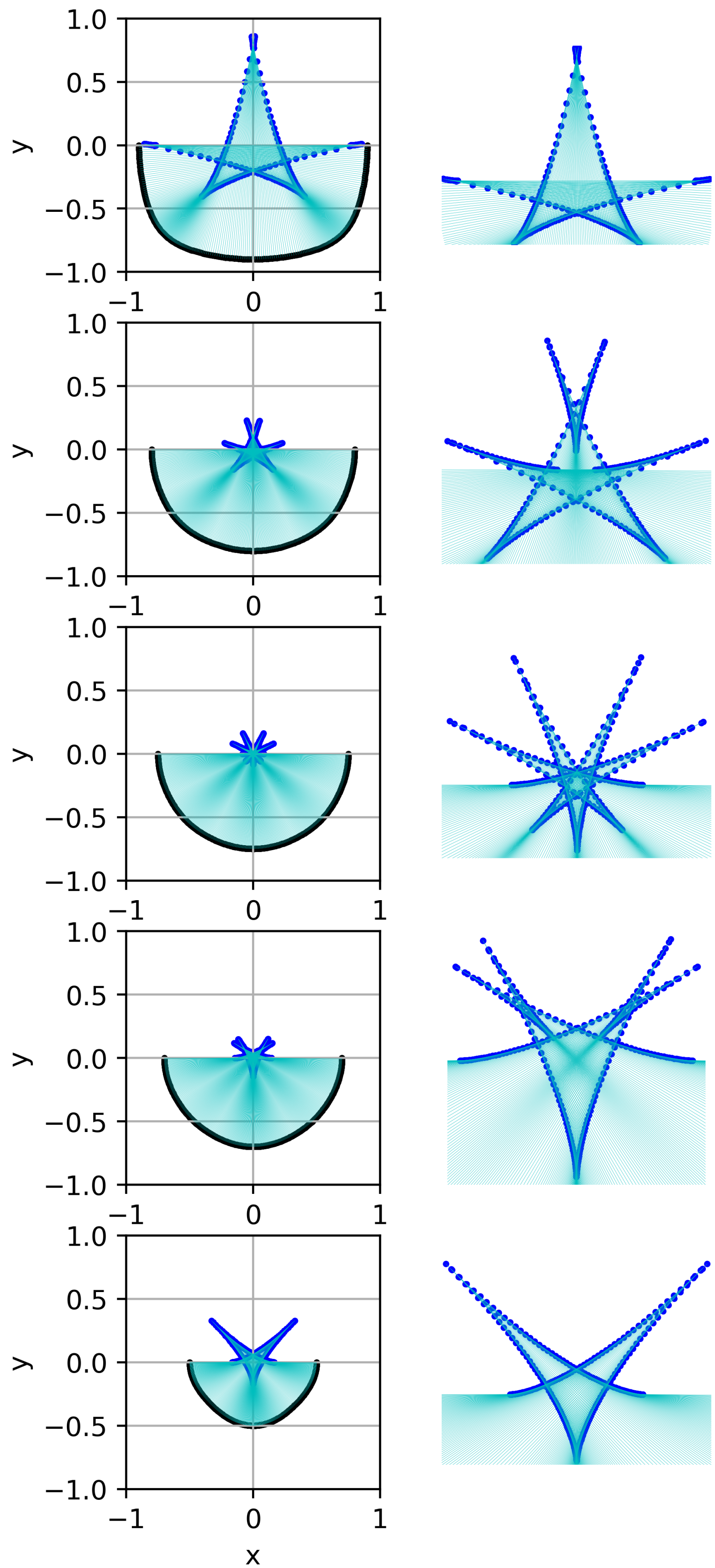

Figure 6: Buoyancy loci (black) and their evolutes, the metacentric loci (dark blue), along with the lines drawn to find the evolute (cyan). Top to bottom: relative densities 0.1, 0.2, 0.25, 0.3, 0.5. The right side has zoomed in on the metacentric loci to make them more readable.

4 Conclusions

All three methods of analysis allow points of stable equilibrium and of vanishing equilibrium to be found, as well as describing the behaviour of the object in between. Although catastrphe theory is very beautiful, GZ curves seem to me to be most practical in real use, both since it is easy to interpret the results (including the magnitude of the restoring force that brings the object back into equilibrium) as well as having potential to easily add other forces into the model, such as wind loading.

5 Appendix

All the code I used in this project is written in Python and can be found at `https://github.com/suspicious-salmon/hydrostability-playground`